Quantum Mechanics and Brownian/Stochastic Motion Part II

Francesco R. Ruggeri Hanwell, N.B. Jan. 4, 2021

In Part I of this note, we compared an approach using conditional single momentum probability P(p/x)= a(p)exp(ipx)/W(x), based on free particle conditional probability exp(ipx), together with conservation of average energy to an approach using stochastic equations presented in (1). A force equation is established in(1) with the usual dp/dt expression replaced with D p where D is an operator containing Brownian motion type terms i.e. grad.grad and p expressed in terms of derivatives of spatial density etc.

In this note, we continue the comparison of the two approaches and argue that a classical Brownian motion approach applied to spatial density (which is the probability P(x)) may be applied to relative conditional probability, which leads to a Schrodinger type equation.

Brownian Motion

In classical physics (2), an equation for Brownian motion is developed by considering changes in spatial density in a tiny interval of time. During this time, there are “Brownian” fluctuations in space based on a probability P(b)db where b=dx. A change in spatial density in dt is equated to Brownian changes in space i.e.

Density (t+dt) = density(t) + dt d density/ dt = Integral density (x+b,t) P(b) db

= density(t) + (d/dx d/dx density) Integral bb/2 P(b) db   ((1))

The linear term Integral b P(b) db =0 because there is equal probability for b and -b.

It seems the above approach may be applied to relative conditional probability as well as to full probability which both exist in quantum mechanics. Full probability is P(x)=spatial density, but conditional probability P(p/x) = a(p)exp(ipx)/W(x) where W(x) is the wavefunction and also relative conditional probability. If the potential V(x) is stochastic (and only V(x) on average), one may argue for a Brownian type of motion which creates a momentum distribution at each x point. Applying the Brownian motion arguments of ((1)) to W(x) instead of spatial density yields:

d/dt (partial) W(x,t) = D d/dx d/dx W(x,t)   ((2a)) for a free particle and

d/dt (partial) W(x,t) = D d/dx d/dx W(x,t) + V(x)W(x)   ((2b)) for a particle in a potential

Thus from ((2a), there seems to be Brownian motion even for a free particle in addition to the bound one which receives stochastic hits. Experimentally, free particle Brownian type behaviour manifests itself when a free particle interacts with the potential of a two slit experiment (for example). In other words, a free particle within a region with a potential has the same relative conditional probability exp(ipx) as one in a region with no potential, the difference being that there is a distribution of exp(ipx) in the region with a potential and a single p in the free region. It is almost as if the constant free momentum p is an average of fluctuations.

Furthermore, one may note that ((2a)) has two solutions, a forward moving p and a backward moving one, both with the same energy. There is only one type of solution, however, to ((2b)),. The Brownian motion term D d/dx /dx W must then include the two solutions from ((2a)). This motion, however, is linked to the stochastic hits from the potential V(x) which itself is interacting with W(x) as seen by the term V(x)W(x). This Brownian motion seems to be generated by V(x). Nevertheless, there is also an intrinsic Brownian motion expressed in ((2a)) which has no potential. There seems to be a coupling of two Brownian motions, one leading to exp(ipx), the second to a(p)'s in P(p/x).

This picture is very different from classical physics in which a bound particle has a certain time interval in which it moves forward and a separate interval in which it is moving backwards. The two solutions of ((2a)) at first appear to map to this, but they don't as Brownian motion in the bound state does not allow one to distinguish between separate forward and backward motion intervals. Thus, there is "interference" in terms of the two solutions of ((2a)).

The key difference here with usual stochastic theories is that the starting point is the relative conditional probability W(x) and not P(x) the spatial density as seens in ((2a)) and ((2b)).

A solution to dW/dt partial = D d/dx d/dx W is W(x)=exp(ipx-iEt) with D=-1/2m(-i)

Thus, a second issue arises in this treatment- namely the conservation of energy. In other words, in quantum mechanics, Brownian motion is not simply connected with a background stochastic field described by temperature. Here, there is an "intrinsic" Brownian motion governing the conditional probability in the free particle case, but for the bound state, a further Brownian motion is imposed by the potential V(x). This second Brownian motion imposes the values a(p) in P(p/x) = a(p)exp(ipx) / W(x) or W(x)=Sum over p a(p)exp(ipx). Using the idea of Brownian motion to even describe a particle moving in one direction allows one to see the periodic nature of quantum mechanics linked to momentum i.e. exp(ipx). This periodic nature arises from the form of ((2a)) which includes the idea of energy conservation. (It is not only the bound equation ((2b)) which makes use of this conservation.) So in the free quantum particle situation, it is Brownian motion together with conservation of energy which yields the exp(ipx) solution.

The modulus of exp(ipx) is 1 everywhere suggesting this is a statistical result (maximizing entropy), yet there is no entropy in the Brownian motion equation ((2a)). ((2a)), however, treats all points of space the same for the solution exp(ipx) and so one finds that it is a periodic equation that satisfies the equation under this condition.

Comparison with Stochastic Theory (1)

In (1), a stochastic theory is developed by defining two velocities v(x,t) and u(x,t) and combining these to form an overall momentum m[v(x,t)+ d u(x,t)] where d is a constant. In addition, a generalized d/dt operator is used which includes v.d/dx, ud/x and D d/dx d/dx terms, the last being a Brownian motion term. A Newton's second law equation is developed which in the simplest case has force= -d/dx V(x).

d/dt p (partial) + p.d/dx p +bD d/dx d/dx p = -d/dx V ((3)) with p=m[v(x,t)+ d u(x,t)]

The next step is to link v(x,t) and u(x,t) to spatial density. Thus, a priori, the relative conditional probability does not appear.

v(x,t) is found from solving: d/dt (partial) density + d/dx (v(x,t)density(x,t)) = 0 ((4))

and u(x,t) = D d/dx density / density ((5))

This leads (as shown in (1)) to: v(x,t)= iD(d/dx ln(W*) - d/dx(ln(W))) and u(x,t)=D(d/dx(lnW*) + d/dx(lnW)) ((6))

For exp(ipx) u(x,t)=0 and v(x,t)=-2DP (P=constant momentum). For a bound state: v(x,t)=0 and u(x,t)=2iD d/dx (lnW).

We note that: W(x+dx) = W(x) exp(i (-id/dx lnW)) with a similar expression holding for spatial density. Quantum mechanical bound states are usually characterized by humps so it is u(x,t) which causes them.

If one considers a Newton second law type equation, which should apply to exp(ipx), one finds:

d/dt (partial) v(x,t) + v.grad v + b D grad.grad v = 0 ((6))

v(x,t) = constant for exp(ipx) so ((6)) is 0=0 which does not demonstrate the motion of exp(ipx). Thus, we think it is clearer to have W(x)=exp(ipx) used directly in a Brownian motion equation.

Brownian Motion and Quantum Hump Behaviour

Quantum mechanical bound states are characterized by humps in W(x) which lead to humps in spatial density. These are already a direct consequence of the d/dx d/dx Brownian motion operator in the free quantum particle equation ((2a)). The conditional probability solution exp(ipx) exhibits humping features in two dimensions (cos(px), isin(px)) as it varies in value in space. Classical physics only deals with values at each (x,t) i.e. velocity, acceleration, so it is very different. It is as if a stochastic Brownian motion (together with a second Brownian motion if a potential exists) is required to create average motion which mimics classical physics. Thus, there is a third velocity not included in (1), namely an rms velocity which follows from Sum over p pp/2m P(p/x) for a bound quantum system which matches the classical velocity exactly.

If the above ideas about Brownian motion and quantum mechanics hold, it seems one should be able to create classical systems which vibrate classical particles in a Brownian motion manner, but still preserve an average conservation of energy law. Such classical systems should mimic quantum mechanics.(3)

Conclusion

In conclusion, we attempt to apply Brownian motion to relative conditional probability W(x) (wavefunction) instead of spatial density as usually done in the literature. We find that both the free quantum particle and quantum bound particle follow a Brownian-motion like equation which applies to conditional probability. Why should one use conditional probability? We argue that it contains p (momentum) and x which are the variables of interest. One may argue that p is constant for a free quantum particle, (but it may only be constant on average). A particle in a box with infinite potential walls has a solution C(exp(ipave x) + exp(ipave x))/2, but pave is only an average momentum. In addition, we have argued in a previous note (4), that W(x) satisfies a momentum hydrodynamical equation (instead of spatial density). Thus, we suggest that quantum relative conditional probability follows Brownian motion type equations combined with conservation of average energy (for both the free and bound quantum particle). The solutions in turn exhibit periodic or hump like behaviour characteristic to quantum mechanics.

References

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