

# Mathematical Philology in the Treatise on Double False Position in an Arabic Manuscript at Columbia University\*

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## Abstract

This article examines an Arabic mathematical manuscript at Columbia University's Rare Book and Manuscript Library (or. 45), focusing on a previously unpublished set of texts: the treatise on the mathematical method known as Double False Position, as supplemented by Jābir ibn Ibrāhīm al-Ṣābī (tenth century?), and the commentaries by Aḥmad ibn al-Sarī (d. 548/1153–4) and Sa'd al-Dīn As'ad ibn Sa'īd al-Hamadhānī (12th/13th century?), the latter previously unnoticed. The article sketches the contents of the manuscript, then offers an *editio princeps*, translation, and analysis of the treatise. It then considers how the Swiss historian of mathematics Heinrich Suter (1848–1922) read Jābir's treatise (as contained in a different manuscript) before concluding with my own proposal for how to go about reading this mathematical text: as a witness of multiple stages of a complex *textual* tradition of teaching, extending, and rethinking mathematics — that is, we should read it philologically.

“A woman dies, leaving her husband, a son, and three daughters.”<sup>1</sup> You are tasked with dividing the property among her heirs according to Islamic law. The rules in this case are that the husband inherits one quarter of the estate, and that in dividing what remains, the son's share is twice each daughter's share. Suppose her estate is 100 dinars. How much does the son inherit?

You have never studied basic algebra. Or if you have, pretend for a moment that you have not. You know how to add, subtract, multiply, and divide integers and even integral fractions. So to answer this question, you begin guessing. You guess that the son's share

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<sup>1</sup>The example is from al-Khwārizmī's *Algebra*, trans. Solomon Gandz, “The Algebra of Inheritance: A Rehabilitation of Al-Khuwarizmi,” *Osiris* 5 (1938): 327.

is 50. Since the husband's share is  $\frac{100}{4} = 25$ , and each daughter's share is half of the son's share, this would yield a total estate of  $25 + 50 + 3 \cdot \frac{50}{2} = 150$ . Too large. What about if the son's share is 20? Then the total estate would be  $25 + 20 + 3 \cdot \frac{20}{2} = 75$ . Too small. Maybe 40? But then the estate would be  $25 + 40 + 3 \cdot \frac{40}{2} = 125$ . Too large again. You could keep going, but you start to wonder if there is a better way to do this. Fortunately, you come upon a manuscript that includes a treatise that describes a method called "calculation by two errors" (*ḥisāb al-khaṭa'ayn*). You've already made three errors, so this seems promising. You read on.

The present article is about just such a manuscript and just such a treatise. The manuscript is New York, Columbia University, or. 45 (ca. thirteenth century).<sup>2</sup> The treatise is the *Explication of the Demonstration of Calculation by Two Errors, Improved Edition (iṣlāḥ)* by *Abū Sa'd Jābir ibn Ibrāhīm al-Ṣābī*. "Calculation by two errors," known in English as the method of Double False Position, appears as a minor chapter in the history of mathematics, especially when conceived as a linear history of progress from primitive problem-solving and limited understanding to sophisticated techniques and more complete theorems. Double False Position is a somewhat sophisticated technique but one that was at least at first glance entirely superseded by algebra.

To solve the above inheritance problem by Double False Position, we define  $x_1$  and  $x_2$  as the first and second guesses,  $y_1$  and  $y_2$  as the resulting outputs, where  $y = 100$  is the desired output. The aim is to find  $x$  such that operating on  $x$  as stipulated in the problem yields the desired output  $y$ . Based on the above calculations, we can assign these terms the following values:  $x_1 = 50$ ,  $y_1 = 150$ ;  $x_2 = 20$ ,  $y_2 = 75$ . We further define two "errors"  $e_1 = y_1 - y = 50$  and  $e_2 = y_2 - y = -25$ . Finally, we plug these values into the formula

$$x = \frac{x_1 e_2 - x_2 e_1}{e_2 - e_1} \quad (1)$$

in order to obtain

$$x = \frac{50 \cdot (-25) - 20 \cdot 50}{-25 - 50} = \frac{1250 + 1000}{75} = 30.$$

And indeed,  $25 + 30 + 3 \cdot \frac{30}{2} = 55 + 45 = 100$ , which was the deceased's total estate as stipulated in the problem. So the son's share is 30 dinars, making each daughter's share 15 dinars. Double False Position has allowed us to come to this conclusion without knowledge of algebra.

Once a systematic algebra of polynomials (albeit restricted to positive rational numbers and quadratic equations) had been developed by al-Khwārizmī in the ninth century,<sup>3</sup> one might even expect Double False Position to have been abandoned altogether as superfluous.

But it was not. Part of the reason must be that Double False Position can come in handy even if one knows algebra.<sup>4</sup> As Randy Schwartz has pointed out, not only is it an

<sup>2</sup>See Gūrgīs 'Awwād, "Al-Makhtūṭāt al-'arabiyyah fī dūr al-kutub al-Amīrikiyyah" [Arabic Manuscripts in American Libraries], *Sumer* 7 (1951): 262–63, esp. 262 (date, based on the script).

<sup>3</sup>Roshdi Rashed, *The Development of Arabic Mathematics: Between Arithmetic and Algebra* (Dordrecht: Kluwer, 1994), 8–21.

<sup>4</sup>Indeed, it still has a place in modern mathematics as a standard way to approximate solutions to equations whose algebraic solutions are unknown.

accessible method for a wide range of tradesmen with limited education to use; for some types of problems, it is in fact a quicker and simpler method than first expressing the problem as an algebraic equation and then solving for  $x$ . Schwartz's example is from a twelfth-century Latin treatise and can be summarized as follows: you carry some apples through three gates but at each gate must give up half of what you are carrying plus two apples to the gatekeeper; at the end you have one apple; how many did you start with? In the time that you will take to write out the algebraic expression corresponding to this problem, I can make two guesses, run them through the procedure, and have an answer from Double False Position.<sup>5</sup>

Indeed, the manuscript at the center of the present article contains not only a treatise on Double False Position with commentary but also Omar Khayyam's treatise on algebra, which built on al-Khwārizmī and his successors to produce a more systematic treatment that included cubic equations.<sup>6</sup> This perhaps surprising juxtaposition offers the opportunity for the present inquiry, which is primarily focused not on the history of mathematics but on the history of mathematical *philology*: the scholarly and textual practices used to preserve, communicate, and explore mathematics and its history.<sup>7</sup>

This article will move between several chronological layers. As its title makes clear, the treatise on Double False Position in question was written, at least in its current form, by Abū Sa'd Jābir ibn Ibrāhīm al-Šābī. In particular, it indicates that Jābir edited a pre-existing text to improve its clarity, fill in gaps in its reasoning or exposition, or standardize its technical vocabulary.<sup>8</sup> To judge from Jābir's name, he may be the son of the

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<sup>5</sup>Randy K. Schwartz, "Issues in the Origin and Development of Hisab al-Khata'ayn (Calculation by Double False Position)," in *Actes du huitième colloque maghrébin sur l'histoire des mathématiques arabes: Tunis, les 18-19-20 décembre 2004* (Tunis: Association Tunisienne des sciences mathématiques, 2004), 2–3. In this particular example, one might obtain the answer even faster by "working backwards":  $((1 + 2) \cdot 2 + 2) \cdot 2 + 2 = 36$ . But that is beside the point, which is that to solve this problem by *algebra* is exceedingly cumbersome compared to either of these numerical methods.

<sup>6</sup>Rashed, *Development of Arabic Mathematics*, 43–50.

<sup>7</sup>For definitions of philology — or *Zukunftphilologie*, to which this journal is devoted — and some of the stakes involved, see, e.g., Sheldon Pollock, "Philology and Freedom," *Philological Encounters* 1 (2016): 4–30; Sheldon Pollock et al., eds., *World Philology* (Cambridge, MA: Harvard University Press, 2015). Such a philology liberates the intellectual historian from the distorted extremes of a seamless but wholly anachronistic history of linear progress on the one hand and a disconnected string of solipsistic "original" texts on the other. In its place, a historical philology, by highlighting the iterated attempts to understand and *make present* with each new generation texts and ideas from the past, offers an intellectual history that emphasizes thinking subjects — whether author, reader, scribe, or commentator — and their interactions, between contemporaries and across time, potentially all the way up to our present day. Not only does such a philology seek to bridge the gap between historicism (meaning of a text when it was first composed) and presentism (what it means for me today). It also conceives of each thinking subject, at least potentially, as a fellow philologist, who likewise might have read texts — for his or her own purposes of course — as a historicist, as a presentist, and also, perhaps especially, with an eye to the intervening tradition.

<sup>8</sup>The title indicates that it is a treatise, whose author remains unnamed, that Jābir subsequently revised, producing a new, revised edition (*islāh*) of the text. As Mohammed Abbattouy has shown, the term *islāh* was typically used to describe the product of correcting, clarifying, and filling in the gaps in mathematical texts (often early translations from Greek into Arabic) that were faulty, unclear, or lacunose. This was often carried out by a scholar with technical, rather than linguistic, expertise, although *islāh* can also refer to revisions of a primarily stylistic or linguistic nature. See Mohammed Abbattouy, "La tradition arabe de *Maqāla fī al-mīzān*, un traité sur la théorie du levier attribué à Euclide," *Mirror of Heritage (Ayene-ye Miras): Quarterly Journal of Book Review, Bibliography and Text Information* (Tehran), n.s., 4, no. 4 (2007):

Sabian physician and mathematician Abū Ishāq Ibrāhīm ibn Sinān ibn Thābit ibn Qurra (909–946), grandson of the famous mathematician, physician, astronomer, and translator Thābit ibn Qurra (born in Ḥarrān; active in Baghdad; d. 901).<sup>9</sup> If this identification should prove correct, it would place Jābir in the tenth century. (Otherwise all we have is the *terminus ante quem* provided by the text’s commentators.) Interspersed with the original text is a commentary by the mathematician and philosopher Ibn al-Sarī (also known as Ibn al-Ṣalāḥ; from Hamadān; active in Baghdad; d. 1153–4),<sup>10</sup> as well as a brief note by one Sa’d al-Dīn al-Hamadhānī. Works by both of these commentators appear elsewhere in the manuscript. As discussed in the following, Ibn al-Sarī’s commentary points out a fatal flaw in Jābir’s geometrical proof of the validity of the method of Double False Position; al-Hamadhānī explains a single aspect of Ibn al-Sarī’s commentary.

In the early twentieth century, this treatise was studied by the Swiss teacher and historian of mathematics, Heinrich Suter (1848–1922).<sup>11</sup> Suter was primarily interested in the text as evidence that the method of Double False Position was known prior to the twelfth century, when it appears in Latin. His secondary interest was to evaluate the mathematical worth of the treatise; concurring with Ibn al-Sarī, he rated it quite low. (Suter subsequently gained access to a different Arabic treatise on Double False Position by the ninth/tenth-century Byzantine Christian scholar Qusṭā ibn Lūqā of Ba‘labakk. He concluded that it was of sufficient worth to merit being published in German translation.)<sup>12</sup>

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67–104, esp. §1. I owe this reference to an anonymous reviewer.

<sup>9</sup>Carl Brockelmann, *Geschichte der arabischen Litteratur*, 1st ed. (Weimar; Berlin: Emil Felber, 1898–1902), 1:219; Heinrich Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig: Teubner, 1900), 69 = no. 162; Fuat Sezgin, *Geschichte des arabischen Schrifttums*, 17 vols. (Leiden: Brill, 1967–2015), 5:254. For Abū Ishāq Ibrāhīm ibn Sinān, see Suter, *Die Mathematiker und Astronomen*, 53–54 = no. 113; Sezgin, *GAS*, 5:292–295. On the other hand, I am not aware of anything that would exclude identifying his father with other Sabians named Ibrāhīm, such as the well-known secretary of the Buyids Abū Ishāq Ibrāhīm ibn Hilāl ibn Ibrāhīm ibn Hārūn (925–994) or his grandfather. See Alexandre M. Roberts, “Being a Sabian at Court in Tenth-Century Baghdad,” *Journal of the American Oriental Society* 137, no. 2 (2017): 253–77; Sezgin, *GAS*, 5:314. Since the Sabians identified themselves as Abrahamic *ḥunafā’*, we might expect the name Ibrāhīm to be common among them.

<sup>10</sup>Suter, *Die Mathematiker und Astronomen*, 120 = no. 287; Theodosius, *Sphaerica: Arabic and Medieval Latin Translations*, ed. and trans. Paul Kunitzsch and Richard Lorch (Stuttgart: Steiner, 2010), 2 (introduction), mentioned because one of the manuscripts of Theodosios’s *Sphairika* says that it was copied from an exemplar that was copied from an exemplar in Ibn al-Sarī’s own hand.

<sup>11</sup>Heinrich Suter, “Einige geometrische Aufgaben bei arabischen Mathematikern,” *Bibliotheca Mathematica*, 3rd ser., 8 (1907–8): 23–36, esp. §1 = pp. 24–27; reprinted (with original pagination) in Heinrich Suter, *Beiträge zur Geschichte der Mathematik und Astronomie im Islam: Nachdruck seiner Schriften aus den Jahren 1892–1922*, ed. Fuat Sezgin, 2 vols. (Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe-Universität, 1986), 2:217–230. On Suter’s life and work, see Julius Ruska, “Heinrich Suter,” *Isis* 5, no. 2 (1923): 409–17.

<sup>12</sup>Heinrich Suter, “Die Abhandlung Qusṭā ben Lūqās und zwei andere anonyme über die Rechnung mit zwei Fehlern und mit der angenommenen Zahl,” *Bibliotheca Mathematica*, 3rd ser., 9 (1908–9): 111–22; reprinted in Suter, *Beiträge: Nachdruck*, 2:231–242. Suter points out a flaw (albeit a less fatal flaw) in this treatise as well, concluding that it is much closer to being a successful proof of Double False Position, such that Jābir cannot have based his treatise on it. — Qusṭā can plausibly be described as a Byzantine Christian because he was a Chalcedonian Christian in communion with the Byzantine church and a native speaker of Greek; see Maria Mavroudi, “Greek Language and Education under Early Islam,” chap. 11 in *Islamic Cultures, Islamic Contexts: Essays in Honor of Professor Patricia Crone*, ed. Behnam Sadeghi et al. (Leiden: Brill, 2014), 306, citing Ibn Abī Uṣaybi‘ah.

Now, in the twenty-first century, I revisit this treatise as a locus for understanding the aims and approaches of premodern mathematical writers, readers of mathematical manuscripts, and twentieth- and twenty-first-century historians of mathematics. First, I sketch the contents of the Columbia manuscript containing Jābir’s treatise (§I). Then I present an *editio princeps* (§II), translation (§III), and analysis (§IV) of the treatise. After considering Suter’s reading of this treatise (§V), I offer my own historical, mathematical, and philological reading of the text, its manuscript context, and its commentators (§VI), arguing for the value of such a mathematical treatise — fatally flawed but nevertheless copied and read — for intellectual history, not only the history of mathematics, but also of how mathematical texts were read.

## I The Columbia Manuscript

New York, Columbia University, or. 45 is a medieval codex containing a significant collection of mathematical texts. It includes texts on astronomy and engineering, but the primary focus is geometry. Most of its texts are in Arabic; one is in Persian. A paper codex, most of it was written by a single scribe (Scribe 1) in a neat *naskh* (with some Persian features such as the shape of initial *hāʾ*), typically with 19 lines per page (texts no. 2–18, on fols. 15<sup>v</sup>–128<sup>v</sup>). Text no. 1 was written by a different scribe (Scribe 2), with no dots, also typically with 19 lines per page (fols. 1<sup>v</sup>–14<sup>v</sup>). The last portion of the manuscript, apparently added later, was written much more informally, in less regular scripts: one hand (Scribe 3) wrote items nos. 19 and 21 with no margins (fols. 129<sup>r</sup>–137<sup>v</sup>, 144<sup>v</sup>–146<sup>r</sup>), and another (Scribe 4) wrote no. 20 with slight margins (139<sup>v</sup>–143<sup>r</sup>).

Two previous descriptions of the manuscript are known to me: a page devoted to the manuscript’s contents in Gūrgīs ‘Awwād’s catalog of Arabic manuscripts in American libraries,<sup>13</sup> and the series of typewritten cards in the unpublished card catalog of Arabic and other Islamicate manuscripts housed at Columbia’s Rare Book and Manuscript Library (RBML).<sup>14</sup> ‘Awwād’s list of the manuscript’s contents is more complete but still quite limited. In what follows, I present a much improved and elaborated description.

My aim here, I should note, is not simply to improve our bibliographic knowledge by supplementing Sezgin’s handbook on the evidence of this manuscript (that is happy side-effect). Instead, I include a description of the manuscript as an integral part of the project to construct a philology that treats manuscripts not merely as warehouses to be mined for the texts they contain but also as evidence for the intellectual milieu that produced and studied them. Like the anonymous treatise on Double False Position, Jābir’s attempt to improve it, Ibn al-Sarī’s critique of that attempt, and al-Hamadhānī’s brief gloss on that critique (and, we might be tempted to add, Suter’s account and critique of the whole assemblage), the Columbia manuscript represents one of many his-

<sup>13</sup>‘Awwād, “Dūr al-kutub al-Amrikiyyah,” 262–63.

<sup>14</sup>I am grateful to Jane Siegel, Rare Books Librarian at RBML, for introducing me to Columbia’s Arabic manuscript collection and pointing me to this card catalog when I first arrived at Columbia in the Fall of 2015. Since then I have benefited from her knowledge of the collection’s history. She kindly provided me with photographs of the manuscript so that I could continue to work on it after my departure from New York.

torical layers of interest in and engagement with a particular mathematical problem.<sup>15</sup> To interpret the manuscript as such, first we must read it.

‘Awwād dated the script to the seventh Hijrī century (thirteenth century CE). Modern bibliographical notes added to the manuscript itself place it in the thirteenth or fourteenth century CE. The manuscript includes Arabic translations of ancient Greek texts, but many of its texts date from the eleventh and especially twelfth centuries.

Two brief notes, one on the treatise on Double False Position, are ascribed to one Sa’d al-Dīn As’ad ibn Sa’id al-Hamadhānī. This may be someone associated with the production of this manuscript or of the collection it contains. The manuscript refers to him as one refers to an acquaintance: “the wise judge (*al-qāḍi al-ḥakīm*) Sa’d al-Dīn As’ad ibn Sa’id al-Hamadhānī, may God preserve his high rank.”

The Persian treatise (no. 8) and features of Scribe 1’s handwriting would tend to situate the manuscript’s production in Iran. Its apparently close connection to al-Hamadhānī, as well as the prominence of another author from the same city, Ibn al-Sarī, points to Hamadān as a possible place of the manuscript’s production and subsequent use.

[Scribe 2: first text only.]

1. Menelaos (fl. ca. 95 CE), *On Spherics* (*Kitāb M[ānālā]wus fī l-Kuriyyāt*, 1<sup>v</sup>–14<sup>v</sup>). Greek original not extant, but various Arabic versions survive.<sup>16</sup>

[15<sup>r</sup>: blank page with brief note.]

[Scribe 1: most of the manuscript.]

2. *Question asked by Shams al-Dīn, the Nizāmī chief emir, of Sharaf al-Dīn Bahā’ al-Islām Ḥujjat al-Zamān Muẓaffar ibn Muḥammad al-Muẓaffar al-Ṭūsī in Hamadān AH 50<6> [= 1112–13 CE] about Dividing a Square... (15<sup>v</sup>–18<sup>v</sup>).*<sup>17</sup>

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<sup>15</sup>It thus expands the possibilities of philology on “plane 2,” to use the terminology of Pollock, “Philology and Freedom,” 19–26.

<sup>16</sup>The author’s name is partly damaged, which prevented ‘Awwād from naming its author. The undamaged portion of the name (a *mīm*, the tops of two *alifs*, the tail of a *waw* or *rā’*, and a *sīn*) suggests that Menelaos is the author, and this is confirmed by the close similarity of the beginning of the text (as well as its diagrams) with the beginning of Menelaos’s text in *Kitāb Mānālāwus fī l-Ashkāl al-kuriyyah*, in the edition (*islāḥ*) of Aḥmad ibn Abī Sa’d al-Harawī (d. ca. 990–1000 CE), appearing after the editor’s preface and the words *qāla Mānālāwus al-muhandīs*, in Leiden, Univ. Library, or. 399, fol. 83<sup>r</sup> (available at <http://hdl.handle.net/1887.1/item:1567441>). Further research might establish which of the Arabic versions described by Sezgin (*GAS*, 5:161–163) is contained in the Columbia manuscript, e.g., whether it is al-Harawī’s edition stripped of its preface, or the otherwise nonextant edition of al-Māhānī upon which al-Harawī’s is based, or one of the others. — In response to this note, an anonymous reviewer kindly alerted me to the recent critical edition and English translation of this very text in *Menelaus’ “Spherics”: Early translation and al-Māhānī, al-Harawī’s Version*, ed. and trans. Roshdi Rashed and Athanase Papadopoulos (Berlin: de Gruyter, 2017), 399–483. The editors identify the text in the Columbia manuscript as the fragment of an early anonymous translation of Menelaos’s *Spherics*, contrasting it with al-Harawī’s *islāḥ* (and indeed with the original translation upon which al-Māhānī’s *islāḥ* was itself based), of which they also provide a critical edition and translation (pp. 485–777).

<sup>17</sup>*Mas’alah sa’alahā Shams al-Dīn amīr al-umarā’ al-Nizāmīyyah ‘an al-imām al-ajall al-awḥad al-‘ālim Sharaf al-Dīn Bahā’ al-Islām Ḥujjat al-Zamān Muẓaffar ibn Muḥammad al-Muẓaffar al-Ṭūsī... bi-balah Hamadhān sanata <sittah> wa-khamsa-mīah hijriyyah ‘an murabba’ mutasāwī al-aḍlā’*. The word *sittah* was omitted by the manuscript’s scribe by haplography (since the previous word, *sanah*, has the same consonantal skeleton). I supply it from the catalog entry of the Leiden manuscript that contains this

3. Ibn al-Haytham (354–430/965–1039), *On Deriving the Altitude of the Pole* (*Maqālah fī istikhraj irtifā' al-quṭb*, 18<sup>v</sup>–24<sup>r</sup>).<sup>18</sup>
4. Abū Rashīd (?), *Useful Shapes for the Book of Archimedes* (*Ashkāl nāfi'ah fī kitāb Arshimīdis*, 24<sup>r</sup>–25<sup>r</sup>).<sup>19</sup>
5. Archimedes (ca. 287–212f BCE), “ascribed,” *The Circle’s Area and the Relation of the Diameter to the Circumference* (*Qawl mansūb ilā Arshimīdhis fī misāḥat al-dā'irah wa-nisbat al-quṭr ilā l-muḥīt*, 25<sup>r</sup>–30<sup>v</sup>).<sup>20</sup>
6. Aristarchus (fl. 280 BCE), *On the Bulks of the Sun and Moon and Their Distances* (*Kitāb Aristā<r>khus fī jirmay al-shams wa-l-qamar wa-ab'ādihimā*, 30<sup>v</sup>–48<sup>v</sup>).<sup>21</sup>
7. 'Alī ibn Aḥmad al-Nasawī (11th century), *Book of the Brilliant concerning the Examples of the Comprehensive Tables* (*Kitāb al-Lāmi' fī amthilat al-Zij al-Jāmi'*, 49<sup>r</sup>–75<sup>v</sup>).<sup>22</sup> The text begins with a detailed preface describing the purpose of the work.<sup>23</sup>

same text (and was used by Suter): P. de Jong and M. J. de Goeje, eds., *Catalogus codicum Orientalium Bibliothecae Academiae Lugduno Batavae*, vol. 3 (Leiden: Brill, 1865), 71. Likewise, the word I have printed as *hujjah* is from the Leiden catalog; this word in the Columbia manuscript is spelled حح without vowels or diacritics. The final letter could be understood as a *tā'*, used (in Persian fashion) in place of a *tā' marbūṭah*.

<sup>18</sup>Sezgin, *GAS*, 5:358–374, esp. 366, work no. 5. Latin and German translations have appeared in print.

<sup>19</sup>Sezgin, 5:156, referring only to 'Awwād's description of this very manuscript. The text begins:

أشكال نافعة في كتاب أرشميدس لأبي الرشيد عبد [...] كل مربع متساوي الأضلاع في دائرة فهو أكبر من نصفها لأن ربع كل المربع الأعظم، أعني مثلث أهدج أصغر من ربع الدائرة.

It ends:

فهو أربعة أمثال الذي على نصف القطر، أعني أد، أعني أب، وإذا نقص مربع اب (كأ) من مربع أج، تبقى منه أربع أمثال  
[[و]] مربع أب، والله أعلم.

<sup>20</sup>Sezgin, 5:130–131, no. 2. Extant in Greek: *Κύκλου μέτρησης*, ed. J. Heiberg.

<sup>21</sup>Sezgin, 6:74–75. Part of the Alexandrian collection known as “the Little Astronomy.” Extant in Greek: *Περὶ μεγεθῶν καὶ ἀποστημάτων ἡλίου καὶ σελήνης*, ed./trans. Thomas Heath (Oxford, 1913).

<sup>22</sup>Sezgin, 6:245–246, no. 1, citing only 'Awwād's description of the Columbia manuscript. Note that this work is *not* entitled *Risālah fī ma'rifat al-taqwīm wa-l-aṣṭurlāb* as Sezgin, transcribing 'Awwād's description of the text, reports. There is no title at the beginning of the text, but al-Nasawī explicitly mentions the title he has given his work at the end of his preface (fol. 49<sup>v</sup><sub>6</sub>). Nor is the work primarily about calendars or astrolabes. Instead, it is a commentary on a set of astronomical tables by Abū l-Ḥasan Kūshyār ibn Labbān al-Jīlī (as al-Nasawī notes in his preface, fol. 49<sup>r</sup><sub>14</sub>), the *Comprehensive Tables* (*al-Zij al-Jāmi'*, compiled ca. 1000 CE); on which see Edward Stewart Kennedy, “A Survey of Islamic Astronomical Tables,” *Transactions of the American Philosophical Society* 46, no. 2 (1956): 125, no. 9, and further pp. 156–57 = §10; Sezgin, *GAS*, 6:246–248.

<sup>23</sup>Al-Nasawī begins his preface (fol. 49<sup>r</sup>) by dividing astral scientists (*'ulamā' al-nujūm*) into four cumulative levels (*tabaqāt*): those who know (1) calendars and the astrolabe; (2) basic astrology like planetary and zodiacal attributes and the astrological verdicts (*aḥkām*) that result from their combination — these he calls the “verdicticians” (*aḥkāmīyyūn*); (3) basic calculations of astral positions and use of astronomical tables and calendars; (4) mathematical astronomy (*hay'ah*) and geometrical proofs of the validity of such calculations — the province of the “complete astronomer” (*al-munajjim al-tāmm*). Most people of “our time,” continues al-Nasawī, only reach the first two levels. Now, [Abū 'Abdallāh Muḥammad ibn Sinān ibn Jābir al-Ṣābī] al-Battānī's astronomical tables (the *Sabian Tables*, ed. Nallino; see Kennedy, “Survey,” 132–33, no. 55) are rightly regarded as the most accurate, but unfortunately they are built upon the Roman (Byzantine) and Hijri calendars, which are difficult to use in combination with the Persian cal-

The text itself, as advertised in the preface, is a series of examples meant to illustrate and supplement each chapter of *al-Zīj al-Jāmi‘* by Kūshyār ibn Labbān.<sup>24</sup> Based on internal evidence, al-Nasawī’s text can be dated to 1047 CE.<sup>25</sup> This should lead us to modify Sezgin’s estimate that al-Nasawī was active in the last quarter of the tenth century and the first quarter of the eleventh.<sup>26</sup>

[76<sup>r</sup>: blank page, still blank today.]

8. Persian treatise entitled *Treatise of Ornamentation on Calculating the Table of the Thirty* (*Risālat al-tazyīn fī ḥisāb jadwal al-thalāthīn*, 76<sup>v</sup>–81<sup>r</sup><sub>5</sub>, rest of page blank).<sup>27</sup>
9. ‘Umar al-Khayyām (d. 1123), *Algebra* (*al-Jabr wa-l-muqābalah*, 81<sup>v</sup>–109<sup>r</sup><sub>12</sub>, rest of page blank). Ed. Rashed and Jabbār.<sup>28</sup>
10. anonymous and Jābir ibn Ibrāhīm al-Ṣābī [with commentary by Aḥmad <ibn> al-Sarī<sup>29</sup> and *al-qāḍī al-ḥakīm* Sa’d al-Dīn As’ad ibn Sa’id al-Hamadhānī], *Explanation of the Demonstration of Calculation by Two Errors, Improved Edition by Abū Sa’d Jābir ibn Ibrāhīm al-Ṣābī* (*Īḍāḥ al-burhān ‘alā ḥisāb al-khaṭa‘ayn, islāḥ Abī Sa’d Jābir ibn Ibrāhīm al-Ṣābī*, 109<sup>v</sup>–113<sup>r</sup>).

endar “because of leap years and fractions” (*bi-sabab al-kabā’is wa-l-kusūr*). And so the late Kūshyār ibn Labbān made his astronomical tables, the *Comprehensive Tables* (*al-Zīj al-Jāmi‘*), using the Persian calendar. This makes them much easier to use. Continuing on the next page (fol. 49<sup>v</sup>), al-Nasawī explains that he has produced a supplement to these tables (or to an abridgment of the tables in 85 chapters) in which he provides explanatory examples for tricky or unclear chapters as a sort of commentary (*sharḥ*). This material is on the third level of astronomical achievement, he explains, but made possible by his knowledge of *al-hay’ah* (the fourth level).

<sup>24</sup>The beginning of the text is indeed about calendars and conversions between them, but this is presumably because *al-Zīj al-Jāmi‘* began with this topic. Al-Nasawī’s text soon proceeds to discussing the geometry and mathematical astronomy necessary for the construction of *zīj*es. Al-Nasawī does not always give an example; for “part 4, chapter 1,” he writes, “You need no example because it is obvious” (*lā tahtāju ilā mithālīn li-annahu zhāhirun*; fol. 55<sup>v</sup><sub>6</sub>).

<sup>25</sup>At one point (fol. 50<sup>v</sup><sub>4</sub>), al-Nasawī gives an example of how to convert a Hijri date to other formats; the example he gives is of his own present day, which he gives as 12 Ṣafar 439 (8 August 1047 CE). A later hand pointed out that the text was composed in 439 AH, in a note in the top margin on the first page of the text (fol. 49<sup>r</sup>). The creator of the card catalog entry housed at Columbia’s RBML seems to have misread this note as 429 AH (rather than 439 AH) and misinterpreted it as the year when the text was *copied* rather than when it was written.

<sup>26</sup>Indeed, an anonymous reviewer has informed me that a recent study (Abū l-Ḥasan ‘Alī ibn Aḥmad al-Nasawī, *Kitāb al-tajrīd fī uṣūl al-handasah*, ed. Muṣṭafā Mawālīdī [London: Mu’assasat al-Furqān li-l-Turāth al-Islāmī, Markaz Dirāsāt al-Makhṭūṭāt al-Islāmiyyah, 2016], 17) establishes al-Nasawī’s birth date as 393/1002 and places his death after 473/1080. I have not been able to consult this book.

<sup>27</sup>The title, reported by ‘Awwād, appears at the end of the preface, fol. 78<sup>r</sup><sub>8</sub>.

<sup>28</sup>Omar Khayyam, *Rasā’il al-Khayyām al-jabriyyah*, ed. and trans. Roshdi Rashed and Aḥmad Jabbār (Aleppo: Jāmi‘at Ḥalab, Ma’had al-Turāth al-‘Ilmī al-‘Arabī, 1981), 1–73<sub>9</sub>; updated (as pointed out by an anonymous reviewer) in three publications by Roshdi Rashed and Bijan Vahabzadeh: *Al-Khayyam mathématicien* (Paris: Albert Blanchard, 2000); *Omar Khayyam, the Mathematician* (New York: Bibliotheca Persica Press, 2000); *Riyāḍiyyāt ‘Umar al-Khayyām* (Beirut: Markaz Dirāsāt al-Waḥdah al-‘Arabiyyah, 2005).

<sup>29</sup>See nos. 17 and 18 below



11. Aḥmad ibn Ibrāhīm al-Sijzī (?), *Introduction to Crafting an Instrument by which Distances Can Be Known* (*Muqaddimah li-ṣanʿat ālatin tuʿraf bihā al-abʿād*, 113<sup>r</sup>–115<sup>r</sup>).<sup>30</sup>
12. Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī (fl. ca. second half of the 10th century),<sup>31</sup> *Treatise on the Knowledge of Straight and Bent Lines* (*Risālah fī maʿrifat al-khaṭṭayn al-mustaqīm wa-l-munḥanī*, 115<sup>v</sup>–118<sup>v</sup>).<sup>32</sup>
13. Muḥammad ibn Aḥmad ibn Muḥammad ibn Kashnah(?) al-Qummī (d. ca. first half of the 11th century), *On the Possibility of Two Lines Existing Which Become Indefinitely Closer but Do Not Meet* (...*imkān wujūd al-khaṭṭayn alladhayn yaqṭaribān abadan wa-lā yaltaqiyān*, 119<sup>r</sup>–121<sup>v</sup>).<sup>33</sup>
14. Ibn al-Haytham, *Treatise on Obtaining the Altitude of Standing Objects, Mountains, and Clouds* (*Maqālah li-l-shaykh Abī ʿAlī ibn al-Haytham fī maʿrifat irtifāʿ al-ashkhāṣ al-qāʾimah wa-aʿmidat al-jibāl wa-rtifāʿ al-ghuyūm*, 121<sup>v</sup>–122<sup>r</sup>).
15. comments of *al-ḥakīm al-fāḍil* Saʿd al-Dīn {ibn}<sup>34</sup> Asʿad ibn Saʿīd al-Hamadhānī (122<sup>r</sup><sub>1-3</sub>). This is the same person who added a brief note to Ibn al-Sarī’s commentary on Jābir’s “improved edition” (*iṣlāḥ*) of the treatise on Double False Position appearing earlier in this manuscript (no. 10).
16. a problem from Abū Naṣr al-Fārābī (d. ca. 950), Music *fann* 1, *maqālah* 1 (*Masʿalah dhakarrahā Abū Naṣr al-Fārābī fī l-maqālah al-ūlā min al-fann al-awwal fī l-mūsīqā*, 122<sup>v</sup>–125<sup>r</sup><sub>2</sub>).
17. Abū l-Futūḥ ibn al-Sarī (d. 548/1153–4), *Problem*, on constructing “a triangle whose sides are equal to the diameter” of a given circle (*Masʿalah min kalām Abī l-Futūḥ ibn al-Sarī*, 125<sup>r</sup><sub>3</sub>–126<sup>r</sup><sub>1</sub>).<sup>35</sup> This is the same Ibn al-Sarī who commented on the treatise on Double False Position (no. 10 above).
18. Abū l-Futūḥ ibn al-Sarī, *Treatise on Constructing an Equilateral Triangle Inside another Equilateral Triangle of a Given Proportion to the First* (*Qawl li-Abī l-Futūḥ Aḥmad ibn Muḥammad ibn al-Sarī fī ʿamal muthallath mutasāwī l-aḍlāʿ fī dākḥil muthallath mutasāwī l-aḍlāʿ lahu nisbah ilayhi mafrūdah...*, 126<sup>r</sup><sub>2</sub>–128<sup>v</sup><sub>14</sub>, rest of the page blank). Again, this is the same Ibn al-Sarī.

<sup>30</sup>Sezgin, *GAS*, 5:333, no. 26, citing only ʿAwwād’s description. ʿAwwād only mentioned the author’s *nisbah*, so Sezgin treats this as a work of Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī (explicitly named as the author of the next text in the Columbia manuscript), but the scribe here seems deliberately to have named someone else — whom is not clear to me. Someone by this name studied with ʿAbd al-Qāhir al-Jurjānī (11th century): Yāqūt al-Rūmī, *Muʿjam al-udabāʾ*, ed. Iḥsān ʿAbbās, 6 vols., continuous pagination (Beirut, 1993), 187, no. 53. This is probably a coincidence.

<sup>31</sup>Sometimes called al-Saḥarī, but here (on fol. 119<sup>r</sup><sub>6</sub>) the name is marked with diacritics, as al-Sijzī.

<sup>32</sup>Sezgin, *GAS*, 5:333, no. 25, citing only ʿAwwād’s description of the Columbia manuscript.

<sup>33</sup>Sezgin, 5:336. Following Suter, Sezgin considers al-Qummī a “younger contemporary” of Aḥmad ibn Muḥammad ibn ʿAbd al-Jalīl al-Sijzī.

<sup>34</sup>This “ibn” is a mistaken addition that should be suppressed to make the name correspond with that of the same personage in no. 10.

<sup>35</sup>See n. 10 above.

[Scribe 3: new, messier hand, fills up whole pages with no margin or regular ruling.]

19. Commentary on the First Book of Euclid's *Uṣūl*.<sup>36</sup> Incipit: *Ikhtilāf wuqū' li-shakli B min al-maqālah al-ūlā min Kitāb al-Uṣūl li-Iwqḷīdus...* (129<sup>r</sup>–137<sup>v</sup>).

[138<sup>r</sup>: blank.]

[138<sup>v</sup>: originally blank, with two sketches of hyperbolas added later. These may be related to the following commentary on Apollonios.]

[139<sup>r</sup>: blank.]

[Scribe 4: new hand, likewise unruled but leaves a slight margin. Begins in brown ink then after a few lines changes to black ink, with figures in brown ink. Much of it is without dots.]

20. Banū Mūsā ibn Shākir (9th century), extracts, *What we need by way of introduction to the Book of Apollonios from the introductory remarks of the Banū Mūsā at the beginning of their improved edition (Mā naḥtāj ilā taqdīmihi 'alā Kitāb Abuluniyūs mimmā qaddamahu Banī [sic] Mūsā 'alayhi fī ṣadr [iṣlāḥi]him, 139<sup>v</sup>–143<sup>r</sup>).*<sup>37</sup>

[143<sup>v</sup>: blank, now with a brief note.]

[144<sup>r</sup>: a few later notes and drawings, written sideways.]

[Scribe 3, again.]

21. Notes, beginning: *Idhā ukhrija quṭru GEB min dā'irat BGDZ 'alā istiḳāmatihi...* (144<sup>v</sup>–146<sup>r</sup>).

[147<sup>v</sup>: final page, blank with two diagrams at the top.]

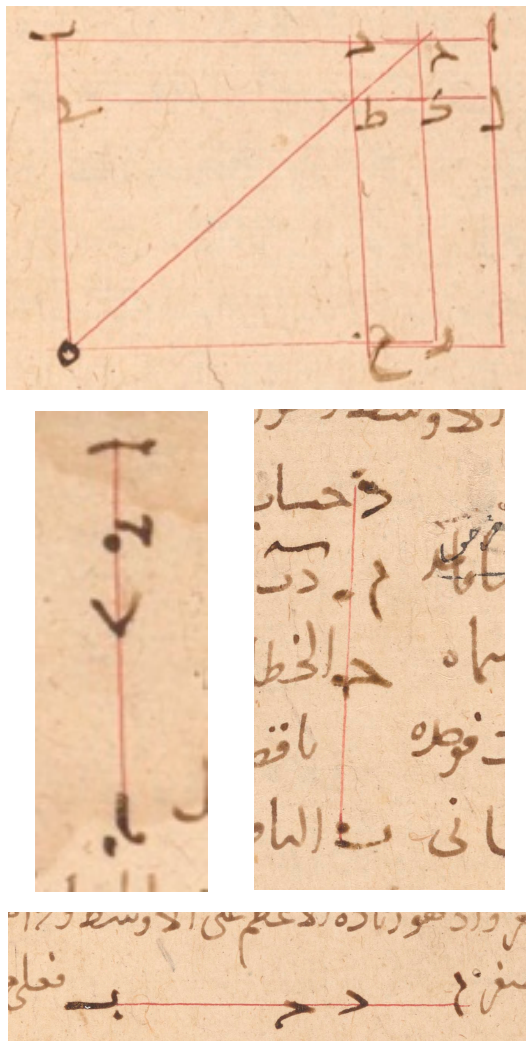
## II Text

In what follows, I present an *editio princeps* of the treatise on Double False Position as it appears in New York, Columbia University, or. 45 (no. 10; fols. 109<sup>v</sup>–113<sup>r</sup>). As with the sketch of the manuscript's contents in the previous section, I carry out this work, philological in the narrower sense (textual criticism), in the spirit and in the service of a more ambitious philology that shows curiosity for and invests resources in texts preserved by a textual tradition that may at first sight appear irrelevant to modern critics.

Within the text, diagrams (see Figure 1) illustrate the geometrical definitions and proofs on fol. 110<sup>v</sup> after line 3 (DIAGRAM 1: two-dimensional diagram illustrating the geometric proof in ¶2), fol. 111<sup>r</sup> in the upper margin extending into line 1 (DIAGRAM 2: vertical

<sup>36</sup>As 'Awwād calls it: *Ta'liqāt 'alā l-maqālah al-ūlā min Kitāb al-Uṣūl li-Uqḷīdis*.

<sup>37</sup>At the very bottom of fol. 143<sup>r</sup>, it looks like the scribe has written *tammāt*, signalling the end of the text, suggesting that what follows are other notes. As an anonymous reviewer pointed out to me, this text was extracted from *Muqaddamāt Kitāb al-makhrūṭāṭ li-Banī Mūsā*, ed./trans. in *Apollonius de Perge, Coniques: Texte grec et arabe*, ed. Roshdi Rashed, multiple vols. (Berlin: de Gruyter, 2008), 1.1:500–533, beginning after the preface at Lemma 1 (p. 509). This edition used the Columbia manuscript, among others, to establish the text; see p. 498. The folio numbers I have indicated do not match Rashed's; this may be because Rashed's numbering follows page numbers rather than the folio numbers that have quite recently been added in pencil in the top left corner of each *recto*.



**Figure 1:** Diagrams in the treatise on Double False Position in New York, Columbia University, or. 45, no. 10, on fols. 110<sup>v</sup>, 111<sup>r</sup>, 112<sup>v</sup>. Top: DIAGRAM 1. Middle left: DIAGRAM 2. Middle right: DIAGRAM 3. Bottom: DIAGRAM 4.

line  $A—G—D—B$ ), fol. 111<sup>r</sup><sub>8-12</sub> with the lines of text written around it (DIAGRAM 3: vertical line  $D—A—G—B$ ), and fol. 112<sup>v</sup> in the midst of the last line of the page (DIAGRAM 4: horizontal line  $A—D—G—B$ ). There is also a space of about 5 lines left blank at the end of fol. 111<sup>r</sup> after 14 lines of text, as if for a diagram that was never added; at the top of the next page a modern hand pencilled in the heading *ḥisāb al-khaṭaʿayn*, but the original text simply began at the top of the page (with the beginning of Ibn al-Sarī commentary) with no new heading.

The commentaries by Ibn al-Sarī and al-Hamadhānī were originally copied as a single block of text visually undifferentiated from the main text of the treatise. Marginal and interlinear labels were subsequently added to distinguish commentary from focus text (Figure 2). This suggests that an ancestor of the Columbia manuscript had the commentaries (or at least Ibn al-Sarī's commentary) in the margin; that a more proximate ancestor that descended from the first then incorporated the marginalia into the body of the text, probably rubricated or visually differentiated from the focus text some other



**Figure 2:** New York, Columbia University, or. 45, fol. 111<sup>v</sup>: example of later marginal and inter-linear labels distinguishing the commentary (*hāshiyah*) from the focus text (*matn*).

way; and that a subsequent scribe, perhaps the scribe of the Columbia manuscript, then copied this ancestor without rubrication.

In the Arabic text, I supply *hamzas* where necessary to suit modern orthography. I do not emend the consonantal skeleton without indicating it (e.g., with angle brackets), except that wherever the manuscript says *الخطائين*, I write *الخطأين*. When the letter *jīm* is used as a mathematical symbol, I print ج, even when that is not precisely the shape used in the manuscript.

In the translation, I render the Arabic letters used to represent geometric points with the English letter corresponding to the *abjad* order, i.e., آ is *A*, ب is *B*, ج is *G*, د is *D*, هـ is *E*, و is *W*, ز is *Z*, ح is *H*, ط is *T*, ي/ے is *Y*, ك/ك is *K*, ل is *L*, م is *M*, ن is *N*, س is *S*, ع is *O*, ف is *F*.

(ص ١٠٩ ب)

إيضاح البرهان على حساب الخطأين  
إصلاح أبي سعد جابر بن إبراهيم الصبائي

(١) إذا أردت حساب شيء من فنون هذا الباب فاقضيت مقداراً من الجنس الذي تسحأـل عنه، أي مقدار كان، كالعدد أو الخط أو السطح أو غير ذلك مما يقع عليه الحساب، ويتم ذلك المقدار المال الأول. ثم افعل به مثل ما قيل لك في السؤال، فإن اتفق لك أن تصب، فهو الجواب، والإصابة على هذه السبيل لا يعتد بها. وإن أخطأ ما أردت، فجد مقدار ما أخطأت به وسمه الخطأ الأول. وإن كان العمل أنتج لك زيادة بذلك المقدار عما يوجبه السؤال، فسمه الخطأ الزائد، وإن كان أنتج نقصاناً، فسمه الخطأ الناقص. ثم اقتضب بعد ذلك مقداراً آخر مخالفاً للأول، وسمه المال الثاني. وافعل به كما فعلت بالمال الأول سواء. فإن أخطأ فجد مقدار الخطأ، وسمه الخطأ الثاني. وإن كان أيضاً زائداً، فسمه الزائد، وإن كان ناقصاً، فسمه الناقص. ثم انظر، فإن كان الخطآن زائدين معاً أو ناقصين معاً، فألني الأقل من الأكثر، وإن كانا مختلفين زائداً وناقصاً، فاجمعهما، وسم أي ذلك عملت الجزء، وهو الذي

عليه القسمة. ثم اضرب المال الأول في الخطأ الثاني، واضرب المال الثاني في الخطأ الأول. وانظر، فإن كنت جمعت الخطأين، فاجمع هذين أيضاً، وإن كنت نقصت أقل الخطأين من أكثرهما، فأنقص أقل هذين من أكثرهما، وأقسم أي ذلك حصل لك على الجزء، فما خرج فهو المطلوب.

(ص 110)

(٢) تعليل ذلك: كل خط يقسم بثلاثة أقسام، فإن ضرب ذلك بأسره في الأوسط من أقسامه مع <...><sup>١</sup> ضرب القسم الأوسط مع أحد القسمين اللذين عن جنبيه مجموعين لخط في القسم الأوسط بعينه مع القسم الآخر مجموعين لخط واحد. مثال ذلك أن خط  $\overline{اب}$  مقسوم بثلاثة أقسام، وهي  $\overline{اج}$   $\overline{جد}$   $\overline{دب}$ ، فأقول إن ضرب  $\overline{اب}$  بأسره في  $\overline{جد}$ ، وضرب  $\overline{اج}$  في  $\overline{دب}$  مجموعين، مثل ضرب خطي  $\overline{جد}$   $\overline{اج}$  مجموعين، وهما  $\overline{اد}$ ، في خطي  $\overline{جد}$   $\overline{دب}$  مجموعين، وهما  $\overline{جب}$ . برهانه أن ترسم على خط  $\overline{جب}$  مربع  $\overline{جب}$   $\overline{دب}$  متساوي الأضلاع، قائم الزوايا، ونخرج قطره، وهو  $\overline{جد}$ ، ونخرج من نقطة  $\overline{د}$  عموداً على ضلع  $\overline{جب}$  يقطع قطر المربع على نقطة  $\overline{ط}$ ، ويلقاء ضلع  $\overline{زه}$  على نقطة  $\overline{ح}$ ، ونجبر (٤) على  $\overline{ط}$  خط  $\overline{طك}$  موازياً لضلع  $\overline{جب}$ ، ونتم سطح  $\overline{اه}$  المستطيل، ونخرج خط  $\overline{طك}$  على الاستقامة، حتى يلقي ضلع  $\overline{اه}$  المستطيل على نقطة  $\overline{ل}$ . فن البين أن سطح  $\overline{جط}$  مربع متساوي الأضلاع قائم الزوايا، وكذلك سطح  $\overline{طه}$  مربع متساوي الأضلاع قائم الزوايا، وسطح  $\overline{دك}$   $\overline{طز}$  متساويان، لأنهما المتممان اللذان عن جنبيه قطر مربع  $\overline{جب}$   $\overline{زه}$ . وإذا كان ذلك كذلك، فإننا نقول أن سطحي  $\overline{ال}$   $\overline{ز}$  مجموعين مساويين<sup>١</sup> لسطح  $\overline{اح}$ . برهانه أن سطح  $\overline{دس}$  مساو لسطح  $\overline{طز}$  فيجعل سطحي  $\overline{اط}$   $\overline{لز}$  مشتركة، فيكون سطح  $\overline{ال}$   $\overline{ز}$  مساوية<sup>٢</sup> لسطح  $\overline{اح}$ . لكن سطح  $\overline{ال}$  هو من ضرب خط  $\overline{اب}$  في خط  $\overline{جد}$ ، لأن خط  $\overline{جد}$  مساو لخط  $\overline{دب}$ ، وسطح  $\overline{لز}$  هو من ضرب خط  $\overline{اج}$  (ص 110) في خط  $\overline{دب}$ ، لأن خط  $\overline{ل}$   $\overline{ك}$  مساو لخط  $\overline{اج}$ ، وخط  $\overline{كز}$  مساو لخط  $\overline{دب}$ . فأما سطح  $\overline{اح}$ ، فإنه من ضرب  $\overline{اد}$  في  $\overline{جب}$ ، لأن خط  $\overline{جب}$  مساو لخط  $\overline{دح}$ . فحضر خط  $\overline{اب}$  بأسره في خط  $\overline{جد}$  مع ضرب  $\overline{اج}$  (رسم ١) في  $\overline{دب}$  مثل ضرب خط  $\overline{اد}$  في خط  $\overline{جب}$ ، وذلك ما أردنا أن نبين.

(٣) كل ثلاثة خطوط أو أعداد مختلفة فإن ضرب الأعظم في زيادة الأوسط على الأصغر مع ضرب الأصغر في زيادة الأعظم على الأوسط مجموعين مثل ضرب الأوسط في زيادة الأعظم على الأصغر. فليكن ثلاثة خطوط أو أعداد مختلفة عليها  $\overline{اب}$   $\overline{اد}$   $\overline{اج}$ ، و  $\overline{اب}$  أعظمها، و  $\overline{اج}$  أصغرها. فأقول إن ضرب الأعظم، وهو  $\overline{اب}$ ، في زيادة الأوسط على الأصغر، وهي  $\overline{جد}$ ، مع ضرب الأصغر، وهو  $\overline{اج}$ ، في زيادة الأعظم على الأوسط، وهي  $\overline{دب}$ ، مجموعين مثل <ضرب> الأوسط، وهو  $\overline{اد}$ ، في زيادة الأعظم على الأصغر، وهو  $\overline{جب}$ . برهانه أن خط  $\overline{اب}$  قسم بثلاثة أقسام، وهي  $\overline{اج}$   $\overline{جد}$   $\overline{دب}$ . فعلى ما بينا فيما تقدم يكون ضرب  $\overline{اب}$  بأسره في  $\overline{جد}$  الأوسط مع ضرب (ص 111)  $\overline{اج}$  في  $\overline{دب}$  مثل (رسم ٢) ضرب  $\overline{اد}$  في  $\overline{جب}$ ، وذلك ما أردنا أن نبين.

(٤) وإذ قد بوط (٤٤) هذه المعاني، فلنضع ثلاثة خطوط أو أعداد مختلفة، وهي  $\overline{دب}$   $\overline{اب}$   $\overline{جب}$ ، ولكن أولاً  $\overline{دب}$  أعظم من  $\overline{اب}$ ، و  $\overline{جب}$  أصغر من  $\overline{اب}$ ، فزيادة  $\overline{دب}$  على  $\overline{اب}$  هي  $\overline{دا}$ ، ونقصان  $\overline{جب}$  عن  $\overline{اب}$  هو  $\overline{اج}$ ، فنخطو  $\overline{اب}$   $\overline{دب}$   $\overline{جب}$  مختلفة. فحضر الأعظم، وهو  $\overline{دب}$ ، في زيادة الأوسط على الأصغر، وهو  $\overline{اج}$ ، مع ضرب الأصغر، وهو  $\overline{جب}$ ، في زيادة الأعظم على الأوسط، وهو  $\overline{دا}$ ، مثل ضرب الأوسط، وهو  $\overline{اب}$ ، في زيادة الأعظم على الأصغر، وهو  $\overline{جد}$ .

(٥) فصاحب (رسم ٣) حساب الخطأين لما كان يلتبس مطلوباً بمنزلة  $\overline{اب}$  هاهنا فأخذ<sup>١٠</sup>  $\overline{دب}$ ، فوجده زائداً عليه، وأخذ مقدارا ما بينهما، وهو  $\overline{دا}$ ، وسماه الخطأ الأول الزائد. ثم رجع فاقترض مقدارا آخر، فوقع له  $\overline{جب}$ ، فوجده ناقصاً عن  $\overline{اب}$ . فأخذ مقدارا ما بينهما، وهو  $\overline{اج}$ ، وسماه الخطأ الثاني الناقص. ثم ضرب الذي أخذه أولاً، وهو يسميه المال الأول، وهو في مثالنا هذا  $\overline{دب}$ ، في الخطأ الثاني، وهو  $\overline{اج}$ . ثم ضرب الذي يسميه المال الثاني، وهو  $\overline{جب}$ ، ...

(بياض حوالي ٥ سطور)

(ص 111 ب)

<حاشية السري، ج ١: قال أحمد السري إن هذا الكلام الذي يذكره أبو سعد الصابي من أن «صاحب حساب الخطأين لما كان يلتبس مطلوباً بمنزلة  $\overline{اب}$  فأخذ  $\overline{دب}$ ، فوجد زائداً عليه، وأخذ مقداراً ما بينهما، وهو  $\overline{دا}$ » وباقي كلامه إلى آخر كتابه ليس هو الذي يستعمله صاحب حساب الخطأين. وذلك لأنه إذا طلب  $\overline{اب}$  ووجد  $\overline{دب}$  زائداً عليه وأخذ مقداراً ما بينهما، وهو  $\overline{دا}$ ، وجب أن يكون  $\overline{اب}$  المجهول معلوماً من غير ضرب أو قسمة. وذلك لأن  $\overline{دب}$  معلوم وقد أخذ منه  $\overline{دا}$  المعلوم، فبقي  $\overline{اب}$  المعلوم... >  
<متن جابر، استقرار فقرة ٥: >... في الخطأ الأول، وهو  $\overline{دا}$ . فلما كان الخطآن مختلفين أحدهما زائد والآخر ناقص، جمع هذين المضروبين، فكان الجميع مثل ضرب الأوسط، وهو  $\overline{اب}$ ، في زيادة الأعظم على الأصغر، وهي  $\overline{جد}$ ، عمد إلى  $\overline{دج}$ ، الذي <هو> مجموع<sup>١١</sup> الخطأين، فجعل القسمة عليه، أعني قسمة مجموع المضروبين، فنخرج له من القسمة مقدار  $\overline{اب}$ ، وهو المطلوب.

(٦) وأيضاً فليكن على الوجه...

<حاشية السري، ج ٢: >... ولا يحتاج فيه إلى عمل آخر. وهذا الرجل إنما يستخرج بهذا العمل<sup>١٢</sup> المفروض له في المسألة<sup>١٣</sup> لا المطلوب. مثال ذلك أنا التمسنا منه معرفة أي مال إذا زدنا عليه نصفه وثلثه كان أحد عشر. فكأنه فرض لأحد عشر أب، ثم فرض عددا، وهو تسعة، فزاد عليه نصفه وثلثه، فصار ذلك ستة عشر ونصف. ففرضه عدد دب، ثم قاس دب إلى الأحد عشر، وهي أب، فوجد > زائداً على الأحد عشر خمسة ونصف لعدد دأ، فسمى دأ الخطأ الأول، وهو كذلك لعمرى! ثم فرض عددا هو أربعة، وزاد (ص ١١٢) عليه نصفه وثلثه...

<متن جابر، فقرة ٦، ج ٢: >... الثاني كل واحد من دب جد<sup>١٤</sup> أعظم من أب فيكون أيضاً خطوط أو أعداد أب جد<sup>١٥</sup> دب الثلاثة مختلفة، ودب أعظمها، وأب أصغرهما. ولتكن زيادة الأوسط، وهو جب، على الأصغر، وهو أب، هي جأ، ولتكن زيادة الأعم، وهو دب، على الأصغر، وهو أب، هي أد. فإِن أجل ما تقدم تكون ضرب الأعم، وهو دب، في زيادة الأوسط على الأصغر، وهو جأ، مع<sup>١٦</sup> ضرب أب...

<حاشية السري، ج ٣: >... فصار سبعة وثلث، ففرضه عدد جب، ثم قاسه إلى الأحد عشر، فوجده ينقص عنه ثلاثة وثلثان لعدد جأ، فسماه الخطأ الثاني، وضرب السبعة وثلث، وهي بجا، في أد الخطأ الأول، وقسم مجموع المرتفعين على جد مجموع الخطأين، فنخرج له أب، هو الأحد عشر المفروض أولاً، لا العدد المطلوب، وهو فلم يستعمل في حسابه طريق الخطأين، وذلك لأن العدد الأول هو التسعة لا الستة عشر...

<متن جابر، فقرة ٦، ج ٣: >... وهو الأصغر، في دج، وهو زيادة الأعم على الأوسط، > مثل ضرب الأوسط، > وهو بجا<sup>١٧</sup>، في زيادة الأعم على الأصغر، وهو أد. فليحفظ ذلك. فصاحب حساب الخطأين، لما كان مطلوبه بمنزلة أب<sup>١٨</sup>، فأخذ أولاً دب وسماه المال الأول، فوجده زائداً...

<حاشية السري، ج ٤: >... ونصف، والعدد الثاني هو الأربعة لا السبعة وثلث. وهذه الأعداد الثلثة<sup>١٩</sup>، أعني الأحد عشر والستة عشر والنصف والسبعة وثلث، هي على نسبة الأعداد الثلثة، أعني على نسبة التسعة والأربعة والعدد المجهول المطلوب منا. ولو فعلنا طريق الخطأين وقسمنا المرتفع على (ص ١١٢ ب) جد بعينه، لخرج لنا المجهول: ستة. فقد بان أن كلامه ليس يبيح حساب الخطأين، وكذلك في باقي الشكلين اللذين بعده. فافهم ذلك، أعني أنها تلغو<sup>٢٠</sup> بمثل هذا الكلام.

<حاشية الهمداني: > قال القاضي الحكيم سعد الدين أسعد بن سعيد الهمداني أدام الله علوه: طريق الخطأين أن يضرب بدل دب التسعة التي هي على نسبتته في آج، وبدل بجا الأربعة التي هي على نسبتته في أد فيكون مجموعهما ٥٥ ٢١، ثم يقسمه على الجزء، هو تسعة وسدس، فيخرج المطلوب.

<متن جابر، فقرة ٦، ج ٤: >... على أب بمقدار دأ، فسمى دأ الخطأ الأول الزائد. ثم رجع فاقترض مقدار آخر، فوقع له جب، فسماه المال الثاني، ووجد أيضاً زائداً على ما يُطلب وهو أب، فأخذ مقدار ما أخطأ به، وهو جأ، وسمي الخطأ الثاني الزائد أيضاً. ثم ضرب المال الثاني، وهو جب، في الخطأ الأول، وهو أد، وأسقط من ذلك ضرب المال الأول، وهو دب، في الخطأ الثاني، وهو جأ، فالخطأين حينئذ زائدان. فبقي له مقداران مساويان لضرب أب في جد<sup>٢٣</sup>. فيجعل قسمة هذه البقية على جد<sup>٢٤</sup>، الذي هو فضل ما بين الخطأ الأول والخطأ الثاني، فنخرج له من القسمة مقدار أب، وهو المطلوب، التي<sup>٢٥</sup> كان يلتبس عليه.

(٧) وأيضاً فليكن على الوجه الثالث كل واحد من دب جب أصغر من أب، فيكون أيضاً خطوط أو أعداد أب دب جب الثلاثة مختلفة، وأب أعظمها، ودب الأوسط، وحب الأصغر، فدج هو زيادة الأوسط على الأصغر، وأد هو زيادة الأعم على الأوسط، وجأ هو زيادة الأعم على الأصغر (رم ٤) فعلى ما بيننا فيما تقدم (ص ١١٣) يكون ضرب أب في دج مع ضرب جب في دأ مجموعين مثل ضرب دب في دأ، فليحفظ ذلك. وصاحب حساب الخطأين هاهنا أيضاً لما كان يلتبس وجود أب، فأخذ أولاً دب وسماه المال الأول، وأخطأ فيه بدأ، فسمى دأ الخطأ الأول الناقص. ثم رجع فاقترض ما لا ثانياً، فاتفق له جب وأخطأ فيه بجا، فسمى جأ الخطأ الثاني الناقص أيضاً. ثم ضرب دب، وهو المال الثاني، في دأ، وهو الخطأ الأول، فبقي له مقدار مساوٍ لضرب أب في دج. فإذا قسم هذه البقية على دج، الذي هو فضل ما بين الخطأ الأول والخطأ الثاني، خرج له من القسمة أب، وهو المطلوب الذي كان يلتبس عليه. وذلك ما أردنا أن نبين.

تم والحمد لله رب العالمين، وصلواته على سيدنا محمد وآله أجمعين.

<sup>١</sup> مع >...> وقد يكون الصحيح: مثل؛ وقد يكون أن أكثر من ذلك سقط من نسخة نيويورك<sup>٢</sup> جب هـ: ويمكن الصحيح: جب هـ<sup>٣</sup> زه: ليس للزاء نقطة في الأصل<sup>٤</sup> قائم: [[قام]] قائم: ك<sup>٥</sup> دج: والصحيح: دك<sup>٦</sup> مساويين: أي: مساويان<sup>٧</sup> مساوية: كذا في الأصل<sup>٨</sup> جب: جب (بدون خط): ك<sup>٩</sup> وطب (٢٢): أي: وضعت؟<sup>١٠</sup> هاهنا فأخذ: شطب عليه قارئ متأخر<sup>١١</sup> مجموع: كتب الكاتب «مجموعين» أولاً، ثم صحح<sup>١٢</sup> العمل: صححه، في الأصل: للعمل<sup>١٣</sup> المسألة: في الأصل: المسألة<sup>١٤</sup> جد: كذا في الأصل، والصحيح: جب<sup>١٥</sup> جد: كذا في الأصل، والصحيح: جب<sup>١٦</sup> مع: في الأصل كأنه: بغ<sup>١٧</sup> بجا: في الأصل يبدو أنه تصحح متأخر<sup>١٨</sup> أب: بلا خط في الأصل<sup>١٩</sup> الثلثة: في الأصل: الثلث، والصحيح: الثلثة، كما ورد فيما بعد<sup>٢٠</sup> أنها تلغو بمثل: (٢) بلا نقط في الأصل<sup>٢١</sup> ٥٥: مكتوب بأعداد عشرية في الأصل: ٥٥: ثم لم يكن في الأصل أولاً، فأضيف متأخر<sup>٢٢</sup> جد: والصحيح: جب<sup>٢٤</sup> جد: والصحيح: جب<sup>٢٥</sup> التي: كذا في الأصل، والصحيح: الذي، كما ورد في نفس العبارة في ما بعد، في نهاية فقرة ٧

### III Translation

*Explication of the Demonstration of Calculation by Two Errors, Improved Edition by Abū Saʿd Jābir ibn Ibrāhīm al-Šābī*

1. If you wish to calculate this sort of thing, then you will come up with an amount of the kind about which you inquire, whatever amount it may be, such as number, line, surface, or other things that can be calculated. That amount becomes the *first estate*.<sup>38</sup> Then operate upon it as you were instructed in the question. If you happen to be right, then that is the answer. Getting it right this way is unreliable. If it errs from what you were seeking, find the amount by which you erred, and call it the *first error*. If the operation yielded an excess by that amount above what the question requires, then call it the *excessive error*; if it yielded a deficit, then call it the *deficient error*. Then after that come up with another amount different from the first, and call it the *second estate*. Do the same to it as you did to the first estate. If it errs, find the amount of the error and call it the *second error*. If it too is excessive, call it excessive; if deficient, call it deficient. Then look, and if the two errors are both excessive or both deficient, displace [i.e., subtract] the lesser from the greater; but if they are different, one excessive and the other deficient, then take their sum. Call the result of either operation *the part*; this will be the divisor.<sup>39</sup> Then multiply the first estate by the second error, and the second estate by the first error. Look, and if you summed the two errors, then sum these two [products] too; and if you subtracted the lesser of the two errors from the greater of the two, then subtract the lesser of these two [products] from the greater of the two. Either way, divide the result by *the part*. The result is the answer.

2. The justification of this method: Each line is divisible into three segment.<sup>40</sup> That line in its entirety multiplied by its middle segment,<sup>41</sup> plus  $\langle \dots \rangle$ <sup>42</sup> the product of multiplying the middle segment plus one of the two adjacent segments joined into a line by the middle segment itself plus the other segment joined into a single line.<sup>43</sup> For example, the line  $AB$  is divided into three segments, namely  $AG$ ,  $GD$ ,  $DB$ . I'm saying that  $AB$  in its entirety multiplied by  $GD$  and  $AG$  multiplied by  $DB$ , summed together, are congruent with the two lines  $GD$ ,  $AG$  summed together, which is  $AD$ , multiplied by the two lines  $GD$ ,  $DB$  summed together, which is  $GB$ .<sup>44</sup> The proof of this is for us to draw, upon the line  $GB$ , an equilateral right quadrangle<sup>45</sup>  $GBED$  [read:  $GBEZ$ ]; determine its diagonal, which is  $GE$ ; extend from point  $D$  a vertical line upon side  $GB$  such that it intersects the square's diagonal at point  $T$  and meets side  $ZE$  at point  $H$ ; place line  $YTK$  on  $T$  parallel to side  $GB$ ; complete the oblong<sup>46</sup> surface  $AE$  and ex-

<sup>38</sup>I translate *māl* here literally as the "estate." Here it refers to the unknown quantity  $x$  that one is seeking to find by Double False Position (not, as is frequent in Arabic algebra, the square of the unknown,  $x^2$ ).

<sup>39</sup>Literally, "that over which the division" will take place.

<sup>40</sup>Call them  $a$ ,  $b$ ,  $c$ .

<sup>41</sup> $b(a + b + c) = ab + b^2 + bc$ .

<sup>42</sup>The text here appears corrupt; for *maʿ* perhaps read *mithl*, "is like," i.e., equal to.

<sup>43</sup> $(b + c)(b + a) = b^2 + bc + ab + ac$ . As the text stands, it is not a complete sentence. If we emend *maʿ* to *mithl*, it still isn't quite right because the first half of the equation would be missing a term:  $ac$ .

<sup>44</sup>i.e.,  $AB \times GD + AG \times DB = (GD + AG) \times (GD + DB) = AD \times GB$ .

<sup>45</sup>i.e., a square.

<sup>46</sup>Here, rectangular in particular.

tend line  $YTK$  straight until it meets the side of oblong surface  $AE$  at point  $L$ . [See Figures 1 and 3.] It is clear that the surface  $GT$  is an equilateral right quadrangle [a square], and so is the surface  $TE$ . The two surfaces  $DY$  and  $TZ$  are equal because they are the two complements that are on either side of the diagonal of the square  $GBEZ$ .<sup>47</sup> If that is so, then we say that the two surfaces  $AY, LZ$  taken together are equal to the surface  $AH$ . The proof is that surface  $DY$  is equal to surface  $TZ$ , so we consider<sup>48</sup> surfaces  $AT, LZ$  to be a shared <portion?>.<sup>49</sup> Surfaces  $AY, LZ$  are thus equal to surface  $AH$ . But surface  $AY$  is from the multiplication of line  $AB$  by line  $GD$  because line  $GD$  is equal to line  $YB$ ;<sup>50</sup> and surface  $LZ$  is from the multiplication of line  $AG$  by line  $DB$  because line  $LK$  is equal to line  $AG$  and line  $KZ$  is equal to line  $DB$ .<sup>51</sup> As for surface  $AH$ , it is from the multiplication of  $AD$  by  $GB$  because line  $GB$  equals line  $DH$ .<sup>52</sup> And so the multiplication of line  $AB$  in its entirety by line  $GD$ , along with the multiplication of  $AG$  [DIAGRAM 1] by  $DB$  is congruent to the multiplication of line  $AD$  by line  $GB$ .<sup>53</sup> Q.E.D.

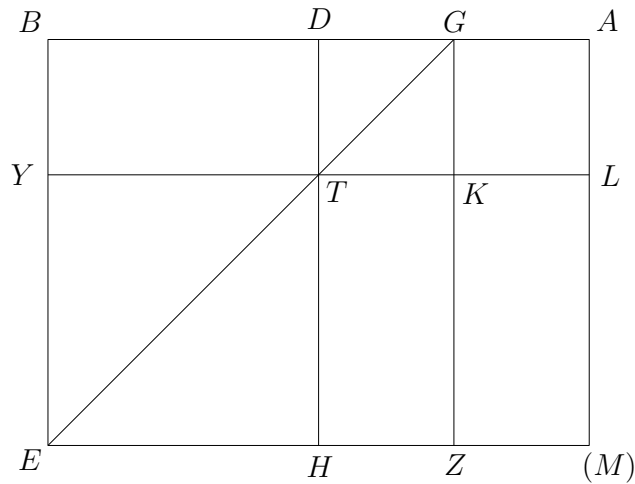


Figure 3: Arabic DIAGRAM 1 redrawn using the corresponding Latin letters.

3. For any three different lines or numbers,<sup>54</sup> the product of the greatest and the excess of the intermediate above the least,<sup>55</sup> along with the product of the least and

<sup>47</sup>Cf. Euclid, *Elements* 1.43; cited by Linda Hand Noel, “The Fundamental Theorem of Algebra: A Survey of History and Proofs” (EdD diss., Oklahoma State University, 1991), 26 (reading “43” for “4”). In particular, the wording here (لأنهما المثلثان اللذان عن جنبي قطر مربع جـهـز) is reminiscent of the eighth/ninth-century Arabic translation of Euclid’s *Elements* by al-Ḥajjāj (on which see Sezgin, *GAS*, 5:90), ed. R. O. Besthorn and J. L. Heiberg (Copenhagen, 1897), 1:166: عن جنبي القطر ط متساويان (كـا) فإن السطحين المتممين اللذين (كـا). For the case of a square, this result is visually apparent, but the theorem applies more generally to parallelograms.

<sup>48</sup>? *najʿal*.

<sup>49</sup>i.e.,  $AH$  and  $AY + LZ$  each include  $AT + LZ$ . Their remaining portions are, respectively,  $TZ$  and  $DY$ .

<sup>50</sup>i.e.,  $AY = AB \times YB = AB \times GD$ .

<sup>51</sup>i.e.,  $LZ = LK \times KZ = AG \times DB$ .

<sup>52</sup>i.e.,  $AH = AD \times DH = AD \times GB$ .

<sup>53</sup> $AB \times GD + AG \times DB [= AY + LZ = AH] = AD \times GB$ .

<sup>54</sup> $a > b > c$ .

<sup>55</sup> $a(b - c)$ .



the excess of the greatest over the intermediate,<sup>56</sup> summed together, is congruent to the product of the intermediate and the excess of the greatest above the least.<sup>57</sup> Let there be three different lines or numbers  $AB$ ,  $AD$ ,  $AG$ , where  $AB$  is the greatest of them and  $AG$  is the least.<sup>58</sup> I am saying that the greatest,  $AB$ , multiplied by the excess of the intermediate above the least, namely  $GD$ , along with the least,  $AG$ , multiplied by the excess of the greatest above the intermediate, namely  $DB$ , summed together, is congruent with the intermediate,  $AD$ , multiplied by the excess of the greatest above the least, namely  $GB$ . The proof of this is that the line or number  $AB$  is divided into three sections,  $AG$ ,  $GD$ ,  $DB$ .<sup>59</sup> According to what we have shown above [¶2],  $AB$  in its entirety multiplied by  $GD$ , the intermediate, along with //fol. 111r//  $AG$  multiplied by  $DB$  is congruent to [DIAGRAM 2]  $AD$  multiplied by  $GB$ . Q.E.D.

4. Since these concepts have been made clear [??], let us posit three different lines or numbers,  $DB$ ,  $AB$ ,  $GB$ . But first,  $DB > AB$ , and  $GB < AB$ , so the excess of  $DB$  above  $AB$  is  $DA$ ,<sup>60</sup> and the deficiency of  $GB$  from  $AB$  is  $AG$ .<sup>61</sup> The lines  $AB$ ,  $DB$ ,  $GB$  are distinct. Thus the greatest,  $DB$ , multiplied by the excess of the intermediate over the least, which is  $AG$ , along with the least,  $GB$ , multiplied by the excess of the greatest over the intermediate, which is  $DA$ , is congruent to the intermediate,  $AB$ , multiplied by the excess of the greatest over the least, which is  $GD$ .<sup>62</sup>

5. The author [DIAGRAM 3] of *Calculation by Two Errors*, when he sought the desired result in the place of  $AB$  here, he took  $DB$  and found it in excess over the former. Taking the amount between the two,  $DA$ , he called it the *first, excessive error*. Then he went back and came up with another amount, so now he had  $GB$ . He found it short of  $AB$ , so he took the amount between them,  $AG$ , and called it the *second, deficient error*. Then he multiplied what he had first taken — which he calls the *first estate* — by the second error,  $AG$ . Then he multiplied the one that he calls the *second estate*, namely  $GB$ ...

[ca. 5 blank lines]

//fol. 111r//

[Comment, part 1.] *Aḥmad <ibn> al-Sarī said: This wording that Abū Sa‘d al-Ṣābī mentions, that “the author of Calculation by Two Errors, when he sought the desired result in the place of  $AB$  here, he took  $DB$  and found it in excess over the former. Taking the amount between the two,  $DA$ ...” and so on to the end of what he wrote is not what the author of Calculation by Two Errors in fact uses. This is because if he seeks  $AB$  and finds  $DB$  in excess over it and chooses some amount between the two, namely  $DA$ , then the unknown  $AB$  must become known without multiplication or division, since  $DB$  is known, and the known  $DA$  is subtracted from it, leaving the known  $AB$ ...*

[Jābir, ¶5 continued.] ...by the first error, which is  $DA$ . Since the two errors are different, one of them excessive and the other deficient, he summed these two products of multiplication (*maḍrūbayn*). The sum was congruent to the intermediate,  $AB$ , multi-

<sup>56</sup> $c(a - b)$ .

<sup>57</sup> $b(a - c)$ .

<sup>58</sup> $a = AB$ ,  $b = AD$ ,  $c = AG$ .

<sup>59</sup>As depicted in the diagram already used to illustrate the geometrical proof in ¶2, Figure 1.

<sup>60</sup> $DB - AB = DA$ .

<sup>61</sup> $AB - GB = AG$ .

<sup>62</sup>i.e.,  $DB \times AG + GB \times DA = AB \times GD$ .

plied by the excess of the greatest over the least, namely  $GD$ , [by which] he meant  $DG$ , which is the sum of the two errors. Then he performed the division over it — I mean the division of the sum of the two products (*maḍrūbayn*). From the division came out the amount  $AB$ , which is the desired result.

6. And also, let there be, in the [second]<sup>63</sup> place...

[Comment, part 2.] ...and for that [i.e., finding  $AB$ ] he needs no other operation. Thus what this man is in fact doing with this operation<sup>64</sup> is computing the amount stipulated for him in the question, not the desired result. Suppose for example we ask him for knowledge of what estate is such that if we increase it by half of itself plus a third of itself, it becomes eleven.<sup>65</sup> Then it is as if he stipulated that eleven is  $AB$ , then stipulated [as his first guess] a number, say nine, and added to it half of itself plus a third of itself, resulting in sixteen and a half.<sup>66</sup> Next he stipulates that this is the number  $DB$ <sup>67</sup> then compares  $DB$  to eleven, which is  $AB$ , and finds [it] in excess over eleven by five and a half [which he assigns] to the number  $DA$ .<sup>68</sup> And so he calls  $DA$  the first error — really, I swear! Then he stipulates [another] number, say four, and adds //fol. 112<sup>r</sup>// to it half of itself plus a third of itself..<sup>69</sup>

[Jābir, ¶6 continued.] ...[in the] second [place],<sup>70</sup>  $DB$  and  $GD$  [read:  $GB$ ], each greater than  $AB$ , such that the lines or numbers  $AB$ ,  $GD$  [read:  $GB$ ],  $DB$  are all three different, with  $DB$  the greatest of them and  $AB$  the least of them.<sup>71</sup> And let the excess of the intermediate,  $GB$ , above the least,  $AB$ , be  $GA$ ; and let the excess of the greatest,  $DB$ , above the least,  $AB$ , be  $AD$ .<sup>72</sup> On account of the foregoing, the greatest,  $DB$ , multiplied by the excess of the intermediate above the least, which is  $GA$ , along with the product of  $AB$ ...

[Comment, part 3.] ...and so it becomes seven and a third. He posits that it is the number  $GB$  then compares it to eleven and finds that it's short of [eleven] by three and two thirds [which he assigns] to the number  $GA$ , which he calls the second error. He multiplies seven and one third, which is  $BG$ , by  $AD$ , the first error, and divides the sum of the two dividends<sup>73</sup> by  $GD$ , the sum of the two errors. Thus he obtains  $AB$ , which is the eleven posited to begin with, not the desired result. Thus he does not calculate by the method of two errors because the first number is nine, not sixteen...

[Jābir, ¶6 continued.] ...which is the least, multiplied by  $DG$ , the excess of the greatest over the least, <is congruent to the multiplication-product of the intermedi-

<sup>63</sup>The word 'second' only appears below, after the intervening comment.

<sup>64</sup>Reading *bi-hādhā l-ʿamal* for *bi-hādhā li-l-ʿamal*.

<sup>65</sup>i.e., find  $x$  where  $x + \frac{x}{2} + \frac{x}{3} = 11$ . (By algebra:  $\implies \frac{11}{6}x = 11 \implies x = 6$ .)

<sup>66</sup>i.e., he says let  $AB = 11$ , and let  $x_1 = 9$ , then calculates  $9 + \frac{9}{2} + \frac{9}{3} = 16\frac{1}{2}$ .

<sup>67</sup>i.e., let  $DB = 16\frac{1}{2}$ .

<sup>68</sup>i.e.,  $DB - AB = 16\frac{1}{2} - 11 = 5\frac{1}{2}$ .

<sup>69</sup>i.e., he says let  $x_2 = 4$ , then calculates  $4 + \frac{4}{2} + \frac{4}{3} = 7\frac{1}{3}$ .

<sup>70</sup>The phrase "in the second place" (*ʿalā l-wajhi l-thānī*) straddles the intervening comment; this block of focus text begins with *al-thānī*.

<sup>71</sup>i.e., let  $DB > GB > AB$ . The reading  $GD$ , though it appears twice, must be a scribal error for  $GB$  (as appears in the next line).

<sup>72</sup>i.e.,  $GD - AB = GA$  and  $DB - AB = AD$ .

<sup>73</sup>*murtafaʿayn*, lit., "raised [numbers]."

ate,<sup>74</sup> which is  $BG$ , by the excess of the greatest over the least, which is  $AD$ .<sup>75</sup> Let that [amount] be remembered. Now, the author of *Calculation by Two Errors*, since the result he was seeking was in the place of (*bi-manzilāt*)  $AB$ , he first took  $DB$  and called it the *first estate* and found that it was in excess...

[Comment, part 4.] ...and a half. The second number is four, not seven and a third. These three numbers — I mean eleven, sixteen and a half, and seven and a third — are proportional to the [other] three numbers — I mean proportional to nine, four, and the unknown number asked of us. If we had operated according to the method of two errors and had divided the dividend by  $\frac{112}{10}$  the same  $GD$ , then we would have obtained the unknown: six. It has thus become clear that what he says tells us nothing about calculation by Two Errors, and the same goes for the two figures that come after it. Therefore understand this, I mean that this kind of talk is a bunch of nonsense (?).

[Comment by al-Hamadhānī.] The wise judge Saʿd al-Dīn Asʿad ibn Saʿīd al-Hamadhānī, may God preserve his high rank, said: The method of Two Errors is to multiply, in place of  $BD$ , nine, which is related to it, by  $AG$ ; and in place of  $BG$ , four, which is related to it, by  $AD$ , so that the sum of the two is 55.<sup>76</sup> Then he divides it by the part, which is nine and a sixth, and the desired result emerges.

[Jābir, ¶6 continued.]<sup>77</sup> ...above  $AB$  by the amount  $DA$ , and he called  $DA$  the first, excessive error. Then he went back and tried another amount, coming up with  $GB$ , which he called the *second estate*. It too was found to be in excess above what was sought, namely  $AB$ . So he took the amount by which he had erred, which is  $AG$ , and it was called the second error, also excessive. Then he multiplied the second estate,  $GB$ , by the first error,  $AD$ , and he subtracted from that the product of the first estate,  $DB$ , multiplied by the second error,  $GA$ , for the errors in this case are both excessive.<sup>78</sup> So he is left with two amounts equal to the product of  $AB$  multiplied by  $GD$  [read:  $GB?$ ]. He carries out the division of this remainder by  $GD$  [read:  $GB?$ ], which is the surplus between the first and second errors, and so he obtains from the division the amount  $AB$ , the desired result that he sought to know.

7. Also, in the third place, let  $DB$ ,  $GB$  each be less than  $AB$ , such that the lines or numbers  $AB$ ,  $DB$ ,  $GB$  are also all three different, where  $AB$  is the greatest,  $DB$  is the intermediate, and  $GB$  is the least. Then  $DG$  is the excess of the intermediate over the least;  $AD$  is the excess of the greatest over the intermediate; and  $GA$  is the excess of the greatest over the least. [DIAGRAM 4] According to what we have demonstrated in the foregoing,  $\frac{113}{10}$   $AB \times DG + GB \times DA$  summed together is congruent to  $DB \times DA$ . Let that [amount] be remembered. At this point too, the author of *Calculation by Two Errors*, since he was seeking to find  $AB$ , first took  $DB$  and called it the first estate. Using it, he erred by  $DA$ , so he called  $DA$  the first, deficient error.

<sup>74</sup>This rather interventionist emendation is meant to make the sentence make sense both grammatically and mathematically. On mathematical grounds, this is clearly the meaning that the text originally conveyed; I base the wording of this emendation on how the text describes similar terms in other equations.

<sup>75</sup>i.e.,  $DB \times GA + AB \times DG = BG \times AD$ .

<sup>76</sup>The manuscript expresses this number using decimal 'Arabic' numerals.

<sup>77</sup>The manuscript indicates the continuity between this text and the previous portion of the *matn* with a small circle that appears above both the last word of that portion (*zāʿidan*) and the first word of this continuation (*ʿalā*).

<sup>78</sup>i.e., this is why he subtracted here instead of adding.

Then he went back and tried a second estate, and he happened to get  $GB$ , in which he erred by  $GA$ , so he called  $GA$  the second error, also deficient. Then he multiplied  $DB$ , the second estate, by  $DA$ , the first error, and he was left with an amount equal to  $AB \times DG$ . When he divided this remainder by  $DG$ , which is the surplus between the first error and the second error, he obtained  $AB$  from the division, being the result which he was seeking. Q.E.D.

The end. Praise be to God, lord of the worlds, and his blessings upon our *sayyid* Muḥammad and all of his family.

## IV Analysis

The text does not explicitly distinguish between the original (anonymous) text and Jābir's revisions and additions. Nevertheless, ¶4 begins by indicating that the foregoing text (¶1–3) may require further elaboration, and then ¶5 explicitly refers to the author of the *Calculation by Two Errors*, which seems to be a short version of the title of the work that Jābir undertook to revise and improve. This strongly suggests that Jābir's contribution begins there, probably with ¶4 and definitely with ¶5. As for the comments by Ibn al-Sarī and Sa'd al-Dīn al-Hamadhānī, these are clearly labeled in the text.

In the present section, I will provide a rather detailed mathematical paraphrase and analysis of the treatise. Though mathematicians and historians of mathematics may find it excessive to spell out every step, my hope is that this will make explicit more of my interpretive reasoning. In other words, I aim to foreground the philology — the self-critical interpretation of a text and a textual tradition — involved in rendering a medieval mathematical text into modern mathematical modes of expression, rather than elide it.

### Method (¶1)

Let  $f(x) = a_1x \pm a_2x \pm \cdots \pm a_Nx \pm b_1 \pm b_2 \pm \cdots \pm b_P$ , where  $N, P \in \mathbb{Z}^+$  and

$$x, a_1, \dots, a_N, b_1, \dots, b_P \in \mathbb{R}^+.$$

Find  $x$  such that  $f(x) = y$  for some  $y \in \mathbb{R}^+$ .

Let  $x_1$  be the first guess (*māl*, “estate”). Let  $y_1 = f(x_1)$ . If  $y_1 = y$ , then  $x = x_1$ . Lucky guess. If not, then let  $e_1 = y_1 - y$  be the first error. If  $y_1 > y$ , then  $e_1$  is an excessive error (i.e.,  $e_1 > 0$ ). If  $y_1 < y$ , then  $e_1$  is a deficient error (i.e.,  $e_1 < 0$ ).<sup>79</sup>

Let  $x_2$  be the second guess. Let  $y_2 = f(x_2)$ . Assuming  $y_2 \neq y$  (so that the answer  $x$  is not simply  $x_2$ ), let  $e_2 = y_2 - y$  be the second error, excessive if  $y_2 > y$  and deficient if  $y_2 < y$ .

Now if  $e_1 < 0$  and  $e_2 < 0$  or  $e_1 > 0$  and  $e_2 > 0$  (i.e., if  $e_1e_2 > 0$ ), then let the ‘part’ (*juz'*) be  $j = |e_2 - e_1|$ . Otherwise (if  $e_1e_2 < 0$ ),  $j = |e_1| + |e_2|$ . (In either case, we can express this as  $j = |e_2 - e_1|$ , since when  $e_1e_2 < 0$ ,  $|e_2 - e_1| = |e_1| + |e_2|$ .) Compute  $x_1e_2$  and  $x_2e_1$ . If  $e_1e_2 < 0$ , then sum them together:  $|x_1e_2| + |x_2e_1|$ . (Since

<sup>79</sup>Thus the words ‘excessive’ and ‘deficient’ are used to represent a positive and negative result in the absence of the concept of negative numbers.

$x_1 > 0$  and  $x_2 > 0$  by assumption, for the text does not employ negative numbers, this is just  $|x_1e_2 - x_2e_1|$ . If  $e_1e_2 > 0$ , then subtract:  $|x_1e_2 - x_2e_1|$ . (Thus, either way we are finding  $|x_1e_2 - x_2e_1|$ .) Now divide by  $j$  to obtain

$$x = \frac{|x_1e_2 - x_2e_1|}{|e_2 - e_1|}. \quad (2)$$

Using modern algebraic computation (including the concept of negative numbers), it is trivial to justify this method by expressing  $x$  in terms of  $x_1, x_2, f(x_1), f(x_2)$  then expressing  $f(x)$  as  $ax + b$  (i.e., reducing it to linear and constant terms) and reducing the result to

$$x = \frac{y - b}{a},$$

which is the algebraic solution to the equation

$$ax + b = y.$$

Perhaps more intuitively, working in the other direction, the method of Double False Position can be derived from a basic result of linear algebra, namely that two points define a line, whose slope is thus known (see Figure 4). Once we know  $(x_1, y_1)$  and  $(x_2, y_2)$  — by choosing guesses  $x_1$  and  $x_2$  arbitrarily then calculating the corresponding outputs  $y_1$  and  $y_2$  — we can express them in terms of the two (possibly negative) errors  $e_1 = y_1 - y$  and  $e_2 = y_2 - y$ :  $(x_1, y + e_1)$  and  $(x_2, y + e_2)$ . The line's slope is then

$$a = \frac{(y + e_2) - (y + e_1)}{x_2 - x_1} = \frac{e_2 - e_1}{x_2 - x_1}.$$

Now we start at point  $(x_1, y_1)$ . Since  $(x, y)$  is on the same line, we know that

$$\frac{y - y_1}{x - x_1} = \frac{-e_1}{x - x_1}$$

also =  $a$ , so

$$-e_1 = \frac{e_2 - e_1}{x_2 - x_1}(x - x_1)$$

or

$$x - x_1 = \frac{-e_1(x_2 - x_1)}{e_2 - e_1}.$$

As a result,

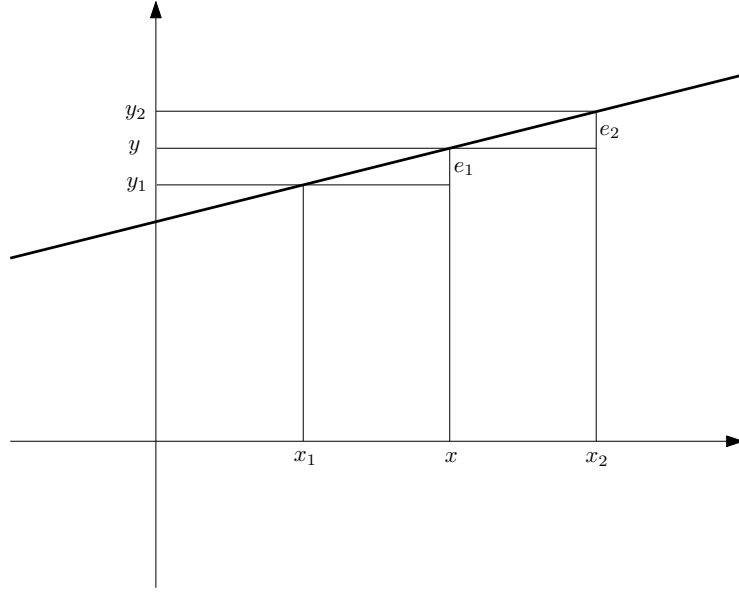
$$x = \frac{-e_1(x_2 - x_1) + x_1(e_2 - e_1)}{e_2 - e_1} = \frac{x_1e_2 - x_2e_1}{e_2 - e_1}.$$

But of course this is not how the text proceeds.

## Geometrical Proof of a Relation Between Line Segments (¶2)

Let  $AB$  be a line (segment) subdivided by two points along it,  $G$  and  $D$ :  $A—G—D—B$ . The resulting line segments are related as follows:

$$AB \times GD + AG \times DB = (GD + AG) \times (GD + DB) = AD \times GB \quad (3)$$



**Figure 4:** Double False Position visualized using Cartesian coordinates.

*Proof:* Construct DIAGRAM 1 (see Figures 1 and 3). The rectangles  $DY$  and  $TZ$  are equal because they are complements about the diagonal of the square  $GBEZ$  [Euclid, *Elements* 1.43]. This lets us equate the gnomon  $BAMZKY$  [supposing we label the lower-right corner of the diagram  $M$ ] — which the text calls  $AY$  plus  $LZ$  — with rectangle  $AH$ , since the only difference between the two is that  $AH$  contains  $TZ$  rather than  $DY$ , but as we just saw,  $DY = TZ$ . The rectangle  $AY$  can be expressed as  $AB \times YB = AB \times GD$ , and rectangle  $LZ$  is  $LK \times KZ = AG \times DB$ , so the gnomon is  $AB \times GD + AG \times DB$ . On the other hand, the rectangle  $AH$  is  $AD \times DH = AD \times GB$ . Therefore,  $AB \times GD + AG \times DB = AD \times GB$ . Q.E.D.

Suter says that this step is flawed because Jābir unnecessarily restricts his result by using a square rather than a rectangle in the proof's geometrical construction. But this part of the text is not purporting to be the entire proof; it is simply proving a geometrical relation between lines and the numbers corresponding to their lengths. It is only when we arrive at ¶5 that Suter's critique hits home. Indeed, it is there that Ibn al-Sārī critiques Jābir — a critique that I believe amounts to the same one that Suter makes.

### Generalize this result to any three numbers (¶3)

Let  $a > b > c > 0$ , where  $a, b, c \in \mathbb{R}^+$ . Then

$$a(b - c) + c(a - b) = b(a - c). \quad (4)$$

In today's algebra, this is trivial to prove. The text, however, offers a geometric proof that rests on the proof in ¶2:

Assign the three numbers to segments of the original line in Figure 1:  $a = AB$ ,  $b = AD$ ,  $c = AG$ . In ¶2, we showed that  $AB \times GD + AG \times DB = AD \times GB$ , where  $GD = AD - AG$ ,  $DB = AB - AD$ , and  $GB = AB - AG$ . Thus

$$AB \times (AD - AG) + AG \times (AB - AD) = AD \times (AB - AG). \quad (5)$$

Substitute  $a, b, c$  for  $AB, AD, AG$  to obtain  $a(b - c) + c(a - b) = b(a - c)$ . Q.E.D.

### Restating the result in different terms (¶4)

This result can be restated in terms of three other line segments, called  $DB, AB, GB$ , where the points  $A, B, G, D$  do *not* correspond to those used in ¶2–3. In particular, their relative sizes are different:  $DB > AB > GB$  (whereas in ¶2–3 the relation was  $AB > GB > DB$ ). This means that we should draw the points as follows:  $D—A—G—B$ . Thus we now have  $a = DB, b = AB, c = GB$ . By Equation 4,

$$DB \times (AB - GB) + GB \times (DB - AB) = AB \times (DB - GB).$$

Furthermore, we can define all the differences more simply as their own line segments:  $DB - AB = DA, AB - GB = AG$ . (Also, though the text doesn't mention this relation explicitly,  $DB - GB = GD$ .) Substituting in these simpler expressions, we obtain

$$DB \times AG + GB \times DA = AB \times GD. \quad (6)$$

### Relating this result to Double False Position (¶5)

According to Jābir, the method of *Calculation by Two Errors* (Double False Position) can be mapped onto Equation 6.

In the first case (one error excessive, the other deficient):  $x = AB, x_1 = DB, e_1^+ = x_1 - x = DB - AB = DA, x_2 = GB, e_2^- = x - x_2 = AB - GB = AG$ .

[This step, if I have understood it correctly, is Jābir's misstep: by defining the two errors as differences between the unknown,  $x$ , and, respectively, the two guesses  $x_1$  and  $x_2$ , Jābir has entirely changed their definition as it appears in the method of Double False Position, namely  $e_1 = y_1 - y$  and  $e_2 = y_2 - y$ .]

Thus, continues Jābir,  $x_1 e_2 = DB \times AG$ , and  $x_2 e_1 = GB \times DA$ . Since one error was excessive and the other deficient, the method says to sum them:  $x_1 e_2 + x_2 e_1 = DB \times AG + GB \times DA$ . But (by Equation 6) we know that this equals  $AB \times GD$ . Since  $GD = DG = DB - GB$ , we can write  $GD = DB - GB = (DB - AB) + (AB - GB) = e_1 + e_2$ . Thus  $AB \times GD = x(e_1 + e_2)$ . Therefore,

$$x_1 e_2 + x_2 e_1 = x(e_1 + e_2). \quad (7)$$

Then, Jābir tells us, the author of *Calculation by Two Errors* divided the left side of this equation by  $e_1 + e_2$  to obtain the result

$$\frac{x_1 e_2 + x_2 e_1}{e_1 + e_2} = AB [= x] \quad (8)$$

### The same, for the case where both errors are excessive (¶6)

[Instead of continuing in the order of the text and translation, I will skip the comments of Ibn al-Sarī and al-Hamadhānī for now and get back to them after finishing the analysis of Jābir.]

In the second case, both guesses produce an output that is greater than the target:  $DB > GB > AB$ , so that we can again return to Equation 4, this time making the substitutions  $a = DB, b = GB, c = AB$  to obtain

$$DB \times (GB - AB) + AB \times (DB - GB) = GB \times (DB - AB).$$

(Though the text does not offer a diagram at this point, the relations imply that the points should be arranged like this:  $D—G—A—B$ .) From our diagram we observe that  $GA = GB - AB, AD = DB - AB$  (and, though again the text does not mention it,  $DG = DB - GB$ ). This allows us to simplify the equation to

$$DB \times GA + AB \times DG = GB \times AD.$$

Now again Jābir correlates this with the method of Double False Position, with the same problem described above.

### The same, for the case where both errors are deficient (¶7)

In the third case, both guesses are low:  $AB > DB > GB$ . So again, in Equation 4, we substitute  $a = AB, b = DB, c = GB$ , and so obtain

$$AB \times (DB - GB) + GB \times (AB - DB) = DB \times (AB - GB).$$

As DIAGRAM 4 illustrates, the points are arranged like this:  $A—D—G—B$ . Therefore,  $DG = DB - GB, AD = AB - DB, GA = AB - GB$ . So again we can simplify the equation to

$$AB \times DG + GB \times AD = DB \times GA.$$

And again, Jābir correlates this with the method of Double False Position, with the same fatal flaw.

### Ibn al-Sarī's Critique of ¶5–7

[The critique focuses on ¶5, but as Ibn al-Sarī points out, it applies just as much to ¶6–7.]

Ibn al-Sarī begins by pointing out that Jābir has misrepresented what the method of Double False Position entails. Jābir, Ibn al-Sarī explains, has defined  $x = AB, x_1 = DB, e_1 = DA = DB - AB = x_1 - x$ , and so on. Thus  $x = AB$  is the unknown quantity sought but it is used to calculate  $e_1$ ; thus it has “become known without multiplication or division,” since  $x_1$  (our first guess) is known and, apparently,  $e_1$  is known as well. Thus all one has to do to find  $x$ , as Jābir has defined it, is to calculate  $x = x_1 - e_1$ . Therefore, Ibn al-Sarī continues, in effect Jābir is simply “computing the amount stipulated for him in the question” ( $y$ ), “not the desired result” ( $x$ ).

Ibn al-Sarī is not saying that Jābir does not know how to use the method of Double False Position in practice, but rather that Jābir's purported proof implies the faulty method Ibn al-Sarī describes.

Ibn al-Sarī proceeds to explain his critique by means of an example: suppose we want to know  $x$  such that

$$x + \frac{x}{2} + \frac{x}{3} = 11. \tag{9}$$



The method implied by Jābir's proof would be to say  $AB [= x] = 11$  (even though in fact  $y$ , not  $x$ , is supposed to equal 11), then make a guess, 9, then plug it into the left side of Equation 9 to obtain

$$9 + \frac{9}{2} + \frac{9}{3} = 16\frac{1}{2}.$$

But instead of defining  $DB$  (which is supposed to be  $x_1$ ) as 9, the first guess, now Jābir would have us define  $DB$  as  $16\frac{1}{2}$  (which is actually the *output* of the first guess,  $y_1$ ) and then compares  $DB$  to  $AB$ , finding that as he has defined them  $DB - AB = DA = 16\frac{1}{2} - 11 = 5\frac{1}{2}$ .

Then, Ibn al-Sarī continues, Jābir would have us guess another number  $x_2 = 4$ . Plugging this second guess into Equation 9 produces

$$4 + \frac{4}{2} + \frac{4}{3} = 7\frac{1}{3}.$$

This result is then called  $GB$  (or  $BG$ ), so  $BG = 7\frac{1}{3}$ , and then compared to 11; the difference between them, defined as  $GA$ , is  $GA = 11 - 7\frac{1}{3} = 3\frac{2}{3}$ , which is then called the second error.

To continue our erroneous calculation, we compute  $BG \times AD$  (where  $AD$  is the same as  $DA$ ) to obtain  $7\frac{1}{3} \cdot 5\frac{1}{2} = 40\frac{1}{3}$ . [Ibn al-Sarī skips the next step, presumably because it is obvious, namely taking the other product,  $DB \times AG$ , where  $AG$  is  $GA$ , which is  $16\frac{1}{2} \cdot 3\frac{2}{3} = 60\frac{1}{2}$ .] Then we sum these two products to obtain  $40\frac{1}{3} + 60\frac{1}{2} = 100\frac{5}{6}$ , and divide that sum by  $GD$ , where  $GD$  is defined as “the sum of the two errors,” namely  $DA + GA = 5\frac{1}{2} + 3\frac{2}{3} = 9\frac{1}{6}$ . And so this means that we calculate  $100\frac{5}{6} \div 9\frac{1}{6} = \frac{605}{6} \times \frac{6}{55} = 11$ . This, observes Ibn al-Sarī, is nothing but the desired output initially stipulated in the question ( $y$ ), not the unknown that was to produce it ( $x$ ).

Thus, Ibn al-Sarī concludes, Jābir's proof is not about Double False Position at all because “the first number” (i.e.,  $x_1$ ) should be 9, not  $16\frac{1}{2}$  (the first output  $y_1$ ); and “the second number” (i.e.,  $x_2$ ) should be 4, not  $7\frac{1}{3}$  (the second output  $y_2$ ). These numbers stand in a relation of proportionality to each other:

$$16\frac{1}{2} : 9 = 7\frac{1}{3} : 4 = 11 : x,$$

or, more generally,

$$y_1 : x_1 = y_2 : x_2 = y : x. \quad (10)$$

(As Ibn al-Sarī seems to be pointing out here, this problem does not require *double* false position: a single guess would have sufficed, since then by Equation 10,  $x = \frac{x_1}{y_1}y$ , or, in this example,

$$x = \frac{9}{16\frac{1}{2}} \cdot 11 = 6.$$

In any case, this is not the main point he is trying to make.)

If we had used Double False Position properly and calculated  $x_1e_2 + x_2e_1 = 9 \cdot 3\frac{2}{3} + 4 \cdot 5\frac{1}{2} = 55$  and then divided by the same  $GD [= e_1 + e_2 = 9\frac{1}{6}]$ , we would have arrived at the correct answer:  $x = 55 \div 9\frac{1}{6} = 55 \cdot \frac{6}{55} = 6$ .

The next two parts of Jābir's proof (for the cases where the two errors have the same sign, positive or negative) follow the example of the first part, so Ibn al-Sarī doesn't deal with them individually; instead, he dismisses Jābir's proof as insufficient to tell us anything about Double False Position.

## Al-Hamadhānī's Comment

Here a brief comment by one Sa'd al-Dīn As'ad ibn Sa'īd al-Hamadhānī appears, spelling out the calculation implied by Ibn al-Sarī's statement (indeed, al-Hamadhānī does precisely what I just did in my paraphrase of Ibn al-Sarī). He says that in the formula  $BD \times AG + BG \times AD$  one should replace  $BD$  (as Jābir had defined it) with the proportional number 9 and  $BG$  (as Jābir had defined it) with the proportional number 4, in order to obtain  $9 \cdot 3\frac{2}{3} + 4 \cdot 5\frac{1}{2} = 55$ . Then divide that by the *part*, which is  $9\frac{1}{6}$ , to get "the desired result." (Al-Hamadhānī doesn't spell out what that result is, presumably leaving it to the reader to perform the calculation.)

## V Suter as a Reader of the Treatise

Suter did not have a high opinion of Jābir's treatise. He consulted the text contained in Leiden, Univ. Library, or. 14, nos. 3–4 (pp. 218–223).<sup>80</sup> To judge from his description of the text, it was very similar to the version contained in the Columbia manuscript, including the intermingled commentary of (Ibn) al-Sarī.<sup>81</sup> Suter did not deign to publish the text or a translation: "Since [the text's] proof itself is a bit flawed, it would be a waste of effort to wish to provide a complete word-by-word translation of it..."<sup>82</sup> Instead, he summarizes the proof and points out its flaw. In the process, he says, "I... avail myself as often as possible of our present-day manner of representation," that is, modern mathematical notation.<sup>83</sup>

Jābir's first step is correct, Suter remarks, namely his statement and geometrical proof of a relation between three arbitrary, consecutive segments of a line: given the line  $AB$  and two points  $G$  and  $D$  between  $A$  and  $B$ , in the order  $A—G—D—B$ , Jābir shows (¶2) that

$$AB \times GD + AG \times BD = AD \times BG. \quad (11)$$

But after that he starts to go wrong. "Now," continues Suter, "Jābir sets  $AG$  equal to the first guess  $\alpha_1$  [=  $x_1$ ] and  $GD$  equal to the first error  $f(\alpha_1)$  [=  $e_1$ ], and further  $AB = \alpha_2$  [=  $x_2$ ] and  $BD = f(\alpha_2)$  [=  $e_2$ ]; then from the equation above [Equation 11] he obtains the following expression for the unknown magnitude  $AD$ :

$$AD = x = \frac{\alpha_1 f(\alpha_2) + \alpha_2 f(\alpha_1)}{f(\alpha_1) + f(\alpha_2)} \left[ = \frac{x_1 e_2 + x_2 e_1}{e_1 + e_2} \right],$$

which is correct in the case where the errors  $f(\alpha_1)$  and  $f(\alpha_2)$  [ $e_1$  and  $e_2$ ] have different signs but is here taken absolutely."<sup>84</sup> Here Suter suggests that Jābir has already gone

<sup>80</sup>Suter, "Einige geometrische Aufgaben," 23–24.

<sup>81</sup>Item no. 4 of the Leiden manuscript, Suter describes, "contains not only... the commentary but also the text's continuation mixed together with glosses" (

"Nr. 4 enthält nämlich nicht nur... den Kommentar, sondern die Fortsetzung des Textes mit Glossen untermischt"); Suter, 24. Like the Columbia manuscript, the Leiden manuscript calls the commentator "al-Sarī" rather than Ibn al-Sarī.

<sup>82</sup>Suter, 24: "Da der Beweis selbst etwas verfehlt ist, so wäre es eine unnütze Mühe, eine vollständige wörtliche Übersetzung desselben geben zu wollen..."

<sup>83</sup>Suter, 24: "...und bediene mich so oft als möglich unserer heutigen Darstellungsweise."

<sup>84</sup>Suter, 24–25.

astray by claiming generality for a result that only applies in a special case. But in fact, this is only the first of the three parts of Jābir’s proof, each addressing one of the three possible cases: the errors have opposite signs, i.e.,  $e_1 e_2 < 0$  (§5); both errors are “excessive,” or positive, i.e.,  $e_1 > 0, e_2 > 0$  (§6); or both errors are “deficient,” or negative, i.e.,  $e_1 < 0, e_2 < 0$  (§7).

Suter’s next critique should be taken more seriously: as he mentions, it is the same critique that Ibn al-Sarī himself undertook to make. It is not clear how closely Suter read Jābir’s text, but we can be sure that Ibn al-Sarī had read it carefully. As Suter puts it:

Jābir seems not to have recognized, however, that this proof is valid only for a very special case, namely for the case where  $e_1 + e_2$  is exactly equal to  $BG = x_2 - x_1$ . This was also recognized by the glossator Aḥmad ibn al-Surrī [i.e., Ibn al-Sarī] when he remarks that for calculating the unknown here there would of course be no need at all for any multiplication or division, since  $x$  would of course be simply  $= AG + DG = x_1 + e_1$ , or  $= AB - BD = x_2 - e_2$ .<sup>85</sup>

Indeed, this is precisely the point that Ibn al-Sarī makes.

But Suter’s next remark seems to misread the rest of Ibn al-Sarī’s commentary. Suter writes:

Another error that the glossator [Ibn al-Sarī] accuses the author [Jābir] of making is, however, unfounded [i.e., the accusation is unfounded]. He seems to have overlooked the fact that when Jābir al-Šābī applies his geometrical equation to the Rule of Two Errors, he adopts different letters from those in the figure accompanying the proof [of that geometrical equation]...<sup>86</sup>

This suggests that Suter did not realize that Ibn al-Sarī’s entire commentary (assuming it is the same in the Leiden and Columbia manuscripts) is devoted to addressing the same fatal flaw in Jābir’s proof that Suter identified. As described in §IV above, Ibn al-Sarī begins by noting this fatal flaw then devotes the rest of his note to *illustrating* that flaw with a numerical example. So Ibn al-Sarī is not pointing out “another error” at all, as Suter thought, but simply seeking to make clear to his reader why Jābir’s proof fails. Suter’s explanation of Ibn al-Sarī’s purported error indicates that Suter must have read Ibn al-Sarī’s commentary very cursorily, since he imagines that Ibn al-Sarī was confused by Jābir’s repeated redefinition of the line segments corresponding to the underlying quantities in question (§4–5, 6, and 7).

<sup>85</sup>Suter, 25: “Ĝābir scheint aber nicht erkannt zu haben, daß dieser Beweis nur für einen ganz speziellen Fall zutrifft, nämlich für den Fall, wo  $f(\alpha_1) + f(\alpha_2)$  genau gleich  $bg = \alpha_2 - \alpha_1$  ist. Das hat auch der Glossator Aḥmed b. el-Surrī eingesehen, indem er bemerkt, daß es hier zur Berechnung der Unbekannten ja gar keiner Multiplikation und Division bedürfe, denn  $x$  wäre ja einfach  $= ag + dg = \alpha_1 + f(\alpha_1)$ , oder  $= ab - bd = \alpha_2 - f(\alpha_2)$ .”

<sup>86</sup>Suter, 25: “Ein anderer Fehler, den der Glossator dem Verfasser vorwirft, ist aber unbegründet, er scheint übersehen zu haben, daß Ĝābir el-Šābī bei der Anwendung seines geometrischen Satzes auf die Regel der beiden Fehler andere Buchstaben annimmt als in der Beweisfigur...”

Suter's subsequent remark confirms his cursory reading not only of Ibn al-Sarī but of Jābir's text as well. According to Suter, Ibn al-Sarī "also seems not to have correctly construed the sense of some admittedly obscure passages." Here Suter opens the possibility that he himself has overlooked something, continuing,

we at least found no other error than the one already discussed, including in the continuation of the treatise, where the author [Jābir] gives the proofs for the cases where the errors  $e_1$  and  $e_2$  have the same sign, so that  $x_1$  and  $x_2$  are either both greater or both smaller than  $AD [= x]$ .<sup>87</sup>

It is not clear which passages of Jābir's text Suter found "obscure," since he correctly understood that the rest of Jābir's treatise repeats the proof for the other two cases (§6–7). It is even less clear, then, which part of Ibn al-Sarī he thought might be misinterpreting those obscurities. To his credit, Suter does not claim here to have a full understanding of either Jābir's or Ibn al-Sarī's text. In spite of this, he is nonetheless inclined to view Ibn al-Sarī's commentary as flawed.

Suter's subsequent discussion embraces the assumption that he, Suter, has understood the texts in question sufficiently to be able to evaluate Jābir's (and presumably also Ibn al-Sarī's) worth as a mathematician. He introduces his own corrections to Jābir's proof with the words

Jābir certainly cannot have been much of a mathematical mind; otherwise, he would have recognized his own error and would easily have figured out how to come to his own aid: he could have generalized his proof in the following way...<sup>88</sup>

What then follows is Suter's revised version of the geometrical proof that omits certain constraints. In particular, he constructs the same diagram with the same labels (see Figure 3), but without requiring the rectangles  $BZ (= BGZE)$ ,  $DK (= DGKT)$ , or  $YH (= YTHE)$  to be squares (i.e., he does not set  $BG = GZ$ ,  $DG = GK$ , or  $YT = TH$ ). He then makes the figure correspond to Double False Position as follows (using my rather than Suter's notation):  $x_1 = AG = KL$  (first guess),  $e_1 = GK = BY$  (first error),  $x_2 = AB = LY$  (second guess), and  $e_2 = YE = KZ$  (second error). [Furthermore,  $x = AD$  (the quantity we wish to find).]<sup>89</sup> Finally, he points out that the

<sup>87</sup>Suter, "Einige geometrische Aufgaben," 25: "...und scheint auch den Sinn einiger allerdings undeutlicher Stellen nicht richtig aufgefaßt zu haben; wir wenigstens haben keinen andern Fehler als den eben besprochenen gefunden, auch nicht in der Fortsetzung der Abhandlung, wo der Verfasser die Beweise für die Fälle gibt, wo die Fehler  $f(\alpha_1)$  und  $f(\alpha_2)$  beide gleiches Zeichen haben, also  $\alpha_1$  und  $\alpha_2$  entweder beide größer oder beide kleiner als  $ad$  sind; auf diese Beweise treten wir hier aber nicht mehr ein, sie sind leicht aus dem ersten abzuleiten."

<sup>88</sup>Suter, 25: "Ein bedeutender mathematischer Kopf kann Ĝābir allerdings nicht gewesen sein, sonst hätte er seinen Fehler erkannt und sich leicht zu helfen gewußt, er hätte seinen Beweis in folgender Weise verallgemeinern können..."

<sup>89</sup>The geometrical diagram does constrain these quantities such that  $EH : HT = TK : GK$  (to ensure that the diagonal goes through point  $T$ ). This means that  $EH = BD = AB - AD$ ,  $HT = YE$ ,  $TK = DG$ , and  $GK = BY$ ; and so  $(AB - AD) : YE = DG : BY = (AD - AG) : BY$ , or  $(x_2 - x) : e_2 = (x - x_1) : e_1$ , i.e., that there is a fixed proportion between how far off the guess is (where we are dealing with the case in which  $x_2 > x$  and  $x_1 < x$ ) and how far off the resulting output is. Since we are dealing only with linear functions, this constraint poses no problem; the inverse of that fixed proportion is (the magnitude of) the line's slope.

gnomon  $BAMZKY$  (i.e., the sum of the rectangles  $BALY$  and  $KLMZ$ ) is still equal to the rectangle  $DAMH$  (because  $BDTY = TKZH$ , since the two rectangles are complements about the diagonal).<sup>90</sup> As a result,  $AB \times BY + KL \times KZ = AD \times DH$ ; since  $DH = BY + YE$ , this becomes  $x_2e_1 + x_1e_2 = x(e_1 + e_2)$ , or

$$x = \frac{x_2e_1 + x_1e_2}{e_1 + e_2},$$

Q.E.D.

Having completed the proof, Suter then notes that like Jābir, Ibn al-Sarī too failed to recognize the generalized version of Jābir’s proof.<sup>91</sup> In a certain sense, this is true: Ibn al-Sarī does not provide the generalized proof offered by Suter, or indeed any other proof, in place of Jābir’s flawed proof. But perhaps Ibn al-Sarī’s only aim in the commentary was to show why Jābir’s proof fails to work. It seems a bit hasty to say that Ibn al-Sarī failed to come up with a correct version of the proof when he may simply have chosen not to present it here.

## VI Mathematical Philology

Suter was a prolific historian and philologist of Arabic mathematics and did much to advance the field. The son of a farmer and postmaster in a village outside of Zurich, he was remembered as an “unpretentious man,” a hardworking and humble scholar who resolved to learn Arabic at the age of forty out of a fascination with the Islamic world, and a “free thinker” who believed in the similarity of world religions and the common humanity of all.<sup>92</sup> He worked at a time when even less (much less) of the relevant source material was available outside of manuscripts and when photographs of distant manuscripts were much harder to come by. He read many mathematical texts attentively and with great discernment. His publications on the topic are a vast repository of information and astute analysis and remain key references today. In the case of Jābir’s text and Ibn al-Sarī’s response, he does not pretend to have dwelt on it at length or captured every nuance of the text.

For all these reasons, it would be rash, unproductive, and entirely unfair to hurl back Suter’s insults at him, calling him not much of a philological mind, just as he called Jābir not much of a mathematical mind, and thus generalize about Suter based on a single section of a single scholarly article. Nevertheless, the rapidity of Suter’s reading of the treatise was driven by the overarching priorities and methodological principles embraced by Suter and his fellow historians of mathematics. For this reason — combined with his warm and conscientious attitude toward Arabic mathematical texts, which rules out any facile dismissal of his work — it is perhaps worth dwelling for a moment on the characteristics of Suter’s reading of the treatise before considering what alternative mode might best suit a different set of priorities.

<sup>90</sup>This exploits Euclid, *Elements* 1.43, to make a point that is less obvious than the one Jābir had used that theorem to make; see n. 47 above.

<sup>91</sup>Suter, “Einige geometrische Aufgaben,” 26: “Dies hat auch der Glossator Aḥmed b. el-Surrî nicht erkannt.”

<sup>92</sup>Ruska, “Heinrich Suter,” esp. 409 (“diesem anspruchlosen Manne”), 411 (“ein freier Denker”), 411–412.

Suter was working within the tradition that approaches the history of science and mathematics by asking who first discovered things that we now know to be true and when. Information not pertaining to this line of inquiry was accordingly unimportant — hence his decision to refrain from publishing Jābir’s faulty proof verbatim.

Tellingly, after he mentions the treatise’s (correct) proof of Equation 11 in ¶2, Suter continues by regretting that he could not answer the question that presumably he could expect his reader to be asking: who, and in particular which nation, first came up with that correct proof? Suter writes: “Whether this theorem belongs properly to the Greeks or the Arabs, we cannot decide; it is not to be found in Euclid to our knowledge.”<sup>93</sup> Likewise, to conclude his discussion of Jābir’s treatise and Ibn al-Sarī’s commentary, Suter writes that he cannot help but mention that the existence of this treatise and Ibn al-Nadīm’s references to other works on Double False Position refute the view expressed by some of his contemporaries that the method of Double False Position was first discovered in the twelfth century by European mathematicians.<sup>94</sup>

To avoid any misunderstanding, it is worth emphasizing here that the response to Suter that I propose is *not* a critique of “Orientalist thought,” a vindication of Arabic or Islamic mathematics in the face of European bias or ignorance. That vindication is precisely what Suter was eager to carry out. Instead, Suter’s blind spot is connected to the approach to the history of science and mathematics that he embraced, one in which the wheat must be separated from the chaff — according to simple scientific or mathematical criteria, not hermeneutically recursive historical or textual criteria — so that the historian could avoid wasting too much effort on the chaff. In other words, it is a historical approach in which the historian adopts the criteria of his own contemporaries in the natural and mathematical sciences and uses them as *historical* criteria.<sup>95</sup> The question of most interest to historians embracing this approach is when each aspect of modern science or mathematics was first “discovered.” In addressing an individual scholar of the past, the questions then become how much he knew and understood of (modern) science or mathematics, and how much credit he deserves for uncovering some part of that modern body of knowledge.<sup>96</sup>

While quite powerful in its own way, this approach tends to downplay or omit altogether an account of how mathematicians of the past thought about, discussed, arrived at, and communicated their results. With an emphasis on what they knew and when they knew it, in other words, it tends to skip over false starts, flawed proofs, and critiques of such errors, thus suppressing valuable evidence for the aims of mathematicians and

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<sup>93</sup>Suter, “Einige geometrische Aufgaben,” 25: “Ob dieser Satz griechisches oder arabisches Eigentum sei, können wir nicht entscheiden, bei Euklid findet er sich unseres Wissens nicht.”

<sup>94</sup>Suter, 26–27: “Man entschuldige uns, wenn wir hier folgende Bemerkung nicht unterlassen können...”

<sup>95</sup>This should not be confused with using one’s contemporary scientific criteria as *scientific* criteria for assessing the past, which entails using what we know or think we know today in order to gain a perspective on past scientific work that might not have been available to past scientists themselves. The difference is crucial: when these scientific criteria are used as a substitute for historical criteria, we allow present-day scientific concerns to warp our understanding of how and why ideas developed. See further Richard Rorty et al., *Philosophy in History: Essays on the Historiography of Philosophy* (Cambridge: Cambridge University Press, 1984), 1–14, who frame similar debates in terms of “history of philosophy” versus “intellectual history.”

<sup>96</sup>See Thomas S. Kuhn, *The Structure of Scientific Revolutions*, 3rd ed. (Chicago: University of Chicago Press, 1996), chs. 1–2.

the conceptual frameworks that conditioned those aims and how they were pursued and that were in turn shaped by all aspects of mathematical production, not only the statements and proofs admired by modern mathematicians.

A philology of mathematics that takes such evidence into account will be the best equipped to produce the kind of deeper history of mathematics that Roshdi Rashed has advocated, a history not only of methods available and theorems proven (or at least exploited) but also of conceptual framing, modes of understanding, and notions of the possible directions available to a given field of mathematics in a given time and place.<sup>97</sup>

In the case of Jābir's treatise on Double False Position and the Columbia manuscript, such a mode of reading, applicable to flawed and flawless mathematical texts alike, allows us to return to the juxtaposition with which this article began. What was a flawed proof of a numerical method that today's mathematicians would regard as hopelessly elementary doing in the same manuscript, copied by the same scribe, as Omar Khayyam's pathbreaking treatise on algebra and, indeed, coming right after it?

As already mentioned, Double False Position could be very useful in practice. But this manuscript was not a manual for traders; clearly this collection was produced by and for mathematicians, focused on theoretical texts and demonstration of theorems, not practical numerical methods and their applications. Why, then, include the treatise on Double False Position?

The answer, I propose, lies in precisely what Suter found unsatisfactory about the text: the faulty proof that Jābir added to the basic description of the method of Double False Position, along with Ibn al-Sarī's critique of that proof. This may seem like an odd proposal: why would working mathematicians wish to preserve and even study a misguided, incorrect proof? But the Columbia manuscript is evidence of just that wish: mathematicians and students of mathematics in the medieval Islamic world — in particular Iran, probably Hamadān — were interested in understanding what was wrong with Jābir's proof.<sup>98</sup> This would have offered them a lesson in how to catch a proof's fault while preserving an episode in the history of their discipline.

Nor was this episode necessarily lodged exclusively in the past from the perspective of the scholars who used this manuscript. After all, there were plenty of other treatises on Double False Position. As already mentioned, that of the Arabophone Byzantine Christian scholar Qusṭā ibn Lūqā (d. ca. 912–13) sparked Suter's interest because it contained a more nearly *valid* geometric proof of why Double False Position works.<sup>99</sup> Vari-

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<sup>97</sup>Rashed, *Development of Arabic Mathematics*, ch. 1, esp. pp. 14–16. Such an approach is related more broadly to the methods practiced and advocated, for example, by Kuhn and the historians and sociologists who have taken inspiration from aspects of his approach; see Barry Barnes, *T. S. Kuhn and Social Science* (London: Macmillan, 1982).

<sup>98</sup>A single scribe (Scribe 1) copied texts no. 2–18 in the manuscript, including the treatise on Double False Position (no. 10). Even if the compilation represented by this subset of the manuscript had already been compiled piecemeal over time (such that Scribe 1 would not be the *compiler* of this compilation, only its copyist), nevertheless it was still the scribe's choice to copy a pre-existing compilation *in its entirety* — a choice that suggests an interest in studying the text on Double False Position alongside the other texts.

<sup>99</sup>See n. 12 above. As Suter points out (Suter, "Die Abhandlung Qosṭā ben Lūqās," 119–21), Qusṭā's treatise (at least as translated by Suter) sets up the correspondence between the line segments in its geometric proof and Double False Position's parameters in such a way as to assume implicitly that the equation in question is of the form  $ax = y$ , i.e., that the  $y$ -intercept is zero. Suter is puzzled that a mathematician like Qusṭā would have missed this and suggests that the attribution may be false. But if the correspondence is tweaked, the proof is successful; indeed Suter also suggests that an error of transmission could

ous other treatises on the topic are known today only by their titles. Sezgin lists treatises entitled (*Ḥisāb*) *al-Khaṭaʿayn*, or (*Calculation by*) *Two Errors*, by Abū Kāmil (whom Sezgin tentatively places in the second half of the ninth century), Abū Yūsuf al-Rāzī and Abū Yūsuf al-Miṣṣīṣī (both probably active in the first half of the tenth century according to Sezgin), al-Karajī (active ca. tenth/eleventh century), and Ibn al-Haytham (965–1039).<sup>100</sup> (There is no significance to the fact that Ibn al-Haytham is the latest author in this list; Sezgin’s multivolume biobibliographical reference work stops at ca. 430 AH/1038 CE, so it would automatically have excluded any treatises on Double False Position that might have been composed after the mid-eleventh century.) In other words, there seems to have been enduring interest in this algorithm and its mathematical justification. Further research into such treatises — especially if any of them should turn up in the vast number of uncatalogued and undercatalogued Arabic manuscripts around the world — might help us understand the context of Jābir’s treatise. For example, if indeed he was working later than Qusṭā, as Suter thought, we might imagine that Jābir was seeking to produce a simpler proof, or else that he sought to reproduce Qusṭā’s proof from memory and ended up getting it wrong without realizing his mistake. Similarly, if indeed Jābir did not have much of a head for math, as Suter claimed, it would be interesting to know what social and cultural incentives impelled him to take up the task of proving Double False Position nevertheless. Or, if other works by the same Jābir turn up showing him to be more of a mathematical mind than Suter thought, we might ask what led him astray in this one treatise — or we might reconsider what he was trying to do in this treatise and ask why subsequent readers from Ibn al-Sarī to Suter to the present author misunderstood his aims.<sup>101</sup>

In any case, we must still contend with the widespread interest in proofs of Double False Position. Jābir’s purported proof was clearly something that Ibn al-Sarī considered worth his time to refute in the twelfth century, and his refutation was still being studied closely when Sa’d al-Dīn al-Hamadhānī subsequently explained it (presumably

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have introduced the error into the text. To be sure of what is going on, it will be necessary to consult the original Arabic of Qusṭā’s treatise anew.

<sup>100</sup>Abū Kāmil: Sezgin, *GAS*, 5:277–281, esp. 277 (date) and 281 (*al-Khaṭaʿayn*). Abū Yūsuf al-Rāzī (*Ḥisāb al-khaṭaʿayn*): Sezgin, 5:300. Abū Yūsuf al-Miṣṣīṣī (*al-Khaṭaʿayn*): Sezgin, 5:297. Al-Karajī (who dedicated one of his works to a patron who died in 1014), *al-Khaṭaʿayn*: Sezgin, 5:329; on whether his name was al-Karajī or al-Karkhī, see Rashed, *Development of Arabic Mathematics*, 22. Ibn al-Haytham, *Ḥisāb al-khaṭaʿayn*: Sezgin, *GAS*, 5:374.

<sup>101</sup>As an anonymous reviewer generously informed me, Jābir is named as the author of astronomical works in Oxford, Bodleian, Thurston 3 (13th century; José Bellver, on *Ptolemaeus Arabus et Latinus*, <https://ptolemaeus.badw.de/ms/672>, entry updated 10 November 2018) and Oxford, Bodleian, Marsh 720 (17th century; José Bellver, <http://ptolemaeus.badw.de/ms/685>, entry updated 10 November 2018): *Maqālah fī hayʿat aḥlāk ʿUṭarīd wa-khtilāf marākizihā wa-masīrihā* (<https://ptolemaeus.badw.de/work/225>; Thurston 104<sup>v</sup>, Marsh 207<sup>r-v</sup>) and *Maqaddimāt fī biḍʿ ashkāl min al-Majisṭī* (<https://ptolemaeus.badw.de/work/221>; Thurston 105<sup>r</sup>–107<sup>r</sup>, Marsh 208<sup>r</sup>–211<sup>v</sup>), of which *Burhān mā qālahu Baṭlamīyūs fī l-shakl al-rābīʿ min al-maqālah al-thānīyah ʿashar <min al-Majisṭī>* (not explicitly ascribed to Jābir, <https://ptolemaeus.badw.de/work/226>; Thurston 107<sup>r</sup>, Marsh 212<sup>r</sup>) is probably a continuation (according to the author of the *Ptolemaeus* entry on this work, José Bellver). These manuscripts also contain Jābir’s treatise on Double False Position and Ibn al-Sarī’s commentary on it (Thurston 136<sup>v</sup>–137<sup>r</sup>, Marsh 271<sup>r</sup>–272<sup>v</sup>), followed by Qusṭā’s treatise on Double False Position. Bellver, in his entry on the Thurston manuscript, mentions that “a note on f. 105r indicates that this group of works by Jābir b. Ibrāhīm al-Ṣābī was copied from a first generation copy from an autograph by Jābir b. Ibrāhīm al-Ṣābī.” All this suggests that these manuscripts, especially the Thurston manuscript, would be a promising avenue for future research.



to students) and when the Columbia manuscript was produced. This concern for refuting a bad proof of Double False Position might have stemmed in part from the numerical method's widespread use, but ultimately it must have been part of medieval Arabic mathematicians' broader project. Perhaps it was precisely because Double False Position was clearly applicable to many of the same problems that the new algebra subsumed, it was important to study it not simply as a handy numerical method but as a theorem to be demonstrated by a satisfactory and revealing geometric proof and thus properly integrated into the new mathematics.<sup>102</sup>

Suter's observation that Jābir's treatise attests to the existence of the method of Double False Position already in early Arabic mathematics, then, is only the beginning of the historian's task. Rather than stop there and dismiss the treatise as otherwise useless because mathematically incorrect, philologically-minded historians of mathematics might ask how the treatise, its commentary, its subsequent study, and other treatises like it on Double False Position can be reconciled and integrated into the picture of medieval mathematics that continues to emerge, one newly edited mathematical text at a time.

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<sup>102</sup>J. Murdoch asked a question of Roshdi Rashed at a conference after the latter's talk on the social context of algebra's development. As the discussion continued, Murdoch asked an open-ended question about "false position" and its place in this history of algebra, as an example of a topic for future historical research. See Rashed, *Development of Arabic Mathematics*, 61. Texts like Jābir's would presumably be at the heart of such an inquiry.

# Bibliography

## Primary Sources

- Apollonios of Perga. *Apollonius de Perge, Coniques: Texte grec et arabe*, ed. Roshdi Rashed. multiple vols. Berlin: de Gruyter, 2008.
- Euclid. *Euclidis Elementa ex interpretatione al-Hadschdschadschii cum commentariis al-Narizii: Codex Leidensis 399,1*, ed. R. O. Besthorn and J. L. Heiberg. Copenhagen, 1897.
- Khayyam, Omar. *Al-Khayyam mathématicien*, ed./trans. Roshdi Rashed and Bijan Vahabzadeh. Paris: Albert Blanchard, 1999.
- . *Omar Khayyam, the Mathematician*. Ed./trans. Roshdi Rashed and Bijan Vahabzadeh. New York: Bibliotheca Persica Press, 2000.
- . *Rasā'il al-Khayyām al-jabriyyah*, ed./trans. Roshdi Rashed and Aḥmad Jabbār. Aleppo: Jāmi'at Ḥalab, Ma'had al-Turāth al-'Ilmī al-'Arabī, 1981.
- . *Riyāḍiyyāt 'Umar al-Khayyām*, ed./trans. Roshdi Rashed and Bijan Vahabzadeh. Beirut: Markaz Dirāsāt al-Waḥdah al-'Arabiyyah, 2005.
- Menelaos. *Menelaus' "Spherics": Early translation and al-Māhānī, al-Harawī's Version*. Ed./trans. Roshdi Rashed and Athanase Papadopoulos. Berlin: de Gruyter, 2017.
- al-Nasawī, Abū l-Ḥasan 'Alī ibn Aḥmad. *Kitāb al-tajrīd fī uṣūl al-handasah*, ed. Muṣṭafā Mawālīdī. London: Mu'assasat al-Furqān li-l-Turāth al-Islāmī, Markaz Dirāsāt al-Makhtūṭāt al-Islāmiyyah, 2016.
- Theodosius. *Sphaerica: Arabic and Medieval Latin Translations*. Ed./trans. Paul Kunitzsch and Richard Lorch. Stuttgart: Steiner, 2010.
- Yāqūt al-Rūmī. *Mu'jam al-udabā'*. Ed. Iḥsān 'Abbās. 6 vols. Continuous pagination. Beirut, 1993.

## Secondary Literature

- Abattouy, Mohammed. "La tradition arabe de *Maqāla fī al-mīzān*, un traité sur la théorie du levier attribué à Euclide." *Mirror of Heritage (Ayene-ye Miras): Quarterly Journal of Book Review, Bibliography and Text Information* (Tehran), n.s., 4, no. 4 (2007): 67–104.
- 'Awwād, Gūrgīs. "Al-Makhtūṭāt al-'arabiyyah fī dūr al-kutub al-Amīrikiyyah" [Arabic Manuscripts in American Libraries]. *Sumer* 7 (1951): 237–77.
- Barnes, Barry. *T. S. Kuhn and Social Science*. London: Macmillan, 1982.
- Brockelmann, Carl. *Geschichte der arabischen Litteratur*. 1st ed. Weimar; Berlin: Emil Felber, 1898–1902.
- de Jong, P., and M. J. de Goeje, eds. *Catalogus codicum Orientalium Bibliothecae Academiae Lugduno Batavae*. Vol. 3. Leiden: Brill, 1865.
- Gandz, Solomon. "The Algebra of Inheritance: A Rehabilitation of Al-Khuwarizmi." *Osiris* 5 (1938): 319–91.
- Kennedy, Edward Stewart. "A Survey of Islamic Astronomical Tables." *Transactions of the American Philosophical Society* 46, no. 2 (1956): 123–77.
- Kuhn, Thomas S. *The Structure of Scientific Revolutions*. 3rd ed. Chicago: University of Chicago Press, 1996.
- Mavroudi, Maria. "Greek Language and Education under Early Islam." Chap. 11 in *Islamic Cultures, Islamic Contexts: Essays in Honor of Professor Patricia Crone*, ed. Behnam Sadeghi et al., 295–342. Leiden: Brill, 2014.
- Noel, Linda Hand. "The Fundamental Theorem of Algebra: A Survey of History and Proofs." EdD diss., Oklahoma State University, 1991.

- Pollock, Sheldon. "Philology and Freedom." *Philological Encounters* 1 (2016): 4–30.
- Pollock, Sheldon, Benjamin A. Elman, and Ku-ming Kevin Chang, eds. *World Philology*. Cambridge, MA: Harvard University Press, 2015.
- Rashed, Roshdi. *The Development of Arabic Mathematics: Between Arithmetic and Algebra*. Dordrecht: Kluwer, 1994.
- Roberts, Alexandre M. "Being a Sabian at Court in Tenth-Century Baghdad." *Journal of the American Oriental Society* 137, no. 2 (2017): 253–77.
- Rorty, Richard, J. B. Schneewind, and Quentin Skinner. *Philosophy in History: Essays on the Historiography of Philosophy*. Cambridge: Cambridge University Press, 1984.
- Ruska, Julius. "Heinrich Suter." *Isis* 5, no. 2 (1923): 409–17.
- Schwartz, Randy K. "Issues in the Origin and Development of Hisab al-Khata'ayn (Calculation by Double False Position)." In *Actes du huitième colloque maghrébin sur l'histoire des mathématiques arabes: Tunis, les 18-19-20 décembre 2004*. Tunis: Association Tunisienne des sciences mathématiques, 2004.
- Sezgin, Fuat. *Geschichte des arabischen Schrifttums*. 17 vols. Leiden: Brill, 1967–2015.
- Suter, Heinrich. *Beiträge zur Geschichte der Mathematik und Astronomie im Islam: Nachdruck seiner Schriften aus den Jahren 1892–1922*, ed. Fuat Sezgin. 2 vols. Veröffentlichungen des Institutes für Geschichte der Arabisch-Islamischen Wissenschaften. Reihe B, Nachdrucke. Abteilung Mathematik 2. Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe-Universität, 1986.
- . "Die Abhandlung Qostā ben Lūqās und zwei andere anonyme über die Rechnung mit zwei Fehlern und mit der angenommenen Zahl." *Bibliotheca Mathematica*, 3rd ser., 9 (1908–9): 111–22.
- . *Die Mathematiker und Astronomen der Araber und ihre Werke*. Abhandlungen zur Geschichte der mathematischen Wissenschaften 10. Leipzig: Teubner, 1900.
- . "Einige geometrische Aufgaben bei arabischen Mathematikern." *Bibliotheca Mathematica*, 3rd ser., 8 (1907–8): 23–36.