Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021,
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# Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021

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**W E L C O M E - 2021**
***Mathematical Style***

**Abstract**

*The idea of **bordered magic squares** is well known in the literature. In this work, **bordered magic squares** are constructed in such a way that the final magic sum of each **bordered magic square** is 2021. The work is for the orders 3 to 26. The work include fractional and decimal numbers entries having positive and/or negative signs. In some cases, the sum-magic sums lead us to **Pythagorean triples**. It happens with the even order magic squares starting from order 10, such as, orders 10, 12, ..., 24, 26.*

## Contents



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# 1 Introduction

Based on the work of H. White [6], recently, author [11, 12, 13, 14, 15, 16, 17, 18, 19] worked on the **bordered magic squares** in different ways. Some of these ways are specified in following two subsections.

## 1.1 Odd Ordered Natural Number Entries

Author [12] studied the **bordered magic squares** for the **consecutive odd numbers**. The summary is given in the following result.

**Result 1.1.** *[12] For **bordered magic squares** for **consecutive odd numbers**, the total entries sums are given by*

$$T_{k\times m} := k^2 \times m^2,$$

*where k is the order of **bordered magic squares**, and m is the **order of each bordered sub-magic square**. This lead us to very interesting connection with **Pythagoras theorem**.*

In particular, the bordered magic squares constructed with odd order consecutive natural numbers starting from 1, the total sum entries are as follows:

- order 24, $k = 24$, $T_{24\times m} := 24^2 \times m^2$, $m = 4, 6, 8, 10, 12, 14, 16, 18, 20, 22$ and 24;
- order 25, $k = 25$, $T_{25\times m} := 25^2 \times m^2$, $m = 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$ and 25.

## 1.2 Consecutive Natural Number Entries

Author [13] studied the **bordered magic squares** for the **consecutive natural numbers**. The summary is given in the following result.

**Result 1.2.** *[13] The **bordered magic squares** constructed for the consecutive natural numbers starting from 1 satisfy the following properties:*

1. $S_{k\times k} := k \times L$;
2. $T_{k\times k} := k^2 \times L$;
3. $C_{k\times k} := (k-1) \times 4 \times L$;
4. $d_{border} := 8 \times L$.

*where $k$ is the order of bordered magic square and*

$$L := T_{1\times 1}, \quad \textit{odd order magic squares}$$
$$L := \frac{T_{2\times 2}}{4}, \quad \textit{even order magic squares}$$

*and*

$$\begin{aligned} S_{k\times k} &\longrightarrow \textit{magic square sums;}\\ T_{k\times k} &\longrightarrow \textit{total entries sums;}\\ C_{k\times k} &\longrightarrow \textit{borders entries sums;}\\ d_{border} &\longrightarrow \textit{difference among borders value.}\end{aligned}$$

In particular, for the orders 24 and 25, we have

1. For the **bordered magic square** of order 24 for the consecutive entries 1 to 576, has the following symmetric results:
   1. $S_{k\times k} := \frac{k}{2} \times \frac{T_{2\times 2}}{2}$;
   2. $T_{k\times k} := \left(\frac{k}{2}\right)^2 \times T_{2\times 2}$;
   3. $C_{k\times k} := (k-1) \times T_{2\times 2}$.
   4. $d_{border} := 2 \times T_{2\times 2}$.

   where $k = 4, 6, \ldots 20, 22$ and 24 orders of magic squares appearing **bordered magic square** of order 24, and $T_{2\times 2} := 1154$ is the sum of four central values of magic square.
2. For the **bordered magic square** of order 25 for the consecutive entries 1 to 625, has the following symmetric results:

1. $S_{k\times k} := k \times T_{1\times 1}$;
2. $T_{k\times k} := k^2 \times T_{1\times 1}$;
3. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times 1}$.
4. $d_{border} := 8 \times T_{1\times 1}$.

where $k = 3, 5, 7, \ldots$ 21, 23 and 25 orders of magic squares appearing **bordered magic square** of order 25, and $T_{1\times 1} := 313$ is the central value of the magic square.

## 1.3 Square of Order Sum

Here we shall write bordered magic squares in such a way that the total sum is the square of order of magic squares. For example, for the bordered magic square of order 9, the total sum is $9^2$, etc. This study include decimal entries as well as negative numbers.

**Result 1.3.** *[15] The general formula the **magic sum** of each **sub-magic square** is as follows:*

$$S_{k\times k} := k \times m,$$

*where m is the order of **magic square** and k is the order of each **sub-magic squares**.*

For example,

- order 24, $k = 24$, $S_{24\times m} := 24 \times m$, $m = 4, 6, 8, 10, 12, 14, 16, 18, 20, 22$ and 24;
- order 25, $k = 25$, $S_{25\times m} := 25 \times m$, $m = 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$ and 25.

More results in this direction can be seen in the [1, 2, 4, 3, 5, 6, 7]. Some results on general sum can be seen in author's work [14]. In [15], author wrote different **bordered magic squares** with magic sum always 2020.

In this work, we shall write **bordered magic squares** in such a way that the final sum is 2020. The work is for the **bordered magic squares** of orders 3 to 26. We observe that the **magic sum** of **sub-magic square** give us a symmetric result. The work include fractional, decimal and whole numbers with positive and negative signs.

# 2 Bordered Magic Squares With Magic Sum 2021

The author [14] wrote **bordered magic squares** for the general sum as a natural number $n$. Also in 2020, the author [20] wrote **bordered magic squares** of orders 3 to 25 with magic sum 2020. Based on this idea, the subsections below give **bordered magic squares** where magic sum is always 2021. The entries of magic squares are either fractional numbers or decimal numbers. These entries are positive and/or negative values to giving sum 2021. Some of the sub-magic squres sums lead us to **Pythagorean triples**. These are given for the **bordered magic squares** of even orders.

For calculating the sub-magic squares sums, the formula given in the following result is applied:

**Result 2.1.** *The **sub-magic squares sums** of **boredred magic square** are given by*

$$S_{k\times k} := 2021 \times \frac{k}{m}, \tag{1}$$

*where k is the order of each **sub-magic square** and m is the order of **bordered magic squares**.*

For example,

(i) Let k=25, then according to (1),

$$S_{k\times k} := 2021 \times \frac{k}{25},$$

where $k = 3, 5, 7 \ldots 23, 25$.

(ii) Let k=16, then according to (1),

$$S_{k\times k} := 2021 \times \frac{k}{16},$$

where $k = 4, 6, 8 \ldots 14, 16$.

Moreover, even order magic squares starting from magic square of order 10, lead us to **Pythagorean triples** with sub-magic square sums, such as, order 10, 12,..., 22, 24. From order 20 onward, there are double **Pythagorean triples**.

## 2.1 Bordered Magic Square of Order 3

The **bordered magic square** of order 3 for the magic sum 2021 is given by

| | | | |
|---|---|---|---|
| | | | 2021 |
| 674 2/3 | 669 2/3 | 676 2/3 | 2021 |
| 675 2/3 | 673 2/3 | 671 2/3 | 2021 |
| 670 2/3 | 677 2/3 | 672 2/3 | 2021 |
| 2021 | 2021 | 2021 | 2021 |

## 2.2 Bordered Magic Square of Order 4

The **bordered magic square** of order 4 for the magic sum 2021 is given by

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| | | | | |
|---|---|---|---|---|
| | | | | 2021 |
| 510,75 | 497,75 | 500,75 | 511,75 | 2021 |
| 503,75 | 508,75 | 505,75 | 502,75 | 2021 |
| 507,75 | 504,75 | 501,75 | 506,75 | 2021 |
| 498,75 | 509,75 | 512,75 | 499,75 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 |

In this case the sum of internal four entries is also the same as of magic square sum, i.e., $S_{4\times 4} := 2020$.

## 2.3 Bordered Magic Square of Order 5

The **bordered magic square** of order 5 for the magic sum 2021 is given by

| | | | | | |
|---|---|---|---|---|---|
| | | | | | 2021 |
| 395,2 | 392,2 | 412,2 | 410,2 | 411,2 | 2021 |
| 415,2 | 405,2 | 400,2 | 407,2 | 393,2 | 2021 |
| 414,2 | 406,2 | 404,2 | 402,2 | 394,2 | 2021 |
| 399,2 | 401,2 | 408,2 | 403,2 | 409,2 | 2021 |
| 397,2 | 416,2 | 396,2 | 398,2 | 413,2 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{5} = 1212.6$$
$$S_{5\times 5} := 2021 \times \frac{5}{5} = 2021.$$

## 2.4 Bordered Magic Square of Order 6

The **bordered magic square** of order 6 for the magic sum 2021 is given by

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| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | 2021 |
| 350 1/3 | 348 1/3 | 321 1/3 | 354 1/3 | 322 1/3 | 324 1/3 | 2021 |
| 320 1/3 | 342 1/3 | 329 1/3 | 332 1/3 | 343 1/3 | 353 1/3 | 2021 |
| 326 1/3 | 335 1/3 | 340 1/3 | 337 1/3 | 334 1/3 | 347 1/3 | 2021 |
| 328 1/3 | 339 1/3 | 336 1/3 | 333 1/3 | 338 1/3 | 345 1/3 | 2021 |
| 346 1/3 | 330 1/3 | 341 1/3 | 344 1/3 | 331 1/3 | 327 1/3 | 2021 |
| 349 1/3 | 325 1/3 | 352 1/3 | 319 1/3 | 351 1/3 | 323 1/3 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

$$S_{4\times 4} := 2021 \times \frac{4}{6} = \frac{4042}{3}.$$
$$S_{6\times 6} := 2021 \times \frac{6}{6} = 2021.$$

## 2.5 Bordered Magic Square of Order 7

The **bordered magic square** of order 7 for the magic sum 2021 is given by

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | 2021 |
| 271 5/7 | 275 5/7 | 273 5/7 | 308 5/7 | 309 5/7 | 311 5/7 | 269 5/7 | 2021 |
| 312 5/7 | 279 5/7 | 276 5/7 | 296 5/7 | 294 5/7 | 295 5/7 | 264 5/7 | 2021 |
| 310 5/7 | 299 5/7 | 289 5/7 | 284 5/7 | 291 5/7 | 277 5/7 | 266 5/7 | 2021 |
| 270 5/7 | 298 5/7 | 290 5/7 | 288 5/7 | 286 5/7 | 278 5/7 | 306 5/7 | 2021 |
| 272 5/7 | 283 5/7 | 285 5/7 | 292 5/7 | 287 5/7 | 293 5/7 | 304 5/7 | 2021 |
| 274 5/7 | 281 5/7 | 300 5/7 | 280 5/7 | 282 5/7 | 297 5/7 | 302 5/7 | 2021 |
| 307 5/7 | 301 5/7 | 303 5/7 | 268 5/7 | 267 5/7 | 265 5/7 | 305 5/7 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{7} = \frac{6063}{7}$$
$$S_{5\times 5} := 2021 \times \frac{5}{7} = \frac{10105}{7}$$
$$S_{7\times 7} := 2021 \times \frac{7}{7} = 2021.$$

## 2.6 Bordered Magic Square of Order 8

The **bordered magic square** of order 8 for the magic sum 2021 is given by

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 2021 |
| 228.125 | 222.125 | 282.125 | 284.125 | 271.125 | 233.125 | 273.125 | 227.125 | 2021 |
| 225.125 | 266.125 | 264.125 | 237.125 | 270.125 | 238.125 | 240.125 | 280.125 | 2021 |
| 226.125 | 236.125 | 258.125 | 245.125 | 248.125 | 259.125 | 269.125 | 279.125 | 2021 |
| 231.125 | 242.125 | 251.125 | 256.125 | 253.125 | 250.125 | 263.125 | 274.125 | 2021 |
| 281.125 | 244.125 | 255.125 | 252.125 | 249.125 | 254.125 | 261.125 | 224.125 | 2021 |
| 276.125 | 262.125 | 246.125 | 257.125 | 260.125 | 247.125 | 243.125 | 229.125 | 2021 |
| 275.125 | 265.125 | 241.125 | 268.125 | 235.125 | 267.125 | 239.125 | 230.125 | 2021 |
| 278.125 | 283.125 | 223.125 | 221.125 | 234.125 | 272.125 | 232.125 | 277.125 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{8} = 1010.50$$
$$S_{6\times 6} := 2021 \times \frac{6}{8} = 1515.75$$
$$S_{8\times 8} := 2021 \times \frac{8}{8} = 2021.$$

## 2.7 Bordered Magic Square of Order 9

The **bordered magic square** of order 9 for the magic sum 2021 is given by

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 2021 |
| 191 5/9 | 263 5/9 | 261 5/9 | 259 5/9 | 258 5/9 | 195 5/9 | 197 5/9 | 199 5/9 | 193 5/9 | 2021 |
| 184 5/9 | 207 5/9 | 211 5/9 | 209 5/9 | 244 5/9 | 245 5/9 | 247 5/9 | 205 5/9 | 264 5/9 | 2021 |
| 186 5/9 | 248 5/9 | 215 5/9 | 212 5/9 | 232 5/9 | 230 5/9 | 231 5/9 | 200 5/9 | 262 5/9 | 2021 |
| 188 5/9 | 246 5/9 | 235 5/9 | 225 5/9 | 220 5/9 | 227 5/9 | 213 5/9 | 202 5/9 | 260 5/9 | 2021 |
| 256 5/9 | 206 5/9 | 234 5/9 | 226 5/9 | 224 5/9 | 222 5/9 | 214 5/9 | 242 5/9 | 192 5/9 | 2021 |
| 254 5/9 | 208 5/9 | 219 5/9 | 221 5/9 | 228 5/9 | 223 5/9 | 229 5/9 | 240 5/9 | 194 5/9 | 2021 |
| 252 5/9 | 210 5/9 | 217 5/9 | 236 5/9 | 216 5/9 | 218 5/9 | 233 5/9 | 238 5/9 | 196 5/9 | 2021 |
| 250 5/9 | 243 5/9 | 237 5/9 | 239 5/9 | 204 5/9 | 203 5/9 | 201 5/9 | 241 5/9 | 198 5/9 | 2021 |
| 255 5/9 | 185 5/9 | 187 5/9 | 189 5/9 | 190 5/9 | 253 5/9 | 251 5/9 | 249 5/9 | 257 5/9 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{9} = \frac{2021}{3}$$

$$S_{5\times 5} := 2021 \times \frac{5}{9} = \frac{10105}{9}$$

$$S_{7\times 7} := 2021 \times \frac{7}{9} = \frac{14147}{9}$$

$$S_{9\times 9} := 2021 \times \frac{9}{9} = 2021.$$

## 2.8 Bordered Magic Square of Order 10

The **bordered magic square** of order 10 for the magic sum 2021 is given by

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | 2021 |
| 242,6 | 237,6 | 167,6 | 235,6 | 169,6 | 165,6 | 155,6 | 249,6 | 153,6 | 243,6 | 2021 |
| 164,6 | 177,6 | 171,6 | 231,6 | 233,6 | 220,6 | 182,6 | 222,6 | 176,6 | 239,6 | 2021 |
| 240,6 | 174,6 | 215,6 | 213,6 | 186,6 | 219,6 | 187,6 | 189,6 | 229,6 | 163,6 | 2021 |
| 162,6 | 175,6 | 185,6 | 207,6 | 194,6 | 197,6 | 208,6 | 218,6 | 228,6 | 241,6 | 2021 |
| 247,6 | 180,6 | 191,6 | 200,6 | 205,6 | 202,6 | 199,6 | 212,6 | 223,6 | 156,6 | 2021 |
| 152,6 | 230,6 | 193,6 | 204,6 | 201,6 | 198,6 | 203,6 | 210,6 | 173,6 | 251,6 | 2021 |
| 244,6 | 225,6 | 211,6 | 195,6 | 206,6 | 209,6 | 196,6 | 192,6 | 178,6 | 159,6 | 2021 |
| 158,6 | 224,6 | 214,6 | 190,6 | 217,6 | 184,6 | 216,6 | 188,6 | 179,6 | 245,6 | 2021 |
| 246,6 | 227,6 | 232,6 | 172,6 | 170,6 | 183,6 | 221,6 | 181,6 | 226,6 | 157,6 | 2021 |
| 160,6 | 166,6 | 236,6 | 168,6 | 234,6 | 238,6 | 248,6 | 154,6 | 250,6 | 161,6 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{10} = 808.4 \qquad S_{8\times 8} := 2021 \times \frac{8}{10} = 1616.8$$

$$S_{6\times 6} := 2021 \times \frac{6}{10} = 1212.6 \qquad S_{10\times 10} := 2021 \times \frac{10}{10} = 2021.$$

In this case, there is a **Pythagorean triples** with magic sums:

$$S^2_{6\times 6} + S^2_{8\times 8} := S^2_{10\times 10}$$

## 2.9 Bordered Magic Square of Order 11

The **bordered magic square** of order 11 for the magic sum 2021 is given by

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 2021 |
| 134 8/11 | 142 8/11 | 140 8/11 | 138 8/11 | 136 8/11 | 235 8/11 | 236 8/11 | 238 8/11 | 240 8/11 | 242 8/11 | 132 8/11 | 2021 |
| 243 8/11 | 150 8/11 | 222 8/11 | 220 8/11 | 218 8/11 | 217 8/11 | 154 8/11 | 156 8/11 | 158 8/11 | 152 8/11 | 123 8/11 | 2021 |
| 241 8/11 | 143 8/11 | 166 8/11 | 170 8/11 | 168 8/11 | 203 8/11 | 204 8/11 | 206 8/11 | 164 8/11 | 223 8/11 | 125 8/11 | 2021 |
| 239 8/11 | 145 8/11 | 207 8/11 | 174 8/11 | 171 8/11 | 191 8/11 | 189 8/11 | 190 8/11 | 159 8/11 | 221 8/11 | 127 8/11 | 2021 |
| 237 8/11 | 147 8/11 | 205 8/11 | 194 8/11 | 184 8/11 | 179 8/11 | 186 8/11 | 172 8/11 | 161 8/11 | 219 8/11 | 129 8/11 | 2021 |
| 133 8/11 | 215 8/11 | 165 8/11 | 193 8/11 | 185 8/11 | 183 8/11 | 181 8/11 | 173 8/11 | 201 8/11 | 151 8/11 | 233 8/11 | 2021 |
| 135 8/11 | 213 8/11 | 167 8/11 | 178 8/11 | 180 8/11 | 187 8/11 | 182 8/11 | 188 8/11 | 199 8/11 | 153 8/11 | 231 8/11 | 2021 |
| 137 8/11 | 211 8/11 | 169 8/11 | 176 8/11 | 195 8/11 | 175 8/11 | 177 8/11 | 192 8/11 | 197 8/11 | 155 8/11 | 229 8/11 | 2021 |
| 139 8/11 | 209 8/11 | 202 8/11 | 196 8/11 | 198 8/11 | 163 8/11 | 162 8/11 | 160 8/11 | 200 8/11 | 157 8/11 | 227 8/11 | 2021 |
| 141 8/11 | 214 8/11 | 144 8/11 | 146 8/11 | 148 8/11 | 149 8/11 | 212 8/11 | 210 8/11 | 208 8/11 | 216 8/11 | 225 8/11 | 2021 |
| 234 8/11 | 224 8/11 | 226 8/11 | 228 8/11 | 230 8/11 | 131 8/11 | 130 8/11 | 128 8/11 | 126 8/11 | 124 8/11 | 232 8/11 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{11} = \frac{6063}{11}$$

$$S_{5\times 5} := 2021 \times \frac{5}{11} = \frac{10105}{11}$$

$$S_{7\times 7} := 2021 \times \frac{7}{11} = \frac{14147}{11}$$

$$S_{9\times 9} := 2021 \times \frac{9}{11} = \frac{18189}{11}$$

$$S_{11\times 11} := 2021 \times \frac{11}{11} = 2021.$$

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## 2.10 Bordered Magic Square of Order 12

The **bordered magic square** of order 12 for the magic sum 2021 is given by

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | 2021 |
| 229 11/12 | 238 11/12 | 98 11/12 | 236 11/12 | 100 11/12 | 96 11/12 | 117 11/12 | 219 11/12 | 115 11/12 | 221 11/12 | 113 11/12 | 228 11/12 | 2021 |
| 111 11/12 | 208 11/12 | 203 11/12 | 133 11/12 | 201 11/12 | 135 11/12 | 131 11/12 | 121 11/12 | 215 11/12 | 119 11/12 | 209 11/12 | 224 11/12 | 2021 |
| 110 11/12 | 130 11/12 | 143 11/12 | 137 11/12 | 197 11/12 | 199 11/12 | 186 11/12 | 148 11/12 | 188 11/12 | 142 11/12 | 205 11/12 | 225 11/12 | 2021 |
| 226 11/12 | 206 11/12 | 140 11/12 | 181 11/12 | 179 11/12 | 152 11/12 | 185 11/12 | 153 11/12 | 155 11/12 | 195 11/12 | 129 11/12 | 109 11/12 | 2021 |
| 227 11/12 | 128 11/12 | 141 11/12 | 151 11/12 | 173 11/12 | 160 11/12 | 163 11/12 | 174 11/12 | 184 11/12 | 194 11/12 | 207 11/12 | 108 11/12 | 2021 |
| 234 11/12 | 213 11/12 | 146 11/12 | 157 11/12 | 166 11/12 | 171 11/12 | 168 11/12 | 165 11/12 | 178 11/12 | 189 11/12 | 122 11/12 | 101 11/12 | 2021 |
| 223 11/12 | 118 11/12 | 196 11/12 | 159 11/12 | 170 11/12 | 167 11/12 | 164 11/12 | 169 11/12 | 176 11/12 | 139 11/12 | 217 11/12 | 112 11/12 | 2021 |
| 105 11/12 | 210 11/12 | 191 11/12 | 177 11/12 | 161 11/12 | 172 11/12 | 175 11/12 | 162 11/12 | 158 11/12 | 144 11/12 | 125 11/12 | 230 11/12 | 2021 |
| 104 11/12 | 124 11/12 | 190 11/12 | 180 11/12 | 156 11/12 | 183 11/12 | 150 11/12 | 182 11/12 | 154 11/12 | 145 11/12 | 211 11/12 | 231 11/12 | 2021 |
| 232 11/12 | 212 11/12 | 193 11/12 | 198 11/12 | 138 11/12 | 136 11/12 | 149 11/12 | 187 11/12 | 147 11/12 | 192 11/12 | 123 11/12 | 103 11/12 | 2021 |
| 102 11/12 | 126 11/12 | 132 11/12 | 202 11/12 | 134 11/12 | 200 11/12 | 204 11/12 | 214 11/12 | 120 11/12 | 216 11/12 | 127 11/12 | 233 11/12 | 2021 |
| 107 11/12 | 97 11/12 | 237 11/12 | 99 11/12 | 235 11/12 | 239 11/12 | 218 11/12 | 116 11/12 | 220 11/12 | 114 11/12 | 222 11/12 | 106 11/12 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{12} = \frac{2021}{3}$$

$$S_{6\times 6} := 2021 \times \frac{6}{12} = 1010.5$$

$$S_{8\times 8} := 2021 \times \frac{8}{12} = \frac{4042}{3}$$

$$S_{10\times 10} := 2021 \times \frac{10}{12} = \frac{10105}{6}$$

$$S_{12\times 12} := 2021 \times \frac{12}{12} = 2020.$$

In this case, there is a **Pythagorean triples** with magic sums:

$$S^2_{6\times 6} + S^2_{8\times 8} := S^2_{10\times 10}$$

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## 2.11 Bordered Magic Square of Order 13

The **bordered magic square** of order 13 for the magic sum 2021 is given by

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | 2021 |
| 226 6/13 | 217 6/13 | 219 6/13 | 221 6/13 | 223 6/13 | 225 6/13 | 227 6/13 | 79 6/13 | 77 6/13 | 75 6/13 | 73 6/13 | 71 6/13 | 82 6/13 | 2021 |
| 72 6/13 | 106 6/13 | 114 6/13 | 112 6/13 | 110 6/13 | 108 6/13 | 207 6/13 | 208 6/13 | 210 6/13 | 212 6/13 | 214 6/13 | 104 6/13 | 238 6/13 | 2021 |
| 74 6/13 | 215 6/13 | 122 6/13 | 194 6/13 | 192 6/13 | 190 6/13 | 189 6/13 | 126 6/13 | 128 6/13 | 130 6/13 | 124 6/13 | 95 6/13 | 236 6/13 | 2021 |
| 76 6/13 | 213 6/13 | 115 6/13 | 138 6/13 | 142 6/13 | 140 6/13 | 175 6/13 | 176 6/13 | 178 6/13 | 136 6/13 | 195 6/13 | 97 6/13 | 234 6/13 | 2021 |
| 78 6/13 | 211 6/13 | 117 6/13 | 179 6/13 | 146 6/13 | 143 6/13 | 163 6/13 | 161 6/13 | 162 6/13 | 131 6/13 | 193 6/13 | 99 6/13 | 232 6/13 | 2021 |
| 80 6/13 | 209 6/13 | 119 6/13 | 177 6/13 | 166 6/13 | 156 6/13 | 151 6/13 | 158 6/13 | 144 6/13 | 133 6/13 | 191 6/13 | 101 6/13 | 230 6/13 | 2021 |
| 81 6/13 | 105 6/13 | 187 6/13 | 137 6/13 | 165 6/13 | 157 6/13 | 155 6/13 | 153 6/13 | 145 6/13 | 173 6/13 | 123 6/13 | 205 6/13 | 229 6/13 | 2021 |
| 224 6/13 | 107 6/13 | 185 6/13 | 139 6/13 | 150 6/13 | 152 6/13 | 159 6/13 | 154 6/13 | 160 6/13 | 171 6/13 | 125 6/13 | 203 6/13 | 86 6/13 | 2021 |
| 222 6/13 | 109 6/13 | 183 6/13 | 141 6/13 | 148 6/13 | 167 6/13 | 147 6/13 | 149 6/13 | 164 6/13 | 169 6/13 | 127 6/13 | 201 6/13 | 88 6/13 | 2021 |
| 220 6/13 | 111 6/13 | 181 6/13 | 174 6/13 | 168 6/13 | 170 6/13 | 135 6/13 | 134 6/13 | 132 6/13 | 172 6/13 | 129 6/13 | 199 6/13 | 90 6/13 | 2021 |
| 218 6/13 | 113 6/13 | 186 6/13 | 116 6/13 | 118 6/13 | 120 6/13 | 121 6/13 | 184 6/13 | 182 6/13 | 180 6/13 | 188 6/13 | 197 6/13 | 92 6/13 | 2021 |
| 216 6/13 | 206 6/13 | 196 6/13 | 198 6/13 | 200 6/13 | 202 6/13 | 103 6/13 | 102 6/13 | 100 6/13 | 98 6/13 | 96 6/13 | 204 6/13 | 94 6/13 | 2021 |
| 228 6/13 | 93 6/13 | 91 6/13 | 89 6/13 | 87 6/13 | 85 6/13 | 83 6/13 | 231 6/13 | 233 6/13 | 235 6/13 | 237 6/13 | 239 6/13 | 84 6/13 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{13} = \frac{6063}{13}$$

$$S_{5\times 5} := 2021 \times \frac{5}{13} = \frac{10105}{13}$$

$$S_{7\times 7} := 2021 \times \frac{7}{13} = \frac{14147}{13}$$

$$S_{9\times 9} := 2021 \times \frac{9}{13} = \frac{18189}{13}$$

$$S_{11\times 11} := 2021 \times \frac{11}{13} = \frac{22231}{13}$$

$$S_{13\times 13} := 2021 \times \frac{13}{13} = 2021.$$

Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021,
**Zenodo**, December 16, 2020, pp. 1-33, *http://doi.org/10.5281/zenodo.4327333*

## 2.12 Bordered Magic Square of Order 14

The **bordered magic square** of order 14 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | 2021 |
| 59 6/7 | 53 6/7 | 233 6/7 | 55 6/7 | 231 6/7 | 57 6/7 | 241 6/7 | 52 6/7 | 227 6/7 | 61 6/7 | 225 6/7 | 63 6/7 | 223 6/7 | 229 6/7 | 2021 |
| 240 6/7 | 205 6/7 | 214 6/7 | 74 6/7 | 212 6/7 | 76 6/7 | 72 6/7 | 93 6/7 | 195 6/7 | 91 6/7 | 197 6/7 | 89 6/7 | 204 6/7 | 47 6/7 | 2021 |
| 48 6/7 | 87 6/7 | 184 6/7 | 179 6/7 | 109 6/7 | 177 6/7 | 111 6/7 | 107 6/7 | 97 6/7 | 191 6/7 | 95 6/7 | 185 6/7 | 200 6/7 | 239 6/7 | 2021 |
| 238 6/7 | 86 6/7 | 106 6/7 | 119 6/7 | 113 6/7 | 173 6/7 | 175 6/7 | 162 6/7 | 124 6/7 | 164 6/7 | 118 6/7 | 181 6/7 | 201 6/7 | 49 6/7 | 2021 |
| 50 6/7 | 202 6/7 | 182 6/7 | 116 6/7 | 157 6/7 | 155 6/7 | 128 6/7 | 161 6/7 | 129 6/7 | 131 6/7 | 171 6/7 | 105 6/7 | 85 6/7 | 237 6/7 | 2021 |
| 236 6/7 | 203 6/7 | 104 6/7 | 117 6/7 | 127 6/7 | 149 6/7 | 136 6/7 | 139 6/7 | 150 6/7 | 160 6/7 | 170 6/7 | 183 6/7 | 84 6/7 | 51 6/7 | 2021 |
| 222 6/7 | 210 6/7 | 189 6/7 | 122 6/7 | 133 6/7 | 142 6/7 | 147 6/7 | 144 6/7 | 141 6/7 | 154 6/7 | 165 6/7 | 98 6/7 | 77 6/7 | 65 6/7 | 2021 |
| 216 6/7 | 199 6/7 | 94 6/7 | 172 6/7 | 135 6/7 | 146 6/7 | 143 6/7 | 140 6/7 | 145 6/7 | 152 6/7 | 115 6/7 | 193 6/7 | 88 6/7 | 71 6/7 | 2021 |
| 70 6/7 | 81 6/7 | 186 6/7 | 167 6/7 | 153 6/7 | 137 6/7 | 148 6/7 | 151 6/7 | 138 6/7 | 134 6/7 | 120 6/7 | 101 6/7 | 206 6/7 | 217 6/7 | 2021 |
| 218 6/7 | 80 6/7 | 100 6/7 | 166 6/7 | 156 6/7 | 132 6/7 | 159 6/7 | 126 6/7 | 158 6/7 | 130 6/7 | 121 6/7 | 187 6/7 | 207 6/7 | 69 6/7 | 2021 |
| 68 6/7 | 208 6/7 | 188 6/7 | 169 6/7 | 174 6/7 | 114 6/7 | 112 6/7 | 125 6/7 | 163 6/7 | 123 6/7 | 168 6/7 | 99 6/7 | 79 6/7 | 219 6/7 | 2021 |
| 220 6/7 | 78 6/7 | 102 6/7 | 108 6/7 | 178 6/7 | 110 6/7 | 176 6/7 | 180 6/7 | 190 6/7 | 96 6/7 | 192 6/7 | 103 6/7 | 209 6/7 | 67 6/7 | 2021 |
| 66 6/7 | 83 6/7 | 73 6/7 | 213 6/7 | 75 6/7 | 211 6/7 | 215 6/7 | 194 6/7 | 92 6/7 | 196 6/7 | 90 6/7 | 198 6/7 | 82 6/7 | 221 6/7 | 2021 |
| 58 6/7 | 234 6/7 | 54 6/7 | 232 6/7 | 56 6/7 | 230 6/7 | 46 6/7 | 235 6/7 | 60 6/7 | 226 6/7 | 62 6/7 | 224 6/7 | 64 6/7 | 228 6/7 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times4} := 2021 \times \frac{4}{14} = \frac{4042}{7}$$

$$S_{6\times6} := 2021 \times \frac{6}{14} = \frac{6063}{7}$$

$$S_{8\times8} := 2021 \times \frac{8}{14} = \frac{8084}{7}$$

$$S_{10\times10} := 2021 \times \frac{10}{14} = \frac{10105}{7}$$

$$S_{12\times12} := 2021 \times \frac{12}{14} = \frac{12126}{7}$$

$$S_{14\times14} := 2021 \times \frac{14}{14} = 2021.$$

In this case, there is a **Pythagorean triples** with magic sums:

$$S^2_{6\times6} + S^2_{8\times8} := S^2_{10\times10}$$

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**Zenodo**, December 16, 2020, pp. 1-33, *http://doi.org/10.5281/zenodo.4327333*

## 2.13 Bordered Magic Square of Order 15

The **bordered magic square** of order 15 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | 2021 |
| 231 11/15 | 220 11/15 | 222 11/15 | 224 11/15 | 226 11/15 | 228 11/15 | 230 11/15 | 232 11/15 | 32 11/15 | 30 11/15 | 28 11/15 | 26 11/15 | 24 11/15 | 22 11/15 | 35 11/15 | 2021 |
| 23 11/15 | 205 11/15 | 196 11/15 | 198 11/15 | 200 11/15 | 202 11/15 | 204 11/15 | 206 11/15 | 58 11/15 | 56 11/15 | 54 11/15 | 52 11/15 | 50 11/15 | 61 11/15 | 245 11/15 | 2021 |
| 25 11/15 | 51 11/15 | 85 11/15 | 93 11/15 | 91 11/15 | 89 11/15 | 87 11/15 | 186 11/15 | 187 11/15 | 189 11/15 | 191 11/15 | 193 11/15 | 83 11/15 | 217 11/15 | 243 11/15 | 2021 |
| 27 11/15 | 53 11/15 | 194 11/15 | 101 11/15 | 173 11/15 | 171 11/15 | 169 11/15 | 168 11/15 | 105 11/15 | 107 11/15 | 109 11/15 | 103 11/15 | 74 11/15 | 215 11/15 | 241 11/15 | 2021 |
| 29 11/15 | 55 11/15 | 192 11/15 | 94 11/15 | 117 11/15 | 121 11/15 | 119 11/15 | 154 11/15 | 155 11/15 | 157 11/15 | 115 11/15 | 174 11/15 | 76 11/15 | 213 11/15 | 239 11/15 | 2021 |
| 31 11/15 | 57 11/15 | 190 11/15 | 96 11/15 | 158 11/15 | 125 11/15 | 122 11/15 | 142 11/15 | 140 11/15 | 141 11/15 | 110 11/15 | 172 11/15 | 78 11/15 | 211 11/15 | 237 11/15 | 2021 |
| 33 11/15 | 59 11/15 | 188 11/15 | 98 11/15 | 156 11/15 | 145 11/15 | 135 11/15 | 130 11/15 | 137 11/15 | 123 11/15 | 112 11/15 | 170 11/15 | 80 11/15 | 209 11/15 | 235 11/15 | 2021 |
| 34 11/15 | 60 11/15 | 84 11/15 | 166 11/15 | 116 11/15 | 144 11/15 | 136 11/15 | 134 11/15 | 132 11/15 | 124 11/15 | 152 11/15 | 102 11/15 | 184 11/15 | 208 11/15 | 234 11/15 | 2021 |
| 229 11/15 | 203 11/15 | 86 11/15 | 164 11/15 | 118 11/15 | 129 11/15 | 131 11/15 | 138 11/15 | 133 11/15 | 139 11/15 | 150 11/15 | 104 11/15 | 182 11/15 | 65 11/15 | 39 11/15 | 2021 |
| 227 11/15 | 201 11/15 | 88 11/15 | 162 11/15 | 120 11/15 | 127 11/15 | 146 11/15 | 126 11/15 | 128 11/15 | 143 11/15 | 148 11/15 | 106 11/15 | 180 11/15 | 67 11/15 | 41 11/15 | 2021 |
| 225 11/15 | 199 11/15 | 90 11/15 | 160 11/15 | 153 11/15 | 147 11/15 | 149 11/15 | 114 11/15 | 113 11/15 | 111 11/15 | 151 11/15 | 108 11/15 | 178 11/15 | 69 11/15 | 43 11/15 | 2021 |
| 223 11/15 | 197 11/15 | 92 11/15 | 165 11/15 | 95 11/15 | 97 11/15 | 99 11/15 | 100 11/15 | 163 11/15 | 161 11/15 | 159 11/15 | 167 11/15 | 176 11/15 | 71 11/15 | 45 11/15 | 2021 |
| 221 11/15 | 195 11/15 | 185 11/15 | 175 11/15 | 177 11/15 | 179 11/15 | 181 11/15 | 82 11/15 | 81 11/15 | 79 11/15 | 77 11/15 | 75 11/15 | 183 11/15 | 73 11/15 | 47 11/15 | 2021 |
| 219 11/15 | 207 11/15 | 72 11/15 | 70 11/15 | 68 11/15 | 66 11/15 | 64 11/15 | 62 11/15 | 210 11/15 | 212 11/15 | 214 11/15 | 216 11/15 | 218 11/15 | 63 11/15 | 49 11/15 | 2021 |
| 233 11/15 | 48 11/15 | 46 11/15 | 44 11/15 | 42 11/15 | 40 11/15 | 38 11/15 | 36 11/15 | 236 11/15 | 238 11/15 | 240 11/15 | 242 11/15 | 244 11/15 | 246 11/15 | 37 11/15 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{15} = \frac{2021}{5}$$

$$S_{5\times 5} := 2021 \times \frac{5}{15} = \frac{2021}{3}$$

$$S_{7\times 7} := 2021 \times \frac{7}{15} = \frac{14147}{15}$$

$$S_{9\times 9} := 2021 \times \frac{9}{15} = \frac{6063}{5}$$

$$S_{11\times 11} := 2021 \times \frac{11}{15} = \frac{22231}{15}$$

$$S_{13\times 13} := 2021 \times \frac{13}{15} = \frac{26273}{15}$$

$$S_{15\times 15} := 2021 \times \frac{15}{15} = 2021.$$

## 2.14 Bordered Magic Square of Order 16

The **bordered magic square** of order 16 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | 2021 |
| 239 13/16 | 19 13/16 | 233 13/16 | 17 13/16 | 16 13/16 | 236 13/16 | 237 13/16 | 246 13/16 | 231 13/16 | 11 13/16 | 10 13/16 | 242 13/16 | 8 13/16 | 244 13/16 | 6 13/16 | 13 13/16 | 2021 |
| 252 13/16 | 41 13/16 | 35 13/16 | 215 13/16 | 37 13/16 | 213 13/16 | 39 13/16 | 223 13/16 | 34 13/16 | 209 13/16 | 43 13/16 | 207 13/16 | 45 13/16 | 205 13/16 | 211 13/16 | - 3/16 | 2021 |
| 13/16 | 222 13/16 | 187 13/16 | 196 13/16 | 56 13/16 | 194 13/16 | 58 13/16 | 54 13/16 | 75 13/16 | 177 13/16 | 73 13/16 | 179 13/16 | 71 13/16 | 186 13/16 | 29 13/16 | 251 13/16 | 2021 |
| 250 13/16 | 30 13/16 | 69 13/16 | 166 13/16 | 161 13/16 | 91 13/16 | 159 13/16 | 93 13/16 | 89 13/16 | 79 13/16 | 173 13/16 | 77 13/16 | 167 13/16 | 182 13/16 | 221 13/16 | 1 13/16 | 2021 |
| 2 13/16 | 220 13/16 | 68 13/16 | 88 13/16 | 101 13/16 | 95 13/16 | 155 13/16 | 157 13/16 | 144 13/16 | 106 13/16 | 146 13/16 | 100 13/16 | 163 13/16 | 183 13/16 | 31 13/16 | 249 13/16 | 2021 |
| 248 13/16 | 32 13/16 | 184 13/16 | 164 13/16 | 98 13/16 | 139 13/16 | 137 13/16 | 110 13/16 | 143 13/16 | 111 13/16 | 113 13/16 | 153 13/16 | 87 13/16 | 67 13/16 | 219 13/16 | 3 13/16 | 2021 |
| 4 13/16 | 218 13/16 | 185 13/16 | 86 13/16 | 99 13/16 | 109 13/16 | 131 13/16 | 118 13/16 | 121 13/16 | 132 13/16 | 142 13/16 | 152 13/16 | 165 13/16 | 66 13/16 | 33 13/16 | 247 13/16 | 2021 |
| -1 3/16 | 204 13/16 | 192 13/16 | 171 13/16 | 104 13/16 | 115 13/16 | 124 13/16 | 129 13/16 | 126 13/16 | 123 13/16 | 136 13/16 | 147 13/16 | 80 13/16 | 59 13/16 | 47 13/16 | 253 13/16 | 2021 |
| 27 13/16 | 198 13/16 | 181 13/16 | 76 13/16 | 154 13/16 | 117 13/16 | 128 13/16 | 125 13/16 | 122 13/16 | 127 13/16 | 134 13/16 | 97 13/16 | 175 13/16 | 70 13/16 | 53 13/16 | 224 13/16 | 2021 |
| 225 13/16 | 52 13/16 | 63 13/16 | 168 13/16 | 149 13/16 | 135 13/16 | 119 13/16 | 130 13/16 | 133 13/16 | 120 13/16 | 116 13/16 | 102 13/16 | 83 13/16 | 188 13/16 | 199 13/16 | 26 13/16 | 2021 |
| 25 13/16 | 200 13/16 | 62 13/16 | 82 13/16 | 148 13/16 | 138 13/16 | 114 13/16 | 141 13/16 | 108 13/16 | 140 13/16 | 112 13/16 | 103 13/16 | 169 13/16 | 189 13/16 | 51 13/16 | 226 13/16 | 2021 |
| 227 13/16 | 50 13/16 | 190 13/16 | 170 13/16 | 151 13/16 | 156 13/16 | 96 13/16 | 94 13/16 | 107 13/16 | 145 13/16 | 105 13/16 | 150 13/16 | 81 13/16 | 61 13/16 | 201 13/16 | 24 13/16 | 2021 |
| 23 13/16 | 202 13/16 | 60 13/16 | 84 13/16 | 90 13/16 | 160 13/16 | 92 13/16 | 158 13/16 | 162 13/16 | 172 13/16 | 78 13/16 | 174 13/16 | 85 13/16 | 191 13/16 | 49 13/16 | 228 13/16 | 2021 |
| 229 13/16 | 48 13/16 | 65 13/16 | 55 13/16 | 195 13/16 | 57 13/16 | 193 13/16 | 197 13/16 | 176 13/16 | 74 13/16 | 178 13/16 | 72 13/16 | 180 13/16 | 64 13/16 | 203 13/16 | 22 13/16 | 2021 |
| 21 13/16 | 40 13/16 | 216 13/16 | 36 13/16 | 214 13/16 | 38 13/16 | 212 13/16 | 28 13/16 | 217 13/16 | 42 13/16 | 208 13/16 | 44 13/16 | 206 13/16 | 46 13/16 | 210 13/16 | 230 13/16 | 2021 |
| 238 13/16 | 232 13/16 | 18 13/16 | 234 13/16 | 235 13/16 | 15 13/16 | 14 13/16 | 5 13/16 | 20 13/16 | 240 13/16 | 241 13/16 | 9 13/16 | 243 13/16 | 7 13/16 | 245 13/16 | 12 13/16 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{16} = 505.25$$

$$S_{6\times 6} := 2021 \times \frac{6}{16} = 757.875$$

$$S_{8\times 8} := 2021 \times \frac{8}{16} = 1010.5$$

$$S_{10\times 10} := 2021 \times \frac{10}{16} = 1263.125$$

$$S_{12\times 12} := 2021 \times \frac{12}{16} = 1515.75$$

$$S_{14\times 14} := 2021 \times \frac{14}{16} = 1768.375$$

$$S_{16\times 16} := 2021 \times \frac{16}{16} = 2021.$$

In this case, there is a **Pythagorean triples** with magic sums:

$$S_{6\times 6}^2 + S_{8\times 8}^2 := S_{10\times 10}^2$$

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## 2.15 Bordered Magic Square of Order 17

The **bordered magic square** of order 17 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | 2021 |
| -10 2/17 | 261 15/17 | 259 15/17 | 257 15/17 | 255 15/17 | 253 15/17 | 251 15/17 | 249 15/17 | 248 15/17 | -6 2/17 | -4 2/17 | -2 2/17 | - 2/17 | 1 15/17 | 3 15/17 | 5 15/17 | -8 2/17 | 2021 |
| -25 2/17 | 215 15/17 | 204 15/17 | 206 15/17 | 208 15/17 | 210 15/17 | 212 15/17 | 214 15/17 | 216 15/17 | 16 15/17 | 14 15/17 | 12 15/17 | 10 15/17 | 8 15/17 | 6 15/17 | 19 15/17 | 262 15/17 | 2021 |
| -23 2/17 | 7 15/17 | 189 15/17 | 180 15/17 | 182 15/17 | 184 15/17 | 186 15/17 | 188 15/17 | 190 15/17 | 42 15/17 | 40 15/17 | 38 15/17 | 36 15/17 | 34 15/17 | 45 15/17 | 229 15/17 | 260 15/17 | 2021 |
| -21 2/17 | 9 15/17 | 35 15/17 | 69 15/17 | 77 15/17 | 75 15/17 | 73 15/17 | 71 15/17 | 170 15/17 | 171 15/17 | 173 15/17 | 175 15/17 | 177 15/17 | 67 15/17 | 201 15/17 | 227 15/17 | 258 15/17 | 2021 |
| -19 2/17 | 11 15/17 | 37 15/17 | 178 15/17 | 85 15/17 | 157 15/17 | 155 15/17 | 153 15/17 | 152 15/17 | 89 15/17 | 91 15/17 | 93 15/17 | 87 15/17 | 58 15/17 | 199 15/17 | 225 15/17 | 256 15/17 | 2021 |
| -17 2/17 | 13 15/17 | 39 15/17 | 176 15/17 | 78 15/17 | 101 15/17 | 105 15/17 | 103 15/17 | 138 15/17 | 139 15/17 | 141 15/17 | 99 15/17 | 158 15/17 | 60 15/17 | 197 15/17 | 223 15/17 | 254 15/17 | 2021 |
| -15 2/17 | 15 15/17 | 41 15/17 | 174 15/17 | 80 15/17 | 142 15/17 | 109 15/17 | 106 15/17 | 126 15/17 | 124 15/17 | 125 15/17 | 94 15/17 | 156 15/17 | 62 15/17 | 195 15/17 | 221 15/17 | 252 15/17 | 2021 |
| -13 2/17 | 17 15/17 | 43 15/17 | 172 15/17 | 82 15/17 | 140 15/17 | 129 15/17 | 119 15/17 | 114 15/17 | 121 15/17 | 107 15/17 | 96 15/17 | 154 15/17 | 64 15/17 | 193 15/17 | 219 15/17 | 250 15/17 | 2021 |
| 246 15/17 | 18 15/17 | 44 15/17 | 68 15/17 | 150 15/17 | 100 15/17 | 128 15/17 | 120 15/17 | 118 15/17 | 116 15/17 | 108 15/17 | 136 15/17 | 86 15/17 | 168 15/17 | 192 15/17 | 218 15/17 | -9 2/17 | 2021 |
| 244 15/17 | 213 15/17 | 187 15/17 | 70 15/17 | 148 15/17 | 102 15/17 | 113 15/17 | 115 15/17 | 122 15/17 | 117 15/17 | 123 15/17 | 134 15/17 | 88 15/17 | 166 15/17 | 49 15/17 | 23 15/17 | -7 2/17 | 2021 |
| 242 15/17 | 211 15/17 | 185 15/17 | 72 15/17 | 146 15/17 | 104 15/17 | 111 15/17 | 130 15/17 | 110 15/17 | 112 15/17 | 127 15/17 | 132 15/17 | 90 15/17 | 164 15/17 | 51 15/17 | 25 15/17 | -5 2/17 | 2021 |
| 240 15/17 | 209 15/17 | 183 15/17 | 74 15/17 | 144 15/17 | 137 15/17 | 131 15/17 | 133 15/17 | 98 15/17 | 97 15/17 | 95 15/17 | 135 15/17 | 92 15/17 | 162 15/17 | 53 15/17 | 27 15/17 | -3 2/17 | 2021 |
| 238 15/17 | 207 15/17 | 181 15/17 | 76 15/17 | 149 15/17 | 79 15/17 | 81 15/17 | 83 15/17 | 84 15/17 | 147 15/17 | 145 15/17 | 143 15/17 | 151 15/17 | 160 15/17 | 55 15/17 | 29 15/17 | -1 2/17 | 2021 |
| 236 15/17 | 205 15/17 | 179 15/17 | 169 15/17 | 159 15/17 | 161 15/17 | 163 15/17 | 165 15/17 | 66 15/17 | 65 15/17 | 63 15/17 | 61 15/17 | 59 15/17 | 167 15/17 | 57 15/17 | 31 15/17 | 15/17 | 2021 |
| 234 15/17 | 203 15/17 | 191 15/17 | 56 15/17 | 54 15/17 | 52 15/17 | 50 15/17 | 48 15/17 | 46 15/17 | 194 15/17 | 196 15/17 | 198 15/17 | 200 15/17 | 202 15/17 | 47 15/17 | 33 15/17 | 2 15/17 | 2021 |
| 232 15/17 | 217 15/17 | 32 15/17 | 30 15/17 | 28 15/17 | 26 15/17 | 24 15/17 | 22 15/17 | 20 15/17 | 220 15/17 | 222 15/17 | 224 15/17 | 226 15/17 | 228 15/17 | 230 15/17 | 21 15/17 | 4 15/17 | 2021 |
| 245 15/17 | -24 2/17 | -22 2/17 | -20 2/17 | -18 2/17 | -16 2/17 | -14 2/17 | -12 2/17 | -11 2/17 | 243 15/17 | 241 15/17 | 239 15/17 | 237 15/17 | 235 15/17 | 233 15/17 | 231 15/17 | 247 15/17 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} : 2021 \times \frac{3}{17} = \frac{6063}{17}$$

$$S_{5\times 5} := 2021 \times \frac{5}{17} = \frac{10105}{17}$$

$$S_{7\times 7} := 2021 \times \frac{7}{17} = \frac{14147}{17}$$

$$S_{9\times 9} := 2021 \times \frac{9}{17} = \frac{18189}{17}$$

$$S_{11\times 11} := 2021 \times \frac{11}{17} = \frac{22231}{17}$$

$$S_{13\times 13} := 2021 \times \frac{13}{17} = \frac{26273}{17}$$

$$S_{15\times 15} := 2021 \times \frac{15}{17} = \frac{30315}{17}$$

$$S_{17\times 17} := 2021 \times \frac{17}{17} = 2021.$$

## 2.16 Bordered Magic Square of Order 18

The **bordered magic square** of order 18 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | 2021 |
| 256 7/9 | -25 2/9 | 250 7/9 | -27 2/9 | 252 7/9 | -29 2/9 | 254 7/9 | -31 2/9 | 265 7/9 | -49 2/9 | 258 7/9 | -35 2/9 | 260 7/9 | -37 2/9 | 262 7/9 | -39 2/9 | 264 7/9 | -33 2/9 | 2021 |
| 247 7/9 | 225 7/9 | 5 7/9 | 219 7/9 | 3 7/9 | 2 7/9 | 222 7/9 | 223 7/9 | 232 7/9 | 217 7/9 | -2 2/9 | -3 2/9 | 228 7/9 | -5 2/9 | 230 7/9 | -7 2/9 | - 2/9 | -23 2/9 | 2021 |
| -22 2/9 | 238 7/9 | 27 7/9 | 21 7/9 | 201 7/9 | 23 7/9 | 199 7/9 | 25 7/9 | 209 7/9 | 20 7/9 | 195 7/9 | 29 7/9 | 193 7/9 | 31 7/9 | 191 7/9 | 197 7/9 | -14 2/9 | 246 7/9 | 2021 |
| 245 7/9 | -13 2/9 | 208 7/9 | 173 7/9 | 182 7/9 | 42 7/9 | 180 7/9 | 44 7/9 | 40 7/9 | 61 7/9 | 163 7/9 | 59 7/9 | 165 7/9 | 57 7/9 | 172 7/9 | 15 7/9 | 237 7/9 | -21 2/9 | 2021 |
| -20 2/9 | 236 7/9 | 16 7/9 | 55 7/9 | 152 7/9 | 147 7/9 | 77 7/9 | 145 7/9 | 79 7/9 | 75 7/9 | 65 7/9 | 159 7/9 | 63 7/9 | 153 7/9 | 168 7/9 | 207 7/9 | -12 2/9 | 244 7/9 | 2021 |
| 243 7/9 | -11 2/9 | 206 7/9 | 54 7/9 | 74 7/9 | 87 7/9 | 81 7/9 | 141 7/9 | 143 7/9 | 130 7/9 | 92 7/9 | 132 7/9 | 86 7/9 | 149 7/9 | 169 7/9 | 17 7/9 | 235 7/9 | -19 2/9 | 2021 |
| -18 2/9 | 234 7/9 | 18 7/9 | 170 7/9 | 150 7/9 | 84 7/9 | 125 7/9 | 123 7/9 | 96 7/9 | 129 7/9 | 97 7/9 | 99 7/9 | 139 7/9 | 73 7/9 | 53 7/9 | 205 7/9 | -10 2/9 | 242 7/9 | 2021 |
| 241 7/9 | -9 2/9 | 204 7/9 | 171 7/9 | 72 7/9 | 85 7/9 | 95 7/9 | 117 7/9 | 104 7/9 | 107 7/9 | 118 7/9 | 128 7/9 | 138 7/9 | 151 7/9 | 52 7/9 | 19 7/9 | 233 7/9 | -17 2/9 | 2021 |
| -16 2/9 | -15 2/9 | 190 7/9 | 178 7/9 | 157 7/9 | 90 7/9 | 101 7/9 | 110 7/9 | 115 7/9 | 112 7/9 | 109 7/9 | 122 7/9 | 133 7/9 | 66 7/9 | 45 7/9 | 33 7/9 | 239 7/9 | 240 7/9 | 2021 |
| -24 2/9 | 13 7/9 | 184 7/9 | 167 7/9 | 62 7/9 | 140 7/9 | 103 7/9 | 114 7/9 | 111 7/9 | 108 7/9 | 113 7/9 | 120 7/9 | 83 7/9 | 161 7/9 | 56 7/9 | 39 7/9 | 210 7/9 | 248 7/9 | 2021 |
| -42 2/9 | 211 7/9 | 38 7/9 | 49 7/9 | 154 7/9 | 135 7/9 | 121 7/9 | 105 7/9 | 116 7/9 | 119 7/9 | 106 7/9 | 102 7/9 | 88 7/9 | 69 7/9 | 174 7/9 | 185 7/9 | 12 7/9 | 266 7/9 | 2021 |
| 267 7/9 | 11 7/9 | 186 7/9 | 48 7/9 | 68 7/9 | 134 7/9 | 124 7/9 | 100 7/9 | 127 7/9 | 94 7/9 | 126 7/9 | 98 7/9 | 89 7/9 | 155 7/9 | 175 7/9 | 37 7/9 | 212 7/9 | -43 2/9 | 2021 |
| -44 2/9 | 213 7/9 | 36 7/9 | 176 7/9 | 156 7/9 | 137 7/9 | 142 7/9 | 82 7/9 | 80 7/9 | 93 7/9 | 131 7/9 | 91 7/9 | 136 7/9 | 67 7/9 | 47 7/9 | 187 7/9 | 10 7/9 | 268 7/9 | 2021 |
| 269 7/9 | 9 7/9 | 188 7/9 | 46 7/9 | 70 7/9 | 76 7/9 | 146 7/9 | 78 7/9 | 144 7/9 | 148 7/9 | 158 7/9 | 64 7/9 | 160 7/9 | 71 7/9 | 177 7/9 | 35 7/9 | 214 7/9 | -45 2/9 | 2021 |
| -46 2/9 | 215 7/9 | 34 7/9 | 51 7/9 | 41 7/9 | 181 7/9 | 43 7/9 | 179 7/9 | 183 7/9 | 162 7/9 | 60 7/9 | 164 7/9 | 58 7/9 | 166 7/9 | 50 7/9 | 189 7/9 | 8 7/9 | 270 7/9 | 2021 |
| 271 7/9 | 7 7/9 | 26 7/9 | 202 7/9 | 22 7/9 | 200 7/9 | 24 7/9 | 198 7/9 | 14 7/9 | 203 7/9 | 28 7/9 | 194 7/9 | 30 7/9 | 192 7/9 | 32 7/9 | 196 7/9 | 216 7/9 | -47 2/9 | 2021 |
| -48 2/9 | 224 7/9 | 218 7/9 | 4 7/9 | 220 7/9 | 221 7/9 | 1 7/9 | 7/9 | -8 2/9 | 6 7/9 | 226 7/9 | 227 7/9 | -4 2/9 | 229 7/9 | -6 2/9 | 231 7/9 | -1 2/9 | 272 7/9 | 2021 |
| 257 7/9 | 249 7/9 | -26 2/9 | 251 7/9 | -28 2/9 | 253 7/9 | -30 2/9 | 255 7/9 | -41 2/9 | 273 7/9 | -34 2/9 | 259 7/9 | -36 2/9 | 261 7/9 | -38 2/9 | 263 7/9 | -40 2/9 | -32 2/9 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times4} := 2021 \times \frac{4}{18} = \frac{4042}{9} \qquad S_{10\times10} := 2021 \times \frac{10}{18} = \frac{10105}{9} \qquad S_{16\times16} := 2021 \times \frac{16}{18} = \frac{16168}{9}$$

$$S_{6\times6} := 2021 \times \frac{6}{18} = \frac{2021}{3} \qquad S_{12\times12} := 2021 \times \frac{12}{18} = \frac{4042}{3} \qquad S_{18\times18} := 2021 \times \frac{18}{18} = 2021$$

$$S_{8\times8} := 2021 \times \frac{8}{18} = \frac{8084}{9} \qquad S_{14\times14} := 2021 \times \frac{14}{18} = \frac{14147}{9}$$

In this case, there is a **Pythagorean triples** with magic sums:

$$S_{6\times6}^2 + S_{8\times8}^2 := S_{10\times10}^2$$

## 2.17 Bordered Magic Square of Order 19

The **bordered magic square** of order 19 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | 2021 |
| -54 12/19 | -38 12/19 | -40 12/19 | -42 12/19 | -44 12/19 | -46 12/19 | -48 12/19 | -50 12/19 | -52 12/19 | 270 7/19 | 271 7/19 | 273 7/19 | 275 7/19 | 277 7/19 | 279 7/19 | 281 7/19 | 283 7/19 | 285 7/19 | -56 12/19 | 2021 |
| 286 7/19 | -22 12/19 | 249 7/19 | 247 7/19 | 245 7/19 | 243 7/19 | 241 7/19 | 239 7/19 | 237 7/19 | 236 7/19 | -18 12/19 | -16 12/19 | -14 12/19 | -12 12/19 | -10 12/19 | -8 12/19 | -6 12/19 | -20 12/19 | -73 12/19 | 2021 |
| 284 7/19 | -37 12/19 | 203 7/19 | 192 7/19 | 194 7/19 | 196 7/19 | 198 7/19 | 200 7/19 | 202 7/19 | 204 7/19 | 4 7/19 | 2 7/19 | 7/19 | -1 12/19 | -3 12/19 | -5 12/19 | 7 7/19 | 250 7/19 | -71 12/19 | 2021 |
| 282 7/19 | -35 12/19 | -4 12/19 | 177 7/19 | 168 7/19 | 170 7/19 | 172 7/19 | 174 7/19 | 176 7/19 | 178 7/19 | 30 7/19 | 28 7/19 | 26 7/19 | 24 7/19 | 22 7/19 | 33 7/19 | 217 7/19 | 248 7/19 | -69 12/19 | 2021 |
| 280 7/19 | -33 12/19 | -2 12/19 | 23 7/19 | 57 7/19 | 65 7/19 | 63 7/19 | 61 7/19 | 59 7/19 | 158 7/19 | 159 7/19 | 161 7/19 | 163 7/19 | 165 7/19 | 55 7/19 | 189 7/19 | 215 7/19 | 246 7/19 | -67 12/19 | 2021 |
| 278 7/19 | -31 12/19 | - 12/19 | 25 7/19 | 166 7/19 | 73 7/19 | 145 7/19 | 143 7/19 | 141 7/19 | 140 7/19 | 77 7/19 | 79 7/19 | 81 7/19 | 75 7/19 | 46 7/19 | 187 7/19 | 213 7/19 | 244 7/19 | -65 12/19 | 2021 |
| 276 7/19 | -29 12/19 | 1 7/19 | 27 7/19 | 164 7/19 | 66 7/19 | 89 7/19 | 93 7/19 | 91 7/19 | 126 7/19 | 127 7/19 | 129 7/19 | 87 7/19 | 146 7/19 | 48 7/19 | 185 7/19 | 211 7/19 | 242 7/19 | -63 12/19 | 2021 |
| 274 7/19 | -27 12/19 | 3 7/19 | 29 7/19 | 162 7/19 | 68 7/19 | 130 7/19 | 97 7/19 | 94 7/19 | 114 7/19 | 112 7/19 | 113 7/19 | 82 7/19 | 144 7/19 | 50 7/19 | 183 7/19 | 209 7/19 | 240 7/19 | -61 12/19 | 2021 |
| 272 7/19 | -25 12/19 | 5 7/19 | 31 7/19 | 160 7/19 | 70 7/19 | 128 7/19 | 117 7/19 | 107 7/19 | 102 7/19 | 109 7/19 | 95 7/19 | 84 7/19 | 142 7/19 | 52 7/19 | 181 7/19 | 207 7/19 | 238 7/19 | -59 12/19 | 2021 |
| -55 12/19 | 234 7/19 | 6 7/19 | 32 7/19 | 56 7/19 | 138 7/19 | 88 7/19 | 116 7/19 | 108 7/19 | 106 7/19 | 104 7/19 | 96 7/19 | 124 7/19 | 74 7/19 | 156 7/19 | 180 7/19 | 206 7/19 | -21 12/19 | 268 7/19 | 2021 |
| -53 12/19 | 232 7/19 | 201 7/19 | 175 7/19 | 58 7/19 | 136 7/19 | 90 7/19 | 101 7/19 | 103 7/19 | 110 7/19 | 105 7/19 | 111 7/19 | 122 7/19 | 76 7/19 | 154 7/19 | 37 7/19 | 11 7/19 | -19 12/19 | 266 7/19 | 2021 |
| -51 12/19 | 230 7/19 | 199 7/19 | 173 7/19 | 60 7/19 | 134 7/19 | 92 7/19 | 99 7/19 | 118 7/19 | 98 7/19 | 100 7/19 | 115 7/19 | 120 7/19 | 78 7/19 | 152 7/19 | 39 7/19 | 13 7/19 | -17 12/19 | 264 7/19 | 2021 |
| -49 12/19 | 228 7/19 | 197 7/19 | 171 7/19 | 62 7/19 | 132 7/19 | 125 7/19 | 119 7/19 | 121 7/19 | 86 7/19 | 85 7/19 | 83 7/19 | 123 7/19 | 80 7/19 | 150 7/19 | 41 7/19 | 15 7/19 | -15 12/19 | 262 7/19 | 2021 |
| -47 12/19 | 226 7/19 | 195 7/19 | 169 7/19 | 64 7/19 | 137 7/19 | 67 7/19 | 69 7/19 | 71 7/19 | 72 7/19 | 135 7/19 | 133 7/19 | 131 7/19 | 139 7/19 | 148 7/19 | 43 7/19 | 17 7/19 | -13 12/19 | 260 7/19 | 2021 |
| -45 12/19 | 224 7/19 | 193 7/19 | 167 7/19 | 157 7/19 | 147 7/19 | 149 7/19 | 151 7/19 | 153 7/19 | 54 7/19 | 53 7/19 | 51 7/19 | 49 7/19 | 47 7/19 | 155 7/19 | 45 7/19 | 19 7/19 | -11 12/19 | 258 7/19 | 2021 |
| -43 12/19 | 222 7/19 | 191 7/19 | 179 7/19 | 44 7/19 | 42 7/19 | 40 7/19 | 38 7/19 | 36 7/19 | 34 7/19 | 182 7/19 | 184 7/19 | 186 7/19 | 188 7/19 | 190 7/19 | 35 7/19 | 21 7/19 | -9 12/19 | 256 7/19 | 2021 |
| -41 12/19 | 220 7/19 | 205 7/19 | 20 7/19 | 18 7/19 | 16 7/19 | 14 7/19 | 12 7/19 | 10 7/19 | 8 7/19 | 208 7/19 | 210 7/19 | 212 7/19 | 214 7/19 | 216 7/19 | 218 7/19 | 9 7/19 | -7 12/19 | 254 7/19 | 2021 |
| -39 12/19 | 233 7/19 | -36 12/19 | -34 12/19 | -32 12/19 | -30 12/19 | -28 12/19 | -26 12/19 | -24 12/19 | -23 12/19 | 231 7/19 | 229 7/19 | 227 7/19 | 225 7/19 | 223 7/19 | 221 7/19 | 219 7/19 | 235 7/19 | 252 7/19 | 2021 |
| 269 7/19 | 251 7/19 | 253 7/19 | 255 7/19 | 257 7/19 | 259 7/19 | 261 7/19 | 263 7/19 | 265 7/19 | -57 12/19 | -58 12/19 | -60 12/19 | -62 12/19 | -64 12/19 | -66 12/19 | -68 12/19 | -70 12/19 | -72 12/19 | 267 7/19 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{19} = \frac{6063}{19}$$

$$S_{5\times 5} := 2021 \times \frac{5}{19} = \frac{10105}{19}$$

$$S_{7\times 7} := 2021 \times \frac{7}{19} = \frac{14147}{19}$$

$$S_{9\times 9} := 2021 \times \frac{9}{19} = \frac{18189}{19}$$

$$S_{11\times 11} := 2021 \times \frac{11}{19} = \frac{22231}{19}$$

$$S_{13\times 13} := 2021 \times \frac{13}{19} = \frac{26273}{19}$$

$$S_{15\times 15} := 2021 \times \frac{15}{19} = \frac{30315}{19}$$

$$S_{17\times 17} := 2021 \times \frac{17}{19} = \frac{34357}{19}$$

$$S_{19\times 19} := 2021 \times \frac{19}{19} = 2021.$$

## 2.18 Bordered Magic Square of Order 20

The **bordered magic square** of order 20 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | 2021 |
| -80 9/20 | 271 11/20 | -68 9/20 | 269 11/20 | -66 9/20 | 267 11/20 | -64 9/20 | 265 11/20 | -62 9/20 | 263 11/20 | 300 11/20 | 292 11/20 | -91 9/20 | 294 11/20 | -93 9/20 | 296 11/20 | -95 9/20 | 298 11/20 | -97 9/20 | -79 9/20 | 2021 |
| 290 11/20 | 245 11/20 | -36 9/20 | 239 11/20 | -38 9/20 | 241 11/20 | -40 9/20 | 243 11/20 | -42 9/20 | 254 11/20 | -60 9/20 | 247 11/20 | -46 9/20 | 249 11/20 | -48 9/20 | 251 11/20 | -50 9/20 | 253 11/20 | -44 9/20 | -88 9/20 | 2021 |
| -87 9/20 | 236 11/20 | 214 11/20 | -5 9/20 | 208 11/20 | -7 9/20 | -8 9/20 | 211 11/20 | 212 11/20 | 221 11/20 | 206 11/20 | -13 9/20 | -14 9/20 | 217 11/20 | -16 9/20 | 219 11/20 | -18 9/20 | -11 9/20 | -34 9/20 | 289 11/20 | 2021 |
| 288 11/20 | -33 9/20 | 227 11/20 | 16 11/20 | 10 11/20 | 190 11/20 | 12 11/20 | 188 11/20 | 14 11/20 | 198 11/20 | 9 11/20 | 184 11/20 | 18 11/20 | 182 11/20 | 20 11/20 | 180 11/20 | 186 11/20 | -25 9/20 | 235 11/20 | -86 9/20 | 2021 |
| -85 9/20 | 234 11/20 | -24 9/20 | 197 11/20 | 162 11/20 | 171 11/20 | 31 11/20 | 169 11/20 | 33 11/20 | 29 11/20 | 50 11/20 | 152 11/20 | 48 11/20 | 154 11/20 | 46 11/20 | 161 11/20 | 4 11/20 | 226 11/20 | -32 9/20 | 287 11/20 | 2021 |
| 286 11/20 | -31 9/20 | 225 11/20 | 5 11/20 | 44 11/20 | 141 11/20 | 136 11/20 | 66 11/20 | 134 11/20 | 68 11/20 | 64 11/20 | 54 11/20 | 148 11/20 | 52 11/20 | 142 11/20 | 157 11/20 | 196 11/20 | -23 9/20 | 233 11/20 | -84 9/20 | 2021 |
| -83 9/20 | 232 11/20 | -22 9/20 | 195 11/20 | 43 11/20 | 63 11/20 | 76 11/20 | 70 11/20 | 130 11/20 | 132 11/20 | 119 11/20 | 81 11/20 | 121 11/20 | 75 11/20 | 138 11/20 | 158 11/20 | 6 11/20 | 224 11/20 | -30 9/20 | 285 11/20 | 2021 |
| 284 11/20 | -29 9/20 | 223 11/20 | 7 11/20 | 159 11/20 | 139 11/20 | 73 11/20 | 114 11/20 | 112 11/20 | 85 11/20 | 118 11/20 | 86 11/20 | 88 11/20 | 128 11/20 | 62 11/20 | 42 11/20 | 194 11/20 | -21 9/20 | 231 11/20 | -82 9/20 | 2021 |
| 283 11/20 | 230 11/20 | -20 9/20 | 193 11/20 | 160 11/20 | 61 11/20 | 74 11/20 | 84 11/20 | 106 11/20 | 93 11/20 | 96 11/20 | 107 11/20 | 117 11/20 | 127 11/20 | 140 11/20 | 41 11/20 | 8 11/20 | 222 11/20 | -28 9/20 | -81 9/20 | 2021 |
| -70 9/20 | -27 9/20 | -26 9/20 | 179 11/20 | 167 11/20 | 146 11/20 | 79 11/20 | 90 11/20 | 99 11/20 | 104 11/20 | 101 11/20 | 98 11/20 | 111 11/20 | 122 11/20 | 55 11/20 | 34 11/20 | 22 11/20 | 228 11/20 | 229 11/20 | 272 11/20 | 2021 |
| -89 9/20 | -35 9/20 | 2 11/20 | 173 11/20 | 156 11/20 | 51 11/20 | 129 11/20 | 92 11/20 | 103 11/20 | 100 11/20 | 97 11/20 | 102 11/20 | 109 11/20 | 72 11/20 | 150 11/20 | 45 11/20 | 28 11/20 | 199 11/20 | 237 11/20 | 291 11/20 | 2021 |
| -78 9/20 | -53 9/20 | 200 11/20 | 27 11/20 | 38 11/20 | 143 11/20 | 124 11/20 | 110 11/20 | 94 11/20 | 105 11/20 | 108 11/20 | 95 11/20 | 91 11/20 | 77 11/20 | 58 11/20 | 163 11/20 | 174 11/20 | 1 11/20 | 255 11/20 | 280 11/20 | 2021 |
| -77 9/20 | 256 11/20 | 11/20 | 175 11/20 | 37 11/20 | 57 11/20 | 123 11/20 | 113 11/20 | 89 11/20 | 116 11/20 | 83 11/20 | 115 11/20 | 87 11/20 | 78 11/20 | 144 11/20 | 164 11/20 | 26 11/20 | 201 11/20 | -54 9/20 | 279 11/20 | 2021 |
| 278 11/20 | -55 9/20 | 202 11/20 | 25 11/20 | 165 11/20 | 145 11/20 | 126 11/20 | 131 11/20 | 71 11/20 | 69 11/20 | 82 11/20 | 120 11/20 | 80 11/20 | 125 11/20 | 56 11/20 | 36 11/20 | 176 11/20 | - 9/20 | 257 11/20 | -76 9/20 | 2021 |
| 277 11/20 | 258 11/20 | -1 9/20 | 177 11/20 | 35 11/20 | 59 11/20 | 65 11/20 | 135 11/20 | 67 11/20 | 133 11/20 | 137 11/20 | 147 11/20 | 53 11/20 | 149 11/20 | 60 11/20 | 166 11/20 | 24 11/20 | 203 11/20 | -56 9/20 | -75 9/20 | 2021 |
| -74 9/20 | -57 9/20 | 204 11/20 | 23 11/20 | 40 11/20 | 30 11/20 | 170 11/20 | 32 11/20 | 168 11/20 | 172 11/20 | 151 11/20 | 49 11/20 | 153 11/20 | 47 11/20 | 155 11/20 | 39 11/20 | 178 11/20 | -2 9/20 | 259 11/20 | 276 11/20 | 2021 |
| 275 11/20 | 260 11/20 | -3 9/20 | 15 11/20 | 191 11/20 | 11 11/20 | 189 11/20 | 13 11/20 | 187 11/20 | 3 11/20 | 192 11/20 | 17 11/20 | 183 11/20 | 19 11/20 | 181 11/20 | 21 11/20 | 185 11/20 | 205 11/20 | -58 9/20 | -73 9/20 | 2021 |
| -72 9/20 | -59 9/20 | 213 11/20 | 207 11/20 | -6 9/20 | 209 11/20 | 210 11/20 | -9 9/20 | -10 9/20 | -19 9/20 | -4 9/20 | 215 11/20 | 216 11/20 | -15 9/20 | 218 11/20 | -17 9/20 | 220 11/20 | -12 9/20 | 261 11/20 | 274 11/20 | 2021 |
| 273 11/20 | 246 11/20 | 238 11/20 | -37 9/20 | 240 11/20 | -39 9/20 | 242 11/20 | -41 9/20 | 244 11/20 | -52 9/20 | 262 11/20 | -45 9/20 | 248 11/20 | -47 9/20 | 250 11/20 | -49 9/20 | 252 11/20 | -51 9/20 | -43 9/20 | -71 9/20 | 2021 |
| 281 11/20 | -69 9/20 | 270 11/20 | -67 9/20 | 268 11/20 | -65 9/20 | 266 11/20 | -63 9/20 | 264 11/20 | -61 9/20 | -98 9/20 | -90 9/20 | 293 11/20 | -92 9/20 | 295 11/20 | -94 9/20 | 297 11/20 | -96 9/20 | 299 11/20 | 282 11/20 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{20} = 404.2 \qquad S_{10\times 10} := 2021 \times \frac{10}{20} = 1010.5 \qquad S_{16\times 16} := 2021 \times \frac{16}{20} = 1616.8$$
$$S_{6\times 6} := 2021 \times \frac{6}{20} = 606.3 \qquad S_{12\times 12} := 2021 \times \frac{12}{20} = 1212.6 \qquad S_{18\times 18} := 2021 \times \frac{18}{20} = 1818.9$$
$$S_{8\times 8} := 2021 \times \frac{8}{20} = 808.4 \qquad S_{14\times 14} := 2021 \times \frac{14}{20} = 1414.7 \qquad S_{20\times 20} := 2021 \times \frac{20}{20} = 2021.$$

In this case, there are **Pythagorean triples** with magic sums:

$$S_{6\times 6}^2 + S_{8\times 8}^2 := S_{10\times 10}^2$$
$$S_{12\times 12}^2 + S_{16\times 16}^2 := S_{20\times 20}^2.$$

## 2.19 Bordered Magic Square of Order 21

The **bordered magic square** of order 21 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 295 5/21 | 278 5/21 | 280 5/21 | 282 5/21 | 284 5/21 | 286 5/21 | 288 5/21 | 290 5/21 | 292 5/21 | 294 5/21 | 296 5/21 | -107 16/21 | -109 16/21 | -111 16/21 | -113 16/21 | -115 16/21 | -117 16/21 | -119 16/21 | -121 16/21 | -123 16/21 | -104 16/21 | 2021 |
| -122 16/21 | 259 5/21 | -49 16/21 | -51 16/21 | -53 16/21 | -55 16/21 | -57 16/21 | -59 16/21 | -61 16/21 | -63 16/21 | -65 16/21 | 262 5/21 | 264 5/21 | 266 5/21 | 268 5/21 | 270 5/21 | 272 5/21 | 274 5/21 | 276 5/21 | -64 16/21 | 315 5/21 | 2021 |
| -120 16/21 | 241 5/21 | 225 5/21 | -17 16/21 | -19 16/21 | -21 16/21 | -23 16/21 | -25 16/21 | -27 16/21 | -29 16/21 | -31 16/21 | 228 5/21 | 230 5/21 | 232 5/21 | 234 5/21 | 236 5/21 | 238 5/21 | 240 5/21 | -30 16/21 | -48 16/21 | 313 5/21 | 2021 |
| -118 16/21 | 243 5/21 | 209 5/21 | 195 5/21 | 181 5/21 | 183 5/21 | 185 5/21 | 187 5/21 | 189 5/21 | 191 5/21 | -3 16/21 | -4 16/21 | -6 16/21 | -8 16/21 | -10 16/21 | -12 16/21 | -14 16/21 | 193 5/21 | -16 16/21 | -50 16/21 | 311 5/21 | 2021 |
| -116 16/21 | 245 5/21 | 211 5/21 | 10 5/21 | 23 5/21 | 12 5/21 | 14 5/21 | 16 5/21 | 18 5/21 | 20 5/21 | 168 5/21 | 166 5/21 | 164 5/21 | 162 5/21 | 160 5/21 | 158 5/21 | 167 5/21 | 182 5/21 | -18 16/21 | -52 16/21 | 309 5/21 | 2021 |
| -114 16/21 | 247 5/21 | 213 5/21 | 8 5/21 | 179 5/21 | 45 5/21 | 36 5/21 | 38 5/21 | 40 5/21 | 42 5/21 | 146 5/21 | 144 5/21 | 142 5/21 | 140 5/21 | 138 5/21 | 145 5/21 | 13 5/21 | 184 5/21 | -20 16/21 | -54 16/21 | 307 5/21 | 2021 |
| -112 16/21 | 249 5/21 | 215 5/21 | 6 5/21 | 177 5/21 | 155 5/21 | 127 5/21 | 57 5/21 | 59 5/21 | 61 5/21 | 62 5/21 | 125 5/21 | 123 5/21 | 121 5/21 | 129 5/21 | 37 5/21 | 15 5/21 | 186 5/21 | -22 16/21 | -56 16/21 | 305 5/21 | 2021 |
| -110 16/21 | 251 5/21 | 217 5/21 | 4 5/21 | 175 5/21 | 153 5/21 | 122 5/21 | 113 5/21 | 73 5/21 | 75 5/21 | 76 5/21 | 111 5/21 | 109 5/21 | 115 5/21 | 70 5/21 | 39 5/21 | 17 5/21 | 188 5/21 | -24 16/21 | -58 16/21 | 303 5/21 | 2021 |
| -108 16/21 | 253 5/21 | 219 5/21 | 2 5/21 | 173 5/21 | 151 5/21 | 124 5/21 | 110 5/21 | 105 5/21 | 101 5/21 | 86 5/21 | 85 5/21 | 103 5/21 | 82 5/21 | 68 5/21 | 41 5/21 | 19 5/21 | 190 5/21 | -26 16/21 | -60 16/21 | 301 5/21 | 2021 |
| -106 16/21 | 255 5/21 | 221 5/21 | 5/21 | 171 5/21 | 149 5/21 | 126 5/21 | 112 5/21 | 90 5/21 | 93 5/21 | 98 5/21 | 97 5/21 | 102 5/21 | 80 5/21 | 66 5/21 | 43 5/21 | 21 5/21 | 192 5/21 | -28 16/21 | -62 16/21 | 299 5/21 | 2021 |
| -105 16/21 | -67 16/21 | -33 16/21 | -1 16/21 | 170 5/21 | 148 5/21 | 128 5/21 | 114 5/21 | 88 5/21 | 100 5/21 | 96 5/21 | 92 5/21 | 104 5/21 | 78 5/21 | 64 5/21 | 44 5/21 | 22 5/21 | 194 5/21 | 226 5/21 | 260 5/21 | 298 5/21 | 2021 |
| 293 5/21 | -68 16/21 | -34 16/21 | 198 5/21 | 27 5/21 | 49 5/21 | 60 5/21 | 74 5/21 | 108 5/21 | 95 5/21 | 94 5/21 | 99 5/21 | 84 5/21 | 118 5/21 | 132 5/21 | 143 5/21 | 165 5/21 | -5 16/21 | 227 5/21 | 261 5/21 | -100 16/21 | 2021 |
| 291 5/21 | -70 16/21 | -36 16/21 | 200 5/21 | 29 5/21 | 51 5/21 | 58 5/21 | 72 5/21 | 89 5/21 | 91 5/21 | 106 5/21 | 107 5/21 | 87 5/21 | 120 5/21 | 134 5/21 | 141 5/21 | 163 5/21 | -7 16/21 | 229 5/21 | 263 5/21 | -98 16/21 | 2021 |
| 289 5/21 | -72 16/21 | -38 16/21 | 202 5/21 | 31 5/21 | 53 5/21 | 56 5/21 | 77 5/21 | 119 5/21 | 117 5/21 | 116 5/21 | 81 5/21 | 83 5/21 | 79 5/21 | 136 5/21 | 139 5/21 | 161 5/21 | -9 16/21 | 231 5/21 | 265 5/21 | -96 16/21 | 2021 |
| 287 5/21 | -74 16/21 | -40 16/21 | 204 5/21 | 33 5/21 | 55 5/21 | 63 5/21 | 135 5/21 | 133 5/21 | 131 5/21 | 130 5/21 | 67 5/21 | 69 5/21 | 71 5/21 | 65 5/21 | 137 5/21 | 159 5/21 | -11 16/21 | 233 5/21 | 267 5/21 | -94 16/21 | 2021 |
| 285 5/21 | -76 16/21 | -42 16/21 | 206 5/21 | 35 5/21 | 47 5/21 | 156 5/21 | 154 5/21 | 152 5/21 | 150 5/21 | 46 5/21 | 48 5/21 | 50 5/21 | 52 5/21 | 54 5/21 | 147 5/21 | 157 5/21 | -13 16/21 | 235 5/21 | 269 5/21 | -92 16/21 | 2021 |
| 283 5/21 | -78 16/21 | -44 16/21 | 208 5/21 | 25 5/21 | 180 5/21 | 178 5/21 | 176 5/21 | 174 5/21 | 172 5/21 | 24 5/21 | 26 5/21 | 28 5/21 | 30 5/21 | 32 5/21 | 34 5/21 | 169 5/21 | -15 16/21 | 237 5/21 | 271 5/21 | -90 16/21 | 2021 |
| 281 5/21 | -80 16/21 | -46 16/21 | - 16/21 | 11 5/21 | 9 5/21 | 7 5/21 | 5 5/21 | 3 5/21 | 1 5/21 | 196 5/21 | 197 5/21 | 199 5/21 | 201 5/21 | 203 5/21 | 205 5/21 | 207 5/21 | -2 16/21 | 239 5/21 | 273 5/21 | -88 16/21 | 2021 |
| 279 5/21 | -82 16/21 | 223 5/21 | 210 5/21 | 212 5/21 | 214 5/21 | 216 5/21 | 218 5/21 | 220 5/21 | 222 5/21 | 224 5/21 | -35 16/21 | -37 16/21 | -39 16/21 | -41 16/21 | -43 16/21 | -45 16/21 | -47 16/21 | -32 16/21 | 275 5/21 | -86 16/21 | 2021 |
| 277 5/21 | 257 5/21 | 242 5/21 | 244 5/21 | 246 5/21 | 248 5/21 | 250 5/21 | 252 5/21 | 254 5/21 | 256 5/21 | 258 5/21 | -69 16/21 | -71 16/21 | -73 16/21 | -75 16/21 | -77 16/21 | -79 16/21 | -81 16/21 | -83 16/21 | -66 16/21 | -84 16/21 | 2021 |
| 297 5/21 | -85 16/21 | -87 16/21 | -89 16/21 | -91 16/21 | -93 16/21 | -95 16/21 | -97 16/21 | -99 16/21 | -101 16/21 | -103 16/21 | 300 5/21 | 302 5/21 | 304 5/21 | 306 5/21 | 308 5/21 | 310 5/21 | 312 5/21 | 314 5/21 | 316 5/21 | -102 16/21 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{21} = \frac{2021}{7}$$
$$S_{5\times 5} := 2021 \times \frac{5}{21} = \frac{10105}{21}$$
$$S_{7\times 7} := 2021 \times \frac{7}{21} = \frac{2021}{3}$$
$$S_{9\times 9} := 2021 \times \frac{9}{21} = \frac{6063}{7}$$
$$S_{11\times 11} := 2021 \times \frac{11}{21} = \frac{22231}{21}$$
$$S_{13\times 13} := 2021 \times \frac{13}{21} = \frac{26273}{21}$$
$$S_{15\times 15} := 2021 \times \frac{15}{21} = \frac{10105}{7}$$
$$S_{17\times 17} := 2021 \times \frac{17}{21} = \frac{34357}{21}$$
$$S_{19\times 19} := 2021 \times \frac{19}{21} = \frac{38399}{21}$$
$$S_{21\times 21} := 2021 \times \frac{21}{21} = 2021.$$

## 2.20 Bordered Magic Square of Order 22

The **bordered magic square** of order 22 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | 2021 |
| -129 7/11 | 322 4/11 | -137 7/11 | 320 4/11 | -135 7/11 | 318 4/11 | -133 7/11 | 316 4/11 | -131 7/11 | 314 4/11 | -149 7/11 | 323 4/11 | -127 7/11 | 310 4/11 | -125 7/11 | 308 4/11 | -123 7/11 | 306 4/11 | -121 7/11 | 304 4/11 | -119 7/11 | 312 4/11 | 2021 |
| -117 7/11 | 272 4/11 | 264 4/11 | -81 7/11 | 266 4/11 | -83 7/11 | 268 4/11 | 269 4/11 | -86 7/11 | -87 7/11 | -98 7/11 | -79 7/11 | 274 4/11 | 275 4/11 | -92 7/11 | 277 4/11 | -94 7/11 | 279 4/11 | -96 7/11 | 281 4/11 | -89 7/11 | 301 4/11 | 2021 |
| 300 4/11 | -78 7/11 | 236 4/11 | -45 7/11 | 230 4/11 | -47 7/11 | 232 4/11 | -49 7/11 | 234 4/11 | -51 7/11 | 245 4/11 | -69 7/11 | 238 4/11 | -55 7/11 | 240 4/11 | -57 7/11 | 242 4/11 | -59 7/11 | 244 4/11 | -53 7/11 | 262 4/11 | -116 7/11 | 2021 |
| -115 7/11 | 261 4/11 | 227 4/11 | 204 4/11 | 198 4/11 | -15 7/11 | 200 4/11 | 201 4/11 | -18 7/11 | -19 7/11 | -28 7/11 | -13 7/11 | 206 4/11 | 207 4/11 | -24 7/11 | 209 4/11 | -26 7/11 | 211 4/11 | -21 7/11 | -43 7/11 | -77 7/11 | 299 4/11 | 2021 |
| 298 4/11 | -76 7/11 | -42 7/11 | -12 7/11 | 176 4/11 | 12 4/11 | 172 4/11 | 10 4/11 | 174 4/11 | 8 4/11 | 183 4/11 | -5 7/11 | 178 4/11 | 4 4/11 | 180 4/11 | 2 4/11 | 182 4/11 | 6 4/11 | 196 4/11 | 226 4/11 | 260 4/11 | -114 7/11 | 2021 |
| -113 7/11 | 259 4/11 | 225 4/11 | 195 4/11 | 169 4/11 | 152 4/11 | 148 4/11 | 149 4/11 | 33 4/11 | 32 4/11 | 25 4/11 | 36 4/11 | 154 4/11 | 155 4/11 | 27 4/11 | 157 4/11 | 30 4/11 | 14 4/11 | -11 7/11 | -41 7/11 | -75 7/11 | 297 4/11 | 2021 |
| 296 4/11 | -74 7/11 | -40 7/11 | -10 7/11 | 15 4/11 | 37 4/11 | 132 4/11 | 127 4/11 | 57 4/11 | 125 4/11 | 59 4/11 | 55 4/11 | 45 4/11 | 139 4/11 | 43 4/11 | 133 4/11 | 146 4/11 | 168 4/11 | 194 4/11 | 224 4/11 | 258 4/11 | -112 7/11 | 2021 |
| -111 7/11 | 257 4/11 | 223 4/11 | 193 4/11 | 167 4/11 | 145 4/11 | 54 4/11 | 117 4/11 | 122 4/11 | 62 4/11 | 60 4/11 | 73 4/11 | 111 4/11 | 71 4/11 | 116 4/11 | 129 4/11 | 38 4/11 | 16 4/11 | -9 7/11 | -39 7/11 | -73 7/11 | 295 4/11 | 2021 |
| 294 4/11 | -72 7/11 | -38 7/11 | -8 7/11 | 17 4/11 | 39 4/11 | 130 4/11 | 114 4/11 | 104 4/11 | 80 4/11 | 107 4/11 | 74 4/11 | 106 4/11 | 78 4/11 | 69 4/11 | 53 4/11 | 144 4/11 | 166 4/11 | 192 4/11 | 222 4/11 | 256 4/11 | -110 7/11 | 2021 |
| -109 7/11 | 255 4/11 | 221 4/11 | 191 4/11 | 165 4/11 | 143 4/11 | 52 4/11 | 115 4/11 | 101 4/11 | 97 4/11 | 84 4/11 | 87 4/11 | 98 4/11 | 82 4/11 | 68 4/11 | 131 4/11 | 40 4/11 | 18 4/11 | -7 7/11 | -37 7/11 | -71 7/11 | 293 4/11 | 2021 |
| 292 4/11 | -70 7/11 | -36 7/11 | -6 7/11 | 19 4/11 | 41 4/11 | 137 4/11 | 120 4/11 | 83 4/11 | 90 4/11 | 95 4/11 | 92 4/11 | 89 4/11 | 100 4/11 | 63 4/11 | 46 4/11 | 142 4/11 | 164 4/11 | 190 4/11 | 220 4/11 | 254 4/11 | -108 7/11 | 2021 |
| 302 4/11 | -107 7/11 | -44 7/11 | -35 7/11 | 13 4/11 | 20 4/11 | 42 4/11 | 70 4/11 | 81 4/11 | 94 4/11 | 91 4/11 | 88 4/11 | 93 4/11 | 102 4/11 | 113 4/11 | 141 4/11 | 163 4/11 | 170 4/11 | 219 4/11 | 228 4/11 | 291 4/11 | -118 7/11 | 2021 |
| 324 4/11 | -99 7/11 | -62 7/11 | -29 7/11 | - 7/11 | 24 4/11 | 134 4/11 | 65 4/11 | 75 4/11 | 85 4/11 | 96 4/11 | 99 4/11 | 86 4/11 | 108 4/11 | 118 4/11 | 49 4/11 | 159 4/11 | 184 4/11 | 213 4/11 | 246 4/11 | 283 4/11 | -140 7/11 | 2021 |
| -141 7/11 | 284 4/11 | 247 4/11 | 214 4/11 | 185 4/11 | 160 4/11 | 48 4/11 | 64 4/11 | 105 4/11 | 103 4/11 | 76 4/11 | 109 4/11 | 77 4/11 | 79 4/11 | 119 4/11 | 135 4/11 | 23 4/11 | -1 7/11 | -30 7/11 | -63 7/11 | -100 7/11 | 325 4/11 | 2021 |
| 326 4/11 | -101 7/11 | -64 7/11 | -31 7/11 | -2 7/11 | 22 4/11 | 136 4/11 | 67 4/11 | 61 4/11 | 121 4/11 | 123 4/11 | 110 4/11 | 72 4/11 | 112 4/11 | 66 4/11 | 47 4/11 | 161 4/11 | 186 4/11 | 215 4/11 | 248 4/11 | 285 4/11 | -142 7/11 | 2021 |
| -143 7/11 | 286 4/11 | 249 4/11 | 216 4/11 | 187 4/11 | 162 4/11 | 50 4/11 | 56 4/11 | 126 4/11 | 58 4/11 | 124 4/11 | 128 4/11 | 138 4/11 | 44 4/11 | 140 4/11 | 51 4/11 | 21 4/11 | -3 7/11 | -32 7/11 | -65 7/11 | -102 7/11 | 327 4/11 | 2021 |
| 328 4/11 | -103 7/11 | -66 7/11 | -33 7/11 | -4 7/11 | 153 4/11 | 35 4/11 | 34 4/11 | 150 4/11 | 151 4/11 | 158 4/11 | 147 4/11 | 29 4/11 | 28 4/11 | 156 4/11 | 26 4/11 | 31 4/11 | 188 4/11 | 217 4/11 | 250 4/11 | 287 4/11 | -144 7/11 | 2021 |
| -145 7/11 | 288 4/11 | 251 4/11 | 218 4/11 | 177 4/11 | 171 4/11 | 11 4/11 | 173 4/11 | 9 4/11 | 175 4/11 | 4/11 | 189 4/11 | 5 4/11 | 179 4/11 | 3 4/11 | 181 4/11 | 1 4/11 | 7 4/11 | -34 7/11 | -67 7/11 | -104 7/11 | 329 4/11 | 2021 |
| 330 4/11 | -105 7/11 | -68 7/11 | 205 4/11 | -14 7/11 | 199 4/11 | -16 7/11 | -17 7/11 | 202 4/11 | 203 4/11 | 212 4/11 | 197 4/11 | -22 7/11 | -23 7/11 | 208 4/11 | -25 7/11 | 210 4/11 | -27 7/11 | -20 7/11 | 252 4/11 | 289 4/11 | -146 7/11 | 2021 |
| -147 7/11 | 290 4/11 | 237 4/11 | 229 4/11 | -46 7/11 | 231 4/11 | -48 7/11 | 233 4/11 | -50 7/11 | 235 4/11 | -61 7/11 | 253 4/11 | -54 7/11 | 239 4/11 | -56 7/11 | 241 4/11 | -58 7/11 | 243 4/11 | -60 7/11 | -52 7/11 | -106 7/11 | 331 4/11 | 2021 |
| 332 4/11 | 273 4/11 | -80 7/11 | 265 4/11 | -82 7/11 | 267 4/11 | -84 7/11 | -85 7/11 | 270 4/11 | 271 4/11 | 282 4/11 | 263 4/11 | -90 7/11 | -91 7/11 | 276 4/11 | -93 7/11 | 278 4/11 | -95 7/11 | 280 4/11 | -97 7/11 | -88 7/11 | -148 7/11 | 2021 |
| -128 7/11 | -138 7/11 | 321 4/11 | -136 7/11 | 319 4/11 | -134 7/11 | 317 4/11 | -132 7/11 | 315 4/11 | -130 7/11 | 333 4/11 | -139 7/11 | 311 4/11 | -126 7/11 | 309 4/11 | -124 7/11 | 307 4/11 | -122 7/11 | 305 4/11 | -120 7/11 | 303 4/11 | 313 4/11 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times 4} := 2021 \times \frac{4}{22} = \frac{4042}{11} \qquad S_{12\times 12} := 2021 \times \frac{12}{22} = \frac{12126}{11} \qquad S_{18\times 18} := 2021 \times \frac{18}{22} = \frac{18189}{11}$$

$$S_{6\times 6} := 2021 \times \frac{6}{22} = \frac{6063}{11} \qquad S_{14\times 14} := 2021 \times \frac{14}{22} = \frac{14147}{11} \qquad S_{20\times 20} := 2021 \times \frac{20}{22} = \frac{20211}{11}$$

$$S_{8\times 8} := 2021 \times \frac{8}{22} = \frac{8084}{11} \qquad S_{16\times 16} := 2021 \times \frac{16}{22} = \frac{16168}{11} \qquad S_{22\times 22} := 2021 \times \frac{22}{22} = 2021$$

$$S_{10\times 10} := 2021 \times \frac{10}{22} = \frac{10105}{11}$$

In this case, there are **Pythagorean triples** with magic sums:

$$S^2_{6\times 6} + S^2_{8\times 8} := S^2_{10\times 10}$$
$$S^2_{12\times 12} + S^2_{16\times 16} := S^2_{20\times 20}.$$

## 2.21 Bordered Magic Square of Order 23

The **bordered magic square** of order 23 for the magic sum 2021 is given by

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 335 4/23 | -129 19/23 | -131 19/23 | -133 19/23 | -135 19/23 | -137 19/23 | -139 19/23 | -141 19/23 | -143 19/23 | -145 19/23 | -147 19/23 | -149 19/23 | 338 4/23 | 340 4/23 | 342 4/23 | 344 4/23 | 346 4/23 | 348 4/23 | 350 4/23 | 352 4/23 | 354 4/23 | 356 4/23 | -148 19/23 | 2120 |
| 313 4/23 | 291 4/23 | 274 4/23 | 276 4/23 | 278 4/23 | 280 4/23 | 282 4/23 | 284 4/23 | 286 4/23 | 288 4/23 | 290 4/23 | 292 4/23 | -111 19/23 | -113 19/23 | -115 19/23 | -117 19/23 | -119 19/23 | -121 19/23 | -123 19/23 | -125 19/23 | -127 19/23 | -108 19/23 | -128 19/23 | 2120 |
| 315 4/23 | -126 19/23 | 255 4/23 | -53 19/23 | -55 19/23 | -57 19/23 | -59 19/23 | -61 19/23 | -63 19/23 | -65 19/23 | -67 19/23 | -69 19/23 | 258 4/23 | 260 4/23 | 262 4/23 | 264 4/23 | 266 4/23 | 268 4/23 | 270 4/23 | 272 4/23 | -68 19/23 | 311 4/23 | -130 19/23 | 2120 |
| 317 4/23 | -124 19/23 | 237 4/23 | 221 4/23 | -21 19/23 | -23 19/23 | -25 19/23 | -27 19/23 | -29 19/23 | -31 19/23 | -33 19/23 | -35 19/23 | 224 4/23 | 226 4/23 | 228 4/23 | 230 4/23 | 232 4/23 | 234 4/23 | 236 4/23 | -34 19/23 | -52 19/23 | 309 4/23 | -132 19/23 | 2120 |
| 319 4/23 | -122 19/23 | 239 4/23 | 205 4/23 | 191 4/23 | 177 4/23 | 179 4/23 | 181 4/23 | 183 4/23 | 185 4/23 | 187 4/23 | -7 19/23 | -8 19/23 | -10 19/23 | -12 19/23 | -14 19/23 | -16 19/23 | -18 19/23 | 189 4/23 | -20 19/23 | -54 19/23 | 307 4/23 | -134 19/23 | 2120 |
| 321 4/23 | -120 19/23 | 241 4/23 | 207 4/23 | 6 4/23 | 19 4/23 | 8 4/23 | 10 4/23 | 12 4/23 | 14 4/23 | 16 4/23 | 164 4/23 | 162 4/23 | 160 4/23 | 158 4/23 | 156 4/23 | 154 4/23 | 163 4/23 | 178 4/23 | -22 19/23 | -56 19/23 | 305 4/23 | -136 19/23 | 2120 |
| 323 4/23 | -118 19/23 | 243 4/23 | 209 4/23 | 4 4/23 | 175 4/23 | 41 4/23 | 32 4/23 | 34 4/23 | 36 4/23 | 38 4/23 | 142 4/23 | 140 4/23 | 138 4/23 | 136 4/23 | 134 4/23 | 141 4/23 | 9 4/23 | 180 4/23 | -24 19/23 | -58 19/23 | 303 4/23 | -138 19/23 | 2120 |
| 325 4/23 | -116 19/23 | 245 4/23 | 211 4/23 | 2 4/23 | 173 4/23 | 151 4/23 | 123 4/23 | 53 4/23 | 55 4/23 | 57 4/23 | 58 4/23 | 121 4/23 | 119 4/23 | 117 4/23 | 125 4/23 | 33 4/23 | 11 4/23 | 182 4/23 | -26 19/23 | -60 19/23 | 301 4/23 | -140 19/23 | 2120 |
| 327 4/23 | -114 19/23 | 247 4/23 | 213 4/23 | 4/23 | 171 4/23 | 149 4/23 | 118 4/23 | 109 4/23 | 69 4/23 | 71 4/23 | 72 4/23 | 107 4/23 | 105 4/23 | 111 4/23 | 66 4/23 | 35 4/23 | 13 4/23 | 184 4/23 | -28 19/23 | -62 19/23 | 299 4/23 | -142 19/23 | 2120 |
| 329 4/23 | -112 19/23 | 249 4/23 | 215 4/23 | -1 19/23 | 169 4/23 | 147 4/23 | 120 4/23 | 106 4/23 | 101 4/23 | 97 4/23 | 82 4/23 | 81 4/23 | 99 4/23 | 78 4/23 | 64 4/23 | 37 4/23 | 15 4/23 | 186 4/23 | -30 19/23 | -64 19/23 | 297 4/23 | -144 19/23 | 2120 |
| 331 4/23 | -110 19/23 | 251 4/23 | 217 4/23 | -3 19/23 | 167 4/23 | 145 4/23 | 122 4/23 | 108 4/23 | 86 4/23 | 89 4/23 | 94 4/23 | 93 4/23 | 98 4/23 | 76 4/23 | 62 4/23 | 39 4/23 | 17 4/23 | 188 4/23 | -32 19/23 | -66 19/23 | 295 4/23 | -146 19/23 | 2120 |
| -151 19/23 | -109 19/23 | -71 19/23 | -37 19/23 | -5 19/23 | 166 4/23 | 144 4/23 | 124 4/23 | 110 4/23 | 84 4/23 | 96 4/23 | 92 4/23 | 88 4/23 | 100 4/23 | 74 4/23 | 60 4/23 | 40 4/23 | 18 4/23 | 190 4/23 | 222 4/23 | 256 4/23 | 294 4/23 | 336 4/23 | 2120 |
| -152 19/23 | 289 4/23 | -72 19/23 | -38 19/23 | 194 4/23 | 23 4/23 | 45 4/23 | 56 4/23 | 70 4/23 | 104 4/23 | 91 4/23 | 90 4/23 | 95 4/23 | 80 4/23 | 114 4/23 | 128 4/23 | 139 4/23 | 161 4/23 | -9 19/23 | 223 4/23 | 257 4/23 | -104 19/23 | 337 4/23 | 2120 |
| -154 19/23 | 287 4/23 | -74 19/23 | -40 19/23 | 196 4/23 | 25 4/23 | 47 4/23 | 54 4/23 | 68 4/23 | 85 4/23 | 87 4/23 | 102 4/23 | 103 4/23 | 83 4/23 | 116 4/23 | 130 4/23 | 137 4/23 | 159 4/23 | -11 19/23 | 225 4/23 | 259 4/23 | -102 19/23 | 339 4/23 | 2120 |
| -156 19/23 | 285 4/23 | -76 19/23 | -42 19/23 | 198 4/23 | 27 4/23 | 49 4/23 | 52 4/23 | 73 4/23 | 115 4/23 | 113 4/23 | 112 4/23 | 77 4/23 | 79 4/23 | 75 4/23 | 132 4/23 | 135 4/23 | 157 4/23 | -13 19/23 | 227 4/23 | 261 4/23 | -100 19/23 | 341 4/23 | 2120 |
| -158 19/23 | 283 4/23 | -78 19/23 | -44 19/23 | 200 4/23 | 29 4/23 | 51 4/23 | 59 4/23 | 131 4/23 | 129 4/23 | 127 4/23 | 126 4/23 | 63 4/23 | 65 4/23 | 67 4/23 | 61 4/23 | 133 4/23 | 155 4/23 | -15 19/23 | 229 4/23 | 263 4/23 | -98 19/23 | 343 4/23 | 2120 |
| -160 19/23 | 281 4/23 | -80 19/23 | -46 19/23 | 202 4/23 | 31 4/23 | 43 4/23 | 152 4/23 | 150 4/23 | 148 4/23 | 146 4/23 | 42 4/23 | 44 4/23 | 46 4/23 | 48 4/23 | 50 4/23 | 143 4/23 | 153 4/23 | -17 19/23 | 231 4/23 | 265 4/23 | -96 19/23 | 345 4/23 | 2120 |
| -162 19/23 | 279 4/23 | -82 19/23 | -48 19/23 | 204 4/23 | 21 4/23 | 176 4/23 | 174 4/23 | 172 4/23 | 170 4/23 | 168 4/23 | 20 4/23 | 22 4/23 | 24 4/23 | 26 4/23 | 28 4/23 | 30 4/23 | 165 4/23 | -19 19/23 | 233 4/23 | 267 4/23 | -94 19/23 | 347 4/23 | 2120 |
| -164 19/23 | 277 4/23 | -84 19/23 | -50 19/23 | -4 19/23 | 7 4/23 | 5 4/23 | 3 4/23 | 1 4/23 | - 19/23 | -2 19/23 | 192 4/23 | 193 4/23 | 195 4/23 | 197 4/23 | 199 4/23 | 201 4/23 | 203 4/23 | -6 19/23 | 235 4/23 | 269 4/23 | -92 19/23 | 349 4/23 | 2120 |
| -166 19/23 | 275 4/23 | -86 19/23 | 219 4/23 | 206 4/23 | 208 4/23 | 210 4/23 | 212 4/23 | 214 4/23 | 216 4/23 | 218 4/23 | 220 4/23 | -39 19/23 | -41 19/23 | -43 19/23 | -45 19/23 | -47 19/23 | -49 19/23 | -51 19/23 | -36 19/23 | 271 4/23 | -90 19/23 | 351 4/23 | 2120 |
| -168 19/23 | 273 4/23 | 253 4/23 | 238 4/23 | 240 4/23 | 242 4/23 | 244 4/23 | 246 4/23 | 248 4/23 | 250 4/23 | 252 4/23 | 254 4/23 | -73 19/23 | -75 19/23 | -77 19/23 | -79 19/23 | -81 19/23 | -83 19/23 | -85 19/23 | -87 19/23 | -70 19/23 | -88 19/23 | 353 4/23 | 2120 |
| -170 19/23 | 293 4/23 | -89 19/23 | -91 19/23 | -93 19/23 | -95 19/23 | -97 19/23 | -99 19/23 | -101 19/23 | -103 19/23 | -105 19/23 | -107 19/23 | 296 4/23 | 298 4/23 | 300 4/23 | 302 4/23 | 304 4/23 | 306 4/23 | 308 4/23 | 310 4/23 | 312 4/23 | -106 19/23 | 355 4/23 | 2120 |
| 333 4/23 | 314 4/23 | 316 4/23 | 318 4/23 | 320 4/23 | 322 4/23 | 324 4/23 | 326 4/23 | 328 4/23 | 330 4/23 | 332 4/23 | 334 4/23 | -153 19/23 | -155 19/23 | -157 19/23 | -159 19/23 | -161 19/23 | -163 19/23 | -165 19/23 | -167 19/23 | -169 19/23 | -171 19/23 | -150 19/23 | 2120 |
| 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 | 2120 |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{23} = \frac{6063}{23}$$
$$S_{5\times 5} := 2021 \times \frac{5}{23} = \frac{10105}{23}$$
$$S_{7\times 7} := 2021 \times \frac{7}{23} = \frac{14147}{23}$$
$$S_{9\times 9} := 2021 \times \frac{9}{23} = \frac{18189}{23}$$
$$S_{11\times 11} := 2021 \times \frac{11}{23} = \frac{22231}{23}$$
$$S_{13\times 13} := 2021 \times \frac{13}{23} = \frac{26273}{23}$$
$$S_{15\times 15} := 2021 \times \frac{15}{23} = \frac{30315}{23}$$
$$S_{17\times 17} := 2021 \times \frac{17}{23} = \frac{34357}{23}$$
$$S_{19\times 19} := 2021 \times \frac{19}{23} = \frac{38399}{23}$$
$$S_{21\times 21} := 2021 \times \frac{21}{23} = \frac{42441}{23}$$
$$S_{23\times 23} := 2021 \times \frac{23}{23} = 2021.$$

## 2.22 Bordered Magic Square of Order 24

The **bordered magic square** of order 24 for the magic sum 2021 is given by

Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021,
**Zenodo**, December 16, 2020, pp. 1-33, *http://doi.org/10.5281/zenodo.4327333*

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | | 2021 |
| 349 17/24 | -170 7/24 | 339 17/24 | -172 7/24 | 341 17/24 | -174 7/24 | 343 17/24 | -176 7/24 | -177 7/24 | 346 17/24 | 347 17/24 | 360 17/24 | 337 17/24 | -182 7/24 | -183 7/24 | 352 17/24 | -185 7/24 | 354 17/24 | -187 7/24 | 356 17/24 | -189 7/24 | 358 17/24 | -191 7/24 | -180 7/24 | 2021 |
| 370 17/24 | -137 7/24 | 314 17/24 | -145 7/24 | 312 17/24 | -143 7/24 | 310 17/24 | -141 7/24 | 308 17/24 | -139 7/24 | 306 17/24 | -157 7/24 | 315 17/24 | -135 7/24 | 302 17/24 | -133 7/24 | 300 17/24 | -131 7/24 | 298 17/24 | -129 7/24 | 296 17/24 | -127 7/24 | 304 17/24 | -202 7/24 | 2021 |
| -201 7/24 | -125 7/24 | 264 17/24 | 256 17/24 | -89 7/24 | 258 17/24 | -91 7/24 | 260 17/24 | 261 17/24 | -94 7/24 | -95 7/24 | -106 7/24 | -87 7/24 | 266 17/24 | 267 17/24 | -100 7/24 | 269 17/24 | -102 7/24 | 271 17/24 | -104 7/24 | 273 17/24 | -97 7/24 | 293 17/24 | 369 17/24 | 2021 |
| 368 17/24 | 292 17/24 | -86 7/24 | 228 17/24 | -53 7/24 | 222 17/24 | -55 7/24 | 224 17/24 | -57 7/24 | 226 17/24 | -59 7/24 | 237 17/24 | -77 7/24 | 230 17/24 | -63 7/24 | 232 17/24 | -65 7/24 | 234 17/24 | -67 7/24 | 236 17/24 | -61 7/24 | 254 17/24 | -124 7/24 | -200 7/24 | 2021 |
| -199 7/24 | -123 7/24 | 253 17/24 | 219 17/24 | 196 17/24 | 190 17/24 | -23 7/24 | 192 17/24 | 193 17/24 | -26 7/24 | -27 7/24 | -36 7/24 | -21 7/24 | 198 17/24 | 199 17/24 | -32 7/24 | 201 17/24 | -34 7/24 | 203 17/24 | -29 7/24 | -51 7/24 | -85 7/24 | 291 17/24 | 367 17/24 | 2021 |
| 366 17/24 | 290 17/24 | -84 7/24 | -50 7/24 | -20 7/24 | 168 17/24 | 4 17/24 | 164 17/24 | 2 17/24 | 166 17/24 | 17/24 | 175 17/24 | -13 7/24 | 170 17/24 | -3 7/24 | 172 17/24 | -5 7/24 | 174 17/24 | -1 7/24 | 188 17/24 | 218 17/24 | 252 17/24 | -122 7/24 | -198 7/24 | 2021 |
| -197 7/24 | -121 7/24 | 251 17/24 | 217 17/24 | 187 17/24 | 161 17/24 | 144 17/24 | 140 17/24 | 141 17/24 | 25 17/24 | 24 17/24 | 17 17/24 | 28 17/24 | 146 17/24 | 147 17/24 | 19 17/24 | 149 17/24 | 22 17/24 | 6 17/24 | -19 7/24 | -49 7/24 | -83 7/24 | 289 17/24 | 365 17/24 | 2021 |
| 364 17/24 | 288 17/24 | -82 7/24 | -48 7/24 | -18 7/24 | 7 17/24 | 29 17/24 | 124 17/24 | 119 17/24 | 49 17/24 | 117 17/24 | 51 17/24 | 47 17/24 | 37 17/24 | 131 17/24 | 35 17/24 | 125 17/24 | 138 17/24 | 160 17/24 | 186 17/24 | 216 17/24 | 250 17/24 | -120 7/24 | -196 7/24 | 2021 |
| -195 7/24 | -119 7/24 | 249 17/24 | 215 17/24 | 185 17/24 | 159 17/24 | 137 17/24 | 46 17/24 | 109 17/24 | 114 17/24 | 54 17/24 | 52 17/24 | 65 17/24 | 103 17/24 | 63 17/24 | 108 17/24 | 121 17/24 | 30 17/24 | 8 17/24 | -17 7/24 | -47 7/24 | -81 7/24 | 287 17/24 | 363 17/24 | 2021 |
| 362 17/24 | 286 17/24 | -80 7/24 | -46 7/24 | -16 7/24 | 9 17/24 | 31 17/24 | 122 17/24 | 106 17/24 | 96 17/24 | 72 17/24 | 99 17/24 | 66 17/24 | 98 17/24 | 70 17/24 | 61 17/24 | 45 17/24 | 136 17/24 | 158 17/24 | 184 17/24 | 214 17/24 | 248 17/24 | -118 7/24 | -194 7/24 | 2021 |
| -193 7/24 | -117 7/24 | 247 17/24 | 213 17/24 | 183 17/24 | 157 17/24 | 135 17/24 | 44 17/24 | 107 17/24 | 93 17/24 | 89 17/24 | 76 17/24 | 79 17/24 | 90 17/24 | 74 17/24 | 60 17/24 | 123 17/24 | 32 17/24 | 10 17/24 | -15 7/24 | -45 7/24 | -79 7/24 | 285 17/24 | 361 17/24 | 2021 |
| -203 7/24 | 284 17/24 | -78 7/24 | -44 7/24 | -14 7/24 | 11 17/24 | 33 17/24 | 129 17/24 | 112 17/24 | 75 17/24 | 82 17/24 | 87 17/24 | 84 17/24 | 81 17/24 | 92 17/24 | 55 17/24 | 38 17/24 | 134 17/24 | 156 17/24 | 182 17/24 | 212 17/24 | 246 17/24 | -116 7/24 | 371 17/24 | 2021 |
| -158 7/24 | 294 17/24 | -115 7/24 | -52 7/24 | -43 7/24 | 5 17/24 | 12 17/24 | 34 17/24 | 62 17/24 | 73 17/24 | 86 17/24 | 83 17/24 | 80 17/24 | 85 17/24 | 94 17/24 | 105 17/24 | 133 17/24 | 155 17/24 | 162 17/24 | 211 17/24 | 220 17/24 | 283 17/24 | -126 7/24 | 326 17/24 | 2021 |
| 327 17/24 | 316 17/24 | -107 7/24 | -70 7/24 | -37 7/24 | -8 7/24 | 16 17/24 | 126 17/24 | 57 17/24 | 67 17/24 | 77 17/24 | 88 17/24 | 91 17/24 | 78 17/24 | 100 17/24 | 110 17/24 | 41 17/24 | 151 17/24 | 176 17/24 | 205 17/24 | 238 17/24 | 275 17/24 | -148 7/24 | -159 7/24 | 2021 |
| -160 7/24 | -149 7/24 | 276 17/24 | 239 17/24 | 206 17/24 | 177 17/24 | 152 17/24 | 40 17/24 | 56 17/24 | 97 17/24 | 95 17/24 | 68 17/24 | 101 17/24 | 69 17/24 | 71 17/24 | 111 17/24 | 127 17/24 | 15 17/24 | -9 7/24 | -38 7/24 | -71 7/24 | -108 7/24 | 317 17/24 | 328 17/24 | 2021 |
| 329 17/24 | 318 17/24 | -109 7/24 | -72 7/24 | -39 7/24 | -10 7/24 | 14 17/24 | 128 17/24 | 59 17/24 | 53 17/24 | 113 17/24 | 115 17/24 | 102 17/24 | 64 17/24 | 104 17/24 | 58 17/24 | 39 17/24 | 153 17/24 | 178 17/24 | 207 17/24 | 240 17/24 | 277 17/24 | -150 7/24 | -161 7/24 | 2021 |
| -162 7/24 | -151 7/24 | 278 17/24 | 241 17/24 | 208 17/24 | 179 17/24 | 154 17/24 | 42 17/24 | 48 17/24 | 118 17/24 | 50 17/24 | 116 17/24 | 120 17/24 | 130 17/24 | 36 17/24 | 132 17/24 | 43 17/24 | 13 17/24 | -11 7/24 | -40 7/24 | -73 7/24 | -110 7/24 | 319 17/24 | 330 17/24 | 2021 |
| 331 17/24 | 320 17/24 | -111 7/24 | -74 7/24 | -41 7/24 | -12 7/24 | 145 17/24 | 27 17/24 | 26 17/24 | 142 17/24 | 143 17/24 | 150 17/24 | 139 17/24 | 21 17/24 | 20 17/24 | 148 17/24 | 18 17/24 | 23 17/24 | 180 17/24 | 209 17/24 | 242 17/24 | 279 17/24 | -152 7/24 | -163 7/24 | 2021 |
| -164 7/24 | -153 7/24 | 280 17/24 | 243 17/24 | 210 17/24 | 169 17/24 | 163 17/24 | 3 17/24 | 165 17/24 | 1 17/24 | 167 17/24 | -7 7/24 | 181 17/24 | -2 7/24 | 171 17/24 | -4 7/24 | 173 17/24 | -6 7/24 | - 7/24 | -42 7/24 | -75 7/24 | -112 7/24 | 321 17/24 | 332 17/24 | 2021 |
| 333 17/24 | 322 17/24 | -113 7/24 | -76 7/24 | 197 17/24 | -22 7/24 | 191 17/24 | -24 7/24 | -25 7/24 | 194 17/24 | 195 17/24 | 204 17/24 | 189 17/24 | -30 7/24 | -31 7/24 | 200 17/24 | -33 7/24 | 202 17/24 | -35 7/24 | -28 7/24 | 244 17/24 | 281 17/24 | -154 7/24 | -165 7/24 | 2021 |
| -166 7/24 | -155 7/24 | 282 17/24 | 229 17/24 | 221 17/24 | -54 7/24 | 223 17/24 | -56 7/24 | 225 17/24 | -58 7/24 | 227 17/24 | -69 7/24 | 245 17/24 | -62 7/24 | 231 17/24 | -64 7/24 | 233 17/24 | -66 7/24 | 235 17/24 | -68 7/24 | -60 7/24 | -114 7/24 | 323 17/24 | 334 17/24 | 2021 |
| 335 17/24 | 324 17/24 | 265 17/24 | -88 7/24 | 257 17/24 | -90 7/24 | 259 17/24 | -92 7/24 | -93 7/24 | 262 17/24 | 263 17/24 | 274 17/24 | 255 17/24 | -98 7/24 | -99 7/24 | 268 17/24 | -101 7/24 | 270 17/24 | -103 7/24 | 272 17/24 | -105 7/24 | -96 7/24 | -156 7/24 | -167 7/24 | 2021 |
| -168 7/24 | -136 7/24 | -146 7/24 | 313 17/24 | -144 7/24 | 311 17/24 | -142 7/24 | 309 17/24 | -140 7/24 | 307 17/24 | -138 7/24 | 325 17/24 | -147 7/24 | 303 17/24 | -134 7/24 | 301 17/24 | -132 7/24 | 299 17/24 | -130 7/24 | 297 17/24 | -128 7/24 | 295 17/24 | 305 17/24 | 336 17/24 | 2021 |
| 348 17/24 | 338 17/24 | -171 7/24 | 340 17/24 | -173 7/24 | 342 17/24 | -175 7/24 | 344 17/24 | 345 17/24 | -178 7/24 | -179 7/24 | -192 7/24 | -169 7/24 | 350 17/24 | 351 17/24 | -184 7/24 | 353 17/24 | -186 7/24 | 355 17/24 | -188 7/24 | 357 17/24 | -190 7/24 | 359 17/24 | -181 7/24 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times4} := 2021 \times \frac{4}{24} = \frac{2021}{6}$$
$$S_{6\times6} := 2021 \times \frac{6}{24} = 505.25$$
$$S_{8\times8} := 2021 \times \frac{8}{24} = \frac{2021}{3}$$
$$S_{10\times10} := 2021 \times \frac{10}{24} = \frac{10105}{12}$$
$$S_{12\times12} := 2021 \times \frac{12}{21} = 1010.5$$
$$S_{14\times14} := 2021 \times \frac{14}{24} = \frac{14147}{12}$$
$$S_{16\times16} := 2021 \times \frac{16}{24} = \frac{4042}{3}$$
$$S_{18\times18} := 2021 \times \frac{18}{24} = 1515.75$$
$$S_{20\times20} := 2021 \times \frac{20}{24} = \frac{10105}{6}$$
$$S_{22\times22} := 2021 \times \frac{22}{24} = \frac{22231}{12}$$
$$S_{24\times24} := 2021 \times \frac{24}{24} = 2021.$$

In this case, there are **Pythagorean triples** with magic sums:

$$S^2_{6\times 6} + S^2_{8\times 8} := S^2_{10\times 10}$$
$$S^2_{12\times 12} + S^2_{16\times 16} := S^2_{20\times 20}.$$

## 2.23 Bordered Magic Square of Order 25

The **bordered magic square** of order 23 for the magic sum 2021 is given by

Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021,
**Zenodo**, December 16, 2020, pp. 1-33, *http://doi.org/10.5281/zenodo.4327333*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | | | | 2021 |
| -206,16 | 392,84 | 390,84 | 388,84 | 386,84 | 384,84 | 382,84 | 380,84 | 378,84 | 376,84 | 374,84 | 372,84 | -207,16 | -205,16 | -203,16 | -201,16 | -199,16 | -197,16 | -195,16 | -193,16 | -191,16 | -189,16 | -187,16 | -185,16 | 369,84 | 2021 |
| -184,16 | 323,84 | -141,16 | -143,16 | -145,16 | -147,16 | -149,16 | -151,16 | -153,16 | -155,16 | -157,16 | -159,16 | -161,16 | 326,84 | 328,84 | 330,84 | 332,84 | 334,84 | 336,84 | 338,84 | 340,84 | 342,84 | 344,84 | -160,16 | 345,84 | 2021 |
| -186,16 | 301,84 | 279,84 | 262,84 | 264,84 | 266,84 | 268,84 | 270,84 | 272,84 | 274,84 | 276,84 | 278,84 | 280,84 | -123,16 | -125,16 | -127,16 | -129,16 | -131,16 | -133,16 | -135,16 | -137,16 | -139,16 | -120,16 | -140,16 | 347,84 | 2021 |
| -188,16 | 303,84 | -138,16 | 243,84 | -65,16 | -67,16 | -69,16 | -71,16 | -73,16 | -75,16 | -77,16 | -79,16 | -81,16 | 246,84 | 248,84 | 250,84 | 252,84 | 254,84 | 256,84 | 258,84 | 260,84 | -80,16 | 299,84 | -142,16 | 349,84 | 2021 |
| -190,16 | 305,84 | -136,16 | 225,84 | 209,84 | -33,16 | -35,16 | -37,16 | -39,16 | -41,16 | -43,16 | -45,16 | -47,16 | 212,84 | 214,84 | 216,84 | 218,84 | 220,84 | 222,84 | 224,84 | -46,16 | -64,16 | 297,84 | -144,16 | 351,84 | 2021 |
| -192,16 | 307,84 | -134,16 | 227,84 | 193,84 | 179,84 | 165,84 | 167,84 | 169,84 | 171,84 | 173,84 | 175,84 | -19,16 | -20,16 | -22,16 | -24,16 | -26,16 | -28,16 | -30,16 | 177,84 | -32,16 | -66,16 | 295,84 | -146,16 | 353,84 | 2021 |
| -194,16 | 309,84 | -132,16 | 229,84 | 195,84 | -5,16 | 7,84 | -3,16 | -1,16 | 0,84 | 2,84 | 4,84 | 152,84 | 150,84 | 148,84 | 146,84 | 144,84 | 142,84 | 151,84 | 166,84 | -34,16 | -68,16 | 293,84 | -148,16 | 355,84 | 2021 |
| -196,16 | 311,84 | -130,16 | 231,84 | 197,84 | -7,16 | 163,84 | 29,84 | 20,84 | 22,84 | 24,84 | 26,84 | 130,84 | 128,84 | 126,84 | 124,84 | 122,84 | 129,84 | -2,16 | 168,84 | -36,16 | -70,16 | 291,84 | -150,16 | 357,84 | 2021 |
| -198,16 | 313,84 | -128,16 | 233,84 | 199,84 | -9,16 | 161,84 | 139,84 | 111,84 | 41,84 | 43,84 | 45,84 | 46,84 | 109,84 | 107,84 | 105,84 | 113,84 | 21,84 | -0,16 | 170,84 | -38,16 | -72,16 | 289,84 | -152,16 | 359,84 | 2021 |
| -200,16 | 315,84 | -126,16 | 235,84 | 201,84 | -11,16 | 159,84 | 137,84 | 106,84 | 97,84 | 57,84 | 59,84 | 60,84 | 95,84 | 93,84 | 99,84 | 54,84 | 23,84 | 1,84 | 172,84 | -40,16 | -74,16 | 287,84 | -154,16 | 361,84 | 2021 |
| -202,16 | 317,84 | -124,16 | 237,84 | 203,84 | -13,16 | 157,84 | 135,84 | 108,84 | 94,84 | 89,84 | 85,84 | 70,84 | 69,84 | 87,84 | 66,84 | 52,84 | 25,84 | 3,84 | 174,84 | -42,16 | -76,16 | 285,84 | -156,16 | 363,84 | 2021 |
| -204,16 | 319,84 | -122,16 | 239,84 | 205,84 | -15,16 | 155,84 | 133,84 | 110,84 | 96,84 | 74,84 | 77,84 | 82,84 | 81,84 | 86,84 | 64,84 | 50,84 | 27,84 | 5,84 | 176,84 | -44,16 | -78,16 | 283,84 | -158,16 | 365,84 | 2021 |
| 370,84 | -163,16 | -121,16 | -83,16 | -49,16 | -17,16 | 154,84 | 132,84 | 112,84 | 98,84 | 72,84 | 84,84 | 80,84 | 76,84 | 88,84 | 62,84 | 48,84 | 28,84 | 6,84 | 178,84 | 210,84 | 244,84 | 282,84 | 324,84 | -209,16 | 2021 |
| 371,84 | -164,16 | 277,84 | -84,16 | -50,16 | 182,84 | 11,84 | 33,84 | 44,84 | 58,84 | 92,84 | 79,84 | 78,84 | 83,84 | 68,84 | 102,84 | 116,84 | 127,84 | 149,84 | -21,16 | 211,84 | 245,84 | -116,16 | 325,84 | -210,16 | 2021 |
| 373,84 | -166,16 | 275,84 | -86,16 | -52,16 | 184,84 | 13,84 | 35,84 | 42,84 | 56,84 | 73,84 | 75,84 | 90,84 | 91,84 | 71,84 | 104,84 | 118,84 | 125,84 | 147,84 | -23,16 | 213,84 | 247,84 | -114,16 | 327,84 | -212,16 | 2021 |
| 375,84 | -168,16 | 273,84 | -88,16 | -54,16 | 186,84 | 15,84 | 37,84 | 40,84 | 61,84 | 103,84 | 101,84 | 100,84 | 65,84 | 67,84 | 63,84 | 120,84 | 123,84 | 145,84 | -25,16 | 215,84 | 249,84 | -112,16 | 329,84 | -214,16 | 2021 |
| 377,84 | -170,16 | 271,84 | -90,16 | -56,16 | 188,84 | 17,84 | 39,84 | 47,84 | 119,84 | 117,84 | 115,84 | 114,84 | 51,84 | 53,84 | 55,84 | 49,84 | 121,84 | 143,84 | -27,16 | 217,84 | 251,84 | -110,16 | 331,84 | -216,16 | 2021 |
| 379,84 | -172,16 | 269,84 | -92,16 | -58,16 | 190,84 | 19,84 | 31,84 | 140,84 | 138,84 | 136,84 | 134,84 | 30,84 | 32,84 | 34,84 | 36,84 | 38,84 | 131,84 | 141,84 | -29,16 | 219,84 | 253,84 | -108,16 | 333,84 | -218,16 | 2021 |
| 381,84 | -174,16 | 267,84 | -94,16 | -60,16 | 192,84 | 9,84 | 164,84 | 162,84 | 160,84 | 158,84 | 156,84 | 8,84 | 10,84 | 12,84 | 14,84 | 16,84 | 18,84 | 153,84 | -31,16 | 221,84 | 255,84 | -106,16 | 335,84 | -220,16 | 2021 |
| 383,84 | -176,16 | 265,84 | -96,16 | -62,16 | -16,16 | -4,16 | -6,16 | -8,16 | -10,16 | -12,16 | -14,16 | 180,84 | 181,84 | 183,84 | 185,84 | 187,84 | 189,84 | 191,84 | -18,16 | 223,84 | 257,84 | -104,16 | 337,84 | -222,16 | 2021 |
| 385,84 | -178,16 | 263,84 | -98,16 | 207,84 | 194,84 | 196,84 | 198,84 | 200,84 | 202,84 | 204,84 | 206,84 | 208,84 | -51,16 | -53,16 | -55,16 | -57,16 | -59,16 | -61,16 | -63,16 | -48,16 | 259,84 | -102,16 | 339,84 | -224,16 | 2021 |
| 387,84 | -180,16 | 261,84 | 241,84 | 226,84 | 228,84 | 230,84 | 232,84 | 234,84 | 236,84 | 238,84 | 240,84 | 242,84 | -85,16 | -87,16 | -89,16 | -91,16 | -93,16 | -95,16 | -97,16 | -99,16 | -82,16 | -100,16 | 341,84 | -226,16 | 2021 |
| 389,84 | -182,16 | 281,84 | -101,16 | -103,16 | -105,16 | -107,16 | -109,16 | -111,16 | -113,16 | -115,16 | -117,16 | -119,16 | 284,84 | 286,84 | 288,84 | 290,84 | 292,84 | 294,84 | 296,84 | 298,84 | 300,84 | -118,16 | 343,84 | -228,16 | 2021 |
| 391,84 | 321,84 | 302,84 | 304,84 | 306,84 | 308,84 | 310,84 | 312,84 | 314,84 | 316,84 | 318,84 | 320,84 | 322,84 | -165,16 | -167,16 | -169,16 | -171,16 | -173,16 | -175,16 | -177,16 | -179,16 | -181,16 | -183,16 | -162,16 | -230,16 | 2021 |
| -208,16 | -231,16 | -229,16 | -227,16 | -225,16 | -223,16 | -221,16 | -219,16 | -217,16 | -215,16 | -213,16 | -211,16 | 368,84 | 366,84 | 364,84 | 362,84 | 360,84 | 358,84 | 356,84 | 354,84 | 352,84 | 350,84 | 348,84 | 346,84 | 367,84 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | |

According to (1), the sub-magic squares sums are as given by

$$S_{3\times 3} := 2021 \times \frac{3}{25} = 242.52 \qquad S_{11\times 11} := 2021 \times \frac{11}{25} = 889.24 \qquad S_{19\times 19} := 2021 \times \frac{19}{25} = 1535.96$$

$$S_{5\times 5} := 2021 \times \frac{5}{25} = 404.20 \qquad S_{13\times 13} := 2021 \times \frac{13}{25} = 1050.92 \qquad S_{21\times 21} := 2021 \times \frac{21}{25} = 1697.64$$

$$S_{7\times 7} := 2021 \times \frac{7}{25} = 565.88 \qquad S_{15\times 15} := 2021 \times \frac{15}{25} = 1212.60 \qquad S_{23\times 23} := 2021 \times \frac{23}{25} = 1859.32$$

$$S_{9\times 9} := 2021 \times \frac{9}{25} = 727.56 \qquad S_{17\times 17} := 2021 \times \frac{17}{25} = 1374.28 \qquad S_{25\times 25} := 2021 \times \frac{25}{25} = 2021.$$

## 2.24 Bordered Magic Square of Order 26

The **bordered magic square** of order 23 for the magic sum 2021 is given by

Fractional and Decimal Type Bordered Magic Squares With Magic Sum 2021,
**Zenodo**, December 16, 2020, pp. 1-33, *http://doi.org/10.5281/zenodo.4327333*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | | | | 2021 |
| -235 10/13 | 402 3/13 | -245 10/13 | 400 3/13 | -243 10/13 | 398 3/13 | -241 10/13 | 396 3/13 | -239 10/13 | 394 3/13 | -237 10/13 | 392 3/13 | -259 10/13 | 403 3/13 | -233 10/13 | 388 3/13 | -231 10/13 | 386 3/13 | -229 10/13 | 384 3/13 | -227 10/13 | 382 3/13 | -225 10/13 | 380 3/13 | -223 10/13 | 390 3/13 | 2021 |
| -221 10/13 | -186 10/13 | -197 10/13 | 352 3/13 | -195 10/13 | 350 3/13 | -193 10/13 | 348 3/13 | -191 10/13 | 346 3/13 | -189 10/13 | -188 10/13 | 331 3/13 | 354 3/13 | 341 3/13 | 340 3/13 | -183 10/13 | -182 10/13 | 337 3/13 | -180 10/13 | 335 3/13 | -178 10/13 | 333 3/13 | -176 10/13 | 343 3/13 | 377 3/13 | 2021 |
| 376 3/13 | -208 10/13 | 298 3/13 | 287 3/13 | -130 10/13 | 285 3/13 | -128 10/13 | 283 3/13 | -126 10/13 | 281 3/13 | -124 10/13 | 279 3/13 | -122 10/13 | -132 10/13 | -154 10/13 | 311 3/13 | -156 10/13 | 313 3/13 | -158 10/13 | 315 3/13 | -160 10/13 | 317 3/13 | -162 10/13 | 299 3/13 | 364 3/13 | -220 10/13 | 2021 |
| -219 10/13 | 363 3/13 | -133 10/13 | -103 10/13 | 267 3/13 | -110 10/13 | 265 3/13 | -108 10/13 | 263 3/13 | -106 10/13 | 261 3/13 | 260 3/13 | -93 10/13 | -112 10/13 | -101 10/13 | -100 10/13 | 255 3/13 | 254 3/13 | -97 10/13 | 252 3/13 | -95 10/13 | 250 3/13 | 258 3/13 | 289 3/13 | -207 10/13 | 375 3/13 | 2021 |
| 374 3/13 | -206 10/13 | 290 3/13 | 248 3/13 | 222 3/13 | -59 10/13 | 216 3/13 | -61 10/13 | 218 3/13 | -63 10/13 | 220 3/13 | -65 10/13 | 231 3/13 | -83 10/13 | 224 3/13 | -69 10/13 | 226 3/13 | -71 10/13 | 228 3/13 | -73 10/13 | 230 3/13 | -67 10/13 | -92 10/13 | -134 10/13 | 362 3/13 | -218 10/13 | 2021 |
| -217 10/13 | 361 3/13 | -135 10/13 | -91 10/13 | 213 3/13 | -35 10/13 | 182 3/13 | -25 10/13 | 180 3/13 | -23 10/13 | 178 3/13 | -21 10/13 | 176 3/13 | 205 3/13 | 199 3/13 | -44 10/13 | 201 3/13 | -46 10/13 | 203 3/13 | -48 10/13 | -34 10/13 | -57 10/13 | 247 3/13 | 291 3/13 | -205 10/13 | 373 3/13 | 2021 |
| 372 3/13 | -204 10/13 | 292 3/13 | 246 3/13 | -56 10/13 | 197 3/13 | -7 10/13 | 168 3/13 | -11 10/13 | 166 3/13 | -9 10/13 | 164 3/13 | -19 10/13 | 169 3/13 | -5 10/13 | 160 3/13 | -3 10/13 | 158 3/13 | -1 10/13 | 162 3/13 | -41 10/13 | 212 3/13 | -90 10/13 | -136 10/13 | 360 3/13 | -216 10/13 | 2021 |
| -215 10/13 | 359 3/13 | -137 10/13 | -89 10/13 | 211 3/13 | -40 10/13 | 3/13 | 139 3/13 | 21 3/13 | 20 3/13 | 136 3/13 | 137 3/13 | 144 3/13 | 133 3/13 | 15 3/13 | 14 3/13 | 142 3/13 | 12 3/13 | 17 3/13 | 155 3/13 | 196 3/13 | -55 10/13 | 245 3/13 | 293 3/13 | -203 10/13 | 371 3/13 | 2021 |
| 370 3/13 | -202 10/13 | 294 3/13 | 244 3/13 | -54 10/13 | 195 3/13 | 154 3/13 | 148 3/13 | 119 3/13 | 115 3/13 | 39 3/13 | 117 3/13 | 32 3/13 | 127 3/13 | 35 3/13 | 121 3/13 | 33 3/13 | 37 3/13 | 7 3/13 | 1 3/13 | -39 10/13 | 210 3/13 | -88 10/13 | -138 10/13 | 358 3/13 | -214 10/13 | 2021 |
| -213 10/13 | 357 3/13 | -139 10/13 | -87 10/13 | 209 3/13 | -38 10/13 | 2 3/13 | 8 3/13 | 29 3/13 | 102 3/13 | 55 3/13 | 54 3/13 | 49 3/13 | 99 3/13 | 104 3/13 | 105 3/13 | 52 3/13 | 126 3/13 | 147 3/13 | 153 3/13 | 194 3/13 | -53 10/13 | 243 3/13 | 295 3/13 | -201 10/13 | 369 3/13 | 2021 |
| 368 3/13 | -200 10/13 | 296 3/13 | 242 3/13 | -52 10/13 | 193 3/13 | 152 3/13 | 146 3/13 | 125 3/13 | 57 3/13 | 91 3/13 | 89 3/13 | 62 3/13 | 95 3/13 | 63 3/13 | 65 3/13 | 98 3/13 | 30 3/13 | 9 3/13 | 3 3/13 | -37 10/13 | 208 3/13 | -86 10/13 | -140 10/13 | 356 3/13 | -212 10/13 | 2021 |
| -211 10/13 | 355 3/13 | -141 10/13 | -85 10/13 | 207 3/13 | 192 3/13 | 4 3/13 | 10 3/13 | 31 3/13 | 97 3/13 | 61 3/13 | 71 3/13 | 80 3/13 | 76 3/13 | 83 3/13 | 94 3/13 | 58 3/13 | 124 3/13 | 145 3/13 | 151 3/13 | -36 10/13 | -51 10/13 | 241 3/13 | 297 3/13 | -199 10/13 | 367 3/13 | 2021 |
| 366 3/13 | 365 3/13 | 309 3/13 | 240 3/13 | -50 10/13 | -27 10/13 | 150 3/13 | 6 3/13 | 41 3/13 | 59 3/13 | 67 3/13 | 82 3/13 | 77 3/13 | 81 3/13 | 70 3/13 | 88 3/13 | 96 3/13 | 114 3/13 | 149 3/13 | 5 3/13 | 183 3/13 | 206 3/13 | -84 10/13 | -153 10/13 | -209 10/13 | -210 10/13 | 2021 |
| 378 3/13 | 320 3/13 | -163 10/13 | 277 3/13 | -58 10/13 | -42 10/13 | 156 3/13 | 27 3/13 | 45 3/13 | 46 3/13 | 69 3/13 | 85 3/13 | 74 3/13 | 78 3/13 | 73 3/13 | 86 3/13 | 109 3/13 | 110 3/13 | 128 3/13 | - 10/13 | 198 3/13 | 214 3/13 | -121 10/13 | 319 3/13 | -164 10/13 | -222 10/13 | 2021 |
| 404 3/13 | -165 10/13 | 300 3/13 | 269 3/13 | -76 10/13 | -33 10/13 | 170 3/13 | 129 3/13 | 111 3/13 | 48 3/13 | 87 3/13 | 72 3/13 | 79 3/13 | 75 3/13 | 84 3/13 | 68 3/13 | 107 3/13 | 44 3/13 | 26 3/13 | -14 10/13 | 189 3/13 | 232 3/13 | -113 10/13 | -144 10/13 | 321 3/13 | -248 10/13 | 2021 |
| -249 10/13 | 322 3/13 | -145 10/13 | -114 10/13 | 233 3/13 | -32 10/13 | -15 10/13 | 25 3/13 | 43 3/13 | 108 3/13 | 90 3/13 | 66 3/13 | 93 3/13 | 60 3/13 | 92 3/13 | 64 3/13 | 47 3/13 | 112 3/13 | 130 3/13 | 171 3/13 | 188 3/13 | -77 10/13 | 270 3/13 | 301 3/13 | -166 10/13 | 405 3/13 | 2021 |
| 406 3/13 | -167 10/13 | 302 3/13 | 271 3/13 | -78 10/13 | 187 3/13 | 172 3/13 | 131 3/13 | 113 3/13 | 103 3/13 | 100 3/13 | 101 3/13 | 106 3/13 | 56 3/13 | 51 3/13 | 50 3/13 | 53 3/13 | 42 3/13 | 24 3/13 | -16 10/13 | -31 10/13 | 234 3/13 | -115 10/13 | -146 10/13 | 323 3/13 | -250 10/13 | 2021 |
| -251 10/13 | 324 3/13 | -147 10/13 | -116 10/13 | 235 3/13 | 186 3/13 | -17 10/13 | 23 3/13 | 118 3/13 | 40 3/13 | 116 3/13 | 38 3/13 | 123 3/13 | 28 3/13 | 120 3/13 | 34 3/13 | 122 3/13 | 36 3/13 | 132 3/13 | 173 3/13 | -30 10/13 | -79 10/13 | 272 3/13 | 303 3/13 | -168 10/13 | 407 3/13 | 2021 |
| 408 3/13 | -169 10/13 | 304 3/13 | 273 3/13 | -80 10/13 | -29 10/13 | 174 3/13 | 138 3/13 | 134 3/13 | 135 3/13 | 19 3/13 | 18 3/13 | 11 3/13 | 22 3/13 | 140 3/13 | 141 3/13 | 13 3/13 | 143 3/13 | 16 3/13 | -18 10/13 | 185 3/13 | 236 3/13 | -117 10/13 | -148 10/13 | 325 3/13 | -252 10/13 | 2021 |
| -253 10/13 | 326 3/13 | -149 10/13 | -118 10/13 | 237 3/13 | 184 3/13 | -6 10/13 | -12 10/13 | 167 3/13 | -10 10/13 | 165 3/13 | -8 10/13 | 175 3/13 | -13 10/13 | 161 3/13 | -4 10/13 | 159 3/13 | -2 10/13 | 157 3/13 | 163 3/13 | -28 10/13 | -81 10/13 | 274 3/13 | 305 3/13 | -170 10/13 | 409 3/13 | 2021 |
| 410 3/13 | -171 10/13 | 306 3/13 | 275 3/13 | -82 10/13 | 190 3/13 | -26 10/13 | 181 3/13 | -24 10/13 | 179 3/13 | -22 10/13 | 177 3/13 | -20 10/13 | -49 10/13 | -43 10/13 | 200 3/13 | -45 10/13 | 202 3/13 | -47 10/13 | 204 3/13 | 191 3/13 | 238 3/13 | -119 10/13 | -150 10/13 | 327 3/13 | -254 10/13 | 2021 |
| -255 10/13 | 328 3/13 | -151 10/13 | -120 10/13 | 223 3/13 | 215 3/13 | -60 10/13 | 217 3/13 | -62 10/13 | 219 3/13 | -64 10/13 | 221 3/13 | -75 10/13 | 239 3/13 | -68 10/13 | 225 3/13 | -70 10/13 | 227 3/13 | -72 10/13 | 229 3/13 | -74 10/13 | -66 10/13 | 276 3/13 | 307 3/13 | -172 10/13 | 411 3/13 | 2021 |
| 412 3/13 | -173 10/13 | 308 3/13 | -102 10/13 | -111 10/13 | 266 3/13 | -109 10/13 | 264 3/13 | -107 10/13 | 262 3/13 | -105 10/13 | -104 10/13 | 249 3/13 | 268 3/13 | 257 3/13 | 256 3/13 | -99 10/13 | -98 10/13 | 253 3/13 | -96 10/13 | 251 3/13 | -94 10/13 | 259 3/13 | -152 10/13 | 329 3/13 | -256 10/13 | 2021 |
| -257 10/13 | 330 3/13 | -143 10/13 | -131 10/13 | 286 3/13 | -129 10/13 | 284 3/13 | -127 10/13 | 282 3/13 | -125 10/13 | 280 3/13 | -123 10/13 | 278 3/13 | 288 3/13 | 310 3/13 | -155 10/13 | 312 3/13 | -157 10/13 | 314 3/13 | -159 10/13 | 316 3/13 | -161 10/13 | 318 3/13 | -142 10/13 | -174 10/13 | 413 3/13 | 2021 |
| 414 3/13 | -187 10/13 | 353 3/13 | -196 10/13 | 351 3/13 | -194 10/13 | 349 3/13 | -192 10/13 | 347 3/13 | -190 10/13 | 345 3/13 | 344 3/13 | -175 10/13 | -198 10/13 | -185 10/13 | -184 10/13 | 339 3/13 | 338 3/13 | -181 10/13 | 336 3/13 | -179 10/13 | 334 3/13 | -177 10/13 | 332 3/13 | 342 3/13 | -258 10/13 | 2021 |
| -234 10/13 | -246 10/13 | 401 3/13 | -244 10/13 | 399 3/13 | -242 10/13 | 397 3/13 | -240 10/13 | 395 3/13 | -238 10/13 | 393 3/13 | -236 10/13 | 415 3/13 | -247 10/13 | 389 3/13 | -232 10/13 | 387 3/13 | -230 10/13 | 385 3/13 | -228 10/13 | 383 3/13 | -226 10/13 | 381 3/13 | -224 10/13 | 379 3/13 | 391 3/13 | 2021 |
| 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 | 2021 |

According to (1), the sub-magic squares sums are as given by

$$S_{4\times4} := 2021 \times \frac{4}{26} = \frac{4042}{13} \qquad S_{12\times12} := 2021 \times \frac{12}{21} = \frac{12126}{13} \qquad S_{20\times20} := 2021 \times \frac{20}{26} = \frac{20210}{13}$$
$$S_{6\times6} := 2021 \times \frac{6}{26} = \frac{6063}{13} \qquad S_{14\times14} := 2021 \times \frac{14}{26} = \frac{14147}{13} \qquad S_{22\times22} := 2021 \times \frac{22}{26} = \frac{22231}{13}$$
$$S_{8\times8} := 2021 \times \frac{8}{26} = \frac{8084}{13} \qquad S_{16\times16} := 2021 \times \frac{16}{26} = \frac{16168}{13} \qquad S_{24\times24} := 2021 \times \frac{24}{26} = \frac{24252}{13}$$
$$S_{10\times10} := 2021 \times \frac{10}{26} = \frac{10105}{13} \qquad S_{18\times18} := 2021 \times \frac{18}{26} = \frac{18189}{13} \qquad S_{24\times24} := 2021 \times \frac{26}{26} = 2021.$$

In this case, there are **Pythagorean triples** with magic sums:

$$S^2_{6\times6} + S^2_{8\times8} := S^2_{10\times10}$$
$$S^2_{12\times12} + S^2_{16\times16} := S^2_{20\times20}$$
$$S^2_{10\times10} + S^2_{24\times24} := S^2_{26\times26}.$$

## 3 Pythagorean Triples

In above Section, we wrote **Pythagorean triples** in case of magic squares of orders 10, 12, 14, 16, 18, 20, 22 and 24. According to formula (1), below is a general formula for **Pythagorean triples** with sub-magic sums:

(i)

$$\begin{aligned} S^2_{6\times6} + S^2_{8\times8} &:= \left(2021 \times \frac{6}{m}\right)^2 + \left(2021 \times \frac{8}{m}\right)^2 \\ &:= \left(\frac{2021}{m}\right)^2 \times (6^2 + 8^2) \\ &:= \left(\frac{2021}{m}\right)^2 \times 10^2 \\ &:= S^2_{10\times10} \end{aligned}$$

(ii)

$$\begin{aligned} S^2_{12\times 12} + S^2_{16\times 16} &:= \left(2021 \times \frac{12}{m}\right)^2 + \left(2021 \times \frac{16}{m}\right)^2 \\ &:= \left(\frac{2021}{m}\right)^2 \times (12^2 + 16^2) \\ &:= \left(\frac{2021}{m}\right)^2 \times 20^2 \\ &:= S^2_{20\times 20} \end{aligned}$$

(iii)

$$\begin{aligned} S^2_{10\times 10} + S^2_{24\times 24} &:= \left(2021 \times \frac{10}{m}\right)^2 + \left(2021 \times \frac{24}{m}\right)^2 \\ &:= \left(\frac{2021}{m}\right)^2 \times (10^2 + 24^2) \\ &:= \left(\frac{2021}{m}\right)^2 \times 26^2 \\ &:= S^2_{26\times 26} \end{aligned}$$

Among the orders studied in Section 2, there are only two **Pythagorean triples** of even numbers that fits here. The other **Pythagorean triples** triples are with even and odd numbers together, such as

$$[3,4,5]; [5,12,13]; [7,24,25]; [8,15,17]; [9,12,15]; [15,20,25]$$

These triples are not applicable to **bordered magic squares**. In case, we write bordered magic squares of higher orders, then we can get more **Pythagorean triples** with even order **bordered magic squares**.

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