AUTOMORPHISM GROUPS OF CERTAIN ENRIQUES SURFACES

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Abstract. We calculate the automorphism group of certain Enriques surfaces. The Enriques surfaces that we investigate include very general n-nodal Enriques surfaces and very general cuspidal Enriques surfaces. We also describe the action of the automorphism group on the set of smooth rational curves and on the set of elliptic fibrations.

1. Introduction

A central theme in algebraic geometry is to study varieties using convex geometry. The cone of curves of a variety is the convex hull of the numerical equivalence classes of curves. Its dual is the cone of nef line bundles. Much of the birational geometry of a variety is encoded in these cones and their interplay with the canonical divisor. While for Fano varieties the nef cone is rational polyhedral [15, Theorem 3.7], in general the nef cone is not well understood. For instance it can have infinitely many faces or be round.

The Morrison-Kawamata cone conjecture [20, 12] gives a clear picture of the effective nef cone of a Calabi-Yau variety. It predicts that the action of the automorphism group on the effective nef cone admits a fundamental domain which is a rational polyhedral cone.

The conjecture is wide open in dimension three and beyond [18]. But it has been verified for $K3$ surfaces by Sterk [33], and for Enriques surfaces by Namikawa [21]. It follows that an Enriques surface admits up to the action of the automorphism group only finitely many *smooth rational curves*, finitely many *elliptic fibrations*, finitely many *projective models* of a given degree and its automorphism group is finitely generated and in fact finitely presented [19, Corollaries 4.15, 4.16].

Naturally, enumerative questions arise:

- Can one explicitly describe a fundamental domain?
- *How many* smooth rational curves, elliptic fibrations or projective models are there up to the action of the automorphism group?
- Can one give generators for the automorphism group?

This has been answered for very general Enriques surfaces by Barth–Peters [2] and for very general nodal Enriques surfaces by Cossec–Dolgachev [8] (see also the works of Allcock [1] and Peters–Sterk [25]).

To generalize the aforementioned results of Barth, Peters, Cossec and Dolgachev to Enriques surfaces with more nodes, we introduce the notion of $(\tau, \bar{\tau})$-generic

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Enriques surfaces. This notion is closely related to the root invariant introduced by Nikulin [24]. See the next subsection for the precise definition. For instance the very general Enriques surface is $(0, 0)$-generic, a very general nodal Enriques surface is (A_1, A_1)-generic and if Y is very general in the moduli of Enriques surfaces containing n disjoint smooth rational curves, then Y is (nA_1, nA_1)-generic. If Y is very general in the moduli of Enriques surfaces containing two smooth rational curves whose dual graph is ○—○ (that is, Y is a very general cuspidal Enriques surface), then Y is (A_2, A_2)-generic.

In this work we consider enumerative aspects of the Morrison-Kawamata cone conjecture for complex Enriques surfaces. Our first main result is Theorem 3.3. It provides a general formula for the volume of the fundamental domain of the action on the nef cone on an Enriques surface Y under mild genericity assumptions on Y. Next we give algorithms to compute generators for the automorphism group $\mathrm{Aut}(Y)$, a fundamental domain for $\mathrm{Aut}(Y)$ on the nef and big cone $\mathrm{Nef}(Y)$ and orbit representatives for its action on

$$\begin{aligned} \mathcal{R}(Y) &:= \text{the set of smooth rational curves on } Y, \\ \mathcal{E}(Y) &:= \text{the set of elliptic fibrations } Y \to \mathbb{P}^1. \end{aligned}$$

We apply Theorem 3.3 and the aforementioned algorithms to $(\tau, \bar{\tau})$-generic Enriques surfaces. This results in our second, series of main results: Theorem 1.15 expresses the volume of the fundamental domain of $\mathrm{Aut}(Y)$ on the nef cone $\mathrm{Nef}(Y)$ in terms of the Weyl group of τ, Theorem 1.16 relates the orbits of $\mathrm{Aut}(Y)$ on the set of smooth rational curves $\mathcal{R}(Y)$ to the connected components of the Dynkin diagram τ and Theorem 1.18 counts the $\mathrm{Aut}(Y)$-orbits of the set of elliptic fibrations $\mathcal{E}(Y)$ and their fiber types.

Our new idea is the lattice theoretic result obtained in [6] (see also Dolgachev–Kondo [9, Chapter 10]). For a lattice L with the intersection form $\langle -, - \rangle$, let $L(m)$ denote the lattice with the same underlying $\mathbb{Z}$-module as L and with the intersection form $m \langle -, - \rangle$. A lattice L of rank $n > 1$ is said to be *hyperbolic* if the signature is $(1, n-1)$. For a positive integer n with $n \bmod 8 = 2$, let L_n denote an even unimodular hyperbolic lattice of rank n, which is unique up to isomorphism. Borcherds [4], [5] developed a method to calculate the orthogonal group of an even hyperbolic lattice S by embedding S primitively into L_{26} and using the result of Conway [7]. This method has been applied to the study of automorphism groups of $K3$ surfaces by many authors. This method, however, often requires impractically heavy computation (see, for example, [11] and [28]).

On the other hand, in [6], we have classified all primitive embeddings of $L_{10}(2)$ into L_{26} and showed that they have a remarkable property (see Theorems 4.2 and 4.3). This property enables us to calculate the automorphism group $\mathrm{Aut}(Y)$ efficiently and explicitly for $(\tau, \bar{\tau})$-generic Enriques surfaces Y. See Remark 6.1.

1.1. **Definition of $(\tau, \bar{\tau})$-generic Enriques surfaces.** First, we define $(\tau, \bar{\tau})$-generic Enriques surfaces. Let L be a lattice. We let the group $\mathrm{O}(L)$ of isometries of L act on L from the right, and write the action as $v \mapsto v^g$ for $v \in L \otimes \mathbb{R}$ and $g \in \mathrm{O}(L)$. We have a natural identification $\mathrm{O}(L) = \mathrm{O}(L(m))$ for any non-zero integer m. A vector v of a lattice is called a *k-vector* if $\langle v, v \rangle = k$. A (-2)-vector is called a *root*.

Definition 1.1. An ADE-*lattice* is an even negative definite lattice generated by roots. An ADE-lattice R has a basis consisting of roots whose dual graph is a Dynkin diagram of an ADE-type. This ADE-type is denoted by $\tau(R)$.

A *positive half-cone* of a hyperbolic lattice L is one of the two connected components of $\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$. Let $\mathcal{P}$ be a positive half-cone of a hyperbolic lattice L. We put

$$\mathrm{O}^{\mathcal{P}}(L) := \{ g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P} \}.$$

In [29], we classified the ADE-sublattices of L_{10} up to the action of $\mathrm{O}^{\mathcal{P}}(L_{10})$. Let R be an ADE-sublattice of L_{10}, and $\overline{R}$ the primitive closure of R in L_{10}. It turned out that $\overline{R}$ is also an ADE-sublattice of L_{10}.

Proposition 1.2 ([29])**.** (1) *Let R' be another* ADE-*sublattice of L_{10} with the primitive closure $\overline{R}'$. Then R and R' are in the same orbit under the action of $\mathrm{O}^{\mathcal{P}}(L_{10})$ if and only if $(\tau(R), \tau(\overline{R})) = (\tau(R'), \tau(\overline{R}'))$.*

(2) *The pair $(\tau, \bar{\tau})$ of* ADE-*types is equal to $(\tau(R), \tau(\overline{R}))$ of an* ADE-*sublattice R of L_{10} if and only if $(\tau, \bar{\tau})$ is one of the* 184 *pairs in Table 1.1.* □

Let R be an ADE-sublattice of L_{10}. We denote by $\iota_R \colon R \hookrightarrow L_{10}$ the inclusion. We define M_R to be the $\mathbb{Z}$-submodule of $(L_{10}(2) \oplus R(2)) \otimes \mathbb{Q}$ generated by $L_{10}(2)$ and $(\iota_R(v), \pm v)/2 \in (L_{10} \oplus R) \otimes \mathbb{Q}$, where v runs through R, and equip M_R with an intersection form by extending the intersection form of $L_{10}(2) \oplus R(2)$. By definition, M_R is an even hyperbolic lattice with a canonical primitive embedding $\varpi_R \colon L_{10}(2) \hookrightarrow M_R$. If R' is another ADE-sublattice of L_{10} such that $(\tau(R'), \tau(\overline{R}')) = (\tau(R), \tau(\overline{R}))$, then, by Proposition 1.2, we have an isometry $g \colon L_{10} \xrightarrow{\sim} L_{10}$ that induces an isometry $g|_R \colon R \xrightarrow{\sim} R'$, and hence we obtain an isometry $\tilde{g} \colon M_R \xrightarrow{\sim} M_{R'}$ induced by $g \oplus g|_R$, which makes the following diagram commutative:

$$\begin{array}{ccc} L_{10}(2) & \overset{\varpi_R}{\hookrightarrow} & M_R \\ {\scriptstyle g} \downarrow \wr & & {\scriptstyle \tilde{g}} \downarrow \wr \\ L_{10}(2) & \overset{\varpi_{R'}}{\hookrightarrow} & M_{R'}. \end{array}$$

By an explicit calculation, we obtain the following:

Proposition 1.3. *Let R be an* ADE-*sublattice of L_{10}. Then the orthogonal complement of $\varpi_R \colon L_{10}(2) \hookrightarrow M_R$ is isomorphic to $\widetilde{R}(2)$ for some* ADE-*lattice $\widetilde{R}$. In the* 4*th column of Table 1.1, we give the* ADE-*type $\tau(\widetilde{R})$ of $\widetilde{R}$.* □

Let Y be an Enriques surface. We denote by S_Y the lattice of numerical equivalence classes of divisors of Y. It is well-known that S_Y is isomorphic to L_{10}. Let $\pi \colon X \to Y$ be the universal covering of Y, and let S_X denote the lattice of numerical equivalence classes of divisors of the $K3$ surface X. Then the étale double covering π induces a primitive embedding

$$\pi^* \colon S_Y(2) \hookrightarrow S_X.$$

Definition 1.4. Let $(\tau, \bar{\tau})$ be one of the 184 pairs in Table 1.1. An Enriques surface Y is said to be $(\tau, \bar{\tau})$-*generic* if the following conditions are satisfied.

(i) Let T_X be the transcendental lattice of X, and ω a non-zero holomorphic 2-form of X, so that we have $\mathbb{C}\omega = H^{2,0}(X) \subset T_X \otimes \mathbb{C}$. Then the group

$$\mathrm{O}(T_X, \omega) := \{ g \in \mathrm{O}(T_X) \mid g \text{ preserves } \mathbb{C}\omega \subset T_X \otimes \mathbb{C} \}$$

No.	$\tau(R)$	$\tau(\overline{R})$	$\tau(\widetilde{R})$	exist	$c_{(\tau,\bar{\tau})}$	rat	irec
1	A_1	A_1	A_1		1	1	96C
2	$2A_1$	$2A_1$	$2A_1$		1	2	96C
3	A_2	A_2	A_2		1	1	96C
4	$3A_1$	$3A_1$	$3A_1$		1	3	96C
5	A_2+A_1	A_2+A_1	A_2+A_1		1	2	96C
6	A_3	A_3	A_3		1	1	96C
7	$4A_1$	$4A_1$	$4A_1$		1	4	96C
8	$4A_1$	D_4	D_4		1	4	96C
9	A_2+2A_1	A_2+2A_1	A_2+2A_1		1	3	96C
10	A_3+A_1	A_3+A_1	A_3+A_1		1	2	96C
11	$2A_2$	$2A_2$	$2A_2$		1	2	96C
12	A_4	A_4	A_4		1	1	40E
13	D_4	D_4	D_4		1	1	96A
14	$5A_1$	$5A_1$	$5A_1$		1	5	96C
15	$5A_1$	D_4+A_1	D_4+A_1		1	5	96C
16	A_2+3A_1	A_2+3A_1	A_2+3A_1		1	4	96C
17	A_3+2A_1	A_3+2A_1	A_3+2A_1		1	3	96C
18	A_3+2A_1	D_5	D_5		1	3	96C
19	$2A_2+A_1$	$2A_2+A_1$	$2A_2+A_1$		1	3	96C
20	A_4+A_1	A_4+A_1	A_4+A_1		1	2	40E
21	D_4+A_1	D_4+A_1	D_4+A_1		1	2	96A
22	A_3+A_2	A_3+A_2	A_3+A_2		1	2	96C
23	A_5	A_5	A_5		1	1	40E
24	D_5	D_5	D_5		1	1	40A
25	$6A_1$	D_4+2A_1	D_4+2A_1		1	6	96C
26	$6A_1$	D_6	D_6	×	1	6	96C
27	A_2+4A_1	A_2+4A_1	A_2+4A_1		1	5	96C
28	A_2+4A_1	D_4+A_2	D_4+A_2		1	5	96C
29	A_3+3A_1	A_3+3A_1	A_3+3A_1		1	4	96C
30	A_3+3A_1	D_5+A_1	D_5+A_1		1	4	96C
31	$2A_2+2A_1$	$2A_2+2A_1$	$2A_2+2A_1$		1	4	96C
32	A_4+2A_1	A_4+2A_1	A_4+2A_1		1	3	40E
33	D_4+2A_1	D_4+2A_1	D_4+2A_1		1	3	96A
34	D_4+2A_1	D_6	D_6		1	3	96A
35	$A_3+A_2+A_1$	$A_3+A_2+A_1$	$A_3+A_2+A_1$		1	3	96C
36	A_5+A_1	A_5+A_1	A_5+A_1		1	2	40E
37	A_5+A_1	E_6	E_6		1	2	40E
38	D_5+A_1	D_5+A_1	D_5+A_1		1	2	40A
39	$3A_2$	$3A_2$	$3A_2$		1	3	96C
40	$3A_2$	E_6	$3A_2$		1	3	96C
41	A_4+A_2	A_4+A_2	A_4+A_2		1	2	40E
42	D_4+A_2	D_4+A_2	D_4+A_2		1	2	96A
43	$2A_3$	$2A_3$	$2A_3$		1	2	96A
44	$2A_3$	D_6	D_6		1	2	96C
45	A_6	A_6	A_6		1	1	40C
46	D_6	D_6	D_6		1	1	40A
47	E_6	E_6	E_6		1	1	20E
48	$7A_1$	D_6+A_1	D_6+A_1	×	1	7	96C
49	$7A_1$	E_7	E_7	×	1	7	96A
50	A_2+5A_1	$D_4+A_2+A_1$	$D_4+A_2+A_1$		1	6	96C
51	A_3+4A_1	D_5+2A_1	D_5+2A_1		1	5	96C
52	A_3+4A_1	D_4+A_3	D_4+A_3		1	5	96A
53	A_3+4A_1	D_7	D_7	×	1	5	96C
54	$2A_2+3A_1$	$2A_2+3A_1$	$2A_2+3A_1$		1	5	96C
55	A_4+3A_1	A_4+3A_1	A_4+3A_1		1	4	40E
56	D_4+3A_1	D_6+A_1	D_6+A_1		1	4	96A
57	D_4+3A_1	E_7	E_7	×	1	4	96A
58	$A_3+A_2+2A_1$	$A_3+A_2+2A_1$	$A_3+A_2+2A_1$		1	4	96C
59	$A_3+A_2+2A_1$	D_5+A_2	D_5+A_2		1	4	96C
60	A_5+2A_1	A_5+2A_1	A_5+2A_1		1	3	40E
61	A_5+2A_1	E_6+A_1	E_6+A_1		1	3	40E
62	D_5+2A_1	D_5+2A_1	D_5+2A_1		1	3	40A
63	D_5+2A_1	D_7	D_7		1	3	40A
64	$3A_2+A_1$	$3A_2+A_1$	$3A_2+A_1$		1	4	96C
65	$3A_2+A_1$	E_6+A_1	$3A_2+A_1$		1	4	96C
66	$A_4+A_2+A_1$	$A_4+A_2+A_1$	$A_4+A_2+A_1$		1	3	40E
67	$D_4+A_2+A_1$	$D_4+A_2+A_1$	$D_4+A_2+A_1$		1	3	96A
68	$2A_3+A_1$	$2A_3+A_1$	$2A_3+A_1$		1	3	96A
69	$2A_3+A_1$	D_6+A_1	D_6+A_1		1	3	96C
70	$2A_3+A_1$	E_7	D_6+A_1		1	3	96C

TABLE 1.1. ADE-sublattices of L_{10} (continues)

No.	$\tau(R)$	$\tau(\overline{R})$	$\tau(\widetilde{R})$	exist	$c_{(\tau,\bar{\tau})}$	rat	irec
71	A_6+A_1	A_6+A_1	A_6+A_1		1	2	40C
72	D_6+A_1	D_6+A_1	D_6+A_1		1	2	40A
73	D_6+A_1	E_7	E_7		1	2	40A
74	E_6+A_1	E_6+A_1	E_6+A_1		1	2	20E
75	A_3+2A_2	A_3+2A_2	A_3+2A_2		1	3	96C
76	A_5+A_2	A_5+A_2	A_5+A_2		1	2	40E
77	A_5+A_2	E_7	A_5+A_2		1	2	40E
78	D_5+A_2	D_5+A_2	D_5+A_2		1	2	40A
79	A_4+A_3	A_4+A_3	A_4+A_3		1	2	40E
80	D_4+A_3	D_4+A_3	D_4+A_3		1	2	20F
81	D_4+A_3	D_7	D_7		1	2	96A
82	A_7	A_7	A_7		1	1	20D
83	A_7	E_7	E_7		1	1	40C
84	D_7	D_7	D_7		1	1	20B
85	E_7	E_7	E_7		1	$\times 2$	20A
86	$8A_1$	E_7+A_1	E_7+A_1	$\times$	1	8	96A
87	$8A_1$	D_8	D_8	$\times$	1	8	96B
88	$8A_1$	E_8	E_8	$\times$	2	−	−
89	A_2+6A_1	D_6+A_2	D_6+A_2	$\times$	1	7	96C
90	A_3+5A_1	D_7+A_1	D_7+A_1	$\times$	1	6	96C
91	A_4+4A_1	D_4+A_4	D_4+A_4		1	5	40E
92	D_4+4A_1	E_7+A_1	E_7+A_1	$\times$	1	5	96A
93	D_4+4A_1	D_8	D_8	$\times$	1	5	96A
94	D_4+4A_1	E_8	E_8	$\times$	2	5	96A
95	$A_3+A_2+3A_1$	$D_5+A_2+A_1$	$D_5+A_2+A_1$		1	5	96C
96	A_5+3A_1	E_6+2A_1	E_6+2A_1		1	4	40E
97	D_5+3A_1	D_7+A_1	D_7+A_1		1	4	40A
98	$3A_2+2A_1$	E_6+2A_1	$3A_2+2A_1$		1	5	96C
99	$A_4+A_2+2A_1$	$A_4+A_2+2A_1$	$A_4+A_2+2A_1$		1	4	40E
100	$D_4+A_2+2A_1$	D_6+A_2	D_6+A_2		1	4	96A
101	$2A_3+2A_1$	E_7+A_1	D_6+2A_1		1	4	96C
102	$2A_3+2A_1$	D_5+A_3	D_5+A_3		1	4	96A
103	$2A_3+2A_1$	D_8	D_8	$\times$	1	4	96C
104	$2A_3+2A_1$	E_8	D_8	$\times$	1	4	96C
105	A_6+2A_1	A_6+2A_1	A_6+2A_1		1	3	40C
106	D_6+2A_1	E_7+A_1	E_7+A_1		1	3	40A
107	D_6+2A_1	D_8	D_8		1	3	40A
108	D_6+2A_1	E_8	E_8	$\times$	2	3	40A
109	E_6+2A_1	E_6+2A_1	E_6+2A_1		1	3	20E
110	$A_3+2A_2+A_1$	$A_3+2A_2+A_1$	$A_3+2A_2+A_1$		1	4	96C
111	$A_5+A_2+A_1$	$A_5+A_2+A_1$	$A_5+A_2+A_1$		1	3	40E
112	$A_5+A_2+A_1$	E_7+A_1	$A_5+A_2+A_1$		1	3	40E
113	$A_5+A_2+A_1$	E_6+A_2	E_6+A_2		1	3	40E
114	$A_5+A_2+A_1$	E_8	E_6+A_2		1	3	40E
115	$D_5+A_2+A_1$	$D_5+A_2+A_1$	$D_5+A_2+A_1$		1	3	40A
116	$A_4+A_3+A_1$	$A_4+A_3+A_1$	$A_4+A_3+A_1$		1	3	40E
117	$D_4+A_3+A_1$	D_7+A_1	D_7+A_1		1	3	96A
118	A_7+A_1	A_7+A_1	A_7+A_1		1	2	20D
119	A_7+A_1	E_7+A_1	E_7+A_1		1	2	40C
120	A_7+A_1	E_8	E_7+A_1		1	2	40C
121	D_7+A_1	D_7+A_1	D_7+A_1		1	2	20B
122	E_7+A_1	E_7+A_1	E_7+A_1		1	$\times 3$	20A
123	E_7+A_1	E_8	E_8		2	$\times 3$	20A
124	$4A_2$	E_6+A_2	$4A_2$		1	4	96C
125	$4A_2$	E_8	$4A_2$		1	4	96C
126	A_4+2A_2	A_4+2A_2	A_4+2A_2		1	3	40E
127	$2A_3+A_2$	D_6+A_2	D_6+A_2		1	3	96C
128	A_6+A_2	A_6+A_2	A_6+A_2		1	2	40C
129	D_6+A_2	D_6+A_2	D_6+A_2		1	2	40A
130	E_6+A_2	E_6+A_2	E_6+A_2		1	2	20E
131	E_6+A_2	E_8	E_6+A_2		1	2	20E
132	A_5+A_3	A_5+A_3	A_5+A_3		1	2	40E
133	D_5+A_3	D_5+A_3	D_5+A_3		1	2	20F
134	D_5+A_3	D_8	D_8		1	2	40A
135	D_5+A_3	E_8	D_8		1	2	40A
136	$2A_4$	$2A_4$	$2A_4$		1	2	40E
137	$2A_4$	E_8	$2A_4$		1	2	40E
138	D_4+A_4	D_4+A_4	D_4+A_4		1	2	20F
139	A_8	A_8	A_8		1	1	20D
140	A_8	E_8	A_8		1	1	20D

TABLE 1.1. ADE-sublattices of L_{10} (continued and continues)

No.	$\tau(R)$	$\tau(\overline{R})$	$\tau(\widetilde{R})$	exist	$c_{(\tau,\bar{\tau})}$	rat	irec
141	$2D_4$	D_8	D_8		1	2	20F
142	$2D_4$	E_8	E_8	×	1	×1	96A
143	D_8	D_8	D_8		1	1	12B
144	D_8	E_8	E_8		2	×2	20B
145	E_8	E_8	E_8		2	×4	12A
146	$9A_1$	E_8+A_1	E_8+A_1	×	2	–	–
147	A_2+7A_1	E_7+A_2	E_7+A_2	×	1	8	96A
148	A_3+6A_1	D_9	D_9	×	1	7	96B
149	D_4+5A_1	E_8+A_1	E_8+A_1	×	2	6	96A
150	D_5+4A_1	D_9	D_9	×	1	5	40A
151	$D_4+A_2+3A_1$	E_7+A_2	E_7+A_2	×	1	5	96A
152	$2A_3+3A_1$	E_8+A_1	D_8+A_1	×	1	5	96C
153	D_6+3A_1	E_8+A_1	E_8+A_1	×	2	4	40A
154	$A_5+A_2+2A_1$	E_8+A_1	$E_6+A_2+A_1$		1	4	40E
155	$A_4+A_3+2A_1$	D_5+A_4	D_5+A_4		1	4	40E
156	$D_4+A_3+2A_1$	D_9	D_9	×	1	4	96A
157	A_7+2A_1	E_8+A_1	E_7+2A_1		1	3	40C
158	D_7+2A_1	D_9	D_9		1	3	20B
159	E_7+2A_1	E_8+A_1	E_8+A_1		2	×4	20A
160	$4A_2+A_1$	E_8+A_1	$4A_2+A_1$	×	1	5	40E
161	$2A_3+A_2+A_1$	E_7+A_2	$D_6+A_2+A_1$		1	4	96C
162	$A_6+A_2+A_1$	$A_6+A_2+A_1$	$A_6+A_2+A_1$		1	3	40C
163	$D_6+A_2+A_1$	E_7+A_2	E_7+A_2		1	3	40A
164	$E_6+A_2+A_1$	E_8+A_1	$E_6+A_2+A_1$		1	3	20E
165	$A_5+A_3+A_1$	E_6+A_3	E_6+A_3		1	3	40E
166	$D_5+A_3+A_1$	E_8+A_1	D_8+A_1		1	3	40A
167	$2A_4+A_1$	E_8+A_1	$2A_4+A_1$		1	3	40E
168	A_8+A_1	A_8+A_1	A_8+A_1		1	2	20D
169	A_8+A_1	E_8+A_1	A_8+A_1		1	2	20D
170	$2D_4+A_1$	E_8+A_1	E_8+A_1	×	1	×2	96A
171	D_8+A_1	E_8+A_1	E_8+A_1		2	×3	20B
172	E_8+A_1	E_8+A_1	E_8+A_1		2	×5	12A
173	A_3+3A_2	E_6+A_3	A_3+3A_2		1	4	96C
174	A_5+2A_2	E_7+A_2	A_5+2A_2		1	3	40E
175	A_7+A_2	E_7+A_2	E_7+A_2		1	2	40C
176	E_7+A_2	E_7+A_2	E_7+A_2		1	×3	20A
177	$3A_3$	D_9	D_9	×	1	3	96C
178	D_6+A_3	D_9	D_9		1	2	40A
179	E_6+A_3	E_6+A_3	E_6+A_3		1	2	20E
180	A_5+A_4	A_5+A_4	A_5+A_4		1	2	40E
181	D_5+A_4	D_5+A_4	D_5+A_4		1	2	20F
182	A_9	A_9	A_9		1	1	20D
183	D_5+D_4	D_9	D_9		1	2	20F
184	D_9	D_9	D_9		1	×2	12B

TABLE 1.1. ADE-sublattices of L_{10} (continued)

is equal to $\{\pm 1\}$.

(ii) Let R be an ADE-sublattice of L_{10} with $(\tau(R), \tau(\overline{R})) = (\tau, \bar{\tau})$. Then there exist isometries $g\colon L_{10} \xrightarrow{\sim} S_Y$ and $\tilde{g}\colon M_R \xrightarrow{\sim} S_X$ that make the following commutative diagram

$$\begin{array}{ccc} L_{10}(2) & \overset{\varpi_R}{\hookrightarrow} & M_R \\ {\scriptstyle g}\downarrow\wr & & {\scriptstyle \tilde{g}}\downarrow\wr \\ S_Y(2) & \overset{\pi^*}{\hookrightarrow} & S_X. \end{array} \tag{1.1}$$

The numbering of the ADE-types in Table 1.1 of the present article is the same as the numbering in Table 1.1 of our previous paper [29], and hence the 1st-3rd columns of the two tables are identical. By definition, a $(\tau, \bar{\tau})$-generic Enriques surface exists if and only if the 4th column of the corresponding row of Table 1.1 of [29] contains 0. Hence we obtain the following:

Proposition 1.5. *A $(\tau, \bar{\tau})$-generic Enriques surface exists if and only if the 5th column of the corresponding row in Table 1.1 is not marked by $\times$.* □

Let $\mathcal{P}_Y$ (resp. $\mathcal{P}_X$) be the positive half-cone of S_Y (resp. S_X) containing an ample class. We regard $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by the embedding $\pi^* \otimes \mathbb{R}$. We put

$$\begin{aligned} \mathrm{Nef}_X &:= \{ x \in \mathcal{P}_X \mid \langle x, [\widetilde{C}] \rangle \geq 0 \text{ for all curves } \widetilde{C} \text{ on } X \}, \\ \mathrm{Nef}_Y &:= \{ y \in \mathcal{P}_Y \mid \langle y, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } Y \} = \mathcal{P}_Y \cap \mathrm{Nef}_X, \end{aligned}$$

where $[D]$ is the class of a divisor D. The following will be proved in Section 3.2.

Proposition 1.6. *Let Y and Y' be $(\tau, \bar{\tau})$-generic Enriques surfaces with the universal coverings $\pi\colon X \to Y$ and $\pi'\colon X' \to Y'$, respectively. Then there exist isometries $\psi_X\colon S_X \xrightarrow{\sim} S_{X'}$ and $\psi_Y\colon S_Y \xrightarrow{\sim} S_{Y'}$ that make the diagram*

$$\begin{array}{ccc} S_Y(2) & \xrightarrow{\pi^*} & S_X \\ \psi_Y \downarrow & & \downarrow \psi_X \\ S_{Y'}(2) & \xrightarrow{\pi'^*} & S_{X'} \end{array} \tag{1.2}$$

commutative and that induce $\mathrm{Nef}_X \cong \mathrm{Nef}_{X'}$ *and* $\mathrm{Nef}_Y \cong \mathrm{Nef}_{Y'}$*.*

We denote by $\mathrm{aut}(Y)$ the image of the natural representation $\mathrm{Aut}(Y) \to \mathrm{O}^{\mathcal{P}}(S_Y)$. We embed the set $\mathcal{R}(Y)$ of smooth rational curves C on Y into S_Y by $C \mapsto [C]$, and the set $\mathcal{E}(Y)$ of elliptic fibrations $\phi\colon Y \to \mathbb{P}^1$ into S_Y by $\phi \mapsto [F]/2$, where F is a general fiber of ϕ. In Section 6, we will see that $\mathrm{aut}(Y)$ and its actions on Nef_Y, $\mathcal{R}(Y)$, $\mathcal{E}(Y)$ depend only on the data $\pi^*\colon S_Y(2) \hookrightarrow S_X$ and Nef_X. Therefore we obtain the following:

Corollary 1.7. *Let Y and Y' be as in Proposition 1.6. Then there exist an isomorphism* $\mathrm{aut}(Y) \cong \mathrm{aut}(Y')$ *and bijections* $\mathcal{R}(Y) \cong \mathcal{R}(Y')$ *and* $\mathcal{E}(Y) \cong \mathcal{E}(Y')$ *that are compatible with* $\mathrm{aut}(Y) \cong \mathrm{aut}(Y')$*.* □

Remark 1.8. The root invariant of a $(\tau, \bar{\tau})$-generic Enriques surface (defined by Nikulin [24]) is equal to $(\tau, \mathrm{Ker}\,\xi)$, where $\xi\colon R \otimes \mathbb{F}_2 \to L_{10} \otimes \mathbb{F}_2$ is the linear homomorphism induced by the inclusion $R \hookrightarrow L_{10}$ of the ADE-sublattice R of L_{10} such that $(\tau, \bar{\tau}) = (\tau(R), \tau(\overline{R}))$.

1.2. **Chambers.** Before we state our geometric results, we define the notion of *chambers* of hyperbolic lattices, and recall the classical result of Vinberg [35].

A root r of an even lattice L defines the *reflection* $s_r\colon x \mapsto x + \langle x, r \rangle r$ of L with respect to r. The *Weyl group* $W(L)$ of L is the subgroup of $\mathrm{O}(L)$ generated by all the reflections s_r with respect to the roots of L. Let L be an even hyperbolic lattice with a positive half-cone $\mathcal{P}$. For $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, let $(v)^\perp$ denote the hyperplane of $\mathcal{P}$ defined by $\langle x, v \rangle = 0$. Then we have $W(L) \subset \mathrm{O}^{\mathcal{P}}(L)$, and the action of s_r on $\mathcal{P}$ is the reflection into the mirror $(r)^\perp$. A closed subset D of $\mathcal{P}$ is called a *chamber* if D contains a non-empty open subset of $\mathcal{P}$ and D is defined by inequalities

$$\langle x, v_i \rangle \geq 0 \quad (i \in I),$$

where $\{(v_i)^\perp\}_{i \in I}$ is a locally finite family of hyperplanes of $\mathcal{P}$. A *wall* of a chamber D is a closed subset of D of the form $D \cap (v)^\perp$ such that $(v)^\perp$ is disjoint from the interior of D and that $D \cap (v)^\perp$ contains a non-empty open subset of $(v)^\perp$. We say that a vector $v \in L \otimes \mathbb{R}$ *defines a wall* $D \cap (v)^\perp$ *of* D if $D \cap (v)^\perp$ is a wall of D and $\langle x, v \rangle > 0$ holds for one (and hence any) point x in the interior of D. We say that

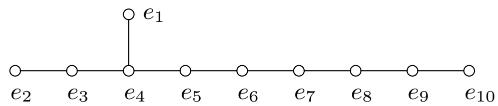

FIGURE 1.1. The basis $e_1, \dots, e_{10}$ of L_{10}

a closed subset A of $\mathcal{P}$ is *tessellated by a set* $\{D_j\}_{j\in J}$ *of chambers* if A is the union of D_j $(j \in J)$ and the interiors of two distinct chambers D_j and $D_{j'}$ in the family $\{D_j\}_{j\in J}$ have no common points.

Definition 1.9. Let L be an even hyperbolic lattice with a positive half-cone $\mathcal{P}$. An L-*chamber* is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \;\setminus\; \bigcup_r (r)^\perp,$$

where r runs through the set of roots of L. For an L-chamber D, we denote the stabilizer of D by

$$\mathrm{O}(L, D) := \{\, g \in \mathrm{O}^{\mathcal{P}}(L) \mid D^g = D \,\}.$$

Remark 1.10. In Section 2.4, we extend the notion of L-chambers to the notion of L/M-chambers in the positive-half cone $\mathcal{P}_M$ of a primitive lattice M of L.

The group $\mathrm{O}^{\mathcal{P}}(L)$ acts on the set of L-chambers. The action of the subgroup $W(L)$ of $\mathrm{O}^{\mathcal{P}}(L)$ on this set is free and transitive. Hence an L-chamber is a standard fundamental domain of the Weyl group $W(L)$. Let D be an L-chamber. Then we have $\mathrm{O}^{\mathcal{P}}(L) = W(L) \rtimes \mathrm{O}(L, D)$, and moreover, $W(L)$ is generated by the reflections s_r with respect to the roots r that define the walls of D.

Recall that L_{10} is an even unimodular hyperbolic lattice of rank 10. Then L_{10} has a basis $e_1, \dots, e_{10}$ consisting of roots whose dual graph is given in Figure 1.1. Let $\mathcal{P}_{10}$ be the positive half-cone of L_{10} containing $e_1^\vee + \dots + e_{10}^\vee$, where $\{e_1^\vee, \dots, e_{10}^\vee\}$ is the basis of $L_{10}^\vee = L_{10}$ dual to $\{e_1, \dots, e_{10}\}$.

Theorem 1.11 (Vinberg [35]). *The chamber* D_0 *in* $\mathcal{P}_{10}$ *defined by* $\langle x, e_i \rangle \geq 0$ *for* $i = 1, \dots, 10$ *is an* L_{10}-*chamber, and* $\{e_1, \dots, e_{10}\}$ *is the set of roots defining walls of* D_0. □

Definition 1.12. We call an L_{10}-chamber a *Vinberg chamber.*

Let D_0 be a Vinberg chamber. Since the dual graph in Figure 1.1 has no non-trivial symmetries, we have $\mathrm{O}(L_{10}, D_0) = \{1\}$ and hence

$$\mathrm{O}^{\mathcal{P}}(L_{10}) = W(L_{10}). \tag{1.3}$$

1.3. **Main results.** We investigate the geometry of a $(\tau, \bar\tau)$-generic Enriques surface Y. In particular, we calculate a finite generating set of $\mathrm{aut}(Y)$ and the action of $\mathrm{aut}(Y)$ on Nef_Y, $\mathcal{R}(Y)$ and $\mathcal{E}(Y)$.

Remark 1.13. Since our approach relies on the interplay beetween lattice theory and hyperbolic geometry, we can, except for the cases Nos. 88 and 146 in Table 1.1, calculate the geometric data of a hypothetical $(\tau, \bar\tau)$-generic Enriques surface even when it is not realized by an actual complex Enriques surface. (See Remark 4.7).

Let Y be an Enriques surface. Recall that $\mathrm{aut}(Y) \subset \mathrm{O}^{\mathcal{P}}(S_Y)$ is the image of the natural homomorphism $\mathrm{Aut}(Y) \to \mathrm{O}^{\mathcal{P}}(S_Y)$. Since S_Y is isomorphic to L_{10}, we have Vinberg chambers in the positive half-cone $\mathcal{P}_Y$. Since Nef_Y is bounded by $([C])^{\perp}$, where C runs through $\mathcal{R}(Y)$, and $\langle [C], [C] \rangle = -2$, the cone Nef_Y is tessellated by Vinberg chambers. We put

$$\mathcal{V}(\mathrm{Nef}_Y) := \text{the set of Vinberg chambers contained in } \mathrm{Nef}_Y,$$

on which $\mathrm{aut}(Y)$ acts, and define

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) := \text{the number of orbits of the action of } \mathrm{aut}(Y) \text{ on } \mathcal{V}(\mathrm{Nef}_Y).$$

An Enriques surface that is very general in the sense of Barth–Peters [2] is $(0,0)$-generic, and its automorphism group was determined by Barth–Peters [2] and Nikulin [23, Theorem 10.1.2 (c)] independently.

Theorem 1.14 (Barth–Peters [2], Nikulin [23])**.** *Let Y_0 be a $(0,0)$-generic Enriques surface. Then $\mathrm{aut}(Y_0) \subset \mathrm{O}^{\mathcal{P}}(S_{Y_0})$ is equal to the kernel of the reduction homomorphism $\mathrm{O}^{\mathcal{P}}(S_{Y_0}) \to \mathrm{O}(S_{Y_0}) \otimes \mathbb{F}_2$. In particular, the index of $\mathrm{aut}(Y_0)$ in $\mathrm{O}^{\mathcal{P}}(S_{Y_0})$ is equal to*

$$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600.$$

Since a $(0,0)$-generic Enriques surface Y_0 contains no smooth rational curves, we have $\mathcal{P}_{Y_0} = \mathrm{Nef}_{Y_0}$. Combining this with (1.3), we obtain bijections

$$\mathrm{O}^{\mathcal{P}}(S_{Y_0}) = W(S_{Y_0}) \cong \mathcal{V}(\mathrm{Nef}_{Y_0}).$$

We define the unit 1_{BP} (BP stands for Barth–Peters) of volume to be

$$1_{\mathrm{BP}} := \mathrm{vol}(\mathrm{Nef}_{Y_0}/\mathrm{aut}(Y_0)) = [\mathrm{O}^{\mathcal{P}}(S_{Y_0}) : \mathrm{aut}(Y_0)] = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31.$$

Our first main result is as follows. For an ADE-type τ, let $W(R_\tau)$ denote the Weyl group of the ADE-lattice R_τ with $\tau(R_\tau) = \tau$, that is, the finite Coxeter group defined by the Dynkin diagram of type τ.

Theorem 1.15. *Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Then we have*

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \frac{c_{(\tau,\bar{\tau})}}{|W(R_\tau)|} \cdot 1_{\mathrm{BP}},$$

where $c_{(\tau,\bar{\tau})} \in \{1, 2\}$ is as given in Table 1.1.

See Section 3.4 for an explanation of the factor $c_{(\tau,\bar{\tau})}$. Theorem 1.15 is obtained from a more general result Theorem 3.3 on $\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y))$. To obtain Theorem 3.3, we prove a result (Proposition 2.1) of the theory of discriminant forms in the spirit of Nikulin [22]. The proof of these theorems does not involve any machine-aided computation. Nevertheless the ability to compute examples played a crucial role in finding the correct statement.

Next, we calculate explicitly a finite generating set of $\mathrm{aut}(Y)$ and a complete set of representatives of the orbits of the action of $\mathrm{aut}(Y)$ on Nef_Y. The algorithms we use for this purpose are variations of a simple algorithm given in Section 4.1, which is an abstraction of the generalized Borcherds' method described in [28]. By means of these computational data, we analyze the action of $\mathrm{aut}(Y)$ on $\mathcal{R}(Y)$ and $\mathcal{E}(Y)$. (Recall that $\mathcal{R}(Y)$ and $\mathcal{E}(Y)$ are embedded into S_Y.)

Our second main result is as follows.

Theorem 1.16. *Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface.*

(1) *There exist smooth rational curves $C_1, \dots, C_m$ on Y whose dual graph Γ is a Dynkin diagram of type τ. Under the action of* $\mathrm{aut}(Y)$*, any smooth rational curve C on Y is in the same orbit as one of $C_1, \dots, C_m$.*

(2) *The size of $\mathcal{R}(Y)/\mathrm{aut}(Y)$ is given in the 7th column* `rat` *of Table 1.1. Except for the cases marked by $\times$ in this column, two curves C_i and C_j are in the same orbit if and only if the vertices of the dual graph Γ corresponding to C_i and C_j belong to the same connected component of Γ, and hence $|\mathcal{R}(Y)/\mathrm{aut}(Y)|$ is equal to the number of connected components of the Dynkin diagram of type τ.*

In [2], Barth and Peters also proved the following.

Theorem 1.17 (Barth–Peters [2])**.** *Let Y_0 be a $(0, 0)$-generic Enriques surface. Then Y_0 has exactly $17 \cdot 31 = 527$ elliptic fibrations modulo* $\mathrm{aut}(Y_0)$.

We calculate $\mathcal{E}(Y)/\mathrm{aut}(Y)$ for $(\tau, \bar{\tau})$-generic Enriques surfaces. Since the tables span 7 pages, we relegate a part of it to the ancillary files.

Theorem 1.18. *Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Then the orbits of the action of* $\mathrm{aut}(Y)$ *on the set $\mathcal{E}(Y)$ of elliptic fibrations of Y are indicated in* Section 6.5 *for* $\mathrm{rank}\,\tau \leq 7$ *and in the ancillary files* [32] *for* $\mathrm{rank}\,\tau \geq 8$.

1.4. **The plan of the paper.** This paper is organized as follows. In Section 2, we prepare basic notions about finite quadratic forms, discriminant forms, lattices and chambers. Proposition 2.1 in Section 2.1 plays a crucial role in the proof of the volume formula in the next section. The notion of L/M-chambers given in Section 2.4 is the main tool of our computation. In Section 3, we investigate the nef-and-big cone Nef_Y of an Enriques surface Y from the point of view of L/M-chambers, and prove Proposition 1.6. Then, by means of Proposition 2.1, we prove a formula (Theorem 3.3) for the volume of $\mathrm{Nef}_Y/\mathrm{aut}(Y)$, and in Section 3.4, we deduce Theorem 1.15 from Theorem 3.3.

In Section 4, we present a computational procedure on a graph (Procedure 4.1), which is an abstraction of the generalized Borcherds' method formulated in [28]. Then we recall the classification of primitive embeddings $L_{10}(2) \hookrightarrow L_{26}$ obtained in [6], and construct primitive embeddings $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ for $(\tau, \bar{\tau})$-generic Enriques surfaces Y. In Section 5, we prepare some geometric algorithms used in the application of the generalized Borcherds' method to $(\tau, \bar{\tau})$-generic Enriques surfaces. In Section 6, we calculate $\mathrm{aut}(Y)$ and $\mathrm{Nef}_Y/\mathrm{aut}(Y)$, and prove Theorems 1.16 and 1.18. The table of elliptic fibrations is given in Section 6.5.

In Section 7, we exhibit some examples. In particular, we treat an (E_6, E_6)-generic Enriques surface (No. 47 of Table 1.1) in detail, because we investigated this surface in [31]. Section 7.1 contains a correction of a wrong assertion made in [31].

In the second author's webpage [32], we put a detailed computation data made by `GAP` [34].

Thanks are due to Professor Igor Dolgachev for his comments on the manuscript of this paper.

2. Finite quadratic forms, lattices and chambers

We fix notions and terminologies about finite quadratic forms, discriminant forms, lattices and chambers.

2.1. **Finite quadratic forms.** A *finite quadratic form* is a finite abelian group A with a quadratic form

$$q_A \colon A \to \mathbb{Q}/2\mathbb{Z}.$$

We say that a finite quadratic form is *non-degenerate* if the bilinear form

$$b_A \colon A \times A \to \mathbb{Q}/\mathbb{Z}$$

associated with q_A is non-degenerate. The automorphism group of a finite quadratic form A is denoted by $\mathrm{O}(A)$, and we let it act on A from the right. For a subgroup $D \subset A$, let $D^\perp$ denote the orthogonal complement of D with respect to b_A, and let $\mathrm{O}(A, D)$ denote the subgroup $\{\, g \in \mathrm{O}(A) \mid D^g = D \,\}$ of $\mathrm{O}(A)$.

The following proposition will play a crucial role in the proof of the volume formula (Theorem 3.3).

Proposition 2.1. *Let (A, q_A) and (B, q_B) be non-degenerate finite quadratic forms, and let $D_A \subset A$ and $D_B \subset B$ be subgroups. Suppose that we have an isomorphism $\phi \colon D_A \xrightarrow{\sim} D_B$ that induces an isometry $(D_A, -q_A|D_A) \cong (D_B, q_B|D_B)$ of finite quadratic forms. Let $\Gamma \subset A \oplus B$ be the graph of ϕ, which is an isotropic subgroup with respect to $q_A \oplus q_B$. We put $C := \Gamma^\perp/\Gamma$. Then $q_A \oplus q_B$ induces a quadratic form q_C on C, and we have a natural homomorphism*

$$\{\, (g, h) \in \mathrm{O}(A) \times \mathrm{O}(B) \mid \Gamma^{(g,h)} = \Gamma \,\} \to \mathrm{O}(C).$$

We denote by K the kernel of this homomorphism. Then the homomorphism

$$i_A \colon K \hookrightarrow \mathrm{O}(A) \times \mathrm{O}(B) \to \mathrm{O}(A), \quad (g, h) \mapsto g$$

is injective, and the image of i_A is equal to the kernel of the natural homomorphism

$$\mathrm{O}(A, D_A) \to \mathrm{O}(D_A^\perp).$$

Proof. First we prove that the natural projection $\Gamma^\perp \to B$ is surjective. Since q_A and q_B are non-degenerate, we have natural isomorphisms $A \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ and $B \cong \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z})$ induced by b_A and b_B. Hence we have natural isomorphisms $\mathrm{Hom}(D_A, \mathbb{Q}/\mathbb{Z}) \cong A/D_A^\perp$ and $\mathrm{Hom}(D_B, \mathbb{Q}/\mathbb{Z}) \cong B/D_B^\perp$. We have an isomorphism

$$-\phi^* \colon \mathrm{Hom}(D_B, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(D_A, \mathbb{Q}/\mathbb{Z})$$

induced by $-\phi \colon D_A \xrightarrow{\sim} D_B$. Combining them, we obtain a homomorphism

$$\psi \colon B \twoheadrightarrow B/D_B^\perp \cong \mathrm{Hom}(D_B, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(D_A, \mathbb{Q}/\mathbb{Z}) \cong A/D_A^\perp. \tag{2.1}$$

For $\alpha \in A$, we put

$$\bar{\alpha} := \alpha \bmod D_A^\perp \quad \in \quad A/D_A^\perp.$$

Then, for $\alpha \in A$ and $\beta \in B$, we have

$$\bar{\alpha} = \psi(\beta) \iff b_A(\alpha, x) = -b_B(\beta, \phi(x)) \text{ for all } x \in D_A \iff (\alpha, \beta) \in \Gamma^\perp. \tag{2.2}$$

In particular, for any $\beta \in B$, we have $\alpha \in A$ such that $(\alpha, \beta) \in \Gamma^\perp$.

Next we prove that $i_A \colon K \to \mathrm{O}(A)$ is injective. Let $(1, h) \in K$ be an element of $\mathrm{Ker}\, i_A$. For $\beta \in B$, we choose $\alpha \in A$ such that $(\alpha, \beta) \in \Gamma^\perp$. Since $(1, h)$ acts on $C = \Gamma^\perp/\Gamma$ trivially, we have $(\alpha, \beta) - (\alpha, \beta^h) = (0, \beta - \beta^h) \in \Gamma$. Since $\Gamma \cap B = 0$, we have $\beta^h = \beta$. Since $\beta \in B$ is arbitrary, we have $h = 1$.

Now we determine the image of i_A. "$\subset$": Suppose that $(g, h) \in K$. Since (g, h) preserves Γ, we see that $g = i_A(g, h)$ preserves the image D_A of the projection $\Gamma \to A$. For any $\alpha \in D_A^\perp$, we have $(\alpha, 0) \in \Gamma^\perp$. Since (g, h) acts on $C = \Gamma^\perp/\Gamma$ trivially, we have $\alpha^g - \alpha \in \Gamma \cap A = 0$. Therefore $\mathrm{Im}\, i_A$ is contained in $\mathrm{Ker}(\mathrm{O}(A, D_A) \to \mathrm{O}(D_A^\perp))$.

"$\supset$": To show the opposite inclusion, we fix $g \in \mathrm{Ker}(\mathrm{O}(A, D_A) \to \mathrm{O}(D_A^\perp))$ and construct $h \in \mathrm{O}(B)$ such that $(g, h) \in K$. Since g acts on $D_A^\perp$ trivially, the linear map

$$l_g\colon A/D_A^\perp \to A, \quad \bar{\alpha} \mapsto \alpha^g - \alpha$$

is well-defined. The image of l_g is contained in $D_A = (D_A^\perp)^\perp$: indeed, for any $\alpha \in A$ and $y \in D_A^\perp$, we have

$$b_A(l_g(\bar{\alpha}), y) = b_A(\alpha^g, y) - b_A(\alpha, y) = b_A(\alpha^g, y^g) - b_A(\alpha, y) = 0.$$

We define $h : B \to B$ by

$$\beta^h := \beta + \phi l_g \psi(\beta),$$

where ψ is given in (2.1). We show that $h \in \mathrm{O}(B)$. We put $\bar{\alpha} = \psi(\beta)$. Then we have

$$q_B(\beta^h) - q_B(\beta) = 2b_B(\beta, \phi l_g(\bar{\alpha})) + q_B(\phi l_g(\bar{\alpha})) = -2b_A(\alpha, \alpha^g - \alpha) - q_A(\alpha^g - \alpha) = 0,$$

because $g \in \mathrm{O}(A)$. It only remains to show that $(g, h) \in \mathrm{O}(A) \times \mathrm{O}(B)$ preserves Γ and acts on $C = \Gamma^\perp/\Gamma$ trivially. Using (2.2) and $\Gamma \subset \Gamma^\perp$, we see that for any $\alpha \in D_A$, we have $\bar{\alpha} = \psi\phi(\alpha)$, and therefore

$$\phi(\alpha)^h = \phi(\alpha) + \phi l_g(\bar{\alpha}) = \phi(\alpha) + \phi(\alpha^g) - \phi(\alpha) = \phi(\alpha^g).$$

Since g preserves D_A, we have $(\alpha, \phi(\alpha))^{(g,h)} = (\alpha^g, \phi(\alpha^g)) \in \Gamma$ for any $\alpha \in D_A$. Therefore (g, h) preserves Γ. Suppose that $(\alpha, \beta) \in \Gamma^\perp$. Then we have $\bar{\alpha} = \psi(\beta)$ by (2.2), and

$$(\alpha^g, \beta^h) - (\alpha, \beta) = (l_g(\bar{\alpha}), \phi l_g(\bar{\alpha})) \quad \in \quad \Gamma.$$

Therefore (g, h) acts on $\Gamma^\perp/\Gamma$ trivially. □

Remark 2.2. Proposition 2.1 holds for non-degenerate finite bilinear forms (A, b_A) and (B, b_B) as well.

2.2. **Discriminant forms and overlattices.** Let L be an even lattice. We put

$$L^\vee := \{\, x \in L \otimes \mathbb{Q} \mid \langle x, v\rangle \in \mathbb{Z} \ \text{ for all } \ v \in L \,\},$$

on which $\mathrm{O}(L)$ acts naturally. The finite abelian group $L^\vee/L$ is called the *discriminant group* of L. Then

$$q(\bar{x}) = \langle x, x\rangle \bmod 2\mathbb{Z} \quad \text{for } x \in L^\vee \text{ and } \bar{x} = x \bmod L$$

defines a finite quadratic form $q\colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z}$, which is called the *discriminant form* of L. An even lattice L' is an *overlattice* of L if we have $L \subset L' \subset L^\vee$ and the intersection form of L' is the extension of that of L. See Nikulin [22] for the details of the theory of discriminant forms and its application to the enumeration of even overlattices of a given even lattice.

To illustrate Proposition 2.1, we apply it to two known extreme cases.

Example 2.3. Let $M, N \subset L$ be primitive sublattices of an even lattice L such that $M \perp N$ and $\operatorname{rank} M + \operatorname{rank} N = \operatorname{rank} L$. Then we have

$$M \oplus N \subset L \subset L^\vee \subset M^\vee \oplus N^\vee,$$

and L is an overlattice of $M \oplus N$. Let $(A, q_A) = (M^\vee/M, q_M)$ and $(B, q_B) = (N^\vee/N, q_N)$ be the respective discriminant forms. Then $\Gamma = L/(M \oplus N)$ is the graph of an anti-isometry $\phi\colon A \supset D_A \to D_B \subset B$ and $\Gamma^\perp/\Gamma \cong L^\vee/L$.

First suppose that L is unimodular. Then, by a result of Nikulin [22], $D_A = A$ and $D_B = B$. Since $L^\vee/L \cong \Gamma^\perp/\Gamma$ is trivial, we have

$$K = \{(g, h) \in \mathrm{O}(A) \times \mathrm{O}(B) : h \circ \phi = \phi \circ g\}.$$

We see that $i_A \colon K \to \mathrm{O}(A)$ is an isomorphism as predicted by Proposition 2.1. Indeed, since $D_A^\perp = A^\perp = 0$, the homomorphism $\mathrm{O}(A, D_A) \to \mathrm{O}(D_A^\perp)$ is trivial.

For the other extreme suppose that $M \oplus N = L$. Then $D_A = 0$, $D_B = 0$, $K = 1$ and $D_A^\perp = A$.

2.3. **Faces of a chamber.** Let L be a hyperbolic lattice with a positive half-cone $\mathcal{P}$, and D a chamber in $\mathcal{P}$. A *face* of D is a closed subset of D that is an intersection of some walls of D. Let f be a face of D. The *dimension* $\dim f$ of f is the dimension of the minimal linear subspace of $L \otimes \mathbb{R}$ containing f, and the *codimension* of f is $\operatorname{rank} L - \dim f$. The walls of D are exactly the faces of D with codimension 1.

Let $\overline{\mathcal{P}}$ and $\overline{D}$ be the closures of $\mathcal{P}$ and D in $L \otimes \mathbb{R}$, respectively. A half-line contained in $(\overline{\mathcal{P}} \setminus \mathcal{P}) \cap \overline{D}$ is called an *isotropic ray* of D.

Suppose that D has only finitely many walls, that they are defined by vectors in $L \otimes \mathbb{Q}$, and that the list of defining vectors of these walls in $L \otimes \mathbb{Q}$ is available. Then we can make the list of faces of D by means of linear programming. For each isotropic ray $\mathbb{R}_{\geq 0} v$, we have a unique primitive vector $v \in L$ that generates $\mathbb{R}_{\geq 0} v$, which we call a *primitive isotropic ray* of D. We can also make the list of primitive isotropic rays of D.

2.4. L/M**-chambers.** Let $(L, \langle\ ,\ \rangle_L)$ and $(M, \langle\ ,\ \rangle_M)$ be even hyperbolic lattices with fixed positive half-cones $\mathcal{P}_L$ and $\mathcal{P}_M$, respectively. Suppose that we have an embedding $M \hookrightarrow L$ that maps $\mathcal{P}_M$ into $\mathcal{P}_L$. We regard $\mathcal{P}_M$ as a subspace of $\mathcal{P}_L$ by this embedding. The notion of L-chambers was introduced in Section 1.2. The following class of chambers plays an important role in this paper.

Definition 2.4. A chamber D_M in $\mathcal{P}_M$ is called an *L/M-chamber* if there exists an L-chamber $D_L \subset \mathcal{P}_L$ such that $D_M = \mathcal{P}_M \cap D_L$. In this case, we say that *D_M is induced by D_L.*

In particular, an L-chamber is an L/L-chamber.

Definition 2.5. Let N be a negative definite even lattice. For a root r of N, let $[r]^\perp$ denote the hyperplane of $N \otimes \mathbb{R}$ defined by $\langle x, r \rangle = 0$. The connected components of $(N \otimes \mathbb{R}) \setminus \bigcup [r]^\perp$, where r runs through the set of roots of N, are called the *Weyl-chambers* of N. The Weyl group $W(N)$ acts simply transitively on the set of Weyl-chambers.

Remark 2.6. Let D_M be an L/M-chamber. Then the number of L-chambers that induce D_M is equal to the number of Weyl-chambers of the orthogonal complement $(M \hookrightarrow L)^\perp$ of M in L. In particular, if $(M \hookrightarrow L)^\perp$ contains no roots, then each L/M-chamber is induced by a unique L-chamber.

Definition 2.7. Two distinct L/M-chambers D_1 and D_2 are *adjacent* if there exists a hyperplane $(v)^\perp$ of $\mathcal{P}_M$ such that $D_1 \cap (v)^\perp$ is a wall of D_1, that $D_2 \cap (v)^\perp$ is a wall of D_2, and that $D_1 \cap (v)^\perp = D_2 \cap (v)^\perp$ holds. In this case, we say that D_2 is adjacent to D_1 *across the wall* $D_1 \cap (v)^\perp$.

Let $\mathrm{pr}\colon L \to M \otimes \mathbb{Q}$ be the orthogonal projection. Then an L/M-chamber is the closure in $\mathcal{P}_M$ of a connected component of

$$\mathcal{P}_M \ \setminus \ \bigcup_r (\mathrm{pr}(r))^\perp,$$

where r runs through the set of roots r of L such that $\langle \mathrm{pr}(r), \mathrm{pr}(r)\rangle_M < 0$ holds, and $(\mathrm{pr}(r))^\perp = \mathcal{P}_M \cap (r)^\perp$ is the hyperplane of $\mathcal{P}_M$ defined by $\mathrm{pr}(r)$. Hence, for each wall $D_M \cap (v)^\perp$ of an L/M-chamber D_M, there exists a unique L/M-chamber adjacent to D_M across the wall $D_M \cap (v)^\perp$.

Since a root of M is mapped to a root of L by the embedding $M \hookrightarrow L$, an M-chamber is tessellated by L/M-chambers. More generally, we have the following proposition, which is easy to prove:

Proposition 2.8. *Suppose that $M_1 \hookrightarrow M_2 \hookrightarrow L$ is a sequence of embeddings of even hyperbolic lattices that induces a sequence of embeddings $\mathcal{P}_{M_1} \hookrightarrow \mathcal{P}_{M_2} \hookrightarrow \mathcal{P}_L$ of fixed positive half-cones. Then each M_2/M_1-chamber is tessellated by L/M_1-chambers.* □

If $\tilde{g} \in \mathrm{O}^{\mathcal{P}}(L)$ satisfies $M^{\tilde{g}} = M$, then $\tilde{g}|M \in \mathrm{O}^{\mathcal{P}}(M)$ preserves the tessellation of $\mathcal{P}_M$ by L/M-chambers.

In general, two distinct L/M-chambers are not isomorphic to each other. See [11] and [28] for examples of $K3$ surfaces X with a primitive embedding $S_X \hookrightarrow L_{26}$ such that $\mathcal{P}_X$ is tessellated by L_{26}/S_X-chambers of various shapes.

Definition 2.9. We say that the tessellation of $\mathcal{P}_M$ by L/M-chambers is *reflexively simple* if, for each wall $D_M \cap (v)^\perp$ of an L/M-chamber D_M, there exists an isometry $\tilde{g}$ of L preserving M such that the restriction $\tilde{g}|M$ of $\tilde{g}$ to M is an involution that fixes every point of the hyperplane $(v)^\perp$. Note that, if this is the case, the isometry $\tilde{g}|M$ of M maps D_M to the L/M-chamber adjacent to D_M across the wall $D_M \cap (v)^\perp$.

The tessellation of $\mathcal{P}_L$ by L/L-chambers is obviously reflexively simple.

3. The cone Nef_Y

Let Y be an Enriques surface with the universal covering $\pi\colon X \to Y$. Let $\varepsilon \in \mathrm{Aut}(X)$ be the deck-transformation of $\pi\colon X \to Y$, and we put

$$S_{X+} := \{\, v \in S_X \mid v^\varepsilon = v\,\}, \quad S_{X-} := \{\, v \in S_X \mid v^\varepsilon = -v\,\}.$$

Then S_{X+} is equal to the image of $\pi^*\colon S_Y(2) \hookrightarrow S_X$, and S_{X-} is the orthogonal complement of S_{X+}. We regard $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$ by $\pi^* \otimes \mathbb{R}$.

3.1. **$S_X/S_Y(2)$-chambers.** It is well-known that Nef_X is an S_X-chamber. Therefore the chamber $\mathrm{Nef}_Y = \mathcal{P}_Y \cap \mathrm{Nef}_X$ is an $S_X/S_Y(2)$-chamber. Since π is étale, the lattice S_{X-} contains no roots, and hence each $S_X/S_Y(2)$-chamber D_Y is induced by a *unique* S_X-chamber D_X, that is, D_Y contains an interior point of D_X.

Proposition 3.1. *The tessellation of $\mathcal{P}_Y$ by $S_X/S_Y(2)$-chambers is reflexively simple. More precisely, every wall of an $S_X/S_Y(2)$-chamber D_Y is defined by a root r of S_Y, and the reflection $s_r \in \mathrm{O}^{\mathcal{P}}(S_Y)$ with respect to the root r is the restriction $s_{\tilde{r}_+}s_{\tilde{r}_-}|S_Y(2)$ of the product of two reflections with respect to roots $\tilde{r}_+, \tilde{r}_-$ of S_X.*

Proof. Let $\langle -,-\rangle_X$ and $\langle -,-\rangle_Y$ be the intersection forms of S_X and S_Y, respectively. We denote by $(u)^{\perp}_X$ the hyperplane of $\mathcal{P}_X$ defined by $u \in S_X \otimes \mathbb{R}$, and by $(v)^{\perp}_Y$ the hyperplane of $\mathcal{P}_Y$ defined by $v \in S_Y \otimes \mathbb{R}$. Let D_Y be an $S_X/S_Y(2)$-chamber, and let $D_Y \cap (v)^{\perp}_Y$ be a wall of D_Y.

By the definition of $S_X/S_Y(2)$-chambers, there exists a root $\tilde{r}$ of S_X such that $(v)^{\perp}_Y = \mathcal{P}_Y \cap (\tilde{r})^{\perp}_X$. We first prove that $\langle \tilde{r}, \tilde{r}^{\varepsilon}\rangle_X = 0$. Let $\tilde{r}$ be written as $v_L + v_R$, where $v_L \in S_Y(2)^{\vee}$ and $v_R \in S^{\vee}_{X-}$. We have $\langle v_L, v_L\rangle_X + \langle v_R, v_R\rangle_X = -2$. Since $\tilde{r}^{\varepsilon} = v_L - v_R$, it is enough to show that $\langle v_L, v_L\rangle_X = -1$. Since

$$\mathcal{P}_Y \cap (\tilde{r})^{\perp}_X = (v_L)^{\perp}_Y$$

is non-empty, we have $\langle v_L, v_L\rangle_Y < 0$. Note that $2v_L \in S_Y$ because $2S_Y(2)^{\vee} = S_Y(2)$. Since S_Y is even, $\langle v_L, v_L\rangle_X = 2\langle v_L, v_L\rangle_Y$ must be an integer. Since S_{X-} is negative definite, we have $\langle v_R, v_R\rangle_X \le 0$ and hence $\langle v_L, v_L\rangle_X$ is -2 or -1. If $\langle v_L, v_L\rangle_X = -2$, then $v_R = 0$ and $\tilde{r} = v_L \in S_Y(2)$, which is absurd.

Let s and s' be the reflections with respect to the roots $\tilde{r}$ and $\tilde{r}^{\varepsilon}$ of S_X, respectively. By $\langle \tilde{r}, \tilde{r}^{\varepsilon}\rangle_X = 0$, we have $ss' = s's$. Since $s' = \varepsilon s \varepsilon$, we see that ss' commutes with ε and hence ss' preserves $\mathcal{P}_Y$. The vector $r := \tilde{r} + \tilde{r}^{\varepsilon}$ is contained in S_Y. Moreover we have $\langle r, r\rangle_Y = -2$ and

$$(v)^{\perp}_Y = \mathcal{P}_Y \cap (\tilde{r})_X = (v_L)^{\perp}_Y = (r)^{\perp}_Y.$$

Therefore the wall $D_Y \cap (v)^{\perp}_Y$ of D_Y is defined by a root r or $-r$ of S_Y. It is easy to confirm that the restriction of ss' to S_Y is equal to the reflection with respect to the root r of S_Y and therefore maps D_Y to the $S_X/S_Y(2)$-chamber D'_Y adjacent to D_Y across the wall $D_Y \cap (v)^{\perp}_Y = D_Y \cap (r)^{\perp}_Y$. □

3.2. **Proof of Proposition 1.6.** We prove Proposition 1.6. By Proposition 1.2, we have isomorphisms ψ_X and ψ_Y that make the diagram (1.2) commutative. By Proposition 3.1, we have $\tilde{g} \in \mathrm{O}^{\mathcal{P}}(S_X)$ commuting with ε such that $\tilde{g}|S_Y(2)$ maps Nef_Y to the inverse image of $\mathrm{Nef}_{Y'}$ by ψ_Y. Then the isometries $\tilde{g} \circ \psi_X \colon S_{X'} \xrightarrow{\sim} S_X$ and $\tilde{g}|S_Y(2) \circ \psi_Y \colon S_{Y'} \xrightarrow{\sim} S_Y$ satisfy the required properties. □

3.3. **The volume of** $\mathrm{Nef}_Y/\mathrm{aut}(Y)$**.** In this subsection, we give a formula (Theorem 3.3) for $\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y))$ under the assumption that

(3.1) the group $\mathrm{O}(T_X, \omega)$ in Definition 1.4 is $\{\pm 1\}$.

We put

(3.2) $G_X := \{\, \tilde{g} \in \mathrm{O}^{\mathcal{P}}(S_X) \mid \tilde{g} \text{ commutes with } \varepsilon \text{ and acts on } S^{\vee}_X/S_X \text{ trivially}\,\}.$

Then $\tilde{g} \mapsto (\tilde{g}|S_{X+}, \tilde{g}|S_{X-})$ embeds G_X into $\mathrm{O}^{\mathcal{P}}(S_{X+}) \times \mathrm{O}(S_{X-})$. Let G_{X+} and G_{X-} denote the images of the projections $G_X \to \mathrm{O}^{\mathcal{P}}(S_{X+})$ and $G_X \to \mathrm{O}(S_{X-})$, respectively. When we regard G_{X+} as a subgroup of $\mathrm{O}^{\mathcal{P}}(S_Y)$ via the identification $S_{X+} = S_Y(2)$ induced by π^*, we write G_Y instead of G_{X+}. Recall that the set $\mathcal{R}(Y)$ of smooth rational curves on Y is embedded into S_Y by $C \mapsto [C]$. The correspondence

$$C \mapsto \mathrm{Nef}_Y \cap ([C])^{\perp}$$

gives a bijection from $\mathcal{R}(Y)$ to the set of walls of the $S_X/S_Y(2)$-chamber Nef_Y. We denote by $W(\mathcal{R}(Y))$ the subgroup of $\mathrm{O}^{\mathcal{P}}(S_Y)$ generated by the reflections $s_{[C]}$ with respect to the roots $[C] \in \mathcal{R}(Y)$. Recall also that $\mathrm{aut}(Y)$ is the image of the natural representation $\mathrm{Aut}(Y) \to \mathrm{O}^{\mathcal{P}}(S_Y)$.

Proposition 3.2. *Suppose that Y satisfies (3.1).*

(1) *The action of G_Y on $\mathcal{P}_Y$ preserves the set of $S_X/S_Y(2)$-chambers, and $\operatorname{aut}(Y)$ is equal to the stabilizer subgroup of Nef_Y in G_Y.*

(2) *The group $W(\mathcal{R}(Y))$ is contained in G_Y as a normal subgroup, and we have $G_Y = W(\mathcal{R}(Y)) \rtimes \operatorname{aut}(Y)$.*

Proof. Since every $g \in G_Y$ lifts to an element $\tilde{g}$ of $G_X \subset \mathrm{O}^{\mathcal{P}}(S_X)$, the action of G_Y on $\mathcal{P}_Y$ preserves the tessellation of $\mathcal{P}_Y$ by $S_X/S_Y(2)$-chambers.

Let $\operatorname{aut}(X)$ be the image of the natural representation $\operatorname{Aut}(X) \to \mathrm{O}^{\mathcal{P}}(S_X)$. By the Torelli theorem for complex $K3$ surfaces ([3, Chapter VIII]), we have a natural embedding

$$\operatorname{Aut}(X) \hookrightarrow \mathrm{O}^{\mathcal{P}}(S_X) \times \mathrm{O}(T_X, \omega), \tag{3.3}$$

and an element $(\tilde{g}, f)$ of $\mathrm{O}^{\mathcal{P}}(S_X) \times \mathrm{O}(T_X, \omega)$ belongs to $\operatorname{Aut}(X)$ if and only if $(\tilde{g}, f)$ preserves the overlattice $H^2(X, \mathbb{Z})$ of $S_X \oplus T_X$ and $\tilde{g}$ preserves Nef_X. The even unimodular overlattice $H^2(X, \mathbb{Z})$ of $S_X \oplus T_X$ induces an isomorphism

$$i_{H(X)} \colon S_X^\vee / S_X \;\cong\; T_X^\vee / T_X$$

of discriminant groups, and $(\tilde{g}, f)$ preserves $H^2(X, \mathbb{Z})$ if and only if the action of $\tilde{g}$ on $S_X^\vee/S_X$ is compatible with the action of f on $T_X^\vee/T_X$ via $i_{H(X)}$ (see [22]). Therefore, by assumption (3.1), an isometry $\tilde{g} \in \mathrm{O}^{\mathcal{P}}(S_X)$ belongs to $\operatorname{aut}(X)$ if and only if $\tilde{g}$ preserves Nef_X and acts on $S_X^\vee/S_X$ as ± 1.

Let $\operatorname{Aut}(X, \varepsilon)$ denote the centralizer of ε in $\operatorname{Aut}(X)$. We have a natural identification $\operatorname{Aut}(Y) \cong \operatorname{Aut}(X, \varepsilon)/\langle \varepsilon \rangle$. Suppose that $g \in \operatorname{aut}(Y)$. We will show that g belongs to the stabilizer subgroup of Nef_Y in G_Y. It is obvious that g preserves Nef_Y. Let $\tilde{\gamma}$ be an element of $\operatorname{Aut}(X, \varepsilon)$ that induces g on S_Y. We write $\tilde{\gamma}$ as $(\tilde{g}, f)$ by (3.3). Note that ε acts on T_X as -1. Hence, replacing $\tilde{\gamma}$ with $\tilde{\gamma}\varepsilon$ if $f = -1$, we can assume $f = 1$. Then the action $\tilde{g} \in \mathrm{O}^{\mathcal{P}}(S_X)$ of $\tilde{\gamma}$ on S_X induces the trivial action on $S_X^\vee/S_X$, which means $\tilde{g} \in G_X$. Hence $g = \tilde{g}|S_Y$ belongs to G_Y.

Conversely, suppose that g is an element of the stabilizer subgroup of Nef_Y in G_Y. We will show that $g \in \operatorname{aut}(Y)$. Let $\tilde{g}$ be an element of G_X such that $g = \tilde{g}|S_Y$. Since Nef_Y contains an interior point of Nef_X, $\tilde{g}$ preserves Nef_X, and hence $\tilde{g}$ belongs to $\operatorname{aut}(X)$. Let $\tilde{\gamma} = (\tilde{g}, f)$ be an element of $\operatorname{Aut}(X)$ that induces $\tilde{g}$. Since $\tilde{g} \in G_X$ commutes with the action of ε on S_X, the first factor of the commutator $[\tilde{\gamma}, \varepsilon] \in \operatorname{Aut}(X)$ is 1. Since $\mathrm{O}(T_X, \omega) = \{\pm 1\}$ is abelian, the second factor of $[\tilde{\gamma}, \varepsilon]$ is also 1. Hence $\tilde{\gamma} \in \operatorname{Aut}(X, \varepsilon)$, and therefore g is induced by an element of $\operatorname{Aut}(Y)$. Thus assertion (1) is proved.

By Proposition 3.1, for each $r \in \mathcal{R}(Y)$, the reflection $s_r = s_{\tilde{r}_+} s_{\tilde{r}_-} | S_Y(2)$ belongs to G_Y, because the reflections $s_{\tilde{r}_+}$ and $s_{\tilde{r}_-}$ act on $S_X^\vee/S_X$ trivially and hence $s_{\tilde{r}_+} s_{\tilde{r}_-} \in G_X$. Therefore we have $W(\mathcal{R}(Y)) \subset G_Y$. Moreover, by Proposition 3.1 again, we see that $W(\mathcal{R}(Y))$ acts on the set of $S_X/S_Y(2)$-chambers transitively.

If $C_1, C_2 \in \mathcal{R}(Y)$ satisfy $\langle C_1, C_2 \rangle_Y > 1$, then the walls $\operatorname{Nef}_Y \cap ([C_1])^\perp$ and $\operatorname{Nef}_Y \cap ([C_2])^\perp$ of Nef_Y do not intersect. Hence each face of Nef_Y with codimension 2 is of the form

$$\operatorname{Nef}_Y \cap ([C_1])^\perp \cap ([C_2])^\perp \quad \text{with} \quad \langle C_1, C_2 \rangle_Y \in \{0, 1\},$$

and we have $(s_{[C_1]} s_{[C_2]})^m = 1$, where $m = 2$ if $\langle C_1, C_2 \rangle_Y = 0$ and $m = 3$ if $\langle C_1, C_2 \rangle_Y = 1$. Therefore, by the standard method of geometric group theory (see, for example, Section 1.5 of [36]), we see that Nef_Y is a standard fundamental domain of the action of $W(\mathcal{R}(Y))$ on $\mathcal{P}_Y$, and $W(\mathcal{R}(Y))$ acts on the set of $S_X/S_Y(2)$-chambers simply-transitively. Recalling that $\mathrm{aut}(Y)$ is the stabilizer subgroup of Nef_Y in G_Y, we have $W(\mathcal{R}(Y)) \cap \mathrm{aut}(Y) = \{1\}$. Moreover G_Y is generated by the union of $W(\mathcal{R}(Y))$ and $\mathrm{aut}(Y)$.

It remains to show that $W(\mathcal{R}(Y))$ is a normal subgroup of G_Y. Let r be a root in $\mathcal{R}(Y)$ and g an arbitrary element of G_Y. It is enough to show that $g^{-1}s_r g$ belongs to $W(\mathcal{R}(Y))$. Note that $g^{-1}s_r g = s_{r^g}$ and r^g defines a wall of the $S_X/S_Y(2)$-chamber $D_Y := {\mathrm{Nef}_Y}^g$. We have an element $w \in W(\mathcal{R}(Y))$ such that $D_Y = {\mathrm{Nef}_Y}^w$. Then $r' := r^{gw^{-1}}$ defines a wall of Nef_Y, and $w s_{r^g} w^{-1} = s_{r'}$ is an element of $W(\mathcal{R}(Y))$. Hence $g^{-1}s_r g = s_{r^g} = w^{-1}s_{r'}w \in W(\mathcal{R}(Y))$. □

Let (A_+, q_+) and (A_-, q_-) be the discriminant forms of $S_{X+} = S_Y(2)$ and S_{X-}, respectively. We put

$$\Gamma_X := S_X/(S_{X+} \oplus S_{X-}) \quad \subset \quad A_+ \oplus A_-,$$

and let $D_+ \subset A_+$ and $D_- \subset A_-$ be the image of the projections of Γ_X. Then Γ_X is the graph of an isometry $(D_+, q_+|D_+) \cong (D_-, -q_-|D_-)$, and the discriminant group of S_X is canonically isomorphic to $\Gamma_X^{\perp}/\Gamma_X$. We denote by $\overline{G}_{X+}$ and $\overline{G}_{X-}$ the images of G_{X+} and G_{X-} by the natural homomorphisms $\mathrm{O}^{\mathcal{P}}(S_{X+}) \to \mathrm{O}(A_+)$ and $\mathrm{O}(S_{X-}) \to \mathrm{O}(A_-)$, respectively, and by $\overline{G}_X$ the image of G_X by the natural homomorphism to $\mathrm{O}(A_+) \times \mathrm{O}(A_-)$.

Theorem 3.3. *Suppose that Y satisfies (3.1). Let $\mathrm{O}(S_{X-}, D_-)$ be the subgroup of $\mathrm{O}(S_{X-})$ consisting of isometries g whose action on A_- preserves D_-. Then we have*

$$G_{X-} = \mathrm{Ker}(\mathrm{O}(S_{X-}, D_-) \to \mathrm{O}(D_-^{\perp})). \tag{3.4}$$

Moreover we have

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \frac{1_{\mathrm{BP}}}{|\overline{G}_{X-}|}.$$

Proof. Note that $\overline{G}_X$ is a subgroup of the kernel K of the natural homomorphism

$$\{\, (g,h) \in \mathrm{O}(A_+) \times \mathrm{O}(A_-) \mid \Gamma_X^{(g,h)} = \Gamma_X \,\} \to \mathrm{O}(\Gamma_X^{\perp}/\Gamma_X) = \mathrm{O}(S_X^{\vee}/S_X).$$

Applying Proposition 2.1(1) to $(A, B) = (A_+, A_-)$ and $(A, B) = (A_-, A_+)$, we have $|\overline{G}_X| = |\overline{G}_{X+}| = |\overline{G}_{X-}|$. Let G_{BP} be the kernel of the natural homomorphism $\mathrm{O}^{\mathcal{P}}(S_{X+}) = \mathrm{O}^{\mathcal{P}}(S_Y(2)) \to \mathrm{O}(A_+)$. Then G_{BP} is equal to $\mathrm{aut}(Y_0)$ by Theorem 1.14 and hence the index of G_{BP} in $\mathrm{O}^{\mathcal{P}}(S_{X+}) = \mathrm{O}^{\mathcal{P}}(S_Y)$ is 1_{BP}. If $g \in G_{\mathrm{BP}}$, then $(g, 1) \in \mathrm{O}^{\mathcal{P}}(S_{X+}) \times \mathrm{O}(S_{X-})$ acts trivially on $A_+ \oplus A_-$, and hence preserves Γ_X and acts on $\Gamma_X^{\perp}/\Gamma_X$ trivially. Therefore the action of $(g, 1)$ on $S_{X+} \oplus S_{X-}$ preserves the overlattice S_X, and $(g, 1)|S_X$ is an element of G_X. Thus G_{BP} is contained in $G_{X+} = G_Y$. Since the natural homomorphism $\mathrm{O}^{\mathcal{P}}(S_Y(2)) \to \mathrm{O}(A_+)$ is surjective (see [2]), the index of G_{BP} in G_Y is equal to $|\overline{G}_{X+}| = |\overline{G}_{X-}|$.

Applying Proposition 2.1 (2) to $(A, B) = (A_-, A_+)$, we see that

$$\overline{G}_{X-} \subset \mathrm{Im}\, i_{A_-} = \mathrm{Ker}(\mathrm{O}(A_-, D_-) \to \mathrm{O}(D_-^{\perp})).$$

Hence the inclusion $\subset$ in (3.4) is proved. Conversely, let f be an element of the right-hand side of (3.4), and denote by $\bar{f} \in \mathrm{O}(A_-)$ the action of f on A_-. By Proposition 2.1, we have $\bar{f} \in \mathrm{Im}\, i_{A_-}$ and hence there exists a unique element $\bar{h} \in K$ such that $i_{A_-}(\bar{h}) = \bar{f}$. We put $\bar{g} := i_{A_+}(\bar{h})$. Since the natural homomorphism $\mathrm{O}^{\mathcal{P}}(S_{X+}) \to \mathrm{O}(A_+)$ is surjective, we have $g \in \mathrm{O}^{\mathcal{P}}(S_{X+})$ that maps to $\bar{g}$. Since

$\bar{h} = (\bar{g}, \bar{f}) \in K$, we have $(g, f) \in G_X$, which implies $f \in G_{X-}$. Thus (3.4) is proved. Moreover we have

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \mathrm{vol}(\mathcal{P}_Y/G_Y) = \frac{1}{[G_Y : G_{\mathrm{BP}}]} \cdot \mathrm{vol}(\mathcal{P}_Y/G_{\mathrm{BP}}) = \frac{1_{\mathrm{BP}}}{|\overline{G}_{X-}|},$$

where the first equality follows from Proposition 3.2. □

Since S_{X-} is negative definite, $\mathrm{O}(S_{X-})$ is a finite group and can be computed easily. Thus this formula enables us to calculate $\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y))$.

3.4. **Proof of Theorem 1.15.** In what follows we calculate the finite group $\overline{G}_{X-}$ of a $(\tau, \bar{\tau})$-generic Enriques surface. It is closely related to the Weyl group $W(R_\tau)$.

For a sublattice L' of a lattice L, we denote by $\mathrm{O}(L, L')$ the group of isometries of L preserving L'. When L is an overlattice of L', then $\mathrm{O}(L, L')$ is the group of isometries of L' preserving the overlattice L, or equivalently the intersection $\mathrm{O}(L) \cap \mathrm{O}(L')$ in $\mathrm{O}(L \otimes \mathbb{Q}) = \mathrm{O}(L' \otimes \mathbb{Q})$, and hence sometimes is written as $\mathrm{O}(L', L)$.

Lemma 3.4. *Let Y be $(\tau, \bar{\tau})$-generic. Recall the commutative diagram (1.1)*

$$\begin{array}{ccc} L_{10}(2) & \overset{\varpi_R}{\hookrightarrow} & M_R \\ {\scriptstyle g}\downarrow\wr & & {\scriptstyle \tilde{g}}\downarrow\wr \\ S_Y(2) & \overset{\pi^*}{\hookrightarrow} & S_X. \end{array}$$

Denote by $\pi_-\colon S_X \to S_{X-}^\vee$ the orthogonal projection. Identify M_R with S_X via $\tilde{g}$. Then the following equalities hold:

(1) $\widetilde{R} = S_{X-}$,
(2) $R = \pi_-(2S_X)$,
(3) $\frac{1}{2}\widetilde{R}^\vee/\widetilde{R} = A_-$,
(4) $\frac{1}{2}R/\widetilde{R} = D_-$,
(5) $R^\vee/\widetilde{R} = D_-^\perp$,
(6) $\mathrm{O}(\widetilde{R}, R) = \mathrm{O}(S_{X-}, D_-)$.

Note that we neglect the quadratic forms in (1)–(5) *and just consider them as equalities of abelian groups.*

Proof. The equality (1) is by the definition.

(2) Note that M_R is spanned by $\mathrm{Im}\,\varpi_R$ and $\{(i_R(v) \pm v)/2 \mid v \in R\}$. Hence $\pi_-(M_R)$ is spanned by 0 and $\frac{1}{2}R$.

(3) As lattices we have $\widetilde{R}(2) = S_{X-}$, and $(\widetilde{R}(2))^\vee = \frac{1}{2}\widetilde{R}^\vee$ yields the claim.

(4) By definition, we have $\pi_-(S_X)/S_{X-} = D_-$.

(5) Let $x \in \frac{1}{2}R$ and $y \in R^\vee$. Then $\langle x, y\rangle_{M_R} = 2\langle x, y\rangle_R \equiv 0 \mod \mathbb{Z}$ and $x + \widetilde{R} \in D_-$. This shows that $R^\vee/\widetilde{R} \subset D_-^\perp$. Conversely let $x + \widetilde{R} \in D_-^\perp$. For $y \in R$ we have $\langle x, y\rangle_R = \frac{1}{2}\langle x, y\rangle_{M_R} = \langle x, \frac{1}{2}y\rangle_{M_R} \equiv 0 \mod \mathbb{Z}$ because $\frac{y}{2} + \widetilde{R} \in D_- = \frac{1}{2}R/\widetilde{R}$. This shows that $x \in R^\vee$.

(6) $\mathrm{O}(\widetilde{R}, \frac{1}{2}R/\widetilde{R}) = \mathrm{O}(\widetilde{R}, \frac{1}{2}R) = \mathrm{O}(\widetilde{R}, R)$. □

Let R be an ADE-lattice and Φ the set of its roots. We fix a subset $\Phi^+ \subset \Phi$ of positive roots. There exists a unique Weyl-chamber C of R (see Definition 2.5) such that for all $r \in \Phi^+$ and $c \in C$ we have $\langle r, c\rangle > 0$. We call C the *fundamental chamber*. The positive roots perpendicular to the walls of C are the so-called *simple roots*. The simple roots form a basis of R whose Dynkin diagram is of ADE-type $\tau(R)$. As before we have $\mathrm{O}(R) = W(R) \rtimes \mathrm{O}(R, C)$, where $\mathrm{O}(R, C)$ is the stabilizer of C in $\mathrm{O}(R)$. Via the action of $\mathrm{O}(R, C)$ on the vertices of the Dynkin diagram, we identify $\mathrm{O}(R, C)$ with the symmetry group $\mathrm{Aut}(\tau(R))$ of the Dynkin diagram $\tau(R)$, that is, we have

$$\mathrm{O}(R) = W(R) \rtimes \mathrm{Aut}(\tau(R)).$$

A lattice is called *irreducible* if it cannot be written as a non-trivial orthogonal sum of two sublattices. Definite lattices admit an orthogonal decomposition into irreducible sublattices which is unique up to reordering (cf. [14, 27.1]).

Lemma 3.5. *Let R be an* ADE*-lattice, and let $\mathrm{O}_0(R)$ be the kernel of the natural homomorphism $\mathrm{O}(R) \to \mathrm{O}(R^\vee/R)$. Then we have*

$$[\mathrm{O}_0(R) : W(R)] = n!,$$

where n is the number of E_8 components of $\tau(R)$.

Proof. Since reflections with respect to roots act trivially on the discriminant group, we have $W(R) \subseteq \mathrm{O}(R)_0$. Thus it suffices to compute the kernel of

$$\psi\colon \mathrm{Aut}(\tau(R)) \to \mathrm{O}(R^\vee/R).$$

If $\tau(R)$ is irreducible, a case by case analysis shows that this map is injective: indeed for A_1, E_7 and E_8, $\mathrm{Aut}(\tau(R)) = 1$; for A_k with $k \geq 1$, D_k with $k > 4$ and E_6 the group $\mathrm{Aut}(\tau(R))$ is of order two. A direct computation shows that it acts faithfully on the discriminant group.

Suppose that the root system $\tau(R)$ is reducible. The decomposition of $\tau(R)$ into connected components corresponds to a decomposition of R into an orthogonal sum of irreducible ADE-lattices, which in turn induces a corresponding decomposition of the discriminant group $R^\vee/R$. The action of $\mathrm{Aut}(\tau(R))$ preserves the three decompositions. Hence the elements of $\mathrm{Ker}\,\psi$ must preserve the components which have a non-trivial discriminant group, that is, all components which are not of type E_8. By the first part, they must act trivially on these components. Finally, since the E_8 diagram has no symmetry, the elements in the kernel act as a permutation of the connected components of $\tau(R)$ of type E_8. □

Lemma 3.6. *Let R be an* ADE*-lattice of rank at most* 10 *and $\widetilde{R}$ an* even *overlattice. Consider the homomorphism*

$$\mathrm{O}(R, \widetilde{R}) \to \mathrm{O}(R^\vee/\widetilde{R}). \tag{3.5}$$

If there is a component $\widetilde{R}_j$ of $\widetilde{R}$ with $\tau(\widetilde{R}_j) = E_8$ and $\tau(\widetilde{R}_j \cap R) = 2D_4$, then the kernel of (3.5) is $W(R) \rtimes \langle h \rangle$ where $h \in \mathrm{Aut}(\tau(R), \widetilde{R})$ is an involution. Otherwise the kernel is just the Weyl group $W(R)$.

Proof. Let $\mathrm{Aut}(\tau(R), \widetilde{R}) \leq \mathrm{Aut}(\tau(R))$ be the stabilizer of $\widetilde{R}$. Since the elements of $W(R)$ act trivially on $R^\vee/R$, they preserve $\widetilde{R}$ and

$$\mathrm{O}(\widetilde{R}, R) = W(R) \rtimes \mathrm{Aut}(\tau(R), \widetilde{R}) \leq W(R) \rtimes \mathrm{Aut}(\tau(R)).$$

The elements of $W(R)$ act trivially on the domain of $R^\vee/R \twoheadrightarrow R^\vee/\widetilde{R}$, so they lie in the kernel of (3.5). Thus it suffices to compute the kernel of

$$\varphi\colon \mathrm{Aut}(\tau(R), \widetilde{R}) \to \mathrm{O}(R^\vee/\widetilde{R}).$$

Indeed, the kernel of (3.5) is given by $W(R) \rtimes \mathrm{Ker}\,\varphi$.

First we suppose that $\tau(R)$ is irreducible. If $R = \widetilde{R}$, then $W(R) = \mathrm{O}_0(R)$ by Lemma 3.5, and hence φ is injective. Otherwise (as $\mathrm{rank}\, R \leq 10$) the pair $(\tau(R), \tau(\widetilde{R})) \in \{(A_7, E_7), (A_8, E_8), (D_8, E_8)\}$. Suppose we are in the case (A_7, E_7).

Then $R^\vee/R \cong \mathbb{Z}/8\mathbb{Z}$ and $\widetilde{R}/R = 4(R^\vee/R)$. Then $\mathrm{Aut}(\tau(R))$ is of order two and acts as ± 1 on $R^\vee/R$ which is non-trivial in $R^\vee/\widetilde{R} \cong \mathbb{Z}/4\mathbb{Z}$. A similar argument applies to (A_8, E_8). Finally the symmetry of the D_8 diagram exchanges the two isotropic vectors of its discriminant. In particular it does not fix any non-trivial even overlattice which implies that $\mathrm{Aut}(\tau(R), \widetilde{R}) = 1$ in the (D_8, E_8) case. In any case φ is injective.

Now suppose that $R = \bigoplus R_i$ has several irreducible components R_i and let $h \in \mathrm{Ker}\,\varphi$. Note that h preserves the decomposition $R^\vee = \bigoplus R_i^\vee$. Let $x \in R_i^\vee$ be a non-zero element.

If x^h lies in the same component $R_i^\vee$ as x, then h must preserve it. Hence we may restrict h to this component and the previous paragraph yields $x^h = x$.

If x and x^h lie in different components $R_i^\vee$ and $R_j^\vee$, then these components are isomorphic and $q(x^h - x) = q(x^h) + q(x) = 2q(x)$. Since $h \in \mathrm{Ker}\,\varphi$, we have $x^h - x \in \widetilde{R}$ Further $\widetilde{R}/R$ is totally isotropic with respect to the discriminant form. Thus $q(x^h - x) = 2q(x) \equiv 0 \mod 2\mathbb{Z}$, i.e. $q(x) \equiv 0 \mod \mathbb{Z}$. If y is any non-trivial element of $R_i^\vee$, then x^h and y^h lie in the same connected component $R_j^\vee$ and the same reasoning applies. In particular

$$\forall y \in R_i^\vee : q(y) \equiv 0 \mod \mathbb{Z}$$

which implies that R_i is 2-elementary and q_{R_i} has values in $\mathbb{Z}/2\mathbb{Z}$. Under the constraint $\operatorname{rank} R \leq 10$, this is possible only if $\tau(R_i) = \tau(R_j) = D_4$. To sum up φ is injective, except possibly if $\tau(R)$ has two D_4 components. We analyse this case in detail.

We may assume that $R = R_1 \oplus R_2$ is of type $2D_4$ and $\tilde{R}$ an overlattice of R. If $\widetilde{R} = R$, then φ is injective by Lemma 3.5. Hence we may further assume that $R \subsetneq \widetilde{R}$. Suppose there exists a non-trivial element h in the kernel of φ. By the previous part this implies that $R_1^h = R_2$.

Let e_1, e_2, e_3, e_4 be the simple roots of R_1 with e_4 giving the central vertex of the Dynkin diagram of type D_4, i.e. $\langle e_4, e_i \rangle = 1$ for $i = 1, 2, 3$. Let $(e_1^\vee, \dots e_4^\vee) \in R_1^\vee$ be the dual basis. The four elements of $R_1^\vee/R_1$ are represented by $e_1^\vee, e_2^\vee, e_3^\vee$ and $e_4^\vee$ representing 0. Set $f_i = e_i^h \in R_2$. Then $f_i^h = e_{\sigma(i)}$ for some permutation $\sigma \in S_4$ with $\sigma(4) = 4$. Since $h \in \mathrm{Ker}\,\varphi$, we have $t_i := e_i^\vee - f_i^\vee \in \widetilde{R}$ for $i \in \{1, 2, 3\}$. Now the cosets of 0, t_1, t_2 and t_3 constitute a maximal totally isotropic subspace of $R^\vee/R$ contained in $\widetilde{R}/R$. Since $\widetilde{R}/R$ is totally isotropic as well, the subspaces must be equal. We conclude that $\tau(\widetilde{R}) = E_8$. By the same reasoning we have $f_i^\vee - e_{\sigma(i)}^\vee \in \widetilde{R}$. As $\widetilde{R}/R$ has only four elements, this is possible only if $\sigma = 1$. Hence h is an involution and uniquely determined by $\widetilde{R}/R$. This shows that the kernel of φ is of order 2. □

Lemma 3.7. *Let $\widetilde{R}$ be an* ADE*-lattice and Φ^+ the set of its positive roots. Then the natural map $\Phi^+ \to \widetilde{R}/2\widetilde{R}$ is injective.*

Proof. We may assume that $\widetilde{R}$ is irreducible. In what follows we explicitly compute $\eta\colon \Phi^+ \to \widetilde{R}/2\widetilde{R}$ for each case using classical constructions of the ADE-lattices (see e.g. [10, Theorem 1.2]).

Let $(\epsilon_1,\dots,\epsilon_{n+1})$ be the standard basis of $\mathbb{Z}^{n+1}$. The $n(n+1)$ roots of the lattice

$$A_n = \left\{(x_i) \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\right\}$$

are given by

$$\Phi(A_n) = \{\alpha_{ij} = \epsilon_i - \epsilon_j \mid 1 \le i \neq j \le n+1\}.$$

Suppose that $\alpha_{ij} \equiv \alpha_{lk} \mod 2A_n \subseteq 2\mathbb{Z}^{n+1}$. Then we have that $\epsilon_i - \epsilon_j + \epsilon_k - \epsilon_l \equiv 0 \mod 2\mathbb{Z}^{n+1}$. This is possible only if each standard basis vector appears twice, i.e. $(i,j) = (k,l)$ or $(i,j) = (l,k)$ which means that $\alpha_{ij} = \pm\alpha_{lk}$. Since either $\alpha_{lk} \in \Phi^+$ or $-\alpha_{lk} \in \Phi^+$, the map η is injective.

Let $(\epsilon_1,\dots,\epsilon_n)$ be the standard basis of $\mathbb{Z}^n$, $n \ge 4$. The $2n(n-1)$ roots of the lattice

$$D_n = \left\{(x_i) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \equiv 0 \mod 2\right\}$$

are given by $\pm(\epsilon_i + \epsilon_j)$ and $\pm(\epsilon_i - \epsilon_j)$ for $1 \le i < j \le n$. Suppose that $\pm\epsilon_i \pm \epsilon_j \equiv \pm\epsilon_k \pm \epsilon_l \mod 2D_n$. As before this implies that $\{i,j\} = \{k,l\}$. Since

$$(\epsilon_i + \epsilon_j) - (\epsilon_i - \epsilon_j) = 2\epsilon_j \notin 2D_4,$$

the map η is injective. We leave the exceptional cases E_6, E_7, E_8 to the reader. □

Lemma 3.8. *Let $\widetilde{R} = \bigoplus_{j\in J} \widetilde{R}_j$ be an* ADE-*lattice with $\widetilde{R}_j$ irreducible. Then the kernel of the natural homomorphism*

$$\psi \colon \mathrm{O}(\widetilde{R}) = \mathrm{O}(\widetilde{R}(2)) \to \mathrm{O}(\tfrac{1}{2}\widetilde{R}^\vee/\widetilde{R}),$$

where $\frac{1}{2}\widetilde{R}^\vee/\widetilde{R}$ is the discriminant form of $\widetilde{R}(2)$, is generated by the elements $\oplus_{j\in J} g_j$ with $g_j = \pm 1_{\widetilde{R}_j}$ if $\widetilde{R}_j$ is unimodular and $g_j = 1_{\widetilde{R}_j}$ otherwise.

Proof. We identify $\frac{1}{2}\widetilde{R}^\vee/\widetilde{R}$ and $\widetilde{R}^\vee/2\widetilde{R}$. Let $g \in \operatorname{Ker}\psi$. Since $\widetilde{R} \subseteq \widetilde{R}^\vee$, g acts trivially on $\widetilde{R}^\vee/2\widetilde{R}^\vee$. The action of $\mathrm{O}(\widetilde{R})$ preserves the decomposition $\widetilde{R} = \bigoplus_{j\in J}\widetilde{R}_j$. In particular g acts on the set J. As $\widetilde{R}^\vee/2\widetilde{R}^\vee = \bigoplus_{j\in J}\widetilde{R}_j^\vee/2\widetilde{R}_j^\vee$ and g is in $\operatorname{Ker}\psi$ we have $j^g = j$. Hence g must fix each connected component of $\widetilde{R}$ and we may and will assume that $\widetilde{R}$ is irreducible.

We tensor the perfect pairing $\widetilde{R}^\vee \times \widetilde{R} \to \mathbb{Z}$ with $\mathbb{F}_2$, to obtain a perfect pairing $\widetilde{R}^\vee/2\widetilde{R}^\vee \times \widetilde{R}/2\widetilde{R} \to \mathbb{F}_2$. Since g acts trivially on the first factor, so does it on the second factor $\widetilde{R}/2\widetilde{R}$. By Lemma 3.7 $\Phi(\widetilde{R})/\{\pm 1\} \cong \Phi^+(\widetilde{R})$ injects into $\widetilde{R}/2\widetilde{R}$, which implies that $g(r) = \pm r$ for every root $r \in \Phi(\widetilde{R})$. As any simple root system of $\widetilde{R}$ is connected, the sign is the same for each simple root. Since the simple roots form a basis, $g = \pm 1$.

Set $\widetilde{R}_\pm = \operatorname{Ker}(g \mp 1) \subset \widetilde{R}$. We apply Proposition 2.1 to the primitive extension

$$\widetilde{R}_+(2) \oplus \widetilde{R}_-(2) \subseteq \widetilde{R}(2).$$

Since g acts trivially on the discriminant group $\frac{1}{2}\widetilde{R}^\vee/\widetilde{R}$ of $\widetilde{R}(2)$, the implication

$$g|_{\frac{1}{2}\widetilde{R}_+^\vee/\widetilde{R}_+} = 1 \implies g|_{\frac{1}{2}\widetilde{R}_-^\vee/\widetilde{R}_-} = 1 \tag{3.6}$$

holds. By definition $g|_{\widetilde{R}_-} = -1_{\widetilde{R}_-}$ and then by the right hand side of (3.6), the lattice $\widetilde{R}_-(2)$ must be 2-elementary, i.e. $\widetilde{R}_-$ is unimodular. In particular $\widetilde{R}_-$ is a direct summand of $\widetilde{R}$. But we assumed the latter to be irreducible, so that $\widetilde{R} \in \{0, \widetilde{R}_-\}$. Thus $g = \pm 1$ if $\widetilde{R}$ is unimodular and $g = 1$ else. □

Remark 3.9. Let R be an irreducible ADE-lattice. By [10, Proposition 1.5], we have $-1 \in W(R)$ if and only if R contains $\operatorname{rank} R$ pairwise orthogonal roots, if and only if $\tau(R)$ is one of A_n $(n \geq 1)$, D_n $(n \geq 4,\ n$ even$)$, E_7, E_8.

Theorem 3.10. *Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface, and let R, $\overline{R}$, $\widetilde{R}$ be as in Table 1.1. Let $\widetilde{R} = \bigoplus_j \widetilde{R}_j$ be the decomposition into irreducible components. Then we have*

$$|\overline{G}_{X-}| = |W(R)|\,\frac{d_{(\tau,\bar{\tau})}}{e_{(\tau,\bar{\tau})}},$$

where $d_{(\tau,\bar{\tau})}, e_{(\tau,\bar{\tau})}$ are given as follows.

$$d_{(\tau,\bar{\tau})} := \begin{cases} 2 & \exists\, j \text{ such that } \tau(\widetilde{R}_j) = E_8 \text{ and } \tau(\widetilde{R}_j \cap R) = 2D_4, \\ 1 & \text{otherwise,} \end{cases}$$

$$e_{(\tau,\bar{\tau})} := \begin{cases} 2 & \exists\, j \text{ such that } \tau(\widetilde{R}_j) = E_8 \text{ and } \widetilde{R}_j \cap R \text{ contains } 8 \text{ orthogonal roots,} \\ 1 & \text{otherwise.} \end{cases}$$

Hence the value of $c_{(\tau,\bar{\tau})}$ in Table 1.1 is equal to $e_{(\tau,\bar{\tau})}/d_{(\tau,\bar{\tau})}$.

Proof. By Theorem 3.3 and Lemma 3.4, we have

$$G_{X-} = \operatorname{Ker}(\mathrm{O}(\widetilde{R}, R) \to \mathrm{O}(R^\vee/\widetilde{R})),$$

which, by Lemma 3.6, is given by $W(R)$, or by $W(R) \rtimes \langle h \rangle$ for some involution $h \in \operatorname{Aut}(\tau(R), \widetilde{R})$ if there is some component $\widetilde{R}_j$ with $\tau(\widetilde{R}_j) = E_8$ and $\tau(\widetilde{R}_j \cap R) = 2D_4$. Consider the natural homomorphism $\psi\colon \mathrm{O}(\widetilde{R}) \to \mathrm{O}(\frac{1}{2}\widetilde{R}^\vee/\widetilde{R})$ in Lemma 3.8. By our dictionary in Lemma 3.4, we have $\overline{G}_{X-} = \psi(G_{X-})$. By Lemma 3.8, the kernel of ψ consists of those $g = \oplus_{j\in J} g_j$ with $g_j = \pm 1_{\widetilde{R}_j}$ if $\widetilde{R}_j$ is unimodular and $g_j = 1_{\widetilde{R}_j}$ else. Further $|\operatorname{Ker}\psi \cap W(R)|$ consists of those g with $g_j = \pm 1$ if $\widetilde{R}_j$ is unimodular and $-1 \in W(R \cap \widetilde{R}_j)$ and $g_j = 1$ else. Now Remark 3.9 yields the condition for $e_{(\tau,\bar{\tau})}$. Since the $g_j = \pm 1$ do not preserve any positive root system, the involution h is not in $\operatorname{Ker}\psi$. This explains the presence of $d_{(\tau,\bar{\tau})}$. □

Remark 3.11. The contribution $e_{(\tau,\bar{\tau})}$ is due to the presence of semi-symplectic numerically trivial automorphisms.

4. Borcherds' method

4.1. **An algorithm on a graph.** The algorithms to prove our main results are variations of the following computational procedure.

Let (V, E) be a simple non-oriented connected graph, where V is the set of vertices and E is the set of edges, which is a set of non-ordered pairs of distinct elements of V. The set V may be infinite. Suppose that a group G acts on (V, E) from the right. We assume the following.

(VE-1) For any vertex $v \in V$, the set $\{\, v' \in V \mid \{v, v'\} \in E \,\}$ of vertices adjacent to v is finite and can be calculated effectively.

(VE-2) For any vertices $v, v' \in V$, we can determine effectively whether the set

$$T_G(v, v') := \{\, g \in G \mid v^g = v' \,\} \tag{4.1}$$

is empty or not, and when it is non-empty, we can calculate an element of $T_G(v, v')$.

(VE-3) For any $v \in V$, the stabilizer subgroup $T_G(v, v)$ of v in G is finitely generated, and a finite set of generators of $T_G(v, v)$ can be calculated effectively.

We define the *G-equivalence relation* $\sim$ on V by

$$v \sim v' \iff T_G(v, v') \neq \emptyset.$$

Suppose that V_0 is a non-empty finite subset of V with the following properties.

(V_0-1) If $v, v' \in V_0$ are distinct, then $v \not\sim v'$.

(V_0-2) We put $\widetilde{V}_0 := \{\, v \in V \mid v \text{ is adjacent to a vertex belonging to } V_0 \,\}$. Then, for each $v \in \widetilde{V}_0$, there exists a vertex $v' \in V_0$ such that $v \sim v'$.

For each $v \in \widetilde{V}_0$, we choose an element $h(v) \in T_G(v, v')$, where v' is the unique vertex in V_0 such that $v \sim v'$, and put

$$\mathcal{H} := \{\, h(v) \mid v \in \widetilde{V}_0 \,\}.$$

We fix an element $v_0 \in V_0$.

Proposition 4.1. *The natural mapping*

$$V_0 \hookrightarrow V \twoheadrightarrow V/\!\sim\, = V/G \tag{4.2}$$

is a bijection, and the group G is generated by the union of $T_G(v_0, v_0)$ and $\mathcal{H}$.

Proof. Let $\langle \mathcal{H} \rangle$ be the subgroup of G generated by $\mathcal{H}$. First we prove that, for any $v \in V$, there exists an element $h \in \langle \mathcal{H} \rangle$ such that $v^h \in V_0$. Let an element $v \in V$ be fixed. A sequence

$$v_{(0)}, v_{(1)}, \dots, v_{(l)} \tag{4.3}$$

of vertices is said to be a *path from V_0 to $v^{\langle \mathcal{H} \rangle}$* if $v_{(i-1)}$ and $v_{(i)}$ are adjacent for $i = 1, \dots, l$, the starting vertex $v_{(0)}$ is in V_0, and the ending vertex $v_{(l)}$ belongs to the orbit $v^{\langle \mathcal{H} \rangle}$ of the fixed vertex v under the action of $\langle \mathcal{H} \rangle$. Since (V, E) is connected and V_0 is non-empty, there exists at least one path from V_0 to $v^{\langle \mathcal{H} \rangle}$. Suppose that the sequence (4.3) is a path from V_0 to $v^{\langle \mathcal{H} \rangle}$ of length $l > 0$. Since $v_{(1)}$ is adjacent to the vertex $v_{(0)}$ in V_0, we have $v_{(1)} \in \widetilde{V}_0$ and there exists an element $h_1 := h(v_{(1)}) \in \mathcal{H}$ that maps $v_{(1)}$ to an element of V_0. Then

$$v_{(1)}^{h_1}, \dots, v_{(l)}^{h_1}$$

is a path from V_0 to $v^{\langle \mathcal{H} \rangle}$ of length $l - 1$. Thus we obtain a path from V_0 to $v^{\langle \mathcal{H} \rangle}$ of length 0, which implies the claim.

The injectivity of (4.2) follows from property (V_0-1) of V_0. The surjectivity follows from the claim above. Suppose that $g \in G$. By the claim, there exists an element $h \in \langle \mathcal{H} \rangle$ such that $v_0^{gh} \in V_0$. By property (V_0-1) of V_0, we have $v_0 = v_0^{gh}$ and hence $gh \in T_G(v_0, v_0)$. Therefore G is generated by the union of $\mathcal{H}$ and $T_G(v_0, v_0)$. □

To obtain V_0 and $\mathcal{H}$, we employ Procedure 4.1. This procedure terminates if and only if $|V/G| < \infty$.

Initialize $V_0 := [v_0]$, $\mathcal{H} := \{\}$, and $i := 0$.
while $i < |V_0|$ **do**
  Let v_i be the $(i+1)$st entry of the list V_0.
  Let $\mathcal{A}(v_i)$ be the set of vertices adjacent to v_i.
  for each vertex v' in $\mathcal{A}(v_i)$ **do**
    Set `flag` := true.
    for each v'' in V_0 **do**
      if $T_G(v', v'') \neq \emptyset$ **then**
        Add an element h of $T_G(v', v'')$ to $\mathcal{H}$.
        Replace `flag` by false.
        Break from the innermost for–loop.
    if `flag` = true **then**
      Append v' to the list V_0 as the last entry.
  Replace i by $i + 1$.

PROCEDURE 4.1. A computational procedure on a graph

4.2. 17 **primitive embeddings.** Recall that L_{26} is an even unimodular hyperbolic lattice of rank 26. The L_{26}-chamber (that is, the standard fundamental domain of $W(L_{26})$) was studied by Conway [7]. He constructed a bijection between the set of walls of an L_{26}-chamber D and the set of vectors of the Leech lattice, and showed that the automorphism group $\mathrm{O}(L_{26}, D)$ of D is isomorphic to the group of *affine* isometries of the Leech lattice. Using this result, Borcherds [4], [5] developed a method to calculate the orthogonal group of an even hyperbolic lattice S by embedding S primitively into L_{26} and investigating the tessellation of an S-chamber (that is, a standard fundamental domain of $W(S)$) by L_{26}/S-chambers.

In [6], we apply this method to $S = L_{10}(2)$. We fix positive half-cones $\mathcal{P}_{10}$ of L_{10} and $\mathcal{P}_{26}$ of L_{26}. In [6], we have proved the following.

Theorem 4.2 ([6])**.** *Up to the action of* $\mathrm{O}(L_{10})$ *and* $\mathrm{O}(L_{26})$*, there exist exactly* 17 *primitive embeddings of* $L_{10}(2)$ *into* L_{26}*.* □

These 17 primitive embeddings of $L_{10}(2)$ into L_{26} are named as

`12A`, `12B`, `20A`, ... , `20F`, `40A`, ... , `40E`, `96A`, `96B`, `96C`, `infty`.

Recall the notion of being reflexively simple from Definition 2.9.

Theorem 4.3 ([6])**.** *Suppose that a primitive embedding* $L_{10}(2) \hookrightarrow L_{26}$ *is not of type* `infty`*, Then each* $L_{26}/L_{10}(2)$*-chamber has only finitely many walls, and they are defined by roots of* L_{10}*. Moreover the tessellation of* $\mathcal{P}_{10}$ *by* $L_{26}/L_{10}(2)$*-chambers is reflexively simple.* □

The explicit description of the 17 primitive embeddings and $L_{26}/L_{10}(2)$-chambers is given in [6] and [30]. From these data, we see the following. Let $L_{10}(2) \hookrightarrow L_{26}$ be a primitive embedding whose type is not `infty`, and D an $L_{26}/L_{10}(2)$-chamber. The automorphism group of D is denoted by

$$\mathrm{O}(L_{10}, D) := \{\, g \in \mathrm{O}^{\mathcal{P}}(L_{10}) \mid D^g = D \,\}.$$

Since the walls of D are defined by roots of L_{10}, the chamber D is tessellated by Vinberg chambers. The volume of D is defined by

$$\mathrm{vol}(D) := \text{the number of Vinberg chambers contained in } D.$$

Let f be a face of D with codimension k. Then the defining roots of the walls of D containing f form a configuration whose dual graph is a Dynkin diagram of an ADE-type. The ADE-*type of* f is the ADE-type of this Dynkin diagram. The closure $\overline{D}$ of D in $S_X \otimes \mathbb{R}$ contains only a finite number of isotropic rays. Let $v \in S_X \cap \overline{D}$ be a primitive isotropic ray (see Section 2.3). Then the defining roots r of walls of D such that $\langle r, v\rangle = 0$ form a configuration whose dual graph is a Dynkin diagram of an *affine* ADE-type. The *affine* ADE-*type* of the isotropic ray $\mathbb{R}_{>0}v$ is the affine ADE-type of this Dynkin diagram.

Example 4.4. Let $L_{10}(2) \hookrightarrow L_{26}$ be the primitive embedding of type `96C`, and D_0 an $L_{26}/L_{10}(2)$-chamber. Then D_0 has exactly 96 walls. The group $\mathrm{O}(L_{10}, D_0)$ is of order $110592 = 2^{12} \cdot 3^3$, and this group acts on the set of walls of D_0 transitively. We have

$$\mathrm{vol}(D_0) = \frac{1_{\mathrm{BP}}}{72} = 652758220800.$$

The $L_{26}/L_{10}(2)$-chamber D_0 has $1728+768+144$ faces of codimension 2, which are decomposed into orbits of size 1728, 768, 144 under the action of $\mathrm{O}(L_{10}, D_0)$. Hence each wall of D_0 is bounded by $36 + 16 + 3 = 55$ faces of codimension 2 of D_0. The ADE-types of faces in these orbits are $2A_1$, $2A_1$, A_2, respectively. The $L_{26}/L_{10}(2)$-chamber D_0 has $18 + 256 + 256 + 864$ isotropic rays, which are decomposed into orbits of size 18, 256, 256, 864 by the action of $\mathrm{O}(L_{10}, D_0)$. The affine ADE-types of isotropic rays of these orbits are $8A_1$, $4A_2$, $4A_2$, $2A_1 + 2A_3$, respectively.

4.3. **Constructing S_X.** Let Y be an Enriques surface with the universal covering $\pi\colon X \to Y$. We consider the following assumption:

(4.4) we have a primitive embedding $S_X \hookrightarrow L_{26}$ such that the composite $S_Y(2) \cong L_{10}(2) \hookrightarrow L_{26}$ of $\pi^*\colon S_Y(2) \hookrightarrow S_X$ and $S_X \hookrightarrow L_{26}$ is not of type `infty`, and we have the list of walls of an $L_{26}/S_Y(2)$-chamber D_0 that is contained in Nef_Y.

Suppose that (4.4) holds. Then $\mathcal{P}_Y$ has the following three tessellations, each of which is a refinement of the one below.

- by Vinberg chambers,
- by $L_{26}/S_Y(2)$-chambers, each of which has only finite number of walls, and
- by $S_X/S_Y(2)$-chambers, one of which is Nef_Y.

The tessellation of Nef_Y by $L_{26}/S_Y(2)$-chambers is very useful in analyzing Nef_Y. Recall that $G_Y \subset \mathrm{O}^{\mathcal{P}}(S_Y)$ is the image of the projection of $G_X \subset \mathrm{O}^{\mathcal{P}}(S_X)$ defined by (3.2).

Proposition 4.5. *Suppose that Y satisfies* (3.1) *and* (4.4)*. Then the action of G_Y on $\mathcal{P}_Y$ preserves the tessellation of $\mathcal{P}_Y$ by $L_{26}/S_Y(2)$-chambers. In particular, the action of* $\mathrm{aut}(Y)$ *on* Nef_Y *preserves the tessellation of* Nef_Y *by $L_{26}/S_Y(2)$-chambers.*

Proof. It is enough to prove that the action of $\tilde{g} \in G_X$ on $\mathcal{P}_X$ preserves the tessellation of $\mathcal{P}_X$ by L_{26}/S_X-chambers. Let id_P be the identity of the orthogonal complement P of S_X in L_{26}. Since the action of $\tilde{g}$ on $S_X^\vee/S_X$ is 1, the action of $(\tilde{g}, \mathrm{id}_P)$ on $S_X \oplus P$ preserves the even unimodular overlattice L_{26} of $S_X \oplus P$. Thus $\tilde{g}$ extends to an isometry of L_{26}, and hence its action on $\mathcal{P}_X$ preserves the L_{26}/S_X-chambers. The second assertion follows from the fact that $\mathrm{aut}(Y)$ is the stabilizer subgroup of Nef_Y in G_Y. $\square$

The purpose of this section is to construct a primitive embedding $S_X \hookrightarrow L_{26}$ for a $(\tau, \bar{\tau})$-generic Enriques surface Y, so that we can assume (4.4). We start from a primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ whose type is not `infty` and which has a fixed $L_{26}/L_{10}(2)$-chamber D_0, and then proceed to the construction of S_X between $L_{10}(2) \cong S_Y(2)$ and L_{26} such that the inclusion of $L_{10}(2) \cong S_Y(2)$ into S_X is the embedding π^*, and that the fixed $L_{26}/L_{10}(2)$-chamber D_0 is contained in Nef_Y.

Recall that, for a $(\tau, \bar{\tau})$-generic Enriques surface Y, the lattice S_X is obtained from $S_Y(2)$ by adding roots of the form $(r+v)/2$, where r is a root of S_Y and v is a (-4)-vector in S_{X-}. To find roots in L_{26} that yield an appropriate extension from $S_Y(2)$ to S_X, we search for pairs $\alpha = (r, v)$ of a root r of L_{10} defining a wall of D_0 and a (-4)-vector v of Q_ι such that $(r+v)/2$ is in L_{26}, where Q_ι is the orthogonal complement of $L_{10}(2)$ in L_{26}. For a finite set $p = \{\alpha_1, \dots, \alpha_m\}$ of such pairs, we consider the sublattice M_p of L_{26} generated by $L_{10}(2)$ and the roots $(r_1+v_1)/2, \dots, (r_m+v_m)/2$ of L_{26}, where $\alpha_i = (r_i, v_i)$. Suppose that $p = \{\alpha_1, \dots, \alpha_m\}$ satisfies the following:

(i) The dual graph of $r_1, \dots, r_m$ is a Dynkin diagram of some ADE-type τ. By Proposition 1.2, the primitive closure $\overline{R}$ of the ADE-sublattice R of L_{10} generated by $r_1, \dots, r_m$ is also an ADE-sublattice of L_{10}. Let $\bar{\tau}$ denote the ADE-type of $\overline{R}$.

By Proposition 1.2, the embedding $L_{10}(2) \hookrightarrow M_p$ is isomorphic to $L_{10}(2) \hookrightarrow M_R$, and hence, by Proposition 1.3, we see that $L_{10}(2)$ is a primitive sublattice of M_p, and the orthogonal complement of $L_{10}(2)$ in M_p contains no roots. We consider the following condition:

(ii) M_p can be embedded primitively into the $K3$ lattice (an even unimodular lattice of rank 22 with signature $(3, 19)$). This condition is checked by calculating the discriminant form of M_p and applying the theory of genera (see [22]).

Suppose that M_p satisfies condition (ii). Since $22 - \operatorname{rank} M_p = 12 - m > 2$, the surjectivity of the period mapping of complex $K3$ surfaces ([3, Chapter VIII]) implies that there exists a $K3$ surface X with $M_p \cong S_X$ such that $\mathrm{O}(T_X, \omega) = \{\pm 1\}$. Moreover, by [13], the $K3$ surface X has a fixed point free involution ε with the quotient morphism $\pi\colon X \to Y = X/\langle\varepsilon\rangle$ to the Enriques surface Y such that, under suitable choices of isometries $M_p \cong S_X$, the embedding $L_{10}(2) \hookrightarrow M_p$ is identified with $\pi^*\colon S_Y(2) \hookrightarrow S_X$. By the construction of M_p, this Enriques surface Y is $(\tau, \bar{\tau})$-generic. Thanks to Proposition 3.1, we can further assume that D_0 is contained in Nef_Y by changing the isometry $M_p \cong S_X$.

Except for the type $(\tau, \bar{\tau})$ of Nos 88 and 146, we can find a set $p = \{\alpha_1, \dots, \alpha_m\}$ satisfying condition (i) above using the primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ given in the 8th column (`irec`) of Table 1.1. If the 5th column (`exist`) is not marked by $\times$, then M_p satisfies condition (ii).

Example 4.6. Let $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ be the primitive embedding of type `96C` (see Example 4.4). Then the even negative definite lattice Q_ι contains 2208 vectors v of square-norm -4, and we have 192 pairs $\alpha = (r, v)$ such that $(r+v)/2 \in L_{26}$. Choosing appropriate subsets from these 192 pairs, we can construct S_X for many types $(\tau, \bar{\tau})$ (Nos. 1, 2, ...).

Remark 4.7. Even when M_p does not satisfy condition (ii), we can use M_p as the Néron-Severi lattice S_X of a "non-existing $K3$ surface" X and run the geometric algorithms below.

5. Geometric algorithms

We prepare some algorithms that will be used in the application of the generalized Borcherds' method to geometric situations.

Let Y be an Enriques surface with the universal covering $\pi\colon X \to Y$. We assume (3.1) and (4.4). First we prepare the following computational data:

(i) an integral interior point $a_{Y0} \in S_Y$ of D_0, which is an ample class of Y,
(ii) the list of roots defining the walls of D_0,
(iii) the finite group $\mathrm{O}^{\mathcal{P}}(S_Y, D_0) = \{ g \in \mathrm{O}^{\mathcal{P}}(S_Y) \mid D_0^g = D_0 \}$,
(iv) the finite group $\mathrm{O}(S_{X-})$, and
(v) the list of (-4)-vectors of S_{X-}.

5.1. **Separating roots.**

Definition 5.1. Let L be an even hyperbolic lattice with a positive half-cone $\mathcal{P}$, and let a_1, a_2 be elements of $\mathcal{P} \cap L$. We say that a hyperplane $(v)^\perp$ of $\mathcal{P}$ *separates* a_1 *and* a_2 if $\langle v, a_1\rangle$ and $\langle v, a_2\rangle$ are non-zero and have different signs. We say that a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v\rangle < 0$ *separates* a_1 *and* a_2 if $(v)^\perp$ separates a_1 and a_2.

By an algorithm given in [27], we can calculate, for any $a_1, a_2 \in \mathcal{P} \cap L$, the set of roots of L that separate a_1 and a_2.

5.2. **Splitting roots.**

Definition 5.2. We say that a root r of S_Y *splits in* S_X if there exists a root $\tilde{r}$ of S_X such that $\pi^*(r) = \tilde{r} + \tilde{r}^\varepsilon$.

A root r of S_Y splits in S_X if and only if there exists a (-4)-vector v of S_{X-} such that $(\pi^*(r) + v)/2 \in S_X$. Hence we can effectively determine whether a given root r of S_Y splits in S_X or not. Moreover, when r splits, we can calculate the roots $\tilde{r} = (\pi^*(r) + v)/2$ and $\tilde{r}^\varepsilon = (\pi^*(r) - v)/2$ of S_X such that $\pi^*(r) = \tilde{r} + \tilde{r}^\varepsilon$.

Suppose that a root r of S_Y satisfies that $\mathrm{Nef}_Y \cap (r)^\perp$ contains a non-empty open subset of $(r)^\perp$ and that $\langle r, a_Y\rangle > 0$ for an ample class a_Y of Y. Then the following are equivalent:

- $\mathrm{Nef}_Y \cap (r)^\perp$ is a wall of Nef_Y (that is, the hyperplane $(r)^\perp$ is disjoint from the interior of Nef_Y),
- r splits in S_X, and
- r is the class of a smooth rational curve C on Y.

In this case, the roots $\tilde{r}$ and $\tilde{r}^\varepsilon$ of S_X are the classes of the smooth rational curves $\widetilde{C}$ and $\widetilde{C}^\varepsilon$ on X such that $\pi^{-1}(C) = \widetilde{C} + \widetilde{C}^\varepsilon$.

5.3. **Membership criterion of G_Y in $\mathrm{O}^{\mathcal{P}}(S_Y)$.** An element g of $\mathrm{O}^{\mathcal{P}}(S_Y)$ belongs to G_Y if and only if there exists an isometry $h \in \mathrm{O}(S_{X-})$ such that the action of (g, h) on $S_{X+} \oplus S_{X-}$ preserves the overlattice S_X and that $\tilde{g} := (g, h)|S_X$ acts on $S_X^\vee / S_X$ trivially. Since we have the list of elements of the finite group $\mathrm{O}(S_{X-})$, we can determine whether an element $g \in \mathrm{O}^{\mathcal{P}}(S_Y)$ belongs to G_Y or not, and if $g \in \mathrm{O}^{\mathcal{P}}(S_Y)$, we can calculate a lift $\tilde{g} \in G_X$ of g.

5.4. **Membership criterion of $\mathrm{aut}(Y)$ in G_Y.** Suppose that $g \in G_Y$, and let $\tilde{g} \in G_X$ be a lift of g. Recall from Proposition 3.2 that g belongs to $\mathrm{aut}(Y)$ if and only if g preserves Nef_Y, or equivalently $\tilde{g}$ preserves Nef_X. Hence $g \in \mathrm{aut}(Y)$ holds if and only if one of the following conditions that are mutually equivalent is satisfied:

- For any ample classes a_X and a'_X of X, there exist no root of S_X separating $a_X^{\tilde{g}}$ and a'_X.
- For any ample classes a_Y and a'_Y of Y, any roots of S_Y separating a_Y^g and a'_Y does not split in S_X.
- There exist ample classes a_X and a'_X of X such that there exist no roots of S_X separating $a_X^{\tilde{g}}$ and a'_X.
- There exist ample classes a_Y and a'_Y of Y such that any root of S_Y separating a_Y^g and a'_Y does not split in S_X.

Thus we can determine effectively whether a given isometry $g \in G_Y$ belongs to $\mathrm{aut}(Y)$ or not, because we have at least one ample class a_{Y0} of Y.

5.5. **Criterion for $\mathrm{aut}(Y)$-equivalence.** Recall from Theorem 4.3 that, for every $L_{26}/S_Y(2)$-chamber D, we have an isometry $g \in \mathrm{O}^{\mathcal{P}}(S_Y)$ such that $D = D_0^g$. Let D_1 and D_2 be $L_{26}/S_Y(2)$-chambers. Suppose that we have isometries $g_1, g_2 \in \mathrm{O}^{\mathcal{P}}(S_Y)$ such that $D_1 = D_0^{g_1}$ and $D_2 = D_0^{g_2}$. Then the set

$$\mathrm{isoms}(D_1, D_2) := \{\, g \in \mathrm{O}^{\mathcal{P}}(S_Y) \mid D_1^g = D_2 \,\} = g_1^{-1} \cdot \mathrm{O}^{\mathcal{P}}(S_Y, D_0) \cdot g_2$$

is finite, and can be explicitly calculated. Therefore we can calculate the set

$$\mathrm{isoms}(Y, D_1, D_2) := \mathrm{aut}(Y) \cap \mathrm{isoms}(D_1, D_2)$$

explicitly, and in particular, we can calculate the group $\mathrm{aut}(Y, D) := \mathrm{isoms}(Y, D, D)$ for an $L_{26}/S_Y(2)$-chamber D.

6. Proofs of main theorems

We present algorithms that prove Theorems 1.16 and 1.18. Let Y be an Enriques surface with the universal covering $\pi\colon X \to Y$. Suppose that Y is $(\tau, \bar{\tau})$-generic, where $(\tau, \bar{\tau})$ is *not* equal to No. 88 nor No. 146 in Table 1.1, so that we can assume (3.1) and (4.4).

6.1. **Generators of $\mathrm{aut}(Y)$ and representatives of $\mathrm{Nef}_Y/\mathrm{aut}(Y)$.** We calculate a finite generating set of $\mathrm{aut}(Y)$ and a complete set of representatives of $\mathrm{Nef}_Y/\mathrm{aut}(Y)$. This calculation affirms Theorem 1.15 computationally. Moreover the results will be used in the proofs of Theorems 1.16 and 1.18 below.

Let (V, E) be the graph where V is the set of $L_{26}/S_Y(2)$-chambers contained in Nef_Y and E is defined by the adjacency relation of $L_{26}/S_Y(2)$-chambers. Let G be the group $\mathrm{aut}(Y)$, and let $v_0 \in V$ be the $L_{26}/S_Y(2)$-chamber D_0 in Nef_Y. Let $D = D_0^g$ be an $L_{26}/S_Y(2)$-chamber contained in Nef_Y, where $g \in \mathrm{O}^{\mathcal{P}}(S_Y)$. Then we can calculate the set of roots defining the walls of D by mapping the set of roots defining the walls of D_0 by the isometry g. For each root r defining a wall of D, the chamber $D^{s_r} = D_0^{g s_r}$ adjacent to D across the wall $D \cap (r)^{\perp}$ of D is contained in Nef_Y if and only if r does *not* split in S_X. Therefore we can determine $D^{s_r} \subset \mathrm{Nef}_Y$ or not by the method in Section 5.2. Therefore condition (VE-1) in Section 4.1 is satisfied. Since we can calculate $\mathrm{isom}(Y, D_0^g, D_0^{g'})$ for any $g, g' \in \mathrm{O}^{\mathcal{P}}(S_Y)$ by Section 5.5, conditions (VE-2) and (VE-3) are also satisfied. Therefore we can apply Procedure 4.1 to the graph (V, E) and the group G, and obtain a complete set V_0 of representatives of orbits of the action of G on V, the stabilizer subgroups $\mathrm{isom}(Y, D, D) = \mathrm{aut}(Y, D)$ of these representatives $D \in V_0$, and a generating set

$$\mathcal{G} := \mathcal{H} \cup \mathrm{aut}(Y, D_0)$$

of $\mathrm{aut}(Y)$. Then we have

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \mathrm{vol}(D_0)\sum_{D\in V_0}\frac{1}{|\mathrm{aut}(Y,D)|}. \tag{6.1}$$

Thus Theorem 1.15 is computationally affirmed.

Remark 6.1. The amount of computation of Procedure 4.1 grows quadratically as $|V/G|$ becomes large, because we have to check $T_G(v,v')=\emptyset$ for all pairs of distinct $v,v'\in V_0$. We could calculate a finite generating set of $\mathrm{aut}(Y)$ by using, naively, the graph (V',E'), where V' is the set of Vinberg chambers contained in Nef_Y and E' is the adjacency relation of Vinberg chambers. However, the size of $V'/\mathrm{aut}(Y)$ is approximately $\mathrm{vol}(D_0)$ times the size of $V/\mathrm{aut}(Y)$. Thus, very roughly speaking, using the primitive embedding $S_Y(2)\hookrightarrow L_{26}$ of type `96C` gives us computational advantage of multiplicative factor the square of $\mathrm{vol}(D_0)=652758220800$.

6.2. **Calculating $\mathcal{R}_{\mathrm{temp}}$, $\mathcal{E}_{\mathrm{temp}}$ and $\mathcal{G}_X$.** From V_0 and $\mathcal{G}$ calculated above, we compute the following data, which will be used in Sections 6.3 and 6.4.

Recall that $\mathcal{R}(Y)$ is embedded in S_Y by $C\mapsto[C]$. For each $D\in V_0$, let $\mathcal{R}(Y,D)$ be the set of roots $r=[C]$ in $\mathcal{R}(Y)$ such that $D\cap(r)^{\perp}$ is a wall of D. Since $D\subset\mathrm{Nef}_Y$, a root r defining a wall of D belongs to $\mathcal{R}(Y)$ if and only if r splits in S_X. Therefore we can calculate $\mathcal{R}(Y,D)$ by the method in Section 5.2. We put

$$\mathcal{R}_{\mathrm{temp}} := \bigcup_{D\in V_0}\mathcal{R}(Y,D).$$

Then the mapping

$$\mathcal{R}_{\mathrm{temp}}\hookrightarrow\mathcal{R}(Y)\twoheadrightarrow\mathcal{R}(Y)/\mathrm{aut}(Y)$$

is surjective. Via the generating set $\mathcal{G}$, we can generate (pseudo-)random elements of $\mathrm{aut}(Y)=\langle\mathcal{G}\rangle$. For $[C],[C']\in\mathcal{R}_{\mathrm{temp}}$, if we find $g\in\mathrm{aut}(Y)$ such that $[C]^g=[C']$, then we remove $[C']$ from $\mathcal{R}_{\mathrm{temp}}$. Repeating this process many times, we obtain a smaller subset $\mathcal{R}'_{\mathrm{temp}}$ of $\mathcal{R}(Y)$ that is mapped to $\mathcal{R}(Y)/\mathrm{aut}(Y)$ surjectively.

Let $\phi\colon Y\to\mathbb{P}^1$ be an elliptic fibration of Y, and F a general fiber of ϕ. Then $f_\phi:=[F]/2\in S_Y$ is a primitive isotropic ray (see Section 2.3 for the definition) contained in the closure of Nef_Y in $\overline{\mathcal{P}}_Y$. For each $D\in V_0$, let $\mathcal{E}(Y,D)$ be the set of primitive isotropic rays contained in the closure $\overline{D}$ of D in $\overline{\mathcal{P}}_Y$. We put

$$\mathcal{E}_{\mathrm{temp}} := \bigcup_{D\in V_0}\mathcal{E}(Y,D).$$

Then the mapping

$$\mathcal{E}_{\mathrm{temp}}\hookrightarrow\mathcal{E}(Y)\twoheadrightarrow\mathcal{E}(Y)/\mathrm{aut}(Y)$$

is surjective. As above, from $\mathcal{E}_{\mathrm{temp}}$ and using $\mathcal{G}$, we obtain a smaller subset $\mathcal{E}'_{\mathrm{temp}}$ of $\mathcal{E}(Y)$ that is mapped to $\mathcal{E}(Y)/\mathrm{aut}(Y)$ surjectively.

Let $\mathrm{Aut}(X,\varepsilon)$ be the centralizer of $\varepsilon\in\mathrm{Aut}(X)$ in $\mathrm{Aut}(X)$, and let $\mathrm{aut}(X,\varepsilon)$ be the image of $\mathrm{Aut}(X,\varepsilon)$ in $\mathrm{aut}(X)$. We write an element $\tilde{\gamma}\in\mathrm{Aut}(X)$ as $(\tilde{g},f)$ by (3.3). Since $\mathrm{O}(T_X,\omega)=\{\pm1\}$ is abelian, we see that $\tilde{\gamma}$ commutes with $\varepsilon\in\mathrm{Aut}(X)$ if and only if $\tilde{g}$ commutes with $\varepsilon\in\mathrm{aut}(X)$. Hence $\mathrm{aut}(X,\varepsilon)$ is equal to the centralizer of $\varepsilon\in\mathrm{aut}(X)$ in $\mathrm{aut}(X)$. By the Torelli theorem (see the proof of Proposition 3.2), an element $\tilde{g}$ of $\mathrm{O}^{\mathcal{P}}(S_X)$ belongs to $\mathrm{aut}(X,\varepsilon)$ if and only if $\tilde{g}$ acts on $S_X^{\vee}/S_X$ as ±1, preserves Nef_X, and commutes with $\varepsilon\in\mathrm{O}^{\mathcal{P}}(S_X)$. Let $\mathrm{aut}(X,\varepsilon)_0$ be the group consisting of elements $\tilde{g}\in\mathrm{aut}(X,\varepsilon)$ that act on $S_X^{\vee}/S_X$ as 1. We have $\mathrm{aut}(X,\varepsilon)_0=\mathrm{aut}(X,\varepsilon)\cap G_X$.

The restriction homomorphism $\tilde{g} \mapsto \tilde{g}|S_Y$ gives a surjective homomorphism $\mathrm{aut}(X, \varepsilon) \to \mathrm{aut}(Y)$. We calculate the kernel

$$K := \mathrm{Ker}(\mathrm{aut}(X, \varepsilon) \to \mathrm{aut}(Y)).$$

The kernel K is naturally embedded into $\mathrm{O}(S_{X-})$ by $\tilde{g} \mapsto \tilde{g}|S_{X-}$. We put

$$K_0 := \mathrm{Ker}(\mathrm{aut}(X, \varepsilon)_0 \to \mathrm{aut}(Y)) \subset G_X.$$

By definition K_0 acts trivially on $S_{X+}^\vee/S_{X+}$ and by Proposition 2.1 it must act trivially on $S_{X-}^\vee/S_{X-}$ as well. Hence, regarded as a subgroup of $G_{X-} \subset \mathrm{O}(S_{X-})$, K_0 is contained in the kernel of

$$\psi\colon G_{X-} \to \mathrm{O}(S_{X-}^\vee/S_{X-}).$$

Conversely the elements of $\mathrm{Ker}\,\psi$ can be extended by the identity on S_{X+} to elements of G_X which trivially preserve Nef_Y. Hence they are induced by automorphisms of Y and we have $K_0 = \mathrm{Ker}\,\psi$. The kernel of ψ is explicitly computed in the proof of Theorem 3.10. Its order is given by $e_{\tau,\bar{\tau}} \in \{1, 2\}$. Suppose that $e_{\tau,\bar{\tau}} = 2$. If $\varepsilon \in K_0$, then $K = K_0 = \langle \varepsilon \rangle$. This is the case if in addition $\tau(\widetilde{R}) = E_8$. Otherwise $K = K_0 \times \langle \varepsilon \rangle$ is of order 4.

For each g in the generating set $\mathcal{G}$ of $\mathrm{aut}(Y)$, we calculate a lift $\tilde{g} \in \mathrm{aut}(X, \varepsilon)$ of g, and put

$$\mathcal{G}_X := \{\, \tilde{g} \mid g \in \mathcal{G} \,\} \;\cup\; K.$$

Then $\mathrm{aut}(X, \varepsilon)$ is generated by $\mathcal{G}_X$.

6.3. **Rational curves on Y.** We prove Theorem 1.16. By the construction of S_X given in Section 4.3, we have a set of splitting roots that define some walls of $D_0 \subset \mathrm{Nef}_Y$ and form the dual graph of ADE-type τ. Therefore the existence of $C_1, \dots, C_m$ in assertion (1) is proved.

Let C be a smooth rational curve on Y, and $r := [C]$ the class of C. Let $\widetilde{V}_C$ be the set of $L_{26}/S_Y(2)$-chambers D such that $D \cap (r)^\perp$ is a wall of D and that D is located on the same side of $(r)^\perp$ as Nef_Y. Let D be an element of $\widetilde{V}_C$, and suppose that $F := D \cap (r)^\perp \cap (r')^\perp$ is a face of codimension 2 of D that is a boundary of the wall $D \cap (r)^\perp$, where r' is a root of S_Y defining a wall of D. Then there exists a unique element D' of $\widetilde{V}_C$ such that $D \cap D' = F$ holds. We say that this chamber D' is *adjacent in* $\widetilde{V}_C$ to D across F. This $L_{26}/S_Y(2)$-chamber D' is calculated as follows. As is seen from the set of faces of $L_{26}/S_Y(2)$-chambers (see [30]), we have $\langle r, r' \rangle = 0$ or $\langle r, r' \rangle = 1$. Let s and s' be the reflections with respect to the roots $r = [C]$ and r', respectively. Then

$$D' = \begin{cases} D^{s'} & \text{if } \langle r, r' \rangle = 0, \\ D^{ss'} & \text{if } \langle r, r' \rangle = 1. \end{cases}$$

Suppose that D is contained in Nef_Y. Then D' is contained in Nef_Y if and only if r' is *not* the class of a smooth rational curve on Y, or equivalently, r' does not split in S_X. We consider the graph (V_C, E_C), where V_C is the set of $L_{26}/S_Y(2)$-chambers $D \in \widetilde{V}_C$ contained in Nef_Y, and E_C is the restriction to $V_C \subset \widetilde{V}_C$ of the adjacency relation on $\widetilde{V}_C$ defined above. Then the stabilizer subgroup

$$G_C := \mathrm{aut}(Y, C) = \{\, g \in \mathrm{aut}(Y) \mid r^g = r \,\}$$

of C in $\mathrm{aut}(Y)$ acts on (V_C, E_C). For $D, D' \in V_C$, we have

$$T_G(D, D') = \{\, g \in \mathrm{isoms}(Y, D, D') \mid r^g = r \,\},$$

where $T_G(D, D') \subset G_C$ is defined by (4.1), and $\mathrm{isoms}(Y, D, D')$ is defined in Section 5.5. Therefore (V_C, E_C) and G_C satisfy conditions (VE-1), …, (VE-3) in Section 4.1. We apply Procedure 4.1 to every $C \in \mathcal{R}'_{\mathrm{temp}}$ and obtain a complete set $V_{C,0}$ of representatives of orbits of the action of G_C on V_C.

Two elements C and C' of $\mathcal{R}'_{\mathrm{temp}}$ are contained in the same orbit under the action of $\mathrm{aut}(Y)$ on $\mathcal{R}(Y)$ if and only if we have one of the following conditions that are mutually equivalent.

- Let D be an arbitrary element of $V_{C,0}$. Then there exists an $L_{26}/S_Y(2)$-chamber D' in $V_{C',0}$ such that $\mathrm{isoms}(Y, D, D')$ contains an isometry g such that $[C]^g = [C']$.
- There exist a pair of $L_{26}/S_Y(2)$-chambers $D \in V_{C,0}$ and $D' \in V_{C',0}$ and an isometry $g \in \mathrm{isoms}(Y, D, D')$ such that $[C]^g = [C']$.

Applying this method to all pairs C, C' of distinct elements of $\mathcal{R}'_{\mathrm{temp}}$, we obtain a complete set of representatives $C'_1, \dots, C'_k$ of orbits of the action of $\mathrm{aut}(Y)$ on $\mathcal{R}(Y)$. We then apply this method to the representatives $C'_1, \dots, C'_k$ and the smooth rational curves $C_1, \dots, C_m$ in assertion (1), and complete the proof of Theorem 1.16.

The algorithm given above is a priori guaranteed to work. A posteriori, Theorem 1.16 can be verified by the following simple strategy. Let $\mathrm{aut}(X, \varepsilon)|S_{X-}$ be the image of the homomorphism

$$\mathrm{aut}(X, \varepsilon) \to \mathrm{O}(S_{X-})$$

given by $\tilde{g} \mapsto \tilde{g}|S_{X-}$. Since we have calculated a finite generating set $\mathcal{G}_X$ of $\mathrm{aut}(X, \varepsilon)$, we can calculate the elements of the finite group $\mathrm{aut}(X, \varepsilon)|S_{X-}$. Let C, C' be elements of $\mathcal{R}(Y)$. If the orbit of $\{\pm v_C\} \subset S_{X-}$ by $\mathrm{aut}(X, \varepsilon)|S_{X-}$ and that of $\{\pm v_{C'}\}$ are disjoint, then the orbits of C and C' by $\mathrm{aut}(Y)$ are disjoint. Even though the converse does not necessarily hold, we know a posteriori that once the size of $\mathcal{R}'_{\mathrm{temp}}$ is small enough, this separates the orbits of $\mathcal{R}'_{\mathrm{temp}}$.

6.4. **Elliptic fibrations of Y.** Let $\phi\colon Y \to \mathbb{P}^1$ be an elliptic fibration of Y. We consider the following graph (V_ϕ, E_ϕ). We define V_ϕ to be the set of $L_{26}/S_Y(2)$-chambers D contained in Nef_Y such that the closure $\overline{D}$ of D in $S_Y \otimes \mathbb{R}$ contains the primitive isotropic ray $f_\phi = [F]/2$, where F is a general fiber of ϕ, and E_ϕ to be the set of pairs of adjacent $L_{26}/S_Y(2)$-chambers in V_ϕ. The stabilizer subgroup

$$G_\phi := \mathrm{aut}(Y, \phi) := \{\, g \in \mathrm{aut}(Y) \mid f_\phi^g = f_\phi \,\}$$

of ϕ in $\mathrm{aut}(Y)$ acts on (V_ϕ, E_ϕ). Then condition (VE-1) is satisfied. Indeed, the set of $L_{26}/S_Y(2)$-chambers in V_ϕ adjacent to $D \in V_\phi$ is the set of all D^{s_r}, where r runs through the set of *non-splitting* roots of S_Y defining walls of D such that $\langle r, f_\phi \rangle = 0$. For $D, D' \in V_\phi$, the subset $T_G(D, D')$ of G_ϕ is the set of isometries belonging to $\mathrm{isoms}(Y, D, D')$ that fixes f_ϕ. Therefore (VE-2) and (VE-3) are also satisfied.

We apply Procedure 4.1 to every $\phi \in \mathcal{E}'_{\mathrm{temp}}$ and obtain a complete set $V_{\phi,0}$ of representatives of orbits of the action of G_ϕ on V_ϕ. We also obtain a finite generating set $\mathcal{G}_\phi$ of the stabilizer subgroup $\mathrm{aut}(Y, \phi)$.

The set Σ_ϕ of classes of smooth rational curves C contained in some fiber of ϕ is calculated as follows. Let a_Y be an ample class of Y. Every class $[C] \in \Sigma_\phi$ satisfies $\langle [C], f_\phi \rangle = 0$ and $0 < \langle [C], a_Y \rangle < 2\langle f_\phi, a_Y \rangle$. We calculate the set Σ' of all roots r of S_Y satisfying $\langle r, f_\phi \rangle = 0$ and $0 < \langle r, a_Y \rangle < 2\langle f_\phi, a_Y \rangle$. Then $r \in \Sigma'$ belongs to Σ_ϕ if and only if r splits in S_X (see Section 5.2) and there exist no roots $r' \in \Sigma_\phi$ such that $\langle r', a_Y \rangle < \langle r, a_Y \rangle$ and $\langle r, r' \rangle < 0$. Therefore we can calculate Σ_ϕ by sorting the elements r of Σ' according to $\langle r, a_Y \rangle$ and applying the above criterion to $r \in \Sigma'$ in this order.

Each connected component of the dual graph of roots in Σ_ϕ corresponds to a reducible fiber of ϕ, and is the Dynkin diagram of an affine ADE-type. Let Γ be a connected component. The weighted sum of roots in Γ with appropriate weights according to the ADE-type of Γ (see, for example, [26, Theorem 5.12]) is either f_ϕ or $2f_\phi$. The former case occurs when the corresponding reducible fiber is a multiple fiber, while the latter occurs when the fiber is non-multiple.

Let $\phi' \colon Y \to \mathbb{P}^1$ be another element of $\mathcal{E}'_{\mathrm{temp}}$. Then ϕ and ϕ' are contained in the same orbit under the action of $\mathrm{aut}(Y)$ on $\mathcal{E}(Y)$ if and only if the following holds. Let D be an element of $V_{\phi,0}$. Then there exists $D' \in V_{\phi',0}$ such that $\mathrm{isoms}(Y, D, D')$ contains an isometry that maps f_ϕ to $f_{\phi'}$. Note that D' can be computed explicitly. Applying this method to all pairs ϕ, ϕ' of distinct elements of $\mathcal{E}'_{\mathrm{temp}}$, we obtain a complete set of representatives of the action of $\mathrm{aut}(Y)$ on $\mathcal{E}(Y)$.

6.5. **Table of elliptic fibrations.** Let $\phi \colon Y \to \mathbb{P}^1$ be an elliptic fibration of an Enriques surface Y. Then ϕ has exactly two multiple fibers, and both of them are of multiplicity 2. In the table below, the first column shows the ADE-types of non-multiple reducible fibers, and the second column shows the ADE-types of multiple reducible fibers. The third column gives the number of elliptic fibrations modulo $\mathrm{aut}(Y)$. See [32] for the cases with $\operatorname{rank} \tau \geq 8$.

No. 1: (A_1, A_1)

none	none	136
A_1	none	255

No. 2: $(2A_1, 2A_1)$

none	none	36
none	A_1	1
A_1	none	128
$2A_1$	none	126

No. 3: (A_2, A_2)

A_1	none	136
A_2	none	119

No. 4: $(3A_1, 3A_1)$

none	none	10
A_1	none	48
A_1	A_1	3
$2A_1$	none	96
$3A_1$	none	60

No. 5: $(A_2 + A_1, A_2 + A_1)$

none	A_1	1
A_1	none	36
$A_2 + A_1$	none	63
$2A_1$	none	63
A_2	none	28

No. 6: (A_3, A_3)

none	A_2	1
$2A_1$	none	36
A_2	none	64
A_3	none	54

No. 7: $(4A_1, 4A_1)$

none	none	3
A_1	none	16
$2A_1$	none	48
$2A_1$	A_1	6
$3A_1$	none	64
$4A_1$	none	25

No. 8: $(4A_1, D_4)$

none	none	10
none	$2A_1$	3
$2A_1$	none	96
$4A_1$	none	60

No. 9: $(A_2 + 2A_1, A_2 + 2A_1)$

A_1	none	10
A_1	A_1	2
$A_2 + A_1$	none	32
$2A_1$	none	32
$A_2 + 2A_1$	none	30
$3A_1$	none	30
A_2	none	6
A_2	A_1	1

No. 10: $(A_3 + A_1, A_3 + A_1)$

A_1	A_1	1
A_1	A_2	1
$A_2 + A_1$	none	32
$A_3 + A_1$	none	30
$2A_1$	none	10
$3A_1$	none	15
A_2	none	16
A_3	none	12

No. 11: $(2A_2, 2A_2)$

none	A_1	1
$A_2 + A_1$	none	56
$2A_1$	none	35
$2A_2$	none	35

No. 12: (A_4, A_4)

none	A_2	1
$A_2 + A_1$	none	36
A_3	none	27
A_4	none	27

No. 13: (D_4, D_4)

none	A_3	3
$4A_1$	none	10
A_3	none	48
D_4	none	20

No. 14: $(5A_1, 5A_1)$

none	none	1
A_1	none	5
$2A_1$	none	20
$3A_1$	none	40
$3A_1$	A_1	10
$4A_1$	none	40
$5A_1$	none	5

No. 15: $(5A_1, D_4 + A_1)$

none	none	3
A_1	none	4
A_1	$2A_1$	3
$2A_1$	none	24
$3A_1$	none	48
$3A_1$	A_1	4
$4A_1$	none	16
$5A_1$	none	24

No. 16: $(A_2 + 3A_1, A_2 + 3A_1)$

A_1	none	3
$A_2 + A_1$	none	12
$A_2 + A_1$	A_1	3
$2A_1$	none	12
$2A_1$	A_1	3
$A_2 + 2A_1$	none	24
$3A_1$	none	24
$A_2 + 3A_1$	none	12
$4A_1$	none	13
A_2	none	1

No. 17: $(A_3 + 2A_1, A_3 + 2A_1)$

$A_2 + A_1$	none	16
$A_3 + A_1$	none	16
$2A_1$	none	3
$2A_1$	A_1	2
$2A_1$	A_2	1
$A_2 + 2A_1$	none	16
$A_3 + 2A_1$	none	13
$3A_1$	none	8
$4A_1$	none	6
A_2	none	4
A_3	none	2
A_3	A_1	1

No. 18: $(A_3 + 2A_1, D_5)$

none	$A_2 + A_1$	1
none	$2A_1$	1
$A_2 + A_1$	none	32
$2A_1$	none	10
$A_3 + 2A_1$	none	30
$4A_1$	none	15
A_3	none	6

No. 19: $(2A_2 + A_1, 2A_2 + A_1)$

A_1	A_1	1
$A_2 + A_1$	none	12
$2A_2 + A_1$	none	15
$2A_1$	none	10
$A_2 + 2A_1$	none	30
$3A_1$	none	15
A_2	A_1	2
$2A_2$	none	10

No. 20: $(A_4 + A_1, A_4 + A_1)$

A_1	A_2	1
$A_2 + A_1$	none	10
$A_3 + A_1$	none	15
$A_4 + A_1$	none	15
$A_2 + 2A_1$	none	15
A_2	A_1	1
A_3	none	6
A_4	none	6

No. 21: $(D_4 + A_1, D_4 + A_1)$

A_1	A_3	3
$A_3 + A_1$	none	24
$D_4 + A_1$	none	12
$3A_1$	A_1	1
$4A_1$	none	3
$5A_1$	none	3
A_3	none	12
D_4	none	4

No. 22: $(A_3 + A_2, A_3 + A_2)$

A_1	A_1	1
$A_2 + A_1$	none	16
$A_3 + A_1$	none	12
$A_2 + 2A_1$	none	6
$3A_1$	none	9
A_2	A_2	1
$A_3 + A_2$	none	18
$2A_2$	none	16

No. 23: (A_5, A_5)

A_1	A_2	1
$A_3 + A_1$	none	15
$2A_2$	none	10
A_4	none	12
A_5	none	15

No. 24: (D_5, D_5)

none	A_3	1
none	A_4	1
$A_3 + 2A_1$	none	10
A_4	none	16
D_4	none	5
D_5	none	10

No. 25: $(6A_1, D_4 + 2A_1)$

none	none	1
A_1	none	2
$2A_1$	none	8
$2A_1$	$2A_1$	3
$3A_1$	none	24
$4A_1$	none	28
$4A_1$	A_1	9
$5A_1$	none	16
$6A_1$	none	3

No. 27: $(A_2 + 4A_1, A_2 + 4A_1)$

A_1	none	1
$A_2 + A_1$	none	4
$2A_1$	none	4
$A_2 + 2A_1$	none	12
$A_2 + 2A_1$	A_1	6
$3A_1$	none	12
$3A_1$	A_1	4
$A_2 + 3A_1$	none	16
$4A_1$	none	16
$A_2 + 4A_1$	none	1
$5A_1$	none	4

No. 28: $(A_2 + 4A_1, D_4 + A_2)$

A_1	none	3
$A_2 + 2A_1$	none	24
$3A_1$	none	24
$3A_1$	A_1	4
$A_2 + 4A_1$	none	12
$5A_1$	none	12
A_2	none	1
A_2	$2A_1$	3

No. 29: $(A_3 + 3A_1, A_3 + 3A_1)$

$A_2 + A_1$	none	6
$A_3 + A_1$	none	6
$A_3 + A_1$	A_1	3
$2A_1$	none	1
$A_2 + 2A_1$	none	12
$A_3 + 2A_1$	none	12
$3A_1$	none	3
$3A_1$	A_1	3
$3A_1$	A_2	1
$A_2 + 3A_1$	none	8
$A_3 + 3A_1$	none	3
$4A_1$	none	6
$5A_1$	none	1
A_2	none	1

No. 30: $(A_3 + 3A_1, D_5 + A_1)$

A_1	$A_2 + A_1$	1
A_1	$2A_1$	1
$A_2 + A_1$	none	8
$A_3 + A_1$	none	4
$A_3 + A_1$	A_1	2
$2A_1$	none	3
$A_2 + 2A_1$	none	16
$A_3 + 2A_1$	none	8
$3A_1$	none	4
$3A_1$	A_1	1
$A_3 + 3A_1$	none	12
$4A_1$	none	4
$5A_1$	none	6
A_3	none	1

No. 31: $(2A_2 + 2A_1, 2A_2 + 2A_1)$

$A_2 + A_1$	none	2
$A_2 + A_1$	A_1	4
$2A_2 + A_1$	none	8
$2A_1$	none	3
$2A_1$	A_1	1
$A_2 + 2A_1$	none	16
$2A_2 + 2A_1$	none	6
$3A_1$	none	8
$A_2 + 3A_1$	none	12
$4A_1$	none	7
$2A_2$	none	3
$2A_2$	A_1	1

No. 32: $(A_4 + 2A_1, A_4 + 2A_1)$

$A_2 + A_1$	none	3
$A_2 + A_1$	A_1	2
$A_3 + A_1$	none	8
$A_4 + A_1$	none	8
$2A_1$	A_2	1
$A_2 + 2A_1$	none	8
$A_3 + 2A_1$	none	7
$A_4 + 2A_1$	none	6
$A_2 + 3A_1$	none	6
A_3	none	1
A_4	none	1
A_4	A_1	1

No. 33: $(D_4 + 2A_1, D_4 + 2A_1)$

$A_3 + A_1$	none	12
$D_4 + A_1$	none	8
$2A_1$	A_3	3
$A_3 + 2A_1$	none	12
$D_4 + 2A_1$	none	3
$4A_1$	none	1
$4A_1$	A_1	2
$5A_1$	none	2
A_3	none	3
D_4	A_1	1

No. 34: $(D_4 + 2A_1, D_6)$

none	$A_3 + A_1$	1
$A_3 + A_1$	none	16
$2A_1$	$2A_1$	1
$2A_1$	A_3	2
$A_3 + 2A_1$	none	8
$D_4 + 2A_1$	none	12
$4A_1$	none	3
$6A_1$	none	3
A_3	none	4
D_4	none	2

No. 35:

$(A_3 + A_2 + A_1, A_3 + A_2 + A_1)$

$A_2 + A_1$	none	4
$A_2 + A_1$	A_1	1
$A_2 + A_1$	A_2	1
$A_3 + A_2 + A_1$	none	6
$2A_2 + A_1$	none	8
$A_3 + A_1$	none	2
$2A_1$	A_1	1
$A_2 + 2A_1$	none	9
$A_3 + 2A_1$	none	7
$3A_1$	none	3
$A_2 + 3A_1$	none	3
$4A_1$	none	3
$A_3 + A_2$	none	6
$2A_2$	none	4
A_3	A_1	1

No. 36: $(A_5 + A_1, A_5 + A_1)$

$2A_2 + A_1$	none	4
$A_3 + A_1$	none	4
$A_4 + A_1$	none	8
$A_5 + A_1$	none	7
$2A_1$	A_2	1
$A_3 + 2A_1$	none	6
$2A_2$	none	3
A_3	A_1	1
A_4	none	2
A_5	none	4

No. 37: $(A_5 + A_1, E_6)$

none	$A_2 + A_1$	1
$A_5 + A_1$	none	15
$A_3 + 2A_1$	none	15
$2A_2$	none	10
A_4	none	6

No. 38: $(D_5 + A_1, D_5 + A_1)$

A_1	A_3	1
A_1	A_4	1
$A_3 + A_1$	A_1	1
$A_4 + A_1$	none	8
$D_4 + A_1$	none	3
$D_5 + A_1$	none	6
$A_3 + 2A_1$	none	3
$A_3 + 3A_1$	none	3
A_4	none	4
D_4	none	1
D_5	none	2

No. 39: $(3A_2, 3A_2)$

$2A_2 + A_1$	none	30
$A_2 + 2A_1$	none	15
$3A_1$	none	10
A_2	A_1	3
$3A_2$	none	5

No. 40: $(3A_2, E_6)$

$2A_2 + A_1$	none	30
$A_2 + 2A_1$	none	15
$3A_1$	none	10
A_2	A_1	3
$3A_2$	none	5

No. 41: $(A_4 + A_2, A_4 + A_2)$

$2A_2 + A_1$	none	6
$A_3 + A_1$	none	6
$A_4 + A_1$	none	6
$A_2 + 2A_1$	none	9
A_2	A_1	1
A_2	A_2	1
$A_3 + A_2$	none	9
$A_4 + A_2$	none	9

No. 42: $(D_4 + A_2, D_4 + A_2)$

$A_3 + A_1$	none	12
$D_4 + A_1$	none	4
$3A_1$	A_1	1
$A_2 + 4A_1$	none	1
$5A_1$	none	2
A_2	A_3	3
$A_3 + A_2$	none	12
$D_4 + A_2$	none	8

No. 43: $(2A_3, 2A_3)$

$2A_1$	A_1	1
$A_2 + 2A_1$	none	8
$A_3 + 2A_1$	none	4
$4A_1$	none	2
$A_3 + A_2$	none	16
$2A_2$	none	8
A_3	A_2	2
$2A_3$	none	9

No. 44: $(2A_3, D_6)$

none	$2A_1$	1
none	$2A_2$	1
$A_3 + 2A_1$	none	12
$4A_1$	none	9
$2A_2$	none	16
$2A_3$	none	18

No. 45: (A_6, A_6)

$A_4 + A_1$	none	6
A_2	A_2	1
$A_3 + A_2$	none	9
A_5	none	6
A_6	none	9

No. 46: (D_6, D_6)

none	A_5	1
$2A_1$	A_3	1
$D_4 + 2A_1$	none	3
$2A_3$	none	3
A_5	none	8
D_5	none	2
D_6	none	6

No. 47: (E_6, E_6)

none	A_4	1
$A_5 + A_1$	none	10
D_5	none	5
E_6	none	5

No. 50:

$(A_2 + 5A_1, D_4 + A_2 + A_1)$

A_1	none	1
$A_2 + A_1$	none	1
$A_2 + A_1$	$2A_1$	3
$2A_1$	none	1
$A_2 + 2A_1$	none	6
$3A_1$	none	6
$A_2 + 3A_1$	none	12
$A_2 + 3A_1$	A_1	4
$4A_1$	none	12
$4A_1$	A_1	5
$A_2 + 4A_1$	none	4
$5A_1$	none	4
$6A_1$	none	3

No. 51: $(A_3 + 4A_1, D_5 + 2A_1)$

$A_2 + A_1$	none	2
$A_3 + A_1$	none	2
$2A_1$	none	1
$2A_1$	$A_2 + A_1$	1
$2A_1$	$2A_1$	1
$A_2 + 2A_1$	none	8
$A_3 + 2A_1$	none	4
$A_3 + 2A_1$	A_1	5
$3A_1$	none	2
$A_2 + 3A_1$	none	8
$A_3 + 3A_1$	none	8
$4A_1$	none	3
$4A_1$	A_1	2
$A_3 + 4A_1$	none	1
$5A_1$	none	4
$6A_1$	none	1

No. 52: $(A_3 + 4A_1, D_4 + A_3)$

$2A_1$	none	1
$A_2 + 2A_1$	none	12
$A_3 + 2A_1$	none	12
$4A_1$	none	6
$4A_1$	A_1	4
$4A_1$	A_2	1
$A_2 + 4A_1$	none	8
$A_3 + 4A_1$	none	3
A_2	none	1
A_3	$2A_1$	3

No. 54: $(2A_2 + 3A_1, 2A_2 + 3A_1)$

$2A_2 + A_1$	none	3
$2A_2 + A_1$	A_1	3
$2A_1$	none	1
$A_2 + 2A_1$	none	6
$A_2 + 2A_1$	A_1	6
$2A_2 + 2A_1$	none	6
$3A_1$	none	3
$3A_1$	A_1	1
$A_2 + 3A_1$	none	12
$4A_1$	none	6
$A_2 + 4A_1$	none	2
$5A_1$	none	3
$2A_2$	none	1

No. 55: $(A_4 + 3A_1, A_4 + 3A_1)$

$A_2 + A_1$	none	1
$A_3 + A_1$	none	3
$A_4 + A_1$	none	3
$A_4 + A_1$	A_1	3
$A_2 + 2A_1$	none	3
$A_2 + 2A_1$	A_1	3
$A_3 + 2A_1$	none	6
$A_4 + 2A_1$	none	6
$3A_1$	A_2	1
$A_2 + 3A_1$	none	6
$A_3 + 3A_1$	none	3
$A_2 + 4A_1$	none	1

No. 56: $(D_4 + 3A_1, D_6 + A_1)$

A_1	$A_3 + A_1$	1
$A_3 + A_1$	none	6
$D_4 + A_1$	none	2
$D_4 + A_1$	A_1	2
$A_3 + 2A_1$	none	10
$D_4 + 2A_1$	none	4
$3A_1$	$2A_1$	1
$3A_1$	A_3	2
$A_3 + 3A_1$	none	4
$D_4 + 3A_1$	none	2
$4A_1$	none	1
$5A_1$	none	1
$5A_1$	A_1	1
$6A_1$	none	1
A_3	none	1

No. 58:

$(A_3 + A_2 + 2A_1, A_3 + A_2 + 2A_1)$

$A_2 + A_1$	none	1
$A_3 + A_2 + A_1$	none	4
$2A_2 + A_1$	none	4
$A_3 + A_1$	A_1	2
$A_2 + 2A_1$	none	4
$A_2 + 2A_1$	A_1	2
$A_2 + 2A_1$	A_2	1
$A_3 + A_2 + 2A_1$	none	1
$2A_2 + 2A_1$	none	4
$A_3 + 2A_1$	none	4
$3A_1$	none	1
$3A_1$	A_1	1
$A_2 + 3A_1$	none	6
$A_3 + 3A_1$	none	2
$4A_1$	none	2
$5A_1$	none	1
$A_3 + A_2$	none	2
$A_3 + A_2$	A_1	1
$2A_2$	none	1

No. 59:

$(A_3 + A_2 + 2A_1, D_5 + A_2)$

$2A_2 + A_1$	none	8
$A_3 + A_1$	none	1
$A_3 + A_1$	A_1	2
$A_2 + 2A_1$	none	9
$A_3 + A_2 + 2A_1$	none	6
$3A_1$	none	3
$3A_1$	A_1	1
$A_3 + 3A_1$	none	6
$A_2 + 4A_1$	none	3
$5A_1$	none	3
A_2	$A_2 + A_1$	1
A_2	$2A_1$	1
$A_3 + A_2$	none	3

No. 60: $(A_5 + 2A_1, A_5 + 2A_1)$

$2A_2 + A_1$	none	2
$A_3 + A_1$	none	1
$A_3 + A_1$	A_1	2
$A_4 + A_1$	none	4
$A_5 + A_1$	none	4
$2A_2 + 2A_1$	none	2
$A_3 + 2A_1$	none	4
$A_4 + 2A_1$	none	4
$A_5 + 2A_1$	none	2
$3A_1$	A_2	1
$A_3 + 3A_1$	none	1
$2A_2$	none	1
A_5	none	1
A_5	A_1	1

No. 61: $(A_5 + 2A_1, E_6 + A_1)$

A_1	$A_2 + A_1$	1
$2A_2 + A_1$	none	4
$A_3 + A_1$	A_1	1
$A_4 + A_1$	none	4
$A_5 + A_1$	none	4
$A_3 + 2A_1$	none	4
$A_5 + 2A_1$	none	6
$A_3 + 3A_1$	none	6
$2A_2$	none	3
A_4	none	1
A_5	A_1	1

No. 62: $(D_5 + 2A_1, D_5 + 2A_1)$

$A_4 + A_1$	none	4
$D_4 + A_1$	none	2
$D_5 + A_1$	none	4
$2A_1$	A_3	1
$2A_1$	A_4	1
$A_3 + 2A_1$	none	1
$A_3 + 2A_1$	A_1	2
$A_4 + 2A_1$	none	4
$D_4 + 2A_1$	none	1
$D_5 + 2A_1$	none	1
$A_3 + 3A_1$	none	2
A_4	none	1
D_5	A_1	1

No. 63: $(D_5 + 2A_1, D_7)$

none	$A_4 + A_1$	1
$A_4 + A_1$	none	8
$2A_1$	A_3	1
$A_3 + 2A_1$	none	3
$D_4 + 2A_1$	none	3
$D_5 + 2A_1$	none	6
$A_3 + 4A_1$	none	3
A_3	$2A_1$	1
D_4	none	1
D_5	none	1

No. 64: $(3A_2 + A_1, 3A_2 + A_1)$

$A_2 + A_1$	A_1	3
$2A_2 + A_1$	none	9
$3A_2 + A_1$	none	3
$A_2 + 2A_1$	none	3
$2A_2 + 2A_1$	none	9
$3A_1$	none	3
$A_2 + 3A_1$	none	9
$4A_1$	none	4
$2A_2$	A_1	3
$3A_2$	none	1

No. 65: $(3A_2 + A_1, E_6 + A_1)$

$A_2 + A_1$	A_1	3
$2A_2 + A_1$	none	9
$3A_2 + A_1$	none	3
$A_2 + 2A_1$	none	3
$2A_2 + 2A_1$	none	9
$3A_1$	none	3
$A_2 + 3A_1$	none	9
$4A_1$	none	4
$2A_2$	A_1	3
$3A_2$	none	1

No. 66:

$(A_4 + A_2 + A_1, A_4 + A_2 + A_1)$

$A_2 + A_1$	A_1	1
$A_2 + A_1$	A_2	1
$A_3 + A_2 + A_1$	none	3
$A_4 + A_2 + A_1$	none	3
$2A_2 + A_1$	none	1
$A_3 + A_1$	none	1
$A_4 + A_1$	none	1
$A_2 + 2A_1$	none	3
$2A_2 + 2A_1$	none	3
$A_3 + 2A_1$	none	4
$A_4 + 2A_1$	none	3
$A_2 + 3A_1$	none	3
$A_3 + A_2$	none	3
$A_4 + A_2$	none	3
$2A_2$	A_1	1
A_4	A_1	1

No. 67:

$(D_4 + A_2 + A_1, D_4 + A_2 + A_1)$

$A_2 + A_1$	A_3	3
$A_3 + A_2 + A_1$	none	6
$A_3 + A_1$	none	3
$A_3 + 2A_1$	none	6
$D_4 + 2A_1$	none	3
$A_2 + 3A_1$	A_1	1
$4A_1$	A_1	1
$5A_1$	none	1
$A_3 + A_2$	none	3
$D_4 + A_2$	none	4
D_4	A_1	1

No. 68: $(2A_3 + A_1, 2A_3 + A_1)$

$A_3 + A_2 + A_1$	none	8
$2A_2 + A_1$	none	4
$A_3 + A_1$	A_1	2
$A_3 + A_1$	A_2	2
$2A_3 + A_1$	none	1
$A_2 + 2A_1$	none	2
$3A_1$	A_1	1
$A_2 + 3A_1$	none	4
$A_3 + 3A_1$	none	2
$4A_1$	none	1
$A_3 + A_2$	none	4
$2A_2$	none	2
$2A_3$	none	4

No. 69: $(2A_3 + A_1, D_6 + A_1)$

A_1	$2A_1$	1
A_1	$2A_2$	1
$2A_2 + A_1$	none	8
$A_3 + A_1$	A_1	2
$2A_3 + A_1$	none	6
$A_3 + 2A_1$	none	2
$A_3 + 3A_1$	none	6
$4A_1$	none	3
$5A_1$	none	3
$2A_2$	none	4
$2A_3$	none	6

No. 70: $(2A_3 + A_1, E_7)$

A_1	$2A_1$	1
A_1	$2A_2$	1
$2A_2 + A_1$	none	8
$A_3 + A_1$	A_1	2
$2A_3 + A_1$	none	6
$A_3 + 2A_1$	none	2
$A_3 + 3A_1$	none	6
$4A_1$	none	3
$5A_1$	none	3
$2A_2$	none	4
$2A_3$	none	6

No. 71: $(A_6 + A_1, A_6 + A_1)$

$A_2 + A_1$	A_2	1
$A_3 + A_2 + A_1$	none	3
$A_4 + A_1$	none	1
$A_5 + A_1$	none	4
$A_6 + A_1$	none	3
$A_4 + 2A_1$	none	3
$A_3 + A_2$	none	3
A_4	A_1	1
A_5	none	1
A_6	none	3

No. 72: $(D_6 + A_1, D_6 + A_1)$

A_1	A_5	1
$2A_3 + A_1$	none	1
$A_5 + A_1$	none	4
$D_4 + A_1$	A_1	1
$D_5 + A_1$	none	2
$D_6 + A_1$	none	2
$D_4 + 2A_1$	none	1
$3A_1$	A_3	1
$2A_3$	none	1
A_5	none	2
D_6	none	2

No. 73: $(D_6 + A_1, E_7)$

A_1	$A_3 + A_1$	1
A_1	A_5	1
$A_5 + A_1$	none	4
$D_6 + A_1$	none	6
$D_4 + 3A_1$	none	3
$2A_3$	none	3
A_5	none	4
D_5	none	1

No. 74: $(E_6 + A_1, E_6 + A_1)$

A_1	A_4	1
$A_5 + A_1$	none	3
$D_5 + A_1$	none	3
$E_6 + A_1$	none	3
$A_5 + 2A_1$	none	3
A_5	A_1	1
D_5	none	1
E_6	none	1

No. 75: $(A_3 + 2A_2, A_3 + 2A_2)$

$A_2 + A_1$	A_1	2
$A_3 + A_2 + A_1$	none	12
$2A_2 + A_1$	none	8
$A_2 + 2A_1$	none	4
$2A_2 + 2A_1$	none	3
$A_3 + 2A_1$	none	1
$4A_1$	none	3
$2A_2$	A_2	1
$3A_2$	none	4
A_3	A_1	1

No. 76: $(A_5 + A_2, A_5 + A_2)$

$A_2 + A_1$	A_2	1
$A_3 + A_2 + A_1$	none	3
$2A_2 + A_1$	none	3
$A_4 + A_1$	none	2
$A_5 + A_1$	none	4
$A_3 + 2A_1$	none	3
$A_4 + A_2$	none	6
$A_5 + A_2$	none	3
$3A_2$	none	1
A_3	A_1	1

No. 77: $(A_5 + A_2, E_7)$

$A_2 + A_1$	A_2	1
$A_3 + A_2 + A_1$	none	3
$2A_2 + A_1$	none	3
$A_4 + A_1$	none	2
$A_5 + A_1$	none	4
$A_3 + 2A_1$	none	3
$A_4 + A_2$	none	6
$A_5 + A_2$	none	3
$3A_2$	none	1
A_3	A_1	1

No. 78: $(D_5 + A_2, D_5 + A_2)$

$A_3 + A_1$	A_1	1
$A_4 + A_1$	none	4
$D_4 + A_1$	none	1
$D_5 + A_1$	none	2
$A_3 + A_2 + 2A_1$	none	1
$A_3 + 3A_1$	none	2
A_2	A_3	1
A_2	A_4	1
$A_4 + A_2$	none	4
$D_4 + A_2$	none	2
$D_5 + A_2$	none	4

No. 79: $(A_4 + A_3, A_4 + A_3)$

$A_2 + A_1$	A_1	1
$A_3 + A_2 + A_1$	none	2
$2A_2 + A_1$	none	4
$A_3 + 2A_1$	none	1
$A_4 + 2A_1$	none	1
$A_2 + 3A_1$	none	2
$A_3 + A_2$	none	4
$A_4 + A_2$	none	4
A_3	A_2	1
$A_4 + A_3$	none	4
$2A_3$	none	5
A_4	A_2	1

No. 80: $(D_4 + A_3, D_4 + A_3)$

$A_3 + 2A_1$	none	3
$4A_1$	A_1	1
$A_2 + 4A_1$	none	1
$A_3 + A_2$	none	6
$D_4 + A_2$	none	4
A_3	A_3	3
$D_4 + A_3$	none	3
$2A_3$	none	6
D_4	A_2	1

No. 81: $(D_4 + A_3, D_7)$

none	$A_3 + A_2$	1
$2A_1$	$2A_1$	1
$A_3 + 2A_1$	none	4
$D_4 + 2A_1$	none	2
$A_3 + 4A_1$	none	1
$6A_1$	none	2
$A_3 + A_2$	none	8
A_3	A_3	2
$D_4 + A_3$	none	8
$2A_3$	none	4

No. 82: (A_7, A_7)

$A_5 + A_1$	none	2
$A_4 + A_2$	none	4
A_3	A_2	1
$2A_3$	none	2
A_6	none	4
A_7	none	5

ADE-type	number
E_6	1
$A_5 + A_1$	5
$3A_2$	1
D_5	1
A_5	1
$A_4 + A_1$	1
$A_3 + 2A_1$	5
$2A_2 + A_1$	1
D_4	1
A_4	1

ADE-type	number
$A_3 + A_1$	1
$2A_2$	1
$A_2 + 2A_1$	1
$4A_1$	5
A_3	1
$A_2 + A_1$	1
$3A_1$	2
A_2	1
$2A_1$	1
A_1	1

TABLE 7.1. RDP-configurations on Y

No. 83: (A_7, E_7)

none	$2A_2$	1
$A_5 + A_1$	none	6
$2A_3$	none	9
A_7	none	9

No. 84: (D_7, D_7)

none	A_6	1
$D_5 + 2A_1$	none	1
A_3	A_3	1
$D_4 + A_3$	none	2
A_6	none	4
D_6	none	1
D_7	none	4

No. 85: (E_7, E_7)

A_1	A_5	1
$D_6 + A_1$	none	3
A_7	none	3
E_6	none	1
E_7	none	3

7. EXAMPLES

7.1. **An (E_6, E_6)-generic Enriques surface.** In [31], we investigated an (E_6, E_6)-generic Enriques surface (No. 47 of Table 1.1). We briefly review the result of [31].

Let $\overline{X} \subset \mathbb{P}^3$ be a quartic Hessian surface associated with a very general cubic homogeneous polynomial, and X the minimal resolution of $\overline{X}$. Then $\overline{X}$ contains 10 lines and has 10 ordinary nodes, and the $K3$ surface X has a fixed-point free involution ε that interchanges the strict transforms of the 10 lines and the exceptional curves over the 10 ordinary nodes. Let $\pi\colon X \to Y$ be the quotient morphism by ε. Then the Enriques surface Y is (E_6, E_6)-generic (see Kondo [17]).

We can construct a sequence of primitive embeddings $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ from the primitive embeddings $L_{10}(2) \hookrightarrow L_{26}$ of type `20E`. We see that D_0 is a fundamental domain of the action of $\mathrm{aut}(Y)$ on Nef_Y, and hence

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \mathrm{vol}(D_0) = \frac{1_{\mathrm{BP}}}{51840} = \frac{1_{\mathrm{BP}}}{|W(R_{E_6})|}.$$

In fact, the $L_{26}/S_Y(2)$-chamber D_0 is equal to the chamber D_Y in [31]. We then obtain the same result as Table 1.1 of [31] for $\mathcal{E}(Y)/\mathrm{aut}(Y)$. We also prove that $\mathrm{aut}(Y)$ acts on $\mathcal{R}(Y)$ transitively.

The last result contradicts Theorem 1.5 of [31], because Table 1.2 of [31] says that there exist 10 orbits of the action of $\mathrm{aut}(Y)$ on $\mathcal{R}(Y)$. In fact, the argument in Section 7.6 of [31] for the calculation of the number of $\mathrm{aut}(Y)$-orbits of RDP-configurations is wrong, and Table 1.2 of [31] should be replaced by Table 7.1 below.

Here we present a correct method for the calculation of $\mathrm{aut}(Y)$-orbits of RDP-configurations. Let $\psi\colon Y \to \overline{Y}$ be a birational morphism to a surface $\overline{Y}$ that has only rational double points as its singularities, and let h_ψ be an ample class of $\overline{Y}$. Since the $L_{26}/S_Y(2)$-chamber D_0 is a fundamental domain of the action of $\mathrm{aut}(Y)$ on Nef_Y, we can assume that $\psi^*(h_\psi) \in S_Y$ belongs to D_0 by composing ψ with an automorphism of Y. Let f be the minimal face of D_0 containing $\psi^*(h_\psi)$. Then the set of the classes of smooth rational curves C contracted by ψ is equal to

$$\Gamma(f) := \{\, [C] \mid C \text{ is a smooth rational curve on } Y \text{ such that } f \subset ([C])^\perp \,\}.$$

For a given face f of D_0, we calculate the set of roots r of S_Y such that $f \subset (r)^\perp$. From this set, we can calculate $\Gamma(f)$ by using the ample class a_Y and the set of (-4)-vectors in S_{X-}. We calculate $\Gamma(f)$ for all faces f of D_0, and obtain 750 RDP-configurations of smooth rational curves. Every RDP-configuration on Y is equal to one of them modulo the action of $\mathrm{aut}(Y)$.

Let Γ be one of the 750 RDP-configurations. We put $\mu := |\Gamma|$, that is, μ is the total Milnor number of the singularities of the surface $\overline{Y}$ corresponding to Γ. The sublattice $\langle\Gamma\rangle$ of S_Y generated by the classes in Γ is negative definite of rank μ, and its orthogonal complement $\langle\Gamma\rangle^\perp$ is hyperbolic of rank $10-\mu$. Let $\mathcal{P}_{\langle\Gamma\rangle^\perp}$ be the positive half-cone of $\langle\Gamma\rangle^\perp$ contained in $\mathcal{P}_Y$. Composing the primitive embedding $\langle\Gamma\rangle^\perp \hookrightarrow S_Y$ with the primitive embedding $S_Y(2) \hookrightarrow L_{26}$ of type `20E`, we have $L_{26}/\langle\Gamma\rangle^\perp(2)$-chambers of $\mathcal{P}_{\langle\Gamma\rangle^\perp}$. The intersection $f_0 := \mathcal{P}_{\langle\Gamma\rangle^\perp} \cap D_0$ is one of the $L_{26}/\langle\Gamma\rangle^\perp(2)$-chambers, and it is the maximal face of D_0 among all the faces f of D_0 such that $\Gamma(f) = \Gamma$. Let (V_Γ, E_Γ) be the graph where V_Γ is the set of $L_{26}/\langle\Gamma\rangle^\perp(2)$-chambers on $\mathcal{P}_{\langle\Gamma\rangle^\perp}$ contained in $\mathcal{P}_{\langle\Gamma\rangle^\perp} \cap \mathrm{Nef}_Y$ and E_Γ is the usual adjacency relation of chambers. Then $D \mapsto \mathcal{P}_{\langle\Gamma\rangle^\perp} \cap D$ gives a bijection to the set V_Γ of vertices from the set of $L_{26}/S_Y(2)$-chambers D contained in Nef_Y such that $\mathcal{P}_{\langle\Gamma\rangle^\perp} \cap D$ is a face of D of dimension $10-\mu$, or equivalently, such that $\mathcal{P}_{\langle\Gamma\rangle^\perp} \cap D$ contains a non-empty open subset of $\mathcal{P}_{\langle\Gamma\rangle^\perp}$. The group

$$G_\Gamma := \{\, g \in \mathrm{aut}(Y) \mid \Gamma^g = \Gamma \,\}$$

acts on the graph (V_Γ, E_Γ). We apply Procedure 4.1 to (V_Γ, E_Γ) and G_Γ, and obtain a complete set $V_{\Gamma,0}$ of representatives of V_Γ/G_Γ. Let Γ' be one of the 750 RDP-configurations with the same ADE-type as Γ. Let $V_{\Gamma',0}$ be a complete set of representatives of $V_{\Gamma'}/G_{\Gamma'}$. Then the RDP-configurations Γ and Γ' are in the same orbit under the action of $\mathrm{aut}(Y)$ if and only if there exists an $L_{26}/\langle\Gamma'\rangle^\perp(2)$-chamber $f' = \mathcal{P}_{\langle\Gamma'\rangle^\perp} \cap D' \in V_{\Gamma',0}$ with $D' \subset \mathrm{Nef}_Y$ such that $\mathrm{isoms}(Y, D_0, D')$ contains an element g satisfying $\Gamma^g = \Gamma'$. Since $|V_{\Gamma',0}|$ is finite, we can determine whether Γ and Γ' are in the same orbit or not. Applying this method to all pairs Γ and Γ' with the same ADE-type, we obtain a complete set of representatives of RDP-configurations modulo $\mathrm{aut}(Y)$.

7.2. **$(4A_1, 4A_1)$-generic and $(4A_1, D_4)$-generic Enriques surfaces.** Let Y be a $(4A_1, 4A_1)$-generic Enriques surface (No. 7 of Table 1.1). We construct a sequence $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ from the primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ of type `96C`. The complete set V_0 of representatives of orbits of the action of $\mathrm{aut}(Y)$ on the set of $L_{26}/S_Y(2)$-chambers contained in Nef_Y consists of 5 elements with the orders of stabilizer subgroups $1, 1, 1, 2, 1$. Since $\mathrm{vol}(D_0) = 1_{\mathrm{BP}}/72$, we have

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \mathrm{vol}(D_0)\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1}\right) = \frac{1_{\mathrm{BP}}}{16} = \frac{1_{\mathrm{BP}}}{|W(R_{4A_1})|}.$$

The set $\mathcal{R}_{\mathrm{temp}}$ is of size 56 and the set $\mathcal{E}_{\mathrm{temp}}$ is of size 6270.

We also construct $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ for a $(4A_1, D_4)$-generic Enriques surface (No. 8 of Table 1.1) from the primitive embedding of type `96C`. The set V_0 consists of

18 elements with the orders of stabilizer subgroups $4, \dots, 4$. We have $|\mathcal{R}_{\mathrm{temp}}| = 154$ and $|\mathcal{E}_{\mathrm{temp}}| = 21452$.

7.3. **A** (D_5, D_5)**-generic Enriques surface.** We have to use the primitive embedding of type `40A` to construct $S_Y(2) \hookrightarrow S_X \hookrightarrow L_{26}$ for a (D_5, D_5)-generic Enriques surface (No. 24 of Table 1.1). The set V_0 consists of 6 elements with the orders of stabilizer subgroups $2, \dots, 2$. In this case, we have $\mathrm{vol}(D_0) = 1_{\mathrm{BP}}/5760$ and

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \mathrm{vol}(D_0)\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right) = \frac{1_{\mathrm{BP}}}{1920} = \frac{1_{\mathrm{BP}}}{|W(R_{D_5})|}.$$

We have $|\mathcal{R}_{\mathrm{temp}}| = 15$ and $|\mathcal{E}_{\mathrm{temp}}| = 758$.

7.4. **Enriques surfaces with finite automorphism group.** Let Y be an Enriques surface with finite automorphism group of type I in Kondo's classification [16]. We assume that Y is chosen very general so that the covering $K3$ surface X is of Picard number 19 and satisfies $\mathrm{O}(T_X, \omega) = \{\pm 1\}$. Then Y is $(E_8{+}A_1, E_8{+}A_1)$-generic (No. 172 of Table 1.1). The automorphism group $\mathrm{Aut}(Y)$ is a dihedral group of order 8, and its image $\mathrm{aut}(Y)$ in $\mathrm{O}^{\mathcal{P}}(S_Y)$ is order 4. The Enriques surface Y has exactly 12 smooth rational curves, and their dual graph is given in [16, Fig. 1.4]. The chamber Nef_Y is isomorphic to an $L_{26}/L_{10}(2)$-chamber D_0 of the primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ of type `12A`, and hence $\mathrm{vol}(D_0) = 1_{\mathrm{BP}}/174182400$ (see [6]). Therefore

$$\mathrm{vol}(\mathrm{Nef}_Y/\mathrm{aut}(Y)) = \frac{\mathrm{vol}(D_0)}{4} = \frac{1_{\mathrm{BP}}}{2^{14}3^5 5^2 7} = \frac{2_{\mathrm{BP}}}{|W(R_{E_8+A_1})|}.$$

The group $\mathrm{aut}(Y)$ decomposes $\mathcal{R}(Y)$ as $2+2+2+2+4$.

For a very general Enriques surface Y with finite automorphism group of type II, the chamber Nef_Y is isomorphic to an $L_{26}/L_{10}(2)$-chamber D_0 of the primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ of type `12B`. We have $\mathrm{vol}(D_0) = 1_{\mathrm{BP}}/3870720$. Note that $3870720 \cdot |\mathfrak{S}_4| = |W(R_{D_9})|$. The Enriques surface Y is (D_9, D_9)-generic (No. 184 of Table 1.1), and we have $\mathrm{Aut}(Y) \cong \mathrm{aut}(Y) \cong \mathfrak{S}_4$. The group $\mathrm{aut}(Y)$ decomposes $\mathcal{R}(Y)$ as $6+6$.

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