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Static Axially Symmetric Vacuum Solutions of Einstein's Equations

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ABSTRACT
Exact analytic solutions of Einstein's equations are difficult because of the high nonlinearity of the equations. Many researchers have shown that field equations for static spherically symmetric and cylindrically symmetric space-times to get physical solutions. Besides, solutions with axial symmetry are less studied. In this paper, we have demonstrated the technique for generating static axially symmetric vacuum solutions from known solutions. Using the technique some new realistic solutions is generated.

Keywords:-*Space-times, spherically symmetric, cylindrically symmetric, axially symmetric, axis of symmetry, spheroidal coordinates.*

INTRODUCTION

Spherically symmetric solutions have attracted attentions of many researchers working in this field due to several reasons. Spherically symmetric perfect fluid solutions [1-10] are interesting because they are first approximations in finding any realistic solution describing a relativistic star. Moreover cylindrical symmetric solutions [11-13] have been shown in the literature. Space-times having symmetries about an axis are said to be axially symmetric. Gravitational fields due to rotating sources are represented by axially symmetric space-times. When rotation of the source is uniform then the space-time is said to be stationary axially symmetric. Static axially symmetric space-times are those for which rotation of the source is zero. Weyl-Lewis-Papapetrou [17-18] form of stationary axially symmetric space-time metric can be written as

$$ds^2 = f\,dt^2 - e^{\mu}(d\rho^2 + dz^2) - l\,d\varphi^2 - 2k\,dt\,d\varphi \qquad (1)$$

where (ρ, φ, z) are cylindrical-polar-like coordinates and f, k, l, μ are functions of ρ and z. It should be noted that in a curved space in general it is not possible to define cylindrical-polar coordinates or Cartesian coordinates etc. Here cylindrical-polar-like coordinate means that for asymptotically symmetric solutions, at a large distance from the source the metric tends to the flat space-time metric

$$ds^2 = dt^2 - d\rho^2 - \rho^2\,d\varphi^2 - dz^2$$

in the familiar cylindrical-polar coordinates related to Cartesian coordinates (x, y, z) by $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, z = z. The metric (1) possesses a symmetry about the line $\rho = 0$, called the axis of symmetry.

We are interested in vacuum solutions so that Einstein's equations reduce to $R_{\mu\nu}=0$. Weyl showed that one can use the field equations to impose the following condition on the functions f, k and l

$$D^2 = fl + k^2 = \rho^2 \qquad (2)$$

The definition of $\sqrt{f\,l + k^2}$ as the new coordinate ρ simplifies the metric (1) reducing it to only three independent metric components instead of four. The metric (1) can be written as

$$ds^2 = f\,dt^2 - f^{-1}e^{2\gamma}(d\rho^2 + dz^2) - l\,d\varphi^2 - 2k\,dt\,d\varphi \qquad (3)$$

Papapetrou found it convenient to use a function ω instead of k, defined by $k=\omega f$. Using this in (2) we obtain

$$l = f^{-1}\rho^2 - \omega^2 f \qquad (4)$$

With (4), metric (3) reduces to

$$ds^2 = f(dt - \omega\,d\varphi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\,d\varphi^2] \qquad (5)$$

The metric form (5) is called the Papapetrou metric.

Rest of this chapter is organized in the following way. In Section-2 a review of Weyl solutions of the field equations for static axially symmetric space-times is provided. In Section-3, general static axially symmetric solution is presented in closed form. In Section-4, a technique for generating new solutions from known solution is presented. The technique is demonstrated by generating some new realistic solutions. Finally in Section-5, some concluding remarks are given.

FIELD EQUATIONS FOR STATIC AXIALLY SYMMETRIC VACUUM SPACETIMES AND WEYL SOLUTIONS

In the absence of rotation $\omega=0$. Thus in the static case metric (5) reduces to

$$ds^2 = e^{2\psi(\rho,z)}\,dt^2 - e^{-2\psi(\rho,z)}[e^{2\gamma(\rho,z)}(d\rho^2 + dz^2) + \rho^2\,d\varphi^2] \qquad (6)$$

where $f=e^{2\psi}$. The metric form (6) is called the Weyl metric. For the metric (6) the equations $R_3^3 - R_4^4 = 0$, $R_1^1 - R_2^2 = 0$, $R_2^1 = 0$, which are valid for vacuum solutions yield respectively

$$\nabla^2\psi = \psi_{zz} + \frac{1}{\rho}\psi_\rho + \psi_{\rho\rho} = 0 \qquad (7)$$

$$\gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2),\ \gamma_z = 2\rho\psi_\rho\psi_z \qquad (8)$$

where $\psi_z = \frac{\partial\psi}{\partial z}$, $\psi_\rho = \frac{\partial\psi}{\partial\rho}$ etc. Equations (7) and (8) are the only independent equations for the two unknown metric functions $\psi(\rho,z)$ and $\gamma(\rho,z)$, which are to be solved. Method of solving the field equations (7), (8) is as follows.

One has to assume a solution of Laplace's equation (7) in two dimensions which specifies $\psi(\rho,z)$. When the equation for $\psi(\rho,z)$ is satisfied, it ensures that equations (8) will be integrable and hence $\gamma(\rho,z)$ will be specified.

A class of exact solutions of equations (7), (8) was found by Weyl and the method of solution is as follows.

Let us introduce prelate spheroidal coordinates (x, y) defined by the transformation equations

$$\rho = k(x^2-1)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}},\ z = kxy \qquad (9)$$

where k is an arbitrary constant. From (9), we have

$$\rho^2 + z^2 + k^2 = k^2(x^2+y^2)$$

$$\Rightarrow (x+y)^2 = \frac{1}{k^2}[\rho^2+(z+k)^2]$$

$$\therefore\ x+y = \frac{1}{k}[\rho^2+(z+k)^2]^{\frac{1}{2}} \qquad (10)$$

Similarly

$$x-y = \frac{1}{k}[\rho^2+(z-k)^2]^{\frac{1}{2}} \qquad (11)$$

From (10) and (11) we get

$$x = \frac{1}{2k}\left[\sqrt{\rho^2+(z+k)^2}+\sqrt{\rho^2+(z-k)^2}\right] \qquad (12)$$

$$y = \frac{1}{2k}\left[\sqrt{\rho^2+(z+k)^2}-\sqrt{\rho^2+(z-k)^2}\right] \qquad (13)$$

In this coordinate system the Laplace operator ∇^2 is given by

$$\nabla^2 = \frac{k^2}{x^2-y^2}\left[\frac{\partial}{\partial x}\left\{(x^2-1)\frac{\partial}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial}{\partial y}\right\}\right]$$

Hence equation (7) reduces to

$$\frac{k^2}{x^2-y^2}\left[\frac{\partial}{\partial x}\left\{(x^2-1)\frac{\partial}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{(1-y^2)\frac{\partial}{\partial y}\right\}\right]\psi = 0 \qquad (14)$$

Let us consider the solution of equation (14) that can be written as a product

$$\psi(x,y) = u(x)v(y) \qquad (15)$$

If we substitute (15) in (14) we obtain

$$\frac{1}{u}\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial u}{\partial x}\right] = \frac{1}{v}\frac{\partial}{\partial y}\left[(y^2-1)\frac{\partial v}{\partial y}\right] = \text{n(n+1)} = \text{Constant}$$

Thus Laplace's equation (14) yields the Legendre equations

$$\frac{\partial}{\partial x}\left[\left(x^2-1\right)\frac{\partial u}{\partial x}\right]-n(n+1)u=0$$
$$\frac{\partial}{\partial y}\left[\left(y^2-1\right)\frac{\partial v}{\partial y}\right]-n(n+1)v=0$$

Weyl solutions are given by

$$\psi=\sum_{n=0}^{\infty}a_n\,Q_n(x)\,P_n(y)$$

where P_n and Q_n are Legendre polynomials of the first and second kinds. As a special case if we let n = 0 then we get

$$\psi=\frac{\delta}{2}\log\frac{x-1}{x+1}$$
$$\gamma=\frac{\delta}{2}\log\frac{x^2-1}{x^2-y^2}$$

where δ is an arbitrary constant. In terms of the coordinates (ρ,z) we have

$$\psi(\rho,z)=\frac{\delta}{2}\log\frac{\sigma_1+\sigma_2-2m}{\sigma_1+\sigma_2+2m}=\frac{\delta}{2}\log\frac{z-m+\sigma_2}{z+m+\sigma_1}$$

where k = m, $\sigma_1=\sqrt{\rho^2+(z+m)^2}$ and $\sigma_2=\sqrt{\rho^2+(z-m)^2}$.

For $\delta=1$ this gives the Schwarzschild solution. For $\delta=1$ we get

$$e^{2\gamma}=\left[(\sigma_1+\sigma_2)^2-4m^2\right]/4\sigma_1\sigma_2$$

ALL STATIC AXIALLY SYMMETRIC VACUUM SOLUTIONS

The general axially symmetric static solution of Einstein's vacuum field equations in closed form has been found by Waylen [14] in the canonical coordinates ρ and z. This general solution depends on a single solution generating function. To see how this is done let us note that equation (7) is a second order linear partial differential equation for $\psi(\rho,z)$. Therefore general solution of equation (7) contains two arbitrary functions, say f and g. In the following we will demonstrate that

$$\psi(\rho,z)=\frac{1}{\pi}\int_0^{\pi}[\,f(u)+g(u)\log\{(\frac{\rho}{\rho_0})\sin^2\theta\}]d\theta \qquad (16)$$

where $u=z+i\rho\cos\theta$, is a solution of (7). Since (16) contains two arbitrary functions f(u) and g(u), it is the general solution of (7). Now (16) can be expressed as

$$\psi(\rho,z)=\varphi_1(\rho,z)+\varphi_2(\rho,z)$$

where

$$\varphi_1(\rho,z)=\frac{1}{\pi}\int_0^{\pi}f(u)d\theta \qquad (17)$$

and

$$\varphi_2(\rho, z) = \frac{1}{\pi}\int_0^{\pi} g(u)\log\{(\frac{\rho}{\rho_0})\sin^2\theta\}d\theta \quad (18)$$

We will show that φ_1 and φ_2 are solutions of equation (7). From (17) we get

$$\frac{1}{\rho}\frac{\partial\varphi_1}{\partial\rho} = \frac{i}{\pi\rho}\int_0^{\pi} f'(u)\cos\theta\, d\theta \quad (19)$$

$$\frac{\partial^2\varphi_1}{\partial\rho^2} = -\frac{1}{\pi}\int_0^{\pi} f''(u)\cos^2\theta\, d\theta \quad (20)$$

$$\frac{\partial^2\varphi_1}{\partial z^2} = \frac{1}{\pi}\int_0^{\pi} f''(u)\, d\theta \quad (21)$$

Now

$$\int_0^{\pi} f'(u)\cos\theta\, d\theta = i\rho\left\{\int_0^{\pi} f''(u)\, d\theta - \int_0^{\pi} f''(u)\cos^2\theta\, d\theta\right\} \quad (22)$$

Putting (22) in (19) we obtain

$$\frac{1}{\rho}\frac{\partial\varphi_1}{\partial\rho} = -\frac{1}{\pi}\int_0^{\pi} f''(u)\, d\theta + \frac{1}{\pi}\int_0^{\pi} f''(u)\cos^2\theta\, d\theta \quad (23)$$

From (20), (21) and (23) we get

$$\frac{\partial^2\varphi_1}{\partial z^2} + \frac{1}{\rho}\frac{\partial\varphi_1}{\partial\rho} + \frac{\partial^2\varphi_1}{\partial\rho^2} = 0 \quad (24)$$

Therefore $\varphi_1(\rho, z)$ is a solution of (7).

Again from (18) we get

$$\frac{1}{\rho}\frac{\partial\varphi_2}{\partial\rho} = \frac{1}{\pi\rho^2}\int_0^{\pi} g(u)\, d\theta + \frac{i}{\pi\rho}\log(\frac{\rho}{\rho_0})\int_0^{\pi} g'(u)\cos\theta\, d\theta$$
$$+\frac{2i}{\pi\rho}\int_0^{\pi} g'(u)\cos\theta\log(\sin\theta)\, d\theta \quad (25)$$

$$\frac{\partial^2\varphi_2}{\partial\rho^2} = -\frac{1}{\pi\rho^2}\int_0^{\pi} g(u)\, d\theta + \frac{2i}{\pi\rho}\int_0^{\pi} g'(u)\cos\theta\, d\theta - \frac{1}{\pi}\log(\frac{\rho}{\rho_0})\int_0^{\pi} g''(u)\cos^2\theta\, d\theta$$
$$-\frac{2}{\pi}\int_0^{\pi} g''(u)\cos^2\theta\log(\sin\theta)\, d\theta \quad (26)$$

$$\frac{\partial^2\varphi_2}{\partial z^2} = \frac{1}{\pi}\log(\frac{\rho}{\rho_0})\int_0^{\pi} g''(u)\, d\theta + \frac{2}{\pi}\int_0^{\pi} g''(u)\log(\sin\theta)\, d\theta \quad (27)$$

Now

$$\frac{2i}{\pi\rho}\int_0^{\pi} g'(u)\cos\theta\,\log(\sin\theta)\, d\theta = -\frac{2}{\pi}\int_0^{\pi} g''(u)\sin^2\theta\log(\sin\theta)\, d\theta$$
$$-\frac{2i}{\pi\rho}\int_0^{\pi} g'(u)\cos\theta\, d\theta \quad (28)$$

and

$$\frac{i}{\pi\rho}\log(\frac{\rho}{\rho_0})\int_0^{\pi} g'(u)\cos\theta\, d\theta = -\frac{1}{\pi}\log(\frac{\rho}{\rho_0})\int_0^{\pi} g''(u)\sin^2\theta\, d\theta \quad (29)$$

From (25) – (29) we get

$$\frac{\partial^2\varphi_2}{\partial z^2}+\frac{1}{\rho}\frac{\partial\varphi_2}{\partial\rho}+\frac{\partial^2\varphi_2}{\partial\rho^2}=0$$

Therefore φ_2 is a solution of equation (7). Thus we have demonstrated that

$$\psi(\rho,z)=\varphi_1(\rho,z)+\varphi_2(\rho,z)$$

is the general solution of (7). To allow the metric function $\psi(\rho,z)$ to assume finite values on the axis of symmetry ρ = 0, which lies in the vacuum, we require g(u) to vanish. Hence the finite general solution of equation (7) is given by

$$\psi(\rho,z)=\frac{1}{\pi}\int_0^{\pi} f(u)\,d\theta \tag{30}$$

Any specification of f(u) generates a solution of equation (7) through equation (30).

Substituting (30) into the pair of equations (8) we get

$$\gamma_\rho=-\frac{\rho}{\pi^2}\left[\{\int_0^{\pi} f'(u)\cos\theta\, d\theta\}^2+\{\int_0^{\pi} f'(u)\,d\theta\}^2\right] \tag{31}$$

$$\gamma_z=\frac{2i\rho}{\pi^2}\{\int_0^{\pi} f'(u)\cos\theta\, d\theta\}\{\int_0^{\pi} f'(u)\,d\theta\} \tag{32}$$

Integrating (31) with respect to ρ we obtain

$$\gamma=k(z)-\frac{1}{\pi^2}\int_0^{\rho}\rho\left[\{\int_0^{\pi} f'(u)\cos\theta\,d\theta\}^2+\{\int_0^{\pi} f'(u)\,d\theta\}^2\right]d\rho \tag{33}$$

Differentiating (33) with respect to z we get

$$\gamma_z=k'(z)-\frac{2}{\pi^2}\int_0^{\rho}\rho\left[\int_0^{\pi} f'(u)\cos\theta\,d\theta\cdot\int_0^{\pi} f''(u)\cos\theta\,d\theta+\int_0^{\pi} f'(u)\,d\theta\cdot\int_0^{\pi} f''(u)\,d\theta\right]d\rho \tag{34}$$

From (22) we have

$$\int_0^{\pi} f''(u)\,d\theta=\int_0^{\pi} f''(u)\cos^2\theta\,d\theta-\frac{i}{\rho}\int_0^{\pi} f'(u)\cos\theta\,d\theta \tag{35}$$

Putting (35) in (34) we obtain

$$\gamma_z=k'(z)-\frac{2}{\pi^2}\int_0^{\rho}\left[\begin{array}{c}\rho\left\{\int_0^{\pi} f'(u)\cos\theta\,d\theta\cdot\int_0^{\pi} f''(u)\cos\theta\,d\theta+\int_0^{\pi} f''(u)\cos^2\theta\,d\theta\cdot\int_0^{\pi} f'(u)\,d\theta\right\}\\ -\int_0^{\pi} f'(u)\cos\theta\,d\theta\cdot\int_0^{\pi} f'(u)\,d\theta\end{array}\right]d\rho$$

or, $$\gamma_z=k'(z)+\frac{2i}{\pi^2}\int_0^{\rho}\frac{\partial}{\partial\rho}\left[\int_0^{\pi} f'(u)\cos\theta\,d\theta\cdot\int_0^{\pi} f'(u)\,d\theta\right]d\rho$$

$$=k'(z)+\frac{2i\rho}{\pi^2}\int_0^{\pi} f'(u)\cos\theta\,d\theta\cdot\int_0^{\pi} f'(u)\,d\theta \tag{36}$$

(36) accords with (32) if k(z) = k = constant. Therefore if k = constant, (33) is the solution of both the equations (31) and (32). Condition of regularity on the axis of symmetry lying in vacuum requires k = 0.

NEW SOLUTIONS FROM KNOWN SOLUTIONS

In Section-3, it has been shown that the finite general solution of equation (7) can be written as

$$\psi(\rho,z)=\frac{1}{\pi}\int_0^{\pi} f(u)\,d\theta \qquad (37)$$

where $u=z+i\rho\cos\theta$ and f(u) is an arbitrary function of u. Now partial differentiation or integration of f(u) with respect to z results in function of u. Let us denote partial differentiation and integration of f(u) with respect to z by F(u) and F(u) respectively,

$$F(u)=\frac{\partial\psi}{\partial z}=\frac{1}{\pi}\int_0^{\pi} f'(u)\,d\theta \text{ and } \mathrm{F}(u) = \int f(u)dz$$

Then from (37) we obtain

$$\psi_1(\rho,z)=\psi_z(\rho,z)=\frac{1}{\pi}\int_0^{\pi} F(u)\,d\theta \qquad (38)$$

and

$$\psi_2(\rho,z)=\frac{1}{\pi}\int_0^{\pi} \mathcal{F}(u)\,d\theta \qquad (39)$$

From (38) and (39) we find that if $\psi(\rho,z)$ is a solution of (7) then $\psi_1(\rho,z)$ and $\psi_2(\rho,z)$ are also solutions of (7) i.e. partial differentiation and integration with respect to z of a solution $\psi(\rho,z)$ of equation (7) are also solutions of (7). This gives a way of generating static axially symmetric vacuum solutions of Einstein's equations from known solutions. The technique is demonstrated below by generating some new solutions from known solutions.

(1) Let us consider the harmonic function

$$\psi(\rho,z)=-\frac{C}{\sqrt{z^2+\rho^2}} \qquad (40)$$

from which we get Curzon's solution [17]. Differentiating (40) with respect to z we obtain

$$\psi_1(\rho,z)=\psi_z(\rho,z)=\frac{Cz}{(\rho^2+z^2)^{\frac{3}{2}}} \qquad (41)$$

From (41) we obtain

$$\frac{\partial\psi_1}{\partial z}=\frac{C(\rho^2-2z^2)}{(\rho^2+z^2)^{\frac{5}{2}}} \text{ and } \frac{\partial\psi_1}{\partial\rho}=-\frac{3C\rho z}{(\rho^2+z^2)^{\frac{5}{2}}} \qquad (42)$$

Putting (42) in equations (8) we obtain

$$\gamma_\rho=C^2\left[\frac{15\rho z^2}{(\rho^2+z^2)^4}-\frac{\rho}{(\rho^2+z^2)^3}-\frac{18\rho z^4}{(\rho^2+z^2)^5}\right] \qquad (43)$$

$$\gamma_z = 6C^2\rho^2\left[\frac{2z}{(\rho^2+z^2)^4} - \frac{3\rho^2 z}{(\rho^2+z^2)^5}\right] \tag{44}$$

From (43) and (44) we get

$$\gamma = \frac{C^2\rho^2(\rho^2-8z^2)}{4(\rho^2+z^2)^4} + k_1(z) = \frac{C^2\rho^2(\rho^2-8z^2)}{4(\rho^2+z^2)^4} + k_2(\rho) \tag{45}$$

From (45) we conclude that, $k_1(z) = k_2(\rho) = k =$ constant. Therefore we have obtained the following new solution by partially differentiating Curzon's solution with respect to z,

$$\psi_1(\rho,z) = \frac{Cz}{(\rho^2+z^2)^{\frac{3}{2}}}, \quad \gamma(\rho,z) = \frac{C^2\rho^2(\rho^2-8z^2)}{4(\rho^2+z^2)^4} + k \tag{46}$$

Solution (46) has finite values on the axis of symmetry and is asymptotically flat.

It should be noted that, integration of $\psi_1(\rho,z)$ with respect to z gives Curzon's solution [19].

(2) Putting C = C_1, C_2, $z = z \mp m$ in (40) we get the following solutions $\overline{\psi}_1(\rho,z)$ and $\overline{\psi}(\rho,z)$ of equation (7)

$$\overline{\psi}_1(\rho,z) = \frac{C_1}{\sqrt{(z-m)^2+\rho^2}}, \quad \overline{\psi}_2(\rho,z) = \frac{C_2}{\sqrt{(z+m)^2+\rho^2}}$$

Integrating $\overline{\psi}_1$ and $\overline{\psi}_2$ with respect to z we get the following solutions of equation (7),

$$\int \overline{\psi}_1 dz = C_1 \log\frac{z-m+\sqrt{(z-m)^2+\rho^2}}{\rho}$$

$$\int \overline{\psi}_2 dz = C_2 \log\frac{z+m+\sqrt{(z+m)^2+\rho^2}}{\rho}$$

Since linear combination of any two solutions of equation (7) is also a solution

$$\overline{\psi}(\rho,z) = \log\frac{\rho^{C_2-C_1}\left\{z-m+\sqrt{(z-m)^2+\rho^2}\right\}^{C_1}}{\left\{z+m+\sqrt{(z+m)^2+\rho^2}\right\}^{C_2}} \tag{47}$$

is a solution of equation (7).

For $C_1 = C_2 = 1$, (47) reduces to

$$\overline{\psi}(\rho,z) = \log\frac{z-m+\sqrt{(z-m)^2+\rho^2}}{z+m+\sqrt{(z+m)^2+\rho^2}} \tag{48}$$

Putting (48) in (8) we obtain

$$\gamma(\rho,z) = \frac{(\rho_1+\rho_2)^2-4m^2}{4\rho_1\rho_2} \tag{49}$$

where

$\rho_1 = z + m + \sqrt{(z+m)^2 + \rho^2}$, $\rho_2 = z - m + \sqrt{(z-m)^2 + \rho^2}$

This is Schwarzschild solution in canonical coordinates ρ , z.

CONCLUSIONS

We have shown that partial differentiation or integration with respect to z of a harmonic function $\psi(\rho, z)$ results in a harmonic function. Using these results we have found some new solutions by differentiating / integrating Curzon's solution. The result can be used to classify all static axially symmetric vacuum solutions of Einstein's equations. This in turn may provide a way of classifying all stationary axially symmetric vacuum solutions.

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