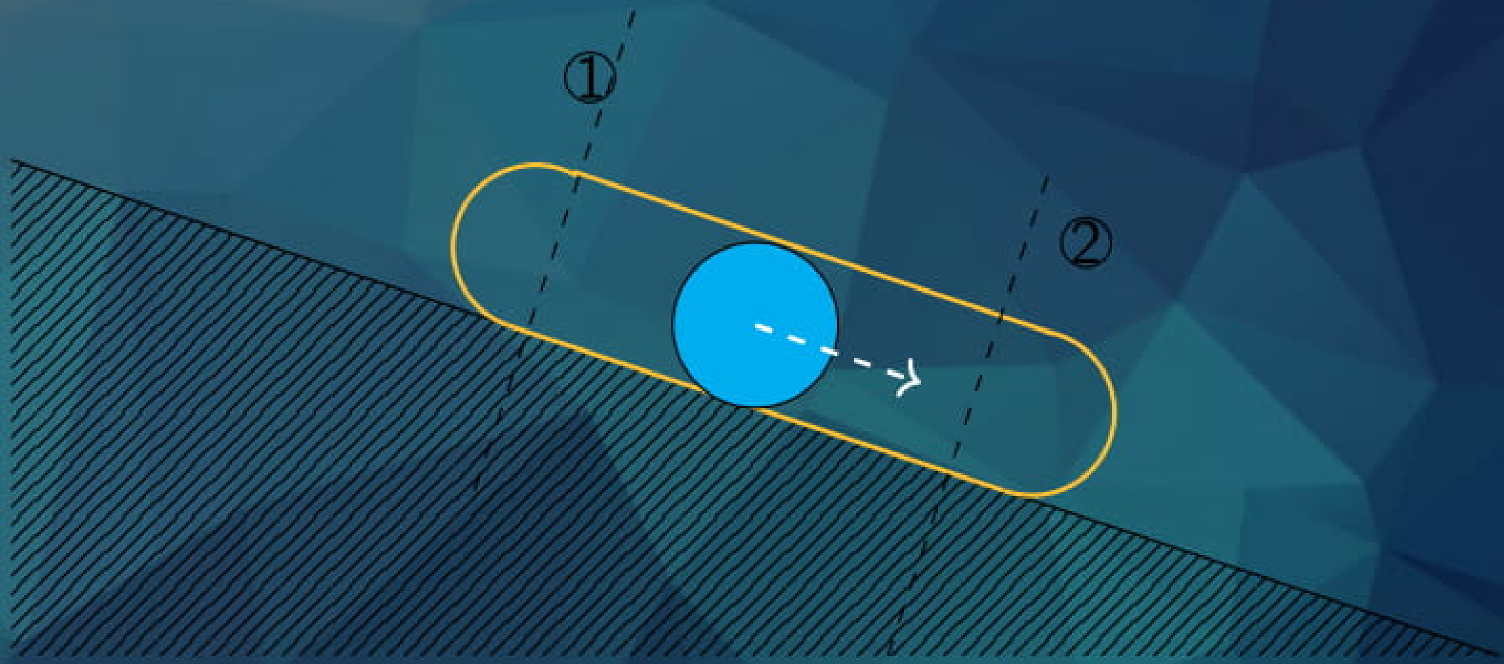
HELLENIC REPUBLIC
National and Kapodistrian
University of Athens

IPT

Jumping Bean

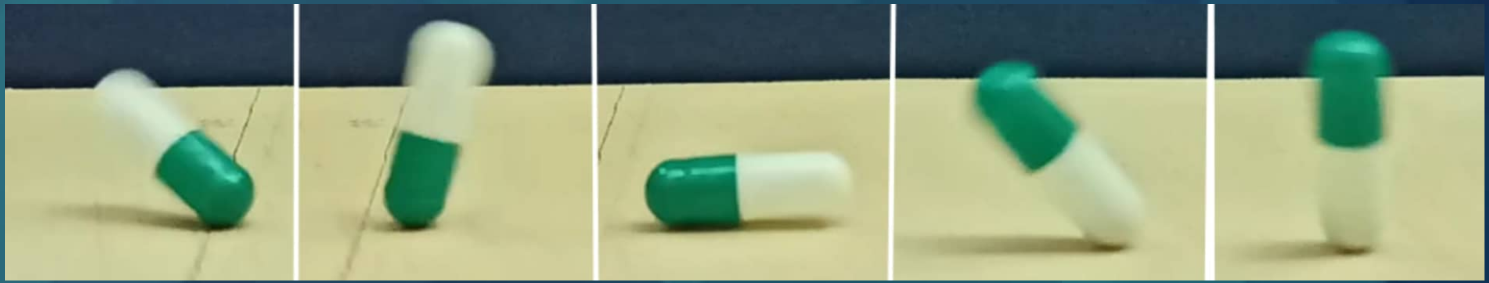
The problem

Let us consider the system of a ball of certain radius (R) and mass (m_B), inside a"bean"-like capsule of mass (m_b), which consists of a cylinder (with length ℓ) and two hemispheres at its ends.



The problem

Placed on an inclined surface at a certain inclination, the jumping bean will tumble down in a rather surprising way, seemingly standing up-right, **flipping end to end, instead of rolling. Investigate its motion.What is the slowest and fastest theoretical bean?**



Preliminary observations

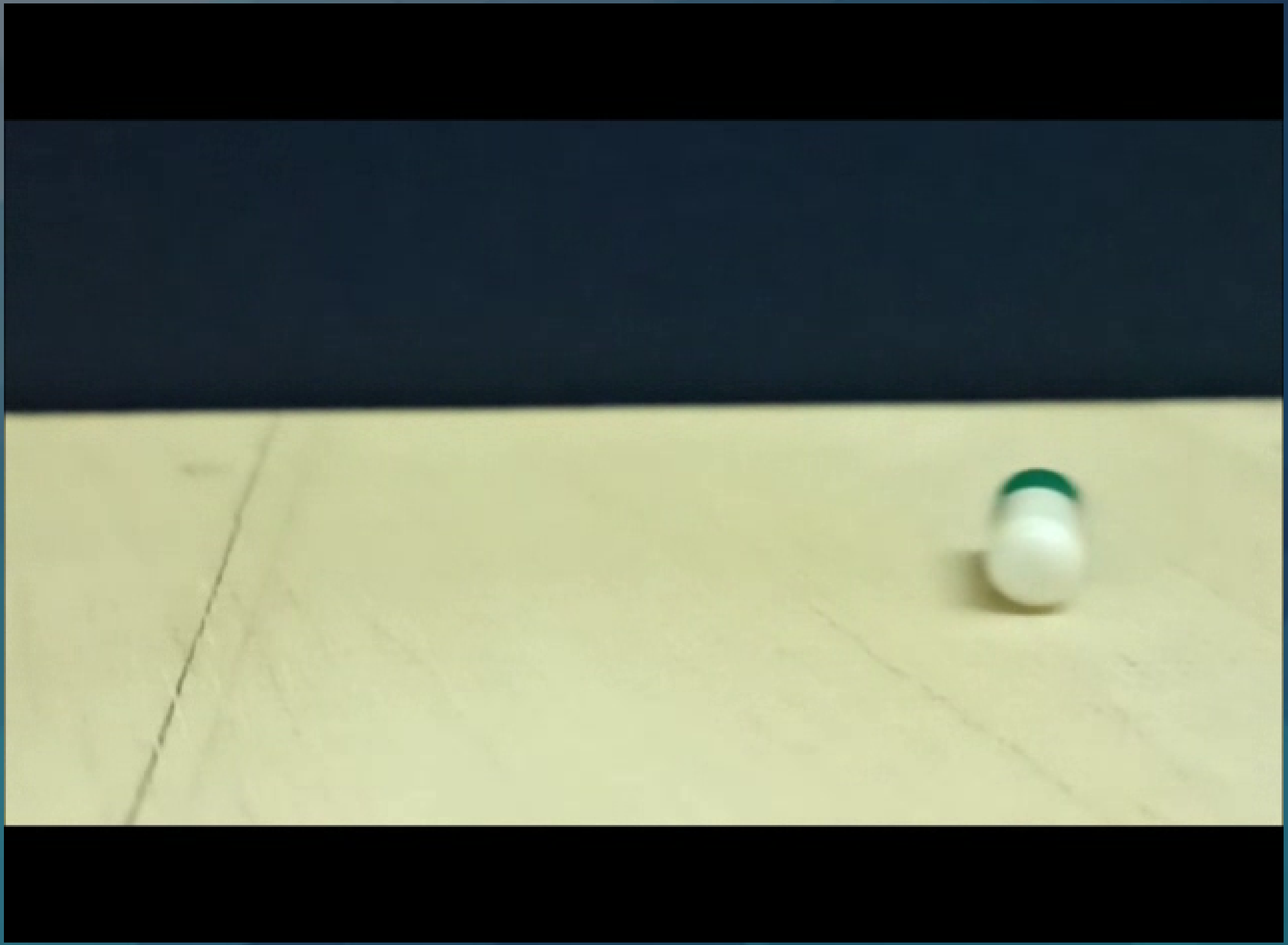
Observing the jumping bean's roll in 3 dimensions we can notice modes of movement:

- jumping bean mode
- waddle mode
- roll mode

Jumping bean mode

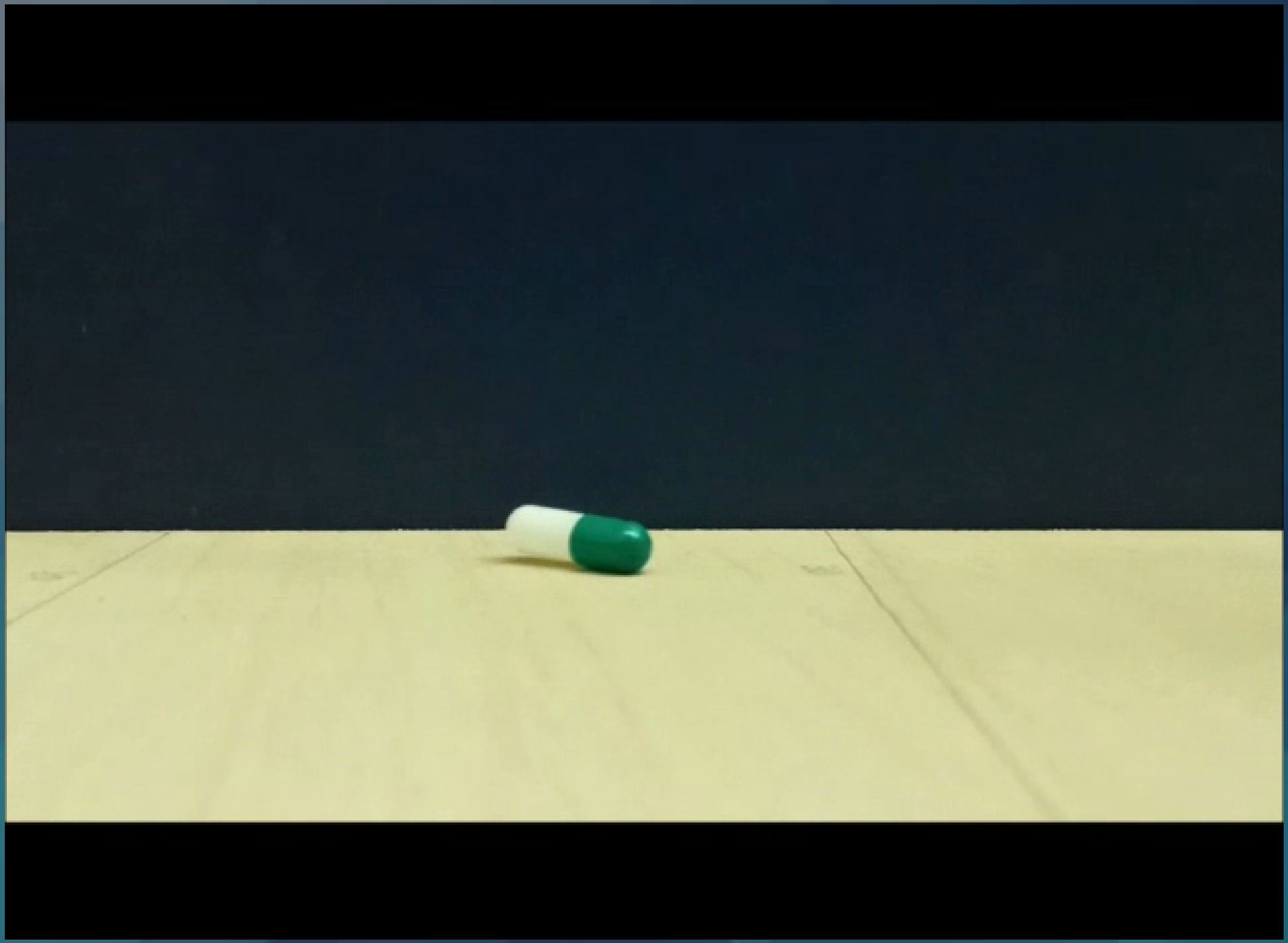
(video)

Roll mode



(video)

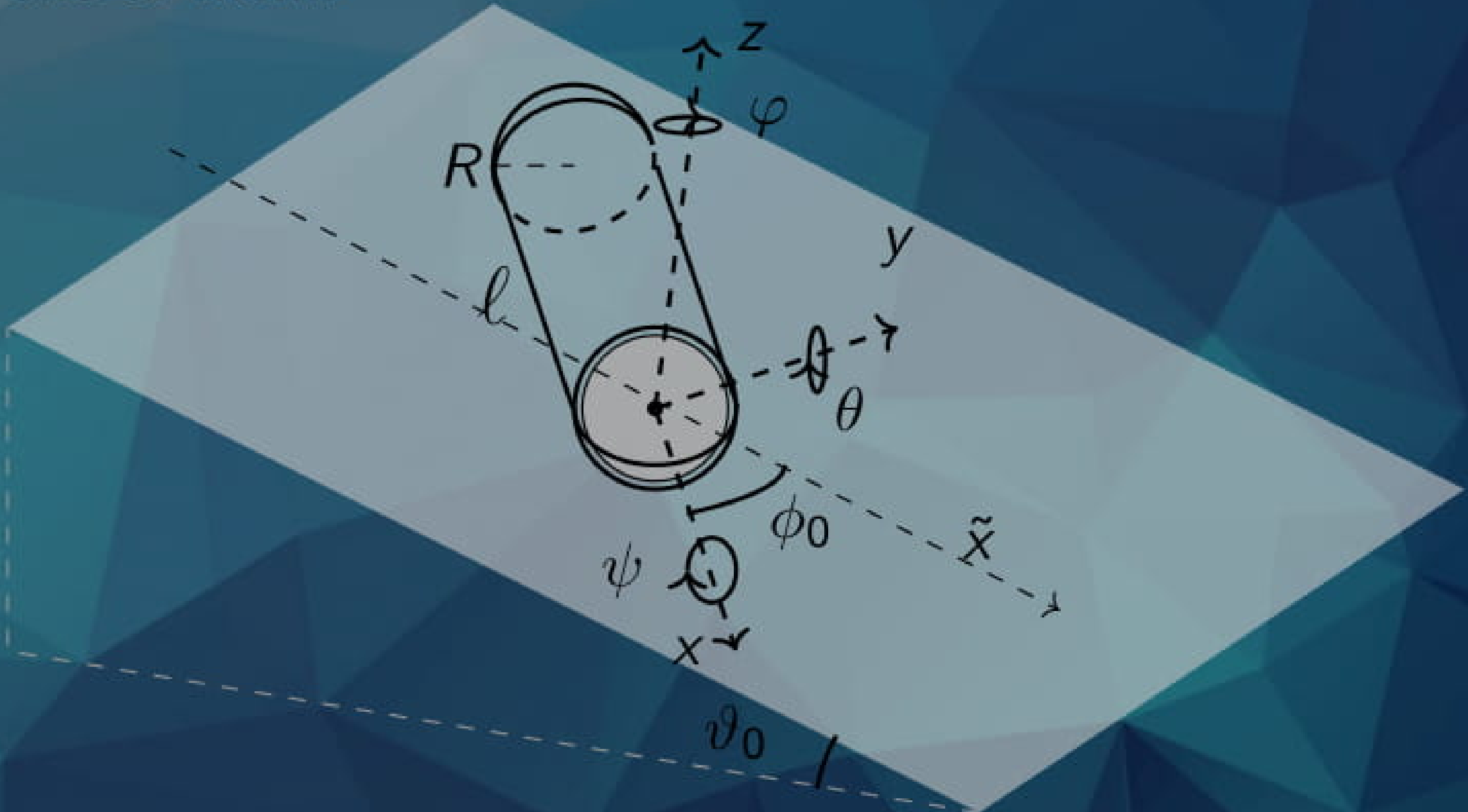
Waddle mode



(video)

Equations of motion

In the three dimensional case we can define the axes (x, y, z), where x is the axis of symmetry of the bean, and the angles (ψ, θ, φ) for the rotation around the three axes, respectively, and construct the Newton equations for each one of them.



Equations of motion

If we assume that $R_{bean} \simeq R_{Ball}$, so that the movement of the Ball is constrained in one direction, then
If the Ball is at the lower end of the bean:

$$\begin{cases} I_y\ddot{\theta} = -\frac{1}{2}m_b g\ell\cos(\theta+\vartheta_0) + (m_B+m_b)gR\cos(\varphi+\phi_0)\sin(\vartheta_0) \\ I_z\ddot{\varphi} = \frac{1}{2}m_b g\ell\sin(\varphi+\phi_0)\sin(\vartheta_0) - \mathcal{T}\delta_{\theta,0} \\ I_x\ddot{\psi} = (m_b+m_B)gR\sin(\varphi+\phi_0)\sin(\vartheta_0) \end{cases}$$

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If the Ball is at distance x_B from the end of the bean:

$$\begin{cases} \tilde{I}_y\ddot{\theta} = -\left(\frac{1}{2}\ell m_b + x_B m_B\right) g\cos(\theta+\vartheta_0) + m_b gR\cos(\varphi+\phi_0)\sin(\vartheta_0) \\ \tilde{I}_z\ddot{\varphi} = \left(\frac{1}{2}\ell m_b + x_B m_B\right) g\sin(\varphi+\phi_0)\sin(\vartheta_0) - \mathcal{T}\delta_{\theta,0} \\ \tilde{I}_x\ddot{\psi} = (m_b+m_B)gR\sin(\varphi+\phi_0)\sin(\vartheta_0) \\ \ddot{x}_B = \frac{5}{2}g\sin(\vartheta_0)\cos(\varphi+\phi_0) \end{cases}$$

With the constraints $\theta \geq 0$, $0 \leq x_B \leq \ell$, $y_B = 0$, $z_B = 0$.

The three types motions

In the above equations we can distinguish three different types of motion. We will consider $\theta(0) = \varphi(0) = \psi(0) = 0$ for each one of them.

- Cylinder motion ($\varphi_0 = \pi/2$, $x_B(0) = \ell/2$) (stable motion)

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- Cylinder motion ($\varphi_0 = \pi/2$, $x_B(0) = \ell/2$) (stable motion)
- Jumping Bean motion ($\varphi_0 = 0$) (unstable motion)
- Waddle motion ($\varphi_0 \neq 0$ and $\neq \pi/2$) (transition from the unstable to the stable motion)

Jumping Bean motion

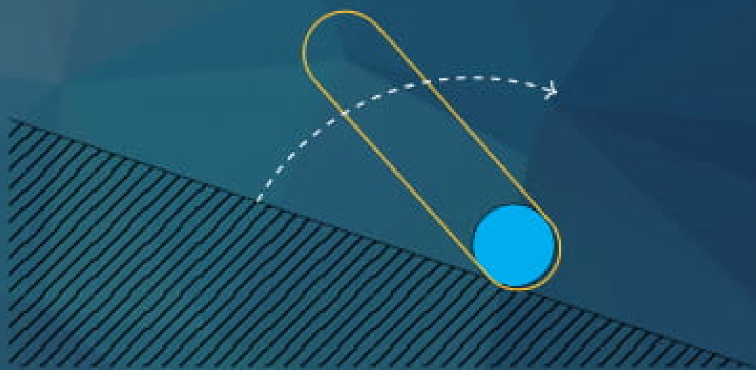
We will define as Jumping Bean the motion where $\varphi_0 = 0$ and thus the previous equations become

$$\begin{cases} \text{(Rotation of the system)} \\ I_y\ddot{\theta} = -\frac{1}{2} m_b g \ell \cos(\theta + \vartheta_0) + (m_B + m_b) g R \sin(\vartheta_0) \quad (1) \\ \text{(Movement of the Ball)} \\ \ddot{x}_B = \frac{5}{2} g \sin(\vartheta_0) \quad (2) \\ \ddot{\varphi}(0) = \dot{\varphi}(0) = \varphi(0) = 0 \Rightarrow \ \varphi(t) = 0 \\ \ddot{\psi}(0) = \dot{\psi}(0) = \psi(0) = 0 \Rightarrow \ \psi(t) = 0 \end{cases}$$

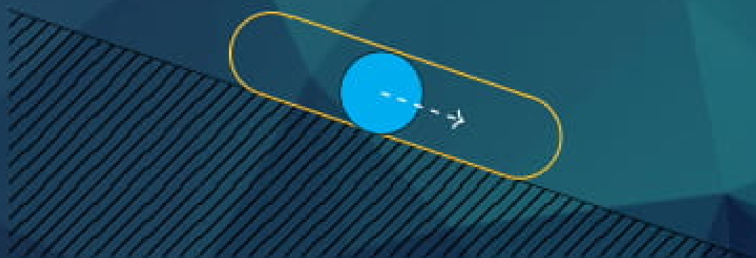
Jumping Bean motion

So the problem becomes two dimensional (only one type of rotation is allowed) and can be separated into two independent parts:

- Rotation of the Ball-bean system.



- The movement of the Ball inside the bean.



Acceleration and mean velocity

Assuming that, the macroscopic, movement of the system has an approximately constant acceleration we can calculate it from the above equations,

- Numerically integrating equation 1 from $\theta(0) = 0, \dot{\theta}(0) = 0$ to $\theta(t1) = \pi$, and computing $t1$ and $u1 = R\dot{\theta}(t1)$

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- Using $u1 = R\dot{\theta}(t1)$ as the initial velocity of the ball we calculate, from equation 2 the velocity $u2$ and time $t2$ in which the Ball moves from one end to the other.
- Integrating again equation 1 from $\theta(t2) = 0, \dot{\theta}(t2) = u2/R$ to $\theta(t3) = \pi$, and computing $t3$ and $u3 = R\dot{\theta}(t3)$

we find that the constant acceleration will be $a = \dfrac{(u3 - u1)}{(t3 - t1)}$.

Acceleration and mean velocity

Finally calculating the macroscopic mean velocity of the system for a given, and constant, distance X, we find

$$\langle u \rangle = \sqrt{\frac{aX}{2}}$$

which gives us a standard algorithm to determine the fastest bean.

Approximate dependence from the physical parameters

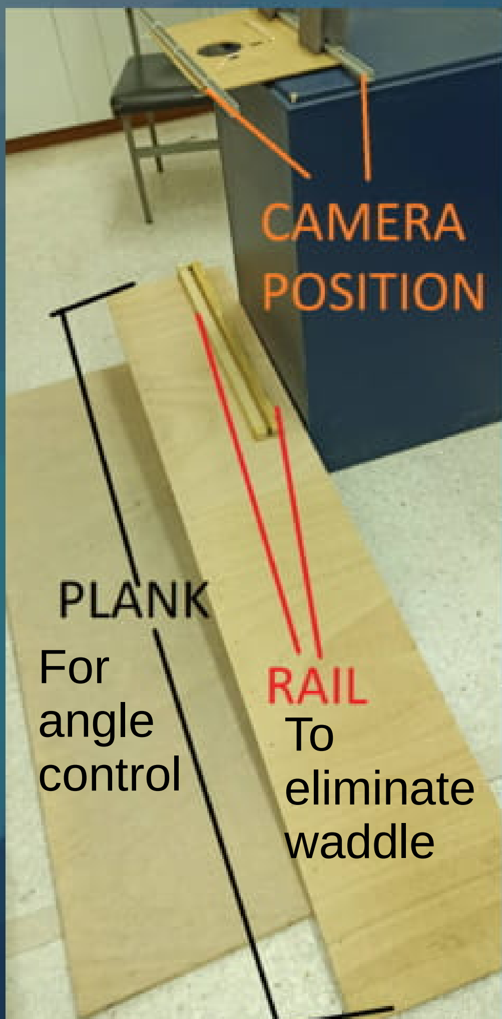
The above algorithm provides us with a way to more accurately calculate the macroscopic velocity of the system, but it does not give us its dependence from the physical parameters. We can have an approximation of this dependence by assuming that the bean with the bigger velocity is the one which rotates faster. Thus we define

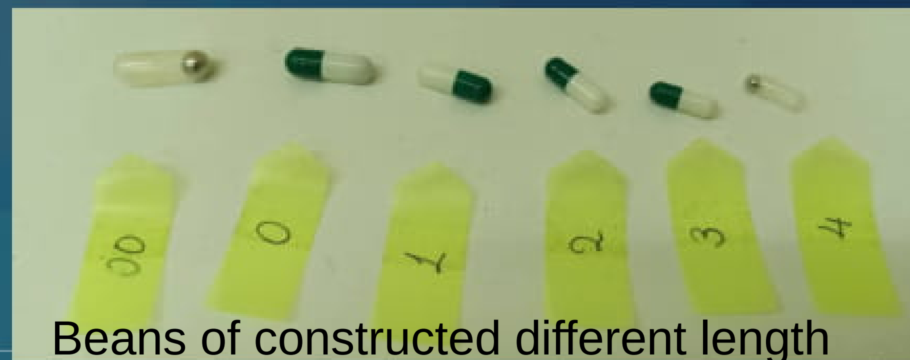
$$b = \frac{(m_B + m_b) g R \sin(\vartheta_0)}{\frac{1}{2} m_b g \ell \cos(\vartheta_0)} = \frac{2(1+\mu)\tan(\vartheta_0)}{\mu d}$$

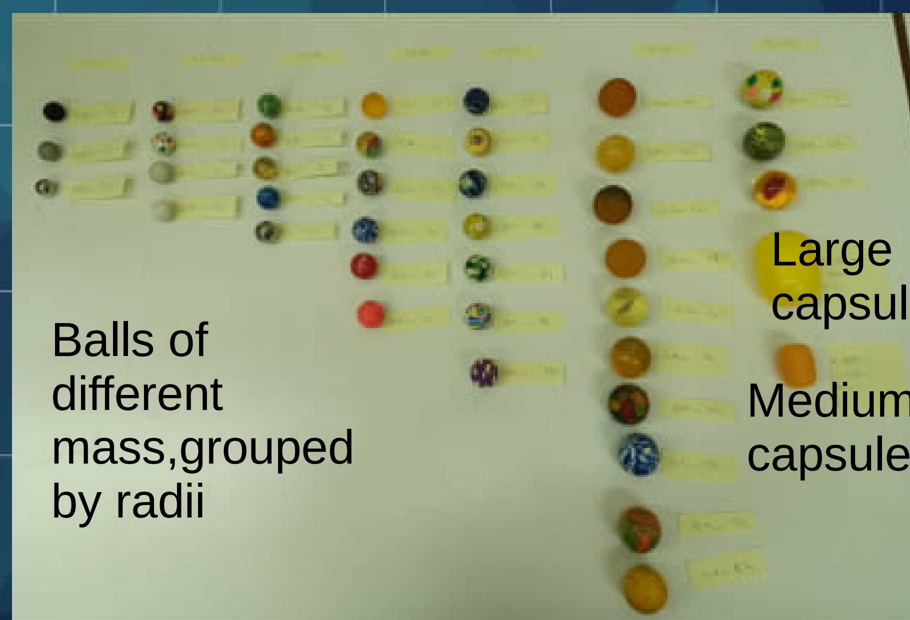
Where $\mu = m_B / m_b$ and $d = \ell / R$. So as bigger this ratio becomes the faster the rotation of the system will be.

Experimental Setup

A simple yet effective setup was devised in order to measure the speed of the "bean" in relation to the parameters of the system.







Experimental Investigation of Important parameters:Inclination and Surface

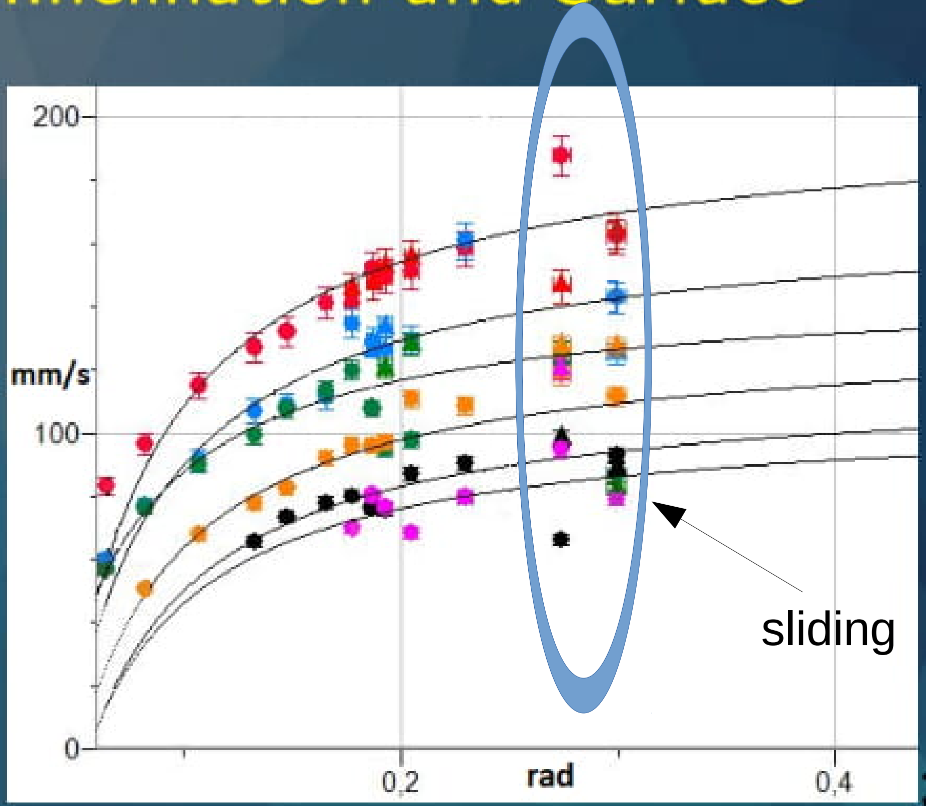
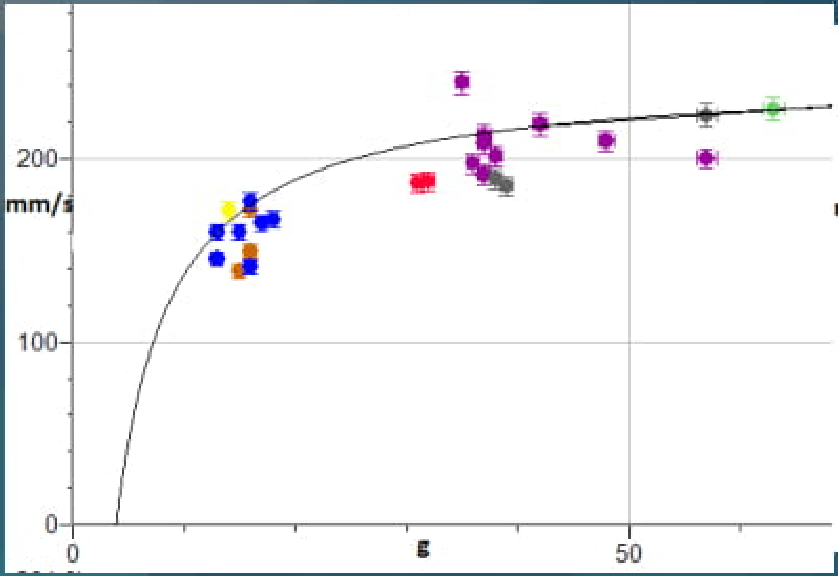


Figure: Velocity (mm/s) to angle of inclination(rad) for the different capsules (types 00,0,1,2,3,4/ red,blue,green,orange,black and purple respectively). The theoretical curves shows the mean velocity $\langle u \rangle$ as a function of the inclination for the given parameters.

Experimental Investigation of Important parameters:Mass and radius of the ball



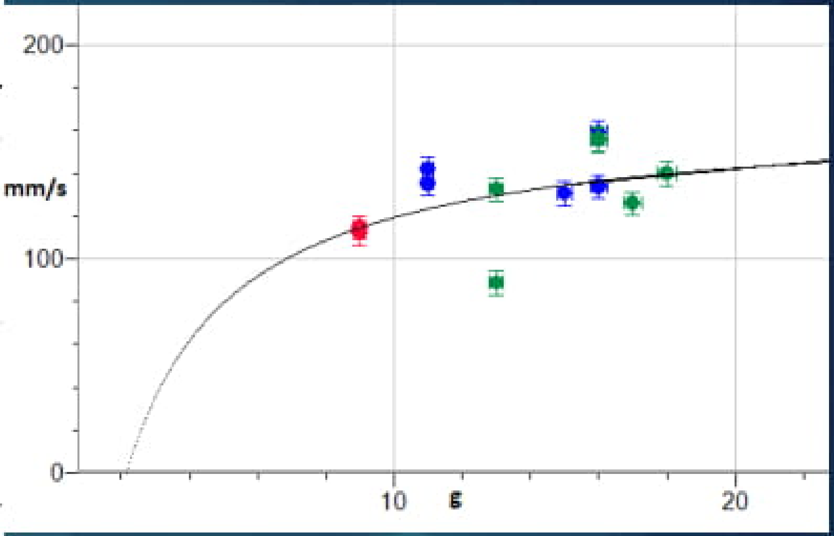
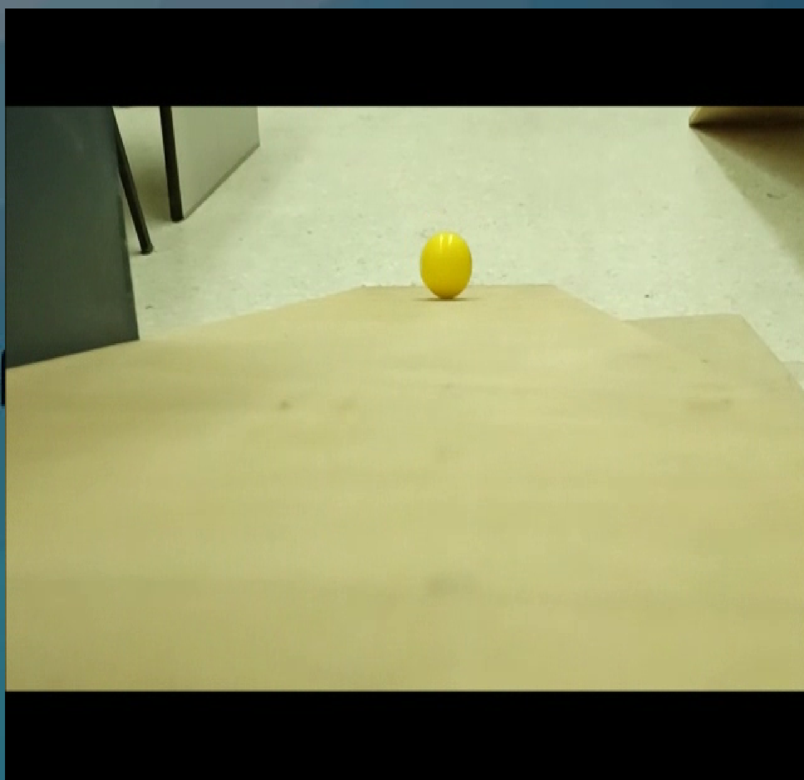


Figure: Mean velocity (mm/s) to weight of ball (g). Left for the larger capsule ($L = \ell + 2R = 79.65mm$, $2R = 59.14mm$, $m_B = 13g$). The mean diameters of the balls are color coded as: $2R = 20.82mm$ (red), $2R = 23.89mm$ (yellow), $2R = 25mm$ (light green), $2R = 28.32mm$ (brown), $2R = 29.09mm$ (blue), $2R = 41.57mm$ (purple) and $2R = 42.35mm$ (dark green). The right diagram is for the medium capsule ($L = \ell + 2R = 45.37mm$, $2R = 32.95mm$, $m_B = 4g$). The color code for the mean diameters of the balls is: $2R = 26.77mm$ (red), $2R = 27.81mm$ (blue), $2R = 28.91mm$ (green). The theoretical curves shows the mean velocity ($\langle u \rangle$) as a function of the balls mass (m_b).

Experimental Investigation of Important parameters:Mass and radius of the ball

Demonstration of effect of radius to waddle and instability



R1ball = Rcapsule

R2ball =1/2* Rcapsule

Video, both m1ball=m2ball

Experimental Investigation of Important parameters:Ratio of length of cylinder part of the capsule to its radius

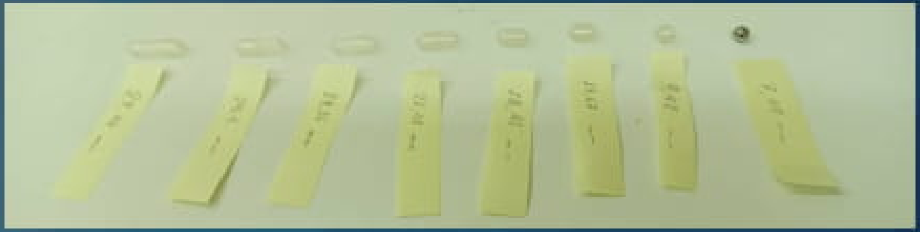
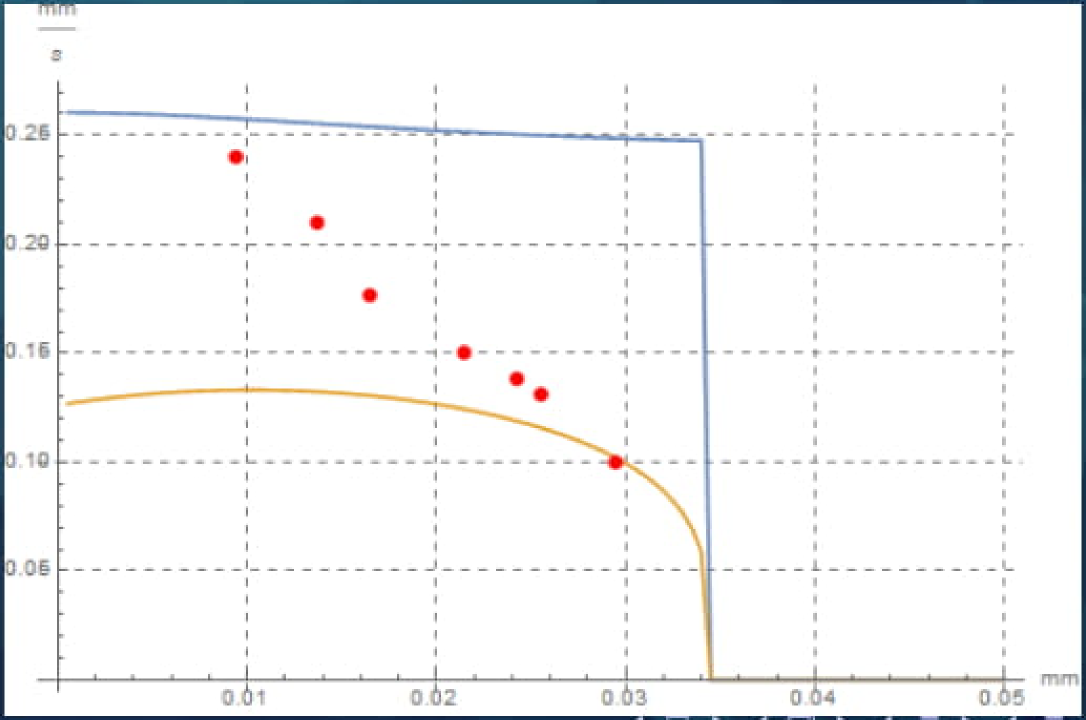


Figure: Beans of constructed different length and the metal ball.



Conclusion

Problem was analyzed to three modes of movement:

- The "jumping bean" mode was studied theoretically and experimentally
- Important parameters related to the bean's velocity were theoretically estimated and experimentally validated
- The ratio of dimensions of the fastest and slowest bean was made easy to understand through the introduction of a factor "b", which gives a measure of how fast the ball-"bean" system is rotating.

Conclusion

- For a given inclination (and assuming that there is no sliding at that inclination) the fastest theoretical bean will have the minimum possible cylinder length to hemisphere radius ratio, and will contain the heaviest possible ball, with a radius matching that of the capsule.
- The theoretically slowest bean will have the maximum possible cylinder length to hemisphere radius ratio, and will contain not only the lightest ball capable of turning it, but also a ball with much less radius than that of the capsule.

Further work

- take into consideration the kinetic friction and the range of parameters where small amounts of sliding occur
- try using more types of balls, as for example hollow balls, due to the difference in moment of inertia
- try to gather the proper materials (eg aerogel balls, metallic capsules etc) to create the "fastest" and "slowest" bean possible

Thank you!

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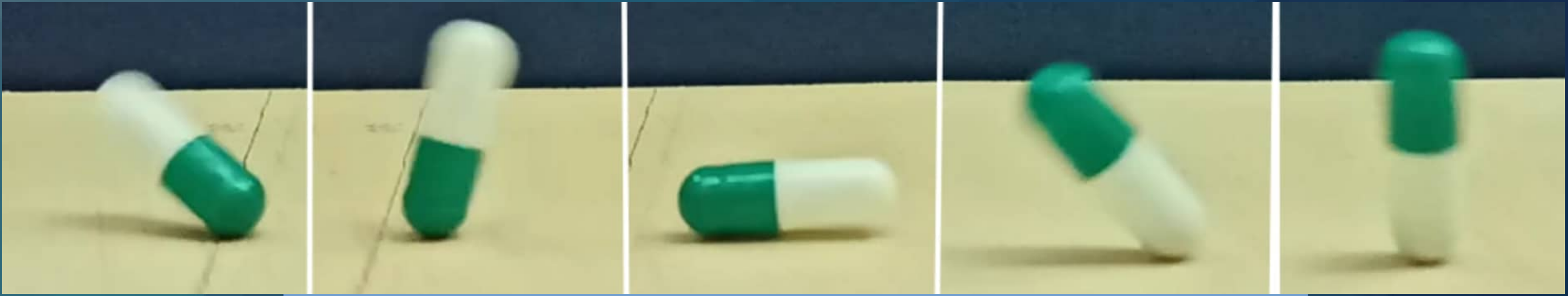

Authors: Andrikopoulos Christos,Dogkas Anargyros, Sofra Christina

Appendix A: The assumption of constant acceleration

Every time the ball moves from the upper end to the lower end it collides with the bean, and because of the friction, between the bean and the surface, the system is losing energy. Therefore $\dot{\theta}(t2) < u2$ and thus the acceleration decreases in the general case.

- So in the limit of small lengths $\ell \rightarrow 0$ the acceleration is approximately as calculated before, as the physical limit of this system should be ball inside a spherical capsule.
- In the opposite limit where $\ell \gg R$ we can consider that after each collusion all the energy is lost and $\dot{\theta}(t2) = 0$. In this case the acceleration will be zero and the constant velocity of the system will be
$$\langle u \rangle = \frac{\pi R + \ell}{t3 - t1}$$

Appendix B: Moments of inertia

$$I_x = \frac{m_b R^2 \left(\ell + \frac{4R}{3}\right)}{\ell + 2R} + \frac{2 m_B R^2}{5}$$

$$I_y = \frac{\ell^2 m_b \left(\frac{\ell}{3} + R\right)}{\ell + 2R} + \frac{m_b R^2 \left(\frac{\ell}{2} + \frac{4R}{3}\right)}{\ell + 2R} + \frac{2 m_B R^2}{5}$$

$$I_z = I_y$$