3. Paper Tube

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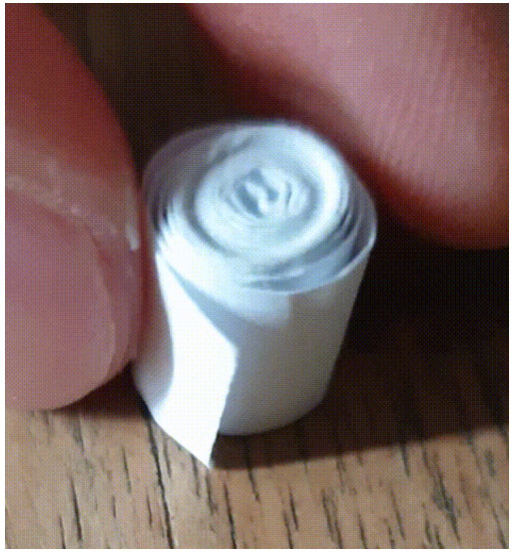
Roll a long paper strip into a tight tube and put it vertically on a table. Why does it often unwind in jerks? What determines the period of the jerks?



What effects are present?

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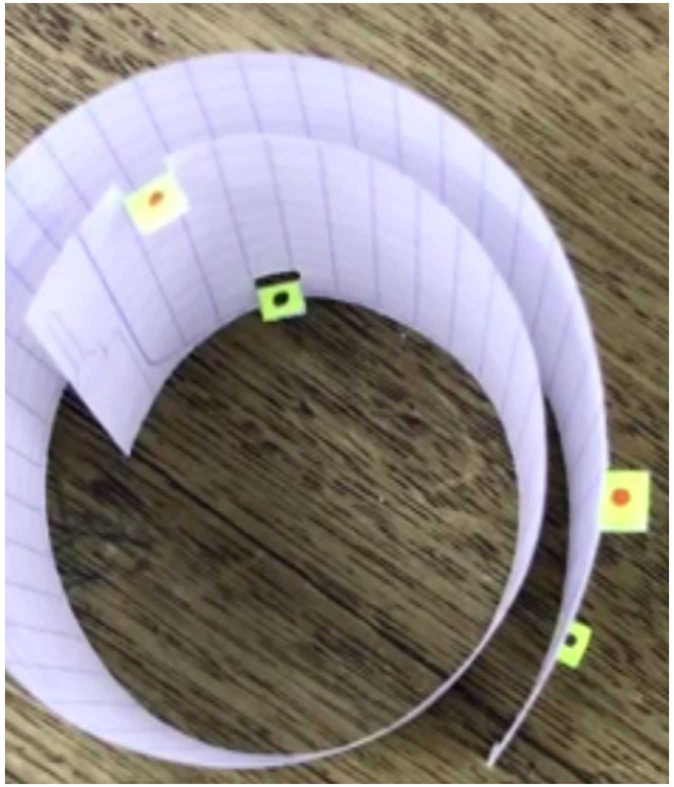

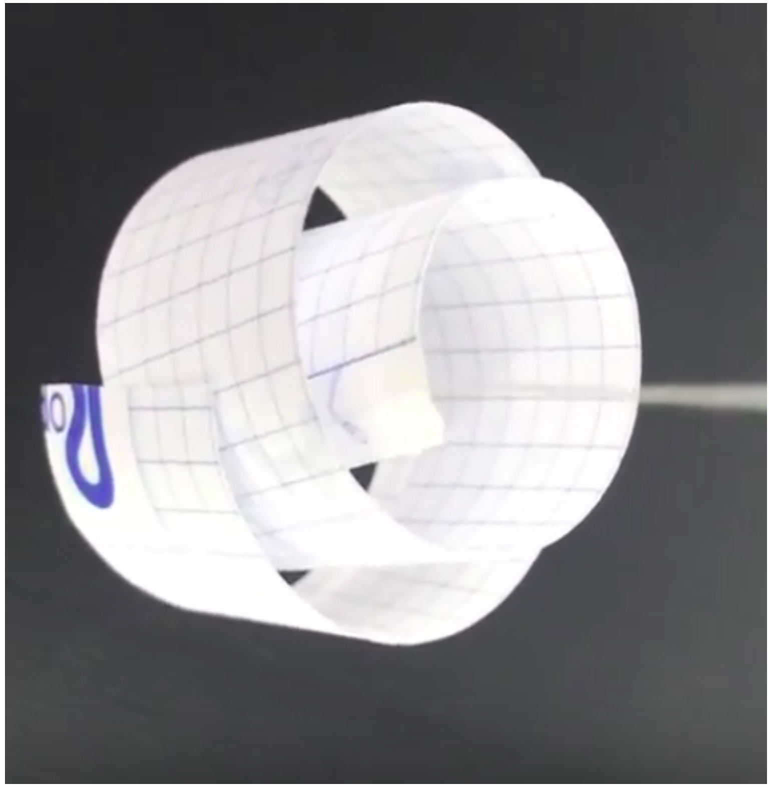

- Restitution forces of the paper.
- Friction:
  - With the surface.
  - Between paper sides.

Causes | Model | Experiments | Conclusions

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Jerks are a simple phenomenon!

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The origin of the phenomenon

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In general:

$$\ddot{x} = g\mu + \frac{k}{m}(x_1 - x)$$

For the dynamic friction domain:

$$\ddot{x} + \frac{k}{m}x = \frac{k}{m}v_1 t + g(\mu_s - \mu_d)$$

Evolution of the stripe.

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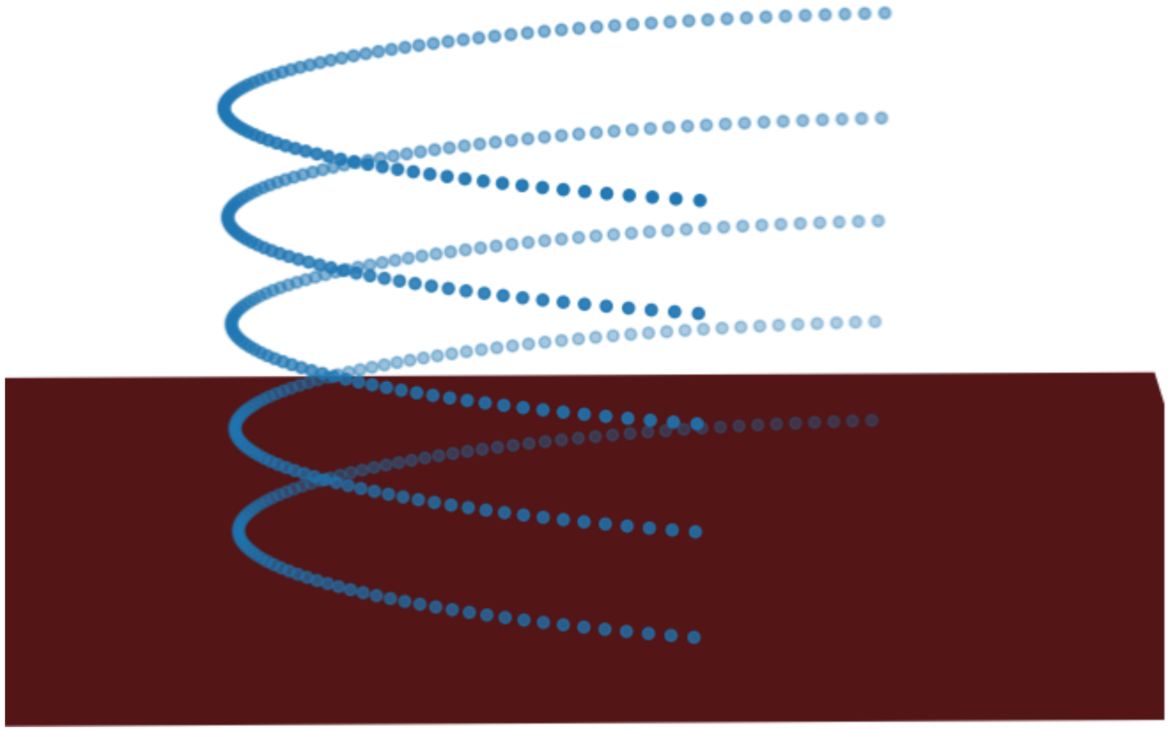
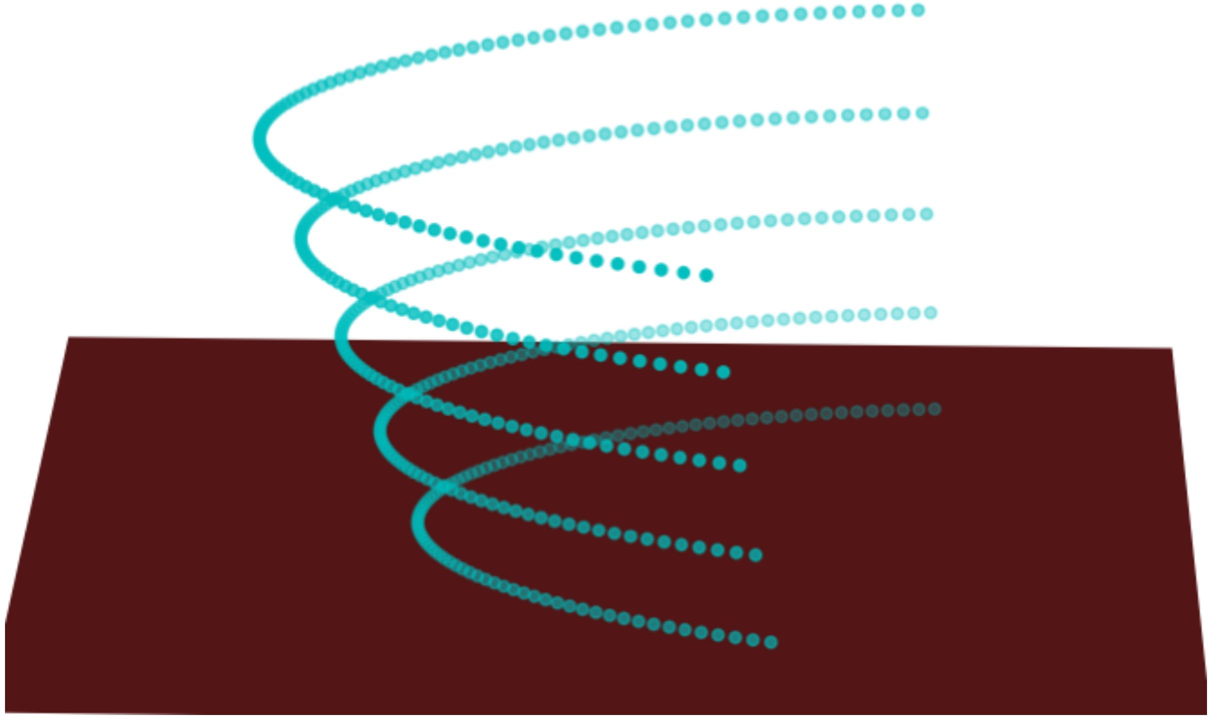
Pure bended paper tube.



Tilted paper tube.



Causes | Model | Experiments | Conclusions

Accumulating tension on the tube

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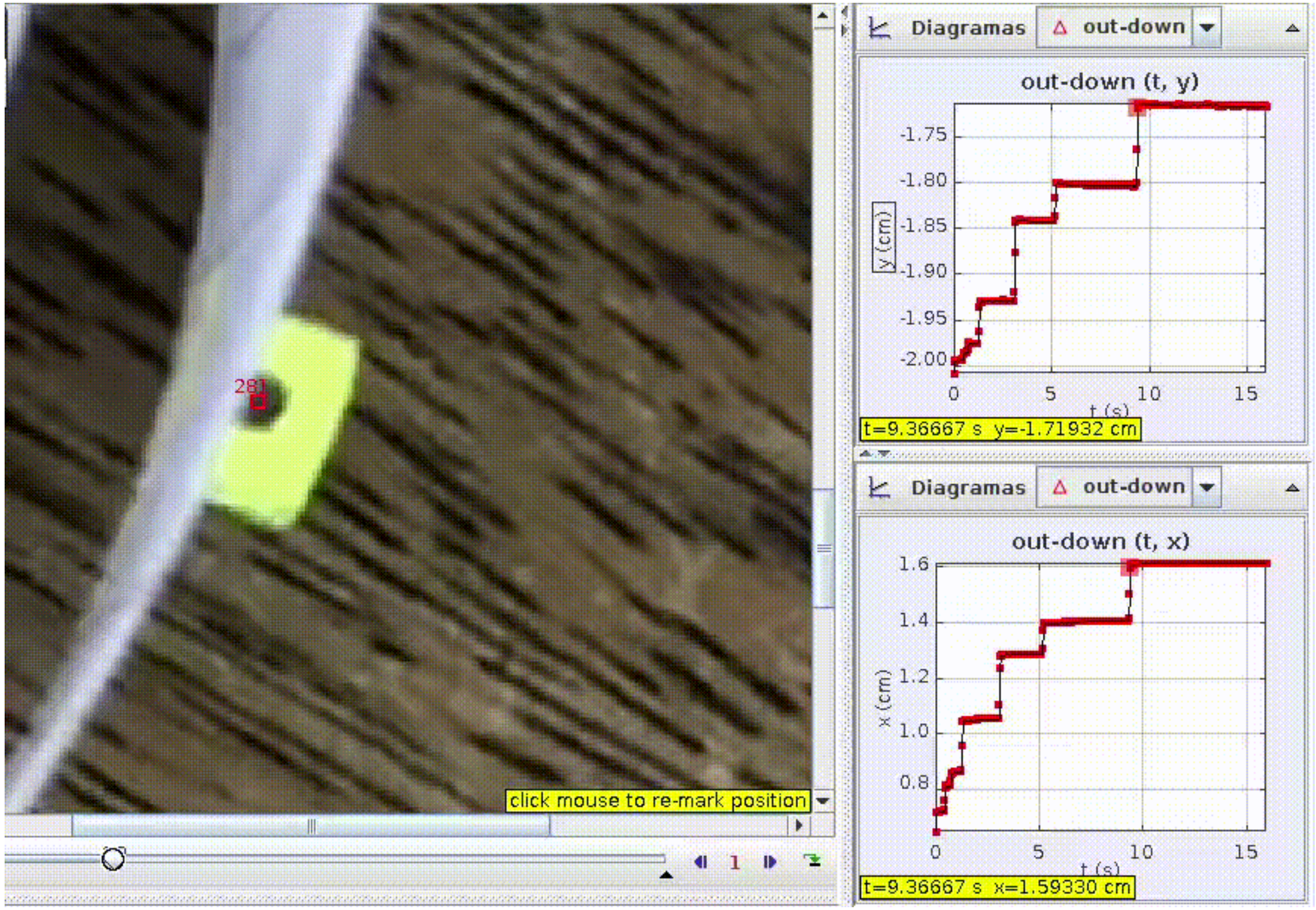

Causes | Model | Experiments | Conclusions

Modeling our problem

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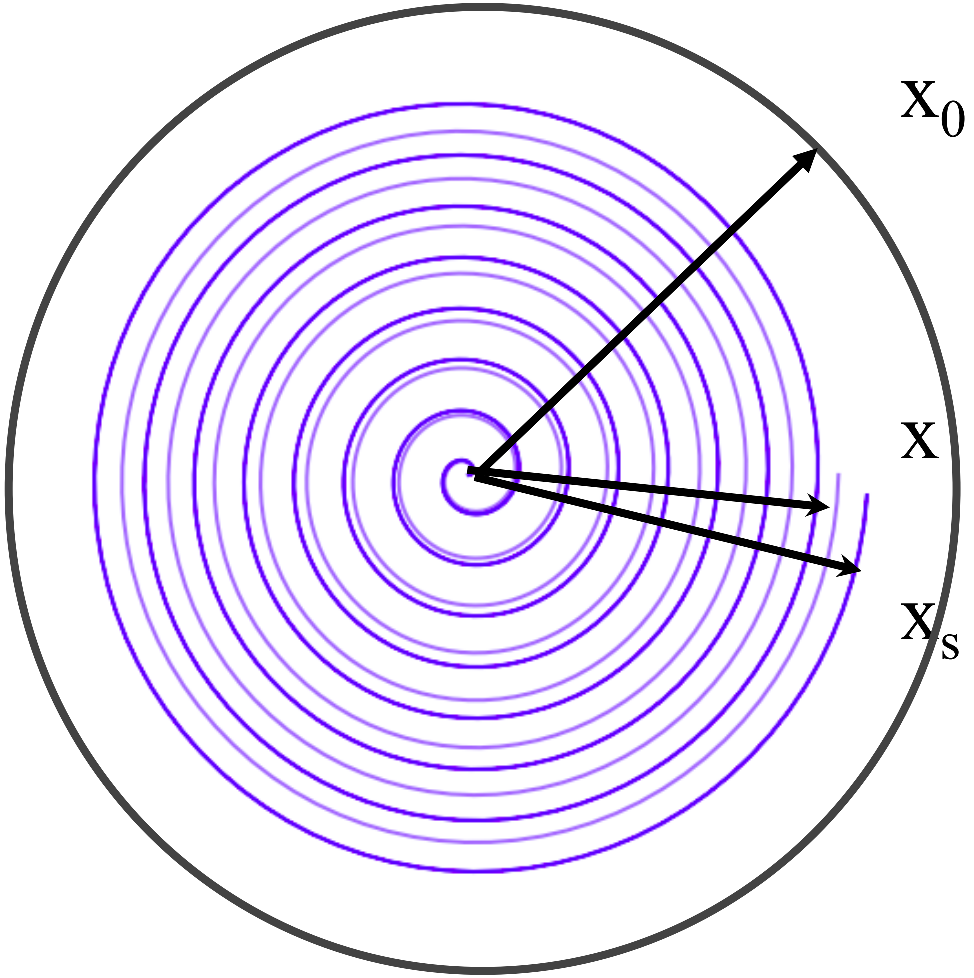

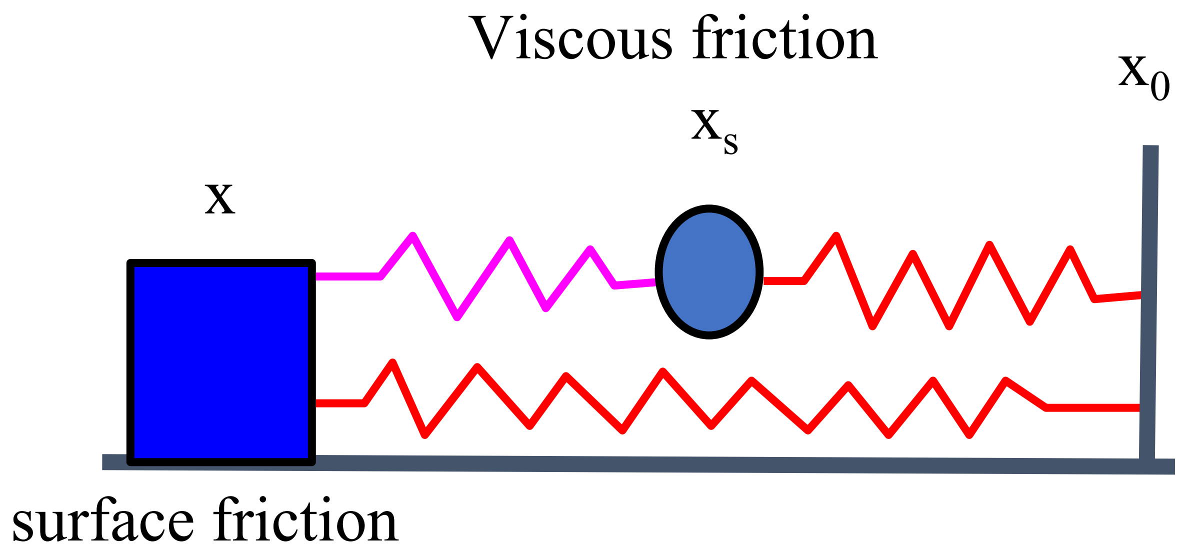

Some considerations on the model...

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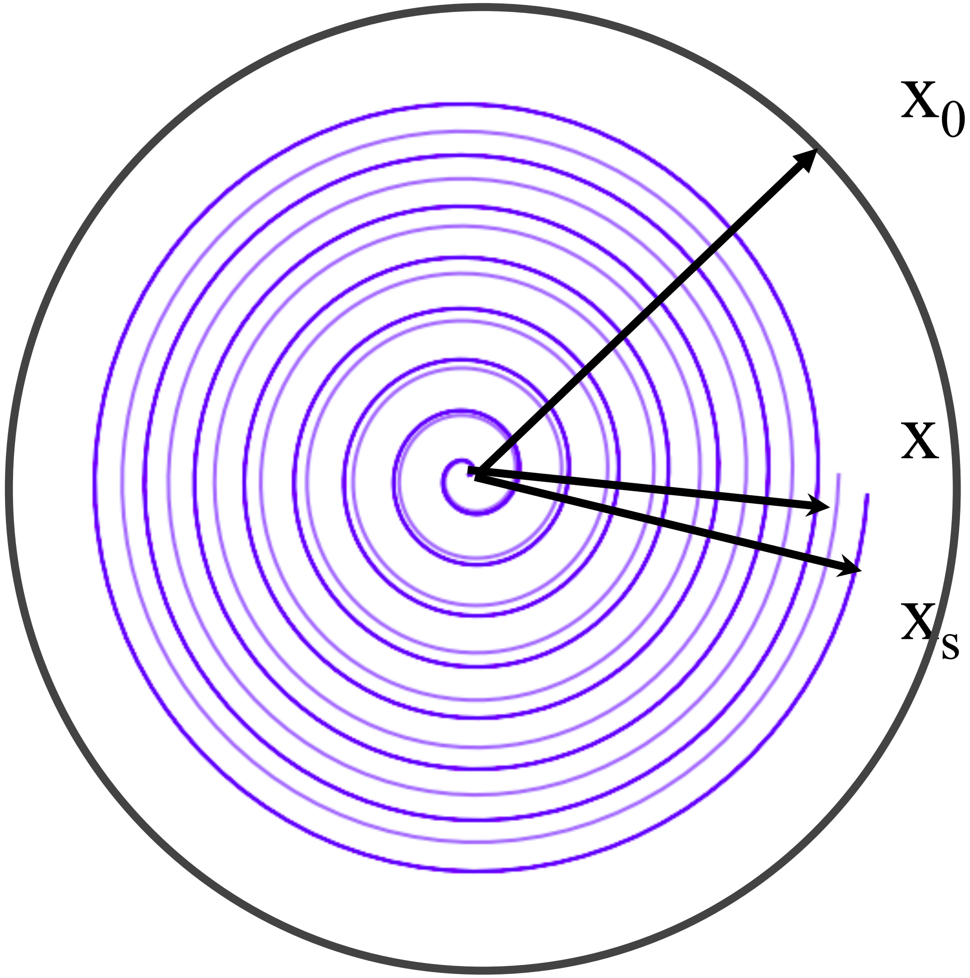


$$L = \int ds :$$

$$r = a\theta$$
$$\alpha \geq \theta \geq 0$$

Parametrization of the roll.

Considering $\alpha \gg 1$:

$$L = \frac{a}{2}\alpha^2$$

Then is obtained by integrating:

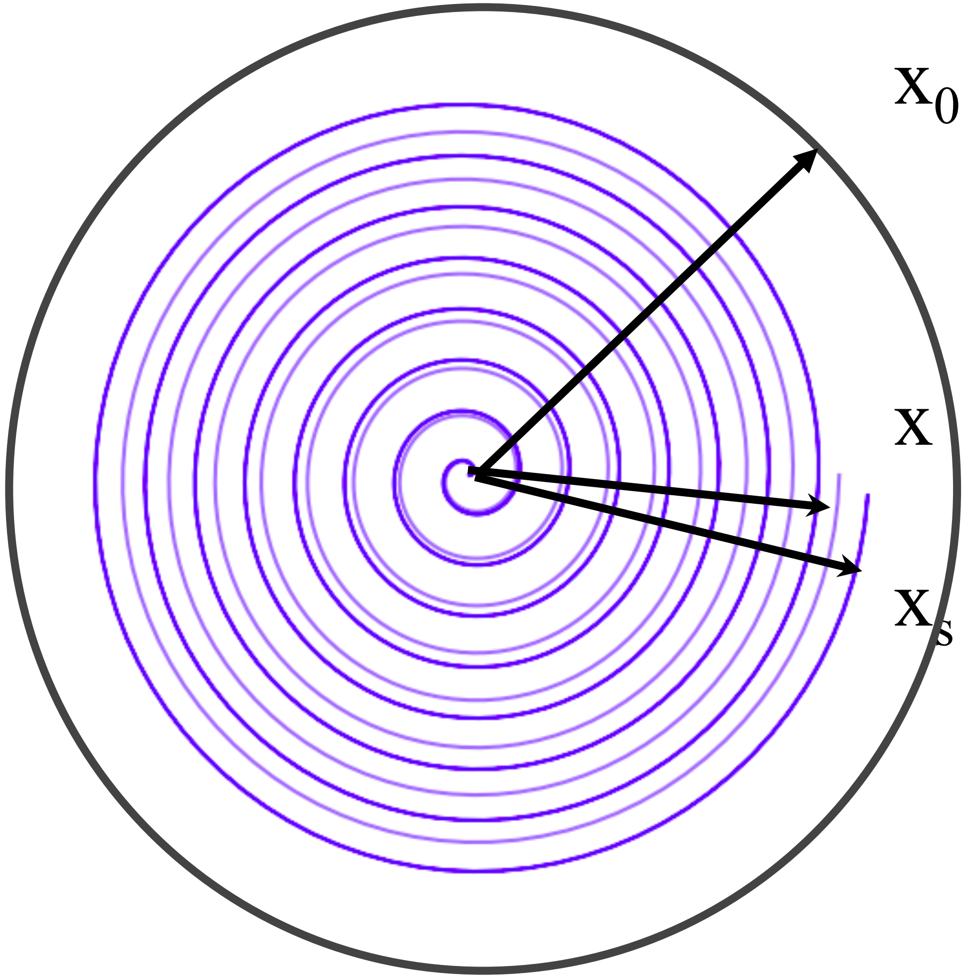
$$L = \frac{a}{2}\{\sqrt{\alpha^2 + 1}\alpha + \sinh^{-1}\alpha\}$$

And considering also $\alpha \gg 1$, then, expanding in a Laurent series and approximating:

$$L = \frac{a}{2}\alpha^2$$

with this relation, the complete form of the roll can be determined with just one number, previously defined as ‘x’, and the lenght, ‘L’.

Model scalable from 1D to 2D. Even 3D.



In conclusion, for this part:

$$r = \frac{2L}{\alpha^2}\theta \qquad \alpha \geq \theta \geq 0$$

$$\alpha = \frac{2L}{x}$$

A relation between a 2D model and a 1D model has been obtained.

Model 2D animation

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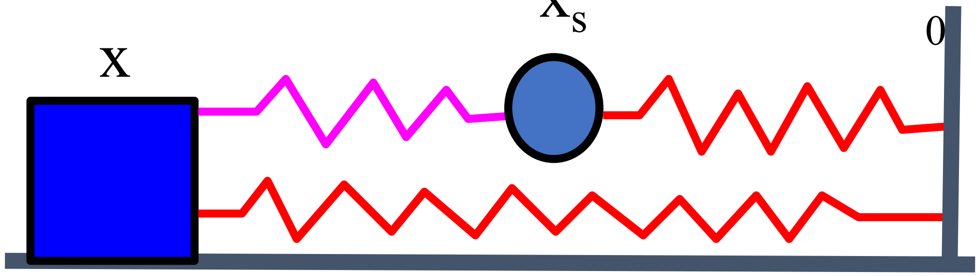
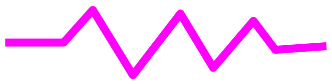
The 1D model of the paper tube



: Force associated with pure bending



: Force associated with tension along the height of the paper tube.

Describing the bending force

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Considering the force in the end of the tube, while pure bended, equal to the force of a flat torsion coil, that is:

$$F_b = k\Delta\alpha$$

And applying the delta operator to the function that relates alpha and x, with an additional approximation, considering that x0 is much greater than the variation of x:

$$-\frac{2L\Delta x}{{x_0}^2} = \Delta\alpha$$

Then:

$$F_b = k_b\Delta x \quad , \quad k_b = \frac{2kL}{{x_0}^2}$$

Equations of motion of the system

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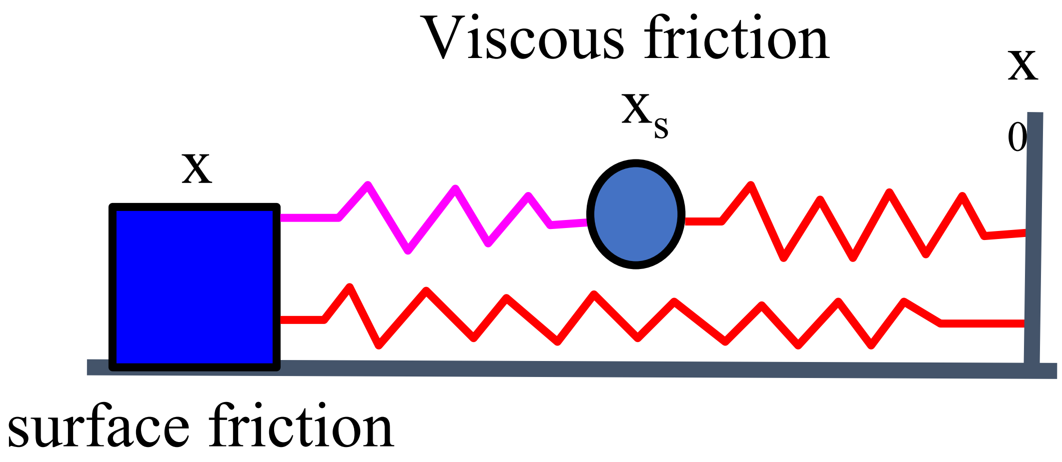

In general, the equations of motion will be given by the following system:

$$\begin{cases} \ddot{x} = -k_b(x - x_0) - k_\tau(x - x_s) + F_{fr} \\ \ddot{x}_s = k_\tau(x - x_s) - k_b(x_s - x_0) - \Gamma \dot{x}_s \end{cases}$$

Analyzing periodicity: solution for $\dot{x} = 0$

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Solving the system for the motionless block (static friction):

$$x_s = \frac{k_\tau x + k_b x_0}{{\omega_0}^2}(1 - \exp\{-\omega_0 t\})$$

where

$${\omega_0}^2 = k_\tau + k_b$$

Evaluating initial and final conditions for this part of the movement:

$$C_1 = k_b\left\{\frac{1}{{\omega_0}^2} + \frac{1}{k_t}\right\}$$

$$T = -\frac{1}{\omega_0}\log\{-C_1 x + C_1 x_0 - \frac{g\mu_s}{k_\tau} + 1\}$$

Solution for the jerk motion

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Differential equations of the jerk motion:

$$\begin{cases} \ddot{x} = -k_b(x - x_0) - k_\tau(x - x_s) - g\mu_s \\ \ddot{x}_s = k_\tau(x - x_s) - k_b(x_s - x_0) - \Gamma\dot{x}_s \end{cases}$$

The velocity dependance difficulties an analytical solution.

Solution for the periodicity

$$T = -\frac{1}{\omega_0}\log(-C_1 x + C_1 x_0')$$

A good description of the movement

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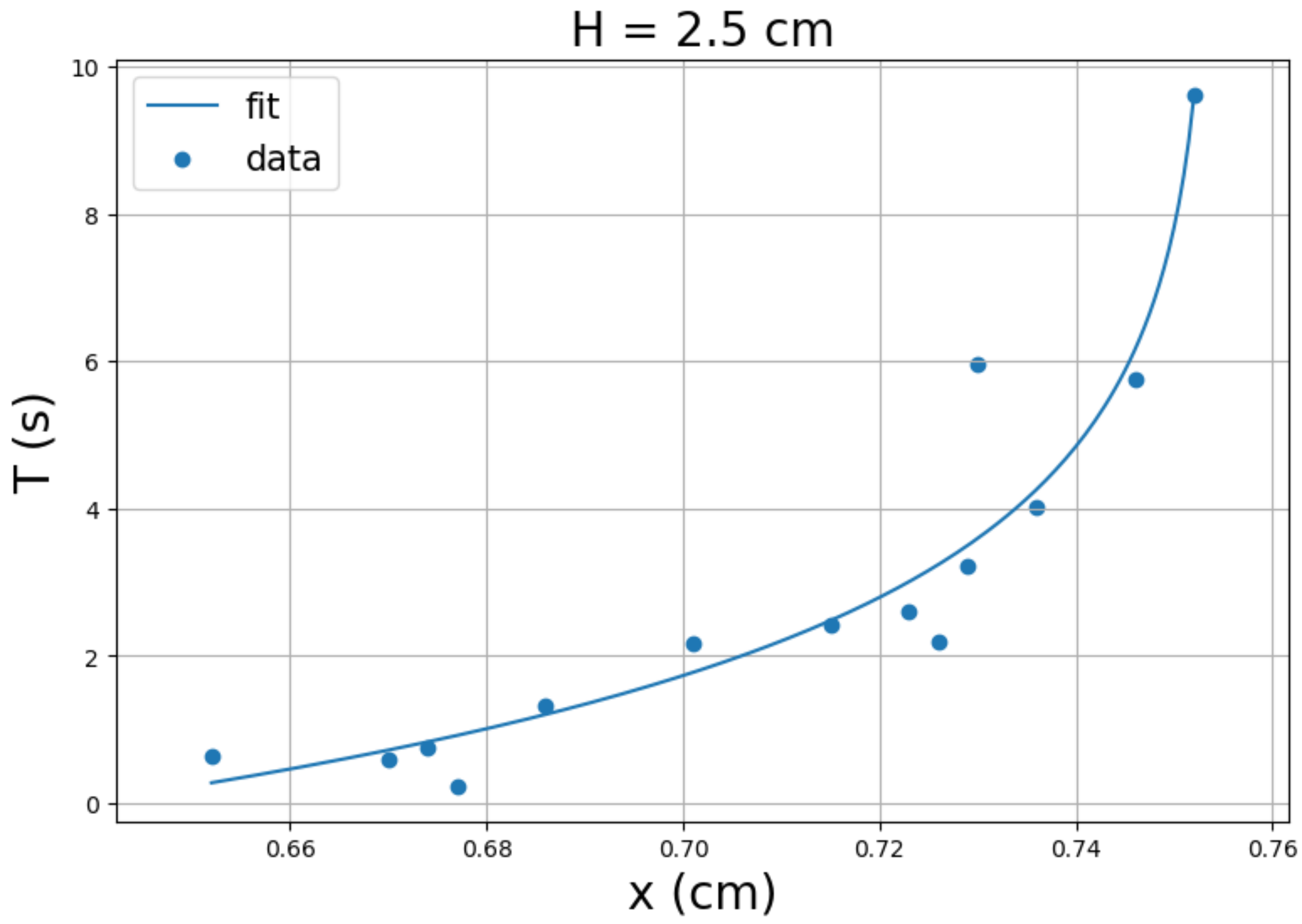

Causes | Model | Experiments | Conclusions

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Experiments.

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- Variation of the height keeping the other parameters fixed.
- Similar measures variating the length of the tube.

Height variation.

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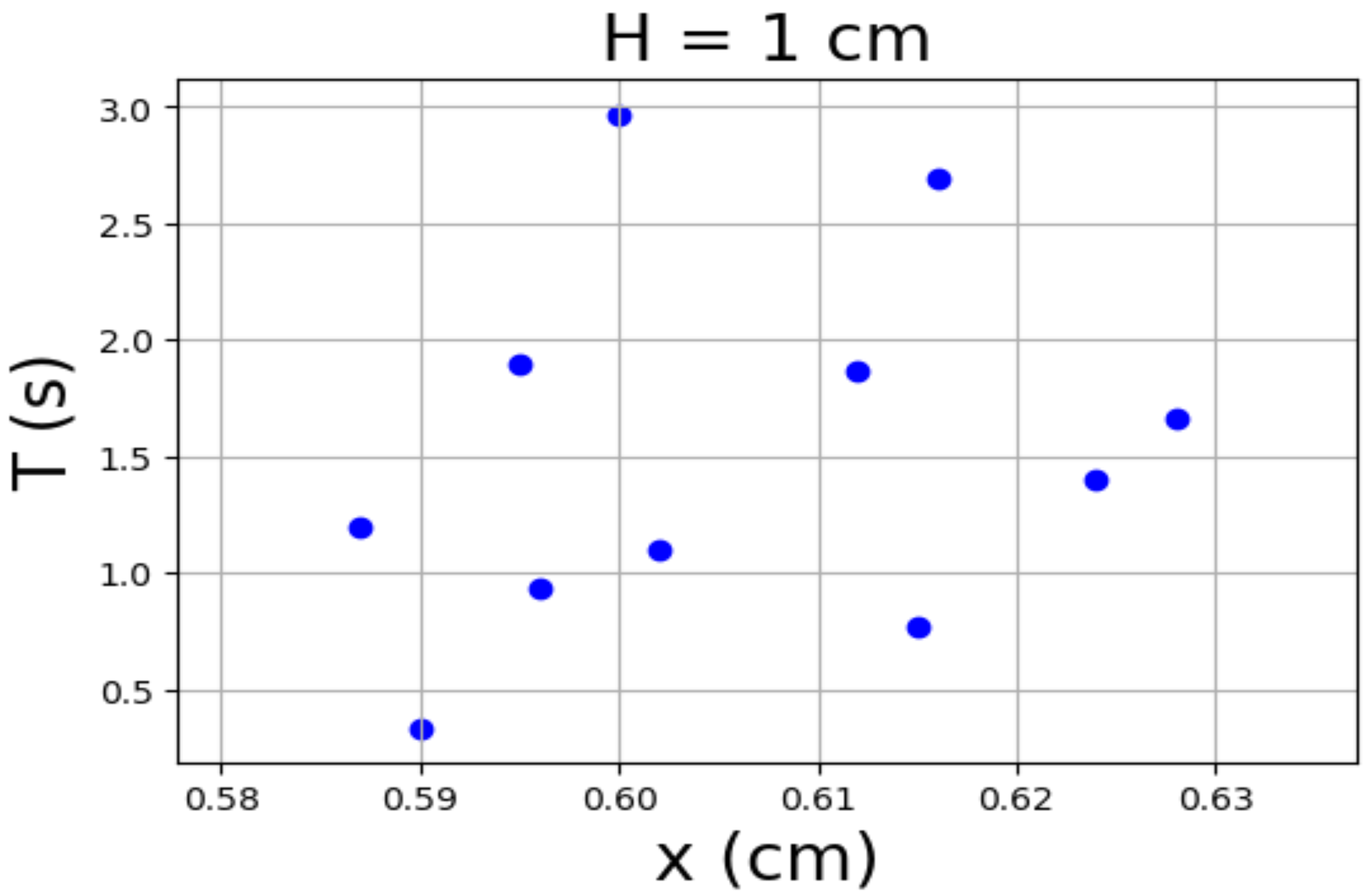
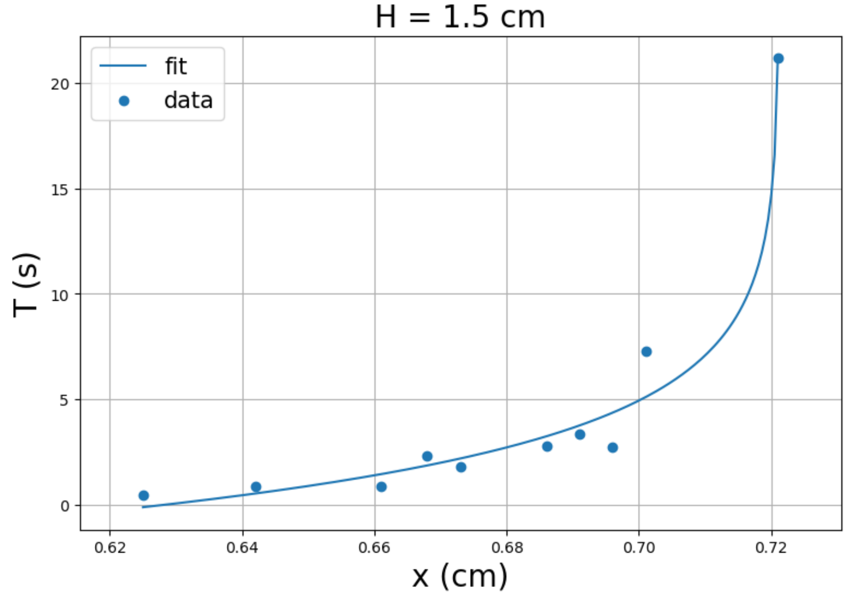
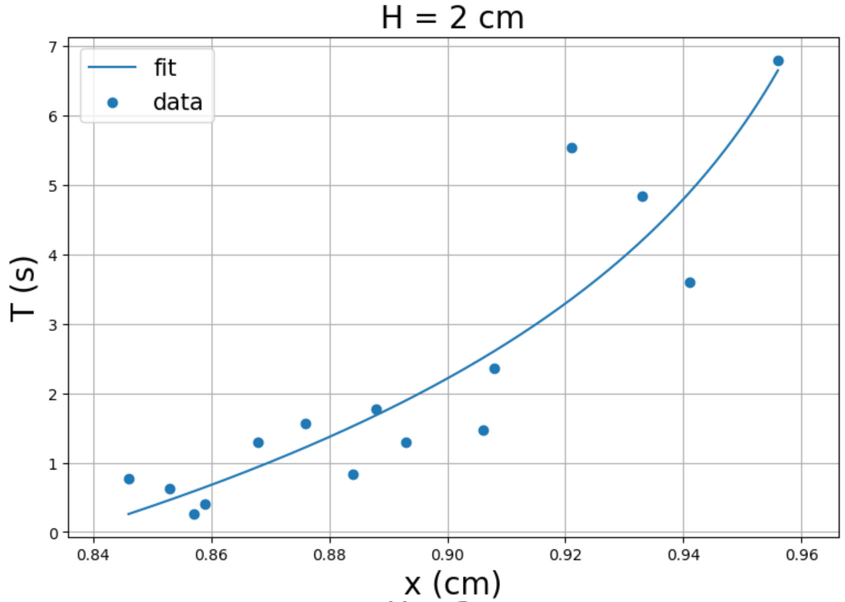
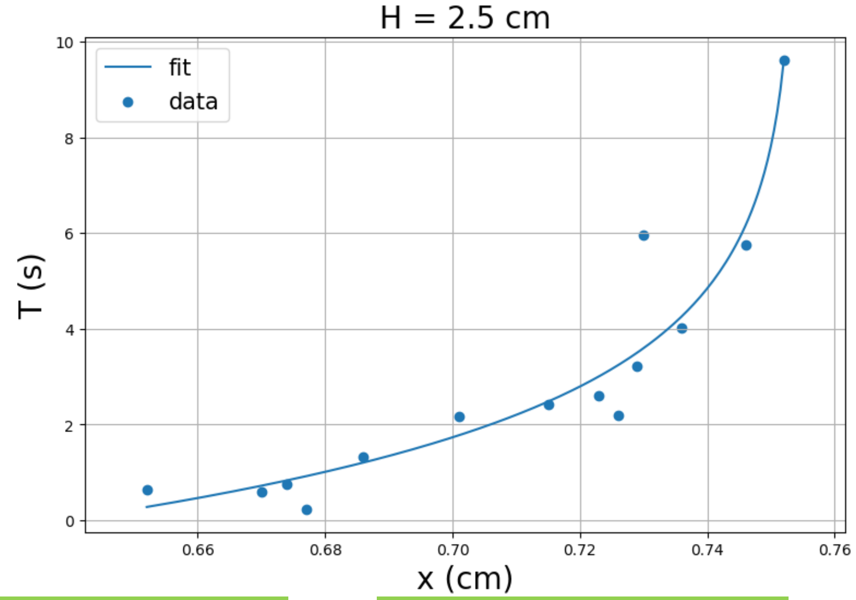
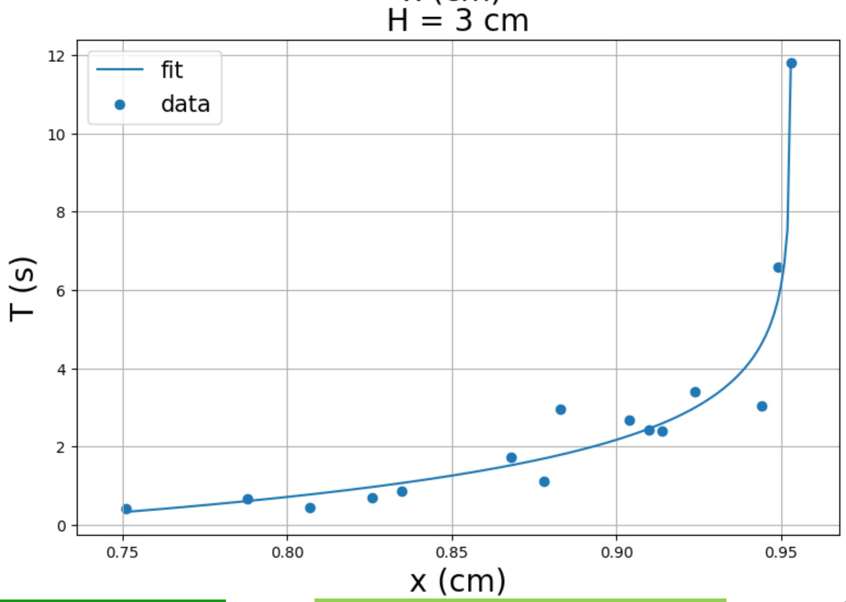



H = 1.5 cm
fit
data
T (s)
x (cm)
0
5
10
15
20
0.62
0.64
0.66
0.68
0.70
0.72


H = 2 cm
fit
data
T (s)
x (cm)
0
1
2
3
4
5
6
7
0.84
0.86
0.88
0.90
0.92
0.94
0.96


H = 2.5 cm
fit
data
T (s)
x (cm)
0
2
4
6
8
10
0.66
0.68
0.70
0.72
0.74
0.76


H = 3 cm
fit
data
T (s)
x (cm)
0
2
4
6
8
10
12
0.75
0.80
0.85
0.90
0.95

Periodicity parameters vs Height

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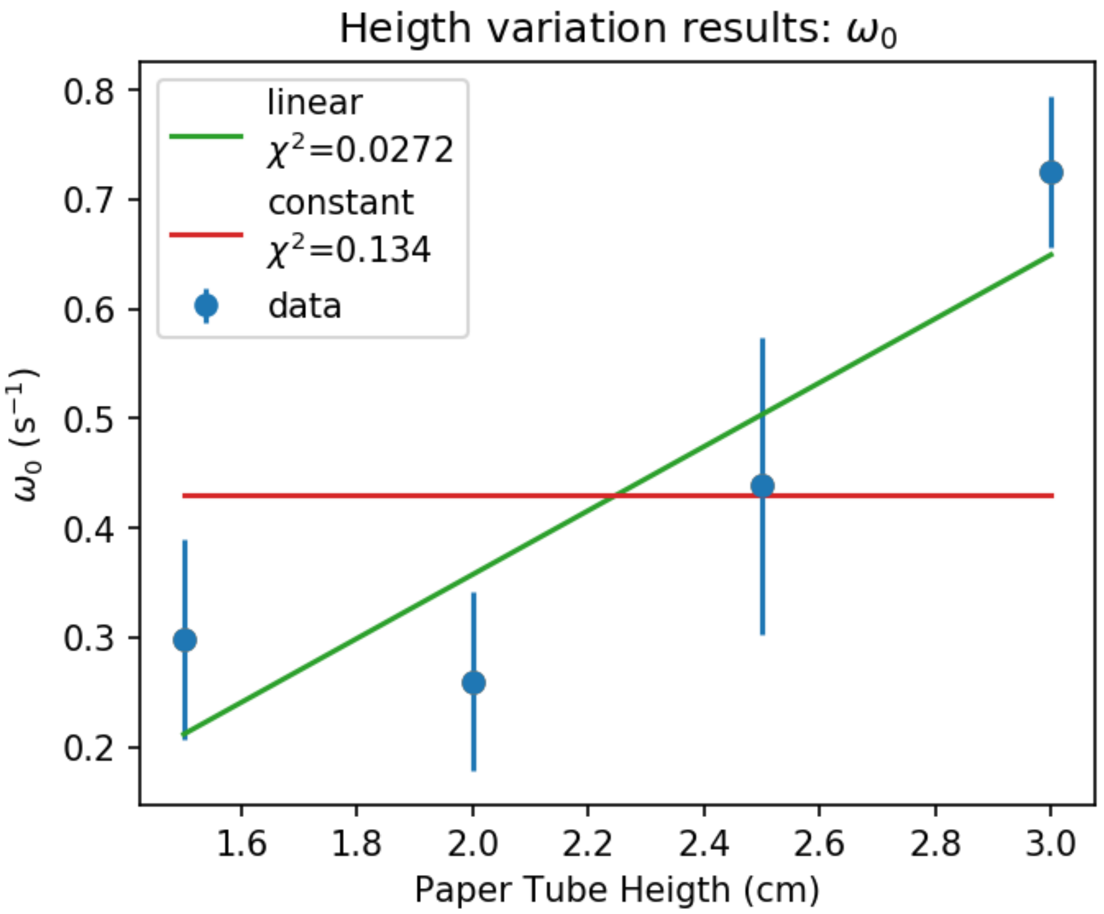

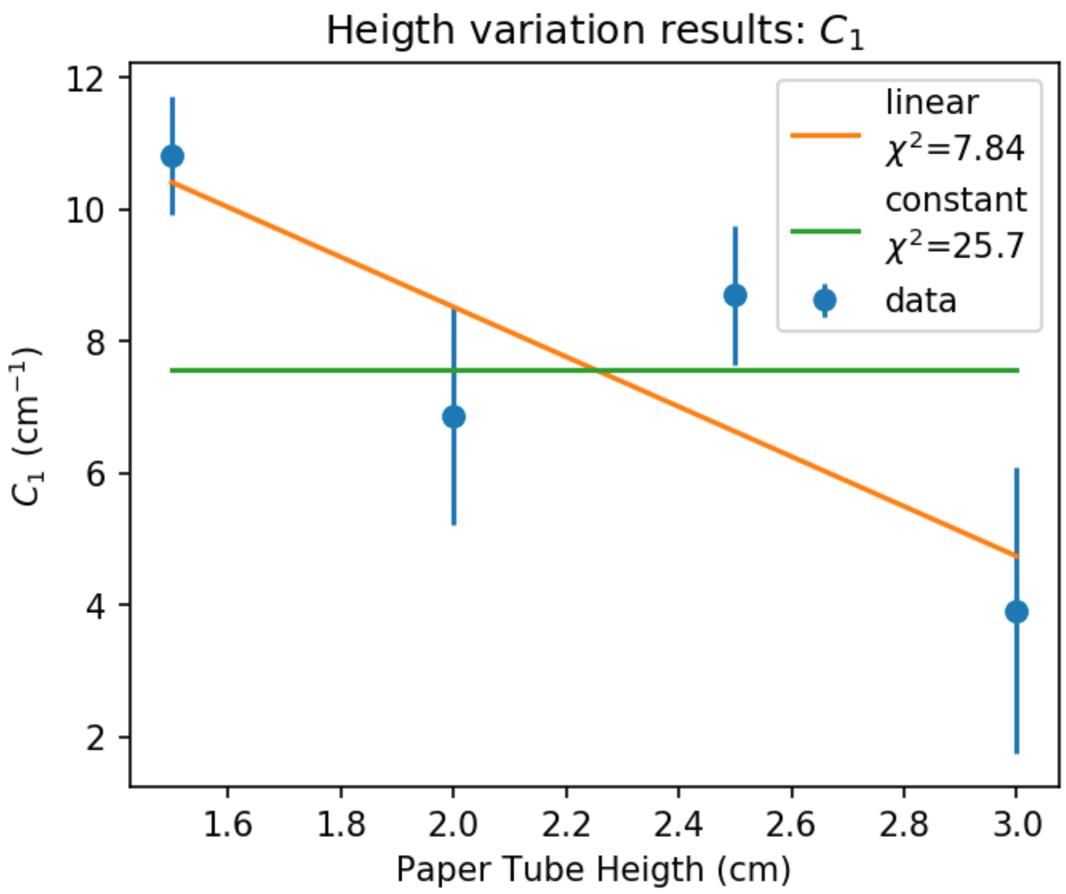

Periodicity parameters vs length

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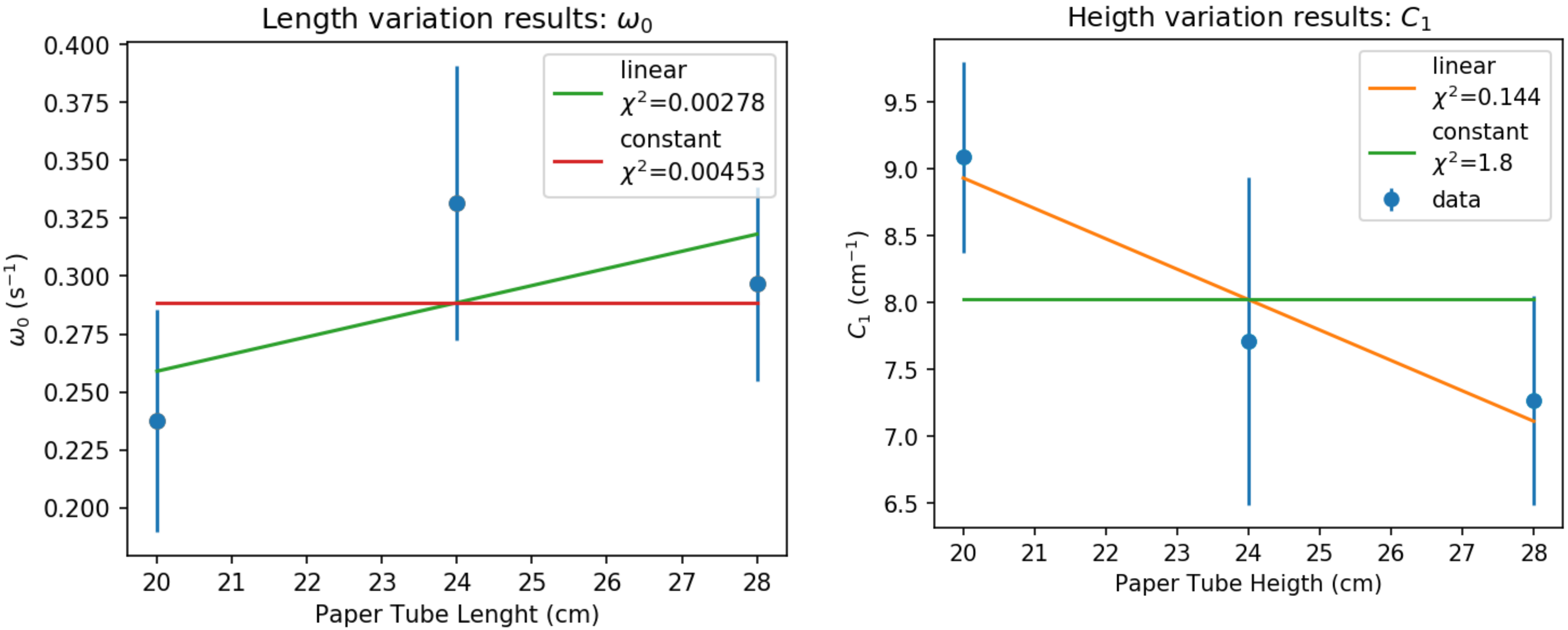

Conclusions

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- The jerks produce due to a combined effect associated with friction properties and an accumulation of tension on the tube.
- The period of a jerk depends on the variable width of the tube:

$$T = -\frac{1}{\omega_0}\log(-C_1 x + C_1 x_0')$$

Conclusions

- Geometrical properties such as length and height also influence periodicity.
- The elastic properties of the paper take importance in the model.

Thank you

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Conclusions

- The difference between dynamic and static friction determines the periodicity.
- Geometrical properties also influence the periodicity
- Pro

Periodicities superposed

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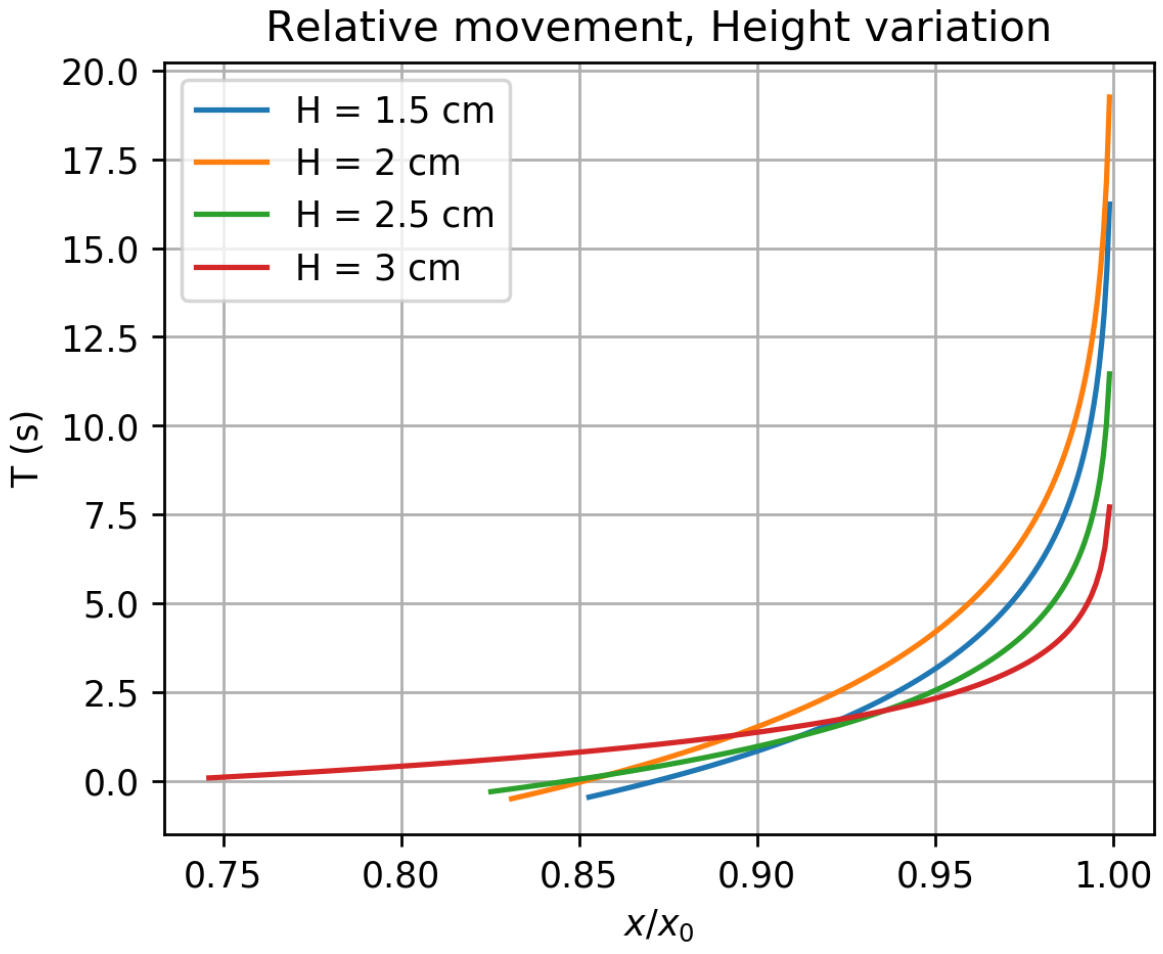

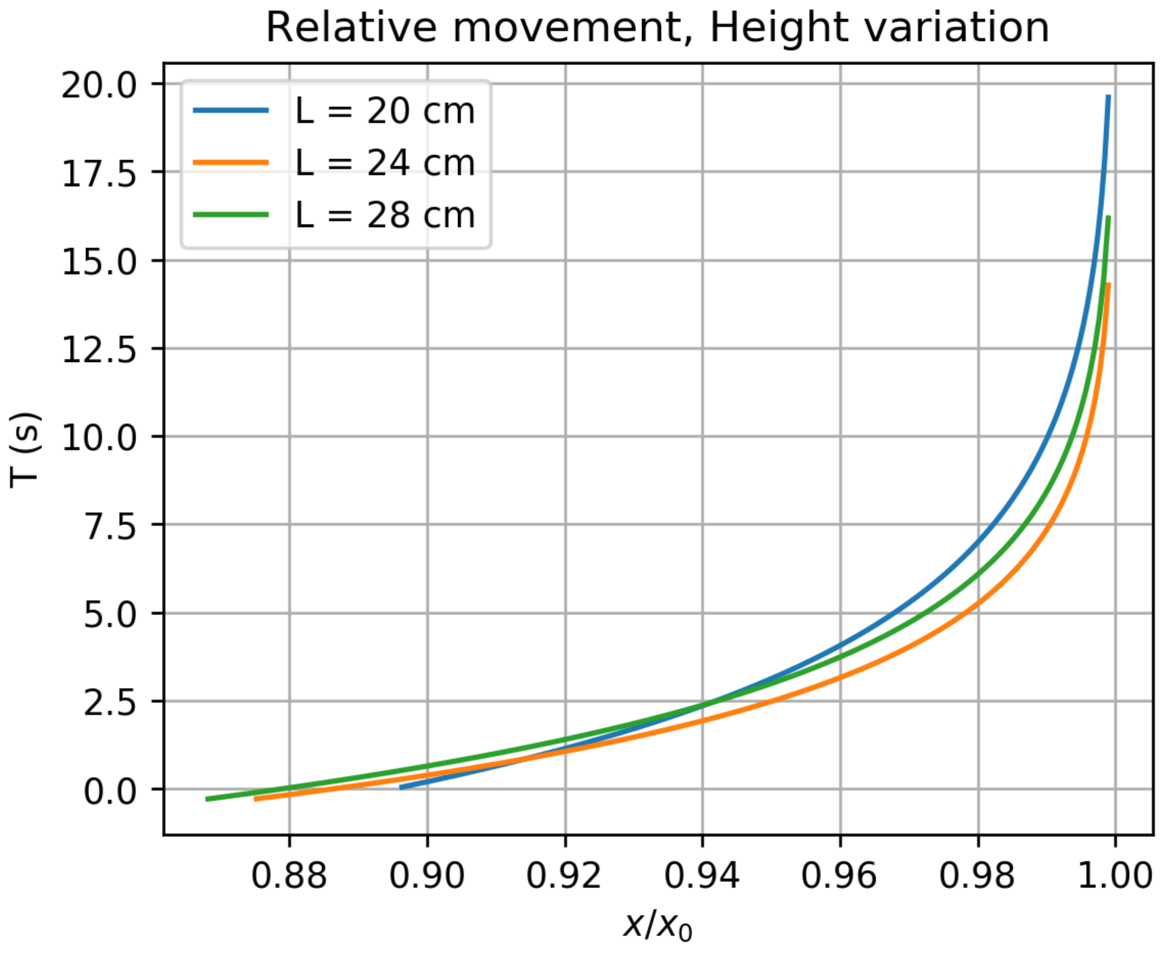

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