A strengthened form of the strong Goldbach conjecture

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Abstract. This paper presents a proof of a strengthened form of the strong Goldbach conjecture. Whereas the traditional approaches focus on the control over the distribution of the prime numbers by means of circle method and sieve theory, we will show that the solution lies in the constructive properties of the primes, reflecting their multiplicative character within the natural numbers.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Theorem (Strengthened strong Goldbach conjecture (SSGB)). *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Proof. We define the set

$$S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q) / 2 \}.$$

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g. The negation $\neg$SSGB means that there is at least one $n \geq 4$ such that $nk \neq mk$ for every $(pk, mk, qk) \in S_g$. Correspondingly, SSGB means there is no such n.

Let us assume $\neg$SSGB now and define S_{g^-} to be S_g, i.e. $\neg$SSGB => $S_g = S_{g^-}$. In the following, we will determine the elements of S_{g^-}, i.e. the elements of S_g under the condition of the existing n.

The whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g, since every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2, where $p \in \mathbb{P}_3$, $k \in \mathbb{N}$.

According to the above three kinds of expression by S_g triple components, for any $n \geq 4$ given by $\neg$SSGB we have the property

(C): $\forall\, k \in \mathbb{N} \quad \exists\, (pk', mk', qk') \in S_g \quad nk = pk' \ \vee\ nk = mk' = 4k'$.

So, every nk given by $\neg$SSGB equals a component of some S_g triple that exists by definition.

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Moreover, since all pairs (p, q) of odd primes with $p < q$ are used in S_g and so all arithmetic means m of two odd primes are generated, we have that an $n \geq 4$ given by ¬SSGB cannot be the arithmetic mean of a pair of odd primes not used in S_g. This results in the property

(M): $\nexists\ p, q \in \mathbb{P}_3,\ p < q \quad n = (p + q) / 2$.

Because the properties (C) and (M) hold for any n given by ¬SSGB, the set S_{g^-} can be written as the union of the following triples, which would otherwise be impossible.

(i) S_g triples of the form (pk' = nk, mk', qk') with k' = k in case n is prime, due to (C)

(ii) S_g triples of the form (pk' = nk, mk', qk') with k' ≠ k in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k', 4k' = nk, 5k') in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk' = nk, mk', qk'), (pk', mk' = nk, qk') or (pk', mk', qk' = nk)

and

(v) S_g triples of the form (pk' ≠ nk, mk' ≠ nk, qk' ≠ nk), i.e. those S_g triples where none of the nk's equals a component.

The triples in (iv) comprise all S_g triples where nk occurs as a component redundantly to the occurrences in (i) - (iii). We can split the triples in (iv) as follows.

(iv, a) S_g triples of the form (pk', mk', qk' = nk) with k' = k in case n is prime

(iv, b) S_g triples of the form (pk', mk' = nk, qk') with k' = k in case n is prime

(iv, c) S_g triples of the form (pk', mk', qk' = nk) with k' ≠ k in case n is composite and not a power of 2

(iv, d) S_g triples of the form (pk', mk' = nk, qk') with k' ≠ k in case n is composite and not a power of 2

(iv, e) S_g triples of the form (pk', mk' = nk, qk') with k' = k in case n is composite

(iv, f) S_g triples of the form (pk' = nk, mk', qk') in case n is a power of 2

(iv, g) S_g triples of the form (pk', mk', qk' = nk) in case n is a power of 2

(iv, h) S_g triples of the form (pk', mk' = nk, qk') with m ≠ 4 in case n is a power of 2.

The types (iv, a) - (iv, h) are of merely informative character. For the sake of completeness also the triples of type (iv, b) and (iv, e) are listed. Of course, they cannot exist due to (M). Also, depending on n and k the triples of some other types may not exist.

Let S_n denote the union of the triples of types (i) to (iv), i.e. S_n is the set of all triples in S_g such that one of the three components is a multiple of n, and let S denote the union of the triples of type (v), i.e. S is the set of all triples in S_g such that none of the three components is a multiple of n.

$S_n = \{ (pk', mk', qk') \in S_g \mid \exists\, k \in \mathbb{N} \quad pk' = nk \,\lor\, mk' = nk \,\lor\, qk' = nk \}$

$S = \{ (pk', mk', qk') \in S_g \mid \forall\, k \in \mathbb{N} \quad pk' \neq nk \,\land\, mk' \neq nk \,\land\, qk' \neq nk \}$

Then, as S_g- consists of all triples of the types (i) to (v) we have S_g- = $S_n \cup S$. Since S_n and S are complementary subsets of S_g, we conclude that S_g- equals S_g where an $n \in \mathbb{N}_4$ different from all m no longer exists.

Note. The above splitting of all the S_g triples into two complementary subsets S_n and S is independent of our information about S_g and it is also independent of the property behind n. The splitting works solely on the basis of the existence of n.

Now, let us assume SSGB and let S_g+ be the set such that SSGB => S_g = S_g+. When we determine the elements of S_g+, i.e. the elements of S_g under the condition of the non-existence of n, we trivially obtain that also S_g+ equals S_g where an $n \in \mathbb{N}_4$ different from all m does not exist.

Therefore, all in all, we have proven

¬SSGB => S_g = S_g- and SSGB => S_g = S_g+, where

S_g- = S_g+ and $\{ m \mid (pk, mk, qk) \in S_g- \} = \{ m \mid (pk, mk, qk) \in S_g+ \} = \mathbb{N}_4$.

This means that we have S_g = S_g- = S_g+ which proves SSGB.

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