SSA FEM NOTES

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Contents

1. Notation

Symbol	Meaning
B	vertically-averaged ice hardness
g	acceleration due to gravity
H	ice thickness
h	ice top surface elevation
n	Glen flow law exponent
N_k	number of trial functions per element
N_q	number of quadrature points per element
q	sliding power law exponent
$\boldsymbol{u} = (u, v)$	horizontal ice velocity
ϵ_β, ϵ_ν, ϵ_η	regularization parameters for $\beta(\boldsymbol{u})$, ν, η
ν	effective viscosity of ice
ϕ	trial functions
ψ	test functions
ρ	ice density
$\boldsymbol{\tau}_b$	basal shear stress
$\boldsymbol{\tau}_d$	driving shear stress

Formulas that appear in the code are $\boxed{\textit{highlighted.}}$

2. The shallow shelf approximation

Define the effective SSA strain rate tensor M [1]:

$$M = \begin{pmatrix} 4u_x + 2v_y & u_y + v_x \\ u_y + v_x & 2u_x + 4v_y \end{pmatrix}.$$

Then the strong form of the SSA system (without boundary conditions) is

$$-\nabla \cdot (\eta\, M) = \boldsymbol{\tau}_b + \boldsymbol{\tau}_d, \tag{1}$$

$$\eta = \boxed{\epsilon_\eta + \nu\, H}. \tag{2}$$

This is equivalent to the more familiar form (found in [3], for example):

$$\begin{aligned} -\Big[(\eta(4u_x + 2v_y))_x + (\eta(u_y + v_x))_y\Big] &= \boldsymbol{\tau}_{b,x} + \boldsymbol{\tau}_{d,x}, \\ -\Big[(\eta(u_y + v_x))_x + (\eta(2u_x + 4v_y))_y\Big] &= \boldsymbol{\tau}_{b,y} + \boldsymbol{\tau}_{d,y}. \end{aligned}$$

Here $\boldsymbol{\tau}_d = \boxed{\rho g H \nabla h}$ is the gravitational driving shear stress; see subsections for definitions of $\boldsymbol{\tau}_b$ and the ice viscosity ν.

2.1. **Ice viscosity.**

Let $U = \{u, v, w\}$ and $X = \{x, y, z\}$.

The three-dimensional strain rate tensor D ([2], equations 3.25 and 3.29) is defined by

$$D_{i,j}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right).$$

We assume that ice is incompressible, so $w_z = -(u_x + v_y)$. Moreover, in the shallow shelf approximation horizontal velocity components do not vary with depth, so $u_z = v_z = 0$.

With these assumptions D becomes

$$D = \begin{pmatrix} u_x & \frac{1}{2}(u_y + v_x) & 0 \\ \frac{1}{2}(u_y + v_x) & v_y & 0 \\ 0 & 0 & -(u_x + v_y) \end{pmatrix}$$

Now the second invariant of D ([2], equation 2.42)

$$\gamma = \operatorname{tr}(D^2) - (\operatorname{tr} D)^2$$

simplifies to

$$\gamma = \boxed{\frac{1}{2}\left((u_x)^2 + (v_y)^2 + (u_x + v_y)^2 + \frac{1}{2}(u_y + v_x)^2\right)}. \tag{3}$$

We define the regularized effective viscosity of ice ν ([3], equation 2.3):

$$\nu = \boxed{\frac{1}{2} B\,(\epsilon_\nu + \gamma)^{(1-n)/(2n)}}, \tag{4}$$

2.2. **Basal shear stress.** The basal shear stress is defined by

$$\boldsymbol{\tau}_b = -\beta(\boldsymbol{u})\boldsymbol{u}, \tag{5}$$

where $\beta = \beta(\boldsymbol{u})$ is a scalar-valued drag coefficient related to the yield stress.

In PISM, $\beta(\boldsymbol{u})$ is defined as follows (see [4]):

$$\beta(\boldsymbol{u}) = \boxed{\frac{\tau_c}{u_{\text{threshold}}^q} \cdot (\epsilon_\beta + |\boldsymbol{u}|^2)^{(q-1)/2}} \tag{6}$$

3. The weak form of the SSA

Multiplying (1) by a test function ψ and integrating by parts, we get the weak form:

$$\begin{aligned}
-\nabla\cdot(\eta M) &= \boldsymbol{\tau}_d + \boldsymbol{\tau}_b,\\
-\int_\Omega \psi\nabla\cdot(\eta M) &= \int_\Omega \psi(\boldsymbol{\tau}_d + \boldsymbol{\tau}_b),\\
-\int_\Omega \nabla\cdot(\psi\,\eta M) + \int_\Omega \nabla\psi\cdot(\eta M) &= \int_\Omega \psi(\boldsymbol{\tau}_d + \boldsymbol{\tau}_b),\\
-\int_{\partial\Omega} (\psi\,\eta M)\cdot\boldsymbol{n}\,ds + \int_\Omega \nabla\psi\cdot(\eta M) &= \int_\Omega \psi(\boldsymbol{\tau}_d + \boldsymbol{\tau}_b)\\
\int_\Omega [\nabla\psi\cdot(\eta M) - \psi(\boldsymbol{\tau}_d + \boldsymbol{\tau}_b)] &= \int_{\partial\Omega} (\psi\,\eta M)\cdot\boldsymbol{n}\,ds. \qquad (7)
\end{aligned}$$

If we ignore the boundary integral (which corresponds to using natural boundary conditions), we can re-write this as follows.

$$\int_\Omega \frac{\partial\psi}{\partial x}\left(\eta(4u_x + 2v_y)\right) + \frac{\partial\psi}{\partial y}\left(\eta(u_y + v_x)\right) - \psi(\boldsymbol{\tau}_{b,x} + \boldsymbol{\tau}_{d,x}) = 0 \tag{8}$$

$$\int_\Omega \frac{\partial\psi}{\partial x}\left(\eta(u_y + v_x)\right) + \frac{\partial\psi}{\partial y}\left(\eta(2u_x + 4v_y)\right) - \psi(\boldsymbol{\tau}_{b,y} + \boldsymbol{\tau}_{d,y}) = 0 \tag{9}$$

This is the system considered in the remainder of these notes.

4. Solving the discretized system

In the following subscripts x and y denote partial derivatives, while subscripts k, l, m denote nodal values of a particular quantity. Also, to simplify notation from here on u, v, etc stand for finite element approximations of corresponding continuum variables.

To build a Galerkin approximation of the SSA system, let ϕ be trial functions and ψ be test functions. Then we have the following basis expansions:

$$\begin{aligned}
u &= \sum_i \phi_i\, u_i,\\
v &= \sum_i \phi_i\, v_i,\\
\frac{\partial u}{\partial x_j} &= \sum_i \frac{\partial \phi_i}{\partial x_j}\; u_i,\\
\frac{\partial v}{\partial x_j} &= \sum_i \frac{\partial \phi_i}{\partial x_j}\; v_i.
\end{aligned} \tag{10}$$

We use a Newton's method to solve the system resulting from discretizing equations (8) and (9). This requires computing residuals and the Jacobian matrix.

4.1. **Residual evaluation.** In this and following sections we focus on element contributions to the residual and the Jacobian. Basis functions used here are defined on the reference element, hence the added determinant of the Jacobian of the map from the reference element to a particular physical element $|J_q|$ appearing in all quadratures.

$$F_{k,1} = \boxed{\sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \left[\eta \left(\frac{\partial \psi_k}{\partial x}(4u_x + 2v_y) + \frac{\partial \psi_k}{\partial y}(u_y + v_x) \right) - \psi_k(\boldsymbol{\tau}_{b,x} + \boldsymbol{\tau}_{d,x}) \right]_{\text{evaluated at } q}} \tag{11}$$

$$F_{k,2} = \boxed{\sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \left[\eta \left(\frac{\partial \psi_k}{\partial x}(u_y + v_x) + \frac{\partial \psi_k}{\partial y}(2u_x + 4v_y) \right) - \psi_k(\boldsymbol{\tau}_{b,y} + \boldsymbol{\tau}_{d,y}) \right]_{\text{evaluated at } q}} \tag{12}$$

4.2. **Jacobian evaluation.** Equations (11) and (12) define a map from $\mathbb{R}^{2\times N}$ to $\mathbb{R}^{2\times N}$, where N is the number of nodes in a FEM mesh. To use Newton's method, we need to be able to compute the Jacobian *of this map.*

It is helpful to rewrite equations defining $F_{k,1}$ and $F_{k,2}$ using basis expansions for u_x, u_y, v_x, and v_y (see (10)), as follows:

$$F_{k,1} = \boxed{\begin{aligned} &\sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[\eta \left(\frac{\partial \psi_k}{\partial x} \left(4 \sum_{m=1}^{N_k} u_m \frac{\partial \phi_m}{\partial x} + 2 \sum_{m=1}^{N_k} v_m \frac{\partial \phi_m}{\partial y} \right) + \frac{\partial \psi_k}{\partial y} \left(\sum_{m=1}^{N_k} u_m \frac{\partial \phi_m}{\partial y} + \sum_{m=1}^{N_k} v_m \frac{\partial \phi_m}{\partial x} \right) \right) \\ &\qquad - \psi_k \left(\boldsymbol{\tau}_{b,x} + \boldsymbol{\tau}_{d,x} \right) \Bigg]_{\text{evaluated at } q} \end{aligned}}$$

$$F_{k,2} = \boxed{\begin{aligned} &\sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[\eta \left(\frac{\partial \psi_k}{\partial x} \left(\sum_{m=1}^{N_k} u_m \frac{\partial \phi_m}{\partial y} + \sum_{m=1}^{N_k} v_m \frac{\partial \phi_m}{\partial x} \right) + \frac{\partial \psi_k}{\partial y} \left(2 \sum_{m=1}^{N_k} u_m \frac{\partial \phi_m}{\partial x} + 4 \sum_{m=1}^{N_k} v_m \frac{\partial \phi_m}{\partial y} \right) \right) \\ &\qquad - \psi_k \left(\boldsymbol{\tau}_{b,y} + \boldsymbol{\tau}_{d,y} \right) \Bigg]_{\text{evaluated at } q} \end{aligned}}$$

The Jacobian has elements

$$J_{k,l,1} = \frac{\partial F_{k,1}}{\partial u_l}, \qquad J_{k,l,2} = \frac{\partial F_{k,1}}{\partial v_l},$$
$$J_{k,l,3} = \frac{\partial F_{k,2}}{\partial u_l}, \qquad J_{k,l,4} = \frac{\partial F_{k,2}}{\partial v_l}.$$

$$\boxed{\begin{aligned} J_{k,l,1} = \sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[&\frac{\partial \eta}{\partial u_l} \cdot \left(\frac{\partial \psi_k}{\partial x}(4u_x + 2v_y) + \frac{\partial \psi_k}{\partial y}(u_y + v_x) \right) \\ &+ \eta \cdot \left(\frac{\partial \psi_k}{\partial x} \cdot 4\frac{\partial \phi_l}{\partial x} + \frac{\partial \psi_k}{\partial y} \cdot \frac{\partial \phi_l}{\partial y} \right) - \psi_k \cdot \frac{\partial \boldsymbol{\tau}_{b,x}}{\partial u_l} \Bigg]_{\text{evaluated at } q} \end{aligned}} \tag{13}$$

$$\boxed{\begin{aligned} J_{k,l,2} = \sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[&\frac{\partial \eta}{\partial v_l} \cdot \left(\frac{\partial \psi_k}{\partial x}(4u_x + 2v_y) + \frac{\partial \psi_k}{\partial y}(u_y + v_x) \right) \\ &+ \eta \cdot \left(\frac{\partial \psi_k}{\partial x} \cdot 2\frac{\partial \phi_l}{\partial y} + \frac{\partial \psi_k}{\partial y} \cdot \frac{\partial \phi_l}{\partial x} \right) - \psi_k \cdot \frac{\partial \boldsymbol{\tau}_{b,x}}{\partial v_l} \Bigg]_{\text{evaluated at } q} \end{aligned}} \tag{14}$$

$$\boxed{\begin{aligned} J_{k,l,3} = \sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[&\frac{\partial \eta}{\partial u_l} \cdot \left(\frac{\partial \psi_k}{\partial x}(u_y + v_x) + \frac{\partial \psi_k}{\partial y}(2u_x + 4v_y) \right) \\ &+ \eta \cdot \left(\frac{\partial \psi_k}{\partial x} \cdot \frac{\partial \phi_l}{\partial y} + \frac{\partial \psi_k}{\partial y} \cdot 2\frac{\partial \phi_l}{\partial x} \right) - \psi_k \cdot \frac{\partial \boldsymbol{\tau}_{b,y}}{\partial u_l} \Bigg]_{\text{evaluated at } q} \end{aligned}} \tag{15}$$

$$\boxed{\begin{aligned} J_{k,l,4} = \sum_{q=1}^{N_q} |J_q| \cdot w_q \cdot \Bigg[&\frac{\partial \eta}{\partial v_l} \cdot \left(\frac{\partial \psi_k}{\partial x}(u_y + v_x) + \frac{\partial \psi_k}{\partial y}(2u_x + 4v_y) \right) \\ &+ \eta \cdot \left(\frac{\partial \psi_k}{\partial x} \cdot \frac{\partial \phi_l}{\partial x} + \frac{\partial \psi_k}{\partial y} \cdot 4\frac{\partial \phi_l}{\partial y} \right) - \psi_k \cdot \frac{\partial \boldsymbol{\tau}_{b,y}}{\partial v_l} \Bigg]_{\text{evaluated at } q} \end{aligned}} \tag{16}$$

In our case the number of trial functions N_k is 4 (Q_1 elements). Our test functions are the same as trial functions (a Galerkin method), i.e. we also have 4 test functions per element. Moreover, each combination of test and trial functions corresponds to 4 values in the Jacobian (2 equations, 2 degrees of freedom). Overall, each element contributes to $4 \times 4 \times 4 = 64$ entries in the Jacobian matrix.

To evaluate $J_{\cdot,\cdot,\cdot}$, we need be able to compute the following:

$$\boldsymbol{\tau}_{b,x}, \quad \boldsymbol{\tau}_{b,y}, \quad \boldsymbol{\tau}_{d,x}, \quad \boldsymbol{\tau}_{d,y}, \quad \frac{\partial \eta}{\partial u_l}, \quad \frac{\partial \eta}{\partial v_l}, \quad \frac{\partial \boldsymbol{\tau}_{b,x}}{\partial u_l}, \quad \frac{\partial \boldsymbol{\tau}_{b,x}}{\partial v_l}, \quad \frac{\partial \boldsymbol{\tau}_{b,y}}{\partial u_l}, \quad \frac{\partial \boldsymbol{\tau}_{b,y}}{\partial v_l}.$$

Subsections that follow describe related implementation details.

4.2.1. *Ice viscosity.* Recall (equation (2)) that $\eta = \epsilon_\eta + \nu H$. We use the chain rule to get

$$\frac{\partial \eta}{\partial u_l} = \boxed{H\,\frac{\partial \nu}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial u_l},} \qquad \frac{\partial \eta}{\partial v_l} = \boxed{H\,\frac{\partial \nu}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial v_l}}.$$

The derivative of ν with respect to γ can be written in terms of ν itself:

$$\begin{aligned} \frac{\partial \nu}{\partial \gamma} &= \frac{1}{2}B \cdot \frac{1-n}{2n} \cdot (\epsilon_\nu + \gamma)^{(1-n)/(2n)-1}, \\ &= \frac{1-n}{2n} \cdot \frac{1}{2}B\,(\epsilon_\nu + \gamma)^{(1-n)/(2n)} \cdot \frac{1}{\epsilon_\nu + \gamma}, \\ &= \boxed{\frac{1-n}{2n} \cdot \frac{\nu}{\epsilon_\nu + \gamma}}. \end{aligned}$$

To compute $\frac{\partial\gamma}{\partial u_l}$ and $\frac{\partial\gamma}{\partial v_l}$ we need to re-write γ (equation (3)) using the basis expansion (10):

$$\gamma = \frac{1}{2}\Bigg(\left(\sum_{m=1}^{N_k} u_m \frac{\partial\phi_m}{\partial x}\right)^2 + \left(\sum_{m=1}^{N_k} v_m \frac{\partial\phi_m}{\partial y}\right)^2 + \left(\sum_{m=1}^{N_k} u_m \frac{\partial\phi_m}{\partial x} + \sum_{m=1}^{N_k} v_m \frac{\partial\phi_m}{\partial y}\right)^2 + \frac{1}{2}\left(\sum_{m=1}^{N_k} u_m \frac{\partial\phi_m}{\partial y} + \sum_{m=1}^{N_k} v_m \frac{\partial\phi_m}{\partial x}\right)^2 \Bigg).$$

So,

$$\begin{aligned}
\frac{\partial\gamma}{\partial u_l} &= u_x\frac{\partial\phi_l}{\partial x} + (u_x+v_y)\frac{\partial\phi_l}{\partial x} + \frac{1}{2}(u_y+v_x)\frac{\partial\phi_l}{\partial y},\\
&= \boxed{(2u_x+v_y)\frac{\partial\phi_l}{\partial x} + \frac{1}{2}(u_y+v_x)\frac{\partial\phi_l}{\partial y}}, \qquad (17)\\
\frac{\partial\gamma}{\partial v_l} &= v_y\frac{\partial\phi_l}{\partial y} + (u_x+v_y)\frac{\partial\phi_l}{\partial y} + \frac{1}{2}(u_y+v_x)\frac{\partial\phi_l}{\partial x},\\
&= \boxed{\frac{1}{2}(u_y+v_x)\frac{\partial\phi_l}{\partial x} + (u_x+2v_y)\frac{\partial\phi_l}{\partial y}}. \qquad (18)
\end{aligned}$$

4.2.2. *Basal drag.* The method `IceBasalResistancePlasticLaw::drag_with_derivative()` computes β and the derivative of β with respect to $\alpha = \frac{1}{2}|\boldsymbol{u}|^2 = \frac{1}{2}(u^2+v^2)$.

Then

$$\frac{\partial\alpha}{\partial u_l} = u\cdot\frac{\partial u}{\partial u_l} = u\cdot\phi_l, \qquad \frac{\partial\alpha}{\partial v_l} = v\cdot\frac{\partial v}{\partial v_l} = v\cdot\phi_l.$$

Recall from equation (5)

$$\boldsymbol{\tau}_{b,x} = \boxed{-\beta(\boldsymbol{u})\cdot u}, \qquad \boldsymbol{\tau}_{b,y} = \boxed{-\beta(\boldsymbol{u})\cdot v}.$$

Using product and chain rules, we get

$$\begin{aligned}
\frac{\partial\boldsymbol{\tau}_{b,x}}{\partial u_l} &= -\left(\beta(\boldsymbol{u})\cdot\frac{\partial u}{\partial u_l} + \frac{\partial\beta(\boldsymbol{u})}{\partial u_l}\cdot u\right)\\
&= -\left(\beta(\boldsymbol{u})\cdot\phi_l + \frac{\partial\beta(\boldsymbol{u})}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial u_l}\cdot u\right)\\
&= \boxed{-\left(\beta(\boldsymbol{u})\cdot\phi_l + \frac{\partial\beta(\boldsymbol{u})}{\partial\alpha}\cdot u^2\phi_l\right)},\\
\frac{\partial\boldsymbol{\tau}_{b,x}}{\partial v_l} &= -\frac{\partial\beta(\boldsymbol{u})}{\partial v_l}\cdot u\\
&= -\frac{\partial\beta(\boldsymbol{u})}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial v_l}\cdot u\\
&= \boxed{-\frac{\partial\beta(\boldsymbol{u})}{\partial\alpha}\cdot u\cdot v\cdot\phi_l},\\
\frac{\partial\boldsymbol{\tau}_{b,y}}{\partial u_l} &= -\frac{\partial\beta(\boldsymbol{u})}{\partial u_l}\cdot v,\\
&= \boxed{-\frac{\partial\beta(\boldsymbol{u})}{\partial\alpha}\cdot u\cdot v\cdot\phi_l},\\
\frac{\partial\boldsymbol{\tau}_{b,y}}{\partial v_l} &= -\left(\beta(\boldsymbol{u})\cdot\frac{\partial v}{\partial v_l} + \frac{\partial\beta(\boldsymbol{u})}{\partial u_l}\cdot v\right)
\end{aligned}$$

$$= -\left(\beta(\boldsymbol{u}) \cdot \phi_l + \frac{\partial \beta(\boldsymbol{u})}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial v_l} \cdot v\right)$$

$$= \boxed{-\left(\beta(\boldsymbol{u}) \cdot \phi_l + \frac{\partial \beta(\boldsymbol{u})}{\partial \alpha} \cdot v^2 \phi_l\right)}.$$

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