THE RIEMANN HYPOTHESIS

FRANK VEGA

Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all sufficiently large n, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where H_n is the n^{th} harmonic number. In this work, we show certain properties of these both inequalities that leave us to a proof of the Riemann Hypothesis.

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [Cho+07]:

$$\sum_{d|n} d.$$

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Let H_n be $\sum_{j=1}^{n} \frac{1}{j}$. Say $\mathsf{Lagarias}(n)$ holds provided

$$\sigma(n) \leq H_n + exp(H_n) \times \log H_n.$$

2010 *Mathematics Subject Classification.* Primary 11M26; Secondary 11A41, 26D15.

Key words and phrases. number theory, inequality, sum-of-divisors function, harmonic number, prime.

This work was supported by another researcher that shall be included as an author after his approval.

The importance of these properties is:

Theorem 1.1. [RH] *If* $\mathsf{Robins}(n)$ *holds for all* $n > 5040$*, then the Riemann Hypothesis is true [Lag02]. If* $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$*, then the Riemann Hypothesis is true [Lag02].*

It is known that $\mathsf{Robins}(n)$ and $\mathsf{Lagarias}(n)$ hold for many classes of numbers n. We known this:

Lemma 1.2. [conditionally] *If* $\mathsf{Robins}(n)$ *holds for some* $n > 5040$*, then* $\mathsf{Lagarias}(n)$ *holds [Lag02].*

Lemma 1.3. [not-divisible] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $2 \nmid n$ *[Cho+07].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [Cho+07]. $\mathsf{Robins}(n)$ holds for all $n > 5040$ that are square free [Cho+07]. Let $\mathsf{core}(n)$ denotes the square free kernel of a natural number n [Cho+07]. We can show this:

Theorem 1.4. [pi] *Let* $\frac{\pi^2}{6} \times \log\log \mathsf{core}(n) \leq \log\log n$ *for some* $n > 5040$*. Then* $\mathsf{Robins}(n)$ *holds.*

Here, it is our main result:

Theorem 1.5. [main] *If* $\mathsf{Lagarias}(4 \times n)$ *holds for some* $n > 5040$*, then* $\mathsf{Lagarias}(n)$ *holds.*

In this way, we finally conclude that

Theorem 1.6. [final] $\mathsf{Lagarias}(n)$ *holds for all* $n \geq 1$ *and thus, the Riemann Hypothesis is true.*

Proof. Every possible counterexample in $\mathsf{Lagarias}(n)$ for some $n > 5040$ must comply that n is divisible by 2 because of lemmas 1.2 [conditionally] and 1.3 [not-divisible]. In addition, $\mathsf{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. Suppose that we need to prove $\mathsf{Lagarias}(n)$ for some $n > 5040$ such that $2^k \| n$. We know by theorem 1.4 [pi], there is some $k' > 1$ such that $\mathsf{Robins}(4^{k'} \times n)$ holds because of $\frac{\pi^2}{6} \times \log\log \mathsf{core}(n) \leq \log\log(4^{k'} \times n)$. Consequently, we can prove with $4^{k'} \times n$ that $\mathsf{Lagarias}(n)$ also holds by a descendant argument using the theorem 1.5 [main]. In addition, we know that $\mathsf{Robins}(n)$ holds for all $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for every integer $1 \leq k \leq 19$ [Her18]: $\mathsf{Lagarias}(n)$ holds for the same numbers due to lemma 1.2 [conditionally]. In conclusion, we show that $\mathsf{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true. ∎

2. Known Results

We use that the following are known:

Lemma 2.1. [sigma-bound]

$$f(n) < \prod_{p|n} \frac{p}{p-1}. \qquad \text{[Cho+07]}$$

Lemma 2.2. [zeta]

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \qquad \text{[Edw01]}$$

Lemma 2.3. [log-bound]

$$\log(e^\gamma \times (n+1)) \geq H_n \geq \log(e^\gamma \times n). \qquad \text{[Lag02]}$$

3. A Central Lemma

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n. This is a key insight.

Lemma 3.1. [pro] *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

Proof. We use that lemma 2.1 [sigma-bound]:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i-1}.$$

Now for $q > 1$,

$$\frac{1}{1-\frac{1}{q^2}} = \frac{q^2}{q^2-1}.$$

So

$$\frac{1}{1-\frac{1}{q^2}} \times \frac{q+1}{q} = \frac{q^2}{q^2-1} \times \frac{q+1}{q}$$
$$= \frac{q}{q-1}.$$

Then by lemma 2.2 [zeta],

$$\prod_{k=1}^{m} \frac{1}{1-\frac{1}{q_k^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} \times \frac{q_i+1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}. \end{aligned}$$

□

4. A Condition on $\mathsf{core}(n)$

4.1. **A Particular Case.** We prove the Robin's inequality for this specific case:

Lemma 4.1. [case] $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $\mathsf{core}(n) \in \{2, 3, 5, 6, 10, 15, 30\}$.

Proof. Let $n > 5040$. Specifically, let $\mathsf{core}(n)$ be the product of the primes $q_1, \ldots, q_m$, such that $\{q_1, \ldots, q_m\} \subseteq \{2, 3, 5\}$. We need to prove that

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

is also true, because of lemma 2.1 [sigma-bound]. Then, we have that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and hence, the proof is completed. □

4.2. **Main Insight.** The next theorem is a main insight. It extends the class of n so that $\mathsf{Robins}(n)$ holds. The key is that the class on n depend on how close n is to $\mathsf{core}(n)$. The usual classes of such n are defined by their prime structure not by an inequality. This is perhaps one of the main insights.

Theorem 4.2. *Let* $\frac{\pi^2}{6} \times \log\log \mathsf{core}(n) \leq \log\log n$ *for some* $n > 5040$. *Then* $\mathsf{Robins}(n)$ *holds.*

Proof. Let $n' = \mathsf{core}(n)$. Let n' be the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

When $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if $n' \notin \{2, 3, 5, 6, 10, 30\}$ [Cho+07]. However, we can ignore this case, since $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $\mathsf{core}(n) \in \{2, 3, 5, 6, 10, 30\}$ because of lemma 4.1 [case]. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^\gamma \times \log\log n'.$$

By previous lemma 3.1 [pro]

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^\gamma \times \log\log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^\gamma \times \log\log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^\gamma \times \log\log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^\gamma \times \log\log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1+1)\times\cdots\times(q_m+1)}{q_1\times\cdots\times q_m}.$$

□

5. Proof of Main Theorem

Theorem 5.1. *If* Lagarias$(4\times n)$ *holds for some* $n > 5040$*, then* Lagarias(n) *holds.*

Proof. We need to prove

$$\sigma(n) \leq H_n + exp(H_n)\times\log H_n.$$

If we multiply the both sides of the inequality by 4, then we obtain that

$$4\times\sigma(n) \leq 4\times H_n + 4\times exp(H_n)\times\log H_n.$$

We know that $4\times\sigma(n) < \sigma(4\times n)$ [Cho+07]. In this way, we have that

$$\sigma(4\times n) \leq 4\times H_n + 4\times exp(H_n)\times\log H_n.$$

Hence, it is enough to prove that

$$H_{4\times n} + exp(H_{4\times n})\times\log H_{4\times n} \leq 4\times H_n + 4\times exp(H_n)\times\log H_n.$$

since Lagarias$(4\times n)$ holds. Using the lemma 2.3 [log-bound], we note that will be equivalent to prove that

$$\log(e^{\gamma}\times(4\times n+1)) + e^{\gamma}\times(4\times n+1)\times\log\log(e^{\gamma}\times(4\times n+1))$$
$$\leq 4\times\log(e^{\gamma}\times n) + 4\times e^{\gamma}\times n\times\log\log(e^{\gamma}\times n).$$

We can check this previous inequality for all $n > 5040$ and thus, the proof is finished. □

References

[Cho+07] YoungJu Choie et al. “On Robin’s criterion for the Riemann hypothesis”. In: *Journal de Théorie des Nombres de Bordeaux* 19.2 (2007), pp. 357–372. DOI: 10.5802/jtnb.591.

[Edw01] Harold M. Edwards. *Riemann’s Zeta Function*. Dover Publications, 2001. ISBN: 0-486-41740-9.

[Her18] Alexander Hertlein. “Robin’s Inequality for New Families of Integers”. In: *Integers* 18 (2018).

[Lag02] Jeffrey C. Lagarias. “An Elementary Problem Equivalent to the Riemann Hypothesis”. In: *The American Mathematical Monthly* 109.6 (2002), pp. 534–543. DOI: 10.2307/2695443.

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
E-mail address: vega.frank@gmail.com