Connections between Bell numbers, Stirling numbers of first and second kind and Partitions. Evaluation of these numbers.

1. Introduction

Bell and Stirling numbers of first and second kind tell the number of ways that *n* objects, or *n* cycles in the case of Stirling numbers of first kind, can be distributed in *k* cells. They are usually obtained through recurrence rules. However, recurrence rules only tell how many distributions are possible, not the specific form of each distribution, so they cannot be used to build the distributions themselves.

Here we present a relationship between these distributions and the *P(n)* and *P(n,k)* partitions, where *P(n,k)* represents the partition of *n* objects in exactly *k* parts. Such a relationship shows the nature of these distributions and provides a quick and direct way to compute them as well.

2. Review and some properties of Bell and Stirling numbers

<u>Bell numbers - B(*n*)</u>

Bell numbers are the number of ways *n* objects can be distributed among *n* cells. The order of the cells or the order of the objects inside each cell doesn't matter. Empty cells are allowed. For instance, B(3) = 5 means that there are 5 ways to distribute 3 objects (A, B, C) in 3 cells.

1st cell	**2nd cell**	**3rd cell**
A	B	C
A B	C	-
A C	B	-
B C	A	-
A B C	-	-

It is important to note, e.g., that, by definition, the 3rd mode (AC, B, –) is identical to (B, AC, –) or (–, CA, B) and to (CA, B, –).

B(*n*) can be obtained through the recurrence rule:

● B(*n*+1) = $\sum_{k=0}^{n}\binom{n}{k}B(k)$ or it can be approximated through Comtet-Dobinsky's formula:

B(*n*) $\approx \frac{1}{e}\sum_{k=1}^{2n}\frac{k^n}{k!}$

For instance, for *n*=5, B(5) is equal to 52 and the Comtet-Dobinsky's formula gives $\frac{141.3460345}{e}$ = 51.99830018 when computed to its 10th term (using 9 significant digits).

● B(*n*) = $\sum_{i=1}^{n}\left\{ {n \atop i} \right\}$ where $\left\{ {n \atop i} \right\}$ are Stirling numbers of second kind.

● The sequence of B(*n*) values is available through OEIS - A000110 where the first few values are:

n	0	1	2	3	4	5	6	7	8	9	10
B(*n*)	1	1	2	5	15	52	203	877	4140	21147	115975

Stirling numbers of the second kind

Stirling numbers of the second kind provide the number of ways that *n* objects can be distributed among *k* cells (boxes). No empty cells are allowed. They are represented as $\left\{ {n \atop k} \right\}$. Examples of values are given in **Table 1**.

● As discussed above, the definition implies B(*n*) = $\sum_{i=1}^{n} \left\{ {n \atop i} \right\}$

For instance, B(4) = $\left\{ {4 \atop 1} \right\}$+$\left\{ {4 \atop 2} \right\}$+$\left\{ {4 \atop 3} \right\}$+$\left\{ {4 \atop 4} \right\}$ = 1+7+6+1=15

● The recurrence rule $\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$ holds, that is useful to compute the value table.

● The values of Stirling *n* numbers of the second kind can be obtained using the formula:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n$$

For instance: $\left\{ {9 \atop 5} \right\} = \frac{1}{5!} \left[\binom{5}{1} 1^9 - \binom{5}{2} 2^9 + \binom{5}{3} 3^9 - \binom{5}{4} 4^9 + \binom{5}{5} 5^9 \right]$ =

$\frac{1}{120} [5 - 10 * 512 + 10 * 19683 - 5 * 262144 + 1953125]$=$\frac{834120}{120} = 6951$

This formula is fast and fairly easy to use but it doesn't tell how to "build" Stirling numbers, i.e. how the 9 objects in the example are distributed among the 5 cells.

● It's important to note that these 'constructions' are not unique; there are k!$\left\{ {n \atop k} \right\}$ of them (M. Fiorentini. Stirling di prima specie – bitman.name/math/article/32).

Stirling numbers of the first kind

Stirling numbers of the first kind provide the number of ways *n* cycles, not objects, can be distributed in *k* non-empty cells. Cycles with only 1 element are allowed. They were initially (~1730) introduced as coefficient in the series expansion of $(x)_n$, i.e. x*(x+1)*(x+2)…..= $\sum_{k=0}^{n} \left| {n \atop k} \right| x^k$

For instance: x(x+1)(x+2)= $\left[{3 \atop 0} \right] + \left[{3 \atop 1} \right] x^1 + \left[{3 \atop 2} \right] x^2 + \left[{3 \atop 3} \right] x^3$=2x+3$x^2 + x^3$

● We remind the reader that *n* elements generate (*n*–1)! cycles. For instance, 4 objects (A, B, C, D) produce the 3!=6 cycles below:

A B C D ≫ A A C B D ≫ A A D C B ≫ A A B D C ≫ A A C D B ≫ A A D B C ≫ A

● Some properties:

The recurrence rule $\left[{n \atop k} \right]$ = $\left[{n-1 \atop k-1} \right] + (n-1) \left[{n-1 \atop k} \right]$ holds, used to compute the values in **Table 2.**

$$\sum_{k=0}^{n} \left|{n \atop k}\right| = n!$$

$$\sum_{p=k}^{n} \begin{bmatrix} n \\ p \end{bmatrix} \binom{p}{k} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \begin{Bmatrix} m \\ k \end{Bmatrix} = \binom{n-1}{k-1} \frac{n!}{k!}$ (Lah numbers)

3. Evaluation of B(*n*), $\begin{Bmatrix} \boldsymbol{n} \\ \boldsymbol{k} \end{Bmatrix}$ and $\begin{bmatrix} \boldsymbol{n} \\ \boldsymbol{k} \end{bmatrix}$ using partitions

To see B(*n*), i.e. to see its form, and to compute its values, we start from the partition of *n*, P(*n*).

P(n) is made of *m* parts: $p_1, p_2, p_3, \ldots, p_m$

Each part p_i is made by many elements $a_1, a_2, a_3, \ldots, a_m$, the sum of which is obviously *n*. Some elements of one part may occur more times, let $R_1, R_2, R_3, \ldots, R_m$ be the number of repetitions of the same element a_i inside the same part p_i .

For instance, for *n* = 4, P(4) is made of 5 parts: p_1 = 4 with $R_1 = 1$, p_2 = 3 1 with $R_1 = 1$ and $R_2 = 1$, p_3 = 2 2 with $R_1 = 2$, p_4 = 2 1 1 with $R_1 = 1$ and $R_2 = 2$, p_5 = 1 1 1 1 with $R_1 = 4$

It is possible to evaluate a contributing amount c_i for each part p_i so that:

$$\sum_{i=1}^{P(n)} c_i = B(n) \qquad \textbf{(1)}$$

For each part p_i the contribution c_i is:

$$c_i = \binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3}\cdots\binom{n-a_1-a_2-\ldots a_{m-1}}{a_m}\frac{1}{R_1!}\frac{1}{R_2!}\cdots\frac{1}{R_m!} \qquad \textbf{(2)}$$

as $\binom{n}{a_1}$ is the number of ways a_1 objects can be chosen among *n* objects, after which *n*–a_1 objects are left, that can be chosen in $\binom{n-a_1}{a_2}$ ways and so on.

For instance, the contribution of the *211* part of B(4) is $\binom{4}{2}\binom{2}{1}\binom{1}{1}\frac{1}{2!}$=6*$\frac{2}{2}$ = 6

In the same example, the two parts *4* and *31* contribute 4 → $\binom{4}{4} = 1$ and 31 → $\binom{4}{3}\binom{1}{1} = 4$, respectively.

Part *22* contributes $\binom{4}{2}\binom{2}{2}\frac{1}{2!} = 3$ and part *1111* contributes $\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}\frac{1}{4!}$= 1

Summing all contributions, we get 1+4+3+6+1 = 15.

Therefore, B(4) = 15.

Because of how binomial coefficients are defined, formula **(2)** can be written:

$$c_i = \frac{n!}{a_1!a_2!a_3!\ldots a_m!}\frac{1}{R_1!}\frac{1}{R_2!}\cdots\frac{1}{R_m!} \qquad \textbf{(2b)}$$

● Example: Let's determine the value and composition of B(5), which we already know to be 52: P(5) is made of the following 7 parts: 5, 41, 32, 311, 221, 2111, 11111. Using **(2b)**:

$5 \rightarrow \frac{5!}{5!} = 1$ $\quad 41 \rightarrow \frac{5!}{(4!1!)} = 5$ $\quad 32 \rightarrow \frac{5!}{(3!2!)} = 10$

$311 \rightarrow \frac{5!}{(3!1!)}\frac{1}{2!} = 10$ $\quad 221 \rightarrow \frac{5!}{(2!2!1!)}\frac{1}{2!} = 15$ $\quad 2111 \rightarrow \frac{5!}{(2!1!1!1!)}\frac{1}{3!}$ =10

$11111 \rightarrow \frac{5!}{(1!1!1!1!1!)}\ \frac{1}{5!} = 1$

Therefore B(5) = 1+5+10+10+15+10+1 = 52, as already established above.

Evaluation of Stirling numbers of the second kind

The construction and evaluation of Stirling numbers of the second kind $\left\{ n \atop k \right\}$ is the same of the one for B(*n*). Formulas **(1)**, **(2)**, and **(2b)** are still valid as long as P(*n,k*) are substituted for P(n), i.e. the partition of *n* in exactly *k* elements is used.

$$\sum_{i=1}^{P(n,k)} c_i = \left\{ n \atop k \right\} \qquad \textbf{(3)}$$

Where the contributions c_i are evaluated with formula **(2)** or **(2b)**.

For instance, let's build $\left\{ 9 \atop 5 \right\}$ = 6951. The parts which make P(9,5), i.e. 51111 42111 33111 32211 22221, and the corresponding contributions are (we are omitting 1!=1):

51111 → $\frac{9!}{5!}\frac{1}{4!}$ =126 $\quad$ 4211 → $\frac{9!}{(4!2!)}\ \frac{1}{3!}$ =1260 $\quad$ 33111 → $\frac{9!}{(3!3!)}\frac{1}{2!}\frac{1}{3!}$ =840

32211 → $\frac{9!}{(3!2!2!)}\frac{1}{2!}\frac{1}{2!}$ =3780 $\quad$ 22221 → $\frac{9!}{(2!2!2!2!)}\ \frac{1}{4!}$ =945

Therefore $\left\{ 9 \atop 5 \right\}$ = 126+1260+840+3780+945 = 6951

Evaluation of Stirling numbers of the first kind

To build Stirling numbers of the first kind we should remember that $\left[n \atop k \right]$ gives the number of ways that *n* cycles, not elements, are distributed among *k* cells. Given that *n* elements generate (*n*–1)! cycles, we just need to multiply each part of the partition P(*n,k*) by $(a_1 - 1)!\,(a_2 - 1)!(a_3 - 1)!$... (where $a_{1,\,2\ldots}$are the elements of the part.

$$\sum_{i=1}^{P(n,k)} t_i = \begin{bmatrix} n \\ k \end{bmatrix} \qquad \textbf{(4)}$$

with

$$t_i = (a_1 - 1)!(a_2 - 1)!(a_3 - 1)!\ldots(a_m - 1)!\, c_i \qquad \textbf{(5)}$$

where the contributions c_i are evaluated with formula **(2)** or **(2b).**

From formula **(5)**, *t* can be directly expressed as:

$$t_i = \frac{n!}{a_1 a_2 a_3 \ldots a_m} \frac{1}{R_1!} \frac{1}{R_2!} \cdots \frac{1}{R_m!} \qquad \textbf{(5b)}$$

This formula is very useful to compute Stirling numbers of the first kind.

For instance, in the case of $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ = 6769, we have P(8,4) = 5111 4211 3311 3221 2222, and, therefore, the various contributions are given by:

5111 $\rightarrow$ $\frac{8!}{5}\frac{1}{3!}$=1344

4211 $\rightarrow$ $\frac{8!}{4*2}\ \frac{1}{2!}$=2520

3311 $\rightarrow$ $\frac{8!}{3*3}\ \frac{1}{2!}\frac{1}{2!} = 1120$

3221 $\rightarrow$ $\frac{8!}{3*2*2}\ \frac{1}{2!} = 1680$

2222 $\rightarrow$ $\frac{8!}{2*2*2*2}\frac{1}{4!}$= 105

Therefore, $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ = 1344+2520+1120+1680+105 = 6769

4. Examples of evaluation and construction

Examples of evaluation of Stirling numbers of the second kind

In addition to knowing in how many ways *n* objects can be distributed among *k* non-empty cells, it is also interesting finding out how this distribution occurs.

For instance, let's consider $\begin{Bmatrix} 5 \\ 4 \end{Bmatrix} = 10$. We know that P(5,4) = 1, i.e. 2111. Because of **(2b),** 2111 gives $\frac{5!}{2!}\frac{1}{3!} = 10$. Therefore, each of the 5 objects ABCDE appears 10 times among all the cells. The 4 cells contain 2 objects, 1 object, 1 object, and 1 object, respectively.

One possible construction is:

1^{st} cell	2^{nd} cell	3^{rd} cell	4^{th} cell
AB	C	D	E
AC	B	D	E
AD	B	C	E
AE	B	C	D
BC	A	D	E
BD	A	C	E
BE	A	C	D
CD	A	B	E
CE	A	B	D
DE	A	B	C

The objects appear in the various cells with different frequencies; such frequencies are listed below each object.

1^{st}	2^{nd}	3^{rd}	4^{th}
ABCDE	ABCDE	ABCDE	ABCDE
44444	63100	03430	00136

It is immediately apparent that, using this construction, the frequency of each element in a cell is irregular: D and E are missing in the 2^{nd} cell, A and E are missing in the 3^{rd} cell, and A and B are missing in the 4^{th} cell. Only the 1^{st} cell is homogenous.

However, the construction showed above for $\left\{ {5 \atop 4} \right\}$ is not the only possibility. For each $\left\{ {n \atop k} \right\}$ there are k! $\left\{ {n \atop k} \right\}$ possible constructions. This means that, in our example, we have 4! $\left\{ {5 \atop 4} \right\}$ = 24*10 = 240 options. Exchanging objects among the 2^{nd}, 3^{rd}, and 4^{th} cell, we get the following:

1^{st}	2^{nd}	3^{rd}	4^{th}
AB	C	D	E
AC	D	E	B
AD	C	B	E
AE	B	C	D
BC	D	E	A
BD	E	A	C
BE	A	C	D
CD	E	A	B
CE	B	D	A
DE	A	B	C

Now, the object frequencies in the 2^{nd}, 3^{rd} and 4^{th} cell become (no changes for the 1^{st} cell):

2^{nd}	3^{rd}	4^{th}
ABCDE	ABCDE	ABCDE
22222	22222	22222

● When $\left\{ n \atop k \right\}$ comes from several parts, as in the case of $\left\{ 5 \atop 3 \right\} = 25$, that is constructed from P(5,3) = 311 and 221, each part has to be optimized individually. As an example, let's focus on the construction related to the 311 part. The contribution to $\left\{ 5 \atop 3 \right\}$ given by the 311 part is $\frac{5!}{3!}\frac{1}{2!}$ =10. Exchanging objects among cells, this contribution can be optimized to:

ABC	E	D
ABD	C	E
ABE	D	C
ACD	B	E
ACE	B	D
ADE	C	B
BCD	E	A
BCE	D	A
BED	A	C
CED	A	B

where the frequencies are:

1st	2nd	3rd
ABCDE	ABCDE	ABCDE
66666	22222	22222

Similarly, for the 221 part (that contributes $\frac{5!}{(2!*2!)}\frac{1}{2!} = \frac{120}{8} = 15$) we get:

AB	CD	E
AC	BE	D
AD	CE	B
AE	CD	B
BC	AE	D
BD	AE	C
BE	CD	A
CD	BE	A
CE	BD	A
DE	AC	B
BC	AD	E
DE	AB	C
DA	BC	E
CE	AB	D
AB	ED	C

with optimized frequencies:

1st	2nd	3rd
ABCDE	ABCDE	ABCDE
66666	66666	33333

Examples of evaluation of Stirling number of the first kind

Let’s consider, for instance $\left|{4 \atop 2}\right| = 11$ P(4,2) = 31 - 22

Using **(5b)**, we get that part 31 contributes $\frac{4!}{3} = \frac{24}{3} = 8$ while part 22 contributes $\frac{4!}{4}\ \frac{1}{2!} = \frac{24}{8} = 3$

The optimized construction of part 31 contribution is:

ABC	D
ACB	D
ACD	B
ADC	B
ABD	C
ADB	C
BCD	A
BDC	A

The 1st cell frequency is ABCD, while the 2nd cell frequency is ABCD
6666 2222

Part 22 contributes in 3 cases:

AB	CD
AC	BD
AD	BC

The 1st cell frequency is 3111, while the 2nd cell frequency is 0222.

Given the small number of elements and cells, the contribution due to part 22 cannot be optimized.

Table 1 $\left\{ {n \atop k} \right\}$

	***k* = 0**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 4**	***k* = 5**	***k* = 6**	***k* = 7**	***k* = 8**	***k* = 9**	***k* = 10**
***n* = 0**	1										
***n* = 1**	0	1									
***n* = 2**	0	1	1								
***n* = 3**	0	1	3	1							
***n* = 4**	0	1	7	6	1						
***n* = 5**	0	1	15	25	10	1					
***n* = 6**	0	1	31	90	65	15	1				
***n* = 7**	0	1	63	301	350	140	21	1			
***n* = 8**	0	1	127	966	1701	1050	266	28	1		
***n* = 9**	0	1	255	3025	7770	6951	2646	462	36	1	
***n* = 10**	0	1	511	9330	34105	42525	22827	5880	750	45	1

Table 2 $\left| {n \atop k} \right|$

	***k* = 0**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 4**	***k* = 5**	***k* = 6**	***k* = 7**	***k* = 8**	***k* = 9**	***k* = 10**
***n* = 0**	1										
***n* = 1**	0	1									
***n* = 2**	0	1	1								
***n* = 3**	0	2	3	1							
***n* = 4**	0	6	11	6	1						
***n* = 5**	0	24	50	35	10	1					
***n* = 6**	0	120	274	225	85	15	1				
***n* = 7**	0	720	1764	1624	735	175	21	1			
***n* = 8**	0	5040	13068	13132	6769	1960	322	28	1		
***n* = 9**	0	40320	109584	118124	67284	22449	4536	546	36	1	
***n* = 10**	0	362880	1026576	1172700	783680	269325	63273	9450	870	45	1