

An Analytic Solution of Benjamin-Ono Equation by Functional Variable Method (FVM): Graphical Approaches on Wave Speed

Md. Monirul Islam¹, Md. Robiul Islam^{2}*

^{1,2}Department of CSE, Green University of Bangladesh, Dhaka, Bangladesh

**Corresponding Author*

E-mail Id:-robiul@cse.green.edu.bd

ABSTRACT

In mathematical physics numerous methods has been applied for finding exact solutions of various nonlinear partial differential equations (NLPDEs). A very powerful solution method is the Functional Variable Method (FVM). In this paper, we discuss the validity & advantages of this method in the Benjamin-Ono equation and achieve graphical representation for different parameters. The Benjamin-Ono equation is the non K-dv type equation and we solve the non K-dv type equation in this paper. By using this useful method, it is shown that the FVM is effective and general than the other method. We presented the graphical representation of the soliton solutions for various wave speed.

Keywords:-Functional variable method; travelling wave; BO equation.

INTRODUCTION

In our real life most of the problems arise in mathematical and engineering fields often described by the described by partial differential equations (PDEs). In physics, for instance, the heat flow and the wave propagation phenomena involve PDEs [1-4]. Most of the population models in ecology [5-6] and the dispersion of a chemically reactive material are characterized by PDEs. Additionally, most physical phenomena of fluid dynamics, quantum physics, electricity, physics, propagation of shallow-water waves, and lots of other models are controlled within its domain of validity by PDEs. PDEs became a useful gizmo for explaining these natural phenomena of science and engineering models. Therefore, it becomes increasingly important to be conversant in all traditional and recently developed methods for solving PDEs and therefore the implementation of those methods.

The simplest wave equation formed as $u_{tt} = c^2 u_{xx}$ has the genral D'Alembert solution

$$u(x, t) = f(x - ct) + g(x + ct)$$

Where c represents wave speed, $u(x, t)$ represents the amplitude of the wave, f and g are two arbitrary functions represent right and left propagating waves respectively and this propagating behaviour propagate without changing identity. For the propagation in the right direction by setting $g = 0$ in the equation $u_t + u_x = 0$ with speed $c = 1$ is

$$u(x, t) = f(x, t),$$

Where $u(x, t)$ represents a disturbance moving in the negative or positive x direction if $c < 0$ or $c > 0$ respectively. If the solution $u(x, t)$ depends only on the difference between the two coordinates of the partial differential equations, then the solution keeps its exact shape and therefore called solitary waves. Hereman [7] introduced solitary wave, one kind of traveling wave as a localized gravity wave that propagate with constant speed and shape.

At the beginning most of the problems were solved analytically. But at present the present problems are so complicated to

solve that analytical solutions becomes impossible. So, the numerical approach becomes very popular within the course of the 20th century. Numerous algorithms have been developed like Newton's method, Lagrange interpolation polynomial, Gaussian elimination or Euler's method. Numerical analysis naturally finds application altogether fields of engineering and therefore the physical sciences, moreover within the 21st century the life sciences, social sciences, medicine, business and even the humanities have also applied elements of scientific calculations.

Numerical methods have many disadvantages whereas analytical techniques have many advantages. Numerical methods give us approximate solution, not exact solution. Numerical methods always work with iteration. When we determine the final answer for each question, it must come with some errors. So the use of analytical method has increased largely in 21st century than initial stage.

There are many analytic methods developed in Mathematical areas to solve nonlinear evolution equations (NLEEs). A variety of powerful methods are discussed by many researchers such as the Sine-Cosine function method [8], the Tanh-Coth method [9], the Rational Sine-Cosine function method [10], the Functional Variable method [11], Jacobi Elliptic function method [12], Backlund Transformation method [13], Homogeneous Balance method [14], Hirota's Bilinear method [15], The Inverse Scattering method [16], etc.

The FVM was further developed by many authors. The FVM is very new method and recently proposed in 2010. The study shows that FVM is more suitable not only for Kdv type equations but also for non Kdv type equations. The reliability of the

method and the reduction in the use of computational domain give this method a wide applicability. We see that it is easier to understand and a different and shortcut process than the other methods. Without the help of a computer algebraic system, all solutions of Benjamin-Ono equation in this work show the efficiency of the Functional Variable Method. We only use MatLab software to get graph of the solutions which we get. The solution procedure is very simple and the traveling wave solutions are expressed by hyperbolic functions and trigonometric functions. It has been shown that the method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics. FVM is easier, quicker and better technique. The main property of this method is that it demonstrates its flexibility and ability to solve nonlinear evolution equations (NLEEs) accurately, efficiently and conveniently. This method is direct, concise and can provide a useful way to efficiently find the exact structures of solutions to a variety of nonlinear wave equations. The performance of the FVM method is reliable, effective and gives the exact solitary wave solutions and periodic wave solutions.

The main objective of this paper is to apply the Functional Variable Method (FVM) in finding the exact solutions of nonlinear evolution equations like Benjamin-Ono equation. We solve the Benjamin-Ono equation which is non Kdv type equation. The obtained solutions may be significant and important for analyzing the nonlinear phenomena arising in applied physical sciences. We can show that the FVM is also more suitable and better technique for non Kdv type equation.

The paper is organized in the following sequence. In Section II, we present the method of FVM. The Application of FVM in terms of BO equation are derived in

Section III. The results along with discussion discuss in section IV. Finally, this paper ends with conclusion in section V.

THE FUNCTIONAL VARIABLE METHOD

For a given nonlinear partial differential equations (NLPDEs), written in several independent variable as

$$P(u, u_t, u_x, u_y, u_z, u_{xy}, u_{yz}, u_{xz}) = 0 \quad (2.1)$$

where P is a polynomial in u and its partial derivatives. To find the travelling wave solution of eq. (2.1), we first introduce the wave variable as

$$\xi = x + y - ct \quad \text{such that } u(x, y, t) = U(\xi). \quad (2.2)$$

Then the NLPDEs (2.1) gets converted to an ODE

$$Q(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, U_{\xi\xi\xi\xi}, \dots) = 0, \quad (2.3)$$

where Q is a polynomial in U and its total derivatives.

Let us make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U) \quad (2.4)$$

and some successive derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \end{aligned} \quad (2.5)$$

where $f' = \frac{df}{du}$. The ODE (2.3) can be reduced in terms of F, U and its derivatives upon using the expressions of eq. (2.5) into eq. (2.3) gives

$$R(U, F, F', F'', F''', \dots) = 0. \quad (2.6)$$

The key idea of this particular form (2.6) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the equation (2.6) provides the expression of F and this in turn together with (2.4) gives the appropriate solutions

to the original problem.

APPLICATION

The Benjamin - Ono equation

The Benjamin-Ono (BO) equation was introduced by Benjamin, 1967 and Ono, 1975 [17,18]. The BO equation is a nonlinear partial integro-differential equation that describes one-dimensional internal waves in deep water. The BO equation arises in the propagation of internal waves in a stratified fluid of great depth. It also has soliton solutions. The BO equation arises interfacial hydrodynamics. The generalized BO equation presents the interesting fact that the dispersive effect is described by a nonlocal operator and is weaker than that exhibited by the generalized Kdv equation. This equation is non Kdv type equation.

Consider the equation

$$u_t + uu_x + hu_{xx} = 0 \quad (3.1)$$

where h is the Hilbert Transform Operator. Again, let the transformation or solution of equation (3.1) be

$$\begin{aligned} u(x, t) &= u(\xi), \quad \xi = x - ct. \\ \frac{\partial u}{\partial t} &= -c \frac{du}{d\xi}, \quad \frac{\partial u}{\partial x} = \frac{du}{d\xi}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{d\xi^2}, \end{aligned} \quad (3.2)$$

Equation (3.2) reduces PDE in equation (3.1) to ODE,

$$-cu_\xi + uu_\xi + hu_{\xi\xi} = 0$$

Integrating w.r.to ξ ,

$$-cu + \frac{u^2}{2} + hu_\xi = 0 \quad (3.3)$$

Now let the functional variable

$$u_\xi = \frac{du}{d\xi} = F(u) \quad (3.4)$$

From equation (3.3) and (3.4) we get,

$$-cu + \frac{u^2}{2} + hF(u) = 0$$

$$\Rightarrow hF(u) = cu - \frac{u^2}{2}$$

$$\Rightarrow \frac{du}{d\xi} = \frac{1}{h}(cu - \frac{u^2}{2})$$

$$\Rightarrow \frac{du}{d\xi} = \frac{-1}{2h}(u^2 - 2cu)$$

$$\begin{aligned} \text{Integrating, } \int \frac{du}{u^2 - 2cu} &= \frac{-1}{2h} \int d\xi \\ \Rightarrow \int \frac{-2du}{u^2 - 2cu + (c)^2 - (c)^2} &= \frac{-1}{2h} \xi \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow \int \frac{du}{(u-c)^2 - (c)^2} = \frac{-1}{2h} \xi \\
 &\Rightarrow \frac{1}{2c} \ln \left(\frac{u-c-c}{u-c+c} \right) = \frac{-1}{2h} \xi \\
 &\Rightarrow \frac{1}{c} \ln \left(\frac{u-2c}{u} \right) = \frac{-1}{h} \xi \\
 &\Rightarrow \ln \left(1 - \frac{2c}{u} \right) = \frac{-c}{h} \xi \\
 &\Rightarrow 1 - \frac{2c}{u} = e^{\frac{-c}{h} \xi} \\
 &\Rightarrow \frac{2c}{u} = 1 - e^{\frac{-c}{h} \xi} \\
 &\Rightarrow \frac{u}{2c} = \frac{1}{1 - e^{\frac{-c}{h} \xi}} \\
 &\Rightarrow u(x, t) = \frac{2c}{1 - e^{\frac{-c}{h} \xi}}
 \end{aligned}$$

Which is the required solution of equation (3.1). Below we show the graphs of

$$u(x, t) = \frac{2c}{1 - e^{\frac{-c}{h} \xi}}; h = 1$$

& $-10 \leq x, t \leq 10$.

RESULT AND DISCUSSION

In the transformation $\xi = x - ct$, defined c as the wave speed. In the solution $u(x, t)$, it is observed that the nonlinear coefficient of the soliton solutions is fully

dependent on the wave speed. The effect of c on the nonlinearity of soliton solution is clearly identified from the graphical representation of the soliton solutions (Figure 1-21) for variation of c .

When $c = 0$, there is no soliton solution of $u(x, t)$. As the value of c increases from $c = 0.1$ to $c = 0.4$, the soliton solutions $u(x, t)$ stay middle and rise or descend in the middle of the graph about the space (x) axes and time (t) axes (Figure 1,2). But when $c = 0.5$, it is seen that the soliton solution $u(x, t)$ gets divided in two portions, one portion is upward and another portion is downward (Figure 3). If we further increase the value of c , from $c = 0.6$ to $c = 0.9$, it is clear that the soliton solutions $u(x, t)$ are similar to Figure 4. But when $c = 1$, in Figure 5, the soliton solution undergoes more rise or descend. These types of solutions are kink solutions. This process carries over upto $c = 3$ (Figure 6-9). But if we more increase the value of c , it is seen that there is no soliton solutions (Figure 10).

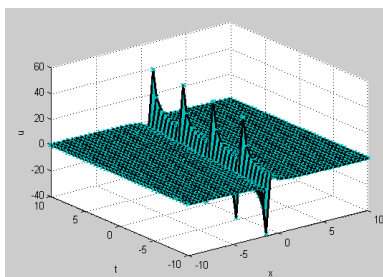


Fig.1:- $c = 0.1$

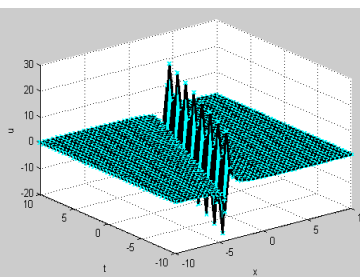


Fig.2:- $c = 0.4$

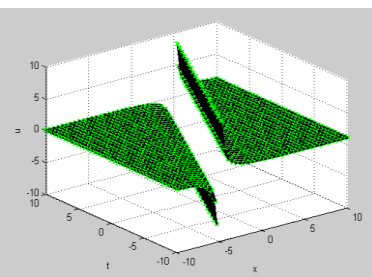


Fig.3:- $c = 0.5$

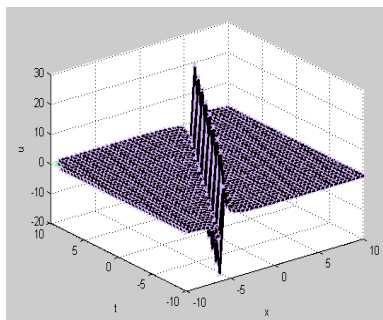


Fig.4:- $c = 0.6$

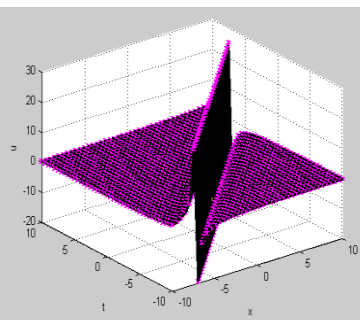


Fig.5:- $c = 1$

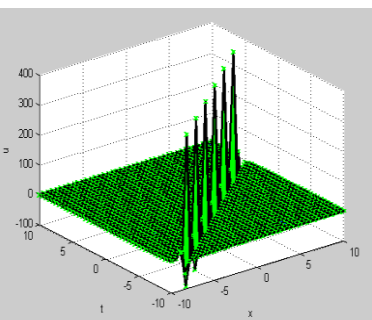


Fig.6:- $c = 1.2$

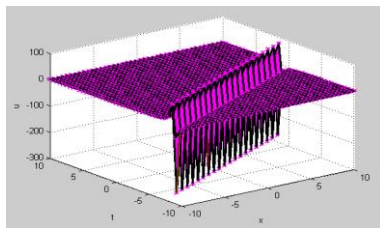


Fig.7:- $c = 2$

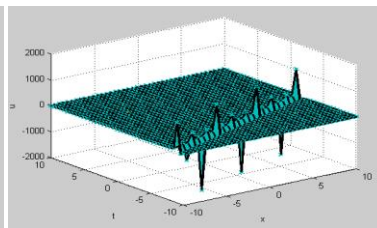


Fig.8:- $c = 2.6$

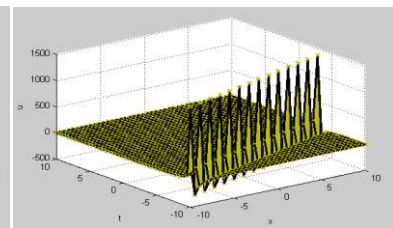


Fig.9:- $c = 3$

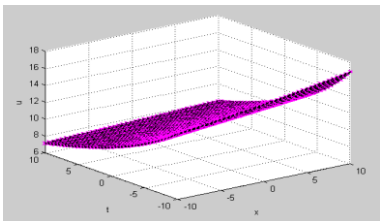


Fig.10:- $c = 5$

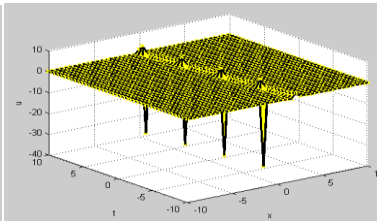


Fig.11:- $c = -0.1$

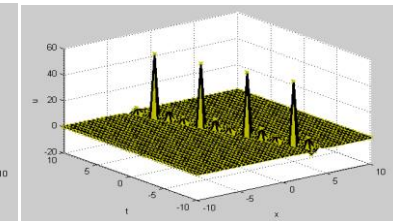


Fig.12:- $c = -0.3$

Again for $c = -0.1$, it is seen that the soliton solution $u(x,t)$ is discrete and stays upwards about the space (x) axis and time (t) axis (Figure 11). But when $c = -0.3$, the soliton solution is also discrete but it stays downwards (Figure12). Again when $c = -0.5$, the soliton solutions $u(x,t)$, stay again upwards about the space

(x) axis and time (t) axis (Figure.13). This process continues upto $c = -5$ which stay upward about the axes (Figure14-18). But if we increase the value of c more, it is evident that the soliton solutions $u(x,t)$ always stay upwards about the axes and constant. These soliton solutions are single soliton solutions (Figure 19-21).

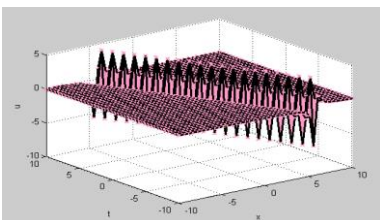


Fig.13:-, $c = -0.5$

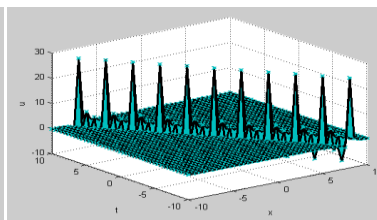


Fig.14:- $c = -0.75$

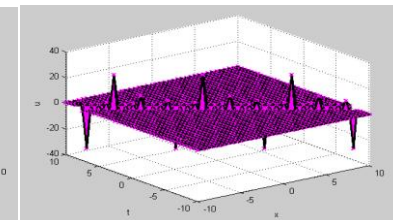


Fig.15:- $c = -1.3$

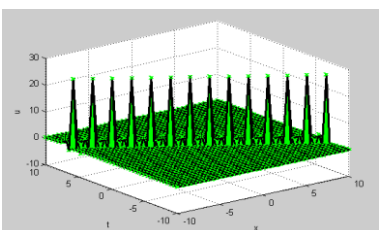


Fig.16:- $c = -1.5$

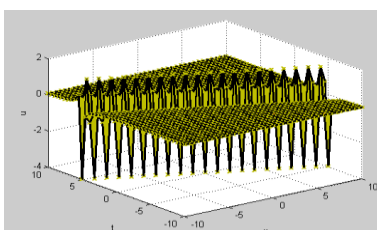


Fig.17:- $c = -2$

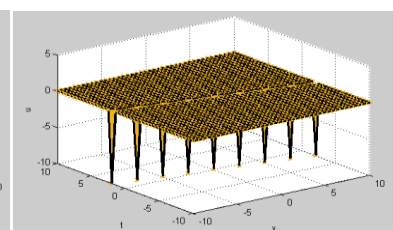


Fig.18:- $c = -5$

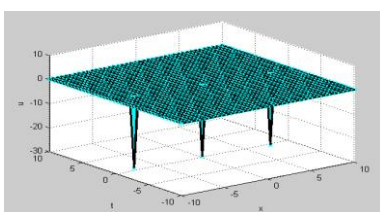


Fig.19:- $c = -15$

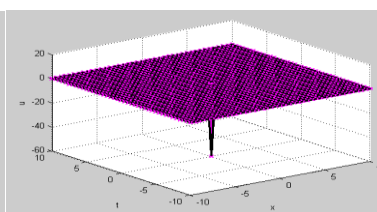


Fig.20:- $c = -30$

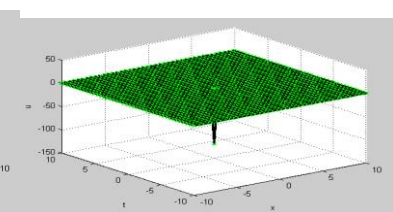


Fig.21:- $c = -60$

CONCLUSION

The Functional Variable Method (FVM) has been successfully used to seek exact travelling wave solutions of the Benjamin-Ono equation which is non Kdv type equation. This work shows that FVM is more suitable not only for Kdv type equations but also for non Kdv type equations. The obtained solutions may be significant and important for analyzing the nonlinear phenomena arising in applied physical sciences. The most benefit of the Functional Variable Method is that the solution is obtained directly by integration but for the other methods, we first find the value of constant then we get the solution by using these values. The FVM is very new method and recently proposed in 2010. So, this method will be more popular and standard in future. We can solve many nonlinear wave equations by this method without much difficulties. Also, we have observed graphs for solutions obtained by FVM come up faster or earlier than the other methods. So, we say that FVM is more direct and concise approach in handling NLPDEs. The method can be used for treating many other nonlinear evolution equations in mathematical physics.

REFERENCES

1. Bleecker, D., & Csordas, G. (1992). *Basic partial differential equations*. CRC Press.
2. Farlow, S. J. (1993). *Partial differential equations for scientists and engineers*. Courier Corporation.
3. John F. (1982). *Partial Differential Equations*, Springer-Verlag, New York.
4. Lam, L. (2000). *Nonlinear Physics for Beginners*.
5. Logan, J. D. (2008). *An introduction to nonlinear partial differential equations* (Vol. 89). John Wiley & Sons.
6. Wazwaz, A. M. (2002). *Partial differential equations*. CRC Press
7. Hereman, W. (2009). Shallow water waves and solitary waves, *Encyclopedia of Complexity and Systems Science*, Ed.: RA Meyers.
8. Wazwaz, A. M. (2004). A sine-cosine method for handling nonlinear wave equations. *Mathematical and Computer modelling*, 40(5-6), 499-508.
9. Malfliet, W., & Hereman, W. (1996). The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta*, 54(6), 563.
10. Alquran, M., Al-Khaled, K., & Ananbeh, H. (2011). New soliton solutions for systems of nonlinear evolution equations by the rational sine-cosine method. *Studies in Mathematical Sciences*, 3(1), 1-9.
11. Zerarka, A., Ouamane, S., & Attaf, A. (2010). On the functional variable method for finding exact solutions to a class of wave equations. *Applied Mathematics and Computation*, 217(7), 2897-2904.
12. Liu, S., Fu, Z., Liu, S., & Zhao, Q. (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Physics Letters A*, 289(1-2), 69-74.
13. Wahlquist, H. D., & Estabrook, F. B. (1975). Prolongation structures of nonlinear evolution equations. *Journal of Mathematical Physics*, 16(1), 1-7.
14. Fan, E., & Zhang, H. (1998). A note on the homogeneous balance method. *Physics Letters A*, 246(5), 403-406.
15. Hirota, R. (1971). Exact solution of the Korteweg—de Vries equation for multiple collisions of solitons. *Physical Review Letters*, 27(18), 1192.
16. Ablowitz, M. J., & Segur, H. (1997). *Solitons and the inverse scattering transform* (SIAM, Philadelphia, 1981). *AC Newell, Solitons in*

- mathematics and physics* (SIAM, Philadelphia, 1985).
17. Benjamin, T. B. (1967). Internal waves of permanent form in fluids of great depth. *Journal of Fluid Mechanics*, 29(3), 559-592.
18. Ono, H. (1975). Algebraic solitary waves in stratified fluids. *Journal of the Physical Society of Japan*, 39(4), 1082-1091.