A shape optimization problem for stationary Navier-Stokes flows in 3D tubes

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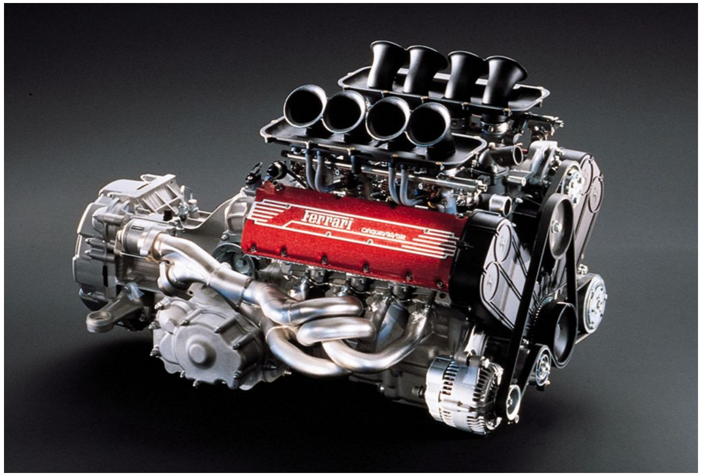
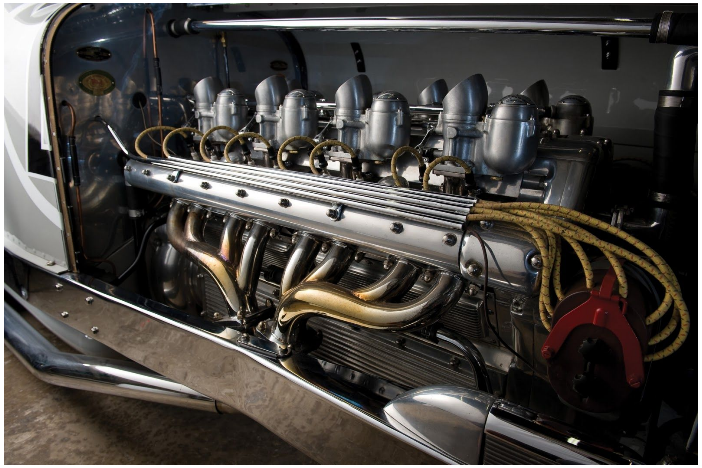
Combustion Engines

(a) lexusenthusiast.com.

(b) BMW Combustion Engine (bmwblog.com).



(c) Ferrari Combustion Engine.



(d) A beautiful combustion engine.

Figure: Various Combustion Engines.

Modeling Air Ducts

A schematic geometry and its boundary:

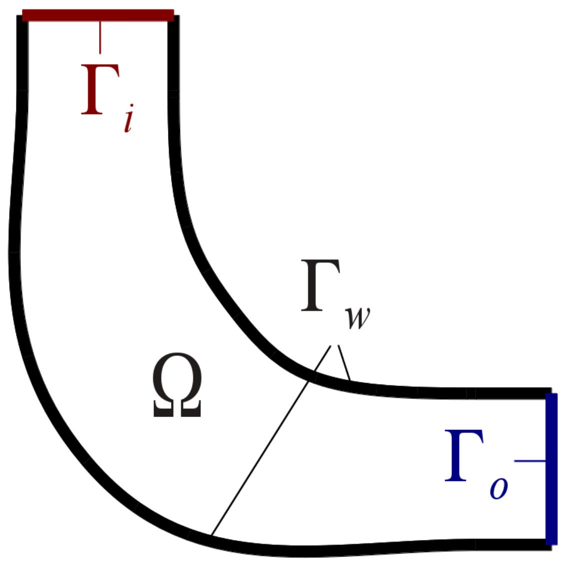


Figure: Simple sketch of an air duct.

Target: *Optimize the shape of air ducts*.

Stationary Navier-Stokes Equations

Stationary NSEs for *velocity* $\mathbf{u}$ and *kinematic pressure* p:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\text{wall}}, \\ -\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{NSEs}$$

where

- ν: kinematic viscosity
- $\mathbf{f}$: source term
- $\mathbf{f}_{\text{in}}$: inflow profile at Γ_{in}

Cost Functionals (1)

Main Problem. Find an $\Omega \in \mathcal{O}_{\mathrm{ad}}$ s.t. 2 criteria are considered:

1. Flow Uniformity at the Outlet.

- The uniformity of the flow upon leaving the outlet plane is an important design criterion of e.g. *automotive air ducts*.
- Other use: Efficiency of distributing fresh air inside the car.

Consider:

$$\mathcal{J}_1(\mathbf{u}(\Omega)) := \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} (\mathbf{u}\cdot\mathbf{n} - u_{\mathrm{d}})^2,$$

where u_{d} is the *desire velocity* in the outlet plane.

Cost Functionals (2)

2. Dissipated Power. Compute power dissipated by a fluid dynamic device as the net inward flux of energy.
I.e., total pressure, through the device boundaries for smooth pressure p:

$$\mathcal{J}_2((\mathbf{u},p)(\Omega)) := -\int_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2\right) \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\Gamma.$$

But $p \in L^2(\Omega)$ only, consider an approximation of $\mathcal{J}_2$ instead:

$$\mathcal{J}_2^\varepsilon((\mathbf{u},p)(\Omega)) := -\frac{|\Gamma_{\text{in}}|}{|\Gamma_{\text{in}}^\varepsilon|}\int_{\Gamma_{\text{in}}^\varepsilon} \left(p + \frac{1}{2}|\mathbf{u}|^2\right) \mathbf{u}\cdot\mathbf{n} - \frac{|\Gamma_{\text{out}}|}{|\Gamma_{\text{out}}^\varepsilon|}\int_{\Gamma_{\text{out}}^\varepsilon} \left(p + \frac{1}{2}|\mathbf{u}|^2\right) \mathbf{u}\cdot\mathbf{n}.$$

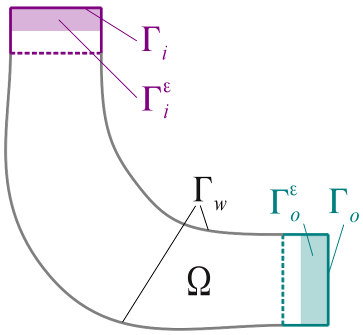


Figure: Simple sketch of Geometry Ω with Modified Inlet $\Gamma_{\text{in}}^\varepsilon$ and Outlet $\Gamma_{\text{out}}^\varepsilon$.

Note. Different measures are applied for Γ_{in} and $\Gamma_{\text{in}}^\varepsilon$.

Mixed Cost Functional & Optimization problem

Mixed cost functional. Take both criteria into effect, with the *weighting parameter* $\gamma \in [0,1]$:

$$\mathcal{J}_{12}^{\varepsilon,\gamma}((\mathbf{u},p)(\Omega)) \coloneqq \gamma \mathcal{J}_1(\mathbf{u}(\Omega)) + (1-\gamma)\mathcal{J}_2^{\varepsilon}((\mathbf{u},p)(\Omega)). \tag{$\mathcal{J}_{12}$}$$

Shape Optimization Problem.

Objective: Minimize the cost functional $\mathcal{J}_{12}^{\varepsilon,\gamma} : \mathcal{O}_{\mathrm{ad}} \to \mathbb{R}$ over some *admissible subset* $\mathcal{O}_{\mathrm{ad}}$ of $2^{\mathbb{R}^d} \coloneqq \{\Omega; \Omega \subset \mathbb{R}^d\}$, i.e.

$$\min_{\Omega \in \mathcal{O}_{\mathrm{ad}}} \mathcal{J}_{12}^{\varepsilon,\gamma}((\mathbf{u},p)(\Omega)) \text{ s.t. } (\mathbf{u},p) \text{ solves (NSEs)}, \ \mathrm{Vol}(\Omega) = V_0. \tag{SOP}$$

Lagrangian & Adjoint Method

Lagrangian:

$$\begin{aligned}\mathcal{L}(\mathbf{u}, p, \Omega, \mathbf{v}, q, \mathbf{v}_{\text{in}}, \mathbf{v}_{\text{wall}}, \mathbf{v}_{\text{out}}, v_{\text{Vol}}) &:= \mathcal{J}_{12}^{\varepsilon,\gamma}((\mathbf{u}, p)(\Omega)) \\ &+ \int_\Omega \mathbf{v} \cdot (-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{f})\mathrm{d}\mathbf{x} + \int_\Omega q\nabla\cdot\mathbf{u}\mathrm{d}\mathbf{x} \\ &+ \int_{\Gamma_{\text{in}}} \mathbf{v}_{\text{in}} \cdot (\mathbf{u} - \mathbf{f}_{\text{in}})\mathrm{d}\Gamma_{\text{in}} + \int_{\Gamma_{\text{wall}}} \mathbf{v}_{\text{wall}} \cdot \mathbf{u}\mathrm{d}\Gamma_{\text{wall}} \\ &+ \int_{\Gamma_{\text{out}}} \mathbf{v}_{\text{out}} \cdot (-\nu\partial_{\mathbf{n}}\mathbf{u} + p\mathbf{n})\mathrm{d}\Gamma_{\text{out}} + v_{\text{Vol}}(\text{Vol}(\Omega) - V_0).\end{aligned} \tag{$\mathcal{L}$}$$

Total variation of $\mathcal{L}$:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\Omega} + \frac{\delta\mathcal{L}}{\delta\mathbf{u}} + \frac{\delta\mathcal{L}}{\delta p}.$$

Adjoint method. Choose Lagrange multiplier $\mathbf{v}, q$ s.t.

$$\frac{\delta\mathcal{L}}{\delta\mathbf{u}} + \frac{\delta\mathcal{L}}{\delta p} = 0,$$

then the total variation $\delta\mathcal{L}$ can be computed *simply* as:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\Omega}.$$

Derive Adjoint Systems

Expand $\frac{\delta \mathcal{L}}{\delta \mathbf{u}} + \frac{\delta \mathcal{L}}{\delta p} = 0$ as

$$\frac{\partial \mathcal{J}_{12}^{\varepsilon,\gamma}}{\partial \mathbf{u}}\delta \mathbf{u} + \frac{\partial \mathcal{J}_{12}^{\varepsilon,\gamma}}{\partial p}\delta p + \int_\Omega \mathbf{v} \cdot [(\delta \mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\delta \mathbf{u} - \nu \Delta \delta \mathbf{u}]\, \mathrm{d}\mathbf{x}$$
$$- \int_\Omega q \nabla \cdot \delta \mathbf{u} \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{v} \cdot \nabla \delta p \mathrm{d}\mathbf{x} + \int_{\Gamma_{\text{in}}} \mathbf{v}_{\text{in}} \cdot \delta \mathbf{u} \mathrm{d}\Gamma_{\text{in}}$$
$$+ \int_{\Gamma_{\text{wall}}} \mathbf{v}_{\text{wall}} \cdot \delta \mathbf{u} \mathrm{d}\Gamma_{\text{wall}} + \int_{\Gamma_{\text{out}}} (-\nu \mathbf{v}_{\text{out}} \cdot \partial_{\mathbf{n}}(\delta \mathbf{u}) + \mathbf{v}_{\text{out}} \cdot \delta p \mathbf{n}) \mathrm{d}\Gamma = 0.$$

Decompose: $\mathcal{J}_{12}^{\varepsilon,\gamma} = \int_\Gamma J_\Gamma \mathrm{d}\Gamma + \int_\Omega J_\Omega \mathrm{d}\Omega$.
Integrate by parts:

$$\int_\Omega (-\nabla \cdot \mathbf{v} + \partial_p \mathcal{J}_\Omega)\delta p \mathrm{d}\mathbf{x}$$
$$+ \int_\Omega [-\nabla \mathbf{v} \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla q + \partial_{\mathbf{u}} \mathcal{J}_\Omega] \cdot \delta \mathbf{u}$$

$+$ Boundary integral terms $= 0,$

for any variation $\delta \mathbf{u}$ and δp.

Adjoint NSEs

Collect terms on each $\int_\Omega(\cdots)\delta\mathbf{u}$, $\int_\Omega(\cdots)\delta p$, $\int_\Gamma(\cdots)\delta\mathbf{u}$, $\int_\Gamma(\cdots)\delta p$, obtain:

Adjoint Navier-Stokes equations.

$$\begin{cases} -(\nabla\mathbf{v})^\top\mathbf{u} - \nabla\mathbf{v}\cdot\mathbf{u} + \nabla q - \nu\Delta\mathbf{v} = (\gamma-1)k_\varepsilon\left[\left(p + \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{n} + (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\right], & \text{in } \Omega, \\ -\nabla\cdot\mathbf{v} = (\gamma-1)k_\varepsilon\mathbf{u}\cdot\mathbf{n} & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{wall}}, \\ -\nu\partial_{\mathbf{n}}\mathbf{v} - \mathbf{n}(\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{n})\mathbf{v} + q\mathbf{n} = \gamma(\mathbf{u}\cdot\mathbf{n} - \bar{u}) & \text{on } \Gamma_{\mathrm{out}}, \end{cases}$$

with

$$k_\varepsilon(x) := \frac{|\Gamma_{\mathrm{in}}|}{|\Gamma^\varepsilon_{\mathrm{in}}|}\chi_{\overline{\Gamma^\varepsilon_{\mathrm{in}}}}(x) + \frac{|\Gamma_{\mathrm{out}}|}{|\Gamma^\varepsilon_{\mathrm{out}}|}\chi_{\overline{\Gamma^\varepsilon_{\mathrm{out}}}}(x),\ \forall x\in\Omega,$$

Shape Derivatives

Describe *perturbed domains* via a *reference domain* Ω:

$$\Omega_t := T_t[V](\Omega) := \{x + tV(x); x \in \Omega\}.$$

Definition (Shape derivative (Sokolowski Zolesio 1992))

Let $D \subset \mathbb{R}^d$. A function $J : 2^D \to \mathbb{R}$ is said to be *shape differentiable*, if the limit

$$dJ(\Omega)[V] := \lim_{t \downarrow 0} \frac{J(T_t[V](\Omega)) - J(\Omega)}{t}$$

exists for all directions V and if the mapping $V \mapsto dJ(\Omega)[V]$ is linear and continuous.

Typical choices for $T_t[V]$.

- *Perturbation of Identity*: $T_t[V](x) = x + tV(x), t \geq 0, x \in \Omega$.
- *Speed method*: $T_t[V](x) = \varphi(t, x)$ where φ solves the ODE

$$\frac{\partial \varphi}{\partial t}(t, x) = V(t, x),\ \varphi(0, x) = x,\ \forall (t, x) \in [0, \infty) \times \Omega.$$

Shape Derivative of $\mathcal{J}_{12}^{\varepsilon,\gamma}$

Applied above formulas, obtain:

$$d\mathcal{J}_{12}^{\varepsilon,\gamma}(\Omega)[V] = \int_{\Gamma_{\text{wall}}} (\partial_{\mathbf{n}}\mathbf{u} \cdot \partial_{\mathbf{n}}\mathbf{v})(V \cdot \mathbf{n}).$$

Shape gradient:

$$D\mathcal{J}_{12}^{\varepsilon,\gamma}(\Omega) = -(\partial_{\mathbf{n}}\mathbf{u} \cdot \partial_{\mathbf{n}}\mathbf{v})\mathbf{n}|_{\Gamma_{\text{wall}}}.$$

Geometrical Constraints (1)

Impose further restrictions on the possible design by geometrical constraints:

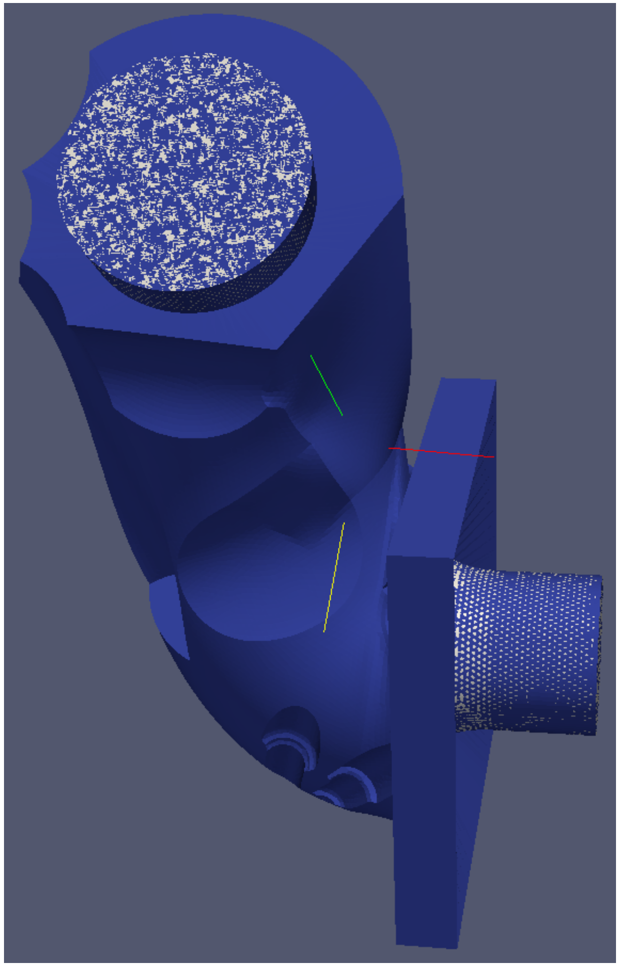


Figure: A design space (blue) contains a tube.

Geometrical Constraints (2)

Minimization problem with geometrical constraint G:

$$\min_{\Omega \subset \mathcal{O}_{\mathrm{ad}} \cap G} \mathcal{J}_{12}^{\varepsilon,\gamma}((\mathbf{u},\mathbf{p})(\Omega)) + \alpha \mathcal{G}(\Omega) \text{ s.t. } (\mathbf{u}, p) \text{ solves (NSEs)}, \ \mathrm{Vol}(\Omega) = V_0,$$

with $\alpha > 0$ and $\mathcal{G}(\Omega) \coloneqq \int_\Omega l_G(x)$.

- **Barrier method.** $l_G(x) \coloneqq |\log d(x, G^c)|$ with $G^c = \mathbb{R}^d \backslash G$.
- **Penalty method.** $l_G(x) \coloneqq (d(x,G))^\beta$ with $\beta \geq 1$ and the distance function $d(x,G) \coloneqq \min_{y \in G} |x - y|$.

Additional shape derivative term:

$$d\mathcal{G}(\Omega)[V] = \int_{\Gamma_{\mathrm{wall}}} (V \cdot \mathbf{n}) l_G(s) ds.$$

Gradient Descent Algorithm (1)

A gradient descent algorithm using Armijo linesearch:

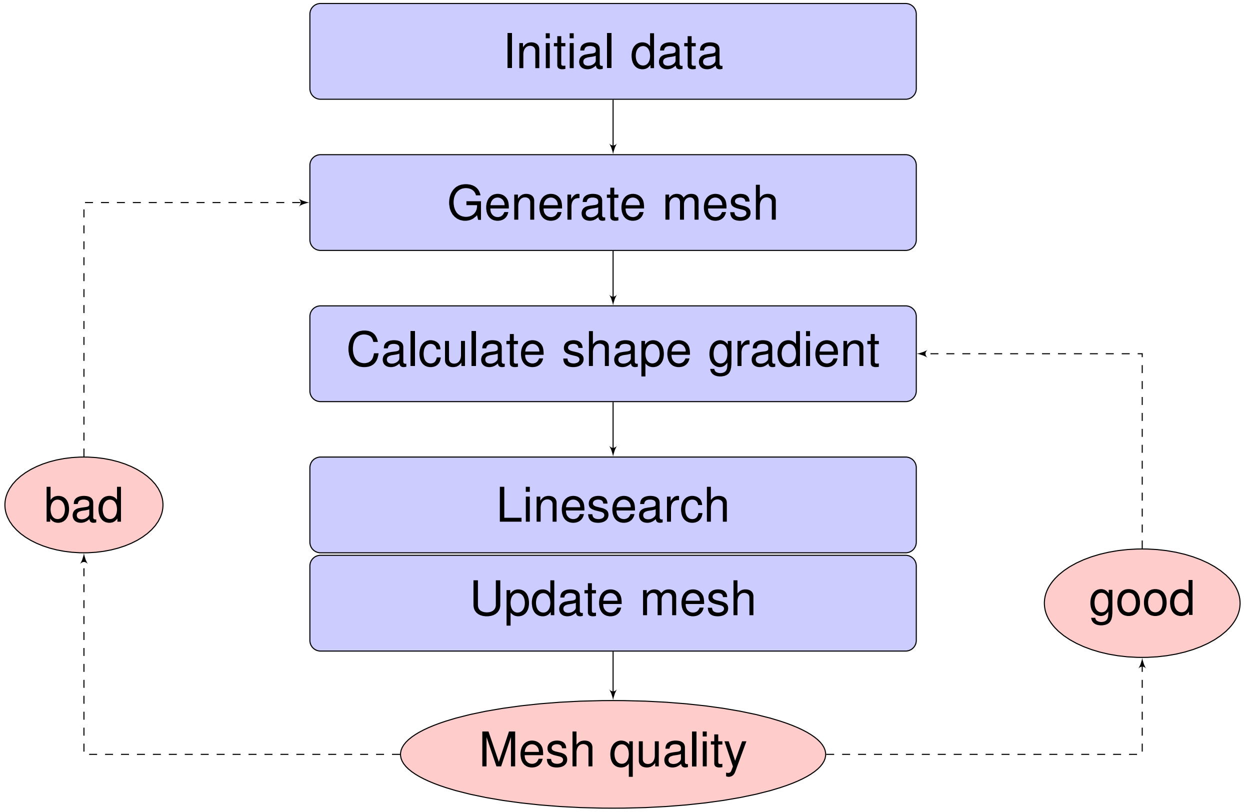


Figure: A Gradient Descent Algorithm.

Data

- Kinematic viscosity $\nu \approx 1.56659 \times 10^{-5}$.

- Weighting parameter $\gamma = 0$: Only consider $\mathcal{J}_2^{\varepsilon}$.

Numerical Results (1)

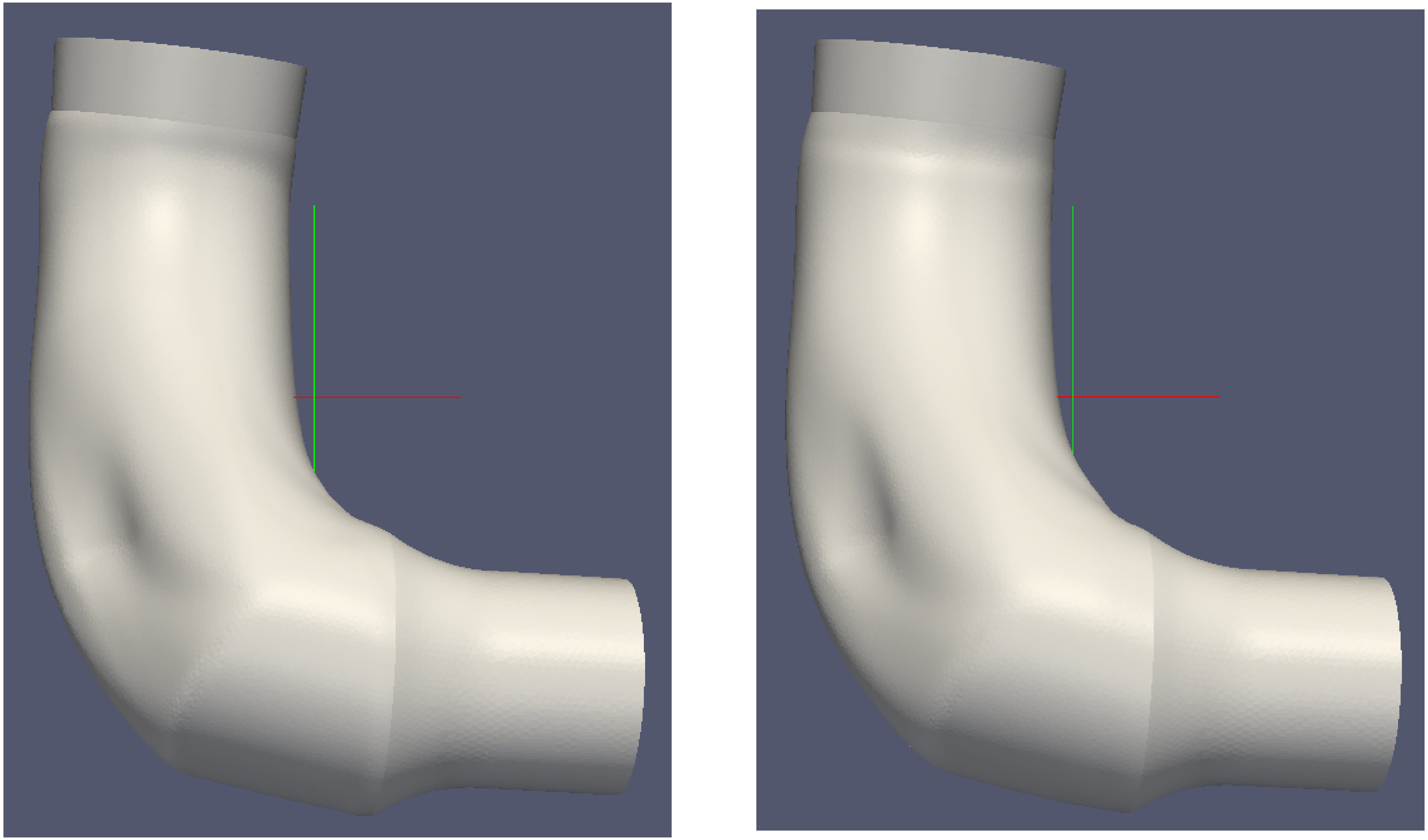
Choose the weighting parameter $\gamma = 0$.
Run 30 gradient-descent iterations (≈ 10 days).

Iteration	$\mathcal{J}_{12}$	Reduce
0	182.07227883754	0%
1	180.929940529294	0.63%
2	180.136772627044	1.06%
3	179.571215644196	1.37%
4	179.091345594335	1.64%
5	178.804927417778	1.79%
6	178.048912220916	2.2%
7	177.935313457448	2.27%
8	177.907822689922	2.29%
9	177.368528982915	2.58%
10	177.354503052929	2.59%
11	177.353679949393	2.59%
12	176.874239498098	2.85%
13	176.872879859059	2.86%
14	176.872793330895	2.86%

Numerical Results (2)

Iteration	$\mathcal{J}_{12}$	Improvement
15	176.397008800941	3.12%
16	176.39201327152	3.12%
17	176.391228177362	3.12%
18	176.027299330372	3.32%
19	176.025503228316	3.32%
20	176.025478930617	3.32%
21	175.583524795214	3.56%
22	175.580539111237	3.57%
23	175.580538614767	3.57%
24	175.377206316433	3.68%
25	175.375696397475	3.68%
26	175.375696266596	3.68%
27	175.117224921322	3.82%
28	175.115844803727	3.82%
29	174.771828640699	4.01%
30	174.656762605995	4.07%

Simulation Results

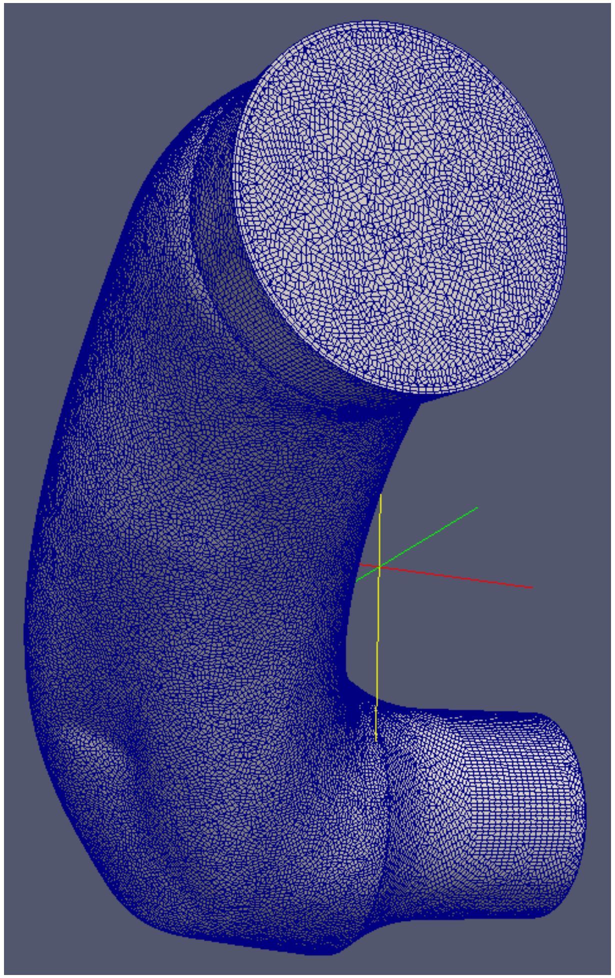


(a) Initial tube.

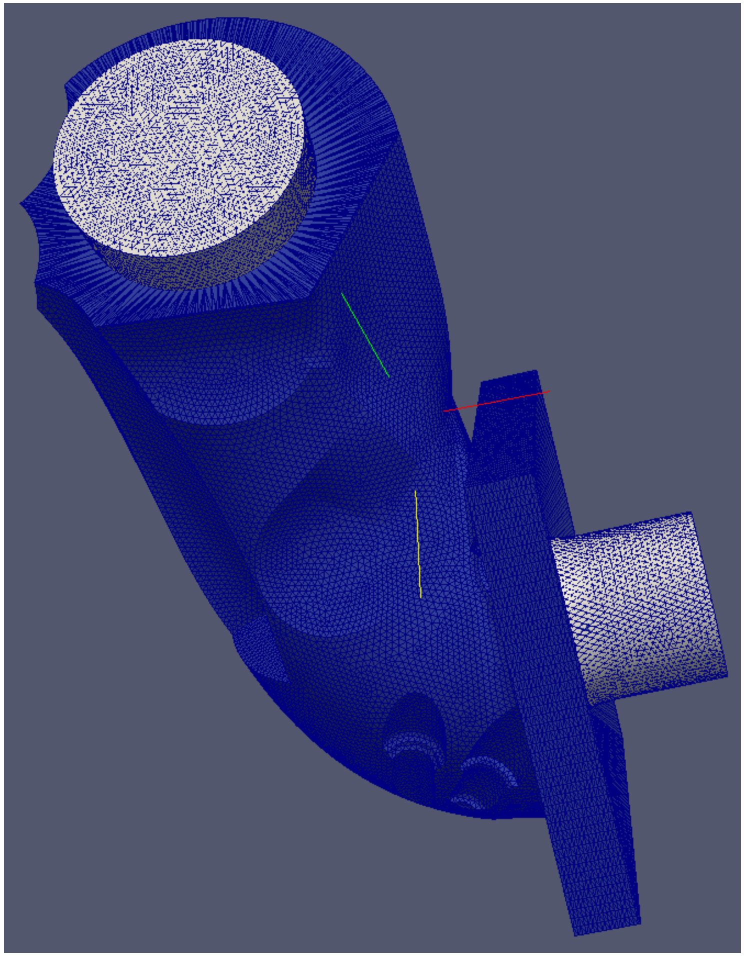
(b) Optimized tube after 30 gradient iterations.

Figure: Initial vs. Optimized Tubes.

Meshes



(a) Tube with Mesh.



(b) Geometrical Constraint with Mesh.

Challenges

- **Model reduction**: Replace NSEs with high Reynolds number with turbulence models in 3D
- Small eddies resolution
- Time dependent case

Literature

- Maz'ya, V.; Rossmann, J. "Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains". *Arch. Ration. Mech. Anal.* 194 (2009), no. 2, 669–712.
- Auer, Naomi; Hintermüller, Michael; Knall, Karl. *Benchmark case for optimal shape design of air ducts in combustion engines*. ROMSOC D5.1, version 3.0, (2020).
- Othmer, C. "A continuous adjoint formulation for the computation of topological and surface sensitivities of ducted flows". In: *Internat. J. Numer. Methods Fluids* 58.8 (2008), pp. 861–877.
- Sokołowski, Jan; Jean-Paul Zolésio. *Introduction to shape optimization*. Vol. 16. Springer Series in Computational Mathematics. Shape sensitivity analysis (1992). Springer-Verlag, Berlin, pp. ii+250.