An Asymptotic Series for an Integral

Michael E. Hoffman, Markus Kuba, Moti Levy, and Guy Louchard

March 10, 2018

Abstract

We obtain an asymptotic series $\sum_{j=0}^{\infty} \frac{I_j}{n^j}$ for the integral $\int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx$ as $n \to \infty$, and compute I_j in terms of alternating (or "colored") multiple zeta value. We also show that I_j is a rational polynomial the ordinary zeta values, and give explicit formulas for $j \leq 12$. As a byproduct, we obtain precise results about the convergence of norms of random variables and their moments. We study $\|(U, 1-U)\|_n$ as n tends to infinity and we also discuss $\|(U_1, U_2, \ldots, U_r)\|_n$ for standard uniformly distributed random variables.

1 Introduction

Let

$$I(n) = \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx. \tag{1}$$

We shall obtain an asymptotic series

$$I(n) = I_0 + \frac{I_1}{n} + \frac{I_2}{n^2} + \frac{I_3}{n^3} + \cdots$$

This integral has been discussed in [9] (together with a different problem proposed by M.D. Ward). Therein, it as treated by a different approach using Euler sums and polylogarithms, leading to the first few terms I_0 up I_7 in terms of multiple zeta values.

Here, we give a complete expansion of $I(n)$. The coefficients I_k can be written in terms of alternating or "colored" multiple zeta values. The multiple zeta values are defined by

$$\zeta(i_1,\ldots,i_k)=\sum_{n_1>\cdots>n_k\geq 1}\frac{1}{n_1^{i_1}\cdots n_k^{i_k}}$$

for positive integers $i_1,\ldots,i_k$ with $i_1>1$. This notation can be extended to alternating or "colored" multiple zeta values by putting a bar over those exponents with an associated sign in the numerator, as in

$$\zeta(\bar{3},\bar{1},1)=\sum_{n_1>n_2>n_3\geq 1}\frac{(-1)^{n_1+n_2}}{n_1^3 n_2 n_3}.$$

Note that $\zeta(a_1,a_2,\ldots,a_k)$ converges unless a_1 is an unbarred 1. We have $\zeta(\bar{1})=-\log 2$ and

$$\zeta(\bar{n})=(2^{1-n}-1)\zeta(n)$$

for $n\geq 2$. Alternating multiple zeta values have been extensively studied, and some identities for them are established in [2]. Our formula for I_k, $k\geq 2$, can be stated as

$$I_k=\frac{(-1)^k}{2}\sum_{j=2}^{k}E_{2\lfloor\frac{j-1}{2}\rfloor+1}(0)\zeta(\bar{j},\underbrace{1,\ldots,1}_{k-j}),\tag{2}$$

where E_n is the nth Euler polynomial. But in fact the right-hand side of Eq. (2) can always be rewritten as a rational polynomial in the ordinary zeta values $\zeta(i)$, $i\geq 2$. This follows from an identity of Kölbig [8] that relates alternating multiple zeta values $\zeta(\bar{n},1,\ldots,1)$ and multiple zeta values $\zeta(n,1,\ldots,1)$.

After our main result, we interpret the integral $I(n)$ as the expected value of a certain random variable Z_n, defined in terms of the nth norm of the random vector $(U,1-U)$. Here, U denotes a standard uniformly distributed random variable. We complement our analysis of $I(n)=\mathbb{E}(Z_n)$ by studying the positive real moments $\mathbb{E}(Z_n^s)$ in terms of (alternating) multiple zeta values, as n tends to infinity. Moreover, we also discuss as a counterpart the nth norm of the random vector $(U_1,U_2,\ldots,U_r)$ for $r\geq 2$ and derive its moments in terms of multiple zeta values and related sums.

2 Main result: a complete expansion of I(n)

Because of the symmetry around $x = \frac{1}{2}$ in (1), one can write

$$I(n) = 2\int_0^{\frac{1}{2}} [x^n + (1-x)^n]^{\frac{1}{n}} dx = 2\int_0^{\frac{1}{2}} (1-x)\left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{1}{n}} dx.$$

Now let $u = \frac{x}{1-x}$, or $x = \frac{u}{1+u}$. Then $dx = \frac{du}{(1+u)^2}$, and we have

$$I(n) = 2\int_0^1 \left(1 - \frac{u}{1+u}\right)(1+u^n)^{\frac{1}{n}} \frac{du}{(1+u)^2} = 2\int_0^1 (1+u^n)^{\frac{1}{n}} \frac{du}{(1+u)^3}.$$

Writing $(1+u^n)^{\frac{1}{n}}$ as $\exp\left(\frac{1}{n}\log(1+u^n)\right)$ and expanding the exponential in series, we have

$$I(n) = 2\int_0^1 \left(1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{1}{n}\log(1+u^n)\right)^k\right) \frac{du}{(1+u)^3}.$$

Now we can write (see [5, p. 351])

$$(\log(1+x))^k = k! \sum_{m=1}^{\infty} \frac{x^m}{m!} s(m,k), \tag{3}$$

where the $s(m,k)$ are (signed) Stirling numbers of the first kind. Hence

$$\begin{aligned} I(n) &= 2\int_0^1 \frac{du}{(1+u)^3} + 2\sum_{k=1}^{\infty} \int_0^1 k! \sum_{m=1}^{\infty} \frac{u^{mn} s(m,k)}{m! n^k k!} \frac{du}{(1+u)^3} \\ &= \frac{3}{4} + 2\sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{m=1}^{\infty} \frac{s(m,k)}{m!} \int_0^1 \frac{u^{mn}}{(1+u)^3} du. \end{aligned}$$

If we let $\zeta_r(i_1, \ldots, i_k)$ denote the truncated multiple zeta value

$$\zeta_r(i_1, \ldots, i_k) = \sum_{r \geq n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

then we have the following relation, which is well-known although perhaps not in this notation (cf. [1]).

Lemma 1. *For positive integers $m \geq k$,*

$$s(m,k) = (-1)^{m-k}(m-1)!\zeta_{m-1}(\{1\}_{k-1}),$$

where $\{1\}_m$ means 1 repeated m times.

Proof. From the relation

$$x(x-1)\cdots(x-n+1) = \sum_{n=0}^{n} s(n,k)x^k$$

it follows that $s(n,k) = (-1)^{n-k}e_{n-k}(1,2,\ldots,n-1)$, were e_j is the jth elementary symmetric function. Divide by $(n-1)!$ to get

$$\frac{s(n,k)}{(n-1)!} = (-1)^{n-k}\frac{e_{n-k}(1,2,\ldots,n-1)}{(n-1)!} = (-1)^{n-k}e_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right),$$

and the conclusion follows since evidently $\zeta_{n-1}(\{1\}_{n-k}) = e_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)$. □

Thus

$$I(n) = \frac{3}{4} + 2\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{m=1}^{\infty}(-1)^{m-k}\frac{\zeta_{m-1}(\{1\}_{k-1})}{m}\int_0^1 \frac{u^{mn}}{(1+u)^3}du. \qquad (4)$$

If we write

$$\int_0^1 \frac{u^r}{(1+u)^3}du = \sum_{j=1}^{\infty}\frac{\beta_{j-1}}{r^j},$$

then the β_j can be computed explicitly as follows.

Lemma 2.

$$\beta_j = \frac{(-1)^j}{4}(E_{j+1}(-1) + E_{j+2}(-1)),$$

where the E_j are Euler polynomials.

Proof. Making the change of variable $u = e^{-t}$, we have

$$\int_0^1 \frac{u^r}{(1+u)^3}du = \int_0^{\infty}\frac{e^{-t}}{(1+e^{-t})^3}e^{-rt}dt.$$

By direct computation

$$\frac{e^{-t}}{(1+e^{-t})^3} = \frac{1}{4}\left[\frac{d^2}{dt^2}\left(\frac{2e^t}{1+e^{-t}}\right) - \frac{d}{dt}\left(\frac{2e^t}{1+e^{-t}}\right)\right].$$

The generating function of the Euler polynomials is defined by

$$\mathcal{E}(t,x) = \frac{2e^{tx}}{1+e^t} = \sum_{j\geq 0} E_j(x)\frac{t^j}{j!}. \tag{5}$$

Differentiating $\mathcal{E}(-t,-1)$ gives

$$\frac{d}{dt}\left(\frac{2e^t}{1+e^{-t}}\right) = -\sum_{n=0}^{\infty}(-1)^n E_{n+1}(-1)\frac{t^n}{n!}$$

and

$$\frac{d^2}{dt^2}\left(\frac{2e^t}{1+e^{-t}}\right) = \sum_{n=0}^{\infty}(-1)^n E_{n+2}(-1)\frac{t^n}{n!}.$$

Hence

$$\begin{aligned}\int_0^\infty \frac{e^{-t}}{(1+e^{-t})^3}e^{-rt}dt &= \sum_{n=0}^{\infty}\frac{(-1)^n}{4}(E_{n+2}(-1)+E_{n+1}(-1))\int_0^\infty \frac{t^n}{n!}e^{-rt}dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{4}(E_{n+2}(-1)+E_{n+1}(-1))\frac{1}{r^{n+1}},\end{aligned}$$

from which the conclusion follows. □

The well-known identity

$$E_n(x) + E_n(x+1) = 2x^n \tag{6}$$

gives $E_n(-1) = 2(-1)^n - E_n(0)$, so that

$$E_{j+1}(-1) + E_{j+2}(-1) = -E_{j+1}(0) - E_{j+2}(0).$$

But $E_n(0) = 0$ for n even, so we have

$$\beta_j = \begin{cases} \frac{1}{4}E_{j+2}(0), & \text{if } j \text{ is odd,} \\ -\frac{1}{4}E_{j+1}(0), & \text{if } j \text{ is even,} \end{cases}$$

or more succinctly $\beta_j = (-1)^{j+1}\frac{1}{4}E_{2\lfloor\frac{j+1}{2}\rfloor+1}(0)$. If we set $a_n = \frac{1}{2}E_{2n+1}(0)$, then $2(-1)^{j-1}\beta_j = a_{\lfloor\frac{j+1}{2}\rfloor}$. The a_n can be written in terms of Bernoulli numbers as

$$a_n = \frac{(1-2^{2n+2})B_{2n+2}}{2n+2},$$

and we also have the exponential generating function

$$\sum_{n=0}^{\infty} a_n \frac{t^{2n+1}}{(2n+1)!} = -\frac{1}{2}\tanh\frac{t}{2}.$$

The first few a_j are

$$a_0 = -\frac{1}{4},\ a_1 = \frac{1}{8},\ a_2 = -\frac{1}{4},\ a_3 = \frac{17}{16},\ a_4 = -\frac{31}{4},\ a_5 = \frac{691}{8},\ a_6 = -\frac{5461}{4}.$$

Using Eq. (4) we can write

$$\begin{aligned} I(n) &= \frac{3}{4} + 2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{m-k}\frac{\zeta_{m-1}(\{1\}_{k-1})\beta_{j-1}}{n^{j+k}m^{j+1}} \\ &= \frac{3}{4} + 2\sum_{p=2}^{\infty}\frac{1}{n^p}\sum_{m=1}^{\infty}\sum_{k=1}^{p-1}(-1)^{m-k}\frac{\zeta_{m-1}(\{1\}_{k-1})\beta_{p-k-1}}{m^{p-k+1}} \\ &= \frac{3}{4} + 2\sum_{p=2}^{\infty}\frac{1}{n^p}\sum_{k=1}^{p-1}(-1)^k\beta_{p-k-1}\zeta(\overline{p-k+1},\{1\}_{k-1}), \end{aligned}$$

from which we see that $I_0 = \frac{3}{4}$, $I_1 = 0$, and

$$I_p = 2\sum_{k=1}^{p-1}(-1)^k\beta_{p-k-1}\zeta(\overline{p-k+1},\{1\}_{k-1}) = 2\sum_{j=2}^{p}(-1)^{p-j-1}\beta_{j-2}\zeta(\bar{j},\{1\}_{p-j})$$

for $p \geq 2$. We have proved the following result.

Theorem 1. *For $p \geq 2$,*

$$I_p = (-1)^p\sum_{j=2}^{p} a_{\lfloor\frac{j-1}{2}\rfloor}\zeta(\bar{j},\{1\}_{p-j}),$$

where $a_n = \frac{1}{2}E_{2n+1}(0) = (1-2^{2n+2})B_{2n+2}/(2n+2)$.

The first two cases are as follows.

$$I_2 = a_0\zeta(\bar{2}) = \frac{1}{8}\zeta(2)$$
$$I_3 = -a_0\zeta(\bar{2},1) - a_1\zeta(\bar{3}) = \frac{1}{4}\cdot\frac{\zeta(3)}{8} + \frac{1}{8}\cdot\frac{3}{4}\zeta(3) = \frac{1}{8}\zeta(3).$$

In all further computations, expressions for alternating multiple zeta values are simplified using the Multiple Zeta Value Data Mine [3]. By Theorem 1,

$$I_4 = a_0\zeta(\bar{2},1,1) + a_1\zeta(\bar{3},1) + a_2\zeta(\bar{4}) = -\frac{1}{4}\zeta(\bar{2},1,1) + \frac{1}{8}\zeta(\bar{3},1) + \frac{1}{8}\zeta(\bar{4}),$$

and since $\zeta(\bar{4}) = -\frac{7}{8}\zeta(4)$, $\zeta(\bar{2},1,1) = -\frac{1}{16}\zeta(4) + \frac{1}{2}\zeta(\bar{3},1)$, this implies $I_4 = -\frac{3}{32}\zeta(4)$. Similarly,

$$I_5 = -a_0\zeta(\bar{2},1,1,1) - a_1\zeta(\bar{3},1,1) - a_1\zeta(\bar{4},1) - a_2\zeta(\bar{5}) = \frac{1}{4}\zeta(\bar{2},1,1,1) - \frac{1}{8}\zeta(\bar{3},1,1) - \frac{1}{8}\zeta(\bar{4},1) + \frac{1}{4}\zeta(\bar{5}).$$

Now $\zeta(\bar{5}) = -\frac{15}{16}\zeta(5)$, and from [3]

$$\zeta(\bar{4},1) = -\frac{29}{32}\zeta(5) + \frac{1}{2}\zeta(2)\zeta(3)$$
$$\zeta(\bar{2},\{1\}_3) = \frac{31}{64}\zeta(5) - \frac{1}{4}\zeta(2)\zeta(3) + \frac{1}{2}\zeta(\bar{3},1,1),$$

giving the result $I_5 = -\frac{1}{8}\zeta(2)\zeta(3)$.

Here, without further details, are I_j for $j = 6, 7, 8, 9, 10, 11, 12$.

$$
\begin{aligned}
I_6 &= \frac{83}{256}\zeta(6) - \frac{1}{16}\zeta(3)^2 \\
I_7 &= \frac{3}{16}\zeta(7) + \frac{27}{64}\zeta(3)\zeta(4) + \frac{3}{16}\zeta(2)\zeta(5) \\
I_8 &= -\frac{2533}{1536}\zeta(8) + \frac{3}{16}\zeta(3)\zeta(5) + \frac{5}{32}\zeta(2)\zeta(3)^2 \\
I_9 &= -\frac{5}{6}\zeta(9) - \frac{289}{128}\zeta(3)\zeta(6) - \frac{135}{64}\zeta(4)\zeta(5) - \frac{9}{8}\zeta(2)\zeta(7) + \frac{5}{96}\zeta(3)^3 \\
I_{10} &= \frac{293937}{20480}\zeta(10) - \frac{87}{32}\zeta(3)\zeta(7) - \frac{9}{16}\zeta(5)^2 - \frac{81}{64}\zeta(3)^2\zeta(4) - \frac{21}{16}\zeta(2)\zeta(3)\zeta(5) \\
I_{11} &= \frac{63}{8}\zeta(11) + \frac{58007}{3072}\zeta(3)\zeta(8) + \frac{5187}{256}\zeta(5)\zeta(6) + \frac{135}{8}\zeta(4)\zeta(7) + \frac{115}{12}\zeta(2)\zeta(9) \\
&\quad - \frac{13}{48}\zeta(2)\zeta(3)^3 - \frac{21}{32}\zeta(3)^2\zeta(5) \\
I_{12} &= -\frac{2095281645}{11321344}\zeta(12) + \frac{115}{12}\zeta(3)\zeta(9) + \frac{81}{8}\zeta(5)\zeta(7) + \frac{5765}{512}\zeta(3)^2\zeta(6) \\
&\quad + \frac{1323}{64}\zeta(3)\zeta(4)\zeta(5) + \frac{45}{4}\zeta(2)\zeta(3)\zeta(7) + \frac{45}{8}\zeta(2)\zeta(5)^2 - \frac{13}{192}\zeta(3)^4
\end{aligned}
$$

In fact, the I_n are always rational polynomials in the ordinary zeta values $\zeta(i)$, in consequence of the following result.

Theorem 2. *For $p \geq 2$,*

$$I_p = \frac{(-1)^p}{2}\sum_{k=1}^{p-1}(-1)^k\zeta(k+1,\{1\}_{p-k-1})\sum_{j=0}^{k-1}\binom{k-1}{j}a_{\lfloor\frac{p-1-j}{2}\rfloor}.$$

The proof makes use of an identity of Kölbig [8], which is phrased in terms of the integral

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{\log^{n-1}(t)\log^p(1-zt)}{t}dt.$$

But $S_{n,p}(z)$ can be written as a multiple zeta value if $z = 1$, and as an alternating multiple zeta value if $z = -1$. The key is the following result.

Lemma 3. *If $|z| \leq 1$, then*

$$\frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{\log^{n-1}(t)\log^p(1-zt)}{t}dt = \sum_{j_1>j_2>\cdots>j_p\geq 1}\frac{z^{j_1}}{j_1^{n+1}j_2\cdots j_p}.$$

Proof. Since

$$\log(1-zt) = -\sum_{i\geq 1} \frac{z^i t^i}{i} \quad \text{and} \quad \int_0^1 t^{m-1}\log^{n-1}(t)dt = \frac{(n-1)!}{m^n},$$

we have

$$\int_0^1 \frac{\log^{n-1}(t)\log^p(1-zt)}{t}dt$$
$$= (-1)^p \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_p=1}^{\infty}\int_0^1 \frac{z^{i_1+\cdots+i_p}t^{i_1+\cdots+i_p-1}\log^{n-1}(t)}{i_1 i_2\cdots i_p}dt$$
$$= (-1)^p \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_p=1}^{\infty}\frac{(-1)^{n-1}(n-1)!z^{i_1+\cdots+i_p}}{i_1 i_2\cdots i_p(i_1+\cdots+i_p)^n}.$$

By [6, Lemma 4.3], this is

$$(-1)^p \sum_{j_1>j_2>\cdots>j_p\geq 1}\frac{(-1)^{n-1}(n-1)!p!z^{j_1}}{j_1^{n+1}j_2\cdots j_p}$$

and the conclusion follows. □

It then follows from definitions that

$$S_{n,p}(1) = \zeta(n+1,\{1\}_{p-1}) \quad \text{and} \quad S_{n,p}(-1) = \zeta(\overline{n+1},\{1\}_{p-1}).$$

In [8] Kölbig refers to $S_{n,p}(1)$ as $s_{n,p}$ and $S_{n,p}(-1)$ as $(-1)^p\sigma_{n,p}$; the result we need is [8, Theorem 3], which reads

$$\sum_{j=1}^{n}\binom{n+p-j-1}{p-1}\sigma_{j,n+p-j} + \sum_{j=1}^{p}\binom{n+p-j-1}{n-1}\sigma_{j,n+p-j} = s_{n,p}. \qquad (7)$$

Proof of Theorem 2. Note that we can rewrite Theorem 1 as

$$I_p = \sum_{i=1}^{p-1}(-1)^i a_{\lfloor \frac{i}{2}\rfloor}\sigma_{i,p-i}$$

and Eq. (7) as

$$\sum_{i=1}^{p-1}\left(\binom{p-i-1}{p-j-1}+\binom{p-i-1}{j-1}\right)\sigma_{i,p-i} = s_{j,p-j}.$$

If we can find ρ_j so that

$$\sum_{j=1}^{p-1} \rho_j s_{j,p-j} = \sum_{j=1}^{p-1}\sum_{i=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i}$$
$$= \sum_{i=1}^{p-1} \sigma_{i,p-i} \sum_{j=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = I_p,$$

i.e.,

$$\sum_{j=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \qquad (8)$$

for $i = 1, 2, \ldots, p-1$, then I_p can be written in terms of the $s_{m,n}$. Now Eqs. (8) can be written

$$\sum_{j=1}^{p-i} \rho_{p-j} \binom{p-i-1}{j-1} + \sum_{j=1}^{p-i} \rho_j \binom{p-i-1}{j-1} = (-1)^i a_{\lfloor \frac{i}{2} \rfloor},\ 1 \le i \le p-1,$$

and if we make the condition $\rho_{p-j} = \rho_j$, this becomes

$$\sum_{j=1}^{p-i} \rho_j \binom{p-i-1}{j-1} = \frac{(-1)^i}{2} a_{\lfloor \frac{i}{2} \rfloor},\ 1 \le i \le p-1, \qquad (9)$$

Restrict the system (9) to the last $\lfloor \frac{p}{2} \rfloor$ equations ($i = \lfloor \frac{p+1}{2} \rfloor, \ldots, p-1$) and use binomial inversion to get

$$\rho_k = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{k-1}{j},\ 1 \le k \le \lfloor \tfrac{p}{2} \rfloor. \qquad (10)$$

We claim that ρ_k so defined, if the definition is extended to $1 \le k \le p-1$, is also a solution of the first $\lfloor \frac{p-1}{2} \rfloor$ equations of (9). The conclusion then follows.

To prove the claim, it is enough to show that the extension of Eqn. (10) to $1 \le k \le p-1$ is consistent with the condition $\rho_{p-k} = \rho_k$, i.e., that

$$\frac{(-1)^k}{2} \sum_{j=0}^{p-k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{p-k-1}{j} = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{k-1}{j},$$

or, using the definition of a_n,

$$\sum_{j=0}^{p-k-1} E_{2\lfloor \frac{p-j-1}{2} \rfloor+1}(0)\binom{p-k-1}{j} = (-1)^p \sum_{j=0}^{k-1} E_{2\lfloor \frac{p-j-1}{2} \rfloor+1}(0)\binom{k-1}{j}.$$

By considering the cases p odd and p even, we see this can be written

$$\sum_{j=0}^{p-k} E_{p-j}(0)\binom{p-k}{j} = (-1)^p \sum_{j=0}^{k} E_{p-j}(0)\binom{k}{j}.$$

The result then follows from taking $n = p - k$ in Lemma 4 below.

Lemma 4. *For nonnegative integers n, k,*

$$\sum_{j=0}^{n} E_{k+j}(0)\binom{n}{j} = (-1)^{n+k} \sum_{j=0}^{k} E_{n+j}(0)\binom{k}{j}$$

Proof. Start with

$$\sum_{j=0}^{n} E_j(0)\binom{n}{j} = -E_n(0)$$

which follows from setting $x = 0$ in the identity (6). Since $E_n(0) = 0$ for n even, we can write this as

$$\sum_{j=0}^{n} E_j(0)\binom{n}{j} = (-1)^n E_n(0),$$

which is the case $k = 0$ of the conclusion. We can then use it as the base case of a proof of the conclusion by induction on k. We have

$$\begin{aligned}
&(-1)^{n+k+1}\sum_{j=0}^{k+1}E_{n+j}(0)\binom{k+1}{j} = \\
&\qquad (-1)^{n+k+1}\left[\sum_{j=1}^{k+1}E_{n+j}(0)\binom{k}{j-1}+\sum_{j=0}^{k}E_{n+j}(0)\binom{k}{j}\right] = \\
&\qquad (-1)^{n+k+1}\left[\sum_{j=0}^{k}E_{n+1+j}(0)\binom{k}{j}+\sum_{j=0}^{k}E_{n+j}(0)\binom{k}{j}\right] = \\
&\sum_{j=0}^{n+1}E_{k+j}(0)\binom{n+1}{j}-\sum_{j=0}^{n}E_{k+j}(0)\binom{n}{j} = \sum_{j=1}^{n+1}E_{k+j}(0)\binom{n}{j-1} \\
&\qquad = \sum_{j=0}^{n}E_{k+1+j}(0)\binom{n}{j}.
\end{aligned}$$

□

Corollary 1. *For $p \geq 2$, I_p is a rational polynomial in the the $\zeta(i)$.*

Proof. For any positive integers n, m the multiple zeta value $\zeta(n+1, \{1\}_m)$ is a rational polynomial in the $\zeta(i)$, as follows from [2, Eq. (10)]. Then Theorem 2 implies the conclusion. □

3 Applications: convergence of norms

Let $U = \text{Uniform}[0,1]$ denote a standard uniformly distributed random variable. Furthermore, for positive real n we define random variables Z_n by

$$Z_n = \|(U, 1-U)\|_n = \left(U^n + (1-U)^n\right)^{\frac{1}{n}}.$$

From the theory of norms we expect that the limit Z_∞ exists and

$$Z_\infty = \|(U, 1-U)\|_\infty = \max\{U, 1-U\}.$$

It is known that $\max\{U, 1-U\} = \text{Uniform}[\frac{1}{2}, 1]$. It turns out that our previous considerations allow to refine this intuition. The integral $I(n)$ treated in detail before is exactly the expected value of Z_n. In the following we give asymptotic expansion of all positive real moments of Z_n.

Theorem 3. *The random variable Z_n, defined in terms of $U = \text{Uniform}[0,1]$, converges for $n \to \infty$ in distribution and with convergence of all integer moments,*

$$Z_n = \big(U^n + (1-U)^n\big)^{\frac{1}{n}} \to Z_\infty = \max\{U, 1-U\},$$

For positive integer $s \geq 1$ we have

$$\mathbb{E}(Z_n^s) = \frac{2(1-\frac{1}{2^{s+1}})}{s+1} + \\ \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} (-1)^{j-1} E_{p-k+j-1}(0) \zeta(\overline{p+1-k}, \{1\}_{k-1}),$$

where the values $\gamma_{s+1,j}$ are given by $\frac{(-1)^{j-1}\left[{s+1 \atop j}\right]}{s!} = (-1)^{j-1}\zeta_s(\{1\}_{j-1})$.

For arbitrary positive real $s > 0$ we have

$$\mathbb{E}(Z_n^s) = \frac{2(1-\frac{1}{2^{s+1}})}{s+1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \zeta(\overline{p+1-k}, \{1\}_{k-1}) \\ \times \sum_{\ell=1}^{p-k} (s+1)^{\underline{\ell}} B_{p-k,\ell}(E_1(0), \dots, E_{p-k-\ell+1}(0)),$$

where $B_{n,k}(x_1, \dots, x_{n+1-k})$ denote the Bell polynomials.

A first by product of our moment expansions is a rate of convergence.

Corollary 2. *The distribution functions $F_n(x) = \mathbb{P}\{Z_n \leq x\}$ and $F_\infty(x) = \mathbb{P}\{Z_\infty \leq x\}$ satisfy*

$$\sup_{x \in \mathbf{R}} |F_n(x) - F_\infty(x)| \leq \frac{C}{n}.$$

We also can directly strengthen to almost-sure convergence.

Corollary 3. *The random variable $Z_n = \big(U^n + (1-U)^n\big)^{\frac{1}{n}}$ converges almost surely to $Z_\infty = \max\{U, 1-U\}$.*

Remark 1. We obtain in a similar way moment convergence of random variables

$$Z_n = \big(B^n + (1-B)^n\big)^{\frac{1}{n}},$$

with B denoting a $\mathit{Beta}(\alpha, \beta)$ distributed random variable with real $\alpha, \beta > 0$, generalizing our results above (case $\alpha = \beta = 1$).

We note that

$$\mathbb{E}(Z_n^s) = \int_\Omega \left(\left(U^n + (1-U)^n\right)^{\frac{1}{n}}\right)^s d\mathbb{P} = \int_0^1 \left(x^n + (1-x)^n\right)^{\frac{s}{n}} dx.$$

Proceeding as before we use the symmetry of the integrand.

$$\mathbb{E}(Z_n^s) = 2\int_0^{\frac{1}{2}} (1-x)^s \left[1 + \left(\frac{x}{1-x}\right)^n\right]^{\frac{s}{n}} dx.$$

Substituting again $u = \frac{x}{1-x}$, or $x = \frac{u}{1+u}$, leads to

$$\mathbb{E}(Z_n^s) = 2\int_0^1 \left(1 - \frac{u}{1+u}\right)^s (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^2} = 2\int_0^1 (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^3}.$$

Writing $(1+u^n)^{\frac{s}{n}}$ as $\exp\left(\frac{s}{n}\log(1+u^n)\right)$ and expanding the exponential in series, we have

$$\mathbb{E}(Z_n^s) = 2\int_0^1 \left(1 + \sum_{k=1}^\infty \frac{1}{k!}\left(\frac{s}{n}\log(1+u^n)\right)^k\right) \frac{du}{(1+u)^{s+2}}.$$

As before,

$$\begin{aligned}\mathbb{E}(Z_n^s) &= 2\int_0^1 \frac{du}{(1+u)^{s+2}} + 2\sum_{k=1}^\infty \int_0^1 k! \sum_{m=1}^\infty \frac{u^{mn} s(m,k) s^k}{m! n^k k!} \frac{du}{(1+u)^{s+2}} \\ &= \frac{2(1-\frac{1}{2^{s+1}})}{s+1} + \sum_{k=1}^\infty \frac{s^k}{n^k} \sum_{m=1}^\infty (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \int_0^1 \frac{2u^{mn}}{(1+u)^{s+2}} du.\end{aligned}$$

It remains to expand the integral into powers of n. Make the substitution $u = e^{-t}$ and then integrate by parts:

$$\begin{aligned}\int_0^1 \frac{2u^{mn}}{(1+u)^{s+2}} du = \int_0^\infty \frac{2e^{-t}}{(1+e^{-t})^{s+2}} e^{-nmt} dt = \\ -\frac{1}{2^s(s+1)} + \frac{nm}{s+1}\int_0^\infty \frac{2}{(1+e^{-t})^{s+1}} e^{-mnt} dt.\end{aligned}$$

We adapt the previous result for $s = 1$ using derivative polynomials. Changing the sign of the variable t in (5) and evaluation at $x = 0$ gives

$$\mathcal{E}(-t, 0) = \frac{2}{1+e^{-t}} = \sum_{j\geq 0} (-1)^j E_j(0) \frac{t^j}{j!}.$$

Thus, for our base function we choose the logistic function

$$f(t) = \frac{1}{2}\mathcal{E}(-t, 0) = \frac{1}{1 + e^{-t}}.$$

Lemma 5 (Derivative polynomials - logistic function)**.** *For positive integer r the derivative $f_r(z) := \frac{d^{r-1}}{dt^{r-1}} f(t)$ can be written as a polynomial in f:*

$$f_r(z) = \sum_{j=1}^{r} c_{r,j} \cdot f(t)^j = \sum_{j=1}^{r} \frac{c_{r,j}}{(1 + e^{-t})^j}.$$

The numbers $c_{r,j}$ are explicitly given by

$$(-1)^{j-1}(j-1)! \left\{ {r \atop j} \right\},$$

where $\left\{ {n \atop k} \right\}$ is the number of ways to partition $\{1, 2, \ldots, n\}$ into k nonempty subsets (Stirling number of the second kind). In particular, $c_{r,1} = 1$ and $c_{r,r} = (r-1)!(-1)^{r-1}$.

Proof. In [7] a general theory of derivative polynomials is developed: if f is a function such that $f'(t) = P(f(t))$ for a polynomial function P, then evidently $f^{(n)}(t) = P_n(f(t))$ for polynomials P_n, and if we let

$$F(x, t) = \sum_{n \geq 0} \frac{t^n}{n!} P_n(x)$$

then [7, Theorem 1] gives

$$F(x, t) = f(f^{-1}(x) + t). \tag{11}$$

In the case $f(t) = (1 + e^{-t})^{-1}$, Eq. (11) gives

$$\sum_{n \geq 0} \frac{t^n}{n!} P_n(x) = \frac{x}{x + (1 - x)e^{-t}} = \frac{xe^t}{1 + x(e^t - 1)} = xe^t \sum_{m=0}^{\infty} (-1)^m x^m (e^t - 1)^m.$$

Using the identity

$$(e^t - 1)^m = m! \sum_{p \geq m} \left\{ {p \atop m} \right\} \frac{t^p}{p!},$$

this becomes

$$\sum_{n\geq 0}\frac{t^n}{n!}P_n(x) = xe^t\sum_{m=0}^{\infty}(-1)^m x^m m!\sum_{p\geq m}\left\{{p \atop m}\right\}\frac{t^p}{p!} =$$
$$\sum_{q=0}^{\infty}\frac{t^q}{q!}\sum_{p=0}^{\infty}\frac{t^p}{p!}\sum_{m=0}^{p}(-1)^m x^{m+1} m!\left\{{p \atop m}\right\}.$$

Extract the coefficient of $t^n/n!$ on both sides to get

$$P_n(x) = \sum_{p=0}^{n}\binom{n}{p}(-1)^m x^{m+1} m!\left\{{p \atop m}\right\} = \sum_{m=0}^{n}(-1)^m x^{m+1} m!\sum_{p=m}^{n}\binom{n}{p}\left\{{p \atop m}\right\}$$
$$= \sum_{m=0}^{n}(-1)^m x^{m+1} m!\left\{{n+1 \atop m+1}\right\},$$

where we used the identity [5, Eq. (6.15)] in the last step. The conclusion then follows. □

Henceforth $c_{r,j}$ denotes the coefficients of the derivative polynomials discussed above.

Lemma 6. *Define $\gamma_{s+1,r}$ as the solutions of the triangular linear system of equations*

$$\begin{pmatrix} c_{1,1} & c_{2,1} & \dots & c_{s+1,1} \\ 0 & c_{2,2} & \dots & c_{s+1,2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & c_{s+1,s+1} \end{pmatrix} \cdot \begin{pmatrix} \gamma_{s+1,1} \\ \gamma_{s+1,2} \\ \vdots \\ \gamma_{s+1,s+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Then, $\gamma_{s+1,r}$ is given by

$$\gamma_{s+1,r} = \frac{(-1)^{r-1}\left[{s+1 \atop r}\right]}{s!} = (-1)^{r-1}\zeta_s(\{1\}_{r-1}),$$

where $\left[{s+1 \atop r}\right]$ denote the signless Stirling numbers of the first kind. Furthermore,

$$\frac{2}{(1+e^{-t})^{s+1}} = \sum_{k\geq 0}\frac{t^k}{k!}\sum_{j=1}^{s+1}(-1)^{k+j-1}\gamma_{s+1,j}E_{k+j-1}(0).$$

Proof. The system of linear equations can be expressed as

$$\sum_{r=j}^{s+1} c_{r,j}\gamma_{s+1,r} = \delta_{j,s+1}, \quad 1 \le j \le s+1,$$

where $\delta_{j,s+1}$ denotes the Kronecker delta. More explicitly,

$$(-1)^{j-1}(j-1)! \sum_{r=j}^{s+1} \left\{ {r \atop j} \right\} \gamma_{s+1,r} = \delta_{j,s+1}.$$

By the inversion relationships between Stirling numbers we directly observe that

$$\gamma_{s+1,r} = \frac{(-1)^{r-1}\left[{s+1 \atop r} \right]}{s!}.$$

By Lemma 1 we obtain the second expression. □

Remark 2. The generalized Euler polynomials $E_n^{(r)}(x)$, $r \in \mathbb{N}$, are defined by the generating function

$$\mathcal{E}_r(t,x) = \left(\frac{2}{1+e^t}\right)^r e^{xt} = \sum_{k\ge 0} E_k^{(r)}(x)\frac{t^k}{k!},$$

see [12]. The result above implies the formula

$$E_k^{(r)}(0) = 2^{r-1}\sum_{j=1}^{r}(-1)^{j-1}\gamma_{r,j}E_{k+j-1}(0),$$

also leading to a new formula for $E_k^{(r)}(x)$. Cf.

$$E_k^{(r)}(0) = \frac{2^{r-1}}{(r-1)!}\sum_{j=0}^{r} s(r,j)(-1)^{r+j}E_{k+j-1}(0)$$

which follows from [10] and gives an alternative derivation of the $\gamma_{r,j}$.

Proof. By our previous result

$$\frac{1}{(1+e^{-t})^{s+1}} = \sum_{j=1}^{s+1}\gamma_{s+1,j}\frac{d^{j-1}}{dt^{j-1}}f(t),$$

where $f(t) = \frac{1}{2}\mathcal{E}(-t,0) = \frac{1}{1+e^{-t}}$. Then

$$\begin{aligned}\frac{2}{(1+e^{-t})^{s+1}} &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} 2f(t) \\ &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} \sum_{k\geq 0} (-1)^k E_k(0) \frac{t^k}{k!} \\ &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k\geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{t^k}{k!}.\end{aligned}$$

□

Lemma 6 implies that

$$\int_0^\infty \frac{2}{(1+e^{-t})^{s+1}} e^{-nmt} dt = \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k\geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{1}{m^{k+1} n^{k+1}}.$$

Furthermore

$$\begin{aligned}\mathbb{E}(Z_n^s) = \frac{2(1-\frac{1}{2^{s+1}})}{s+1} &+ \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \\ &\times \left(-\frac{1}{2^s(s+1)} + \frac{1}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{\ell\geq 0} (-1)^{\ell+j-1} E_{\ell+j-1}(0) \frac{1}{m^\ell n^\ell} \right).\end{aligned}$$

Setting $t = 0$ in Lemma 6 we get

$$\frac{1}{2^s} = \sum_{j=1}^{s+1} (-1)^{j-1} \gamma_{s+1,j} E_{j-1}(0).$$

Consequently, the first summand cancels and we get

$$\mathbb{E}(Z_n^s) = \frac{2(1-\frac{1}{2^{s+1}})}{s+1} +$$
$$\sum_{k=1}^{\infty}\frac{s^k}{n^k}\sum_{m=1}^{\infty}(-1)^{m-k}\frac{\zeta_{m-1}(\{1\}_{k-1})}{m(s+1)}\sum_{j=1}^{s+1}\gamma_{s+1,j}\sum_{\ell\geq 1}(-1)^{\ell+j-1}E_{\ell+j-1}(0)\frac{1}{m^\ell n^\ell}$$
$$= \frac{2(1-\frac{1}{2^{s+1}})}{s+1} +$$
$$\sum_{p=2}^{\infty}\frac{(-1)^p}{n^p}\sum_{k=1}^{p-1}\frac{s^k}{s+1}\sum_{j=1}^{s+1}\gamma_{s+1,j}(-1)^{j-1}E_{p-k+j-1}(0)\zeta(\overline{p-k+1},\{1\}_{k-1})$$

by changing the order of summation.

Concerning arbitrary positive real $s > 0$ we have to proceed in a slightly different way. Let $B_{n,k}(x_1,\ldots,x_{n-k+1})$ denote the kth Bell polynomial defined by

$$B_{n,k}(x_1,\ldots,x_{n-k+1}) = \sum_{\substack{\sum_{\ell=1}^{n-k+1} j_\ell = k \\ \sum_{\ell=1}^{n-k+1}\ell j_\ell = n}} \frac{n!}{j_1!\cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}. \tag{12}$$

We have

$$\frac{2}{(1+e^{-t})^{s+1}} = \big(\mathcal{E}(-t,0)\big)^{s+1} = \big(1+(\mathcal{E}(-t,0)-1)\big)^{s+1} =$$
$$\sum_{j\geq 0}\frac{(s+1)^{\underline{j}}}{j!}(\mathcal{E}(-t,0)-1)^j = \sum_{j\geq 0}\frac{\sum_{\ell=1}^{j}(s+1)^{\underline{\ell}}B_{j,\ell}(E_1(0),\ldots,E_{j-\ell+1}(0))}{j!}(-1)^j t^j.$$

Consequently,

$$\int_0^\infty \frac{2}{(1+e^{-t})^{s+1}}e^{-mnt}dt = \sum_{j\geq 0}(-1)^j\frac{\sum_{\ell=1}^{j}(s+1)^{\underline{\ell}}B_{j,\ell}(E_1(0),\ldots,E_{j-\ell+1}(0))}{(mn)^{j+1}}.$$

Finally,

$$\begin{aligned}
\mathbb{E}(Z_n^s) &= \frac{2(1-\frac{1}{2^{s+1}})}{s+1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \\
&\quad \times \frac{1}{s+1} \sum_{j\geq 1} (-1)^j \frac{\sum_{\ell=1}^{j} (s+1)^{\underline{\ell}} B_{j,k}(E_1(0),\ldots,E_{j-\ell+1}(0))}{(mn)^j} \\
&= \frac{2(1-\frac{1}{2^{s+1}})}{s+1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \zeta(\overline{p+1-k}, \{1\}_{k-1}) \\
&\quad \times \sum_{\ell=1}^{p-k} (s+1)^{\underline{\ell}} B_{p-k,\ell}(E_1(0),\ldots,E_{p-k-\ell+1}(0)).
\end{aligned}$$

Proof of Corollary 2. We use the general version of the Berry-Esseen inequality [4]:

$$\sup_{x\in\mathbf{R}} |F(x)-G(x)| \leq c_1 \int_{-T}^{T} \left| \frac{\phi_F(t)-\phi_G(t)}{t} \right| dt + c_2 \sup_{x\in\mathbf{R}} \big(G(x+\frac{1}{T}) - G(x)\big).$$

From our moment expansion

$$\mathbb{E}(Z_n^s) = \mathbb{E}(Z_\infty^s) + \mathcal{O}\Big(\frac{s\zeta(2)}{2^{s+2}n^2}\Big),$$

we obtain for the characteristic functions $\phi_n(t) = \mathbb{E}(e^{itZ_n})$ and $\phi_\infty(t) = \mathbb{E}(e^{itZ_\infty})$

$$\frac{|\phi_n(t)-\phi_\infty(t)|}{|t|} \leq \frac{C_1}{n^2}.$$

Choosing $T=n$ this gives a $\frac{1}{n}$ bound for the integral. We get $\sup_{x\in\mathbf{R}} \big(G(x+\frac{1}{T}) - G(x)\big) \leq \frac{C_2}{n}$ leading to the stated result. ◻

Proof of Corollary 3. By the Markov inequality we have

$$\mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \leq \ell^2 \mathbb{E}((Z_n-Z_\infty)^2) = \ell^2\big(\mathbb{E}(Z_n^2) + \mathbb{E}(Z_\infty^2) - 2\mathbb{E}(Z_n Z_\infty)\big).$$

The random variables Z_n and Z_∞ are defined in terms of the same uniform distribution and we readily obtain the expansion of

$$\mathbb{E}(Z_n Z_\infty) = \int_0^1 (x^n + (1-x)^n)^{\frac{1}{n}} \cdot \max\{x, 1-x\} dx = \frac{2}{3}\cdot\frac{7}{8} + \mathcal{O}(\frac{1}{n^2})$$

leading to $\mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \le C \cdot \frac{\ell^2}{n^2}$. Let

$$E_{n,\ell} = \Big\{\omega \in \Omega : \ \big|Z_n - Z_\infty\big| > \frac{1}{\ell}\Big\}, \quad n \in \mathbb{N}, \quad \ell > 0.$$

We have

$$\sum_{n\ge 1} \mathbb{P}\{E_{n,\ell}\} \le \sum_{n\ge 1} \frac{C\ell^2}{n^2} < \infty.$$

Let $E_\ell = \limsup E_{n,\ell}$. By the Borel-Cantelli Lemma we have $\mathbb{P}(E_\ell) = 0$ for any $\ell > 0$, giving the stated result. □

3.1 Independent uniformly distributed random variables

Let U_j denote mutually independent standard uniformly distributed random variables, $1 \le j \le r$ with $r \ge 2$, . Further, let $\mathbf{U}$ denote the random vector

$$\mathbf{U} = (U_1, \dots, U_r).$$

Let Z_n be defined as

$$Z_n = \|\mathbf{U}\|_n = (U_1^n + U_2^n + \dots + U_r^n)^{\frac{1}{n}}$$

A folklore result states that any order statistic for uniform distributions is Beta-distributed. In particular,

$$Z_\infty = \|\mathbf{U}\|_\infty = B(r, 1).$$

We are interested in the asymptotics of Z_n as $n \to \infty$ and derive asymptotics of the moments

$$I_s = \mathbb{E}(Z_n^s) = \int_{[0,1]^r} (x_1^n + \dots + x_r^n)^{\frac{s}{n}} d(x_1, \dots, x_r).$$

The special case $r = 2$, $Z_n = (U_1^n + U_2^n)^{\frac{1}{n}}$ is the direct counterpart of our earlier results for $(U^n + (1-U)^n)^{\frac{1}{n}}$. Our asymptotic series involves for $r \ge 2$ multiple zeta values. Interestingly, for $r \ge 3$ variants of multiple zeta values and Euler sums appear. Let $\zeta_r^\star(i_1, \dots, i_k)$ denote the truncated multiple zeta star value

$$\zeta_r^\star(i_1, \dots, i_k) = \sum_{r \ge n_1 \ge n_2 \ge \dots \ge n_k \ge 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{13}$$

and $\zeta_r^\star(i_1,\dots,i_k;x_1,\dots,x_k)$ denote the truncated weighted multiple zeta star value

$$\zeta_r^\star(i_1,\dots,i_k;x_1,\dots,x_k)=\sum_{r\ge n_1\ge n_2\ge\dots\ge n_k\ge 1}\frac{x_1^{n_1}\dots x_k^{n_k}}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}}. \tag{14}$$

Then $\zeta_r^\star(i_1,\dots,i_k;\{1\}_k)$ is the ordinary zeta value $\zeta_r^\star(i_1,\dots,i_k)$, and

$$\zeta_r^\star(\{1\}_k;\{1\}_{k-1},2)=\sum_{r\ge n_1\ge n_2\ge\dots\ge n_k\ge 1}\frac{2^{n_k}}{n_1n_2\cdots n_k}=\sum_{n_1=1}^{r}\frac{1}{n_1}\sum_{n_2=1}^{n_1}\frac{1}{n_2}\cdots\sum_{n_k=1}^{n_{k-1}}\frac{2^{n_k}}{n_k}.$$

Theorem 4. *The random variable $Z_n=\|\mathbf{U}\|_n$ converges to $Z_\infty=B(r,1)$ with convergence of all positive integer moments.*

$$\mathbb{E}(Z_n^s)=\frac{r}{r-1+s}\Big(1-\frac{s(r-1)}{n^2}\zeta(\bar{2})+\mathcal{O}(\frac{1}{n^3})\Big).$$

In particular, for $r=2$ and $Z_n=(U_1^n+U_2^n)^{\frac{1}{n}}$ we have the exact representation

$$\mathbb{E}(Z_n^s)=\frac{2}{1+s}\Big(1+\sum_{p\ge2}\frac{(-1)^p}{n^p}\sum_{\ell=0}^{p-2}s^{p-\ell-1}(-\zeta(\overline{\ell+2},\{1\}_{p-\ell-2}))\Big).$$

For $r=3$ we have the exact representation

$$\mathbb{E}(Z_n^s)=\frac{3}{2+s}\Big(1+\sum_{p\ge2}\frac{2(-1)^p}{n^p}\sum_{\ell=0}^{p-2}s^{p-\ell-1}(-\zeta(\overline{\ell+2},\{1\}_{p-\ell-2}))\Big)+\frac{3}{2+s}\sum_{k=1}^{\infty}\frac{s^k}{n^k}\sum_{\ell_1,\ell_2\ge0}\frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}}$$
$$\times\Bigg[\sum_{i=1}^{\ell_1+1}\binom{i+\ell_2-1}{\ell_2}\sum_{m=1}^{\infty}\frac{(-1)^{m-k}\zeta_{m-1}(\{1\}_{k-1})\Big(\zeta_m^\star(\{1\}_{\ell_1+2-i};\{1\}_{\ell_1+1-i},2)-\zeta_m^\star(\{1\}_{\ell_1+2-i})-\frac{1}{m^{\ell_1+2-i}}\Big)}{m^{1+i+\ell_2}}$$
$$+\sum_{i=1}^{\ell_2+1}\binom{i+\ell_1-1}{\ell_1}\sum_{m=1}^{\infty}\frac{(-1)^{m-k}\zeta_{m-1}(\{1\}_{k-1})\Big(\zeta_m^\star(\{1\}_{\ell_2+2-i};\{1\}_{\ell_2+1-i},2)-\zeta_m^\star(\{1\}_{\ell_2+2-i})-\frac{1}{m^{\ell_2+2-i}}\Big)}{m^{1+i+\ell_1}}\Bigg].$$

3.2 Exact representations

First, we decompose the hypercube into r parts according to the maximum of the x_i:

$$[0,1]^r=\bigcup_{i=1}^{r}\{x_i\in[0,1],\quad 0\le x_j\le x_i,\ j\in\{1,\dots,r\}\setminus\{i\}\}.$$

These parts are not disjoint, but their intersection is of measure zero. By the symmetry of I_s we get

$$I_s = r\int_0^1 \left(\int_{[0,x_r]^{r-1}} (x_1^n + x_2^n + \cdots + x_r^n)^{\frac{s}{n}} d(x_1,\ldots,x_{r-1})\right) dx_r.$$

We use the substitution $x_j = x_r u_j$, $dx_j = x_r du_j$ to obtain

$$I_s = r\int_0^1 x_r^{r-1}\left(\int_{[0,1]^{r-1}} (x_r^n u_1^n + x_r^n u_2^n + \cdots + x_{r-1}^n u_{r-1}^n + 1)^{\frac{s}{n}} d\mathbf{u}\right) dx_r.$$

This implies that the integrals can be separated:

$$\begin{aligned} I_s &= r\int_0^1 x_r^{r-1+s} dx_r \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n)^{\frac{s}{n}} d\mathbf{u} \\ &= \frac{r}{r-1+s} \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n)^{\frac{s}{n}} d\mathbf{u}. \end{aligned}$$

In order to derive an asymptotic expansion of the remaining integral we use the $\exp - \log$ representation:

$$(1+u_1^n+\cdots+u_{r-1}^n)^{\frac{s}{n}} = \exp\left(\frac{s}{n}\ln(1+u_1^n+\cdots+u_{r-1}^n)\right) = 1+\sum_{k=1}^{\infty}\frac{s^k}{n^k k!}\ln^k(1+u_1^n+\cdots+u_{r-1}^n).$$

Using Eq. (3), this implies

$$I_s = \frac{r}{r-1+s}\cdot\left(1+\sum_{k=1}^{\infty}\int_{[0,1]^{r-1}}\sum_{m=1}^{\infty}\frac{s^k s(m,k)}{n^k m!}(u_1^n+\cdots+u_{r-1}^n)^m d\mathbf{u}\right),$$

where (as above) $s(m,k)$ denotes the signed Stirling numbers of the first kind. Then using Lemma 1, we have

$$I_s = \frac{r}{r-1+s}\cdot\left(1+\sum_{k=1}^{\infty}\frac{s^k(-1)^k}{n^k}\sum_{m=1}^{\infty}\frac{(-1)^m\zeta_{m-1}(\{1\}_{k-1})}{m}\int_{[0,1]^{r-1}}(u_1^n+\cdots+u_{r-1}^n)^m d\mathbf{u}\right).$$

In order to evaluate the remaining integral we substitute $u_j = e^{-t_j}$ and obtain

$$\int_{[0,1]^{r-1}}(u_1^n+\cdots+u_{r-1}^n)^m d\mathbf{u} = \int_{[0,\infty)^{r-1}} e^{-t_1-\cdots-t_{r-1}}(e^{-t_1 n}+\cdots+e^{-t_{r-1}n})^m d\mathbf{t}.$$

We expand the exponentials and use the multinomial theorem. By the symmetry of the integrand and the fact

$$\int_0^\infty u^p e^{-ku} du = \frac{p!}{k^{p+1}}$$

we obtain

$$\int_{[0,\infty)^{r-1}} e^{-t_1-\cdots-t_{r-1}}(e^{-t_1 n}+\cdots+e^{-t_{r-1}n})^m d\mathbf{t}$$
$$=\sum_{a=1}^{r-1}\binom{r-1}{a}\sum_{\substack{j_1+\cdots+j_a=m\\ j_i\geq 1}}\binom{m}{j_1,\ldots,j_a}\sum_{\ell_1,\ldots,\ell_a\geq 0}\frac{(-1)^{\ell_1+\cdots+\ell_a}}{n^{\ell_1+\cdots+\ell_a+a}j_1^{\ell_1+1}\cdots j_a^{\ell_a+1}}.$$

For $r=2$ there is only a single summand and we get

$$\int_{[0,\infty)} e^{-t}e^{-tnm}dt=\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{(nm)^{\ell+1}}.$$

Changing summation gives the desired result. For $r=3$ we get

$$\int_{[0,\infty)^2} e^{-t_1-t_2}(e^{-t_1 n}+e^{-t_2 n})^m d(t_1,t_2)=$$
$$2\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{(nm)^{\ell+1}}+\sum_{\ell_1,\ell_2\geq 0}\frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}}\sum_{j=1}^{m-1}\binom{m}{j}\frac{1}{j^{\ell_1+1}(m-j)^{\ell_2+1}}.$$

In order to simplify the arising sums we use a classical partial fraction decomposition, which appears already in [11],

$$\frac{1}{j^a(m-j)^b}=\sum_{i=1}^{a}\frac{\binom{i+b-2}{b-1}}{m^{i+b-1}j^{a+1-i}}+\sum_{i=1}^{b}\frac{\binom{i+a-2}{a-1}}{m^{i+a-1}(m-j)^{b+1-i}}, \tag{15}$$

Thus,

$$\sum_{\ell_1,\ell_2\geq 0}\frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}}\sum_{j=1}^{m-1}\binom{m}{j}\frac{1}{j^{\ell_1+1}(m-j)^{\ell_2+1}}$$
$$=\sum_{\ell_1,\ell_2\geq 0}\frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}}\left(\sum_{i=1}^{\ell_1+1}\frac{\binom{i+\ell_2-1}{\ell_2}}{m^{i+\ell_2}}\sum_{j=1}^{m-1}\binom{m}{j}\frac{1}{j^{\ell_1+2-i}}+\sum_{i=1}^{\ell_2+1}\frac{\binom{i+\ell_1-1}{\ell_1}}{m^{i+\ell_1}}\sum_{j=1}^{m-1}\binom{m}{j}\frac{1}{j^{\ell_2+2-i}}\right).$$

Lemma 7. *For positive integers r, m we have*

$$\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j^r} = \zeta_m^\star(\{1\}_r; \{1\}_{r-1}, 2) - \zeta_m^\star(\{1\}_r).$$

Proof. We use induction with respect to r. For $r = 1$ we have

$$\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j} = \int_0^1 \frac{(1+t)^m - 1}{t} dt = \int_1^2 \frac{t^m - 1}{t-1} dt =$$
$$\int_1^2 (t^{m-1} + t^{m-2} + \cdots + t + 1) dt = \sum_{k=1}^{m} \frac{2^k}{k} - H_m = \zeta_m^\star(1; 2) - \zeta_m^\star(1).$$

Assuming the result for $r - 1$,

$$\sum_{j=1}^{m} \binom{m}{j} \frac{1}{j^r} = \sum_{j=1}^{m} \sum_{k=1}^{m} \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^{m} \sum_{j=1}^{k} \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^{m} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{j^{r-1}}$$
$$= \sum_{k=1}^{m} \frac{1}{k} \Big(\zeta_k^\star(\{1\}_{r-1}; \{1\}_{r-2}, 2) - \zeta_k^\star(\{1\}_{r-1}) \Big) = \zeta_m^\star(\{1\}_r; \{1\}_{r-1}, 2) - \zeta_m^\star(\{1\}_r).$$

□

This gives

$$\sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1+1}(m-j)^{\ell_2+1}}$$
$$= \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}} \Bigg[\sum_{i=1}^{\ell_1+1} \frac{\binom{i+\ell_2-1}{\ell_2}}{m^{i+\ell_2}} \Big(\zeta_m^\star(\{1\}_{\ell_1+2-i}; \{1\}_{\ell_1+1-i}, 2) - \zeta_m^\star(\{1\}_{\ell_1+2-i}) - \frac{1}{m^{\ell_1+2-i}} \Big)$$
$$+ \sum_{i=1}^{\ell_2+1} \frac{\binom{i+\ell_1-1}{\ell_1}}{m^{i+\ell_1}} \Big(\zeta_m^\star(\{1\}_{\ell_2+2-i}; \{1\}_{\ell_2+1-i}, 2) - \zeta_m^\star(\{1\}_{\ell_2+2-i}) - \frac{1}{m^{\ell_2+2-i}} \Big) \Bigg].$$

4 Outlook and Acknowledgments

It seems that similar phenomena appear when discussing random variables $Z_n = \|(X_1, \ldots, X_n)\|_n$, where the X_i are i.i.d. random variables.

The authors thank C. Vignat for his insightful remark, pointing out the closed form expression for the $\gamma_{s+1,j}$.

The fourth author would like to thank H. Prodinger for helping in computing an Euler sum and W.Wang for providing a useful reference.

References

[1] V. Adamchik, On Stirling number and Euler sums, *J. Comput. Appl. Math.* **79** (1997), 119-130.

[2] J. M. Borwein, D. M. Bradley, and D. J. Broadhurst, Evaluation of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, *Electron. J. Combin.* **4(2)** (1997), res. art. 5.

[3] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren, The multiple zeta value data mine, *Comput. Phys. Commun.* **181** (2010), 583-625.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge, UK, 2009.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, New York, 1994.

[6] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (1992), 275-290.

[7] M. E. Hoffman, Derivative polynomials for tangent and secant, *Amer. Math. Monthly* **102** (1995), 23-30.

[8] K. S. Kölbig, Nielsen's generalized polylogarithms, *SIAM J. Math. Anal.* **17** (1986), 1232-1258.

[9] G. Louchard, Two applications of polylog functions and Euler sums, manuscript, 2017. arXiv:1709.08686.

[10] V. H. Moll and C. Vignat, Generalized Bernoulli numbers and a formula of Lucas, *Fibonacci Quart.* **53** (2015), 349-359.

[11] N. Nielsen, *Handbuch der Theorie der Gammafunktion,* Chelsea Publishing Company, New York, 1965.

[12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C.W. Clark, editors. *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[13] J. Zhao, On a conjecture of Borwein, Bradley and Broadhurst, *J. reine angew. Math.* **639** (2010), 223-233.