THE RIEMANN HYPOTHESIS

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ABSTRACT. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln \ln n$ where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann Hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$. In this way, we demonstrate the Riemann Hypothesis is true.

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2].

The sum-of-divisors function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [7]. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality

$$\sigma(n) < e^{\gamma} \times n \times \ln \ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [4]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true [4]. Using this inequality, we show the Riemann Hypothesis is true.

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2. Results

Theorem 2.1. *Given a natural number*

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

such that $q_1, q_2, \cdots, q_m$ *are prime numbers and* $a_1, a_2, \cdots, a_m$ *are natural numbers, then we obtain the following inequality*

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof. From the article reference [1], we know

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}. \tag{2.1}$$

We can easily prove

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} = \prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

However, we know

$$\prod_{i=1}^{m} \frac{1}{1 - q_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}}$$

where q_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [7]. Consequently, we obtain

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}$$

and thus,

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

□

Theorem 2.2. *For* $x \geq 11$*, we have*

$$\sum_{q \leq x} \frac{1}{q} < \ln \ln x + \gamma - 0.12$$

where $q \leq x$ *means all the primes lesser than or equal to* x*.*

Proof. For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \ln \ln x + B + \frac{1}{\ln^2 x}$$

where

$$B = 0.2614972128 \cdots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [5]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \ln \ln x + \gamma - (C - \frac{1}{\ln^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\ln^2 x})$, then this complies with

$$(C - \frac{1}{\ln^2 x}) > (0.31 - \frac{1}{\ln^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \ln \ln x + \gamma - (C - \frac{1}{\ln^2 x}) < \ln \ln x + \gamma - 0.12.$$

□

Definition 2.3. We recall that an integer n is said to be squarefree if for every prime divisor q of n we have $q^2 \nmid n$, where $q^2 \nmid n$ means that q^2 does not divide n [1].

Theorem 2.4. *Given a squarefree number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1, q_2, \cdots, q_m$ *are odd prime numbers, the greatest prime divisor of* n *is greater than* 7 *and* $3 \nmid n$*, then we obtain the following inequality*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \ln \ln(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in Theorem 1.1 from the article reference [1]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [1]. Put $\omega(n) = m$ [1]. We need to prove the assertion for those integers with $m = 1$. From a squarefree number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{2.2}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [1]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^\gamma \times \ln \ln(2^{19} \times q_i). \tag{2.3}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^\gamma \times \ln \ln(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\ln \ln(2^{19} \times 11) < \ln \ln(2^{19} \times q_i)$$

which clearly implies that the inequality (2.3) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [1]. So let $n = q_1 \times \cdots \times q_m$ be a squarefree number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \times (q_m+1) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \ln\ln(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m+1) \times \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\ln q_m} \geq$$
$$\frac{\ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\ln q_m}.$$

From the reference [1], we have if $0 < a < b$, then

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.4}$$

We can apply the inequality (2.4) to the previous one just using $b = \ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \ln(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \ln(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$
$$\ln \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \ln q_m.$$

In this way, we obtain

$$\frac{q_m \times (\ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\ln q_m} >$$
$$\frac{q_m}{\ln(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\ln\ln(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [1].

Case 2: $q_m < \ln(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \ln\ln(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \ln\ln(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\ln(\frac{\pi^2}{5.32}) + (\ln(3+1) - \ln 3) + \sum_{i=1}^{m}(\ln(q_i + 1) - \ln q_i) \leq \gamma + \ln\ln\ln(2^{19} \times n).$$

From the reference [1], we note

$$\ln(q_1 + 1) - \ln q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\ln(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \ln\ln q_m < \gamma + \ln\ln\ln(2^{19} \times n)$$

since $q_m < \ln(2^{19} \times n)$ and therefore, it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \ln\ln q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \ln\ln q_m - 0.12$$

which is true according to the Theorem 2.2 when $q_m \geq 11$. In this way, we finally show the Theorem is indeed satisfied. □

Theorem 2.5. *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then the Robin's inequality is true for n.

Proof. Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers, we need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \ln\ln n$$

according to the inequality (2.1). Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln\ln(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \ln\ln(5040) < e^{\gamma} \times \ln\ln n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \ln\ln(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$ such that

$$e^{\gamma} \times \ln\ln(7^7) \leq e^{\gamma} \times \ln\ln n$$

and as a consequence, the proof is completed. □

Theorem 2.6. *The Robin's inequality is true for every natural number* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin's inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

Proof. We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the Theorem 2.5. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \ln\ln n$$

according to the Theorem 2.1. Using the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \ln\ln n$$

where $n' = q_1 \times \cdots \times q_m$ is the squarefree kernel of n [1]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [1]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for some integer $1 \leq k \leq 19$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \ln\ln(2^{19} \times \frac{n'}{2}) < e^{\gamma} \times n' \times \ln\ln n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \ln\ln(2^{19} \times \frac{n'}{2}).$$

According to the equation (2.2) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \ln\ln(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \ln\ln(2^{19} \times \frac{n'}{2})$$

that is true according to the Theorem 2.4 when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. □

Theorem 2.7. *The Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by 5. Moreover, we show the Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by 7.*

Proof. Let's define $s(n) = \frac{\sigma(n)}{n}$ [6]. Hence, we need to prove

$$s(n) < e^{\gamma} \times \ln\ln n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$s(2^a \times 3^b \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times m).$$

We know

$$s(2^a \times 3^b \times m) = s(3^b) \times s(2^a \times m)$$

since s is multiplicative [6]. In addition, we know $s(3^b) < \frac{3}{2}$ for every natural number b [6]. In this way, we have

$$s(3^b) \times s(2^a \times m) < \frac{3}{2} \times s(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times s(2^a \times m) = \frac{9}{8} \times s(3) \times s(2^a \times m) = \frac{9}{8} \times s(2^a \times 3 \times m)$$

where $s(3) = \frac{4}{3}$ since s is multiplicative [6]. Nevertheless, we have

$$\frac{9}{8} \times s(2^a \times 3 \times m) < s(5) \times s(2^a \times 3 \times m) = s(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times s(2^a \times 3 \times m) < s(7) \times s(2^a \times 3 \times m) = s(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $s(5) = \frac{6}{5}$ and $s(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Hence, we would have

$$s(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \ln\ln(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times m)$$

and

$$s(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \ln\ln(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times m)$$

when $b \geq 13$. □

Theorem 2.8. *The Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by any prime number $11 \leq q_m \leq 47$.*

Proof. We need to prove

$$s(n) < e^{\gamma} \times \ln\ln n$$

when $(2^{20} \times 3^{13} \times 7^{7}) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$s(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 7^c \times m).$$

We know

$$s(2^a \times 3^b \times 7^c \times m) = s(7^c) \times s(2^a \times 3^b \times m)$$

since s is multiplicative [6]. In addition, we know $s(7^c) < \frac{7}{6}$ for every natural number c [6]. In this way, we have

$$s(7^c) \times s(2^a \times 3^b \times m) < \frac{7}{6} \times s(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times s(7) \times s(2^a \times 3^b \times m) = \frac{49}{48} \times s(2^a \times 3^b \times 7 \times m)$$

where $s(7) = \frac{8}{7}$. In addition, we know

$$\frac{49}{48} \times s(2^a \times 3^b \times 7 \times m) < s(q_m) \times s(2^a \times 3^b \times 7 \times m) = s(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $s(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Hence, we would have

$$s(2^a \times 3^b \times 7 \times q_m \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 7 \times q_m \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 7^c \times m)$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. □

Theorem 2.9. *The Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by any prime number $53 \leq q_m \leq 113$.*

Proof. We know the Robin's inequality is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [3]. We need to prove

$$s(n) < e^{\gamma} \times \ln\ln n$$

when $(2^{20} \times 3^{13} \times 11^{6}) \mid n$. Suppose that $n = 2^a \times 3^b \times 11^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 6$, $2 \nmid m$, $3 \nmid m$, $11 \nmid m$, $q_m \nmid m$ and $53 \leq q_m \leq 113$. Therefore, we need to prove

$$s(2^a \times 3^b \times 11^c \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 11^c \times m).$$

We know

$$s(2^a \times 3^b \times 11^c \times m) = s(11^c) \times s(2^a \times 3^b \times m)$$

since s is multiplicative [6]. In addition, we know $s(11^c) < \frac{11}{10}$ for every natural number c [6]. In this way, we have

$$s(11^c) \times s(2^a \times 3^b \times m) < \frac{11}{10} \times s(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{121}{120} \times s(11) \times s(2^a \times 3^b \times m) = \frac{121}{120} \times s(2^a \times 3^b \times 11 \times m)$$

where $s(11) = \frac{12}{11}$. In addition, we know

$$\frac{121}{120} \times s(2^a \times 3^b \times 11 \times m) < s(q_m) \times s(2^a \times 3^b \times 11 \times m) = s(2^a \times 3^b \times 11 \times q_m \times m)$$

where $q_m \nmid m$, $s(q_m) = \frac{q_m+1}{q_m}$ and $53 \leq q_m \leq 113$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 11 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [3]. Hence, we would have

$$s(2^a \times 3^b \times 11 \times q_m \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 11 \times q_m \times m) < e^{\gamma} \times \ln\ln(2^a \times 3^b \times 11^c \times m)$$

when $c \geq 6$ and $53 \leq q_m \leq 113$. □

Theorem 2.10. *The Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by any prime number $q_m \leq 113$.*

Proof. This is a compendium of the results from the Theorems 2.6, 2.7, 2.8 and 2.9. □

Theorem 2.11. *The Robin's inequality is true for every natural number $n > 5040$ when $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ such that n' is the squarefree kernel of n.*

Proof. We will check the Robin's inequality for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are prime numbers and $a_1, a_2, \cdots, a_m$ are natural numbers. We need to prove

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \ln\ln n$$

according to the Theorem 2.1. Using the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \ln\ln n$$

where $n' = q_1 \times \cdots \times q_m$ is the squarefree kernel of n [1]. However, the Robin's inequality has been proved for all the squarefree integers $n' \notin \{2, 3, 5, 6, 10, 30\}$ [1]. In addition, due to the Theorem 2.5, the Robin's inequality is true for every natural number $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 30\}$, where n' is the squarefree kernel of n. In this way, we have

$$\frac{\sigma(n')}{n'} < e^{\gamma} \times \ln\ln n'$$

and therefore, it is enough to prove

$$\frac{\pi^2}{6} \times e^{\gamma} \times \ln\ln n' \leq e^{\gamma} \times \ln\ln n$$

which is the same as

$$\frac{\pi^2}{6} \times \ln\ln n' \leq \ln\ln n$$

and

$$\ln(\ln n')^{\frac{\pi^2}{6}} \leq \ln\ln n$$

that is true when

$$(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$$

and thus, the proof is completed. □

Theorem 2.12. *The Robin's inequality is true for every natural number* $n > 5040$ *when the greatest prime divisor* q_m *of* n *complies with* $q_m \geq 113$.

Proof. We are going to prove this Theorem for every natural number $n > 5040$ using the following two possible cases under the assumption that the greatest prime divisor q_m of n complies with $q_m \geq 113$.

Case 1: $q_m^{e^\gamma} < \ln n$.

According to the Theorem 2.11, we know the Robin's inequality is true for every natural number $n > 5040$ when $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ such that n' is the squarefree kernel of n. In this way, we need to prove for the remaining case, that is when $(\ln n')^{\frac{\pi^2}{6}} > \ln n$, $q_m \geq 113$ and $q_m^{e^\gamma} < \ln n$. That would equivalent to

$$(\ln n')^{\frac{\pi^2}{6}} > q_m^{e^\gamma}$$

which is the same as

$$\ln n' > q_m^{\frac{6 \times e^\gamma}{\pi^2}}.$$

We denote by $\vartheta(x)$ the logarithm of the product of all primes lesser than or equal to x [5]. We know $\vartheta(q_m) \geq \ln n'$ and thus, we would have

$$\vartheta(q_m) > q_m^{\frac{6 \times e^\gamma}{\pi^2}}.$$

From the article reference [5], we have for $x > 0$

$$\vartheta(x) < 1.01624 \times x.$$

In this way, we obtain

$$1.01624 \times q_m > q_m^{\frac{6 \times e^\gamma}{\pi^2}}$$

and since we know $\frac{6 \times e^\gamma}{\pi^2} > 1$, then we only need to prove

$$1.01624 > q_m^{\frac{6 \times e^\gamma}{\pi^2} - 1}.$$

However, we know

$$1.01624 < 113^{\frac{6 \times e^\gamma}{\pi^2} - 1} \leq q_m^{\frac{6 \times e^\gamma}{\pi^2} - 1}$$

and consequently, we obtain a contradiction just assuming that $(\ln n')^{\frac{\pi^2}{6}} > \ln n$ when $q_m \geq 113$ and $q_m^{e^\gamma} < \ln n$. Hence, this implies that necessarily $(\ln n')^{\frac{\pi^2}{6}} \leq \ln n$ when $q_m \geq 113$ and $q_m^{e^\gamma} < \ln n$ and therefore, the Robin's inequality is true for this case when the greatest prime divisor q_m of n complies with $q_m \geq 113$.

Case 2: $q_m^{e^\gamma} \geq \ln n$.

We need to prove

$$\frac{\sigma(n)}{n} < e^\gamma \times \ln\ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^\gamma \times \ln\ln n$$

according to the Theorem 2.1. Under the assumption of this case, we obtain

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{2\times\gamma} \times \ln q_m.$$

If we apply the logarithm to the both sides of the inequality, then we obtain

$$\ln(\frac{\pi^2}{6}) + \sum_{i=1}^{m} (\ln(q_i + 1) - \ln q_i) \leq 2 \times \gamma + \ln \ln q_m.$$

From the reference [1], we note

$$\ln(q_1 + 1) - \ln q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\gamma - \ln(\frac{\pi^2}{6}) > 0$. Therefore, it is enough to prove

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \ln \ln q_m$$

where $q_m \geq 113$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \ln \ln q_m$$

which is true when $q_m \geq 113$ due to the Lemma 2.1 from the article reference [1]. In conclusion, we show the Theorem is indeed satisfied. □

Theorem 2.13. *The Robin's inequality is true for every natural number* $n > 5040$.

Proof. The Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by any prime number less than or equal to 113 because of Theorem 2.10. Moreover, the Robin's inequality is true for every natural number $n > 5040$ when n is divisible by all the prime numbers between 2 and 113 due to the Theorem 2.12, since in this case the greatest prime divisor q_m of n complies with $q_m \geq 113$. These two cases cover the Robin's inequality for all the natural numbers greater than 5040. Certainly, this result is a consequence of the Theorems 2.10 and 2.12. □

Theorem 2.14. *The Riemann Hypothesis is true.*

Proof. If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann Hypothesis is true [4]. As result, this is true according to the Theorem 2.13. □

3. Conclusions

The practical uses of the Riemann Hypothesis include many propositions which are known true under the Riemann Hypothesis, and some of them can be shown equivalent to the Riemann Hypothesis [2]. Certainly, the Riemann Hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf Hypothesis, the large prime gap conjecture, etc [2]. In this way, this proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [2].

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