A proof of the Collatz conjecture and connection of intervals based on "tetrada".*

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Abstract

This paper presents a proof of the Collatz conjecture, also known as the 3x+1 problem.

The proof is based on numeral systems of rational bases, its modifications and sequences of ordered numeral intervals. The latter provide a new representation of integers.

The proof is obtained by dividing numerals into ordered intervals; in this sense, it is connected to the theory of prime numbers.

Key words and phrases: : proof, Collatz conjecture, $3x+1$ problem, numeral systems of rational bases, sequences of ordered numeral intervals , "tetrada"

1 Introduction

1.1 Motivation

This article explores the Collatz conjecture (CC) and related numerical systems, and reveals the mechanism underlying systems of numbers, which suggests the existence of the Collatz conjecture (transformation).

The main motivation is to understand and describe this mechanism.

The use of the concept of "mechanism" is associated with the mechanistic (as I see it) formulation of the hypothesis itself, namely, cyclic repetition of a set of actions with numbers until a certain result is achieved.

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1.2 Brief and accurate presentation of the questions studied

The Collatz transformation hides some very obvious yet invisible properties, necessitating a fundamental approach to the problem. The large share of "constructivism" in this work does not degrade the rigor of the proof.

1.3 Existing literature on the Collatz transformation.

Since 1937, researchers have attempted to solve the Collatz problem (conjecture)(further) using theories of orbits or trajectories.

Scientific and historical works on the Collatz problem are most comprehensively reviewed in Lagarias [8] .

Terras[14] made significant contributions to the theory of orbits and their characteristics, and

Tao recently interpreted the trajectories of orbits in a probabilistic manner[13].

Significant efforts have been made to solve by computational methods.

For very recent contributions on related topics, see [4],[2],[9],[12],[5],[11], [7],[18],[3],[6].

The series of works by Ren [23] is particularly pertinent to the present study. From Introduce [17]:"Section 5 studies all starting numbers whose **lengths** of reduced dynamics are no more than 7."

Having examined the above literature, I did not find the answer to the question: in what way are the resulting sequences of numbers related to each other when transforming the Collatz from the originally given number?

1.4 Contributions of the present article

Our study reveals a hidden "mechanism" in numerical systems based on the fractional bases $\frac{2^a}{3}$ and $\frac{4}{3}$.

Applying this mechanism, we provide a proof of the CC, which was first formulated in 1937.

We also provide a proof, that all natural numbers get their expansion in a set of numbers with a fractional base $\frac{2^a}{3}$.

1.5 Methods

In this paper, we study the distribution of a number, as a potential, on selected levels (scales) of numeral system. During the Collatz transformation, the characteristics of the levels are set by bases $\frac{2^a}{3}$ and $\frac{4}{3}$.

Unlike previous works, we modify the Collatz transformation by deleting its parity (rather than its "recursive destruction" as in conventional "recursive conservation").

The study builds on previous work, but displaces the research plane in the theory of numeral systems of rational bases.

It should be noted that the method of numeral systems of rational bases appeared as a result of the transition from "recursive destruction" to "recursive conservation" while modifying the Collatz transformation.

1.5.1 Definitions of the employed mathematical objects

Further efforts to solve the Collatz transformation have employed numerical systems based on the fractional bases $\frac{2^a}{3}$ and $\frac{4}{3}$, combined number representations of ordered intervals, iterative transformations of ordered intervals, replacement of "rank" with the "scale" concept, operations SEAM and UNSEAM (required to explain the behavior of potentials (the magnitude of a number) when combined into one whole and disconnected into several the numbers) see def.(2.4), and arithmetic interpretations. Here, we introduce concept of the 'length of the significance part" (body) of the number for set of numbers with rational base $\frac{4}{3}$.

1.5.2 Essential results for the present investigation

Recursive transformations in the study of the Collatz problem are constructed in [10], and the orbit characteristics of the Collatz problem are explained in [14].

1.5.3 Brief presentations of the proposed methods

Ordered intervals can be perceived similarly to musical notations[16]. The same analogy applies to the concepts of "scale" and "rank"[15]. As in music, our study considers the parallel transfer of intervals (or their combination, analogously to a piece of music), and their length (analogously to the distance from a musical row).

1.5.4 Structure of the present study

The proof of the CC and underlying mechanism were determined by the following procedure:

a) the recursive-conditional Collatz transformation f is converted to a system of recursive linear transformations F.;

b) we get and research as a result from a) a numeral system with rational base $\frac{2^a}{3}$;

c) we obtain **sufficient conditions** for the CC;

Definition 1.1. Sufficient conditions for CC, we mean such a record of the number $A \in \mathbf{N}$, which is given using two arithmetic operations indicated in the very formulation of the hypothesis

d) Collatz transformations are investigated in a numeral system with a rational base $\frac{4}{3}$ "close" to a numeral system with a rational base $\frac{2^a}{3}$;

e) we obtain **necessary conditions** for the Collatz conjecture CC by the following procedure:

Definition 1.2. Necessary conditions for CC, is the presence for $\forall A \in \mathbf{N}$ of a limited sequence of mathematical operations such that, as a result of applying these operations, we obtain **sufficient conditions CC for this number**.

— –an arbitrary natural number is written in the number system $\frac{4}{3}$, which acquires the characteristic "length of the significance part" or "body" of the number.

—–during the recursive Collatz transformation, the "length of the significance part" of the number can remain unchanged, or can decrease by at least half-digit (these rules are dictated by the specific set of digits in the $\frac{4}{3}$ system).

- —-The "length of the significance part" reduces when the oddness of the form $4k+1$ appears, and is kept when keeping the oddness of the form $4k+3$

- —-As the oddness of the form $4k+3$ cannot be kept indefinitely, the "length of the significance part" periodically decreases.

—– Therefore, it is inevitable that after an appropriate number of iterations, the number transforms into the form $\frac{2^{p(q)}}{3^q}$.

j) we also get the result that natural numbers are a subset of numeral system with a rational base $\frac{2^a}{3}$

2 Research and converting of the Collatz transformation, obtaining $\frac{2^a}{3}$-expansion of numbers from N

MathWorld [20] describes the Collatz problem as follows:

For some positive integer a_0, the Collatz problem asks whether the operations

$$a_n = \begin{cases} \frac{1}{2}a_{n-1} & for \ \ a_{n-1} \ \ even \\ 3a_{n-1}+1 & for \ \ a_{n-1} \ \ odd \end{cases} \tag{2.1}$$

always return to 1.

2.1 Equivalent convert of the Collatz transformation

We write the Collatz transformation as defined in Margenstern [10]:

$$f(x) = \begin{cases} \frac{3x+1}{2}, & if \ \text{x-}odd, \quad \text{x} \in \mathbf{N}, \\ \frac{x}{2}, & if \ \text{x-}even, \quad \text{x} \in \mathbf{N}. \end{cases} \tag{2.2}$$

$$f(x)\colon \mathbf{N} \to \mathbf{N} \tag{2.3}$$

As implemented in [10], we introduce the recurrent transformation

$$\begin{cases} \mathbf{g(x, k+1) = f(g(x, k))}, \\ \mathbf{g(x, 0) = x}. \end{cases} \tag{2.4}$$

$$g(x,k)\colon \mathbf{N} \to \mathbf{N}. \tag{2.5}$$

k is the iteration step indicating the number of sequential applications of the function f before the solution reaches 1.

The iteration count n terminates when $g(x,n) = 1$.

Note that in a binary numeral system, we have

$$\mathbf{f}(\mathbf{x_2}) = \begin{cases} \frac{\mathbf{11x_2+1}}{\mathbf{10}}, & if\ \text{x-}odd,\quad \text{x}\in \mathbf{N}, \\ \frac{\mathbf{x_2}}{\mathbf{10}}, & if\ \text{x-}even,\quad \text{x}\in \mathbf{N}. \end{cases} \tag{2.6}$$

Analyzing the first and second actions of the Collatz transformation.

The binary number system clarifies that:

- the first action (2.6) pads the end of the significant part of the number $f(x_2)$ with zeros, and

- the second action removes the zeros at the end of the significant part $f(x_2)$ and sets the last significant digit to odd.

The following are also clarified:

- The Collatz transformation can be modified by considering only the significant part of the number. That is, we can ignore the second action above and consider the first action alone.

-In this interpretation, the Collatz transformation reduces to obtaining the unit length of the significant part of the number at the **n**-th step.

To convert $g(x_2,k)$ **into a recursive transformation** $F(x_2,i)$, we remove the second action $\frac{\mathbf{x_2}}{\mathbf{10}}$ from the transformation (2.6).

We get the result:

$$\begin{cases} F(x_2,i) = 11 \cdot F(x_2,i-1) + 10^{a(i-1)}, i \in \mathbf{N}, a(i-1) \in \mathbf{N}, \\ F(x_2,0) = x_2, \\ i \in [\mathbf{0},\mathbf{n}], \end{cases} \tag{2.7}$$

$$F(x_2,i)\colon \mathbf{N} \to \mathbf{N}. \tag{2.8}$$

$a(i-1)$ is the number of padded zeros, which the second action (2.6) deletes in all steps up to $i-1$.

$g(x_2,k)$ is understood as a conversion a binary number.

The functions $g(x_2,k)$ and $F(x_2,i)$ differ only in their treatments of the parities: $g(x_2,k)$ destroys the resulting zeros (parities), whereas $F(x_2,i)$ sequentially stores and accumulates them.

Example: $x = 9_{10} = 1001_2$ The step-wise actions of the recursive transformations $g(x,k)$ and $F(x_2,i)$ are compared in the following examples:

k	$g(x,k)$	i	$F(x_2,i)$	$a(i)$
0	***9***	***0***	***1001***	***0***
1	***28=7*4***	***1***	***11100***	***2***
2	***7***			
3	***22=11*2***	***2***	***1011000***	***3***
4	***11***			
5	***34=17*2***	***3***	***100010000***	***4***
6	***17***			
7	***52=13*4***	***4***	***1101000000***	***6***
8	***13***			
9	***40=5*8***	***5***	***101000000000***	***9***
10	***5***			
11	***16=1*16***	***6***	***10000000000000***	***13***
12	***8***			
13	***4***			
14	***2***			
15	***1***			

(2.9)

In the above examples, the recursive transformation $F(x_2,i)$ terminated in Step 6 ($i = 6$), when the significant length of the number $F(x_2,6)$ became unity.

The steps of the recursive transformations k,i ($g(x,k)$ and $F(x_2,i)$ respectively) are mismatched because the second case $F(x_2,i)$ omits the action of dividing an even number by 2.

Note that $F(x,i)$ stores the following items:

- the values from the first (non-zero) significant digit to the last (non-zero) significant digit;

- the number of zeros between the last significant digit and the radix point [22], or between the radix point and the first significant digit;

- the radix point is in the initial number, and when the transformations F are executed, it (radix point) is transferred into the same place to each subsequent value of the number.

Properties of the recursive transformation $F(x,i)$

From recursive transformation (2.7) it follows that

$$\forall i \in \mathbf{N},$$

$$a(i) \in \mathbf{N},$$

$$a(i-1) < a(i) \tag{2.10}$$

As each recursive step i adds a number of no- significant zeros, we have

$$a(0) < a(1) < \cdots < a(i) < \cdots < a(n). \tag{2.11}$$

To find the mechanism of the recursive Collatz transformation of the number x to unit length, it is sufficient to find the sequence $a(i)$ of x.

The recursive transformation $F(x,i)$ is similarly written for other numeral systems.

2.1.1 Cyclic condition

The recursive transformation F is conditionally periodic if the following hold:

In the binary system

$$F(x_2, n) = 10^{a(n)} \text{ has unit length,} \tag{2.12}$$

which corresponds to the unit number of significant digits.

The length of this transformation is periodic because

$$F(F(x_2,n), n+1) = F(10^{a(n)}, n+1) = 10^{a(n)+2} \text{also has unit length.} \tag{2.13}$$

Similarly, in the decimal system, we have

$$F(x_{10}, n) = 2^{a(n)}. \tag{2.14}$$

Because the number matches its divisor, we also have

$$F(F(x_{10},n), n+1) = F(2^{a(n)}, n+1) = 2^{a(n)+2}. \tag{2.15}$$

Finding the values n for which $F(2^{a(n)}, n+1) = 2^{a(n)+2}$ is equivalent to finding the final member $a(n)$.

2.2 Solution of the system of integer equations

The initial (2.7) (step 0), intermediate (2.7)(step i) and final conditions (2.14)(step n) are combined into a system of recursive transformations, from which we can determine the sequence $a(i)$; $i \in \mathbf{N}$; $i \in [\mathbf{0}, \mathbf{n}]$.

Consider the following system of equations in a generic numeral system:

$$\begin{cases} F(x,0) = x;\ x \in \mathbf{N}, \\ \dots \\ F(x,i) = 3 \cdot F(x,i-1) + 2^{a(i-1)}, \\ \dots \\ F(x,n) = 2^{a(n)}. \end{cases} \tag{2.16}$$

As the system (2.16) has a recursive structure, we can construct its inverse transformation:

$$\begin{cases} F(x,n-1) = \dfrac{F(x,n) - 2^{a(n-1)}}{3}, \\ \dots \\ F(x,i-1) = \dfrac{F(x,i) - 2^{a(i-1)}}{3}, \\ \dots \\ F(x,0) = \dfrac{F(x,1) - 2^{a(0)}}{3};\ \ x \in \mathbf{N}. \end{cases} \tag{2.17}$$

As $F(x,n)$ satisfies (2.16), we can write $F(x,0)$ as follows:

$$\begin{cases} F(x,0) = \dfrac{\dfrac{\dfrac{\dfrac{\frac{2^{a(n)}-2^{a(n-1)}}{3} - 2^{a(n-2)}}{3} - \dots - 2^{a(n-i+1)}}{3} - 2^{a(n-i)} - \dots}{3} - 2^{a(0)}}{3}, \\ \qquad \text{where } \ a(0) < a(1) < \dots < a(i) < \dots < a(n). \end{cases} \tag{2.18}$$

The steps of the recursive transformation (2.18) are written in terms of the system (2.16).

Without loss of generality, we hereafter maintain a downward numbering of the recursion steps $F(x,i)$ from 0(start) to n (final).

$$\begin{cases} F(x,0) = x \\ F(x,i) = \dfrac{\dfrac{\dfrac{\frac{2^{a(n)}-2^{a(n-1)}}{3} - 2^{a(n-2)}}{3} - \dots - 2^{a(i+1)}}{3} - 2^{(i)}}{3} \\ F(x,n) = 2^{a(n)} \\ a(0) < a(1) < \dots < a(i) < \dots < a(n) \end{cases} \tag{2.19}$$

Theorem 2.1. *Conditions (2.16) apply to the system of numbers (2.19)*

Proof. The above theorem is proved by sequentially substituting the numbers from (2.19) to (2.16) in the reverse order, and checking for identical expressions. □

2.2.1 Example:

$$7 = \dfrac{\dfrac{\dfrac{\dfrac{\frac{2^{11}-2^{7}}{3} - 2^{4}}{3} - 2^{2}}{3} - 2^{1}}{3} - 2^{0}}{3}$$

$$a(0) < a(1) < \cdots < a(i) < \cdots < a(n). \tag{2.20}$$

$$F(7,0) = \frac{F(7,1) - 2^0}{3} = 7,$$
$$F(7,1) = \frac{F(7,2) - 2^1}{3} = 22,$$
$$F(7,2) = \frac{F(7,3) - 2^2}{3} = 68,$$
$$F(7,3) = \frac{F(7,4) - 2^4}{3} = 208,$$
$$F(7,4) = \frac{F(7,5) - 2^7}{3} = 640,$$
$$F(7,5) = \frac{F(7,6) - 2^{11}}{3} = 2048.$$

The above example constructs a recursive transformation for the number 7 and recursively calculates $a(i)$.

2.2.2 Options for converting the system of equations.

We first introduce $b(i)$ as

$$b(i) = a(i+1) - a(i)$$

$$b(i) \geq 0 \ \text{ because}(2.11) \tag{2.21}$$

then

$$F(x,0) = \frac{2^{b(0)}}{3}(\frac{2^{b(1)}}{3}(\frac{2^{b(2)}}{3}(\cdots(\frac{2^{b(i}}{3}(\cdots(\frac{2^{b(n-2)}}{3}(\frac{2^{a(n)} - 2^{b(n-1)}}{3})-1)\cdots)-1)\cdots)-1)-1). \tag{2.22}$$

Example:

$$F(7,5) = 2^{11},$$
$$F(7,4) = \frac{1}{3}2^7(2^4 - 1) = 40 * 2^4,$$
$$F(7,3) = \frac{1}{3}2^6(40 - 1) = 52 * 2^2,$$
$$F(7,2) = \frac{1}{3}2^2(52 - 1) = 34 * 2^1,$$
$$F(7,1) = \frac{1}{3}2^1(34 - 1) = 22 * 2^0,$$
$$F(7,0) = \frac{1}{3}2^0(22 - 1) = 7.$$

If $i = 0$, then

$$F(x,0) = \frac{2^{b(0)}}{3}(\frac{2^{b(1)}}{3}(\frac{2^{b(2)}}{3}(\cdots(\frac{2^{b(i)}}{3}(\cdots(\frac{2^{b(n-1)}}{3}(\frac{2^{a(n)-b(n-1)}}{3}-\frac{1}{3})-\frac{1}{3})\cdots)-\frac{1}{3})\cdots)-\frac{1}{3})-\frac{1}{3})-\frac{1}{3}. \tag{2.23}$$

For all other i, we have

$$F(x,i)=\frac{2^{a(n)}}{3^{n-i}}-\sum_{j=i}^{n-1}\frac{2^{b(j)}}{3^{j-i+1}}. \tag{2.24}$$

For example from [2.2.2] is converted as

$$7=\frac{2^{11}}{3^5}-\frac{2^7}{3^5}-\frac{2^4}{3^4}-\frac{2^2}{3^3}-\frac{2^1}{3^2}-\frac{2^0}{3^1}$$

2.3 v-expansion (representation) of numbers from N

To obtain the representation (2.24), we must verify that a number record (2.24) exists in a number system with rational base $\frac{2^a}{3}$.

2.3.1 Horner's transformation and pseudo-numbers in the $\frac{2^a}{3}$ numeral system

- In the standard Horner transformation[21], a number is converted into another numeral system as follows:

$$M=c(0)+p(c(1)+p(\cdots(C(n-1)+p\cdot C(n))\cdots)), \tag{2.25}$$

where p is the base of the system,

$c(i)$ are the digits in the system, and $c(i)<p$.

Horner's scheme reduces to obtaining successive residues:

$$\begin{cases} c(0)=\left\{\dfrac{M(0)}{p}\right\} \\ c(1)=\left\{\dfrac{\left[\frac{M(1)}{p}\right]}{p}\right\} \\ \cdots \\ c(i)=\left\{\dfrac{\left[\frac{M(i)}{p}\right]}{p}\right\} \\ \cdots \\ c(n-1)=\left\{\dfrac{\left[\frac{M(n-1)}{p}\right]}{p}\right\}, \end{cases} \tag{2.26}$$

In Horners scheme, we note that

- the coefficients c(i) in the Horner scheme (2.25) and (2.26) and the record $F(x,n)$ in expression (2.24) are obtained by the same deduction scheme, which is given as

$$c(0)=M(n)-\sum_{i=0}^{n-1}c(i)p^i, \tag{2.27}$$

$$F(x,0) = \frac{2^{a(n)}}{3^n} - \sum_{i=0}^{n-1} \frac{2^{b(i)}}{3^{i+1}}. \tag{2.28}$$

- It remains only to assume

  that in expression (2.28) there exists a rational base of the numeral system $\frac{p}{q}$.

  In a numeral system with rational base $q = 3$, the value of p changes dynamically depending on the iteration step i.

  The digits are forming as $\mathbf{p}(\mathbf{i}) = 2^{\mathbf{b}(\mathbf{i})}$.

  For convenience, we call these digits "the digits of the numeral system of $\mathbf{b}(\mathbf{i})$".

  In this approach, the entries are given as

$$c(0) = \langle M(n), c(n-1), \cdots, c(i), \cdots, c(0)\rangle_p, \tag{2.29}$$

$$F(x,0) = \langle a(n), b(n-1), \cdots, b(i), \cdots, b(0)\rangle_{\frac{2}{3}}. \tag{2.30}$$

- The equivalent pairs of equations are given by 2.29–2.27 and 2.30–2.28.

For example, consider the equivalent pairs

$$\begin{aligned} 87_{10} &= \langle 100, 1, 3\rangle_{10} = 100 - 1 * 10 - 3 * 1 \\ 7_{10} = \langle 11, 7, 4, 2, 1, 0\rangle_{\frac{2^a}{3}} &= \frac{2^{11}}{3^5} - \frac{2^7}{3^5} - \frac{2^4}{3^4} - \frac{2^2}{3^3} - \frac{2^1}{3^2} - \frac{2^0}{3^1}, \end{aligned} \tag{2.31}$$

where $\frac{2^a}{3}$ is a designation of actions described by 2.28.

In addition, we denote by $v = \frac{2^a}{3}$ the membership of numbers in a pseudo-digit system consisting of v **-extensions of integers**.

Due to the dynamic nature of the digits, the indicated number system v -extensions cannot belong to classical number systems of a rational base, where the value of p is constant.

Definition 2.2. $\mathbf{V}_v$ is a set of v-expansions of numbers from $\mathbf{N}$.

2.3.2 Comments to v-expansion number

In the presentation of section (2.3.1), it should be noted that

a) The value $p > 1$ is customarily taken as the basis of the numeral system;

b) The digits of the numeral system must be smaller than the base p of the system;

c) in v-expansion number, the description of the number system $\frac{p}{q}$ given in Section 1 [1] is inapplicable because the alphabet of digits v is given in the form of a power function $2^{b(i)}$, which violates $p > q \geq 1$. ($1 < 3$ and $3 > 1$).

d) To describe the number system $\frac{4}{3}$, we can apply the number system $\frac{p}{q}$ given in [1] because the alphabet of digits can be specified in the form of [1] and the condition $p > q \geq 1$ is satisfied $(4 > 3 > 1)$. Another variant of a rational base of the numeral system $\frac{4}{3}$ will be described below.

2.3.3 Elementary numbers in v-expansion numbers from N and operations with them

After confirming that the number $F(x,n)$ (2.24) exists in the number system $\mathbf{V_v}$, it is necessary to study the elementary operations inherent in the number system.

Operation LOCK

Definition 2.3. Let us calculate the q-th element of a number in (2.28),(2.30). We call this operation the [LOCK] operation.

$$\begin{aligned} \frac{2^p}{3^q} &= \frac{2^{p+1}}{3^{q+1}} + \frac{2^p}{3^{q+1}}, \\ -\frac{2^p}{3^q} &= -\frac{2^{p+2}}{3^q} + \frac{2^p}{3^{q-1}}. \end{aligned} \tag{2.32}$$

Operations SEAM and UNSEAM

Definition 2.4. Operations SEAM and UNSEAM on numbers from v-expansion are respectively given by:

$$\begin{aligned} \langle A\rangle_v &= \langle a(n), a(n-1), \cdots a(i), \cdots a(1)\rangle_v, \\ \langle B\rangle_v &= \langle b(m), b(m-1), \cdots b(j), \cdots b(1)\rangle_v. \end{aligned} \tag{2.33}$$

with

$$\begin{aligned} a(1) &= -\frac{2^{p+2}}{3^q}, \\ b(m) &= \frac{2^p}{3^{q-1}}, \\ a(1) + b(m) &= -\frac{2^p}{3^{q-1}}. \end{aligned} \tag{2.34}$$

Then

$\langle \mathbf{A}\rangle_\mathbf{v}$ **[SEAM]** $\langle \mathbf{B}\rangle_\mathbf{v}$ **is** $\langle \mathbf{C}\rangle_\mathbf{v}$.

$$\langle C\rangle_v = \langle A\rangle_v \cup \langle B\rangle_v = \langle a(n), a(n-1), \cdots, a(i), \cdots, a(1) + b(m), b(m-1), \cdots, b(j), \cdots, b(1)\rangle_v \tag{2.35}$$

$\langle \mathbf{C}\rangle_\mathbf{v}$ **[UNSEAM] to** $\langle \mathbf{A}\rangle_\mathbf{v}$**and**$\langle \mathbf{B}\rangle_\mathbf{v}$ constructs A and B in returning order from C.

Note: if operation [SEAM] gives an unambiguous solution, then operation [UNSEAM] gives ambiguous solutions.

An example of the [UNSEAM] operation is $13_{10} = \langle 7,3,0\rangle_v = \frac{\langle 7,5\rangle_v}{3^1} + \frac{\langle 3,0\rangle_v}{3^0}$

Elementary number in $\mathbf{V}_v$ See 2.2

Definition 2.5. Numbers of the form:

$$\langle E(i)\rangle_v = \langle \alpha(i), \beta(i)\rangle_v = \frac{2^{a(i)}}{3} - \frac{2^{b(i)}}{3} \tag{2.36}$$

are considered as "elementary"

because $\alpha(i) = 2^{a(i)}$ and $\beta(i) = 2^{b(i)}$ and therefore the "elementary" numbers can operated on by **[SEAM] .**

2.4 The total v-expansion $A_{10} \in \mathbf{N}$, as a sum of elementary numbers

$$\langle A\rangle_v = \sum_{i=0}^{n-1} \frac{\langle \alpha(i), \beta(i)\rangle_v}{3^i} = \sum_{i=0}^{n-1} \frac{\langle E(i)\rangle_v}{3^i}. \tag{2.37}$$

For example, $5_{10} = \langle 4,0\rangle_v$,
$13_{10} = \langle 7,3,0\rangle_v = \frac{\langle 7,5\rangle_v}{3^1} + \frac{\langle 3,0\rangle_v}{3^0}$,
$7_{10} = \langle 11,7,4,2,1,0\rangle_v = \frac{\langle 11,9\rangle_v}{3^4} + \frac{\langle 7,6\rangle_v}{3^3} + \frac{\langle 4,4\rangle_v}{3^2} + \frac{\langle 2,3\rangle_v}{3^1} + \frac{\langle 1,0\rangle_v}{3^0}$.

Note that, in the general case, the elementary numbers can be negative.

2.4.1 Obtaining a sequence of elementary numbers from the value of A_{10}

In this subsection, we represent $A_{10} \in N$ in the form of v.

This problem is solved by the following algorithm:

1. Define $b(0)$:

   $A \equiv odd \pmod{2^{b(0)}}$.

   If A is odd then $b(0) = 0$;

   if A is even then $b(0) \geq 1$.

   Assuming parity, the definition of $b(0)$ continues until the factor 2 maximally occurs in the number A.

2. Define $a(1)$ such that
   $(A * 3^1 + 2^{b(0)}) \equiv odd \pmod{2^{a(0)}}$.

3. Define the first elementary number as

   $\langle \alpha(0), \beta(0)\rangle_v = \frac{2^{a(0)}}{3^1} - \frac{2^{b(0)}}{3^1}$.

4. Set $b(i+1) = a(i) + 2$

5. and let $a(i+1)$. Then

   $\left[\left(A-\sum_{j=0}^{i-1}\left(\frac{\langle\alpha(j),\beta(j)\rangle_v}{3^j}\right)\right)3^{i+1}+2^{b(i+1)}\right] \equiv odd \pmod{2^{a(i+1)}}$.

   Then $a(i+1)$ becomes the maximum degree of the factor 2.

The following example decomposes 7_{10} into its
elementary numbers $7_{10}=\bigcup_{i=0}^{n-1}\langle\alpha(i),\beta(i)\rangle_v$.

- If $7 \equiv odd\ \ (2^{b(0)})$, then $b(0)=0$

  $(7*3^1+2^{b(0)}) \equiv odd\ \ (2^{a(0)})$, 22 and the numbers below indicate the intermediate results of the calculations

  $a(0)=1$, and

  $\langle\alpha(0),\beta(0)\rangle_v=\langle 1,0\rangle_v$.

- If $b(1)=a(0)+2$ then $b(1)=3$, and

  $\left[\left(7-\sum_{j=0}^{1-1}\left(\frac{\langle\alpha(j),\beta(j)\rangle_v}{3^j}\right)\right)3^{1+1}+2^{b(1)}\right] \equiv odd\ \ (2^{a(1)})$ 68 ,

  then $a(1)=2$, and

  $\langle\alpha(1),\beta(1)\rangle_v=\langle 2,3\rangle_v$.

- If $b(2)=a(1)+2$ then $b(2)=4$, and

  $\left[\left(7-\sum_{j=0}^{2-1}\left(\frac{\langle\alpha(j),\beta(j)\rangle_v}{3^j}\right)\right)3^{2+1}+2^{b(2)}\right] \equiv odd\ \ (2^{a(2)})$ 208 ,

  then $a(2)=4$, and

  $\langle\alpha(2),\beta(2)\rangle_v=\langle 4,4\rangle_v$.

- If $b(3)=a(2)+2$ then $b(3)=6$, and

  $\left[\left(7-\sum_{j=0}^{3-1}\left(\frac{\langle\alpha(j),\beta(j)\rangle_v}{3^j}\right)\right)3^{3+1}+2^{b(3)}\right] \equiv odd\ \ (2^{a(3)})$ 640,

  then $a(3)=7$, and

  $\langle\alpha(3),\beta(3)\rangle_v=\langle 7,6\rangle_v$.

- If $b(4)=a(3)+2$ then $b(4)=9$ and

  $\left[\left(7-\sum_{j=0}^{4-1}\left(\frac{\langle\alpha(j),\beta(j)\rangle_v}{3^j}\right)\right)3^{4+1}+2^{b(4)}\right] \equiv odd\ \ (2^{a(4)})$ 2048

  then $a(4)=11$, and

  $\langle\alpha(4),\beta(4)\rangle_v=\langle 11,9\rangle_v$.

The operation [SEAM] in v gives
$7_{10}=\langle 11,9\rangle_v\cup\langle 7,6\rangle_v\cup\langle 4,4\rangle_v\cup\langle 2,3\rangle_v\cup\langle 1,0\rangle_v=\langle 11,7,4,2,1,0\rangle_v$.

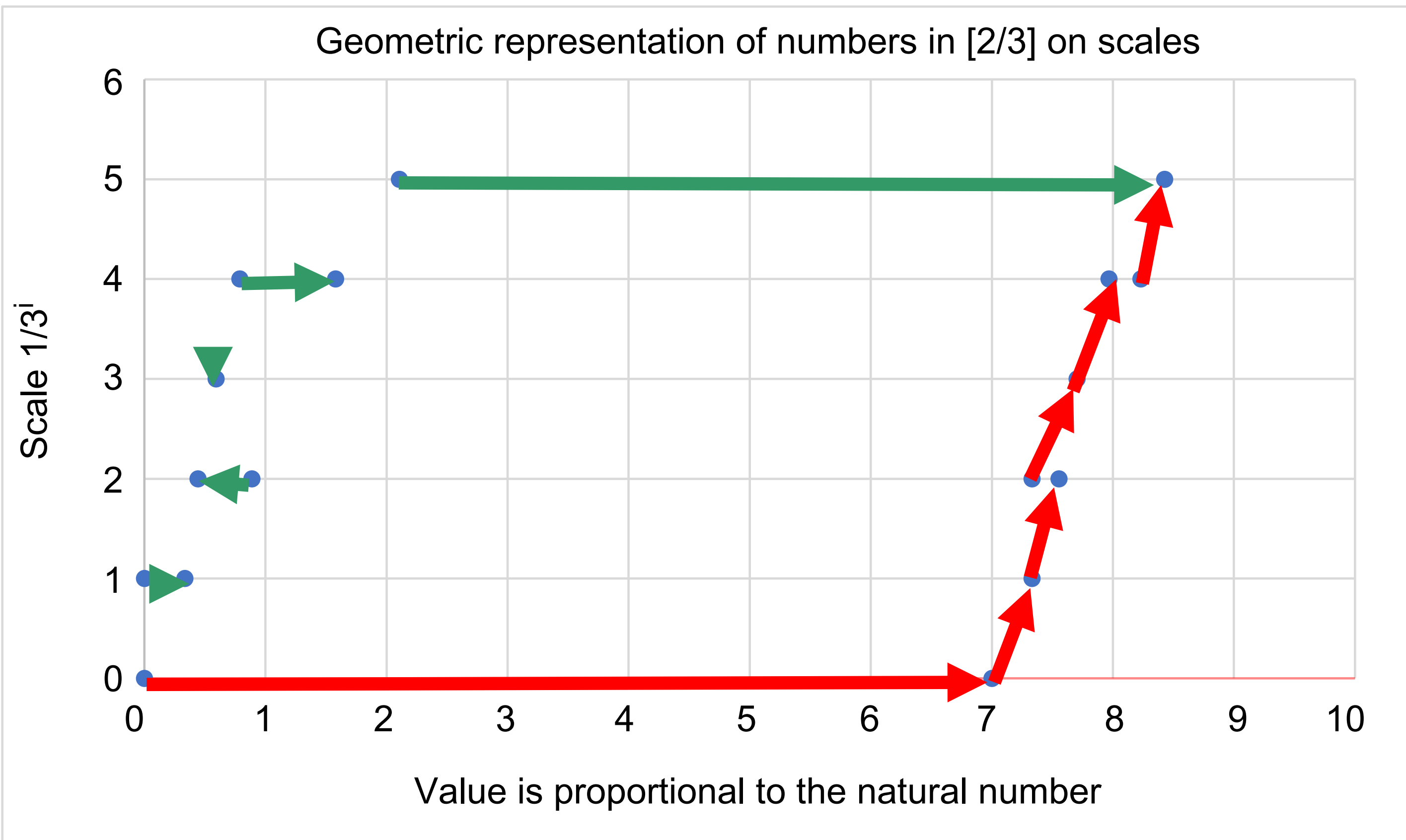

Figure 1: Scales

2.4.2 Geometric representation of v-expansion $\langle A\rangle_v$ numbers on scales

This subsection gives the geometric representation of splitting the number $\langle A\rangle_v$ into its elementary numbers. Figure [1] shows the representation on a linear scale.

Definition 2.6. In numeral systems of rational bases $\frac{p}{q}$ the resulting scales are straight lines of the form $y = \frac{1}{q^i}$.

Definition 2.7. In numeral systems of rational bases $\frac{p}{q}$ the interval on the scale i are spacing of the form $[\frac{\alpha(i)}{q^i}; \frac{\beta(i)}{q^i}]$ and $\alpha(i) < \beta(i)$.

Definition 2.8. In numeral systems with base $\frac{2^a}{3}$ the interval $[\frac{2^{\alpha(i)}}{3^i}; \frac{2^{\beta(i)}}{3^i}]$ on the scale i are acquires a direction or sign as a result of connecting the intervals with each other using the LOCK.

If you connect the interval to the smaller end, you get a plus, and if you get a big one, get a minus.

Remark: to SCALEs. Here we must distinguish between the concept of "scale" and the generally accepted concept of "rank".

This distinction is necessary because in the number system v-expansion, the absolute value of an elementary number can be arbitrarily large, since there are no restrictions on the value of $\beta(i)$ on any scale i.

In the ordinary number system $\frac{p}{q}$ considered in [1], the alphabet A is limited by the number of digits p.

Figure [1] plots the values of $\frac{1}{3^i}$ and $i \in N$ versus the values of $k \in N$.

On these scales, we will represent the elementary number components $\langle A\rangle_v$.

In Figure [1], the composition of the number

$7_{10} = \langle 11,9\rangle_v \cup \langle 7,6\rangle_v \cup \langle 4,4\rangle_v \cup \langle 2,3\rangle_v \cup \langle 1,0\rangle_v = \langle 11,7,4,2,1,0\rangle_v$

is represented on the linear scale as follows:

a) The green arrows and their directions reflect the elementary numbers and their signs, respectively.

scale 0 $\langle 1,0\rangle_v$

scale 1 $\langle 2,3\rangle_v$

scale 2 $\langle 4,4\rangle_v$

scale 3 $\langle 7,6\rangle_v$

scale 4 $\langle 11,9\rangle_v$

b) The red arrows indicate the transitions between coordinates, and their
lengths indicate the value of the [LOCK] operation
while stitching a number. This number is always positive.

The transition between the scales is governed by the [LOCK] rule

$b(i+1) = a(i) + 2$ (figure 1).

3 Proof of sufficient conditions for the Collatz conjecture.

Theorem 3.1. *Proof of sufficient conditions for obtaining a unit length number using Collatz transformation.*

If the number $A_{10} \in \mathbf{N}$ have v-expansions $\langle A\rangle_v$, then in a limited number of iteration steps n of F it will reduce to a number with unit length of the form $\frac{2^{a(n)}}{3^n}$. $\langle A\rangle_v \in \mathbf{V}_q$.

Proof. Let us suppose that the representation of a number $\langle A\rangle_v$ is known.

That is, the number $\langle A\rangle_v$ can be represented in the form (2.37).

Then the action F decreases

the number of scales of $\langle A\rangle_v$ by one or by one of its elementary numbers E.

The first action corresponds to scale 0, and subsequent actions continue

until all scales or all elementary numbers have been deleted.

The final number is $\frac{2^{a(n)}}{3^n}$, which has unit length. □

In Figure [1] above, this process corresponds to the green arrows acting across horizontal intervals, which decrease quantity of the elementary numbers.

The vertical transitions (red) are the direct actions from scales i to $i+1$. They are associated with the formation of the number $\langle A\rangle_v$ as described in Section 2.4.1.

3.1 Necessary conditions

To complete the proof, we consider the necessary conditions for the Collatz transformation, as the existence for $\forall A \in \mathbf{N}$ of its v-expansions in the set $\mathbf{V}_v$ or (in the geometric sense) the limited number of scales for it.

4 Research a recursive transformation F in $\mathbf{W}_q$(the set of $\frac{4}{3}$-expansion of numbers)

As the base of the system $\frac{2^a}{3}$ depends at the iteration step (is dynamic) it follows, what to get a v-expansion of number in the system $\langle \mathbf{V}\rangle_v$ by Horner's method is not possible.

Namely, the iterative process becomes an infinite sequence.

The search of the necessary conditions for CC leads to numeral system with rational base of $\frac{4}{3}$, which "close" to numeral system with base of $\frac{2^a}{3}$.

4.1 Numeral system with rational base $\frac{4}{3}$

The base q of the $\frac{4}{3}$ numeral system is

$$q = \frac{4}{3} > 1 \tag{4.1}$$

Unlike the authors of [1], we compose an alphabet of digits $\gamma(i)$ as follows:

$$\gamma(i) \in \left\{0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\}. \tag{4.2}$$

By Horner's scheme [21] and Theorem 1 of [1], a number $A \in \mathbf{N}$ can be represented in the base-q system as

$$A_q = \sum_{i=0}^{n} \gamma(i) \left(\frac{4}{3}\right)^i = \sum_{i=0}^{n} \gamma(i) \cdot q^i. \tag{4.3}$$

The natural numbers can then be converted to numbers in the q system by the Horner algorithm.

For example, in Figure [2],

the values against the yellow background are the result of converting the number 7 to the q system.

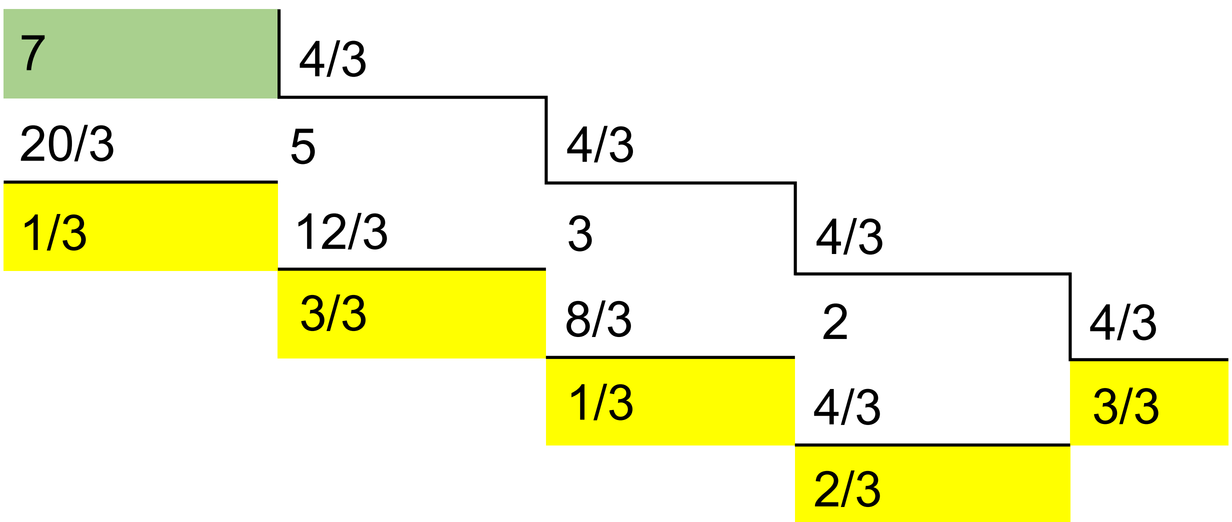

Figure 2: Conversation of the number 7 to the rational base $\frac{4}{3}$

$$7_{10} = \frac{3}{3}\left(\frac{4}{3}\right)^4 + \frac{2}{3}\left(\frac{4}{3}\right)^3 + \frac{1}{3}\left(\frac{4}{3}\right)^2 + \frac{3}{3}\left(\frac{4}{3}\right)^1 + \frac{1}{3}\left(\frac{4}{3}\right)^0. \tag{4.4}$$

An equivalent presentation is

$$\langle A \rangle_q = \sum_{i=0}^{n} \gamma(i) q^i = \sum_{i=0}^{n} \left(\gamma(i)\frac{3}{4}\right) q^{i+1}. \tag{4.5}$$

Shifting by one rank converts the digits to the following form:

$$\left(\gamma(i)\frac{3}{4}\right) \in \left\{0, \frac{1}{3}\frac{3}{4} = \frac{1}{4}, \frac{2}{3}\frac{3}{4} = \frac{2}{4}, \frac{3}{3}\frac{3}{4} = \frac{3}{4}\right\}. \tag{4.6}$$

Definition 4.1. Parallel digit transfer

If

$$\langle A(i)\rangle_q = \sum_{i=s}^{p} \gamma(i)\cdot q^i, \tag{4.7}$$

then

$$\langle A(i)\rangle_q = \left(\frac{4}{3}\right)\cdot \sum_{i=s}^{p} \gamma(i)\cdot\left(\frac{3}{4}\cdot q^i\right) \tag{4.8}$$

or

$$\langle A(i)\rangle_q = \left(\frac{4}{3}\right)\cdot \sum_{i=s-1}^{p-1} \gamma(i+1)\cdot q^i. \tag{4.9}$$

Half-digits

Definition 4.2. Half-digits $\gamma(i2), \gamma(i1)$ are defined as

$$\begin{aligned} \gamma(i) = \alpha\gamma(i2) + \beta\gamma(i1),\\ \text{where } \alpha = \{0,1\},\\ \beta = \{0,1\},\\ \gamma(i2) = \frac{2}{3},\\ \text{and}\\ \gamma(i1) = \frac{1}{3}. \end{aligned} \tag{4.10}$$

Alphabet of half-digits $\gamma(ij)$ as follows:

$$\gamma(ij) \in \left\{\frac{1}{3}, \frac{2}{3}\right\}. \tag{4.11}$$

The fact that the digits of the q-th number system consist of half-digits is an important point, because the Collatz transformation, as will be discussed later, differently changes the last half-digit and other half-digits located to the right of it.

And also half-digits establish a correspondence between alphabets of digits of number systems $\langle \mathbf{V}\rangle_v$ and $\langle \mathbf{W}\rangle_q$

Definition 4.3. Let $\mathbf{W}_q$ be a set of q-expansions of numbers

Definition 4.4. If $i \geq 0$ then $\langle A\rangle_q$ is integer in $\mathbf{W}_q$, then $\langle A\rangle_q = \sum_{i=0}^{n} \gamma(i) q^i$.

Co-factors (divisors)

Definition 4.5. If $\langle d\rangle_q$ is a divisor of $\langle A\rangle_q$, then N summations of $\langle d\rangle_q$ will equal $\langle A\rangle_q$.

$$\underbrace{\langle d\rangle_q + \langle d\rangle_q + \langle d\rangle_q + \ldots + \langle d\rangle_q}_{\langle N\rangle_q} = \langle A\rangle_q \qquad \langle N\rangle_q \in \mathbf{N} \tag{4.12}$$

The division of the q-expansions of the number $\langle A\rangle_q$ is its decomposition into equal counting parts.

Properties $\langle d\rangle_q$ (as divisor)

$$\langle d\rangle_q > 0 \qquad \langle d\rangle_q \leq \langle A\rangle_q \tag{4.13}$$

follows by Def. (4.5) and $N > 1$.

If N is odd and according by Def. (4.5) we have that the last half-digit of the divisor $\langle d\rangle_q$ repeats the last half-digit of the number $\langle A\rangle_q$. The rule is as follows:

A_q	$d_q(1)$
$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{3}$	$\frac{1}{3}$

$$\text{if } d_q(s) = \frac{2}{3}, \text{ then taking it } N = 2n+1\text{-times get out } A_q(s) = \frac{2}{3}, \text{ since } s \text{ is last half-digit}$$
$$\text{if } d_q(s) = \frac{1}{3}, \text{ then taking it } N = 2n+1\text{-times get out } A_q(s) = \frac{1}{3}, \text{ since } s \text{ is last half-digit} \tag{4.14}$$

4.1.1 Fractional positions in $\mathbf{W}_q$

- We can appropriately assume that, like the base-2 or base-10 systems, the $\mathbf{W}_q$ system admits commas and fractional positions:

Definition 4.6. If $i < 0$ that $\langle A\rangle_q$ is fraction in $\mathbf{W}_q$
$\langle A\rangle_q = \sum_{i=-m}^{n} \gamma(i) q^i$
$i \in \mathbf{Z}$
$n, m \in \mathbf{N}$
.

Let us consider the above definition further:

- A shift by one position gives

$$\begin{aligned} \left(0\,,\frac{1}{3}\right)_q &= \frac{1}{3}\left(\frac{4}{3}\right)^{-1} = \frac{1}{3}\left(\frac{3}{4}\right)^{1} = \left(\frac{1}{4}\right)_{10}, \\ \left(0\,,\frac{2}{3}\right)_q &= \frac{2}{3}\left(\frac{4}{3}\right)^{-1} = \frac{2}{3}\left(\frac{3}{4}\right)^{1} = \left(\frac{2}{4}\right)_{10}, \\ \left(0\,,\frac{3}{3}\right)_q &= \frac{3}{3}\left(\frac{4}{3}\right)^{-1} = \frac{3}{3}\left(\frac{3}{4}\right)^{1} = \left(\frac{3}{4}\right)_{10}. \end{aligned} \tag{4.15}$$

- A shift by 2 positions is described as

$$\begin{aligned}
\left(0\,,\,0\,\frac{1}{3}\right)_q &= \frac{1}{3}\left(\frac{4}{3}\right)^{-2} = \frac{1}{4}\left(\frac{4}{3}\right)^{-1} = \frac{3}{16} = \frac{1}{8} + \frac{1}{16},\\
\left(0\,,\,0\,\frac{2}{3}\right)_q &= \frac{2}{3}\left(\frac{4}{3}\right)^{-2} = \frac{2}{4}\left(\frac{4}{3}\right)^{-1} = \frac{3}{8} = \frac{1}{4} + \frac{1}{8},\\
\left(0\,,\,0\,\frac{3}{3}\right)_q &= \frac{3}{3}\left(\frac{4}{3}\right)^{-2} = \frac{3}{4}\left(\frac{4}{3}\right)^{-1} = \frac{9}{16} = \frac{1}{2} + \frac{1}{16}.
\end{aligned} \tag{4.16}$$

- In the number representation of the $\mathbf{W}_q$ system, the whole (integer) part is assembled from thirds and the fractional part is assembled from quarters.

  Meanwhile, factors (dividers) of the integer and fractional part are proportional to thirds and quarters, respectively.

  For instance, consider the number $\delta(-r)_q = 2^{-r}$ in $\mathbf{W}_q$:

  $q^{-1} = \left(\frac{4}{3}\right)^{-1} = \frac{3}{4} = 1 - \frac{1}{4}$,

$$\begin{aligned}
\delta(-1)_q &= \frac{1}{2} = \frac{2}{3}\left(\frac{4}{3}\right)^{-1},\\
\delta(-2)_q &= \frac{1}{4} = \frac{3}{3} - \left(\frac{4}{3}\right)^{-1} = \left(\frac{3}{3} - (q)^{-1}\right)_q,\\
\delta(-3)_q &= \frac{1}{8} = \frac{1}{2}\left(\frac{3}{3} - \left(\frac{4}{3}\right)^{-1}\right) = \left(0\,,\frac{2}{3}\right)_q\left(\frac{3}{3} - (q)^{-1}\right)_q,\\
&\dots\\
\delta(-r)_q &= 2^{-r} = 2^{-r+2}\left(\frac{3}{3} - \left(\frac{4}{3}\right)^{-1}\right) = (\delta(-r+2))_q\left(\frac{3}{3} - (q)^{-1}\right)_q.
\end{aligned} \tag{4.17}$$

- Digits in the fractional part of a number.

  In some situations, there are insufficient digits to demonstrate the result using earlier introduced digits (alphabet).

  In such cases, we can introduce virtual digits.

  Virtual digits of known value can be used in Eq. (4.17), but cannot be presenting(writing) as the components of earlier introduced digits (alphabet).

- The action of adding fractions by (4.17) is illustrated in Figure [3].

4.1.2 Special contradictions in $\mathbf{W}_q$

- In the decimal system, a small number is short in length; for example

$$107 < 1014$$

- In the $\mathbf{W}_q$ system, a smaller number may be longer than the larger designated number.

The length of a number is determined by the number of positions from the radix point; for example,

$$\langle \frac{1}{3} 0 \frac{1}{3}, 0\rangle_q < \langle \frac{3}{3}, 0\rangle_q = 1_{10}.$$

- Below is an example of a disorder in magnitude (numerical mess):

$$\left(\frac{1}{3}\right)_q \cdot \left(\frac{4}{3}\right)_q^{i+1} < \left(\frac{4}{3}\right)_q^{i}.$$

- The disorder table is obtained by computing the ratio of

$$\gamma(i)q < \gamma(i-1). \tag{4.18}$$

In three cases, we obtain

$$\begin{aligned} \frac{1}{3}q &< \frac{2}{3}, \\ \frac{1}{3}q &< \frac{3}{3}, \\ \frac{2}{3}q &< \frac{3}{3}, \end{aligned} \tag{4.19}$$

$$\gamma(i)q^2 < \gamma(i-2). \tag{4.20}$$

In two cases:

$$\begin{aligned} \frac{1}{3}q^2 &< \frac{2}{3}, \\ \frac{1}{3}q^2 &< \frac{3}{3}, \end{aligned} \tag{4.21}$$

$$\gamma(i)q^3 < \gamma(i-3), \tag{4.22}$$

and in one case,

$$\frac{1}{3}q^3 < \frac{3}{3}. \tag{4.23}$$

- The ratio

$$\gamma(i)q^4 < \gamma(i-4) \tag{4.24}$$

has no value, because

$$\frac{1}{3}q^4 > \frac{3}{3}. \tag{4.25}$$

4.2 Positional representation of a number in $\mathbf{W}_q$ and potential

- To properly compare numbers in the disordered $\mathbf{W}_q$ system, we must introduce the concept of potential:

Example:

$$\langle A\rangle_q = \langle\underbrace{\frac{3}{3}\frac{3}{3}\cdots\frac{3}{3}\frac{3}{3}}_{n}\rangle_q,$$

and

$$\langle B\rangle_q = \langle\underbrace{\frac{1}{3}\frac{0}{3}\cdots\frac{0}{3}\frac{0}{3}}_{n+1}\rangle_q,$$

$$\frac{\langle A\rangle_q}{\langle B\rangle_q} = 9 - 9\left(\frac{4}{3}\right)^{-(n+1)} \sim 9.$$

- A number is then understood as a magnitude distribution of the total potential among the scales (i) in $\mathbf{W}_q$.
- For the number system of a rational base q, the concept of a "scale" coincides with the concept of "rank"; differences will arise in further research when the elementary numbers (2.36) appears on "scales" as part of the result of the recursive Collatz transformation F.

  In the further study of the system $\mathbf{W}_q$, we use the concept of a "scale".
-

Definition 4.7. We define the potential $\langle A\rangle_q$ as

$$P(\langle A\rangle_q) = log_q(\langle A\rangle_q). \tag{4.26}$$

4.3 Research of a recursive transformation F in $\mathbf{W}_q$

Lemma 4.8. *The recursive transform of F has the following property:*
$F\left(\langle 2^z \cdot A(i)\rangle_q\right) = 2^z \cdot F(\langle A(i)\rangle_q)$.

Proof. As the factor 2^z is part of the divisor of the number $\langle A(i)\rangle_q$, then by Eqs. (4.12) and (4.28) we have

$$F\left(\langle 2^z \cdot A(i)\rangle_q\right) = 2^z \cdot F(\langle A(i)\rangle_q). \tag{4.27}$$

□

4.3.1 Particular qualities F in $\mathbf{W}_q$

In accordance with (4.3) and (4.12), each iteration of $F(\langle A(i)\rangle_q)$ begins with the decomposition of the number $\langle A(i)\rangle_q$ into divisors of 2 and quantity of this divisors.

We write this expansion on step i in the form $\langle A(i)\rangle_q = \langle N(\langle A(i)\rangle_q)\rangle_q * \langle d(\langle A(i)\rangle_q)\rangle_q$

The equivalent record of converting F over steps i is as follows,

where the expression $\langle N(\langle A(i)\rangle_q)\rangle_q$ and $\langle d(\langle A(i)\rangle_q)\rangle_q$ account for the value of the scale i from the previous transformation

$$\begin{aligned} \langle A(i)\rangle_q = \langle N(\langle A(i)\rangle_q)\rangle_q * \langle d(\langle A(i)\rangle_q)\rangle_q, \\ \langle N(\langle A(i)\rangle_q)\rangle_q \in \mathbf{W}_q. \\ \text{In the number } \langle N(\langle A(i)\rangle_q)\rangle_q, \\ \text{at least one of the numerator or denominator must be odd.} \\ \langle d(\langle A(i)\rangle_q)\rangle_q = 2^{b(i)}; see \;\; Def.(4.5). \end{aligned} \tag{4.28}$$

In iteration i,

the oddness $\langle N(\langle A(i)\rangle_q)\rangle_q$ (before F) is converted over to evenness $\langle n(\langle A(i)\rangle_q)\rangle_q$ (after F) on scale i:

$$\begin{aligned} \langle A(i+1)\rangle_q = F(\langle A(i)\rangle_q) = 3\langle A(i)\rangle_q + \langle d(\langle A(i)\rangle_q)\rangle_q = \\ (3\langle N(\langle A(i)\rangle_q)\rangle_q + 1)\cdot\langle d(\langle A(i)\rangle_q)\rangle_q = \\ \left(\frac{3}{4}\langle N(\langle A(i)\rangle_q)\rangle_q + \frac{1}{4}\right)(4\langle d(\langle A(i)\rangle_q)\rangle_q) = \\ \left(\frac{3}{4}\langle N(\langle A(i)\rangle_q)\rangle_q + \frac{1}{4}\right)2^{b(i)+2} = \\ n(\langle A(i)\rangle_q) * \langle D(\langle A(i)\rangle_q)\rangle_q. \end{aligned} \tag{4.29}$$

$$N(\langle A(i)\rangle_q) \to \frac{3}{4}N(\langle A(i)\rangle_q) + \frac{1}{4} = n(\langle A(i)\rangle_q). \tag{4.30}$$

In this transformation, in the new value of the number of divisors (after F) of $\langle(n(\langle A(i)\rangle_q)\rangle_q$, divisors of two appear. These divisors are separated from the new value of the number of divisors at the beginning of the next iteration.

$$\langle d(\langle A(i)\rangle_q)\rangle_q \to 2^{b(i)+2} = \langle D(\langle A(i)\rangle_q)\rangle_q, \tag{4.31}$$

following step $(i+1)$.

In the new total number of divisors $\langle(n(\langle A(i)\rangle_q)\rangle_q$, the divisors of two are separated and the number of divisors reduces to $\langle N(\langle A(i+1)\rangle_q)\rangle_q \in \mathbf{W}_q$ (4.29). Meanwhile, the selected divisors of two are transferred to the number divider $\langle d(\langle A(i+1)\rangle_q)\rangle_q$.

Equivalent record of converting F over steps $(i+1)$

$$\begin{aligned} \langle A(i+1)\rangle_q = n(\langle A(i)\rangle_q) * \langle D(\langle A(i)\rangle_q)\rangle_q = \\ N(\langle A(i+1)\rangle_q) * \langle d(\langle A(i+1)\rangle_q)\rangle_q \\ N(\langle A(i+1)\rangle_q) \in \mathbf{W}_q \\ \langle d(\langle A(i+1)\rangle_q)\rangle_q = 2^{b(i+1)}. \end{aligned} \tag{4.32}$$

The example of process in working as shown in Figure [4].

	10	4/3							
2 * 3	6	3/3	2/3	1/3	0	2/3			
1 * 3	3			3/3	2/3	1/3			
1/2 * 3	3/2					3/3	,	2/3	
1/4 * 3	3/4					0	,	3/3	
1/8 * 3	3/8					0	,	0	2/3
1/16 * 3	3/16					0	,	0	1/3

Figure 3: Digits in the fractional part of a number

	A_(4/3)								A	D	F(A)
Scale 0	3/3	2/3	1/3	3/3	1/3				7	1	4/3 * 21/4
Scale 1		3/3	2/3	1/3	3/3	,	1/3	4/3	21/4	1/4	22/3
Scale 2		3/3	2/3	1/3	3/3	,	2/3	4/3	11/2	1/8	34/8
Scale 3		3/3	2/3	1/3	0	,	1/3	4/3	17/4	1/16	52/16
Scale 4			3/3	2/3	1/3	,	1/3	4/3	13/4	1/16	40/16
Scale 5				3/3	2/3	,	2/3	4/3	5/2	1/8	16/8
Scale 6				3/3	2/3				2		

Figure 4: Equivalent record of recursively converting 7_{10} over steps (i)

4.3.2 Elementary numbers between iterations

Scale (step) i

We denote by $\langle d(i)\rangle_q$ the divisor of the number $\langle A(i)\rangle_q$:

Definition 4.9. $\langle d(i)\rangle_q = \langle d(\langle A(i)\rangle_q)\rangle_q$.

Clearly, during iteration i on scale i, there remains a difference. This difference is an elementary number $\langle E(i)\rangle_q$.

Theorem 4.10. *Representing the number $A_{10} \in \mathbf{N}$ by $\langle A\rangle_q$, then at each iteration step i in the recursive transformation F, ($\langle A\rangle_q$) will be reduced to an elementary number $\langle E(i)\rangle_q$. $\langle A\rangle_q \in \mathbf{W}_q$*

Proof. Notice that

$$\begin{aligned}
&\langle A(i)\rangle_q - \langle d(i-1)\rangle_q = \\
&\quad = \langle A(i)\rangle_q - \langle d(i)\rangle_q + [\langle d(i)\rangle_q - \langle d(i-1)\rangle_q] + q\cdot\langle d(i)\rangle_q - q\cdot\langle d(i)\rangle_q = \\
&\quad = \left[\langle A(i)\rangle_q + \frac{1}{3}\cdot\langle d(i)\rangle_q\right] - q\cdot\langle d(i)\rangle_q + [\langle d(i)\rangle_q - \langle d(i-1)\rangle_q] = \\
&\quad = q\cdot\left[\langle A(i+1)\rangle_q - \langle d(i)\rangle_q\right] + \langle E(i)\rangle_q.
\end{aligned} \tag{4.33}$$

$$\langle d(i)\rangle_q - \langle d(i-1)\rangle_q = \langle E(i)\rangle_q \rightarrow Elementary\ number\ between\ steps\ of\ iterations \tag{4.34}$$

Next, the value (4.33) is distributed across the scales.

The remainder (4.34) stay on a present scale (i) and

the body of the number is transferred to the next scale, accompanied by a decrease (division) of its value in the base q-system.

Transfer to scale $(i+1)$

Next, we implement a parallel transfer of the numbers from scale i to scale $i+1$:

$$\begin{aligned}
&\left[\langle A(i)\rangle_q + \frac{1}{3}\cdot\langle d(i)\rangle_q\right] - q\cdot\langle d(i)\rangle_q = \\
&\quad = q\cdot F(\langle A(i)\rangle_q) - q\cdot\langle d(i)\rangle_q = q\cdot\left[\langle A(i+1)\rangle_q) - \langle d(i)\rangle_q\right].
\end{aligned} \tag{4.35}$$

The transition to scale (step) $(i+1)$ is given by

$$\langle A(i+1)\rangle_q - \langle d(i)\rangle_q \rightarrow \text{Start next iteration.} \tag{4.36}$$

Clearly, during iteration i on scale i, there remains a difference. This difference is an elementary number $\langle E(i)\rangle_q$.

□

4.3.3 Research of divisor $\langle d(i)\rangle_q$

On the one hand, we have

$$\langle A(i)\rangle_q = \sum_{i=s}^{p} \gamma(i)\cdot q^i; \tag{4.37}$$

and on the other hand,

$$\langle A(i)\rangle_q = N(\langle A(i)\rangle_q) d(\langle A(i)\rangle_q)$$
$$d(\langle A(i)\rangle_q) = 2^{b(i)}. \tag{4.38}$$

This result is possible only when the divisor of the last digit $\gamma(s)\cdot q^s$

also divides a divisor $d(\langle A(i)\rangle_q)$.

The divisor of $\gamma(s)\cdot q^s$ is given by

$$d(\gamma(s)\cdot q^s) = \left(\begin{cases} 1, & if\ \gamma(s) = \{\frac{1}{3};\ \frac{3}{3}\}\ or\ \gamma(s1) = 1 \\ 2, & if\ \gamma(s) = \frac{2}{3}\ or\ \gamma(s1) = 0, \end{cases} \right) \cdot 4^s \tag{4.39}$$

and therefore

$$\langle d(\langle A(i)\rangle_q)\rangle_q \geq \langle d(\gamma(s)\cdot q^s)\rangle_q. \tag{4.40}$$

The ratio of potential dividers is given by

$$log_4\langle d(\langle A(i)\rangle_q)\rangle_q \geq log_4\langle d(\gamma(s)\cdot q^s)\rangle_q. \tag{4.41}$$

Note, that the bottom frame (see subsec.[5.1]) of the potential of the divider is considered as the potential of its last half-digit $\gamma(s2), \gamma(s1)$.

4.4 Conversion of potential of the number $\langle A\rangle_q$ during iterations F

Theorem 4.11. *The transformation F reduces the potential of the number $\langle A\rangle_q \in \langle \mathbf{W}\rangle_q$ at each iteration.*

Proof. Before the next iteration of F, the potential is given by

$$P(\langle A(i)\rangle_q) = log_q(\langle A(i)\rangle_q). \tag{4.42}$$

After the iteration, it becomes

$$P(F(\langle A(i)\rangle_q)) = log_q(F(\langle A(i)\rangle_q)). \tag{4.43}$$

The potentials before and after the transformation are separated by the following interval:

$$\Delta P\left(\frac{F(\langle A(i)\rangle_q)}{\langle A(i)\rangle_q}\right) = P(F(\langle A(i)\rangle_q)) - P(\langle A(i)\rangle_q). \tag{4.44}$$

Specifically, we have

$$\begin{aligned}
&\Delta P\left(\frac{F(\langle A(i)\rangle_q)}{\langle A(i)\rangle_q}\right) = \\
&log_q(F(\langle A(i)\rangle_q)) - log_q(\langle A(i)\rangle_q) = \\
&log_q\left(\frac{F(\langle A(i)\rangle_q)}{\langle A(i)\rangle_q}\right) = \\
&log_q\left(\frac{3}{4} + \frac{1}{4}\cdot\frac{d(\langle A(i)\rangle_q)}{\langle A(i)\rangle_q}\right) = \\
&= log_q\left(\frac{3}{4} + \frac{1}{4}\cdot\frac{1}{N(\langle A(i)\rangle_q)}\right).
\end{aligned} \tag{4.45}$$

As $N(\langle A(i)\rangle_q) \geq 3$, it follows that for any $N(\langle A(i)\rangle_q)$, $\Delta P < 0$.

□

According to the above proof, the potential value decreases in each iteration, and eventually approaches its limit −1.

5 Length of the number

5.1 Potentials and the length of $\langle A(i)\rangle_q$ in $\langle \mathbf{W}\rangle_q$

We will enclose the "body" of the number in a frame (roof frame and bottom frame) and consider the behavior of the body of the number and its frame when converting F, the transformation of the potential of the number should be considered simultaneously with the change in the potential of the divisor.

Definition 5.1. Define by

$l(\langle A(i)\rangle_q) = P(\langle A(i)\rangle_q) - P(d(\gamma(s)4^s))$

the length is quantity (interval) of scales, on which distribute potential of a number

$\langle A(i)\rangle_q$ in the form of significant digits.

The potential (roof frame) $\langle A(i)\rangle_q$ is determined under the conditions of subsect.[4.1.2] and [4.2]:

$$\begin{aligned} \langle A(i)\rangle_q &> \gamma(p)\cdot q^p, \\ log_q(\langle A(i)\rangle_q) &> log_q(\gamma(p)\cdot q^p) = log_q(\gamma(p)) + p, \\ \gamma(p) &< q \\ log_q\gamma(p) &< 0 \\ log_q(\langle A(i)\rangle_q) &\geq p-3. \end{aligned} \tag{5.1}$$

The value of the potential $P(\langle A(i)\rangle_q)$ (4.11) is then determined as

$$P(\langle A(i)\rangle_q) = log_q(\langle A(i)\rangle_q). \tag{5.2}$$

Under the conditions of Section 4.2, the potential $P(d(\gamma(s)4^s))$ (bottom frame) of the last significant digit is determined as $log_4 d(\langle\gamma(s)\cdot 4^s\rangle_q)$

$$\begin{aligned} d(\langle A(i)\rangle_q) &> d(\gamma(s)4^s) = d(\gamma(s))4^s, \\ log_4(d(\langle A(i)\rangle_q)) &> log_4(d(\gamma(s)4^s)) = log_4 d(\gamma(s)) + s, \\ \gamma(s) &< q, \\ log_4 d(\gamma(s))) &= \{0, -\frac{1}{2}\}(4.13). \end{aligned} \tag{5.3}$$

$P(d(\gamma(s)4^s))$ is determined as

$$P(d(\gamma(s)4^s)) = log_4(d(\gamma(s)4^s)). \tag{5.4}$$

The length of the number is write as

$$l(\langle A(i)\rangle_q) = log_q(\langle A(i)\rangle_q) - log_4(d(\gamma(s)4^s)). \tag{5.5}$$

5.2 Length of the number in $\langle \mathbf{W}\rangle_q$ during F

1. In the implementation of F, the quantity of divisor of number becomes even.

2. The divisor of number (before F) contains the last half-digit of number.

3. Multiplication of the last half-digit an even number of times at least doubles it.

4. It says about, that the action of F can show its result only inside the digits of number, or at the level of a half-digits.

5. Therefore, that made compare consecutive iterations along each other in length it is need to take into account these results and the length of number calculate as quantity of significant half-digits using the enumeration, or by introducing the coefficient 2 into the formula of length.

6. To preserve their(half-digits) true quantity, which will be forming (after F) divider $d(\langle A(i+1)\rangle_q)$, we introduce a coefficient of 2 into the length formula.

Definition 5.2. The "big length" L is quantity (half-interval) of half-scales, on which distribute potential of number $\langle A(i)\rangle_q$ in the form of
"significant half-digits" (4.10) and had equation:

$L(\langle A(i)\rangle_q) = 2\cdot l(\langle A(i)\rangle_q) = 2\cdot[log_q(\langle A(i)\rangle_q) - log_4(d(\gamma(s)4^s))]$.

Similarly to (4.44), we consider the length change of a number in the transformation F.

Without loss of substance, we can omit the iteration number (i) and the number system $\langle \mathbf{W} \rangle_q$ when converting the length of a number between iterations.

Before an iteration of F, the length is given by

$$L(A) = 2 \cdot log_q(A) - 2 \cdot log_4(d(\gamma(s)) \cdot 4^s). \tag{5.6}$$

After the iteration, it becomes

$$L(F(A)) = 2 \cdot log_q(F(A)) - 2 \cdot log_4(d(F(\gamma(s) \cdot 4^s))). \tag{5.7}$$

5.2.1 The interval between the lengths $\Delta L\left(\frac{F(A)}{A}\right)$ before and after a transformation is given by

Definition 5.3. $\Delta L\left(\frac{F(A)}{A}\right) = L(F(A)) - L(A)$.

Specifically,

$$\begin{gathered}\Delta L\left(\frac{F(A)}{A}\right) = L(F(A)) - L(A) = \\ [2 \cdot log_q(F(A)) - 2 \cdot log_4(d(F(\gamma(s) \cdot 4^s)))] - \\ [2 \cdot log_q(A) - 2 \cdot log_4(d(\gamma(s)) \cdot 4^s)] = \\ 2 \cdot [log_q(F(A)) - log_q(A)] - \\ 2 \cdot [log_4(d(F(\gamma(s) \cdot 4^s))) - log_4(d(\gamma(s)) \cdot 4^s)] = \\ 2 \cdot \log_q\left(\frac{F(A)}{A}\right) - 2 \cdot log_4\left(\frac{d(F(\gamma(s) \cdot 4^s))}{d(\gamma(s)) \cdot 4^s)}\right).\end{gathered} \tag{5.8}$$

The roof frame of the number then shifts as follows:

$$\Delta L\left(\frac{P(F(A))}{P(A)}\right) = 2 \cdot \log_q\left(\frac{F(A)}{A}\right). \tag{5.9}$$

Accordingly, the bottom frame of the number shifts as

$$\Delta L\left(\frac{P(d(F(\gamma(s) \cdot 4^s)))}{P(d(\gamma(s)) \cdot 4^s))}\right) = 2 \cdot log_4\left(\frac{d(F(\gamma(s) \cdot 4^s))}{d(\gamma(s)) \cdot 4^s)}\right). \tag{5.10}$$

The length change of the "body" of the number between iterations is measured as the difference lengths between the potentials of number and potential of last half-digit before and after transformation F.

5.2.2 Researching $\Delta L\left(\frac{P(d(F(A)))}{P(d(A))}\right)$

Theorem 5.4. *The recursively transformation F increases the length of the number by one half-digit in the bottom frame of the number, if the number consists of $4k+3$ divisors.*

Proof.

$$\Delta L\left(\frac{P(d(F(\gamma(s)\cdot 4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right) = 2\cdot log_4\left(\frac{d(F(\gamma(s)\cdot 4^s))}{d(\gamma(s))\cdot 4^s)}\right) = log_4\left(\frac{d(F(\gamma(s))\cdot 4^s))}{d(\gamma(s))\cdot 4^s))}\right)^2. \quad (5.11)$$

- Properties $\Delta L\left(\frac{P(d(F(\gamma(s)\cdot 4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right)$

When $N(A(i))$ is $4k(i)+3$ and $k(i)$ is even, the transformation F obtains $n(A(i)) = 12k(i)+10 = 2(6k(i)+5)$, and when $N(A(i+1)) = 4k(i+1)+1$, it gives $n(A(i+1)) = 12k(i+1)+4 = 4(3k(i+1)+1)$.

$$\left(\frac{d(F(\gamma(s)\cdot 4^s))}{d(\gamma(s))\cdot 4^s)}\right)^2 = \begin{cases} \frac{1}{4}, N(A) = 4k+3, k\in N \\ 1, N(A) = 4k+1 \\ 4, N(A) = \frac{8k-1}{3} \\ 16, N(A) \\ 64, N(A) \\ \cdots, n(A). \end{cases} \quad (5.12)$$

$$log_4\left(\frac{d(F(\gamma(s)\cdot 4^s))}{d(\gamma(s))\cdot 4^s)}\right)^2 = \begin{cases} -1, N(A) = 4k+3, k\in N \\ 0, N(A) = 4k+1 \\ 1, N(A) = \frac{8k-1}{3} \\ 2, N(A) \\ 4, N(A) \\ \cdots, n(A). \end{cases} \quad (5.13)$$

$$\Delta L\left(\frac{P(d(F(\gamma(s)\cdot 4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right) = \begin{cases} -1, N(A) = 4k+3, k\in N \\ 0, N(A) = 4k+1 \\ 1, N(A) = \frac{8k-1}{3} \\ 2, N(A) \\ 4, N(A) \\ \cdots, N(A). \end{cases} \quad (5.14)$$

A value of -1 indicates an increase in the bottom frame part of the number by one half-digit □

5.2.3 Researching $\Delta L\left(\frac{P(F(A))}{P(A)}\right)$

Theorem 5.5. *The recursive transformation F decreases the length of the significant part of the number in the start part of the number by at least one significant half-digit. That is*

$P(F(A)) < P(A) - 1.$

Proof. According theorem (4.11)

$$\Delta L\left(\frac{P(F(A))}{P(A)}\right) = 2 \cdot \log_q\left(\frac{F(A)}{A}\right) = 2 \cdot log_q\left(\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{N(A)}\right). \tag{5.15}$$

The proof holds because

$$log_q\left(\frac{3}{4} + \frac{1}{4} \cdot \frac{1)}{N(A)}\right) > -1.$$

As N(A) is odd, it holds that

$$log_q\left(\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{N(A)}\right) < log_q\left(\frac{3}{4} + \frac{1}{4 \cdot 3}\right) = log_q(5/6) = -0,63, \tag{5.16}$$

and hence

$$-0,63 > log_q\left(\frac{3}{4} + \frac{1}{4 \cdot 3}\right) > -1$$

$$-1,2 > \Delta L\left(\frac{P(F(A))}{P(A)}\right) > -2.$$

□

Since, the length change of the ”body” of the number between iterations is measured as the difference lengths between the potentials of number and potential of last half-digit before and after transformation F, we write it and **substitute** into it of the possible border’s values(levels):

$$\Delta L\left(\frac{F(A)}{A}\right) = \Delta L\left(\frac{P(F(A))}{P(A)}\right) - \Delta L\left(\frac{P(d(F(\gamma(s) \cdot 4^s)))}{P(d(\gamma(s)) \cdot 4^s))}\right) =$$

$$\begin{cases} -1 \\ -2 \end{cases} - \begin{cases} -1 \\ 0 \\ 1 \\ 2 \\ \dots \end{cases} = \begin{cases} < 0 \\ -1 \\ -2 \\ -3 \\ \dots \end{cases} \tag{5.17}$$

In (5.17), the value of $\Delta L\left(\frac{F(A)}{A}\right)$ may decrease below −1 or be from −1 to 0. The second condition corresponds to storing the length of the ”body” of the number A after the conversion $F(A)$, because the length in this case changes by less of one half-digit or say another: the length was stored.

-

5.3 Possible conditions for $\Delta L\left(\frac{F(A)}{A}\right)$

Consider the conditions, when $\Delta L\left(\frac{P(d(F(\gamma(s)\ cdot4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right)$ is equal to -1, which leads to the preservation of length, and 0, which already according to (5.17) corresponds to the condition of decreasing length during conversion.

In other cases, when $\Delta L\left(\frac{P(d(F(\gamma(s)\ cdot4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right) > 0$, we omit , since the consideration will be similar to the case when $\Delta L\left(\frac{P(d(F(\gamma(s)\ cdot4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right) = 0$.

Theorem 5.6. *Condition of necessity: To maintain the length of the number during the recursive transformation F, the number of divisors must equal $4k+3$.*

Proof. $\Delta L\left(\frac{P(d(F(\gamma(s)\cdot 4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right)$ **equals 0:** then

$$\begin{aligned} N(A) &= 4\cdot k+1 \\ k &\in N \\ &\text{then} \\ log_q\left(\frac{3}{4}+\frac{1}{4}\cdot\frac{1}{4k+1}\right) < log_q\left(\frac{3}{4}+\frac{1}{20}\right) &= \\ log_q\left(0,8\right) &\approx -0,775 \\ &\mathit{and} \\ \Delta L\left(\frac{F(A)}{A}\right) < 2\cdot(-0,75)+0 &< -1. \end{aligned} \tag{5.18}$$

That is, the length of the number is reduced

$\Delta L\left(\frac{P(d(F(\gamma(s)\cdot 4^s)))}{P(d(\gamma(s))\cdot 4^s))}\right)$ **equals -1 then**

$$\begin{aligned} N(A) &= 4\cdot k+3 \\ k &\in N \\ &\textbf{then} \\ log_q\left(\frac{3}{4}+\frac{1}{4}\cdot\frac{1}{4k+3}\right) < log_q\left(\frac{3}{4}+\frac{1}{12}\right) &= \\ log_q\left(\frac{5}{6}\right) &\approx -0.633 \\ &\mathit{and} \\ -1 < \Delta L\left(\frac{F(A)}{A}\right) \le 2\cdot(-0,633)+1 < 0 &\ \textbf{approaches}\ 0. \end{aligned} \tag{5.19}$$

That is, unlike the potential, the number length after an iteration may not be further reduced if it approaches 0.

□

This fact will be investigated further below.

5.4 Not- decrease of $\Delta L\left(\frac{F(A)}{A}\right)$ during a conversion F

Theorem 5.7. *Condition of necessity: To conserve the length of a number during the recursive transformation F, the number of divisors in successive iterations must be $4k+3$. Otherwise, it is proven that $\Delta L\left(\frac{F(A)}{A}\right)$ periodically falls below -1, meaning that the number of half-digits in the number $\langle A(i)\rangle_q$ decreases by one.*

Proof. Only when $N(A) = 4 \cdot k + 3$, we have $\frac{d(F(\gamma(s) \cdot 4^s))}{d(\gamma(s)) \cdot 4^s)} = \frac{1}{2}$, meaning that $\Delta L\left(\frac{F(A)}{A}\right)$ approaches 0:

Under condition 0,

we have

$$N(A(i)) = 4 \cdot k(i) + 3. \text{ Then}$$
$$\frac{d(F(\gamma(s) \cdot 4^s))}{d(\gamma(s)) \cdot 4^s)} = \frac{d(i+1)}{d(i)} = \frac{1}{2}, \text{ and}$$
$$A(i+1) = N(i+1) \cdot d(i+1) = \frac{1}{2} \cdot d(i) \cdot (6 \cdot k(i) + 5), \quad (5.20)$$

Under condition 1, the $4k+3$ property is preserved

If

$$6 \cdot k(i) + 5 = 4 \cdot k(i+1) + 3,$$

then

$$\text{k(i) is } \textbf{odd} \quad (5.21)$$

Under condition 2, the $4k+3$ property is violated,when the new value of $k(i)$ is even.

If

k(i) is **even !!!**

then

$$6 \cdot k(i) + 5 = 4 \cdot k(i+1) + 1 \quad (5.22)$$

The $4k+3$ property is preserved at each iteration if and only if the new value of $k(i+1)$ is odd (this completes the proof) □

5.4.1 Limit of the $4k+3$ property

If conditions 0 are fulfilled once,

then

$$N(A(i)) = 4 \cdot k(i) + 3,$$

and

$$k(i) \text{ is } \textbf{even}. \quad (5.23)$$

If conditions 0 are fulfilled twice,

then

$$N(A(i)) = 8 \cdot k(i) + 7,$$

and

$$k(i) \text{ is } \textbf{even}. \quad (5.24)$$

Theorem 5.8. *Condition of necessity: The length of a number during p recursive transformation F is preserved if and only if the number of divisors in successive iterations is* $2^{p-1} \cdot 4 \cdot k(i) + (2^{p-1} - 1)$.

Proof. The proof is obtained by induction.

When conditions 0 and 1 are fulfilled p times,

then

$$N(A(i)) = 2^{p-1} \cdot 4 \cdot k(i) + (2^{p-1} - 1),$$

and

$$k(i) \text{ is } \textbf{even}. \quad (5.25)$$

□

Some examples are given below:

For

$$175 = 160 + 15 = 16 \cdot 10 + 15,$$

conditions 0 and 1 are fulfilled three times, because $15 = 2^4 - 1$:

$$175 = 4 \cdot 43 + 3 \Rightarrow 263,$$

$$263 = 4 \cdot 65 + 3 \Rightarrow 395,$$

$$395 = 4 \cdot 98 + 1 \Rightarrow \mathbf{593 = 4 \cdot 98 + 1}.$$

For

$$287 = 256 + 31 = 32 \cdot 8 + 31$$

conditions 0 and 1 are fulfilled four times, because $31 = 2^5 - 1$:

$$287 = 4 \cdot 71 + 3 \Rightarrow 431,$$

$$431 = 4 \cdot 107 + 3 \Rightarrow 647,$$

$$647 = 4 \cdot 161 + 3 \Rightarrow 971,$$

$$971 = 4 \cdot 242 + 1 \Rightarrow \mathbf{971 = 4 \cdot 242 + 1}.$$

6 Proof of necessary conditions for the implementation of the Collatz transformations.

6.1 Proof of decreasing length of a number

Lemma 6.1. *In a transformation F, the number of iterations i during which the length of a number $A(i)$ decreases by one half-digit is limited to $P(i)$.*

Proof. Since, for any $N(A(i)) \in N$ there exists $P(i) \in N$ when

$$N(A(i)) \le 2^{P(i)-1} - 1 \tag{6.1}$$

, follows that in less than $P(i)$ steps

$$\Delta L\left(\frac{\overbrace{F(\cdots F(A)\cdots)}^{P(i)}}{A}\right) < -1. \tag{6.2}$$

(overbrace is designation of the application of the operation F many times until the number decreases by 1 in length).

Note, that the quantity of retention(holding) in the corresponding iteration $4k+3$ cannot exceed $P(i)$, since in the extreme case when $N(A(i)) = 2^{P(i)-1} - 1$ will be reduced to a unit length for $P(i)$ steps, since number 1 is number of the form $4k+1$.

is proven □

Theorem 6.2. *The necessary conditions for the implementation of the Collatz transformations.*

Within a limited number of steps, the iterative transformation F transforms an arbitrary natural number $A \in \mathbf{N}$ to a number of unit length $\frac{2^N}{3^Q}$.

Proof. In the Horner scheme, any arbitrary number $A \in \mathbf{N}$ can be represented in $\mathbf{W}_q$.

Therefore, an arbitrary number can be represented as (4.3),

and will contain a limited quantity n of digits in the record, along with quantity half-digits $2n$.

By Lemma (6.1), the length of the number $A(i)$ after a limited quantity $P(i)$ of a transformation F is reduced, as minimum, by one half-digit.

Follow, the quantity P of iterations F required to transform the number A to a number of unit length is calculated as

$$P = \sum_{i=0}^{2\cdot n} P(i) \leq 2 \cdot n * \max P(i) \tag{6.3}$$

The quantity P of iterations F required to transform the number $A \in \mathbf{N}$ to a number of unit- length $\frac{2^{N(A)}}{3^{Q(A)}}$ is **limited**.

This observation remains to be proven.

□

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6.2 On the v-expansions of integer numbers in a rational base $\frac{2^a}{3}$

Theorem 6.3.

For $\forall A \in \mathbf{N}$ there exists an v-expansion A_v in $\mathbf{V_v}$.

Proof. In each transformation $F(A(i))$, there arises an elementary number $E(i)$ that equals the difference between two divisors.

When the conversion is completed, the initial number $A_q \in \mathbf{W}_q$ has become a number $A_v \in \mathbf{V}_v$.

This representation of a number in the system is here called a $\frac{2^a}{3}$-expansion.

Since for $\forall A \in \mathbf{N}$ there exists an q-expansion $A_q \in \mathbf{W}_q$, and $\forall A_q \in \mathbf{W_q}$ there exists an v-expansion it turns out that for $\forall A \in \mathbf{N}$ there exists an v-expansion $A_v \in \mathbf{V}_v$.

This was required to prove to obtain the necessary conditions.

QED □

Thus, sufficient and necessary conditions are obtained for the CC, which leads to its proof.

As a demonstration of the CC mechanism, we present the result of the number 27.

Here, we chose 27 [19] because it is often used to popularize the conjecture.

The mechanism is contrasted with that of Figure [5] in the decimal number system.

Conversion algorithm can see in figure[6]

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i	n(i)	r(i)	-r(i)>1	4k(i)+1	4k(i)+3		k(i)	P(i)
0	**27**	-1			4*6+3	14+13	Even	1
1	41	-2	2	4*10 + 1				
2	31	-1			4*7+3	16+15	Odd	4
3	47	-1			4*11+3		Odd	
4	71	-1			4*17+3		Odd	
5	107	-1			4*26+3		Even	
6	161	-2	2	4*40+1				
7	121	-2	2	4*30+1				
8	91	-1			4*22+3	46+45	Even	1
9	137	-2	2	4*34+1				
10	103	-1			4*25+3	52+51	Odd	2
11	155	-1			4*38+3		Even	
12	233	-2	2	4*58+1				
13	175	-1			4*43+3	88+87	Odd	3
14	263	-1			4*65+3		Odd	
15	395	-1			4*98+3		Even	
16	593	-2	2	4*148+1				
17	445	-3	3	4*111+1				
18	167	-1			4*41+3	84+83	Odd	2
19	251	-1			4*62+3		Even	
20	377	-2	2	4*94+1				
21	283	-1			4*70+3	213+212	Even	1
22	425	-2	2	4*106+1				
23	319	-1			4*79+3	160+159	Odd	5
24	479	-1			4*119+3		Odd	
25	719	-1			4*179+3		Odd	
26	1079	-1			4*269+3		Odd	
27	1619	-1			4*404+3		Even	
28	2429	-3	3	4*607+1				
29	911	-1			4*227+3	456+455	Odd	3
30	1367	-1			4*341+3		Odd	
31	2051	-1			4*512+3		Even	
32	3077	-4	4	4*769+1				
33	577	-2	2	4*144+1				
34	433	-2	2	4*108+1				
35	325	-4	4	4*81+1				
36	61	-3	3	4*15+1				
37	23	-1			4*5+3	12+11	Odd	2
38	35	-1			4*8+3		Even	
39	53	-5	5	4*13+1				
40	5	-4	4	4*1+1				
41	1	0						
	Balance	-70	46					24

Figure 5: Example A=27

1

step 1

Conversion of A_{10} to A_q

$$A_{10} = A_q \quad (0.1)$$

step 2: Iterate F

step 3

If

$$N(A_q(i)) = 4 \cdot k(i) + 1,$$

then

$$A_q(i+1) \text{ —— decrease the number of half-digits} \quad (0.2)$$

step 4

If

$$N(A_q(i)) = 4 \cdot k(i) + 3,$$

then

$$A_q(i+1) \text{ —— do not decrease the number of half-digits} \quad (0.3)$$

step 5 If

$A(i) = 2^a \cdot q^b,$

where $a, b \in Z$

then **terminate the computation** (0.4)

return to step 2

end algorithm

Figure 6: Conversion algorithm

8 Afterword

Below we list the defining properties of prime numbers:

1) A prime has no divisors

2) A prime is represented in the set $\langle A \rangle_q$.

3) A non-prime number can be unpacked into its dividers with the following representation:

$\langle A \rangle_v = \langle B \rangle_v * \langle C \rangle_v;$

$\langle A \rangle_v, \langle B \rangle_v, \langle C \rangle_v \in N.$

4) A prime number satisfies $\langle A \rangle_v = \langle B \rangle_v$ and $\langle C \rangle_v = \{2,0\}_v = 1_{10}$, so can be decomposed by an"arithmetic" in the $\mathbf{V}_v$ system.

Perception of a unit as both a scale of quantity and a scale of number length will likely provide the answer to the famous phrase by Paul Erdos: "[Mathematics may not be ready for such problems.]".

Figure [7] Harmony unifies opposites and connects entities in space. From Pythagorean

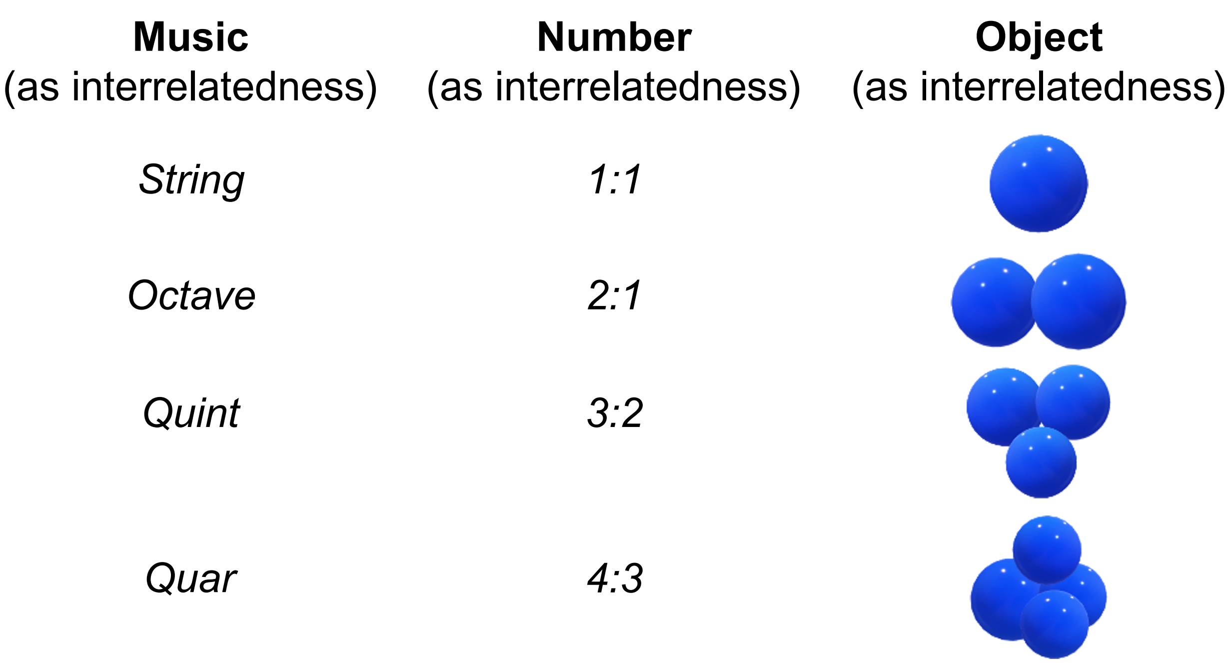

Figure 7: tetrada

theory, consonant intervals are given by $2:1 = (3:2)\cdot(4:3)$. [16]

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List of figures.