The Riemann hypothesis

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Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln \ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$ when $15 \nmid n$, where $15 \nmid n$ means that n is not divisible by 15. More specifically: every counterexample should be divisible by $2^{20} \times 3^{13} \times 5^{8} \times k_1$ or either $2^{20} \times 3^{13} \times k_2$ or $2^{20} \times 5^{8} \times k_3$, where k_1, k_2 and k_3 are not equal to 7 and $15 \nmid k_1$, $3 \nmid k_2$ and $5 \nmid k_3$.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [1]. The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n,

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^{\gamma} \times n \times \ln \ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [3]. Using this inequality, we show an interesting result.

2010 Mathematics Subject Classification: Primary 11M26.
Keywords: number theory, inequality, divisor, prime, counterexample.

2 Results

Theorem 2.1 *Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following inequality*

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

Proof For a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following formula

$$\sigma(n) = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i - 1} \tag{2.1}$$

from the Ramanujan's notebooks [2]. In this way, we have that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)}. \tag{2.2}$$

However, for any prime power $p_i^{a_i}$, we have that

$$\frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)} < \frac{p_i^{a_i+1}}{p_i^{a_i} \times (p_i - 1)} = \frac{p_i}{p_i - 1}.$$

Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

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Theorem 2.2 *Given some prime numbers $p_1, p_2, \ldots, p_m$, then we obtain the following inequality,*

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Proof Given a prime number p_i, we obtain that

$$\frac{p_i}{p_i - 1} = \frac{p_i^2}{p_i^2 - p_i}$$

and that would be equivalent to

$$\frac{p_i^2}{p_i^2 - p_i} = \frac{p_i^2}{p_i^2 - 1 - (p_i - 1)}$$

and that is the same as

$$\frac{p_i^2}{p_i^2 - 1 - (p_i - 1)} = \frac{p_i^2}{(p_i - 1) \times (\frac{p_i^2 - 1}{(p_i - 1)} - 1)}$$

which is equal to

$$\frac{p_i^2}{(p_i-1)\times(\frac{p_i^2-1}{(p_i-1)}-1)} = \frac{p_i^2}{(p_i-1)\times\frac{p_i^2-1}{(p_i-1)}\times(1-\frac{(p_i-1)}{p_i^2-1})}$$

that is equivalent to

$$\frac{p_i^2}{(p_i-1)\times\frac{p_i^2-1}{(p_i-1)}\times(1-\frac{(p_i-1)}{p_i^2-1})} = \frac{p_i^2}{p_i^2-1}\times\frac{1}{1-\frac{(p_i-1)}{p_i^2-1}}$$

which is the same as

$$\frac{p_i^2}{p_i^2-1}\times\frac{1}{1-\frac{(p_i-1)}{p_i^2-1}} = \frac{1}{1-p_i^{-2}}\times\frac{1}{1-\frac{1}{(p_i+1)}}$$

and finally

$$\frac{1}{(1-p_i^{-2})}\times\frac{1}{1-\frac{1}{(p_i+1)}} = \frac{1}{(1-p_i^{-2})}\times\frac{p_i+1}{p_i}.$$

In this way, we have that

$$\prod_{i=1}^{m}\frac{p_i}{p_i-1} = \prod_{i=1}^{m}\frac{1}{1-p_i^{-2}}\times\prod_{i=1}^{m}\frac{p_i+1}{p_i}.$$

However, we know that

$$\prod_{i=1}^{m}\frac{1}{1-p_i^{-2}} < \prod_{j=1}^{\infty}\frac{1}{1-p_j^{-2}}$$

where p_j is the j^{th} prime number and we have that

$$\prod_{j=1}^{\infty}\frac{1}{1-p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [5]. Consequently, we obtain that

$$\prod_{i=1}^{m}\frac{p_i}{p_i-1} < \frac{\pi^2}{6}\times\prod_{i=1}^{m}\frac{p_i+1}{p_i}.$$

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Definition 2.3 We recall that an integer n is said to be squarefree if for every prime divisor p of n we have $p^2 \nmid n$, where $p^2 \nmid n$ means that p^2 does not divide n [3].

Theorem 2.4 *Given a squarefree number $n = q_1 \times \ldots \times q_m$ such that $q_1, q_2, \ldots, q_m$ are odd prime numbers, $3 \nmid n$, $5 \nmid n$ and the greatest prime divisor of n is greater than 7, then we obtain the following inequality*

$$\frac{\pi^2}{6}\times\frac{3}{2}\times\sigma(n) \leq e^{\gamma}\times n\times\ln\ln(2\times n).$$

Proof This proof is very similar with the demonstration in Theorem 1.1 from the article reference [3]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From the equation (2.1), we obtain that

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \ldots \times (q_m + 1) \tag{2.3}$$

when $n = q_1 \times q_2 \times \ldots \times q_m$. In this way, for any prime number $p_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{p_i}) \leq e^{\gamma} \times \ln \ln(2 \times p_i). \tag{2.4}$$

For $p_i = 11$, we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \ln \ln(22)$$

is actually true. For another prime number $p_i > 11$, we have that

$$(1 + \frac{1}{p_i}) < (1 + \frac{1}{11})$$

and

$$e^{\gamma} \times \ln \ln(22) < e^{\gamma} \times \ln \ln(2 \times p_i)$$

which clearly implies that the inequality (2.4) is true for every prime number $p_i \geq 11$. Now, suppose it is true for $m-1$, with $m \geq 1$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [3]. So let $n = q_1 \times \ldots \times q_m$ be a squarefree number and assume that $q_1 < \ldots < q_m$ for $q_m > 7$.

Case 1 :$q_m \geq \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2 \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times \ln \ln(2 \times q_1 q_1 \times \ldots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$
$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2 \times q_1 \times \ldots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show that

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2 \times q_1 \times \ldots \times q_{m-1}) \leq$$
$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times q_m \times \ln \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \ln \ln(2 \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln \ln(2 \times q_1 \times \ldots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln \ln(2 \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} \geq$$

$$\frac{\ln\ln(2\times q_1\times\ldots\times q_{m-1})}{\ln q_m}.$$

From the reference [3], we have that if $0 < a < b$, then

$$\frac{\ln b - \ln a}{b-a} = \frac{1}{(b-a)}\int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.5}$$

We can apply the inequality (2.5) to the previous one just using $b = \ln(2\times q_1\times\ldots\times q_{m-1}\times q_m)$ and $a = \ln(2\times q_1\times\ldots\times q_{m-1})$. Certainly, we have that

$$\ln(2\times q_1\times\ldots\times q_{m-1}\times q_m) - \ln(2\times q_1\times\ldots\times q_{m-1}) =$$

$$\ln\frac{2\times q_1\times\ldots\times q_{m-1}\times q_m}{2\times q_1\times\ldots\times q_{m-1}} = \ln q_m.$$

In this way, we obtain that

$$\frac{q_m\times(\ln\ln(2\times q_1\times\ldots\times q_{m-1}\times q_m) - \ln\ln(2\times q_1\times\ldots\times q_{m-1}))}{\ln q_m} >$$

$$\frac{q_m}{\ln(2\times q_1\times\ldots\times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2\times q_1\times\ldots\times q_m)} \geq \frac{\ln\ln(2\times q_1\times\ldots\times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2\times q_1\times\ldots\times q_{m-1}\times q_m)$ [3].

Case 2 :$q_m < \ln(2\times q_1\times\ldots\times q_{m-1}\times q_m) = \ln(2\times n)$.

We need to prove

$$\frac{\pi^2}{6}\times\frac{3}{2}\times\frac{\sigma(n)}{n} \leq e^\gamma\times\ln\ln(2\times n).$$

We know that $\frac{3}{2} < 1.6 = \frac{4\times 6}{3\times 5}$. Nevertheless, we could have that

$$\frac{3}{2}\times\frac{\sigma(n)}{n}\times\frac{\pi^2}{6} < \frac{4\times 6\times\sigma(n)}{3\times 5\times n}\times\frac{\pi^2}{6} = \frac{\sigma(3\times 5\times n)}{3\times 5\times n}\times\frac{\pi^2}{6} \leq e^\gamma\times\ln\ln(2\times n)$$

where this is possible because of $3\nmid n$ and $5\nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\ln(\frac{\pi^2}{6})+(\ln(3+1)-\ln 3)+(\ln(5+1)-\ln 5)+\sum_{j=i}^{m}(\ln(q_j+1)-\ln q_j) \leq \gamma+\ln\ln\ln(2\times n).$$

From the reference [3], we note that

$$\ln(p_1+1)-\ln p_1 = \int_{p_1}^{p_1+1}\frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note also that $\ln(\frac{\pi^2}{6}) < \frac{1}{2}$. In order to prove this, it is enough to prove that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{q_1}+\ldots+\frac{1}{q_m} \leq \sum_{p\leq q_m}\frac{1}{p} + \leq \gamma+\ln\ln\ln(2\times n)$$

where $p \leq q_m$ means all the prime lesser than or equal to q_m. However, we know that

$$\gamma + \ln\ln q_m < \gamma + \ln\ln\ln(2 \times n)$$

since $q_m < \ln(2 \times n)$ and therefore, we would only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln q_m$$

which is true according to the Lemma 2.1 from the article reference [3]. In this way, we finally show the Theorem is indeed satisfied. ∎

Theorem 2.5 *Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then the Robin's inequality is true for n.*

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that would be the same as

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < e^{\gamma} \times \ln\ln n \tag{2.6}$$

according to Theorem 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln\ln(5040) \approx 3.81.$$

However, we know for $n > 5040$, we have that

$$e^{\gamma} \times \ln\ln(5040) < e^{\gamma} \times \ln\ln n$$

and thus, the proof is completed for that case. Hence, we only need to prove for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $7^k \mid n$ for $1 \leq k \leq 6$ [4] (this article has been published in the journal Integers in the volume 18). Therefore, we need to prove this case for those natural numbers n such that $7^7 \mid n$. In this way, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \ln\ln(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$, we have that

$$e^{\gamma} \times \ln\ln(7^7) \leq e^{\gamma} \times \ln\ln n$$

and thus, the proof is completed. ∎

Theorem 2.6 *The Robin's inequality is true for every natural number* $n > 5040$ *when* $15 \nmid n$*. More specifically: every counterexample should be divisible by* $2^{20} \times 3^{13} \times 5^{8} \times k_1$ *or either* $2^{20} \times 3^{13} \times k_2$ *or* $2^{20} \times 5^{8} \times k_3$*, where* k_1*,* k_2 *and* k_3 *are not equal to* 7 *and* $15 \nmid k_1$*,* $3 \nmid k_2$ *and* $5 \nmid k_3$*.*

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we will check the Robin's inequality for n. We know this true when the greatest prime divisor of n is lesser than or equal to 7 according to Theorem 2.5. Another case is when the greatest prime divisor of n is greater than 7, $3 \nmid n$ and $5 \nmid n$. We need to prove the inequality (2.6) for that case. In addition, the inequality (2.6) would be true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i} < e^{\gamma} \times \ln \ln n$$

according to Theorem 2.2. Using the properties of the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^{\gamma} \times \ln \ln n$$

where $n' = q_1 \times \ldots \times q_m$ is the squarefree representation of n. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we need to prove when $2 \mid n'$. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $2^k \mid n$ for $1 \leq k \leq 19$ [4] (this article has been published in the journal Integers in the volume 18). Consequently, we only need to prove that for all $n > 5040$ such that $2^{20} \mid n$ and thus, we have that

$$e^{\gamma} \times n' \times \ln \ln (2 \times \frac{n'}{2}) < e^{\gamma} \times n' \times \ln \ln n$$

because of $2 \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \ln \ln (2 \times \frac{n'}{2}).$$

According to the equation (2.3) and $2 \mid n'$, we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \ln \ln (2 \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \ln \ln (2 \times \frac{n'}{2})$$

which is true according to the Theorem 2.4. In addition, we know the Robin's inequality is true for every $n > 5040$ such that $3^i \mid n$ and $5^j \mid n$ for $1 \leq i \leq 12$ and $1 \leq j \leq 7$ [4] (this article has been published in the journal Integers in the volume 18). To sum up, we have finally proved this result as the remaining only option. ∎

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