The Riemann hypothesis

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Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln \ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We demonstrate the Robin's inequality is true for every natural number $n > 5040$. Consequently, we show the Riemann hypothesis is true.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [1]. The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n,

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [6]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^{\gamma} \times n \times \ln \ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [3]. Using this inequality, we show that the Riemann hypothesis is true.

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2 Results

Theorem 2.1 *Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following inequality*

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

Proof For a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following formula

$$\sigma(n) = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i - 1} \tag{2.1}$$

from the Ramanujan's notebooks [2]. In this way, we have that

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)}. \tag{2.2}$$

However, for any prime power $p_i^{a_i}$, we have that

$$\frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)} < \frac{p_i^{a_i+1}}{p_i^{a_i} \times (p_i - 1)} = \frac{p_i}{p_i - 1}.$$

Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

■

Theorem 2.2 *Given some prime numbers $p_1, p_2, \ldots, p_m$, then we obtain the following inequality,*

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Proof Given a prime number p_i, we obtain that

$$\frac{p_i}{p_i - 1} = \frac{p_i^2}{p_i^2 - p_i}$$

and that would be equivalent to

$$\frac{p_i^2}{p_i^2 - p_i} = \frac{p_i^2}{p_i^2 - 1 - (p_i - 1)}$$

and that is the same as

$$\frac{p_i^2}{p_i^2 - 1 - (p_i - 1)} = \frac{p_i^2}{(p_i - 1) \times (\frac{p_i^2 - 1}{(p_i - 1)} - 1)}$$

which is equal to

$$\frac{p_i^2}{(p_i-1)\times(\frac{p_i^2-1}{(p_i-1)}-1)} = \frac{p_i^2}{(p_i-1)\times\frac{p_i^2-1}{(p_i-1)}\times(1-\frac{(p_i-1)}{p_i^2-1})}$$

that is equivalent to

$$\frac{p_i^2}{(p_i-1)\times\frac{p_i^2-1}{(p_i-1)}\times(1-\frac{(p_i-1)}{p_i^2-1})} = \frac{p_i^2}{p_i^2-1}\times\frac{1}{1-\frac{(p_i-1)}{p_i^2-1}}$$

which is the same as

$$\frac{p_i^2}{p_i^2-1}\times\frac{1}{1-\frac{(p_i-1)}{p_i^2-1}} = \frac{1}{1-p_i^{-2}}\times\frac{1}{1-\frac{1}{(p_i+1)}}$$

and finally

$$\frac{1}{(1-p_i^{-2})}\times\frac{1}{1-\frac{1}{(p_i+1)}} = \frac{1}{(1-p_i^{-2})}\times\frac{p_i+1}{p_i}.$$

In this way, we have that

$$\prod_{i=1}^{m}\frac{p_i}{p_i-1} = \prod_{i=1}^{m}\frac{1}{1-p_i^{-2}}\times\prod_{i=1}^{m}\frac{p_i+1}{p_i}.$$

However, we know that

$$\prod_{i=1}^{m}\frac{1}{1-p_i^{-2}} < \prod_{j=1}^{\infty}\frac{1}{1-p_j^{-2}}$$

where p_j is the j^{th} prime number and we have that

$$\prod_{j=1}^{\infty}\frac{1}{1-p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [6]. Consequently, we obtain that

$$\prod_{i=1}^{m}\frac{p_i}{p_i-1} < \frac{\pi^2}{6}\times\prod_{i=1}^{m}\frac{p_i+1}{p_i}.$$

■

Theorem 2.3 *For $x > 10^8$, we have*

$$\sum_{p\leq x}\frac{1}{p} < \ln\ln x + \gamma - \ln\frac{3}{2}$$

where $p \leq x$ means all the primes lesser than or equal to x.

Proof For $x \geq 2$, we have

$$\sum_{p\leq x}\frac{1}{p} < \ln\ln x + B + \frac{5}{\ln x}$$

where

$$B = 0.2614972128\ldots$$

is the (Meissel-)Mertens constant, since this is a proven result in Lemma 2.1 from the article reference [3]. This the same as

$$\sum_{p\leq x}\frac{1}{p} < \ln\ln x + \gamma - C + \frac{5}{\ln x}$$

that is the same as

$$\sum_{p\leq x}\frac{1}{p} < \ln\ln x + \gamma - (C - \frac{5}{\ln x})$$

where $C > 1.51957520514$, because of $\gamma > B$. If we analyze $C - \frac{5}{\ln x}$, then this complies with

$$C - \frac{5}{\ln x} >= 1.51957520514 - \frac{5}{\ln x} > 1.51957520514 - \frac{5}{\ln 10^8} > \ln\frac{3}{2}$$

for $x > 10^8$ and thus, we would have

$$\sum_{p\leq x}\frac{1}{p} < \ln\ln x + \gamma - \ln\frac{3}{2}.$$

■

Definition 2.4 We recall that an integer n is said to be squarefree if for every prime divisor p of n we have $p^2 \nmid n$, where $p^2 \nmid n$ means that p^2 does not divide n [3].

Theorem 2.5 *Given a squarefree number $n = q_1 \times \ldots \times q_m$ such that $q_1, q_2, \ldots, q_m$ are odd prime numbers and the greatest prime divisor of n is greater than 7, then we obtain the following inequality*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \ln\ln(2 \times n).$$

Proof This proof is very similar with the demonstration in Theorem 1.1 from the article reference [3]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From the equation (2.1), we obtain that

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \ldots \times (q_m + 1) \tag{2.3}$$

when $n = q_1 \times q_2 \times \ldots \times q_m$. In this way, for any prime number $p_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{p_i}) \leq e^{\gamma} \times \ln\ln(2 \times p_i). \tag{2.4}$$

For $p_i = 11$, we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \ln\ln(22)$$

is actually true. For another prime number $p_i > 11$, we have that

$$(1 + \frac{1}{p_i}) < (1 + \frac{1}{11})$$

and

$$e^{\gamma} \times \ln\ln(22) < e^{\gamma} \times \ln\ln(2 \times p_i)$$

which clearly implies that the inequality (2.4) is true for every prime number $p_i \geq 11$. Now, suppose it is true for $m-1$, with $m \geq 1$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [3]. So let $n = q_1 \times \ldots \times q_m$ be a squarefree number and assume that $q_1 < \ldots < q_m$ for $q_m > 7$.

Case 1 :$q_m \geq \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2 \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \ldots \times (q_{m-1}+1) \leq e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times \ln\ln(2 \times q_1 q_1 \times \ldots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln\ln(2 \times q_1 \times \ldots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show that

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln\ln(2 \times q_1 \times \ldots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times q_m \times \ln\ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \ln\ln(2 \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln\ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln\ln(2 \times q_1 \times \ldots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln\ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln\ln(2 \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} \geq$$

$$\frac{\ln\ln(2 \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}.$$

From the reference [3], we have that if $0 < a < b$, then

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.5}$$

We can apply the inequality (2.5) to the previous one just using $b = \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m)$ and $a = \ln(2 \times q_1 \times \ldots \times q_{m-1})$. Certainly, we have that

$$\ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln(2 \times q_1 \times \ldots \times q_{m-1}) =$$

$$\ln \frac{2 \times q_1 \times \ldots \times q_{m-1} \times q_m}{2 \times q_1 \times \ldots \times q_{m-1}} = \ln q_m.$$

In this way, we obtain that

$$\frac{q_m \times (\ln\ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln\ln(2 \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} >$$

$$\frac{q_m}{\ln(2 \times q_1 \times \ldots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2 \times q_1 \times \ldots \times q_m)} \geq \frac{\ln\ln(2 \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m)$ [3].

Case 2 :$q_m < \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2 \times n)$.

We denote by $\vartheta(x)$ the logarithm of the product of all primes lesser than or equal to x [5]. By definition, we know that $\vartheta(q_m) \geq \ln(2 \times q_1 \times \ldots \times q_{m-1} \times q_m)$. However, we know that $x > \vartheta(x)$ for $0 < x \leq 10^8$ [5]. Hence, we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \ln\ln(2 \times n)$$

when $q_m > 10^8$ is the greatest prime divisor of n. If we apply the logarithm to the both sides of the previous inequality

$$\ln(\frac{\pi^2}{6}) + \sum_{j=i}^{m}(\ln(q_j + 1) - \ln q_j) + \ln\frac{3}{2} \leq \gamma + \ln\ln\ln(2 \times n).$$

From the reference [3], we note that

$$\ln(p_1 + 1) - \ln p_1 = \int_{p_1}^{p_1+1} \frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note also that $\ln(\frac{\pi^2}{6}) < \frac{1}{2}$. In order to prove this, it is enough to prove that

$$\frac{1}{2} + \frac{1}{q_1} + \ldots + \frac{1}{q_m} + \ln\frac{3}{2} \leq \sum_{p \leq q_m} \frac{1}{p} + \ln\frac{3}{2} \leq \gamma + \ln\ln\ln(2 \times n).$$

However, we know that

$$\gamma + \ln\ln q_m < \gamma + \ln\ln\ln(2 \times n)$$

since $q_m < \ln(2 \times n)$ and therefore, we would only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} + \ln\frac{3}{2} \leq \gamma + \ln\ln q_m$$

that is the same as

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln q_m - \ln\frac{3}{2}$$

which is true according the Theorem 2.3 $x = q_m > 10^8$. In this way, we finally show the Theorem is indeed satisfied. ■

Theorem 2.6 *Given a natural number* $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ *such that* $a_1, a_2, a_3, a_4 \geq 0$ *are integers, then the Robin's inequality is true for* n.

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln \ln n$$

that would be the same as

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < e^{\gamma} \times \ln \ln n \tag{2.6}$$

according to Theorem 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln \ln(5040) \approx 3.81.$$

However, we know for $n > 5040$, we have that

$$e^{\gamma} \times \ln \ln(5040) < e^{\gamma} \times \ln \ln n$$

and thus, the proof is completed for that case. Hence, we only need to prove for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every n such that $7^k \mid n$ for $1 \leq k \leq 6$ [4] (this article has been published in the journal Integers in the volume 18). Therefore, we need to prove this case for those natural numbers n such that $7^7 \mid n$. In this way, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \ln \ln(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$, we have that

$$e^{\gamma} \times \ln \ln(7^7) \leq e^{\gamma} \times \ln \ln n$$

and thus, the proof is completed. ■

Theorem 2.7 *The Robin's inequality is true for every natural number $n > 5040$.*

Proof Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we will prove the Robin's inequality is true for n. We know this true when the greatest prime divisor of n is lesser than or equal to 7 according to Theorem 2.6. Therefore, the remaining case is when the greatest prime divisor of n is greater than 7. We need to prove the inequality (2.6) for that case. In addition, the inequality (2.6) would be true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i} < e^{\gamma} \times \ln \ln n$$

according to Theorem 2.2. Using the properties of the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^{\gamma} \times \ln \ln n$$

where $n' = q_1 \times \ldots \times q_m$ is the squarefree representation of n. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we need to prove when $2 \mid n'$. In addition, we know the Robin's inequality is true for every n such that $2^k \mid n$ for $1 \leq k \leq 19$ [4] (this article has been published in the journal Integers in the volume 18). Consequently, we only need to prove that for all $n > 5040$ such that $2^{20} \mid n$ and thus, we have that

$$e^{\gamma} \times n' \times \ln \ln(2 \times \frac{n'}{2}) < e^{\gamma} \times n' \times \ln \ln n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \ln \ln(2 \times \frac{n'}{2}).$$

According to the equation (2.3) and $2 \mid n'$, we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \ln \ln(2 \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \ln \ln(2 \times \frac{n'}{2})$$

which is true according to the Theorem 2.5. To sum up, we have finally proved the Robin's inequality is true for every natural number $n > 5040$. ■

Theorem 2.8 *The Riemann hypothesis is true.*

Proof If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann hypothesis is true [3]. Hence, the Riemann hypothesis is true due to Theorem 2.7. ■

3 Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [1]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [1]. In this way, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [1].

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