ASYMPTOTIC STOCHASTIC ANALYSIS OF PARTIALLY RELAXED DML

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ABSTRACT

The Partial Relaxation approach has recently been proposed to solve the Direction-of-Arrival estimation problem [1, 2]. In this paper, we investigate the outlier production mechanism of the Partially Relaxed Deterministic Maximum Likelihood (PR-DML) Direction-of-Arrival estimator using tools from Random Matrix Theory. An accurate description of the probability of resolution for the PR-DML estimator is provided by analyzing the asymptotic stochastic behavior of the PR-DML cost function, assuming that both the number of antennas and the number of snapshots increase without bound at the same rate. The finite dimensional distribution of the PR-DML cost function is shown to be Gaussian in this asymptotic regime and this result is used to compute the probability of resolution.

1. INTRODUCTION

Direction-of-Arrival (DoA) estimation is a major area of research mainly due to its wide spread applications in radar, sonar, seismology, electronic surveillance and mobile communication [3–6]. Several high resolution algorithms, such as Multiple Signal Classification (MUSIC) [7], the minimum variance method of Capon [8], Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) [9] have been proposed [10, 11]. However, the performance of conventional "low-cost" methods strongly degrades when two or multiple sources are closely spaced [12,13]. This is due to the fact that conventional spectral search based approaches ignore the dependence between the sources, hence the interference, and therefore treat multi-source scenarios as single source scenarios.

The Partial Relaxation (PR) framework was introduced to overcome the aforementioned disadvantages of the conventional spectral-based DoA methods [1, 2]. Instead of ignoring the presence of multiple sources, the PR approach considers both the signal impinging from the current direction of interest as well as the interfering ones. To reduce the computational demand, the manifold structure of the undesired signal components is relaxed, whereas the manifold structure of the desired signal component is kept unchanged. The multi-dimensional optimization problem reduces to a one-dimensional problem that admits a simple spectral based grid search applicable to any array geometry.

The main objective of this paper is the performance characterization of the recently introduced PR-DML technique in the threshold region, whereby both the number of samples per antenna and the Signal-to-Noise Ratio (SNR) take moderate values. This region is typically characterized by a systematic appearance of outliers in the DoA estimates, which are mainly caused by the incapability of resolving closely spaced sources. In [14], the probability of resolution of the PR-DML method was investigated by studying the stochastic behavior of the corresponding cost function for Gaussian distributed observations. The analysis in [14] is asymptotic in both the number of antennas and the number of snapshots. In this paper, we follow a similar approach as in [14] and utilize tools from Random Matrix Theory (RMT) to compute the asymptotic stochastic behavior of the PR-DML cost function whereby both the number of snapshots and the number of antennas are large quantities but their quotient converges to a fixed finite value. In contrast to [14] where the second order asymptotic behavior of the PR-DML cost function is computed numerically, we provide a closed-form expression for the variance and a suitable approximation for the closed-form expression of the covariance.

The paper is organized as follows. The signal model is introduced in Section 2 followed by the description of the PR-DML DoA estimation technique in Section 3. The asymptotic stochastic behavior of the PR-DML cost function is given in Section 4 followed by an expression for the probability of resolution in Section 5. Simulation results are presented in Section 6. Finally, Section 7 concludes this paper.

2. SIGNAL MODEL

Consider an antenna array equipped with M sensors and K impinging narrowband signals that satisfy $M{>}K$. The source signal at time instant n is denoted by $\mathbf{s}(n){=}[s_1(n),...,s_K(n)]^{\mathrm{T}}{\in}\mathbb{C}^K$. The corresponding DoAs of the signals are denoted by $\boldsymbol{\theta}{=}[\theta_1,...,\theta_K]^{\mathrm{T}}$. Furthermore, the full-rank steering matrix is given by $\mathbf{A}(\boldsymbol{\theta}){=}[\mathbf{a}(\theta_1),...,\mathbf{a}(\theta_K)]{\in}\mathbb{C}^{M\times K}$ where $\mathbf{a}(\theta_i){\in}\mathbb{C}^M$ denotes the sensor array response for the i-th impinging signal. The number of sources K is assumed to be known. The received baseband signal $\mathbf{x}(n){\in}\mathbb{C}^M$ at the n-th time instant is given by

$$\mathbf{x}(n){=}\mathbf{A}(\boldsymbol{\theta})\mathbf{s}(n){+}\mathbf{n}(n), \qquad n{=}1,...,N, \tag{1}$$

where N denotes the number of snapshots and $\mathbf{n}(n){\in}\mathbb{C}^M$ the sensor noise. Assuming that signal and noise variables are statistically independent zero-mean circularly symmetric Gaussian distributed, the covariance matrix of the received signal $\mathbf{R}{\in}\mathbb{C}^{M\times M}$ is given by

$$\mathbf{R}{=}\mathbb{E}\Big\{\mathbf{x}(n)\mathbf{x}^{\mathrm{H}}(n)\Big\}{=}\mathbf{A}\mathbf{R}_s\mathbf{A}^{\mathrm{H}}{+}\sigma^2\mathbf{I}_M,$$

where $\mathbf{R}_s{=}\mathbb{E}\big\{\mathbf{s}(n)\mathbf{s}^{\mathrm{H}}(n)\big\}$ denotes the covariance of the transmitted signal and $\sigma^2\mathbf{I}_M$ is the noise covariance matrix. Since the true covariance matrix is unavailable in practice, the sample covariance matrix $\hat{\mathbf{R}}{=}\frac{1}{N}\sum_{n=1}^{N}\mathbf{x}(n)\mathbf{x}^{\mathrm{H}}(n)$ is used instead.

3. PARTIALLY RELAXED DETERMINISTIC MAXIMUM LIKELIHOOD

In the framework of PR, not only the signals from the desired directions but also the dependence between the sources, hence the interference is considered [1]. However, the structure of the interfering signals is relaxed and consequently the computational complexity of the multi-source criteria is greatly reduced. Unlike in conventional Deterministic Maximum Likelihood (DML) and Stochastic Maximum Likelihood (SML) DoA estimation criteria, the steering matrix $\mathbf{A}$ is not described by a fully structured array manifold. Instead, $\mathbf{A}$ describes the partially relaxed array manifold

$$\bar{\mathcal{A}}_K=\Big\{\mathbf{A} \mid \mathbf{A}=[\mathbf{a}(\vartheta),\mathbf{B}],\, \mathbf{a}(\vartheta)\in\mathcal{A}_1,\, \mathbf{B}\in\mathbb{C}^{M\times(K-1)}\Big\},$$

which still retains some geometric structure of the sensor array [2]. Applying the PR approach to the conventional DML method and optimizing with respect to the relaxed signal part $\mathbf{B}$ yields the concentrated PR-DML cost function [1,2]

$$\hat{\eta}(\theta)=\frac{1}{M}\sum_{k=1}^{M-K+1}\hat{\lambda}_k(\theta), \tag{2}$$

where $\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)$ denotes the orthogonal projection matrix and the eigenvalues of the modified sample covariance matrix $\hat{\mathbf{R}}(\theta)=\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)\hat{\mathbf{R}}\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)$ are sorted in non-descending order $0=\hat{\lambda}_1(\theta)\leq\cdots\leq\hat{\lambda}_M(\theta)$. The K DoA estimates are given by the K arguments that correspond to the K deepest local minima of the concentrated objective function in (2). An efficient implementation of the PR-DML method is provided in [1].

The distinct true eigenvalues of the modified true covariance matrix $\mathbf{R}(\theta)=\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)\mathbf{R}\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)$ are denoted by $0=\gamma_0(\theta)<\gamma_1(\theta)<\cdots<\gamma_{\bar{M}(\theta)}(\theta)$ and their corresponding multiplicities are given by $K_m(\theta)$, for $m=0,\ldots,\bar{M}(\theta)$. The number of distinct true eigenvalues of the modified true covariance matrix $\mathbf{R}(\theta)$ amounts to $\bar{M}(\theta)+1$ and the sum of the multiplicities satisfies $\sum_{m=0}^{\bar{M}(\theta)}K_m(\theta)=M$. The non-necessarily Hermitian $M\times M$ positive definite square root of the modified true covariance matrix $\mathbf{R}(\theta)$ can also be expressed using the singular value decomposition

$$\mathbf{R}(\theta)^{1/2}=\mathbf{P}_{\mathbf{a}}^{\perp}(\theta)\mathbf{R}^{1/2}=\sum_{r=0}^{\bar{M}(\theta)}\sqrt{\gamma_r(\theta)}\mathbf{U}_r(\theta)\mathbf{V}_r^{\mathrm{H}}(\theta),$$

where $\mathbf{U}_r(\theta)\in\mathbb{C}^{M\times K_r(\theta)}$ and $\mathbf{V}_r(\theta)\in\mathbb{C}^{M\times K_r(\theta)}$ generate the left and right orthonormal basis of $\mathbf{R}(\theta)^{1/2}$.

4. ASYMPTOTIC BEHAVIOR OF THE PARTIALLY RELAXED DETERMINISTIC MAXIMUM LIKELIHOOD COST FUNCTION

In the following, the asymptotic behavior of the cost function in (2) is derived for the case where $M,N\to\infty$, $M/N\to c$, $0<c<\infty$. The covariance matrix $\hat{\mathbf{R}}(\theta)$ is assumed to have uniformly bounded spectral norm for all M. Furthermore, the covariance matrix can be equivalently expressed as $\hat{\mathbf{R}}(\theta)=\mathbf{R}(\theta)^{1/2}\frac{\mathbf{Z}\mathbf{Z}^{\mathrm{H}}}{N}\left(\mathbf{R}(\theta)^{1/2}\right)^{\mathrm{H}}$, where $\mathbf{Z}$ denotes an $M\times N$ matrix of i.i.d. Gaussian random variables with law $\mathcal{CN}(0,1)$.

In RMT it is well known that under all the previously mentioned assumptions, the empirical eigenvalue distribution of $\hat{\mathbf{R}}(\theta)$ is almost surely close to an asymptotic non-random distribution which is absolutely continuous with density $q_M(x,\theta)$ [15]. With increasing number of snapshots and therefore decreasing $c=M/N$, $q_M(x,\theta)$ tends to concentrate around the true eigenvalues forming different eigenvalue clusters. The number of eigenvalue clusters increases with decreasing c as clusters begin to split [16]. Assuming there are S distinct clusters, the support of the clusters is given by the set of S disjoint compact intervals $\mathcal{S}(\theta)=\left[x_1^-(\theta),x_1^+(\theta)\right]\cup\cdots\cup\left[x_S^-(\theta),x_S^+(\theta)\right]$. Furthermore, it can be observed that each modified true and distinct eigenvalue $\gamma_m(\theta)$ is associated to only one cluster. However, one cluster may be associated to multiple true eigenvalues which results in a non-bijective correspondence [16]. For sufficiently small c, there exist exactly as many clusters as distinct true eigenvalues $\bar{M}(\theta)$ [16].

In order to distinguish between the eigenvalues that are considered by the PR-DML cost function in (2) and the remaining ones it is crucial that the $(M-K+1)$-th modified sample eigenvalue asymptotically splits from the $(M-K+2)$-th one, which can be formalized as follows. We assume that there exists an integer $m(\theta)$ such that $M-K+1=\sum_{r=0}^{m(\theta)}K_r(\theta)$, and the cluster associated to the eigenvalue $\gamma_{m(\theta)}(\theta)$ separates from the one associated to $\gamma_{m(\theta)+1}(\theta)$ in the asymptotic eigenvalue distribution of $\hat{\mathbf{R}}(\theta)$.

Let us consider the asymptotic stochastic behavior of the random real-valued $L\times 1$ vector

$$\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})=\left[\hat{\eta}(\bar{\theta}_1),\ldots,\hat{\eta}(\bar{\theta}_L)\right]^{\mathrm{T}}, \tag{3}$$

where $\bar{\boldsymbol{\theta}}=\left[\bar{\theta}_1,\ldots,\bar{\theta}_L\right]^{\mathrm{T}}$ denotes a set of L points within the Field of View (FoV). Under the previously mentioned assumptions and as $M,N\to\infty$, $M/N\to c$ and $0<c<\infty$, the random vector $\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in (3) converges in distribution to a multivariate standardized Gaussian distribution

$$M\boldsymbol{\Gamma}^{-1/2}(\bar{\boldsymbol{\theta}})\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})-\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})\right)\to\mathcal{N}(\mathbf{0},\mathbf{I}_L). \tag{4}$$

In the following we provide expressions for the asymptotic mean $\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ of the random vector $\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in (3) (first oder asymptotic behavior) and the corresponding $L\times L$ asymptotic covariance matrix $\boldsymbol{\Gamma}(\bar{\boldsymbol{\theta}})/M^2$ (second order asymptotic behavior).

4.1. First Order Asymptotic Behavior

In [14] it was shown shown that the PR-DML cost function $\hat{\eta}(\theta)$ in (2) becomes asymptotically close to its deterministic counterpart

$$\bar{\eta}(\theta)=\frac{1}{M}\sum_{r=1}^{m(\theta)}K_r(\theta)\gamma_r(\theta)\left(1-\frac{1}{N}\sum_{j=m(\theta)+1}^{\bar{M}(\theta)}K_j(\theta)\frac{\gamma_j(\theta)}{\gamma_j(\theta)-\gamma_r(\theta)}\right) \tag{5}$$

in the sense that $|\hat{\eta}(\theta)-\bar{\eta}(\theta)|\to 0$ almost surely pointwise in θ as $M,N\to\infty$ at the same rate. Furthermore, we define the asymptotic mean (first order asymptotic behavior) of the random vector $\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in (3) as

$$\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})=[\bar{\eta}(\bar{\theta}_1),\ldots,\bar{\eta}(\bar{\theta}_L)]^{\mathrm{T}}. \tag{6}$$

4.2. Second Order Asymptotic Behavior

In the following, the nature of the fluctuation of the PR-DML cost function $\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in (3) around its asymptotic mean $\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ in (6) is studied in the asymptotic regime where $M,N\to\infty$ at the same rate. Taking $z\in\mathbb{C}^+\equiv\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$, we define $\omega(z,\theta)$ as the unique solution of

$$z=\omega(z,\theta)\left(1-\frac{1}{N}\mathrm{tr}\left\{\mathbf{R}(\theta)(\mathbf{R}(\theta)-\omega(z,\theta))^{-1}\right\}\right),$$

on the set $\mathbb{C}^+$. Consider the limit of $\omega(z,\theta)$ as z goes to the real axis and the analytical extension to $\mathbb{C}\backslash\{0\}\cup\mathcal{S}(\theta)$. Furthermore, let us introduce the negatively oriented contour $\mathcal{C}(\theta)$ that encloses the $M-K+1$ smallest sample eigenvalues of $\hat{\mathbf{R}}(\theta)$ only. The covariance can be computed according to the following Theorem.

Theorem 1. *According to the above definitions, consider an $L{\times}L$ matrix $\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})$ with entries*

$$\left[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,q}{=}\frac{-1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\frac{\partial z(\omega_1,\bar{\theta}_p)}{\partial\omega_1}\frac{\partial z(\omega_2,\bar{\theta}_q)}{\partial\omega_2}\times \\ \times\log(1{-}\Omega(\omega_1,\omega_2))d\omega_1 d\omega_2, \quad (7)$$

where $p,q{=}1,\ldots,L$, $\mathcal{C}_\omega(\theta){=}\omega(\mathcal{C}(\theta),\theta)$,

$$z(\omega,\theta){=}\omega\left(1{-}\frac{1}{N}\sum_{r=1}^{\bar{M}(\theta)}K_r(\theta)\frac{\gamma_r(\theta)}{\gamma_r(\theta){-}\omega}\right),$$

and

$$\Omega(\omega_1,\omega_2){=}\sum_{i=1}^{\bar{M}(\bar{\theta}_p)}\sum_{j=1}^{\bar{M}(\bar{\theta}_q)}\kappa_{ij}(\bar{\theta}_p,\bar{\theta}_q)\frac{\gamma_i(\bar{\theta}_p)\gamma_j(\bar{\theta}_q)}{\left(\gamma_i(\bar{\theta}_p){-}\omega_1\right)\left(\gamma_j(\bar{\theta}_q){-}\omega_2\right)},$$

with $\kappa_{ij}(\bar{\theta}_p,\bar{\theta}_q){=}\frac{1}{N}\mathrm{tr}\{\mathbf{V}_i(\bar{\theta}_p)\mathbf{V}_i^{\mathrm{H}}(\bar{\theta}_p)\mathbf{V}_j(\bar{\theta}_q)\mathbf{V}_j^{\mathrm{H}}(\bar{\theta}_q)\}{\in}\mathbb{R}$. Assuming that $\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})$ is invertible and that the spectral norm of $\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})^{-1}$ is bounded in M, vector $M\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})^{-1/2}\left(\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}}){-}\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})\right)$ converges in distribution to a multivariate standardized Gaussian random vector for fixed L and $M,N{\to}\infty$, $M/N{\to}c$, $0{<}c{<}\infty$.

Proof. The proof can be obtained by using the approach in [17]. Also see [18, Theorem 2]. ■

In Section 4.2.1 we provide a closed-form expression for the variance of the PR-DML cost function followed by an approximation of the closed-form expression for the covariance that is valid for sufficient small $c{=}M/N$ in Section 4.2.2.

4.2.1. Closed-Form Solution for the Variance

Motivated by the fact that the complex double contour integral in (7) simplifies for the special case where $p{=}q$ we compute a closed-form expression for the variance according to the following Theorem.

Corollary 1.1. *For the special case where $\bar{\theta}_p{=}\bar{\theta}_q$, the computation of the asymptotic variance of the PR-DML cost function simplifies to*

$$\left[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,p}{=}\frac{1}{N}\sum_{r=1}^{m(\bar{\theta}_p)}K_r(\bar{\theta}_p)\gamma_r^2(\bar{\theta}_p) \\ -\frac{1}{N^2}\sum_{r=1}^{m(\bar{\theta}_p)}K_r(\bar{\theta}_p)\gamma_r^2(\bar{\theta}_p)\sum_{l=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\frac{K_r(\bar{\theta}_p)\gamma_r^2(\bar{\theta}_p)}{\left(\gamma_r(\bar{\theta}_p){-}\gamma_l(\bar{\theta}_p)\right)^2}. \quad (8)$$

4.2.2. Approximated Closed-Form Solution for the Covariance

Let us consider the more general case of the covariance in (7) for $p,q{=}1,\ldots,L$. Instead of numerically computing the complex double contour integrals in (7) as it was done in [14], we express the logarithm $\log(1{-}\Omega(\omega_1,\omega_2))$ of the integrand in (7) as a Taylor expansion and only consider the terms of the integrand that involve $1/N$ or $1/N^2$. This is motivated by the closed-form solution of the asymptotic variance obtained in (8) which only involves expressions that vanish with N at rates $1/N$ and $1/N^2$. Using the Taylor expansion of the logarithm, the integrand in (7) is written as a rational function which allows to approximate the closed-form solution of the covariance by using conventional residue calculus [19]. In order to solve the integral in (7), we define $A(\omega,\theta){=}\frac{1}{N}\sum_{r=1}^{\bar{M}(\theta)}K_r(\theta)\frac{\gamma_r^2(\theta)}{(\gamma_r(\theta)-\omega)^2}$ which allows to write $[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})]_{p,q}$ in (7) as

$$\left[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,q}{=}\frac{-1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\left(1{-}A(\omega_1,\bar{\theta}_p)\right)\times \\ \times\left(1{-}A(\omega_2,\bar{\theta}_q)\right)\log(1{-}\Omega(\omega_1,\omega_2))d\omega_1 d\omega_2. \quad (9)$$

The logarithm in the integrand of (9) can be expressed as a Taylor expansion according to the following Lemma.

Lemma 1. *The logarithm in the integrand of (9) can be expressed as Taylor expansion of $\Omega(\omega_1,\omega_2)$ around zero*

$$\log(1{-}\Omega(\omega_1,\omega_2)){=}{-}\sum_{k=1}^{\infty}\frac{\Omega^k(\omega_1,\omega_2)}{k}, \quad (10)$$

which converges since $|\Omega(\omega_1,\omega_2)|{<}1$ for $\omega_1{\in}\mathcal{C}_\omega(\bar{\theta}_p){=}\omega(\mathcal{C}(\bar{\theta}_p),\bar{\theta}_p)$ and $\omega_2{\in}\mathcal{C}_\omega(\bar{\theta}_q){=}\omega(\mathcal{C}(\bar{\theta}_q),\bar{\theta}_q)$.

Proof. See [20, Appendix 1]. ■

Furthermore, we separate between the terms of the integrand in (9) that involve $1/N^r$, $r{=}1{,}2$ and the terms that involve higher order factors $1/N^r$, $r{>}2$. The approximated closed-form solution of the asymptotic covariance in (7) is given by

$$\left[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,q}{=}\frac{1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\Bigg(\Omega(\omega_1,\omega_2){+}\frac{1}{2}\Omega^2(\omega_1,\omega_2) \\ -\left(A(\omega_2,\bar{\theta}_q){+}A(\omega_1,\bar{\theta}_p)\right)\Omega(\omega_1,\omega_2)\Bigg)d\omega_1 d\omega_2{+}\mathcal{O}(c^3) \quad (11)$$

which can be solved in closed-form by integrating term by term using conventional residue calculus [19].

Lemma 2a. *The integration of the first term in (11) yields*

$$\Upsilon_1(\bar{\theta}_p,\bar{\theta}_q){=}\frac{1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\Omega(\omega_1,\omega_2)d\omega_1 d\omega_2 \\ {=}\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\kappa_{ij}(\bar{\theta}_p,\bar{\theta}_q)\gamma_i(\bar{\theta}_p)\gamma_j(\bar{\theta}_q). \quad (12)$$

Lemma 2b. *Introducing $B_{ik}(\theta){=}\frac{\gamma_i(\theta)\gamma_k(\theta)}{\gamma_k(\theta)-\gamma_i(\theta)}$ and $C_{ik}^{jr}(\bar{\theta}_p,\bar{\theta}_q){=}\kappa_{ij}(\bar{\theta}_p,\bar{\theta}_q)\kappa_{kr}(\bar{\theta}_p,\bar{\theta}_q)B_{ik}(\bar{\theta}_p)B_{jr}(\bar{\theta}_q)$ for compact notation, the integration of the second term in (11) yields*

$$\Upsilon_2(\bar{\theta}_p,\bar{\theta}_q){=}\frac{1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\Omega^2(\omega_1,\omega_2)d\omega_1 d\omega_2 \\ {=}\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\sum_{k=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\sum_{r=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}C_{ik}^{jr}(\bar{\theta}_p,\bar{\theta}_q) \\ +\sum_{i=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\sum_{j=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}\sum_{k=1}^{m(\bar{\theta}_p)}\sum_{r=1}^{m(\bar{\theta}_q)}C_{ik}^{jr}(\bar{\theta}_p,\bar{\theta}_q) \quad (13) \\ -\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}\sum_{k=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\sum_{r=1}^{m(\bar{\theta}_q)}C_{ik}^{jr}(\bar{\theta}_p,\bar{\theta}_q) \\ -\sum_{i=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\sum_{k=1}^{m(\bar{\theta}_p)}\sum_{r=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}C_{ik}^{jr}(\bar{\theta}_p,\bar{\theta}_q).$$

Lemma 2c. *Introducing $\mathcal{K}_r(\theta) = K_r(\theta)\gamma_r^2(\theta)$ for compact notation, the integration of the third term in* (11) *yields*

$$
\begin{aligned}
&\Upsilon_3\left(\bar{\theta}_p,\bar{\theta}_q\right)\\
&= \frac{-1}{(2\pi j)^2}\oint_{\mathcal{C}_\omega(\bar{\theta}_p)}\oint_{\mathcal{C}_\omega(\bar{\theta}_q)}\left(A(\omega_1,\bar{\theta}_p)+A(\omega_1,\bar{\theta}_q)\right)\Omega(\omega_1,\omega_2)d\omega_1 d\omega_2\\
&= \frac{1}{N}\sum_{r=1}^{m(\bar{\theta}_p)}\mathcal{K}_r\left(\bar{\theta}_p\right)\sum_{i=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\kappa_{ij}\left(\bar{\theta}_p,\bar{\theta}_q\right)\frac{\gamma_i\left(\bar{\theta}_p\right)\gamma_j\left(\bar{\theta}_q\right)}{\left(\gamma_i\left(\bar{\theta}_p\right)-\gamma_r\left(\bar{\theta}_p\right)\right)^2}\\
&+ \frac{1}{N}\sum_{r=1}^{m(\bar{\theta}_q)}\mathcal{K}_r\left(\bar{\theta}_q\right)\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}\kappa_{ij}\left(\bar{\theta}_p,\bar{\theta}_q\right)\frac{\gamma_i\left(\bar{\theta}_p\right)\gamma_j\left(\bar{\theta}_q\right)}{\left(\gamma_j\left(\bar{\theta}_q\right)-\gamma_r\left(\bar{\theta}_q\right)\right)^2}\\
&- \frac{1}{N}\sum_{r=m(\bar{\theta}_p)+1}^{\bar{M}(\bar{\theta}_p)}\mathcal{K}_r\left(\bar{\theta}_p\right)\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\kappa_{ij}\left(\bar{\theta}_p,\bar{\theta}_q\right)\frac{\gamma_i\left(\bar{\theta}_p\right)\gamma_j\left(\bar{\theta}_q\right)}{\left(\gamma_i\left(\bar{\theta}_p\right)-\gamma_r\left(\bar{\theta}_p\right)\right)^2}\\
&- \frac{1}{N}\sum_{r=m(\bar{\theta}_q)+1}^{\bar{M}(\bar{\theta}_q)}\mathcal{K}_r\left(\bar{\theta}_q\right)\sum_{i=1}^{m(\bar{\theta}_p)}\sum_{j=1}^{m(\bar{\theta}_q)}\kappa_{ij}\left(\bar{\theta}_p,\bar{\theta}_q\right)\frac{\gamma_i\left(\bar{\theta}_p\right)\gamma_j\left(\bar{\theta}_q\right)}{\left(\gamma_r\left(\bar{\theta}_q\right)-\gamma_j\left(\bar{\theta}_q\right)\right)^2}.
\end{aligned}
\tag{14}
$$

Corollary 1.2. *The approximated closed-form solution of the asymptotic covariance in* (7) *yields*

$$
\left[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\right]_{p,q} = \Upsilon_1\left(\bar{\theta}_p,\bar{\theta}_q\right)+\frac{1}{2}\Upsilon_2\left(\bar{\theta}_p,\bar{\theta}_q\right)+\Upsilon_3\left(\bar{\theta}_p,\bar{\theta}_q\right)+\mathcal{O}(c^3), \tag{15}
$$

where $\Upsilon_1\left(\bar{\theta}_p,\bar{\theta}_q\right)$ is given in (12), *$\Upsilon_2\left(\bar{\theta}_p,\bar{\theta}_q\right)$ in* (13) *and $\Upsilon_3\left(\bar{\theta}_p,\bar{\theta}_q\right)$ in* (14).

Remark It can be observed, that the approximated closed-form solution in (15) is equivalent to the closed-form expression of the asymptotic variance in (8) for $\bar{\theta}_p = \bar{\theta}_q$. The approximated closed-form solution of the asymptotic covariance is only valid for sufficient small quotient $c = M/N$ and if there exists separation between $\gamma_{m(\bar{\theta}_p)}\left(\bar{\theta}_p\right)$, respectively $\gamma_{m(\bar{\theta}_q)}\left(\bar{\theta}_q\right)$, and adjacent eigenvalues. However, in contrast to the numerical solution in [14], the approximated closed-form solution of $[\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})]_{p,q}$ in (11) still provides a valid estimate on the asymptotic covariance in case of no separation as illustrated in the simulations.

5. PROBABILITY OF RESOLUTION

To analyze the outlier production mechanism of the PR-DML method we examine the probability of resolution as one possible application of the results derived in the previous section [21]. Let us study a scenario with two sources $K=2$ located at θ_1 and θ_2. Considering the minimization problem in (2), loss of resolution is declared if the cost function evaluated at the mid-angle $(\theta_1+\theta_2)/2$ is lower than evaluated at both true DoAs θ_1 and θ_2 [22]. The constraint to declare resolution can compactly be written as $\mathbf{u}^H\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})<0$ where $\mathbf{u} = [1/2, 1/2, -1]^{\mathrm{T}}$ and $\bar{\boldsymbol{\theta}} = [\theta_1, \theta_2, (\theta_1+\theta_2)/2]^{\mathrm{T}}$, respectively. Utilizing the previously obtained asymptotic stochastic behavior of the PR-DML cost function in (4), the asymptotic probability density function (pdf) of the test quantity $\mathbf{u}^{\mathrm{H}}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})$ can easily be computed for $M,N\rightarrow\infty$ at the same rate [23]

$$
\left(\mathbf{u}^{\mathrm{H}}\mathbf{\Gamma}(\bar{\boldsymbol{\theta}})\mathbf{u}\right)^{-1/2}\left(\mathbf{u}^{\mathrm{H}}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})-\mathbf{u}^{\mathrm{H}}\bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})\right)\rightarrow\mathcal{N}(0,1).
$$

The predicted probability of resolution is therefore obtained by computing the cumulative distribution function (cdf) [24]

$$
P_{\text{res}} = \Pr\left(\mathbf{u}^{\mathrm{H}}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})<0\right) = \int_{-\infty}^{0} f_{\mathbf{u}^{\mathrm{H}}\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})}(x)dx, \tag{16}
$$

where $f_{\mathbf{u}^H\hat{\boldsymbol{\eta}}(\bar{\boldsymbol{\theta}})}(x)$ denotes the asymptotic pdf of the test quantity.

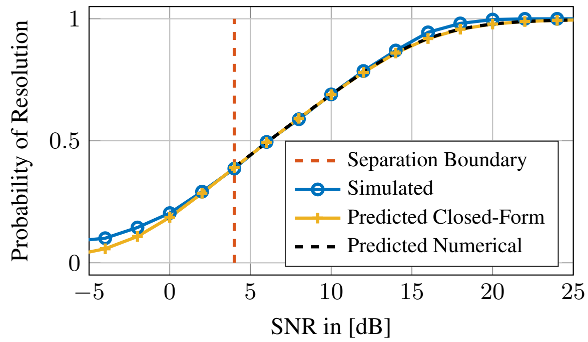

Fig. 1. Uncorrelated Sources, Number of Snapshots $N=10$

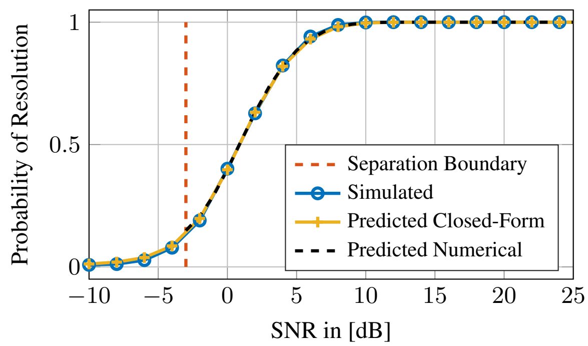

Fig. 2. Uncorrelated Sources, Number of Snapshots $N=100$

6. SIMULATION RESULTS

In this Section, the predicted probability of resolution in (16) is compared to the simulated one. A Uniform Linear Array (ULA) equipped with $M=10$ sensors and two uncorrelated and closely spaced sources at $\boldsymbol{\theta} = [45°, 50°]^T$ are considered. The transmitted signals are zero-mean and statistically independent with unit power and the SNR is given by $\text{SNR} = 1/\sigma_n^2$. The separation boundary is defined as the smallest SNR that provides separation between the eigenvalue clusters associated to the $m\left(\bar{\theta}_l\right)$-th true eigenvalue and larger adjacent true eigenvalues for $l=1,2,3$ and $\bar{\boldsymbol{\theta}} = [45°, 50°, 47.5°]^T$. For SNR values smaller than the separation boundary it is not possible to distinguish between the eigenvalues that are considered by the PR-DML cost-function and the remaining ones.

Figures 1 and 2 depict the probability of resolution versus the SNR for $N=10$ and $N=100$ snapshots. As expected, the probability of declaring resolution increases with increasing number of snapshots. We observe that from our expressions it is possible to predict the probability of resolution remarkably well in both scenarios. Even in case of no separation our prediction of the probability of resolution in (16) that utilizes the approximated closed-form solution of the covariance in (11) is close to the actual one.

7. CONCLUSION

In this paper we have investigated the asymptotic behavior of the PR-DML DoA estimator under the setting of RMT where both the number of snapshots and the number of sensors go to infinity at the same rate. The finite dimensional distribution of the PR-DML cost function has been derived with closed-form expression for the variance and approximated closed-form expression for the covariance. Furthermore, the asymptotic probability distribution of the PR-DML method was used to characterize the probability of resolution in the threshold region, where the generation of outliers causes a performance breakdown.

8. REFERENCES

[1] M. Trinh-Hoang, M. Viberg, and M. Pesavento, "Partial Relaxation Approach: An Eigenvalue-Based DOA Estimator Framework," *IEEE Transactions on Signal Processing*, vol. 66, no. 23, pp. 6190–6203, Dec. 2018.

[2] M. Trinh-Hoang, M. Viberg, and M. Pesavento, "An Improved DoA Estimator Based on Partial Relaxation Approach," *IEEE International Conference on Acoustics, Speech, and Signal Processing*, pp. 3246–3250, Apr. 2018.

[3] A. M. Rembovsky, A. V. Ashikhmin, V. A. Kozmin, and S.M. Smolskiy, *Radio Monitoring: Problems, Methods and Equipment*, Lecture Notes in Electrical Engineering. Springer US, 2009.

[4] H. Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, Detection, Estimation, and Modulation Theory. Wiley, 2004.

[5] H. Krim and M. Viberg, "Two Decades of Array Signal Processing Research: The Parametric Approach," *IEEE Signal Processing Magazine*, vol. 13, no. 4, pp. 67–94, Jul. 1996.

[6] P.-J. Chung, M. Viberg, and J. Yu, "DOA Estimation Methods and Algorithms," in *Academic Press Library in Signal Processing*, A. M. Zoubir, M. Viberg, R. Chellappa, and S. Theodoridis, Eds., vol. 3 of *Academic Press Library in Signal Processing*, pp. 599 – 650. Elsevier, 2014.

[7] R. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation," *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[8] J. Capon, "High-Resolution Frequency-Wavenumber Spectrum Analysis," *Proceedings of the IEEE*, vol. 57, no. 8, pp. 1408–1418, Aug. 1969.

[9] R. Roy and T. Kailath, "ESPRIT-Estimation of Signal Parameters Via Rotational Invariance Techniques," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 7, pp. 984–995, Jul. 1989.

[10] L. C. Godara, "Application of Antenna Arrays to Mobile Communications, Part II: Beam-Forming and Direction-of-Arrival Considerations," *Proceedings of the IEEE*, vol. 85, no. 8, pp. 1195–1245, Aug. 1997.

[11] P. Chung, J. F. Böhme, C. F. Mecklenbräuker, and A. O. Hero, "Detection of the Number of Signals Using the Benjamini-Hochberg Procedure," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2497–2508, Jun. 2007.

[12] P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood and Cramér-Rao Bound: Further Results and Comparisons," *International Conference on Acoustics, Speech, and Signal Processing*, pp. 2605–2608 vol.4, May 1989.

[13] C. Vaidyanathan and K. M. Buckley, "Performance Analysis of the MVDR Spatial Spectrum Estimator," *IEEE Transactions on Signal Processing*, vol. 43, no. 6, pp. 1427–1437, Jun. 1995.

[14] D. Schenck, X. Mestre, and M. Pesavento, "Probability of Resolution of Partially Relaxed DML an Asymptotic Approach," *IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, Dec. 2019.

[15] J. W. Silverstein, "Strong Convergence of the Empirical Distribution of Eigenvalues of Large Dimensional Random Matrices," *Journal of Multivariate Analysis*, vol. 55, no. 2, pp. 331–339, Feb. 1995.

[16] X. Mestre, "Improved Estimation of Eigenvalues and Eigenvectors of Covariance Matrices Using Their Sample Estimates," *IEEE Transactions on Information Theory*, vol. 54, no. 11, pp. 5113–5129, Nov. 2008.

[17] W. Hachem, O. Khorunzhiy, P. Loubaton, J. Najim, and L. Pastur, "A New Approach for Mutual Information Analysis of Large Dimensional Multi-Antenna Channels," *IEEE Transactions on Information Theory*, vol. 54, no. 9, pp. 3987–4004, Sep. 2008.

[18] X. Mestre, P. Vallet, and P. Loubaton, "On the Resolution Probability of Conditional and Unconditional Maximum Likelihood DOA Estimation," *European Signal Processing Conference*, Sep. 2013.

[19] J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, vol. 3, W. H. Freeman, 1999.

[20] X. Mestre, "On the Asymptotic Behavior of the Sample Estimates of Eigenvalues and Eigenvectors of Covariance Matrices," *IEEE Transactions on Signal Processing*, vol. 56, no. 11, pp. 5353 – 5368, Dec. 2008.

[21] F. Athley, "Threshold Region Performance of Maximum Likelihood Direction of Arrival Estimators," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1359–1373, Apr. 2005.

[22] Q. T. Zhang, "Probability of Resolution of the MUSIC Algorithm," *IEEE Transactions on Signal Processing*, vol. 43, no. 4, pp. 978–987, Apr. 1995.

[23] Kun Il Park, *Fundamentals of Probability and Stochastic Processes with Applications to Communications*, Springer International Publishing, 2018.

[24] A. Genz, "Numerical Computation of Multivariate Normal Probabilities," *Journal of Computational and Graphical Statistics*, vol. 1, no. 2, pp. 141–149, 1992.