The Riemann hypothesis

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Abstract. The Robin's inequality is true for every natural number n greater than 5040 if and only if the Riemann hypothesis is true. We demonstrate the Robin's inequality is true for every natural number n greater than 5040. In this way, we prove the Riemann hypothesis is true.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [4]. It is of great interest in number theory because it implies results about the distribution of prime numbers [4]. It was proposed by Bernhard Riemann (1859), after whom it is named [4]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [4]. The divisor function $\sigma(n)$ for n a natural number is defined as the sum of the powers of the divisors of n,

$$\sigma(n) = \sum_{k|n} k$$

where $k \mid n$ means that the natural number k divides n [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^{\gamma} \times n \times \log \log n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [3]. Using this inequality, we show that the Riemann hypothesis is true.

2 Results

Theorem 2.1 *Given a natural number $m = 2^k \times n$ such that k and n are natural numbers and n is odd, we obtain that $\sigma(m) = \sigma(\frac{m}{2}) + 2^k \times \sigma(n)$.*

Proof Certainly, we can separate the divisors in $\sigma(m)$ into two sets such that one set of divisors are divisible by 2^k and the divisors of the other set are

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not divisible by 2^k. The set of divisors which are not divisible by 2^k coincides exactly with the divisors of the number $\frac{m}{2}$. While the divisors of other set are of the form $2^k \times i$ where i is an odd number: The number i will be a divisor of m and since it is odd, then this must be a divisor of n. Consequently, if we sum the divisors of the both sets, then we obtain the value of $\sigma(m)$. The general equation will be $\sigma(m) = \sigma(\frac{m}{2}) + 2^k \times \sigma(n)$ according to the mentioned properties of these sets of divisors. ■

Theorem 2.2 *Given a natural number $m = 2^k \times n$ such that k and n are positive integers and n is odd, we obtain that $\sigma(m) = (2^{k+1} - 1) \times \sigma(n)$.*

Proof From the Theorem 2.1, we obtain that $\sigma(m) = \sigma(\frac{m}{2}) + 2^k \times \sigma(n)$. However, using the same Theorem 2.1, we have that $\sigma(\frac{m}{2}) = \sigma(\frac{m}{4}) + 2^{k-1} \times \sigma(n)$ since $\frac{m}{2} = 2^{k-1} \times n$. Replacing this result in the initial equation, we will have $\sigma(m) = \sigma(\frac{m}{4}) + 2^{k-1} \times \sigma(n) + 2^k \times \sigma(n)$. We can continue with $\sigma(\frac{m}{8})$, $\sigma(\frac{m}{16})$ and so forth until $\sigma(\frac{m}{2^k})$. Consequently, we obtain the formula,

$$\sigma(m) = \sigma(n) + 2 \times \sigma(n) + 4 \times \sigma(n) + \ldots + 2^{k-1} \times \sigma(n) + 2^k \times \sigma(n)$$

which is equal to,

$$\sigma(m) = \sigma(n) \times (1 + 2 + 4 + \ldots + 2^{k-1} + 2^k)$$

where we know that,

$$(2^{k+1} - 1) = (1 + 2 + 4 + \ldots + 2^{k-1} + 2^k).$$

As result, we obtain the general formula $\sigma(m) = (2^{k+1} - 1) \times \sigma(n)$. ■

Theorem 2.3 *Given two naturals number $n \geq 3$ and k such that n is odd and $2^k \times n > 5040$, the inequality,*

$$\sigma(n) \geq e^{\gamma} \times 2^k \times n \times \log\log(2^k \times n) - (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

is true.

Proof The worst case of the previous inequality is when $k = 1$, that is when the right side becomes greater. In this way, if we prove this inequality for $k = 1$, then this will be true for the other possible cases. For $k = 1$, we have that,

$$\sigma(n) \geq e^{\gamma} \times 2 \times n \times \log\log(2 \times n) - (2^{1+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

which is the same as,

$$\sigma(n) \geq e^{\gamma} \times 2 \times n \times \log\log(2 \times n) - 2 \times e^{\gamma} \times n \times \log\log n$$

and finally,

$$\sigma(n) \geq e^{\gamma} \times 2 \times n \times (\log\log(2 \times n) - \log\log n).$$

The smallest value of $\sigma(n)$ is when n is prime where we have $\sigma(n) = n + 1$, because of $n \geq 3$. In this way, this would be the worst case for the previous inequality and if we prove this for the worst case, then we are proving for the other possible cases as well. When $\sigma(n) = n + 1$, then we would have that,

$$n + 1 \geq e^{\gamma} \times 2 \times n \times (\log\log(2 \times n) - \log\log n)$$

that is equivalent to,

$$1 + \frac{1}{n} \geq e^{\gamma} \times 2 \times (\log\log(2 \times n) - \log\log n)$$

if we divide by n and that is trivially true due to,

$$1 > 4 \times (\log\log(2 \times n) - \log\log n) > e^{\gamma} \times 2 \times (\log\log(2 \times n) - \log\log n)$$

since $e^{\gamma} \times 2 < 4$ and $(\log\log(2 \times n) - \log\log n) < \frac{1}{4}$ for $2 \times n > 5040$. In this way, the proposed inequality was proven. ∎

Theorem 2.4 *If the Robin's inequality is false for some natural number $m = 2^k \times n > 5040$ such that k and $n \geq 3$ are positive integers and n is odd, then this will be false for n as well.*

Proof Suppose that the Robin's inequality is false for some natural number $m = 2^k \times n > 5040$ such that k and $n \geq 3$ are positive integers and n is odd. Hence, we would have that,

$$\sigma(m) \geq e^{\gamma} \times m \times \log\log m.$$

However, we know that $\sigma(m) = (2^{k+1} - 1) \times \sigma(n)$ according to Theorem 2.2 and thus,

$$(2^{k+1} - 1) \times \sigma(n) \geq e^{\gamma} \times m \times \log\log m.$$

We know the following inequality,

$$\sigma(n) \geq e^{\gamma} \times m \times \log\log m - (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

is true according to Theorem 2.3. However, the inequality,

$$\sigma(n) \geq e^{\gamma} \times m \times \log\log m - (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

is equivalent to,

$$(2^{k+1} - 1) \times \sigma(n) - (2^{k+1} - 2) \times \sigma(n) \geq$$
$$e^{\gamma} \times m \times \log\log m - (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n.$$

If we subtract the inequality,

$$(2^{k+1} - 1) \times \sigma(n) \geq e^{\gamma} \times m \times \log\log m$$

with,

$$(2^{k+1} - 1) \times \sigma(n) - (2^{k+1} - 2) \times \sigma(n) \geq$$
$$e^{\gamma} \times m \times \log\log m - (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

then, we obtain that,

$$(2^{k+1} - 1) \times \sigma(n) - (2^{k+1} - 1) \times \sigma(n) + (2^{k+1} - 2) \times \sigma(n) \geq$$
$$e^{\gamma} \times m \times \log\log m - e^{\gamma} \times m \times \log\log m + (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

and finally,

$$(2^{k+1} - 2) \times \sigma(n) \geq (2^{k+1} - 2) \times e^{\gamma} \times n \times \log\log n$$

which is the same as,

$$\sigma(n) \geq e^{\gamma} \times n \times \log\log n$$

if we divide by $(2^{k+1} - 2)$. Consequently, we obtain the Robin's inequality will be false for n as well. ■

Theorem 2.5 *If the Robin's inequality is true for every natural number* $m = 2^k \times p > 5040$ *such that* k *is a positive integer and* $p \in \{3, 13\}$*, then the Robin's inequality is true for every natural number* $n > 5040$*.*

Proof In previous Theorem 2.4, we prove if the Robin's inequality is false for some natural number $m > 5040$, then there exists an odd natural number $n \geq 3$ for which the Robin's inequality is false or m is in the form of $2^k \times n > 5040$ where $n = 1$ for $k > 2$. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [2]. The only possible candidates are 1, 3, 5 and 9 [2]. However, we know that $\sigma(1) = 1$, $\sigma(3) = 4$, $\sigma(5) = 2 \times 3$ and $\sigma(9) = 13$. In this way, the possible counterexamples can be of the form $2^k \times n > 5040$ where $n \in \{1, 3, 13\}$ for $k > 2$. We recall that an integer m is said to be squarefull if for every prime divisor p of m we have $p^2 \mid m$ [2]. As result, the numbers in the form $2^k \times n > 5040$ where $n = 1$ for $k > 2$ are squarefull, because of they are divisible by 4. Nevertheless, the only squarefull integers which does not comply with the Robin's inequality are 1, 4, 8, 9, 16 and 36 and they are lesser than 5040 [2]. Hence, the only remaining options are the natural numbers $m = 2^k \times p > 5040$ such that k is a positive integer and $p \in \{3, 13\}$. Under this assumption, we obtain that the Robin's inequality is true for every natural number $n > 5040$. ■

Theorem 2.6 *The Robin's inequality is true for every natural number* $m = 2^k \times p > 5040$ *such that* k *is a positive integer and* $p = 13$*.*

Proof The Robin's inequality is true for every natural number $n = 2^k$ such that $k > 8$ is a positive integer, because it is a squarefull [2]. Hence, we would have that,

$$\sigma(n) < e^{\gamma} \times n \times \log\log n$$

and if we multiply by 13 the two sides of this inequality, then we obtain that,

$$13 \times \sigma(n) < e^{\gamma} \times n \times (13 \times \log\log n).$$

If we sum both inequalities, then we have that,

$$\sigma(n) + 13 \times \sigma(n) < e^{\gamma} \times n \times \log\log n + e^{\gamma} \times n \times (13 \times \log\log n)$$

and that will be equivalent to,

$$\sigma(13 \times n) < e^{\gamma} \times n \times (14 \times \log\log n).$$

Certainly, it is trivial that $\sigma(13 \times n) = \sigma(n) + 13 \times \sigma(n)$ under the properties of the divisor function over the numbers $n = 2^k$ and 13 since they are coprimes (relative primes) [5]. In addition, we know that,

$$e^{\gamma} \times n \times \log\log n + e^{\gamma} \times n \times (13 \times \log\log n)$$

is equal to,

$$e^{\gamma} \times n \times (\log\log n + 13 \times \log\log n)$$

that is finally,

$$e^{\gamma} \times n \times (14 \times \log\log n).$$

In addition, if the Robin's inequality would be true for $13 \times n > 5040$, then we would have that,

$$\sigma(13 \times n) < e^{\gamma} \times (13 \times n) \times \log\log(13 \times n)$$

which is the same that,

$$\sigma(13 \times n) < e^{\gamma} \times n \times (13 \times \log\log(13 \times n)).$$

However, we know that,

$$e^{\gamma} \times n \times (14 \times \log\log n) < e^{\gamma} \times n \times (13 \times \log\log(13 \times n))$$

because we have that,

$$14 \times \log\log n < 13 \times \log\log(13 \times n)$$

is true for $13 \times n > 5040$ and $n = 2^k$ such that $k > 8$ is a positive integer. Consequently, we obtain that,

$$\sigma(13 \times n) < e^{\gamma} \times (13 \times n) \times \log\log(13 \times n)$$

is indeed true and thus, the Robin's inequality is true for $13 \times 2^k > 5040$ such that $k > 8$. To sum up, since $k > 8$ was arbitrarily selected, then we have this Theorem is true. ■

Theorem 2.7 *The Robin's inequality is true for every natural number $m = 2^k \times p > 5040$ such that k is a positive integer and $p = 3$.*

Proof For a natural number $m = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_s^{a_s}$ such that $p_1, p_2, \ldots, p_s$ are prime numbers, then we obtain the following formula,

$$\sigma(m) = \prod_{i=1}^{s} \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

from the Ramanujan's notebooks [1]. In this way, for every natural number $m = 2^k \times 3 > 5040$ such that $k > 10$ is a positive integer, then we have that,

$$\sigma(m) = \frac{2^{k+1} - 1}{2 - 1} \times \frac{3^2 - 1}{3 - 1}$$

which is the same as,

$$\sigma(m) = \frac{(2^{k+1} - 1) \times (3^2 - 1)}{2}.$$

Therefore, we need to prove that,

$$\sigma(m) < e^{\gamma} \times m \times \log\log m$$

which is equivalent to,

$$\frac{(2^{k+1}-1)\times(3^2-1)}{2} < e^{\gamma}\times 2^k \times 3 \times \log\log(2^k \times 3)$$

and thus,

$$\frac{(2^{k+1}-1)\times(3^2-1)}{2}\times\frac{1}{2^k\times 3} < e^{\gamma}\times \log\log(2^k \times 3)$$

but we have that,

$$\frac{(2^{k+1}-1)\times(3^2-1)}{2}\times\frac{1}{2^k\times 3} < \frac{2^{k+1}\times 3^2}{2}\times\frac{1}{2^k\times 3}$$

which also complies that,

$$\frac{2^{k+1}\times 3^2}{2}\times\frac{1}{2^k\times 3} = 3.$$

However, this needs to comply with,

$$e^{\gamma}\times\log\log 5040 < e^{\gamma}\times\log\log(2^k\times 3)$$

and we can easily check that,

$$3 < e^{\gamma}\times\log\log 5040$$

and as a consequence, the Robin's inequality is true for the natural number $m = 2^k \times 3 > 5040$ such that $k > 10$ is a positive integer. In conclusion, since $k > 10$ was arbitrarily selected, then we have this Theorem is true. ■

Theorem 2.8 *The Robin's inequality is true for every natural number $n > 5040$.*

Proof This is a direct consequence of Theorems 2.5, 2.6 and 2.7. ■

Theorem 2.9 *The Riemann hypothesis is true.*

Proof If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann hypothesis is true [3]. Hence, the Riemann hypothesis is true due to Theorem 2.8. ■

3 Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [4]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [4]. In this way, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [4].

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