Acoustic characterization of absorbing materials using dynamic mode decomposition techniques

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MÁSTER EN MATEMÁTICA INDUSTRIAL

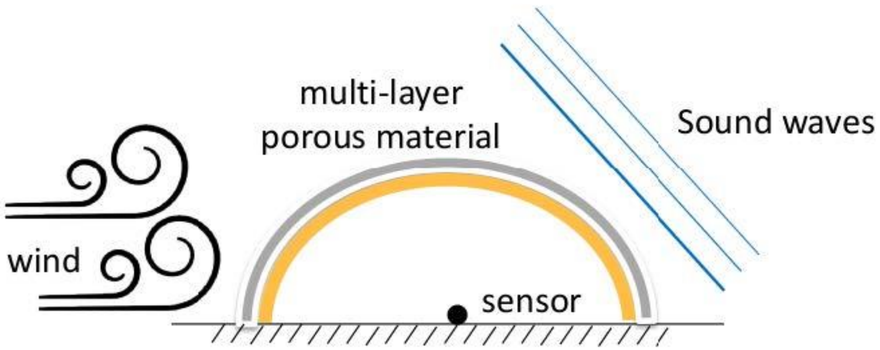
Companies Involved

- **ITMATI** (Thechnological Institute for Industrial Mathematics)
- **Microflown Technologies**
- **ROMSOC Project** (Reduced Order Modeling, Simulation and Optimization of Coupled Systems)

Introduction

Ashwin Nayak's PhD Thesis:
3D unbounded coupled model in the frequency domain

Simplified Problem:
1D acoustic coupled model in the time domain

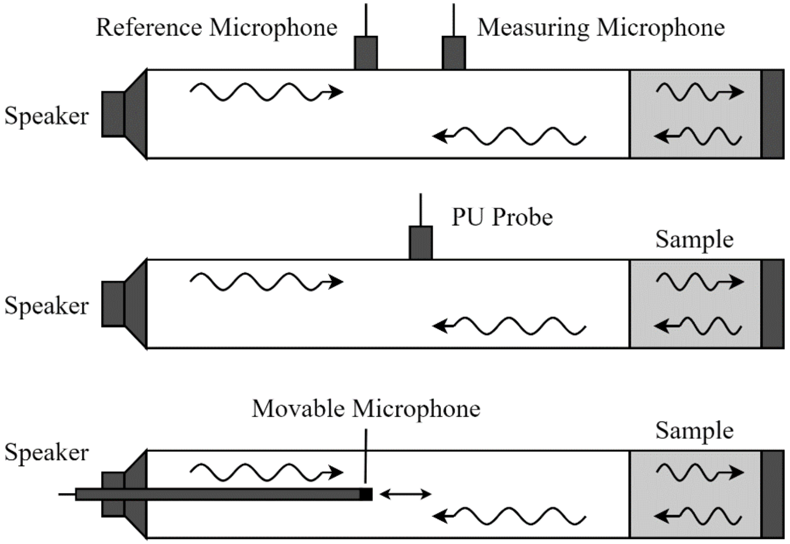


Phases of the project:

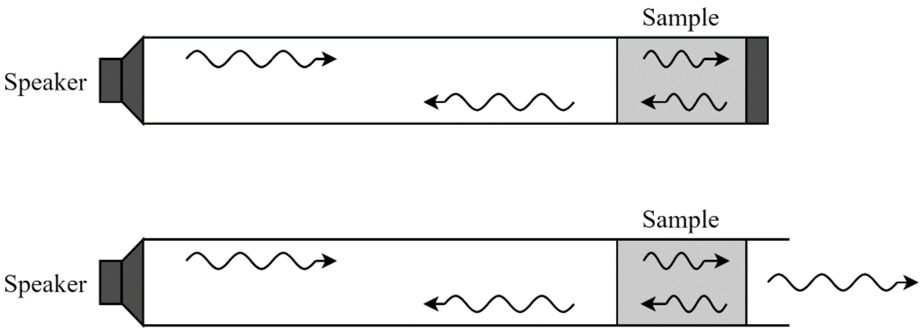
1. Numerical simulation
2. Validating the models
3. Applying DMD

Motivation of the Physical Setting

Experimental Settings



Simulated settings



Holmarc Opto-Mechatronics Impedance tube*

* Holmarc Opto-Mechatronics Pvt. Ltd.: Impedance Tube Apparatus. https://holmarc.com/impedance_tube_apparatus_a.php

Used Models

Fluid Models:

$\mathcal{P}_1$: Fluid with rigid boundaries

$\mathcal{P}_2$: Fluid with rigid-transparent boundaries

Rigid Porous Models:

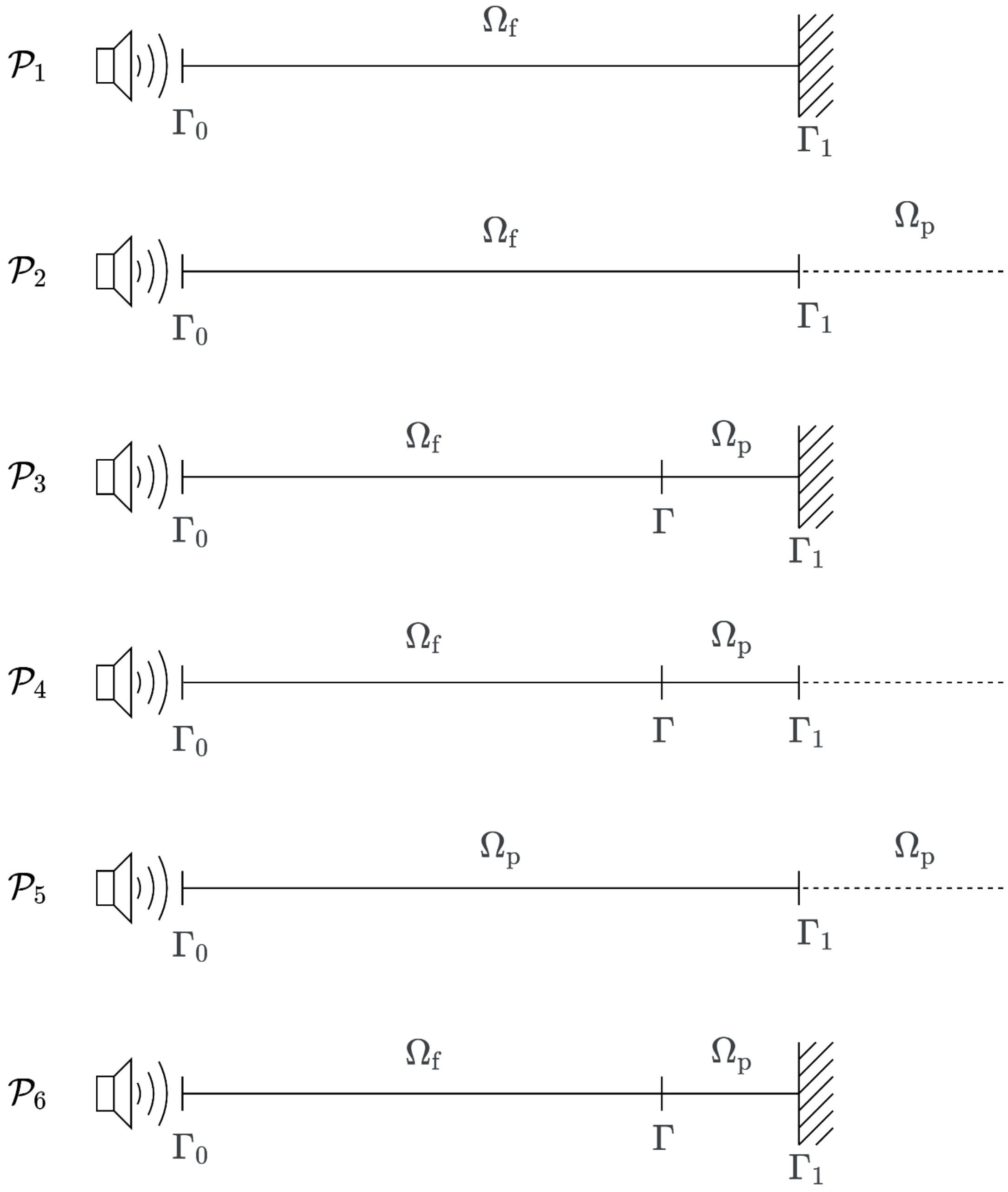
$\mathcal{P}_3$: Coupled with rigid boundaries

$\mathcal{P}_4$: Coupled with rigid-transparent boundaries

Poro-Elastic Models:

$\mathcal{P}_5$: Umnova's low frequency aprox. on porous

$\mathcal{P}_6$: Umnova's low frequency aprox. Coupled



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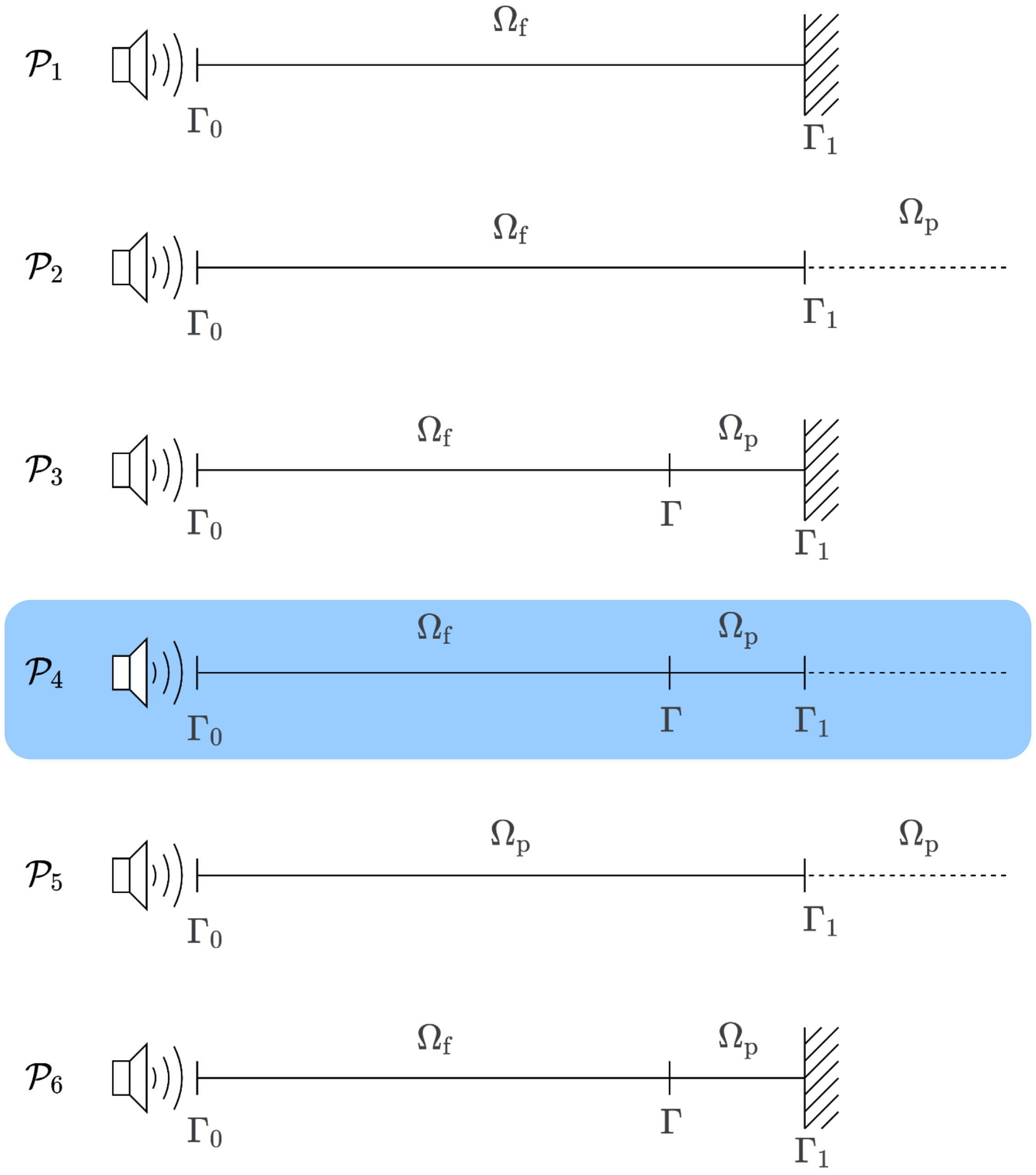
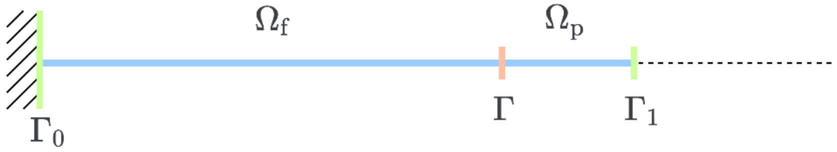


Differential Formulation

Fluid – Rigid Porous Coupled Model



$$
\begin{cases}
\rho_f \partial_t^2 u_f - \rho_f c_f^2 \partial_x^2 u_f = f & \text{in } (0,T) \times \Omega_f, \\
\rho_p \partial_t^2 u_p - \dfrac{\rho_p c_p^2}{\phi \gamma_p} \partial_x^2 u_p + \sigma \partial_t u_p = 0 & \text{in } (0,T) \times \Omega_p, \\
u_f = \phi u_p & \text{on } (0,T) \times \Gamma, \\
\rho_f c_f^2 \partial_x u_f = \dfrac{\rho_p c_p^2}{\phi \gamma_p} \partial_x u_p & \text{on } (0,T) \times \Gamma, \\
u_f = 0 & \text{on } (0,T) \times \Gamma_0, \\
u_f = \phi u_p & \text{on } (0,T) \times \Gamma_1, \\
\rho_f c_f^2 \partial_x u_f = \dfrac{\rho_p c_p^2}{\phi \gamma_p} \partial_x u_p & \text{on } (0,T) \times \Gamma_1, \\
\partial_t u_f + c_f \partial_x u_f = 0 & \text{on } (0,T) \times \Gamma_1.
\end{cases}
$$

Interior domain

Coupling boundary

Exterior boundaries

Differential Formulation

Fluid – Rigid Porous Coupled Model

$$\tilde{u}_p = \phi u_p$$

$$\begin{cases} \rho_f \partial_t^2 u_f - \rho_f c_f^2 \partial_x^2 u_f = f & \text{in } (0,T) \times \Omega_f, \\ \dfrac{\rho_p}{\phi} \partial_t^2 \tilde{u}_p - \dfrac{\rho_p c_p^2}{\phi^2 \gamma_p} \partial_x^2 \tilde{u}_p + \dfrac{\sigma}{\phi} \partial_t \tilde{u}_p = 0 & \text{in } (0,T) \times \Omega_p, \\ u_f = \tilde{u}_p & \text{on } (0,T) \times \Gamma, \\ \rho_f c_f^2 \partial_x u_f = \dfrac{\rho_p c_p^2}{\phi^2 \gamma_p} \partial_x \tilde{u}_p & \text{on } (0,T) \times \Gamma, \\ u_f = 0 & \text{on } (0,T) \times \Gamma_0, \\ \partial_t \tilde{u}_p + \dfrac{\rho_p c_p^2}{\rho_f c_f \phi^2 \gamma_p} \partial_x \tilde{u}_p = 0 & \text{on } (0,T) \times \Gamma_1. \end{cases}$$

Interior domain

Coupling boundary

Exterior boundaries

Variational Formulation

Fluid – Rigid Porous Coupled Model

$$\int_{\Omega_\mathrm{f}} \rho_\mathrm{f} \partial_t^2 u_\mathrm{f} \cdot w \, dx + \int_{\Omega_\mathrm{p}} \frac{\rho_\mathrm{p}}{\phi} \partial_t^2 \tilde{u}_\mathrm{p} \cdot w \, dx$$ → Mass term

$$+ \int_{\Omega_\mathrm{p}} \frac{\sigma}{\phi} \partial_t \tilde{u}_\mathrm{p} \cdot w \, dx + (\rho_\mathrm{f} c_\mathrm{f} \partial_t \tilde{u}_\mathrm{p} \cdot w|_{\Gamma_1}$$ → Damping term

$$+ \int_{\Omega_\mathrm{f}} \rho_\mathrm{f} c_\mathrm{f}^2 \partial_x u_\mathrm{f} \cdot \partial_x w \, dx + \int_{\Omega_\mathrm{p}} \frac{\rho_\mathrm{p} c_\mathrm{p}^2}{\phi^2 \gamma_\mathrm{p}} \partial_x \tilde{u}_\mathrm{p} \cdot \partial_x w \, dx$$ → Stiffness term

$$= \int_{\Omega_\mathrm{f}} f \cdot w \, dx.$$ → External forces

Discretization Algorithms
Finite Element Method

Spatial Discretization: Piecewise linear finite element method

Basis functions

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{\Delta x} & \text{in } x_{i-1} < x \le x_i, \\ -\dfrac{x - x_{i+1}}{\Delta x} & \text{in } x_i < x \le x_{i+1}, \\ 0 & \text{elsewhere }, \end{cases}$$

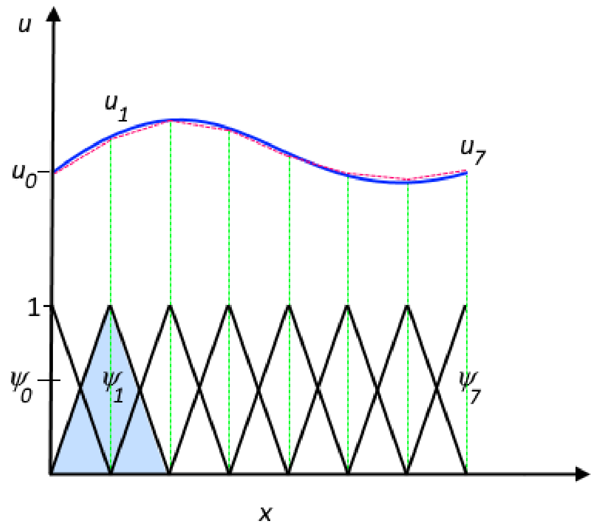
Discrete solution

$$u^h(t,x) = \sum_{i=0}^{N} u_i^h(t)\psi_i(x),$$

$$w^h(x) = \sum_{i=0}^{N} w_i^h \psi_i(x).$$

Finite Element method discretization

$$\underbrace{\sum_{i=0}^{N} M_{j,i}\ddot{u}_i^h}_{\text{Mass}} + \underbrace{\sum_{i=0}^{N} C_{j,i}\dot{u}_i^h}_{\text{Damping}} + \underbrace{\sum_{i=0}^{N} K_{j,i}u_i^h}_{\text{Stiffness}} = \underbrace{f_j}_{\text{External Forces}}$$



Discretization Algorithms
Newmark-Beta Method

Time Discretization: Newmark-beta integration method

$$M\ddot{u} + C\dot{u} + Ku = f$$

$$\begin{cases} u^{n+1} = u^n + \Delta t \dot{u}^n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{u}^n + \beta \Delta t^2 \ddot{u}^{n+1}, \\ \dot{u}^{n+1} = \dot{u}^n + (1-\gamma)\,\Delta t \ddot{u}^n + \gamma \Delta t \ddot{u}^{n+1}, \\ M\ddot{u}^{n+1} + C\dot{u}^{n+1} + Ku^{n+1} = f^{n+1}. \end{cases}$$

In the implementation it is divided in initialization, explicit approximation and prediction.

In order to get accurate results it meets the CFL condition:

$$C = \frac{\Delta t}{\Delta x} c_{\mathrm{f}} \leq C_{\max}.$$

Reduced Order Methods
Singular Value Decomposition

- **Singular Value Decomposition (SVD):** Exact decomposition

$$A = U\Sigma V^T$$

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}_{n\times n}}_{\text{Singular left vectors}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{\min(n,m)} \end{bmatrix}_{n\times m}}_{\text{Singular values}} \underbrace{\begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_m & — \end{bmatrix}_{m\times m}}_{\text{Singular right vectors}} .$$

- **Truncated SVD:** Approximated decomposition

$$A \approx U_r\Sigma_r V_r^T$$

$$A \approx \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_r \\ | & | & & | \end{bmatrix}_{n\times r}}_{\text{Singular left vectors}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}_{r\times r}}_{\text{Singular values}} \underbrace{\begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_r & — \end{bmatrix}_{r\times m}}_{\text{Singular right vectors}} .$$

Reduced Order Methods
Dynamic Mode Decomposition

- **Dynamic Mode Decomposition (DMD):** Approximate prediction capabilities
  1. Define data matrices

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \ldots & x_{m-1} \\ | & | & & | \end{bmatrix}, \quad X' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \ldots & x_m \\ | & | & & | \end{bmatrix}.$$

  2. Perform **Truncated SVD** to get U, Σ and V.
  3. Calculate $\tilde{A} = U^T X' V \Sigma^{-1}$.
  4. Calculate **eigenvalues** (in Λ) and **eigenvectors** (in W) of $\tilde{A}$.
  5. Calculate

$$\Phi = X' V \Sigma^{-1} W,$$ → Modes

$$b = \Phi^* x_1,$$ → Initial amplitude

$$\Omega = \mathrm{diag}(\omega_k), \quad \omega_k = \log(\lambda_k)/\Delta t.$$ → Continuous time eigenvalues

  6. Reconstruct data:

$$x(t) \approx \Phi \; e^{\Omega t} b$$

Modes (Φ) Dynamics ($e^{\Omega t} b$)

Software

- **FEniCS Project**: Finite element method
- **PyDMD**: Dynamic mode decomposition
- **Other Software**: ParaView, scikit-learn package, Docker, MATLAB…

Test Cases

- **Error Control Through Space and Time Step Size**

  Validates discretization methods (FEM and Newmark) using:

  $\mathcal{P}_1$ in smooth impulse response.
  $\mathcal{P}_1$ in sharp impulse response
  $\mathcal{P}_1$ in harmonic regime
  $\mathcal{P}_2$ in smooth impulse response
  $\mathcal{P}_2$ in sharp impulse response
  $\mathcal{P}_2$ in harmonic regime

- **Exact Result Using d'Alembert's Solution**

  Validates fluid models with null Dirichlet condition on Γ_0:

  $\mathcal{P}_1$ in smooth impulse response
  $\mathcal{P}_1$ in sharp impulse response
  $\mathcal{P}_2$ in smooth impulse response
  $\mathcal{P}_2$ in sharp impulse response

- **Exact Result Using Harmonic Solution**

  Validates fluid models with non-zero Dirichlet condition on Γ_0:

  $\mathcal{P}_1$ in harmonic regime
  $\mathcal{P}_2$ in harmonic regime

  Validates fluid - rigid porous coupled models with non-zero Dirichlet condition on Γ_0:

  $\mathcal{P}_3$ in harmonic regime
  $\mathcal{P}_4$ in harmonic regime

- **Umnova's Low Frequency Approximation Comparison**

  Validates poro-elastic model: $\mathcal{P}_5$

Test Cases

- **Error Control Through Space and Time Step Size**
  Validates discretization methods (FEM and Newmark) using:
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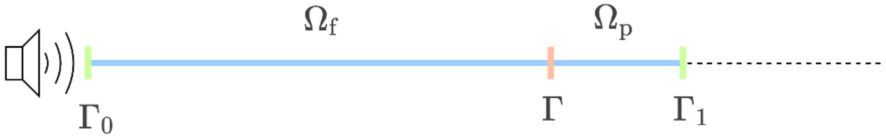
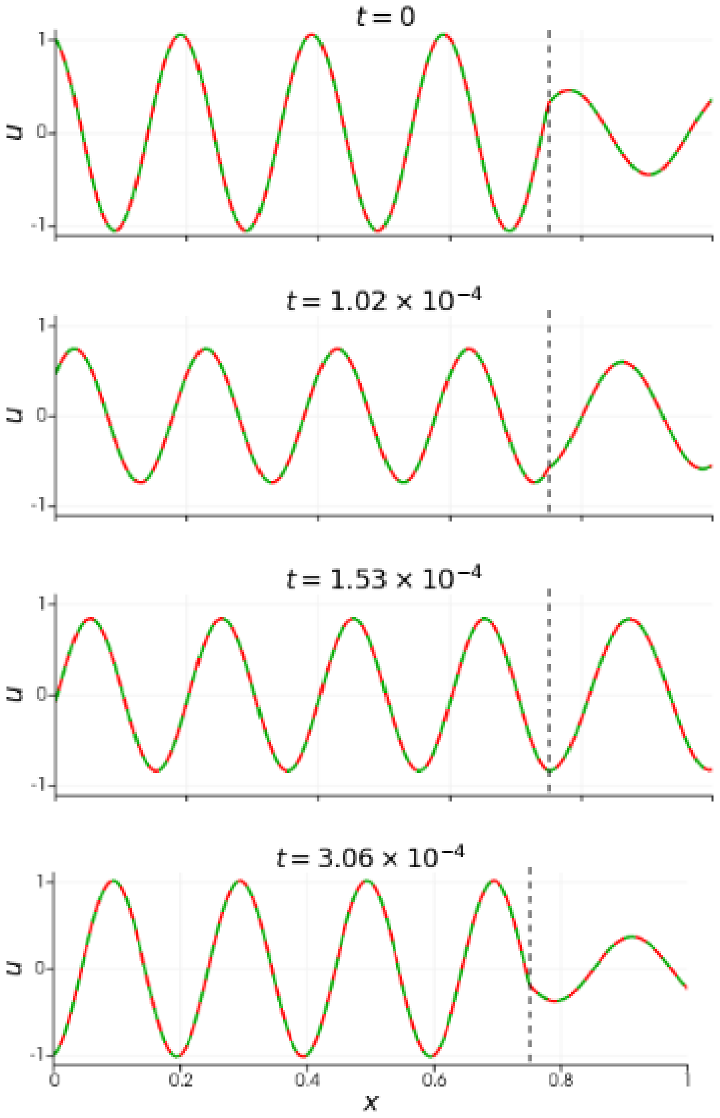
  - $\mathcal{P}_3$ in harmonic regime
  - **$\mathcal{P}_4$ in harmonic regime**

- **Umnova's Low Frequency Approximation Comparison**
  Validates poro-elastic model: $\mathcal{P}_5$

Harmonic Solution



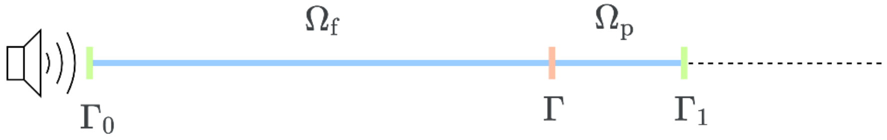


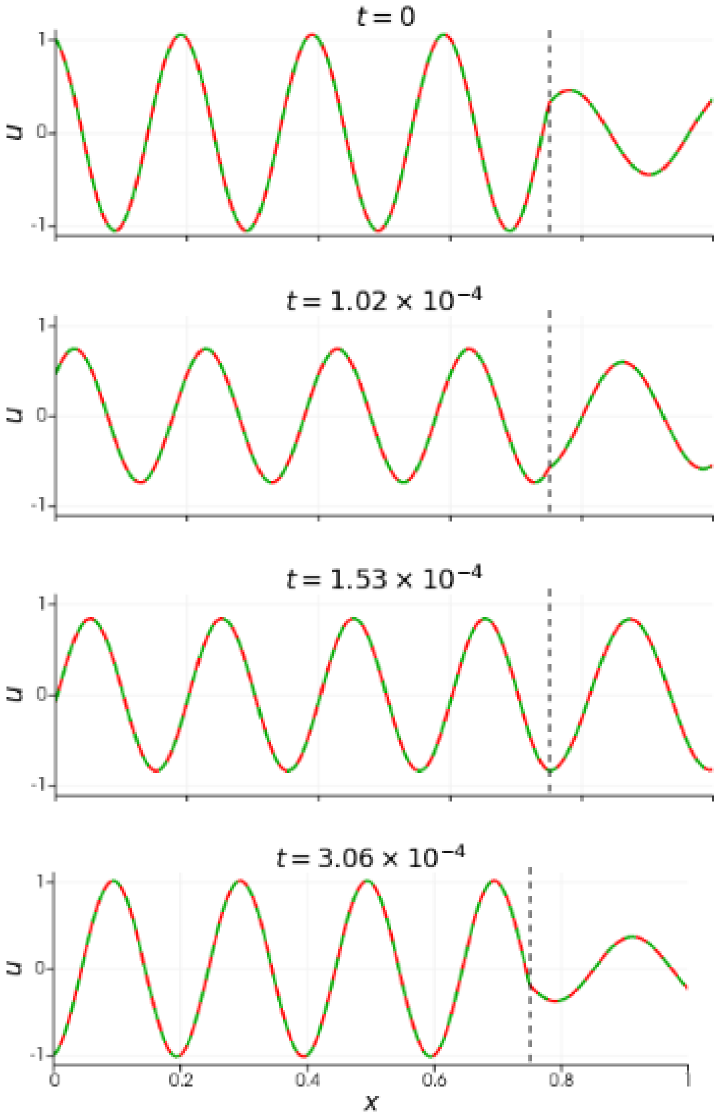
1. Transform the time domain equations to the frequency domain.
2. Solve the Helmholtz problem.
3. Transform back the exact solution to the time domain.
4. Find the initial conditions for the simulation.

$$
\begin{cases}
-\omega^2\rho_\mathrm{f}U_\mathrm{f} - \rho_\mathrm{f}c_\mathrm{f}^2\partial_x^2 U_\mathrm{f} = 0 & \text{in } \Omega_\mathrm{f}, \\
-\left(\omega^2\rho_\mathrm{p} + i\omega\sigma\right)\tilde{U}_\mathrm{p} - \dfrac{\rho_\mathrm{p}c_\mathrm{p}^2}{\phi\gamma_\mathrm{p}}\partial_x^2\tilde{U}_\mathrm{p} = 0 & \text{in } \Omega_\mathrm{p}, \\
U_\mathrm{f} = \tilde{U}_\mathrm{p} & \text{on } \Gamma, \\
\rho_\mathrm{f}c_\mathrm{f}^2\partial_x U_\mathrm{f} = \dfrac{\rho_\mathrm{p}c_\mathrm{p}^2}{\phi\gamma_\mathrm{p}}\partial_x\tilde{U}_\mathrm{p} & \text{on } \Gamma, \\
U_\mathrm{f} = 1 & \text{on } \Gamma_0, \\
-i\omega\tilde{U}_\mathrm{p} + \dfrac{\rho_\mathrm{p}c_\mathrm{p}^2}{\rho_\mathrm{f}c_\mathrm{f}\phi^2\gamma_\mathrm{p}}\partial_x\tilde{U}_\mathrm{p} & \text{on } \Gamma_1.
\end{cases}
$$

Fluid – rigid porous coupled model with rigid-transparent boundaries.

Harmonic Solution





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Sound speed in fluid	c_f	343 m/s
Fluid density	ρ_f	1.21 kg/m^3
Sound speed in porous	c_p	350 m/s
Porous density	ρ_f	1.5 kg/m^3
Porosity	ϕ	0.5
Specific heat capacity ratio	γ_p	1.4
Flux resistivity	σ	100 N s/m^4
Time step	Δt	7.29×10^{-6}s
Space step	Δx	2.5×10^{-3}m

Fluid – rigid porous coupled model with rigid-transparent boundaries.

Numerical Results

- Harmonic Waves
  - Reconstruction and prediction of harmonic waves.
  - Test: Reducing the number of snapshots.
  - Test: Increasing the discretization size.
- Periodic Impulse Responses
  - Reconstruction and prediction of periodic impulse responses.
  - Test: Reducing the number of snapshots.
  - Test: Reducing the DMD rank.
- Non-Periodic Impulse Responses
  - Reconstruction and prediction of non-periodic impulse responses.
- Other Approaches
  - Test: DMD vs. HODMD.
  - Test: DMD vs. SVD.
  - Simulation mixing.
  - Shifted DMD.

Numerical Results

- Harmonic Waves

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  Reconstruction and prediction of non-periodic impulse responses.

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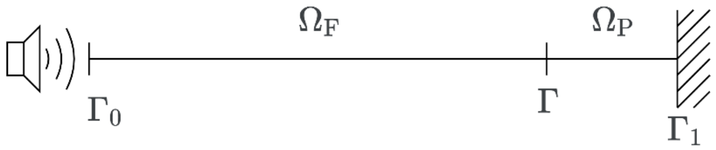
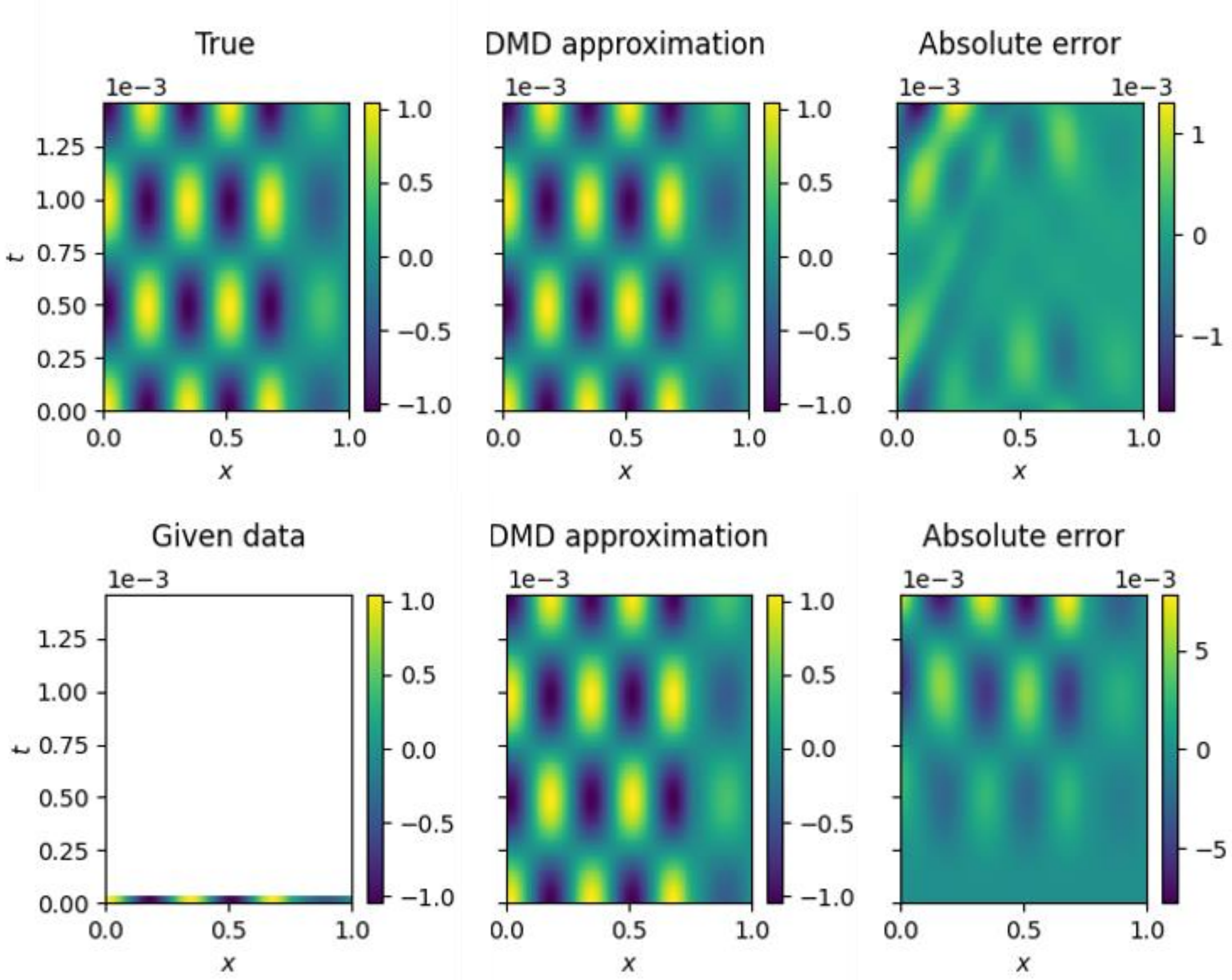
Harmonic Waves



Reconstruction

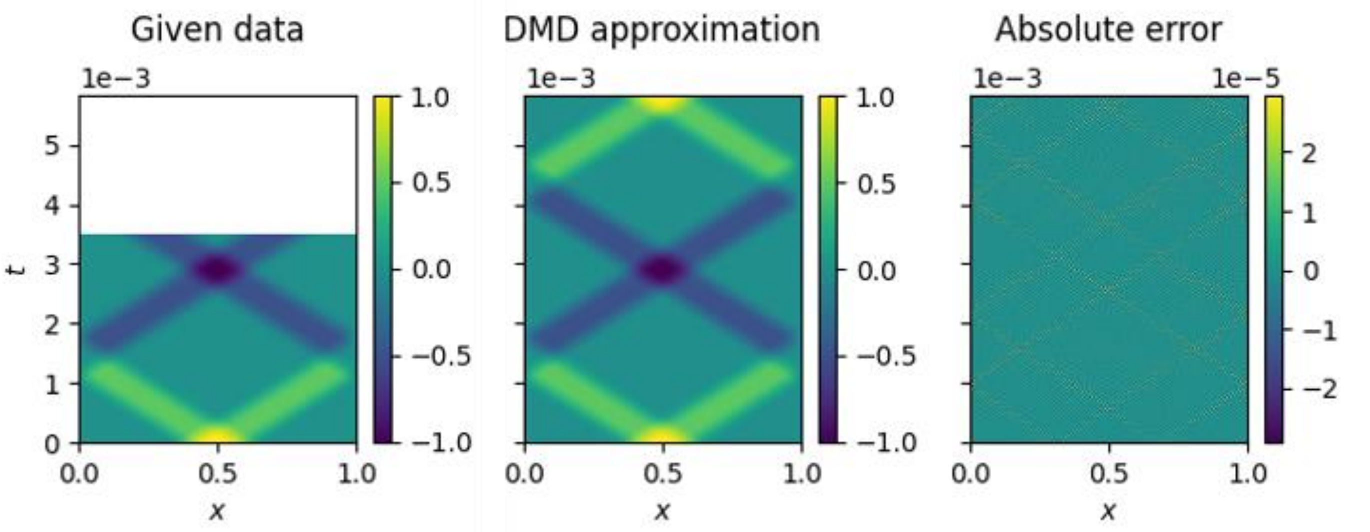
Prediction



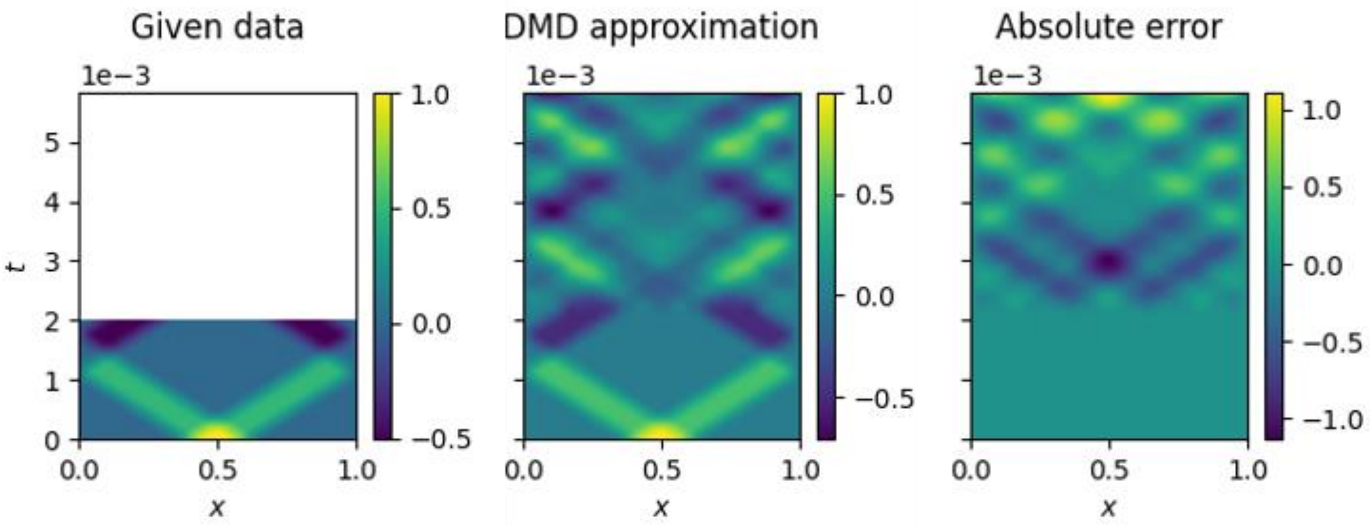
Δt	1.46×10^{-5} s
Δx	5×10^{-3} m
r	2
d	5

Periodic Impulses

Over a period

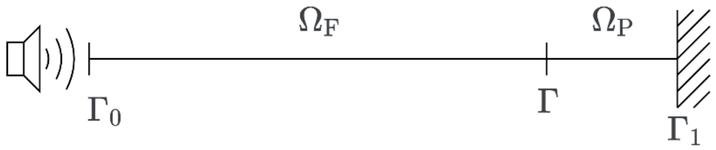


Under a period



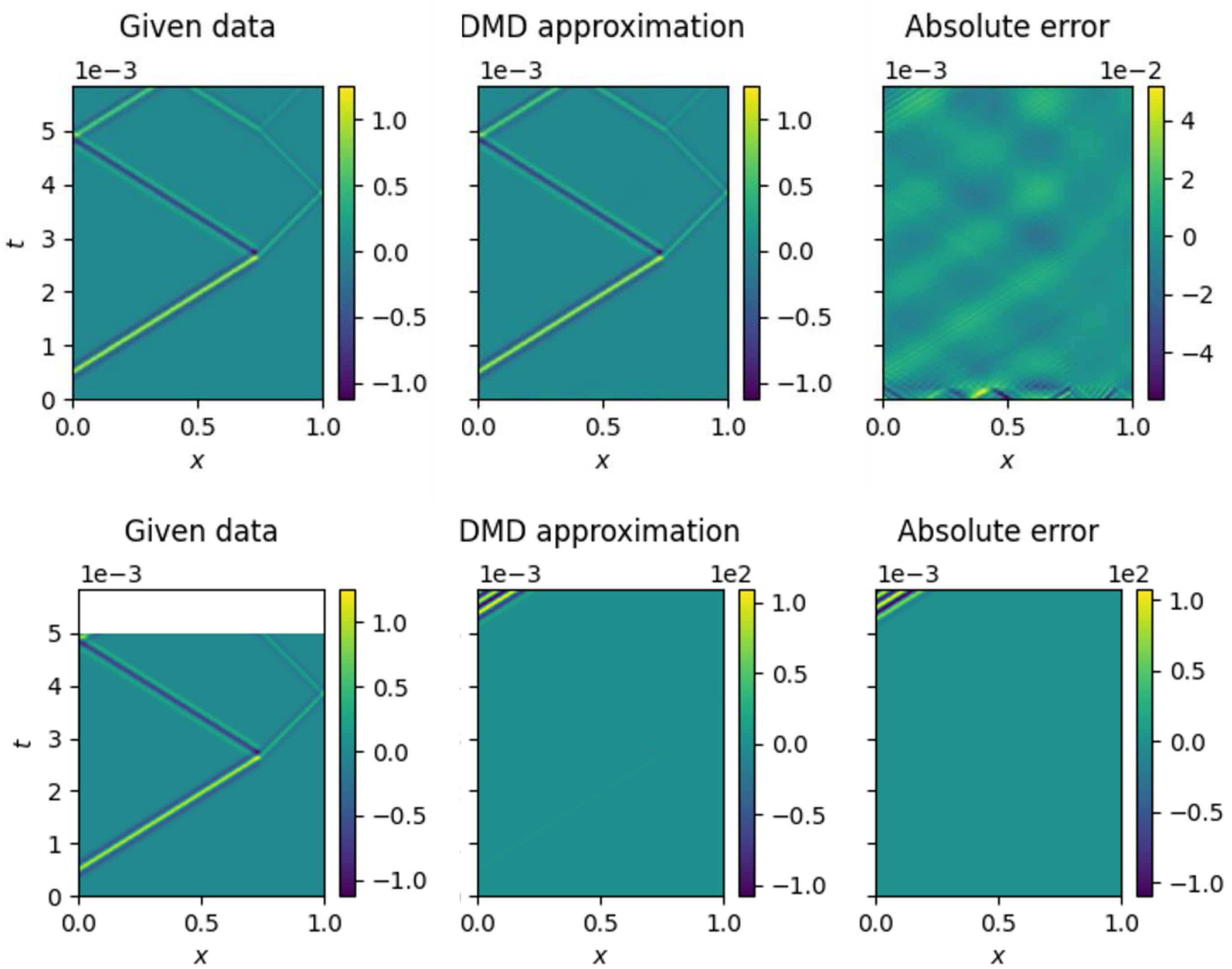
Δt	5.83×10^{-6} s
Δx	2×10^{-3} m
r	100
d	5

Non-Periodic Impulses



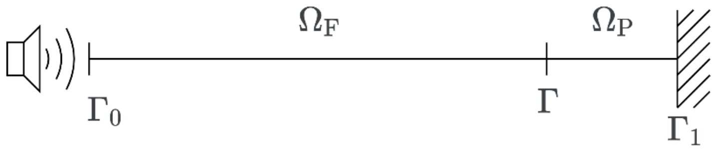
Reconstruction

Prediction

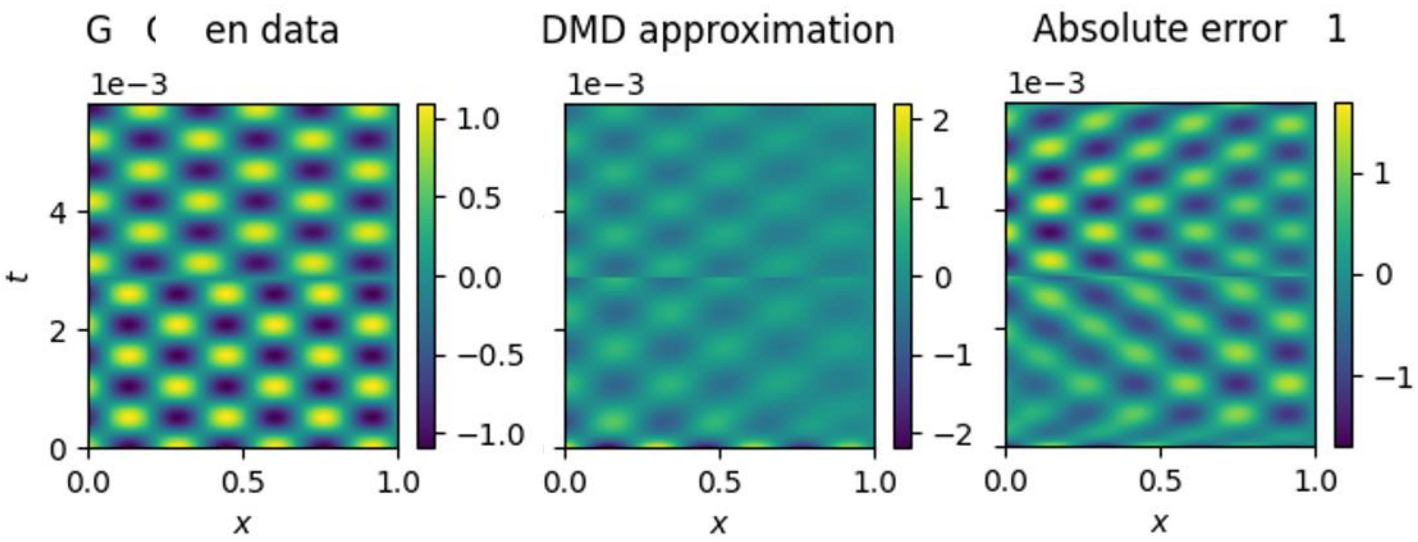


Δt	7.29×10^{-6} s
Δx	2.5×10^{-3} m
r	100
d	5

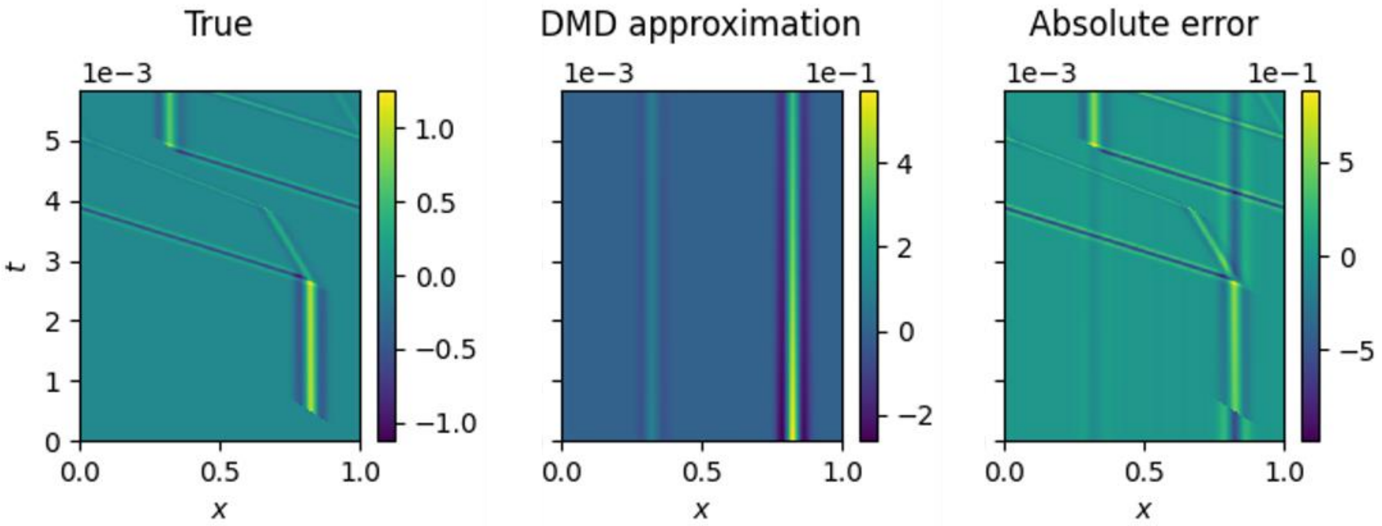
Other approaches



Simulation Mixing



Shifted DMD



Summary

- **Work completed:**

  6 Acoustic models

  15 Test cases

  11 ROMs scenarios

- **Conclusions:**

  Accurate results in **harmonic** cases: only 3 snapshots needed.

  Accurate results in **periodic impulse response** cases: a single wave cycle needed.

  Unfeasible use in **non-periodic impulse response** cases: unable to make predictions.

  Unfeasible use using **simulation mixing**.

- **Other conclusions** (not included in the slides)**:**

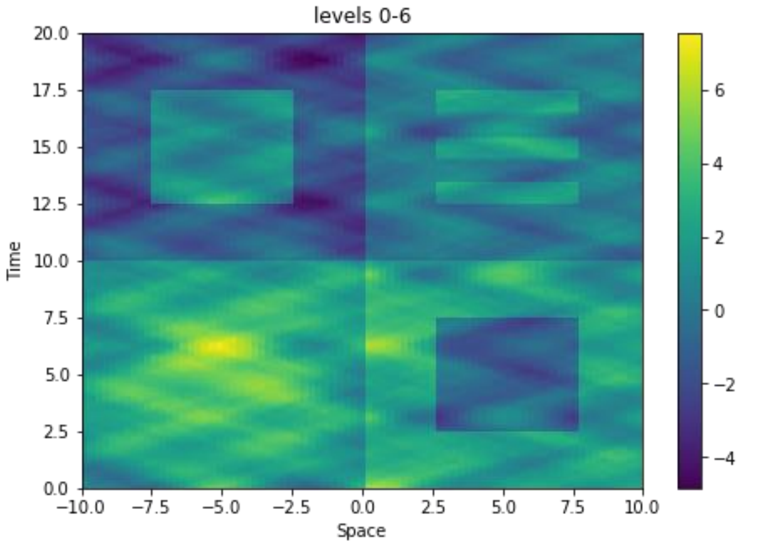
  Discretization size can affect results.

  Increased DMD rank improves predictions.

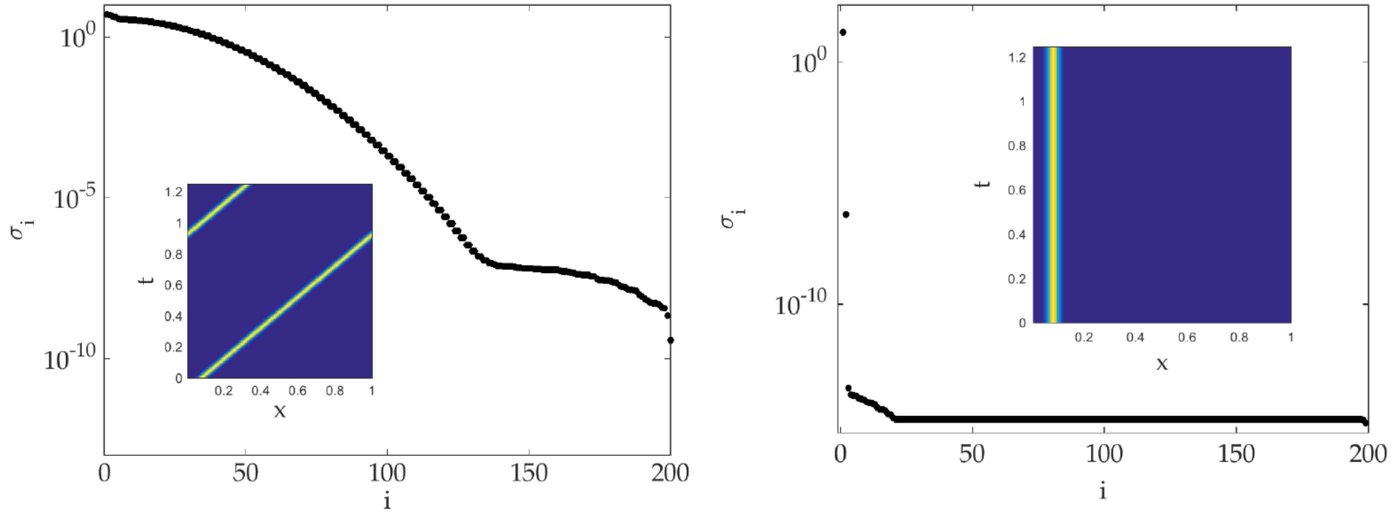
  HODMD is more versatile but requires tuning.

Next Steps

- Explore predictions using MrDMD.
- Extend to 3D models.
- Explore Shifted DMD for non-periodic impulse responses.
- Develop poro-elastic high frequency models.



MrDMD*



Shifted POD**