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A PROOF OF THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

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The classical theorem of the means states that for every sequence of m positive numbers $\{x_i\}$

$$A_m = \frac{1}{m}\sum_{i=1}^{m} x_i \geqslant \prod_{i=1}^{m} x_i^{1/m} = G_m \qquad \text{(AG)}$$

with equality just in case the sequence is constant.

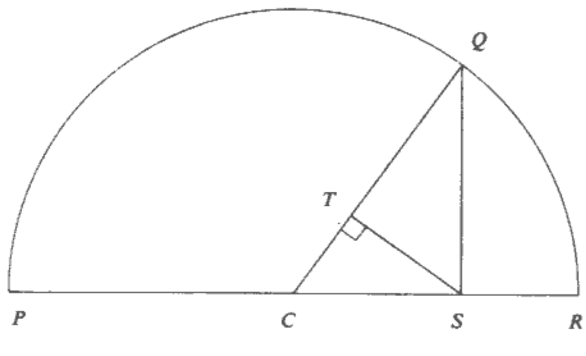

FIG. 1

Many proofs of this fundamental inequality have been discovered in modern times ([**1**], [**2**], [**3**]), but in its simplest setting—for only two numbers—the ancients possessed a singularly neat picture (see Pappus [**4**, Book 3, p. 51]), which makes matters readily apparent (see Fig. 1). Here, $\overline{CQ}$ is the arithmetic mean of the two numbers (segments) $\overline{SP}$ and $\overline{SR}$, $\overline{SQ}$ their geometric mean, and, in addition, $\overline{QT}$ is their harmonic mean.

Is there an analogous picture for three [redacted] numbers?

During an unsuccessful attempt to obtain one, the following iterative procedure suggested itself:

$$x_1+x_2+x_3=\tfrac{1}{2}(x_1+x_2)+\tfrac{1}{2}(x_2+x_3)+\tfrac{1}{2}(x_1+x_3)\geqslant(x_1x_2)^{1/2}$$
$$+(x_2x_3)^{1/2}+(x_1x_3)^{1/2}\geqslant(x_1x_2^2x_3)^{1/4}+(x_1x_2x_3^2)^{1/4}+(x_1^2x_2x_3)^{1/4}\geqslant\cdots.$$

That is, write the first sum cyclically as a sum of averages, apply (AG) for two numbers, and then repeat. After several iterations (and some inspired guessing) the pattern of the resulting monotone sequence became clear: the exponents in the kth term are the entries of the kth power of a certain matrix, namely,

$$V=\tfrac{1}{2}(I+S)=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}.$$

Here I is the 3×3 identity matrix and S the (shift) permutation matrix $S=(\delta_{i+1,j})$. Since V is doubly stochastic, the fundamental theorem for transition matrices (see, for example, [5, Theorem 4.1.4]) implies that $\lim_{n\to\infty}V^n=\frac{1}{3}J$, where J is the matrix of all 1's, so that if we let

$$f_n=f(X,V^n)=\tfrac{1}{3}\sum_{i=1}^{3}\prod_{j=1}^{3}x_j^{(V^n)_{ij}}$$

then $f_0=A_3$ (as usual $V^0=I$), $\lim_{n\to\infty}f_n=G_3$, and (AG) for three numbers obtains as soon as we can show formally that f_n is nonincreasing. Now, the action of the (left) shift S on any matrix is just a cyclic permutation of the rows; so $f(X,V^n)=f(X,SV^n)$. Averaging, then, amounts to nothing more than writing f_n as

$$f_n=\tfrac{1}{2}(f(X,V^n)+f(X,SV^n));$$

that is,

$$f_n=\tfrac{1}{3}\sum_{i=1}^{3}\tfrac{1}{2}\left(\prod_{j=1}^{3}x_j^{(V^n)_{ij}}+\prod_{j=1}^{3}x_j^{(SV^n)_{ij}}\right).$$

Consequently,

$$f_n\geqslant\tfrac{1}{3}\sum_{i=1}^{3}\prod_{j=1}^{3}x_j^{(\frac{1}{2}(I+S)V^n)_{ij}}=f(X,V^{n+1})=f_{n+1}$$

by an application of (AG) for two numbers and the recurrence

$$\tfrac{1}{2}(I+S)V^n=VV^n=V^{n+1}.$$

As for the matter of equality, if $A_3=f_0=\lim_{n\to\infty}f_n=G_3$, then $f_0=f_1$ because f_n is monotone; so

$$\tfrac{1}{2}(x_1+x_2)+\tfrac{1}{2}(x_2+x_3)+\tfrac{1}{2}(x_1+x_3)=(x_1x_2)^{1/2}+(x_2x_3)^{1/2}+(x_1x_3)^{1/2},$$

i.e., $x_1=x_2=x_3$.

To obtain a proof of the (AG) inequality for m numbers, all that need be done is to take the matrix V to be of size m, thereby replacing 3 by m everywhere, and the above argument goes through entirely unchanged.

It is interesting to note that the convergence (V is now $m\times m$) $\lim_{n\to\infty}V^n=(1/m)J$ may also be derived directly from the binomial theorem. Indeed,

$$V^n=2^{-n}(I+S)^n=2^{-n}\sum_{q=0}^{n}\binom{n}{q}S^q;$$

but, since $S^m = I$, the change of variables $q = tm + p$, where $0 \leqslant p \leqslant m-1$ and $0 \leqslant t \leqslant [(n-p)/m]$, shows that $S^q = S^p$. Therefore,

$$V^n = \sum_{p=0}^{m-1}\left[2^{-n}\sum_{t=0}^{\left[\frac{n-p}{m}\right]}\binom{n}{tm+p}\right]S^p;$$

so we will have the desired limit once it is clear that

$$\lim_{n\to\infty} 2^{-n}\sum_{t=0}^{\left[\frac{n-p}{m}\right]}\binom{n}{tm+p} = \frac{1}{m}.$$

For this we again borrow from the past—a combinatorial identity dating back to 1834 ([**6**]; see also [**7**, pp. 19–20] and [**8**, pp. 40–41]):

$$2^{-n}\sum_{t=0}^{\left[\frac{n-p}{m}\right]}\binom{n}{tm+p} = \frac{1}{m}\left(1+\sum_{k=1}^{m-1}\left(\cos\frac{k}{m}\pi\right)^n\cos\frac{k}{m}\pi(n-2p)\right)$$

in which we have only to pass to the limit.

Finally, there is no reason our iterative procedure ought not work in reverse, and it does. We have

$$x_1x_2x_3 = (x_1x_2)^{1/2}(x_2x_3)^{1/2}(x_1x_3)^{1/2} \leqslant \tfrac{1}{2}(x_1+x_2)\tfrac{1}{2}(x_2+x_3)\tfrac{1}{2}(x_1+x_3)$$
$$\leqslant \tfrac{1}{4}(x_1+2x_2+x_3)\tfrac{1}{4}(x_1+x_2+2x_3)\tfrac{1}{4}(2x_1+x_2+x_3) \leqslant \cdots \uparrow \left(\tfrac{1}{3}(x_1+x_2+x_3)\right)^3.$$

More generally, if we let

$$g_n = g(X, V^n) = \left(\prod_{i=1}^{m}\sum_{j=1}^{m}(V^n)_{ij}x_j\right)^{1/m} = \prod_{i=1}^{m}(V^nX)_i^{\,1/m},$$

then, this time, $g_0 = G_m$, $\lim_{n\to\infty} g_n = A_m$, and, for the proper monotonicity,

$$g_n = \left(g(X,V^n)g(X,SV^n)\right)^{1/2} \leqslant g\left(X, \tfrac{1}{2}(I+S)V^n\right) = g_{n+1}.$$

Furthermore, running the iterative procedure from either mean, with the appropriate averaging, yields induction schemes as well, both forward and backward, and, with a bit of care, proofs of the inequality with arbitrary weights may be gotten along similar lines.

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