Journal of Pure and Applied Algebra 19 (1980) 9–20
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MACHINES IN A CATEGORY*

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Dedicated to Saunders MacLane on the occasion of his seventieth birthday

After a brief historical review of early attempts to apply category theory to place sequential machines and control systems in a unified framework, we present three contributions to the theory of machines in a category: the dynamical interpretation functor which relates the response of a machine to a sequence of inputs with the reachability map in the system category; an application of the Krull-Schmidt theorem to provide a parallel decomposition for machines with polynomial state-transition; and a generalized notion of Hankel matrix which enables one to present finiteness conditions for linear systems even when defined over noncommutative rings, and for certain classes of nonlinear systems characterized by adjoint functors.

0. Introduction

Several workers observed that with a suitable definition of transition-preserving homomorphism, sequential machines (cf. Definition 1.5 below) could be regarded as the objects of a category [43, 27, 28, 22], but little use was made here of the theorems of category theory. Impetus toward further development of machines in a category came primarily from the study of tree automata, and from the search for a rapprochement between automata theory and linear systems theory.

The study of tree automata was stimulated by Büchi's observation in lectures around 1960 (but see [15] for a published account of his ideas) that a sequential machine dynamics $\delta : Q \otimes X \to Q$ could be viewed as an X-indexed family $\delta_x : Q \to Q$ of unary operations, and that much of sequential machine theory could be generalized from the string-processing of sequential machines to tree-processing wherein the unary operations above generalize to arbitrary finitary operations. These ideas were brought to fruition by [19] and [48], and received categorical expression when [25] used the algebraic theories of [40] to formalize a number of basic problems in a way that made contact with the growing use of monads/triples/algebraic theories to treat universal algebra with category-theoretic methods. This approach was exploited by [4, 29].

Meanwhile, other authors noted similarities between aspects of sequential machine theory and the linear systems of control theory [49, 35, 2], and a partial

* This research was supported in part by National Science Foundation Grant MCS 76-84477.

unification – emphasizing algebraic but not category-theoretic methods – of these fields was presented in [11, 37, 50] (and indeed, authors have referred to the diagonal fill-in property of an image factorization system in a category [41, 33] as the "Zeiger fill-in lemma"). Kalman's algebraic theory of linear systems and Schützenberger's concern with semigroups and formal power series in the study of automata and languages (see [39] for a recent survey) both made seminal contribution to [23, 24] elegant (non-categorical) synthesis of results on automata, languages and machines.

One response to the problem of unifying sequential machines and linear systems was the study of machines in a closed category [16, 21, 30], with the dynamics $Q\times X_0\to Q$ of a sequential machine generalizing to $Q\otimes X_0\to Q$. Our own approach [5] was to replace $-\otimes X_0$ by a functor $X:\mathscr{K}\to\mathscr{K}$ so that a dynamics has the form $XQ\to Q$. This idea was suggested by algebras over a monad. We required that such X admit free dynamics $XX^@A\to X^@A$ over each $\mathscr{K}$-object A (see 3.6 below). This framework includes tree automata and certain non-linear systems (those of 2.2 as explained in 3.6) which the closed category approach does not. Essentially the same theory as the closed category approach results when X has a right adjoint [7]. Results related to the generation of $X^@$ from X have appeared in the category theory literature [13] [20] and have been extensively pursued by a number of workers in Prague (see [1] for a survey). A related approach was developed in [12].

In some sense 1974 saw the coming of age of the study of machines in a category, with the convening, at the invitation of Saunders Mac Lane, of the First International Symposium: Category Theory Applied to Computation and Control [18] in San Francisco in February of that year. In what follows we have made no attempt to survey the developments since then. Rather, we have chosen three topics from our own work with the objective of demonstrating how the methodology of describing constructions in general categories can lead to a fresh viewpoint on an old problem. We turn now to a specific introduction of the three sections below.

The theory of discrete-time systems deals with state-evolution equations such as

$$q(0)=q_0,$$
$$q(t+1)=\psi(q(t),x(t)),$$
$$y(t)=\beta(q(t))$$

with $x(t)$, $q(t)$, $y(t)$ respectively the input, state and output at time t. Each class of such equations (e.g. "linear") is usually such that the data defining the equations consists of a configuration of morphisms in a particular category (e.g. compare 1.3, 1.2) and most work in "machines in a category" has dealt exclusively with the latter. A very elementary rapprochement between machines in a category and state-evolution equations is presented in the first section. It would have been nice to develop our early papers from this perspective but, alas, we did not formulate the dynamical interpretation functor until very recently [8].

The "system algebras" of Section 2 are natural state-space models for a large class of nonlinear systems. While the Krull-Schmidt theorem of universal algebra

[17] has restrictive hypotheses, it applies nicely to system algebras to prove, in effect, that finite nonlinear systems as in 2.3 admit unique decoupling into non-interacting indecomposable subsystems. This application is a pleasant contradiction to the often-made assumption that universal algebra is at best a convenient way to organize superficial aspects of already-understood structures.

While not pursued in Section 2, we should like to mention here that system n-algebras (by which we mean, in the notation of 2.2, that $F_k=0$ for $k>n$) provide a new extension of the concept of module over a ring. To be specific we will assume $n=2$. The idea is based on the observation that a symmetric n-linear map is determined by its unary specialization. Thus a *system 2-ring* is a real vectorspace S together with a multiplicative monoid structure and two distinguished constants L, B (for "linear" and "bilinear") satisfying the following six axioms with r, s, x, y, z in S, α, β in $\mathbb{R}$:

$$\begin{aligned}
&(r+s)x = rx + sx,\\
&(\alpha r)x = \alpha(rx),\\
&L(x+y) = Lx + Ly,\\
&L(\alpha x) = \alpha Lx,\\
&B(x+y+z) - B(x+y) - B(x+z) - B(y+z) + Bx + By + Bz = 0,\\
&B(\alpha x + \beta y) = \alpha\beta B(x+y) + \alpha(\alpha-\beta)Bx + \beta(\beta-\alpha)By.
\end{aligned}$$

An *S-module* is a real vectorspace X on which S acts subject to the same six axioms except now x, y, z are in X. Then S-module = system 2-algebra. Kalman's [37] module-theoretic approach to linear systems is concerned with the special case $S=\mathbb{R}[z]$, $L=z$, $B=0$. In general, S is only a near-ring. The importance of L and B and the commutativity of addition would seem to say that further development will be largely disjoint from the study of [42].

Turning to Section 3, a well-known result for systems over a commutative ring is that a Hankel matrix has a finitely-generated realization if and only if it satisfies a polynomial recurrence relation. The result does not extend to noncommutative rings owing to its dependence on the Cayley-Hamilton theorem, but some authors have studied the class of rings for which the theorem holds. The contribution of the categorical approach is to point out that "recurrence polynomial" may be modified to 'recurrence morphism' to obtain a general theorem characterizing finite realizeability which then specializes to characterize finitely-generated realizeability for arbitrary rings. Section 3 also considers the question of how to formulate the Hankel matrix of certain nonlinear systems. Some workers have taken the condition $H^n_{m+1} = H^{n+1}_m$ for the linear Hankel matrix as characteristic. We consider this misguided, arguing that H^n_{m+1} and H^{n+1}_m should correspond under adjointness, a fact obscured in the linear setting where the adjoint functors involved are the identity functors.

1. The dynamical interpretation functor

1.1. References. Background, linear systems theory: [36, 37, 50].
Background, linear systems as automata: [2, 11, 23; Ch. XVI, 37, 32].
Background, linear systems over commutative rings: [23; Ch. XVI, 44].
Background, group machines: [3, 14].
Background, systems in a category: [18].
Our work: [5–9].
Related work: [1, 12, 16, 21, 30].

1.2. Definition. A *discrete-time linear system* is (X, G, Q, Y, β) where
Q is a finitely-generated $\mathbb{R}[z]$-module, the *state module*,
$G : X \to Q$ is $\mathbb{R}$-linear, the *input map*; X is finite dimensional,
$\beta : Q \to Y$ is $\mathbb{R}$-linear, the *output map*; Y is finite dimensional.

1.3. Definition. The *dynamical interpretation* of the system of 1.2 is as follows. Let m, n, p be the respective $\mathbb{R}$-dimensions of X, Q, Y. We think of Q as the internal state space of a system with m input lines and p output lines. The endomorphism $F : Q \to Q$ defined as the action of the polynomial z is the state-transition map of the "unforced" system. To "control" the system, an input $x(t)$ in X is to be injected into the system for each time $t = 0, 1, 2, \ldots$. The system then evolves according to the equations

$$
\begin{aligned}
&q(0) = 0,\\
&q(t+1) = Fq(t) + Gx(t), \qquad\qquad (A)\\
&y(t) = \beta q(t)
\end{aligned}
$$

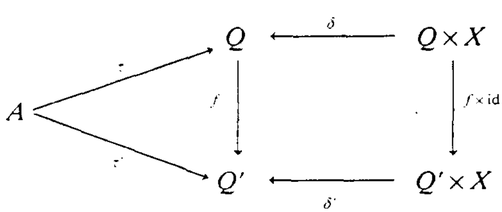
where $q(t) \in Q$ is the state at time t and $y(t) \in Y$ is the output at time t.

The equations model, for example, digitally monitored controlled physical systems governed by linear differential equations with constant coefficients.

1.4. The input/output map. A major reason why linear systems are better understood than many other classes is that there exists a simple closed formula expressing the relationship between the output $y(t)$ and the input sequence $(x(\tau) : 0 \leq \tau < t)$, namely

$$y(t) = \sum_{k=0}^{t-1} F^k G x(t-k-1) \qquad\qquad (B)$$

1.5. Automata. Given sets A, X define the *evolution category* $\mathbf{Ev}(A, X)$ whose morphisms are *action homomorphisms* as in (C).



(C)

If X^* denotes the free monoid generated by X with unit Λ, define

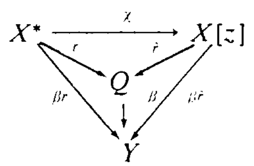
$$\Omega_{A,X} = (A \times X^*,\ \delta_{A,X},\ \tau_{A,X})$$

for $\delta_{A,X}(a, w, x) = (a, wx)$, $\tau_{A,X}(a) = (a, \Lambda)$. This is the initial object of $\mathbf{Ev}(A, X)$. Indeed the two commutativities of (C) defining the unique $r : \Omega_{A,X} \to (Q, \delta, \tau)$ are

$$r(a, \Lambda) = \tau(a), \qquad r(a, wx) = \delta(r(w), x) \tag{D}$$

so that $r(a, w)$ is the state reached in response to input string w with initial state $\tau(a)$ and r is called the *reachability map* of (Q, δ, τ). We may also involve an output map $\beta : Q \to Y$, resulting in the *input/output response map* $\beta r : A \times X^* \to Y$.

1.6. Are linear systems automata? Many authors view linear systems as automata for which Q, X, Y, A are vector spaces, $A = 0$, and all maps are linear. Indeed, since coproducts are products in the category of vector spaces, a linear map $\delta : Q \times X \to Q$ is tantamount to an $\mathbb{R}[z]$-action $F : Q \to Q$ together with a linear map $G : X \to Q$. Given such F, G we have $\delta(q, a) = Fq + Ga$ so that (with $A = 0$) the automaton input/output response $\beta r : X^* \to Y$ is practically the same as (B). To be more precise, let $\hat{r} : X[z] \to (Q, F)$ be the unique $\mathbb{R}[z]$-linear extension of G, giving rise to the $\mathbb{R}$*-linear input/output map* $\beta \hat{r} : X[z] \to Y$. Then, at least at the level of functions, diagram (E) commutes. Here, χ is the function



(E)

defined by $\chi(x_0, \ldots, x_{t-1}) = \sum (x_k z^{t-k-1} : 0 \leq \tau < t)$. The function χ is not an isomorphism (while it is surjective, $\chi(0^n w) = \chi(w)$). Moreover, the 'String-reversing' character of χ is essential.

1.7. Group systems. While the need for diagram (E) provides sufficient motivation for defining χ as we have in the linear system context, even a mild generalization of linear systems creates a challenge. Brockett and Willsky [14] considered systems (A) in which F, G, β are group homomorphisms. Mimicking the linear case, such a group system induces a $(1, X)$-automaton with initial state the group unit and with state-transition formula $\delta(q, x) = (Fq)(Gx)$ in which group multiplication replaces vector addition. In the linear case, the image of r (the "reachability subset") is the same as that of the linear map $\hat{r}$ because χ is surjective, so is a subspace. For group systems, the reachability subset need not be a subgroup. Brockett and Willsky regarded this as pathological because they formulated $r : X^* \to Q$ without seeing a need for $\hat{r}$ or χ. This "pathology" challenges us to define diagram (E) in more general terms.

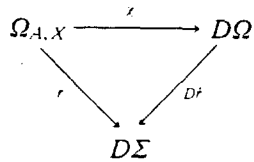
1.8. The dynamical interpretation functor. A *state-space description* [8] is $(\mathscr{A}, A, X, D)$ where

$\mathscr{A}$ is a category (the *systems category*) with an initial object Ω,

A, X are sets,

$D : \mathscr{A} \to \mathbf{Ev}(A, X)$ is a functor, the *dynamical interpretation functor*.

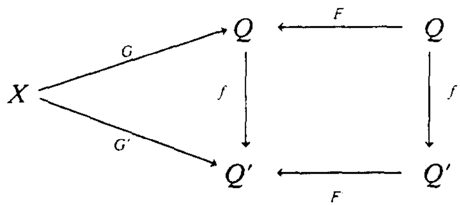
The *encoding map* is the unique action homomorphism $\chi : \Omega_{A,X} \to D\Omega$. For Σ in $\mathscr{A}$, the unique $\mathscr{A}$-morphism $\hat{r} : \Omega \to \Sigma$ is called the *abstract reachability map* of Σ whereas the unique action homomorphism $r : \Omega_{A,X} \to D\Sigma$ provides the *reachability map* of Σ. The abstract version of diagram (E) is



(F)

which commutes because $\Omega_{A,X}$ is initial.

1.9. Example: the linear and group cases. For X an object in any category $\mathscr{X}$ with countable copowers $X^{\S}$, let $\mathscr{A}$ be the category of (G, Q, F) with Q an object of $\mathscr{X}$, $G : X \to Q$, $F : Q \to Q$ and with morphisms as shown in (G). Then $(\mathrm{in}_0, X^{\S}, z)$ is the initial object of $\mathscr{A}$ where $z\ \mathrm{in}_n = \mathrm{in}_{n-1}$ defines



(G)

z. It requires more structure to define a dynamical interpretation functor however. If $\mathscr{X}$ is vector spaces, define $D : \mathscr{A} \to \mathbf{Ev}(1, X)$ by $D(G, Q, F) = (Q, \delta, 0)$, $\delta(q, x) = Fq + Gx$. Similarly for group systems (so that $\mathscr{X}$ = groups) with $D(G, Q, F) = (Q, \delta, e)$ and $\delta(q, x) = (Fq)(Gx)$. Functoriality is easily verified.

The linear encoding map is that of (E). In the group case, $X^{\S}$ is the free product of denumerably-many copies of X and $\chi : X^* \to X^{\S}$ is described by $\chi(x_0, \cdots x_{t-1}) = (x_0, t-1) \ldots (x_{t-1}, 0)$. (We regard $X^{\S}$ as the set of irreducible strings with symbols in $\{x \in X : x \neq e\} \times \mathbb{N}$ in the usual way.)

2. An application of the Krull-Schmidt theorem

2.1. References. [9, 17, 23; Ch. XVI, 37, 46].

2.2. System algebras. A *system algebra* is a pair (Q, F) with Q a vector space and

$F=(F_k : k=1,2,3\ldots)$ a sequence with $F_k : Q^k \to Q$ symmetric and k-linear. We regard system algebras as constituting an equationally-definable class. The operations are those for vector spaces (nullary 0, unary scalar actions, binary $+$) and the F_k together with the equations defining vector spaces and those forcing the F_k to be symmetric and k-linear.

For each vector space X and integer n define a state-space description $(\mathscr{S}, 1, X, D_n)$ as follows. Let $\mathscr{S}$ have objects all (G,Q,F) with $G : X \to Q$ linear and (Q,F) a system algebra. $\mathscr{S}$ is itself an equationally-definable class (add the elements of X as nullary operations and then add the equations that express the linearity of G.) Each such $\mathscr{S}$ is then a category with an initial object Ω (the free algebra generated by the empty set.) Define $D_n(G,Q,F)$ so that the initial state is the zero element $0 : 1 \to Q$ and with state-transition $\delta_n(q,x)=p_n q+Gx$ where $p_n : Q \to Q$ is the polynomial

$$p_n q=F_1 q+F_2(q,q)+\cdots+F_n(q,\ldots,q) \tag{A}$$

Functoriality is easily checked.

2.3. System algebras from differential equations. A broad class of ordinary differential control systems are governed by equations of the form

$$\begin{aligned} \dot{q}(t) &= P(q(t))+G(x(t)), \\ y(t) &= \beta(q(t)) \end{aligned} \tag{B}$$

with state $q(t)$ in $\mathbb{R}^n$, input $x(t)$ in $\mathbb{R}^m$, output $y(t)$ in $\mathbb{R}^p$, G,β linear and P a polynomial satisfying $P(0)=0$. A discrete-time approximation results by fixing a time "quantum" $\Delta : q(t+\Delta) \sim q(t)+\Delta\dot{q}(t)$. The discretized system then evolves according to the equations

$$\begin{aligned} q(t+\Delta) &= p(q(t))+\Delta G(x(t)), \\ y(t) &= \beta(q(t)) \end{aligned} \tag{C}$$

where $p(q)=q+\Delta P(q)$ is again a polynomial satisfying $p(0)=0$. If p has degree n there exist unique symmetric k-linear maps F_k such that

$$p(q)=F_1(q)+F_2(q,q)+\cdots+F_n(q,\ldots,q).$$

Set $F_\nu=0$ for $\nu>n$.

2.4. The parallel decomposition theorem [9]. Let (Q,F) be a system algebra with Q finite-dimensional and $F_k=0$ for all but finitely-many k. Then a decomposition of (Q,F) into a product of system algebras each of which cannot be further so decomposed is unique up to reordering.

This result, well known for linear systems as a corollary of the theorem of Remak-Krull-Schmidt for modules over a principal ideal domain, is basic to any theory of "parallel decoupling". The proof follows quickly from the Krull-Schmidt theorem for universal algebras.

3. Finiteness for state-modules and nonlinear generalizations

3.1. References. Background, systems over a ring: [23; Ch. 10, 38, 44, 45].
Background, Hankel matrix: [14, 26, 34, 37, 47].
Background, free dynamics: [1, 30].
Our work: [7, 8].

3.2. The Hankel matrix. Systems over a ring are linear systems as in 1.2 but replacing $\mathbb{R}$ with an arbitrary ring R. It is standard to analyze such systems in terms of the infinite matrix $H^n_m : X \to Y$ of R-linear maps given by $H^n_m = \beta F^{m+n} G$, called the *Hankel matrix* of (X, G, Q, Y, β). Two standard results are:

3.3. Hankel realization theorem. *For fixed X, Y, an arbitrary matrix $H^n_m : X \to Y$ is the Hankel matrix of some system if and only if $H^n_{m+1} = H^{n-1}_m$ for all m, n.*

3.4. Finiteness theorem. *Assume R is commutative. Then a Hankel matrix $H^n_m : X \to Y$ is realized by a system with finitely-generated state module if and only if there exist $\lambda_0, \ldots, \lambda_{d-1} \in R$ yielding the "recurrence formula"*

$$H^n_d = \sum_{k=0}^{d-1} \lambda_k H^n_k \quad \textit{for all } n. \tag{A}$$

This theorem does not generalize to arbitrary rings because of the failure of the Cayley-Hamilton theorem. In 3.9 below we shall reformulate the notion of recurrence so that the finiteness theorem holds for arbitrary rings. But first we consider an important example which motivates a more general formulation of the Hankel matrix itself.

3.5. Internally-bilinear systems. An *internally-bilinear system* is $(A, \tau, Q, X, \delta, Y, \beta)$ with A, Q, X, Y vector spaces, $\tau : A \to Q$, $\beta : Q \to Y$ linear and $\delta : Q \times X \to Q$ bilinear. For fixed A, X the system category $\mathscr{A}$ has objects (τ, Q, δ) with morphisms as in (1.C) with f linear and the dynamical interpretation functor $D : \mathscr{A} \to \mathbf{Ev}(A, X)$ is the obvious forgetful functor. (The initial object of $\mathscr{A}$ is described more generally in 3.6.) These systems are the subject of much current research. The problem of formulating the Hankel matrix for internally-bilinear systems is far from settled [26, 34, 47] and our solution (E) below differs from others.

3.6. A general class of system categories. Let $X : \mathscr{K} \to \mathscr{K}$ be a functor. An *X-dynamics* is (Q, δ) with $\delta : XQ \to Q$ in $\mathscr{K}$. The morphisms $f : (Q, \delta) \to (Q', \delta')$ of the resulting category $\mathbf{Dyn}(X)$, called *X-dynamorphisms*, satisfy (cf. the right hand squares in (1.C, G):

$$\begin{array}{ccc} XQ & \xrightarrow{\ \delta\ } & Q \\ {\scriptstyle Xf}\downarrow & & \downarrow{\scriptstyle f} \\ XQ' & \xrightarrow[\ \delta'\ ]{} & Q' \end{array} \tag{B}$$

The important functors X are those with the property that $\mathbf{Dyn}(X)$ has a left adjoint. We called these *input processes* in our papers [5, 6] later changing the name to *recursion processes* in [10] for the reasons explained there. These are called *varietors* by Adámek and Trnková [1] who (together with some of their colleagues in Prague) proved the remarkable result that $X : \mathbf{Set} \to \mathbf{Set}$ is a varietor if and only if $\mathrm{card}(XS) \leq \mathrm{card}(S)$ for arbitrarily large sets S. We shall use the term "recursion process" in this section, and will use the notations

$$A \xrightarrow{\eta_A} X^{@}A, \qquad XX^{@}A \xrightarrow{\upsilon_A} X^{@}A$$

for the free dynamics over A. Notice that for each fixed A in $\mathscr{K}$ the comma category of (τ, Q, δ) with (Q, δ) in $\mathbf{Dyn}(X)$, $\tau : A \to Q$ in $\mathscr{K}$ has $(\eta_A, X^{@}A, \upsilon_A)$ as initial object and, so, is a candidate to be the system category of a state-space description. Indeed this occurs in the examples above. In 1.5, take $D = \mathrm{id} : \mathbf{Ev}(A, X_0) \to \mathbf{Ev}(A, X_0)$ (since X is now a functor, the earlier set X must be denoted X_0) and take $XA = A \times X_0$. In 1.9, take X to be the identity functor on vector spaces (and similarly for group systems). For the nth system algebra category of 2.2 take $\mathscr{K}$ to be vector spaces, change the notation of the X of 2.2 to something else, and define the functor X by

$$XV = \bigoplus_{i=1}^{n} \circledS^{i} V$$

where $\circledS$ denotes symmetrized tensor product. For the internally-bilinear systems of 3.5, take $\mathscr{K}$ = vector spaces, $XV = V \otimes X_0$.

3.7. The Hankel matrix of an adjoint system. A very similar theory to the closed-category approach of [16, 21,30] results if the functor X of 3.6 has a right adjoint (although this restriction would exclude the systems algebras example above as well as tree automata [5]). Indeed, assume that $\mathscr{K}$ has countable products and coproducts and provide $\mathscr{K}$ with an image factorization system. Then if $X : \mathscr{K} \to \mathscr{K}$ has a right adjoint $Z : \mathscr{K} \to \mathscr{K}$, X is necessarily a recursion process with $X^{@}A = \coprod(X^{n}A : n \geq 0)$. This subsumes the examples $- \times X_0$, $- \otimes X_0$ mentioned in 3.6 and provides a general prescription for the initial object of the system category induced by A in $\mathscr{K}$ and the functor X.

An *adjoint system*, then, for $X, Z : \mathscr{K} \to \mathscr{K}$ as above, is $\Sigma = (A, \tau, Q, \delta, Y, \beta)$ with

$$A \xrightarrow{\tau} Q, \qquad XQ \xrightarrow{\delta} Q, \qquad Q \xrightarrow{\beta} Y. \tag{C}$$

Given such Σ define $r_m : X^{m}A \to Q$, $\sigma_n : Q \to Z^{n}Y$ by

$$\begin{aligned} r_0 &= \tau, & \sigma_0 &= \beta, \\ r_{m+1} &= XX^{m} \xrightarrow{Xr_m} XQ \xrightarrow{\delta} Q, & \sigma_{n+1} &= Q \xrightarrow{\Delta} ZQ \xrightarrow{Z\sigma_n} ZZ^{n}Y \end{aligned} \tag{D}$$

where Δ corresponds to δ under adjointness. The *Hankel matrix* of Σ is then defined by

$$H_m^n = X^{m}A \xrightarrow{r_m} Q \xrightarrow{\sigma_n} Z^{n}Y \tag{E}$$

3.8. Hankel realization theorem for adjoint systems. *A bisequence of $\mathcal{K}$-morphisms of form $H_m^n : X^m A \to Z^n Y$ is the Hankel matrix of some adjoint system if and only if H_{m-1}^n and H_m^{n+1} correspond under adjointness.*

3.9. Row recurrence. The Hankel matrix H_m^n of an adjoint system is *row-recurrent* (*of degree d*) if there exists a $\mathcal{K}$-morphism ϱ as shown in diagram (F) below. In the R-linear case (so that $X = \mathrm{id} = Z$), (F) is

$$\begin{array}{ccc} \coprod(X^m A : 0 \le m < d) & \xrightarrow{H_m^n : 0 \le m < d)} & Z^n Y \\ {\scriptstyle \varrho}\nwarrow & \text{for all } n & \nearrow {\scriptstyle H_d^n} \\ & X^d A & \end{array} \qquad \text{(F)}$$

equivalent to the existence of R-linear $\varrho_m : A \to A$ $(0 \le m < d)$ satisfying

$$H_d^n(a) = \sum_{m=0}^{d-1} H_m^n \varrho_m(a) \quad \text{for all } n.$$

By specializing a general result for adjoint systems we were able to prove [8]:

3.10. Theorem. *Let R be any ring and assume A is a finitely-generated {projective} {free} R-module. Then the Hankel matrix $H_m^n : A \to Y$ has a realization with {finitely-generated} {free finitely-generated} state module if and only if H_m^n is row recurrent.*

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