The Riemann hypothesis

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Abstract. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. If the Robin's inequality is true for every natural number $n > 5040$, then the Riemann hypothesis is true. We demonstrate if for every natural number $n > 5040$ we have that $d(n) \leq \sqrt{n}$, then the Robin's inequality is true for n, where $d(n)$ is the number of divisors of n. In this way, we found another way of proving that the Riemann hypothesis could be true.

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

$$\sum_{k|n} k < e^{\gamma} \times n \times \log \log n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler's constant and $k \mid n$ means that the natural number k divides n [1]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [1]. Using this inequality, we show a new step forward in proving that the Riemann hypothesis could be true.

2 Results

On the one hand, $d(n)$ is the number of divisors for a natural number n [3]. In general, if n is written as the product of prime factors: $n = p^a \times q^b \times r^c \ldots$ then the number of divisors, $d(n) = (a+1) \times (b+1) \times (c+1) \ldots$ [3]. Eulers totient (phi) function is is the number of integers less than n and co-prime to it, denoted by $\phi(n)$ [3].

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Theorem 2.1 *For every natural number n, we obtain that $n = \phi(n) + d(n) - 1$.*

Proof This is true, because every number lesser than or equal to n complies that is co-prime or divisor of n. In addition, we subtract by 1, because the number 1 is consider as a divisor and co-prime of n at the same time. ■

Theorem 2.2 *For a natural number $n > 5040$, then the Robin's inequality can be restated as*

$$\sum_{k|n} d(k) - n \times (e^{\gamma} \times \log\log n - 1) < d(n).$$

Proof We can transform the Robin's inequality

$$\sum_{k|n} k < e^{\gamma} \times n \times \log\log n$$

as the following inequality

$$\sum_{k|n} (\phi(k) + d(k) - 1) < e^{\gamma} \times n \times \log\log n$$

due to Theorem 2.1. However, that would be equivalent to

$$\sum_{k|n} \phi(k) + \sum_{k|n} d(k) - \sum_{k|n} 1 < e^{\gamma} \times n \times \log\log n$$

where we know that $\sum_{k|n} \phi(k) = n$ and $\sum_{k|n} 1 = d(n)$ [3]. Consequently, we obtain that

$$n + \sum_{k|n} d(k) - d(n) < e^{\gamma} \times n \times \log\log n$$

and thus, we have that

$$\sum_{k|n} d(k) - d(n) < e^{\gamma} \times n \times \log\log n - n$$

and

$$\sum_{k|n} d(k) - n \times (e^{\gamma} \times \log\log n - 1) < d(n).$$

■

Theorem 2.3 *For every natural number $n > 5040$, the inequality*

$$\sum_{k|n} d(k) - n \times (e^{\gamma} \times \log\log n - 1) < d(n)$$

is true when $d(n) \leq \sqrt{n}$.

Proof If we divide by $d(n)$ the inequality

$$\sum_{k|n} d(k) - n \times (e^{\gamma} \times \log\log n - 1) < d(n)$$

then, we obtain that

$$\sum_{k|n} \frac{d(k)}{d(n)} - \frac{n}{d(n)} \times (e^{\gamma} \times \log \log n - 1) < 1.$$

However, we know for every number k that divides n, then $\frac{d(k)}{d(n)} \leq 1$. Hence, we obtain that

$$\sum_{k|n} \frac{d(k)}{d(n)} - \frac{n}{d(n)} \times (e^{\gamma} \times \log \log n - 1) \leq d(n) - \frac{n}{d(n)} \times (e^{\gamma} \times \log \log n - 1).$$

because of $\sum_{k|n} 1 = d(n)$ [3]. In this way, if we prove that

$$d(n) - \frac{n}{d(n)} \times (e^{\gamma} \times \log \log n - 1) < 1$$

then we achieve to show that the Theorem 2.3 is true. For every natural number $n > 5040$ and $d(n) \leq \sqrt{n}$, we have that

$$d(n) - \frac{n}{d(n)} \times (e^{\gamma} \times \log \log n - 1) \leq \sqrt{n} - \sqrt{n} \times (e^{\gamma} \times \log \log n - 1)$$

where

$$\sqrt{n} - \sqrt{n} \times (e^{\gamma} \times \log \log n - 1) = \sqrt{n} \times (2 - e^{\gamma} \times \log \log n)$$

and we know that

$$\sqrt{n} \times (2 - e^{\gamma} \times \log \log n) < 1$$

because of

$$2 - e^{\gamma} \times \log \log n \leq 0$$

when $n > 5040$. In conclusion, we obtain that the Theorem 2.3 is indeed true. ■

Theorem 2.4 *If for every natural number $n > 5040$ we have that $d(n) \leq \sqrt{n}$, then the Riemann hypothesis is true.*

Proof This is a direct consequence of Theorems 2.2 and 2.3. ■

3 Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [2]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [2]. In this way, a possible proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [2].

References

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