Title: A Mathematical Formalization of Qualia Space

Abstract

Our conscious experiences are qualitative and unitary. The qualitative universals given in particular experiences, i.e. qualia, combine into the seamless unity of our conscious experience. The problematics of quality and cohesion are not unique to consciousness studies. In mathematics, the study of qualities (e.g. shape) resulting from quantitative variations in cohesive spaces led to the axiomatization of cohesion and quality. Using the mathematical definition of quality, herein we model qualia space as a categorical product of qualities. Thus modeled qualia space is a codomain space wherein composite qualities (e.g. shape AND color) of conscious experiences can be valued. As part of characterizing the qualia space, we provide a detailed exemplification of the mathematics of quality and cohesion in terms of the categories of idempotents and reflexive graphs. More specifically, with qualities as commutative triangles formed of cohesion-preserving functors, first we calculate the product of commutative triangles. Next, we explicitly show that the category of idempotents is a quality type. Lastly, as part of showing that the category of reflexive graphs is cohesive, we characterize the adjointness between functors relating cohesive graphs to discrete sets. In conclusion, our category theoretic construction of qualia space is a formalization of the binding of qualitative features (colors and shapes) into the cohesive objects (colored-shapes) of conscious experiences. Compared to the feature-vector accounts of conscious experiences, our product-of-qualities account of consciousness is a substantial theoretical advance.

I. Introduction

Consciousness continues to elude the reach of science (Albright et al., 2000; Menon, Sinha and Sreekantan, 2014). How are we going to account for the qualitative character of qualia? The problem of qualia (such as the qualitative feel of seeing brilliant orange sunset as distinct from, say, hearing a soothing lullaby) is a problem of relating descriptions to that which is described. Given that conscious experiences are mediated by the brain, the structure of qualia is to be related not only to the structure of physical stimulus spaces but also to that of the spaces of neural processing. Though we have a reasonably good understanding of the quantitative aspects (cf. increasing / decreasing along a stimulus feature dimension, resultant increases / decreases in neural responses, and the attendant changes in conscious experiences), our understanding of the qualitative nature of qualia is rather rudimentary. It is not clear how to formalize the qualitative distinction between sights and sounds that is so palpable in our everyday experience (Clark, 1993; O'Regan, 2011). This shortcoming, however, is not specific to consciousness studies. Scientific study of qualities such as shapes and types encountered in physics and mathematics is also challenging (Lawvere and Rosebrugh, 2003, p. 232). It is only in the past decade, the notion of QUALITY has been mathematically defined (Lawvere, 2007).

Here we begin with an in-depth study of the mathematics of quality so as to develop a mathematical formalism required for the scientific study of consciousness. Our objective, in the present note, is limited to elaborating the mathematics of cohesion and quality in terms of the categories of reflexive graphs and idempotents (Lawvere and Schanuel, 2009, pp. 135-146), so as to make it accessible to theoretical cognitive neuroscientists.

Brain can be thought of as a universal measurement device (Grossberg, 1983), measuring the physical world, and with conscious experiences as values of the neural measurements. Though there is much within a given conscious experience that lends itself to be quantified (e.g. intensity of pain), there is also much that remains beyond the reach of quantities (i.e. the quality of pain as distinct from, say, pleasure). These qualitative universals given in particular conscious experiences are called qualia (Lewis, 1929, p. 121). Within this framework, we need a space: qualia space (Balduzzi and Tononi, 2009; Stanley, 1999), to serve as codomain in which the neural measures can be valued. In the case of quantitative measurements, for example, the real number line serves as a codomain of values. In the case of brains measuring things in the world, we need a space which can serve as a space of values for qualities such as taste and smell. Analogous to the case of quantitative measurements, wherein additional dimensions are introduced (e.g. plane) to deal with more than one quantity (Lawvere and Rosebrugh, 2003, p. 59), we need a product space of qualities that can serve as a codomain space of values for composite qualities. Once we have a qualia space of composite qualities, we can characterize its geometry (figures and their incidences) and algebra (functions and their determinations). This qualia space can then be related to the physical stimulus spaces and the corresponding spaces of neural processing.

To place the theory of qualia we are developing in perspective, currently conscious experiences are modeled, after reducing qualities of conscious experiences into numbers (cf. intensity of pain), as feature lists and, in turn, as points in a vector space (Stanley, 1999). For example, colored shapes such as ‘red square’ are represented as points (red, square) in a two-dimensional space (Color × Shape). However, color and shape are qualities, which are much more structured than mere points (Hardin, 1988). For example, color is an intensive quality, while shape is an extensive quality. Our approach, building on Lawvere’s definition of qualities (Lawvere, 2007), is a direct formalization of the qualities of conscious experiences.

More broadly, category theory has been put forward as the language of consciousness (Struppa et al., 2002). In line with these suggestions, we are working out the mathematics of composite qualities defined as categorical product of qualities. In the present note, with qualities as commutative triangles formed of cohesion-preserving functors, first we calculate the product of commutative triangles. Next, we show that the category of idempotents is a quality type and has central idempotents required of quality types. Lastly, we characterize the adjointness between functors relating reflexive graphs to discrete sets as part of showing that the category of reflexive graphs is cohesive.

II. Composite Qualities

Quality is that which remains upon identifying all quantitative variations (Lawvere, 1992). A quality is a cohesion-preserving functor q: ***C*** –> ***T*** on a cohesive category ***C*** and valued in a quality type ***T*** (Lawvere, 2007). Cohesion of a category ***C*** is relative to a base category ***S*** of discrete sets and is characterized by an adjoint string:

components ($c_!$) –| **discrete** (c^*) –| **points** (c_*) –| **codiscrete** ($c^!$)

(‘–|’ denotes ‘is left adjoint to’) of four functors:

components $c_!$: ***C*** –> ***S***

discrete c^*: ***S*** –> ***C***

points c_*: ***C*** –> ***S***

codiscrete $c^!$: ***S*** –> ***C***

(see Fig. 1A). The **components** functor maps each cohesive object (in ***C***) to its set of components (in ***S***), while the **points** functor maps each cohesive object to its set of points. The functors **discrete** and **codiscrete** map sets (in ***S***) to corresponding discrete and codiscrete objects (in ***C***). The category of reflexive graphs is an example of a cohesive category (serving as domain of qualities). In the section 'Cohesive Category', we show that the **discrete** functor is left adjoint to **points** functor as part of showing that the category of reflexive graphs is cohesive.

Quality type ***T***, the codomain category of quality, is also relative to the base category ***S*** of sets and is characterized by an adjoint string:

$$\textbf{components}\ (t_!) \dashv \textbf{discrete}\ (t^*) \dashv \textbf{points}\ (t_*) \dashv \textbf{codiscrete}\ (t^!)$$

of four functors collapsed to two functors (Johnstone, 1996):

$$\textbf{components}\ t_! = \textbf{points}\ t_*: \boldsymbol{T} \to \boldsymbol{S}$$

$$\textbf{codiscrete}\ t^! = \textbf{discrete}\ t^*: \boldsymbol{S} \to \boldsymbol{T}$$

as a result of which there is exactly one point in every component of each object of a quality type ***T***. The category of idempotents is an example of quality type. In the section 'Quality Type', we show that the category of idempotents is a quality type and that it has central idempotents required of quality types.

In an effort to systematically characterize the geometry and algebra of qualities, we define a category ***Q*** of qualities. Objects of the category of qualities are cohesion-preserving functors q: $\boldsymbol{C} \to \boldsymbol{T}$ satisfying $t \cdot q = c$, where c: $\boldsymbol{C} \to \boldsymbol{S}$ and t: $\boldsymbol{T} \to \boldsymbol{S}$ are Set-labeled categories and '·' denotes composition. Qualities, in other words, are commutative triangles with functors as edges and categories as vertices (Fig. 1B). Within this formalism, qualities are broadly classified as extensive and intensive. An extensive quality e is a components-preserving functor e: $\boldsymbol{C} \rightarrow \boldsymbol{T}$ satisfying $t_! \cdot e = c_!$, as a result of which the number of components of an extensive quality of a cohesive object is same as the number of components of the cohesive object. An intensive quality, on the other hand, is a points-preserving functor i: $\boldsymbol{C} \rightarrow \boldsymbol{T}$ satisfying $t_* \cdot i = c_*$, as a result of which the number of points of an intensive quality of a cohesive object is same as the number of points of the cohesive object. Both extensive and intensive qualities are thus commutative triangles of functors on cohesive categories and valued in quality types. Morphisms of qualities are commutativity-preserving transformations of one triangle into another.

Composite qualities (cf. colored-shapes) are defined as categorical products of qualities. With color c and shape s as two qualities (two objects in the category $\boldsymbol{Q}$), composite quality colored-shape is an object:

$$p_c: c \times s \rightarrow c,\ p_s: c \times s \rightarrow s$$

in the category $\boldsymbol{Q}_{CS}$ of pairs of maps to the two factors c and s (Lawvere and Schanuel, 2009, pp. 255-256). The natural correspondence:

$$\frac{w \rightarrow c,\ w \rightarrow s}{w \rightarrow c \times s}$$

between pairs of w-shaped figures in the two factors and w-shaped figures in the product object can be used to calculate composite qualities. In order to calculate products of qualities using this method, we need to know the basic shapes of the category $\boldsymbol{Q}$ of qualities. Basic shapes of a category are those objects of the category in terms of which every object of the category can be completely determined. In the category of sets, there is one basic shape, which is the terminal set **1** = {•} consisting of exactly one element. Put differently, every set is completely determined by its elements (Lawvere and Schanuel, 2009, p. 245). The basic shapes of structured categories, unlike that of sets with zero structure, are more structured than mere element, and many categories have more than one basic shape. In the category of graphs, for example, there are two basic shapes: dot and arrow (Lawvere and Schanuel, 2009, p. 250). Since the objects of the category of qualities are commutative triangles of specific functors between chosen categories, the basic shapes of the category of qualities are going to be rather intricate. Once we identify the basic shapes of the category of qualities, we need to enumerate all pairs of figures (of each one of these basic shapes) in the two factors (qualities). These pairs of figures (of specific basic shapes) correspond to figures in the product object. Once we have all the basic-shaped figures in the product object, we need to determine the incidence relations between figures and the projection maps to factors to obtain composite qualities. As a preliminary step towards characterizing composite qualities, we calculated the product of commutative triangles. With vertices, edges, and triangles as basic shapes, the product of two generic triangles consists of a triangulated surface with nine vertices, twenty seven edges, and thirty seven commutative triangles (Fig. 2). In subsequent work, we plan to interpret the vertices and edges as categories and functors, respectively, so as to characterize composite qualities completely.

III. Quality Type

In this section we show that the functor **discrete**: ***S*** –> ***F*** (from the category ***S*** of sets to the category ***F*** of idempotents) has a right adjoint **points**: ***F*** –> ***S***, which is also left adjoint to the **discrete** functor, and hence makes the category ***F*** of idempotents a quality type over the category ***S*** of sets (Lawvere, 2007). First, we show that the functor **discrete** is left adjoint to the **points** functor, and then show that the functor **points** is [also] left adjoint to the **discrete** functor. To show that the **discrete** functor is left adjoint to the **points** functor, we have to show that there is a natural transformation *n*: **discrete · points** –> $1_{\boldsymbol{F}}$ from the composite functor **discrete · points**: ***F*** –> ***S*** –> ***F*** to the identity functor $1_{\boldsymbol{F}}$: ***F*** –> ***F*** (Lawvere and Schanuel, 2009, pp. 372-377).

The functor **points**: ***F*** –> ***S*** assigns to each object in the category ***F*** of idempotents i.e. to each idempotent *e*: X –> X, *e* · *e* = *e* its set of fixed-points Y in the category ***S*** of sets. The set Y of fixed-points can be obtained by splitting the idempotent *e*: X –> X into its retract-section pair X – *r* –> Y – *s* –> X satisfying *s* · *r* = *e* (Lawvere and Schanuel, 2009, p. 102, 117) i.e. **points** (*e*: X –> Y –> X) = Y. The functor **points** assigns to each morphism <*f*, *f*>: *e* –> *e'* of idempotents, i.e. to each commutative diagram satisfying *e'* · *f* = *f* · *e* (Fig. 3A), a function *g*: Y –> Y' (from the set Y of fixed-points of the idempotent *e*: X –> Y –> X to the set Y' of fixed-points of the idempotent *e'*: X' –> Y' –> X') satisfying: *g* · *r* = *r'* · *f* and *s'* · *g* = *f* · *s* (where X' – *r'* –> Y' – *s'* –> X' is the splitting of the idempotent *e'*: X' –> X').

Next, the functor **discrete**: ***S*** –> ***F*** assigns to each set A (in ***S***) its identity function 1_A: A –> A (an idempotent in ***F***) and to each function *v*: A –> B a commutative square satisfying $1_B \cdot v = v \cdot 1_A$ (Fig. 3B), which is a morphism of idempotents in ***F***.

In order to have a natural transformation *n*: **discrete · points** –> $1_{\boldsymbol{F}}$**,** we need, for each idempotent *e*: X –> X, *e* · *e* = *e* (in ***F***) a map n_e: **discrete · points** (*e*) –> $1_{\boldsymbol{F}}$ (*e*) in ***F***. Since **discrete · points** (*e*: X –> Y –> X) = **discrete** (Y) = 1_Y and $1_{\boldsymbol{F}}$ (*e*: X –> Y –> X) = *e*, we need a map *h* from 1_Y: Y –> Y to *e*: X –> X satisfying *e* · *h* = *h* (Fig. 3C). Since Y is the set of fixed-points of the idempotent *e*: X –> X obtained by splitting *e*, i.e. X – *e* –> X = X – *r* –> Y – *s* –> X, we take *h* = *s*: Y –> X and find that *e* · *h* = *s* · *r* · *s* = *s* · 1_Y = *s* = *h* since *r* · *s* = 1_Y (Lawvere and Schanuel, 2009, pp. 108-113). So, we can take sections *s*: Y –> X of the splitting *e* = *s* · *r* as components n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$. Next, for each morphism (in the category ***F*** of idempotents) <*f*, *f*>: *e* –> *e’* (from *e*: X –> X to *e’*: X’ –> X’), we need a commutative square in ***F*** satisfying $n_{e'}$ · **discrete** · **points** (<*f*, *f*>) = $1_{\boldsymbol{F}}$ (<*f*, *f*>) · n_e (Fig. 3D). Since **points** (<*f*, *f*>: *e* –> *e’*) = **points** (*e*) –> **points** (*e’*) = *g*: Y –> Y’ (satisfying *g* · *r* = *r’* · *f* and *s’* · *g* = *f* · *s*, where *s* · *r* = *e* and *s’* · *r’* = *e’*), **discrete** · **points** (<*f*, *f*>) = **discrete** (*g*: Y –> Y’) = <*g*, *g*>: 1_Y –> $1_{Y'}$, and $1_{\boldsymbol{F}}$ (<*f*, *f*>: *e* –> *e’*) = <*f*, *f*>: *e* –> *e’*, we find that we need a commutative diagram satisfying: *f* · n_e = $n_{e'}$ · *g* (Fig. 3E). With sections as components we have the required commutative diagram (Fig. 3F) satisfying *f* · *s* = *s’* · *g*, and in turn a natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$**,** which in turn tells that the functor **discrete**: ***S*** –> ***F*** is left adjoint to the functor **points**: ***F*** –> ***S***.

Next, in order to show that the functor **points**: ***F*** –> ***S*** is left adjoint to the functor **discrete**: ***S*** –> ***F***, we need a natural transformation *u*: $1_{\boldsymbol{F}}$ –> **discrete** · **points** with components u_e: $1_{\boldsymbol{F}}$ (*e*: X –> X) –> **discrete** · **points** (*e*: X –> X) satisfying: $u_{e'}$ · <*f*, *f*> = <*g*, *g*> · u_e (Fig. 4A). Taking the retract *r*: X –> Y of the splitting X – *r* –> Y – *s* –> X of an idempotent X – *e* –> X as the component corresponding to the idempotent i.e. with u_e = *r*: X –> Y we find that *g* · *r* = *r’* · *f* (Fig. 4B). So, we do have a natural transformation from the identity functor $1_{\boldsymbol{F}}$: ***F*** –> ***F*** to the composite functor **discrete** · **points**: ***F*** –> ***F***.

Given both natural transformations **discrete** · **points** –> $1_{\boldsymbol{F}}$ and $1_{\boldsymbol{F}}$ –> **discrete** · **points**, we say that the functor **points**: ***F*** –> ***S*** is both right and left adjoint of the functor **discrete**: ***S*** –> ***F***. Thus the **discrete** functor from the category ***S*** of sets to the category ***F*** of idempotents, with **points** functor as its right and left adjoint, makes the category ***F*** of idempotents a quality type over the category ***S*** of sets (Lawvere, 2007).

Next, we show that the category of idempotents has a central idempotent. A central idempotent is a natural endomorphism (of an identity functor) all of whose components are idempotents (Lawvere, 2004). Given a category ***E***, the identity functor 1_E: ***E*** –> ***E*** maps every object, morphism in ***E*** to the same object, morphism, respectively, in the same ***E*** i.e. 1_E (A) = A and 1_E (*f*: A –> B) = *f*: A –> B. A central idempotent is a natural transformation θ = 1_E –> 1_E assigning to each object A, a map $\theta_A = 1_E$ (A) –> 1_E (A) and to each morphism *f*: A –> B a commutative diagram satisfying $1_E (f) \cdot \theta_A = \theta_B \cdot 1_E (f)$ (Fig. 5A), and with each component an idempotent: $\theta_A \cdot \theta_A = \theta_A$ and $\theta_B \cdot \theta_B = \theta_B$. Since 1_E (A) = A, 1_E (B) = B, and 1_E (*f*: A –> B) = *f*: A –> B, we need commutative squares satisfying $f \cdot \theta_A = \theta_B \cdot f$, $\theta_A \cdot \theta_A = \theta_A$, and $\theta_B \cdot \theta_B = \theta_B$ (Fig. 5B).

Let ***E*** denote the category of idempotents. A morphism *f* from one idempotent (A, α: A –> A; α · α = α) to another idempotent (B, β: B –> B; β · β = β) is a function *f*: A –> B satisfying $f \cdot \alpha = \beta \cdot f$. The identity functor 1_E: ***E*** –> ***E*** maps each object, morphism to itself i.e. 1_E (A, α) = (A, α), 1_E (B, β) = (B, β), and 1_E (*f*: (A, α) –> (B, β)) = *f*: (A, α) –> (B, β). A central idempotent is a natural transformation θ = 1_E –> 1_E assigning to each object (A, α) a map θ_A: (A, α) –> (A, α), and to each morphism *f*: (A, α) –> (B, β) a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$, $\theta_A \cdot \theta_A = \theta_A$, and $\theta_B \cdot \theta_B = \theta_B$ (Fig. 5C). Since α: (A, α) –> (A, α) is a morphism in the category ***E*** of idempotents i.e. satisfies α · α = α · α and α · α = α (Lawvere and Schanuel, 2009, p. 179), we take $\theta_A = \alpha$: A –> A and $\theta_B = \beta$: B –> B. With these components, we obtain a commutative diagram satisfying the required $f \cdot \alpha = \beta \cdot f$ (Fig. 5D), since *f*: (A, α) –> (B, β) is a morphism in the category ***E*** of idempotents i.e. satisfies $f \cdot \alpha = \beta \cdot f$. We also find that the components α: (A, α) –> (A, α) and β: (B, β) –> (B, β) of the natural transformation $\theta = 1_E$ –> 1_E satisfy $\alpha \cdot \alpha = \alpha$ and $\beta \cdot \beta = \beta$ since α: A –> A and β: B –> B are objects of the category ***E*** of idempotents. Thus the natural transformation $\theta = 1_E$ –> 1_E (of the identity functor 1_E: ***E*** –> ***E*** of the category ***E*** of idempotents), each of whose components θ_A: (A, α) –> (A, α) is the corresponding structural map i.e. $\theta_A = \alpha$: A –> A satisfying $\alpha \cdot \alpha = \alpha$, is a central idempotent.

In the next section, as part of characterizing reflexive graphs as a cohesive category (which is the domain of qualities), we show that the functor **discrete**: ***S*** –> ***R*** (from the category ***S*** of sets to the category ***R*** of reflexive graphs) is left adjoint to the functor **points**: ***R*** –> ***S***.

IV. Cohesive Category

The qualities in our product-of-qualities formalization of qualia space are morphisms on cohesive categories (Lawvere, 2007; Lawvere and Menni, 2015). Since calculation of products requires knowledge of basic shapes of the category, we need to be explicit about cohesive categories. Cohesive categories are characterized in terms of adjoint functors to and from the category of discrete sets, relative to which cohesion is measured. In this section, as part of showing that the category of reflexive graphs is a cohesive category, we verify that the functor **discrete**: ***S*** –> ***R*** from the category ***S*** of sets to the category ***R*** of reflexive graphs is left adjoint to the functor **points**: ***R*** –> ***S*** (Lawvere and Schanuel, 2009, pp. 372-377).

A reflexive graph X consists of two component sets: a set X_A of arrows and a set X_D of dots, and three structural maps s_x: X_A –> X_D, t_x: X_A –> X_D, and i_x: X_D –> X_A (assigning to each arrow in X_A its source, target dot in X_D and to each dot in X_D its preferred loop in X_A) satisfying $s_x \cdot i_x = 1_{XD}$ and $t_x \cdot i_x = 1_{XD}$ (Lawvere and Schanuel, 2009, pp. 145-146). A graph morphism *f*: X –> Y (from a graph X to a graph Y) is a pair of set maps f_A: X_A –> Y_A, f_D: X_D –> Y_D satisfying: $s_y \cdot f_A = f_D \cdot s_x$, $t_y \cdot f_A = f_D \cdot t_x$, and $i_y \cdot f_D = f_A \cdot i_x$ (where s_y: Y_A –> Y_D, t_y: Y_A –> Y_D, and i_y: Y_D –> Y_A satisfying $s_y \cdot i_y = 1_{YD}$ and $t_y \cdot i_y = 1_{YD}$ are the structural maps corresponding to the codomain graph Y).

The functor **points**: ***R*** –> ***S*** maps each graph X = (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A) in the domain category ***R*** to its set of dots X_D in the codomain category ***S*** i.e. **points** (X) = X_D, and maps each graph morphism *f*: X –> Y = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> in ***R*** to its dot component f_D: X_D –> Y_D in ***S*** i.e. **points** (*f*: X –> Y) = f_D: X_D –> Y_D.

The functor **discrete**: ***S*** –> ***R*** maps each set P in ***S*** to a graph (also denoted) P in ***R*** with the set P as both of its component sets (set P_A of arrows and set P_D of dots) i.e. $P_A = P_D = P$, and with the identity function on P i.e. 1_P: P –> P as all three structural maps i.e. $s_p = t_p = i_p = 1_P$: P –> P satisfying $s_p \cdot i_p = 1_P$ and $t_p \cdot i_p = 1_P$. Thus **discrete** (P) = (1_P: P –> P, 1_P: P –> P, 1_P: P –> P). A function *z*: P –> Q in ***S*** is mapped by the **discrete** functor to a graph morphism in ***R*** (also denoted) *z*: P –> Q, which has the function *z* as both (arrow and dot) component functions i.e. *z* = <z_A, z_D> = <z, *z*> satisfying the three required commutative conditions: $1_Q \cdot z = z \cdot 1_P$, $1_Q \cdot z = z \cdot 1_P$, and $1_Q \cdot z = z \cdot 1_P$ preserving the source, target, and identity structure of reflexive graphs. Thus **discrete** (*z*: P –> Q) = <z: P –> Q, *z*: P –> Q>.

The composite functor **discrete** · **points** = ***R*** –> ***S*** –> ***R*** maps each graph

X = (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A) to a discrete graph

X_D = (1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D) consisting of preferred loops only on dots, and maps each graph morphism *f*: X –> Y i.e. each *f* = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> (satisfying: $s_y \cdot f_A = f_D \cdot s_x$, $t_y \cdot f_A = f_D \cdot t_x$, and $i_y \cdot f_D = f_A \cdot i_x$) to the graph morphism

f_D = <f_D: X_D –> Y_D, f_D: X_D –> Y_D> satisfying: $1_{YD} \cdot f_D = f_D \cdot 1_{XD}$ (Fig. 6).

To show that the functor **discrete**: ***S*** –> ***R*** is left adjoint to the functor **points**: ***R*** –> ***S*** we have to show that there is a natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{R}}$ from the composite functor **discrete** · **points** to the identity functor $1_{\boldsymbol{R}}$ on the category ***R*** of reflexive graphs. In other words, we have to show that for each graph X = (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A) there is a graph morphism n_X: **discrete** · **points** (X) –> $1_{\boldsymbol{R}}$ (X) i.e. a graph morphism

n_X: **discrete** · **points** (s_x: X_A–>X_D, t_x: X_A–>X_D, i_x: X_D–>X_A) –> $1_{\boldsymbol{R}}$ (s_x: X_A–>X_D, t_x: X_A–>X_D, i_x: X_D–>X_A) which is

n_X: (1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D) –> (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A), which, in turn, is a pair of maps n_X = <n_{XA}: X_D –> X_A, n_{XD}: X_D –> X_D> preserving the source, target, and identity (preferred loop) structure of reflexive graphs. With $n_{XA} = i_X$: X_D –> X_A as the arrow component and with $n_{XD} = 1_{XD}$: X_D –> X_D as the dot component, we have a graph morphism preserving the source, target, and identity structure of reflexive graphs i.e. satisfying the three required commutativity conditions: $s_X \cdot i_X = 1_{XD} \cdot 1_{XD}$ (since $s_X \cdot i_X = 1_{XD}$), $t_X \cdot i_X = 1_{XD} \cdot 1_{XD}$ (since $t_X \cdot i_X = 1_{XD}$), and $i_X \cdot 1_{XD} = i_X \cdot 1_{XD}$. Finally, we have to show that for each graph morphism *f*: X –> Y (in ***R***) the commutativity condition:

$1_{\boldsymbol{R}}$ (*f*: X –> Y) · n_X = n_Y · **discrete** · **points** (*f*: X –> Y) is satisfied. In other words, we have to show that

$1_{\boldsymbol{R}}$ ($<f_A$: $X_A \to Y_A$, f_D: $X_D \to Y_D>$) · n_X = n_Y · **discrete** · **points** ($<f_A$: $X_A \to Y_A$, f_D: $X_D \to Y_D>$). Since $1_{\boldsymbol{R}}$ ($<f_A$: $X_A \to Y_A$, f_D: $X_D \to Y_D>$) = $<f_A$: $X_A \to Y_A$, f_D: $X_D \to Y_D>$ and **discrete** · **points** ($<f_A$: $X_A \to Y_A$, f_D: $X_D \to Y_D>$) = $<f_D$: $X_D \to Y_D$, f_D: $X_D \to Y_D>$, and with n_X = $<n_{XA}$: $X_D \to X_A$, n_{XD}: $X_D \to X_D>$ = $<i_X$: $X_D \to X_A$, 1_{XD}: $X_D \to X_D>$ and n_Y = $<n_{YA}$: $Y_D \to Y_A$, n_{YD}: $Y_D \to Y_D>$ = $<i_Y$: $Y_D \to Y_A$, 1_{YD}: $Y_D \to Y_D>$, we have to show that $f_A \cdot i_X = i_Y \cdot f_D$, which is already given in the identity preserving graph morphism *f* from X to Y. Hence, we have a natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{R}}$ (Fig. 7), and, in turn, the **discrete** functor from sets ***S*** to reflexive graphs ***R*** is left adjoint to the **points** functor from reflexive graphs to sets.

There are two additional axioms that a category has to satisfy in order to be cohesive. They are product-preserving **components** functor and connected truth value object (see axioms 1 and 2 in Lawvere, 2005). The **components** functor (from the category of reflexive graphs to the category of sets, and assigning to each reflexive graph its set of components) preserves products (**components** (A × B) = **components** (A) × **components** (B), where A, B are reflexive graphs). Also, the truth value object **Ω** of the category of reflexive graphs is connected i.e. one component (**components** (Ω) = 1). Thus the category of reflexive graphs is a cohesive category.

V. Conclusion

The problem of conscious experience is particularly challenging in view of the seeming incongruity of the qualities of qualia on one hand, and the seamless cohesion of conscious experience on the other hand. We need a mathematical framework that can capture not only qualitative qualia, but also the combination of these qualities into the unity of our perceptual experiences (Roskies, 1999). There is, within mathematics, an analogous research program directed towards objectification of the unity of mathematics. These foundational investigations seeking to reunite analysis, algebra, combinatorics, geometry, and logic, all arising from the study of qualities resulting from the variation of quantities within cohesive spaces (Lawvere, 2014), led to axiomatization of the hitherto vague notions of cohesion and quality (Lawvere, 2007; Lawvere and Menni, 2015). It is this category theoretic study of cohesion and quality that we are applying to the problem of combining qualities into cohesive consciousness.

We defined composite qualities of our conscious experience as categorical products of qualities. The product-of-qualities account of consciousness serves as an abstract theoretical framework to conceptualize how qualities such as color and shape are combined into the colored-shapes of our visual experience. Our product-of-qualities formalization can be thought of as a refinement of the basic idea of feature conjunctions, wherein the percept of 'red square' is associated with the activation of 'red' neuron and 'square' neuron. The theoretical refinement is essentially in taking into account the structure of qualities (intensive colors *vs.* extensive shapes) and of their composition into the cohesiveness of consciousness. More specifically, with the objective of conceptualizing the binding of extensive shapes and intensive colors into the colored shapes of our visual experience, we focused on the particular case of the product of extensive and intensive qualities. Extensive and intensive qualities are functors, from the domain category (reflexive graphs) of one functor (defining cohesion) to the domain category (idempotents) of another functor (defining quality type). Cohesion and quality type are defined relatively i.e. relative to the category of sets, which is the common codomain of these two functors. Hence qualities (both extensive and intensive) are commutative triangles. With extensive and intensive qualities as commutative triangles, composite qualities are products of commutative triangles. In the present note, as part of characterizing composite qualities, we calculated the product of commutative triangles. We also showed that the category of idempotents is a quality type. As part of showing that the category of reflexive graphs is cohesive, we characterized the adjointness between functors relating reflexive graphs to sets. In our subsequent work, we plan to interpret the vertices and edges of commutative triangles, whose products we calculated, as categories and functors, respectively, so as to calculate products of qualities. Furthermore, we plan to thoroughly characterize the geometry and algebra of composite qualities, which can serve as a basis for precise reasoning and definitive calculations about qualia and consciousness. We also plan to provide, in our subsequent work, an in-depth comparison of Ehresmann's formalization of the binding problem in terms of colimits (Ehresmann and Vanbremeersch, 2007) with our product-of-qualities model of qualia space.

In closing, the main problem with cognitive neuroscience is a lack of good theories to guide neuroscientific investigations (Stevens, 2000). Moreover, the need for an explicit mathematical framework, analogous to the calculus of physics, which can facilitate the advancement of the science of consciousness has long been recognized (Lawvere, 1994, 1999). Our mathematical characterization of qualia space as a categorical product of qualities provides rudiments of the mathematical framework needed for the development of the science of consciousness.

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Figure Legends

Figure 1: Qualities as commutative triangles. (**A**) Quality is defined as a functor *q* on a cohesive category ***C*** and valued in a quality type ***T***. Cohesion of a category ***C*** is relative to a base category ***S*** of sets and is characterized by four functors: **components** $c_!$: ***C*** –> ***S***, **discrete** c^*: ***S*** –> ***C***, **points** c_*: ***C*** –> ***S***, and **codiscrete** $c^!$: ***S*** –> ***C***; with **components** ($c_!$) left adjoint to **discrete** (c^*) left adjoint to **points** (c_*) left adjoint to **codiscrete** ($c^!$). Quality type ***T***, the codomain category of quality, is also relative to the base category ***S*** of sets and is characterized by four functors satisfying: **components** $t_!$ = **points** t_*: ***T*** –> ***S*** and **codiscrete** $t^!$ = **discrete** t^*: ***S*** –> ***T***. (**B**) Quality *q*: ***C*** –> ***T*** is a cohesion-preserving functor satisfying $t \cdot q = c$, where '·' denotes composition. With the functors *c* and *t* as **points** functors c_*: ***C*** –> ***S*** and t_*: ***T*** –> ***S***, respectively, a points-preserving functor *i*: ***C*** –> ***T*** satisfying $t_* \cdot i = c_*$ is an intensive quality. With *c* and *t* as **components** functors $c_!$: ***C*** –> ***S*** and $t_!$: ***T*** –> ***S***, respectively, a components-preserving functor *e*: ***C*** –> ***T*** satisfying $t_! \cdot e = c_!$ is an extensive quality.

Figure 2: Product of commutative triangles. Consider a commutative triangle G with three objects A, B, and C, three maps *f*, *g*, and *h*, and the commutativity equation $gf = h$. The commutative triangle can be modeled as a set V = {A, B, C} of vertices, a set $E = \{1_A, f, 1_B, g, 1_C, h\}$ of edges, and a set $T = \{1_A 1_A = 1_A, f 1_A = f, 1_B f = f, 1_B 1_B = 1_B, g 1_B = g, 1_C g = g, 1_C 1_C = 1_C, h 1_A = h, 1_C h = h, gf = h\}$ of

triangles. The set E of edges includes vertices as identities (e.g. 1_A), while the set T of triangles includes one identity commutative triangle (e.g. $1_A 1_A = 1_A$) for each vertex and two identity commutative triangles (e.g. $f1_A=f$, $1_Bf=f$) for each edge. The product G × G of two commutative triangles consists of nine vertices (V × V = {AA, AB, AC, BA, BB, BC, CA, CB, CC}), thirty six edges (E × E), and hundred triangles (T × T). Of the thirty six edges of the product, nine are identities (corresponding to the nine vertices in V × V), and the remaining twenty seven non-identity edges are displayed above. These twenty seven edges form thirty seven non-identity commutative triangles. Of the hundred commutative triangles in T × T, nine are identities corresponding to the nine vertices and fifty four are identities corresponding to twenty seven edges (two identity commutative triangles for each non-identity edge), with thirty seven non-identity triangles remaining.

Figure 3: The functor discrete: *S* –> *F* is left adjoint to the functor points: *F* –> *S*. (**A**) The functor **points**: ***F*** –> ***S*** assigns to each morphism i.e. to each commutative square (depicted) satisfying $e' \cdot f = f \cdot e$ (in the category ***F*** of idempotents) a function *g*: Y –> Y' (from the set Y of fixed-points of the idempotent *e*: X – *r* –> Y – *s* –> X to the set Y' of fixed-points of the idempotent *e'*: X' – *r'* –> Y' – *s'* –> X') satisfying: $g \cdot r = r' \cdot f$ and $s' \cdot g = f \cdot s$. (**B**) The functor **discrete**: ***S*** –> ***F*** assigns to each function *v*: A –> B a commutative square satisfying $1_B \cdot v = v \cdot 1_A$, which is a morphism of idempotents in ***F***. (**C**) A map *h*: Y –> X making the (depicted) diagram commute i.e. satisfying $e \cdot h = h$ is required as component n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$. Taking the section *s*: Y –> X of the splitting $e = s \cdot r$ as *h*: Y –> X, we find that it satisfies the required $e \cdot h = h$. (**D**) For each morphism (in the category ***F*** of idempotents) <*f*, *f*>: *e* –> *e'* (from *e*: X –> X to *e'*: X' –> X') we need the

displayed square to commute (in ***F***) i.e. satisfy: $n_{e'}$ · **discrete** · **points** ($<f, f>$) = $1_{\boldsymbol{F}}$ ($<f, f>$) · n_e. (**E**) With **points** ($<f, f>$) = *g* and **discrete** · **points** ($<f, f>$) = **discrete** (*g*) = $<g, g>$, and $1_{\boldsymbol{F}}$ ($<f, f>$) = $<f, f>$, we need the square displayed to commute i.e. satisfy: $f \cdot n_e = n_{e'} \cdot g$. (**F**) With sections *s*: Y –> X as components n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$, we have the required commutative diagram satisfying $f \cdot s = s' \cdot g$.

Figure 4: The functor points: *F* –> *S* is left adjoint to the functor discrete: *S* –> *F*. (**A**) A natural transformation *u*: $1_{\boldsymbol{F}}$ –> **discrete** · **points** with components u_e: $1_{\boldsymbol{F}}$ (*e*: X –> X) –> **discrete** · **points** (*e*: X –> X) satisfying: $u_{e'} \cdot <f, f> = <g, g> \cdot u_e$, i.e. making the displayed square commutative, makes the functor **points**: ***F*** –> ***S*** left adjoint to the functor **discrete**: ***S*** –> ***F***. (**B**) With retracts *r*: X –> Y as components u_e: *e* –> 1_Y, i.e. with $u_e = r$: X –> Y, we obtain the required commutative square satisfying $g \cdot r = r' \cdot f$.

Figure 5: Central idempotent. (**A**) A central idempotent is a natural transformation θ assigning to each morphism *f*: A –> B a commutative diagram satisfying $1_E (f) \cdot \theta_A = \theta_B \cdot 1_E (f)$. (**B**) Since the natural transformation is an endomorphism of identity functors with $1_E (f) = f$, we need a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$. (**C**) Central idempotent θ assigns to each object (A, α) in the category ***E*** of idempotents a map θ_A: (A, α) –> (A, α), and to each morphism *f*: (A, α) –> (B, β) a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$. (**D**) With idempotent endomaps as components of the natural transformation, i.e. $\theta_A = \alpha$: A –> A, $\theta_B = \beta$: B –> B, we obtain a commutative diagram satisfying $f \cdot \alpha = \beta \cdot f$.

Figure 6: Composite endofunctor discrete · points on the category of reflexive graphs. Each graph $X = (s_x: X_A \to X_D, t_x: X_A \to X_D, i_x: X_D \to X_A)$ in the category $\boldsymbol{R}$ of reflexive graphs is mapped to a discrete graph $X_D = (1_{XD}: X_D \to X_D, 1_{XD}: X_D \to X_D, 1_{XD}: X_D \to X_D)$ in $\boldsymbol{R}$, and each graph morphism $f: X \to Y$ i.e. each $f = \langle f_A: X_A \to Y_A, f_D: X_D \to Y_D \rangle$ in $\boldsymbol{R}$ is mapped to the graph morphism $f_D = \langle f_D: X_D \to Y_D, f_D: X_D \to Y_D \rangle$ in $\boldsymbol{R}$ by the composite endofunctor **discrete · points:** $\boldsymbol{R} \to \boldsymbol{S} \to \boldsymbol{R}$.

Figure 7: Natural transformation from the composite endofunctor discrete · points to the identity functor on reflexive graphs. Since $f_A \cdot i_X = i_Y \cdot f_D$ (from the definition of graph morphism), we take the inclusion $i_X: X_D \to X_A$ of dots in X_D into X_A as preferred loops as components of the natural transformation n: **discrete · points** $\to 1_{\boldsymbol{R}}$. With

$1_{\boldsymbol{R}} (\langle f_A: X_A \to Y_A, f_D: X_D \to Y_D \rangle) = \langle f_A: X_A \to Y_A, f_D: X_D \to Y_D \rangle$ and

discrete · points $(\langle f_A: X_A \to Y_A, f_D: X_D \to Y_D \rangle) = \langle f_D: X_D \to Y_D, f_D: X_D \to Y_D \rangle$, and taking

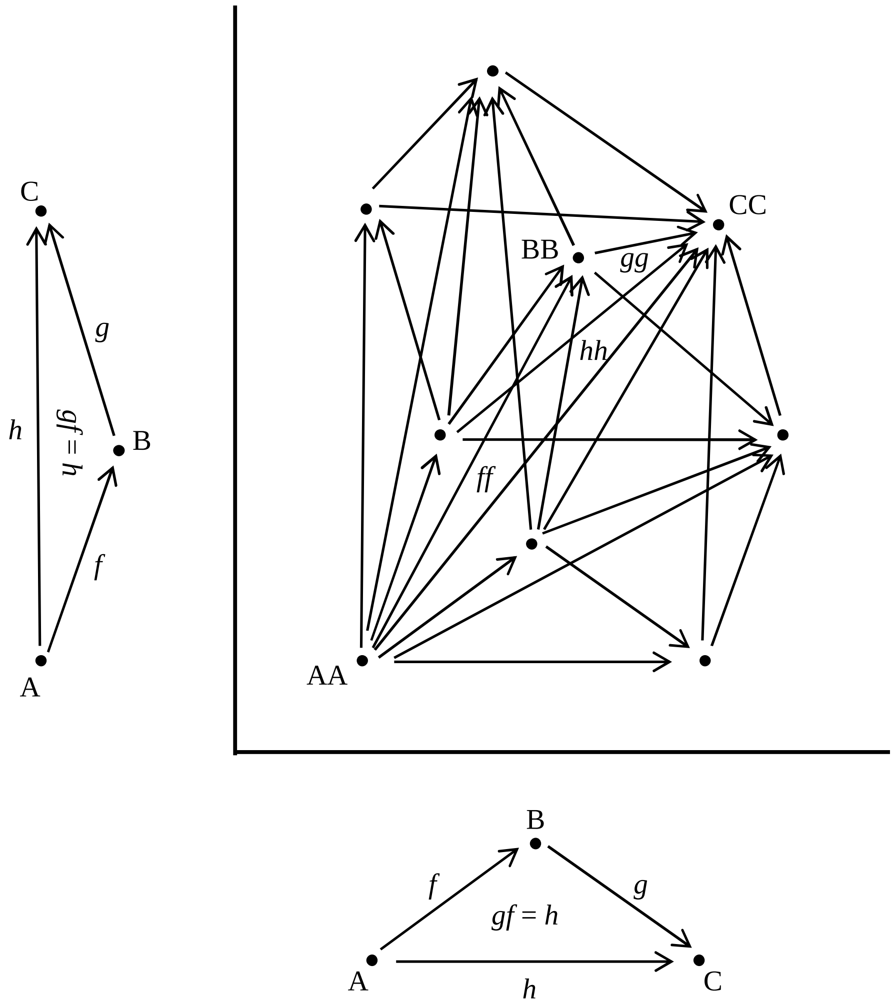
$n_X = \langle n_{XA}: X_D \to X_A, n_{XD}: X_D \to X_D \rangle = \langle i_X: X_D \to X_A, 1_{XD}: X_D \to X_D \rangle$ and

$n_Y = \langle n_{YA}: Y_D \to Y_A, n_{YD}: Y_D \to Y_D \rangle = \langle i_Y: Y_D \to Y_A, 1_{YD}: Y_D \to Y_D \rangle$, we have all the commutativity conditions satisfied and hence a natural transformation from the composite endofunctor **discrete · points** to the identity functor $1_{\boldsymbol{R}}$ on the category of reflexive graphs.

Figure 1

A **B**

Figure 2



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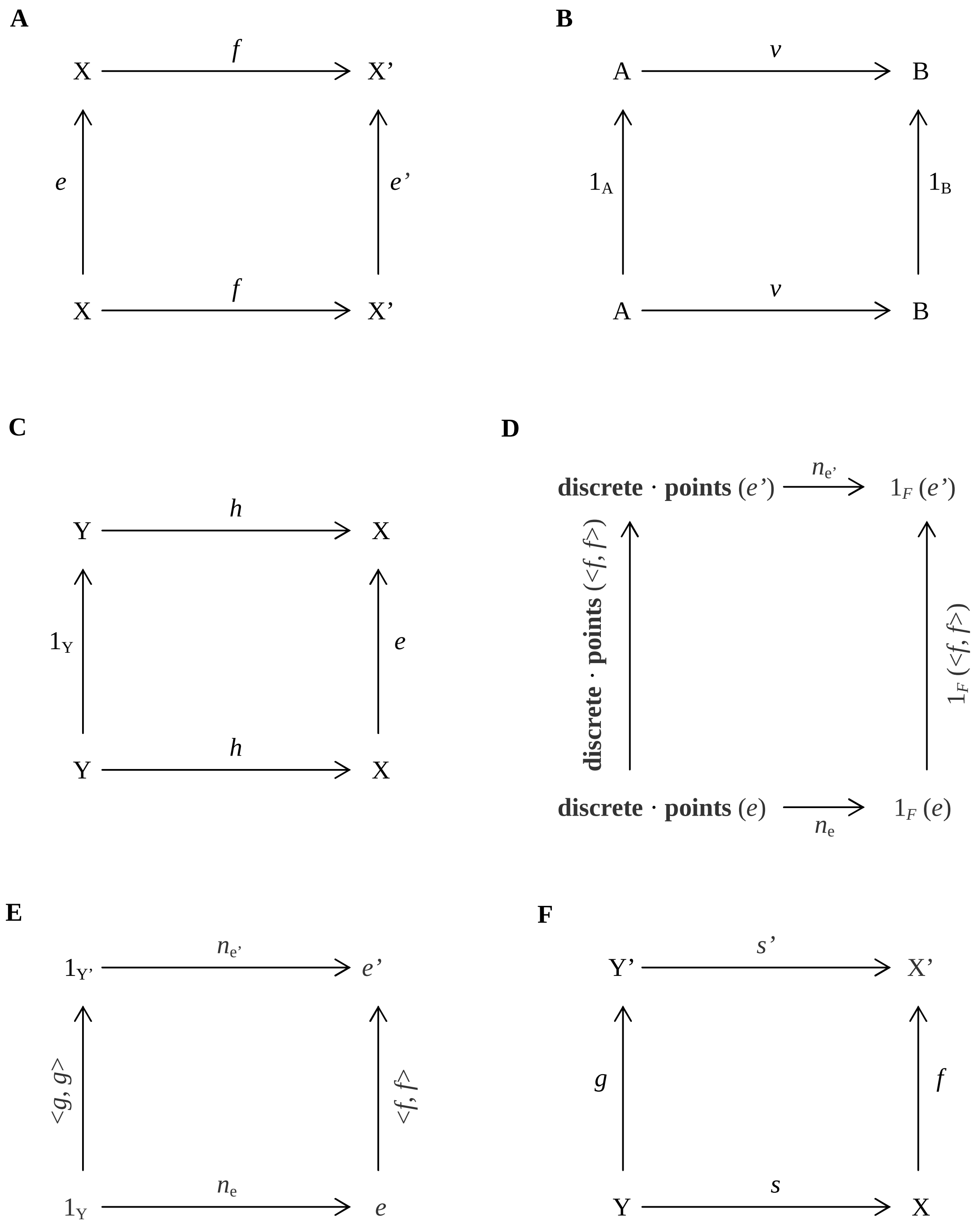
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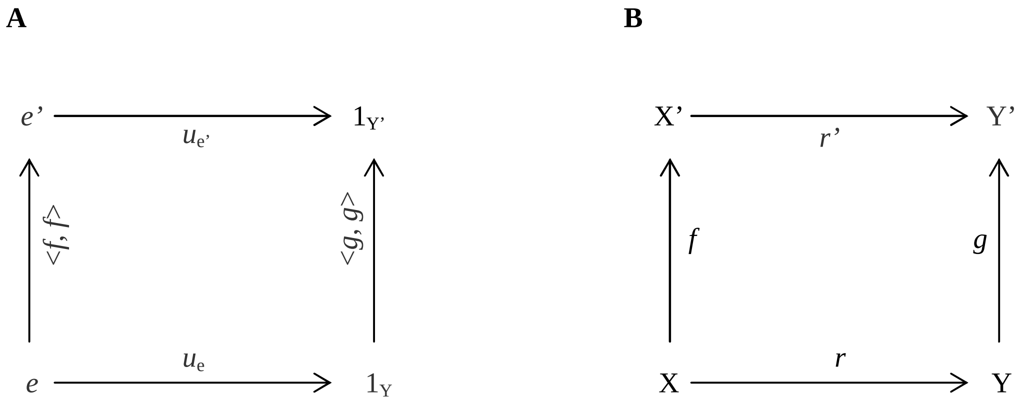
Figure 4



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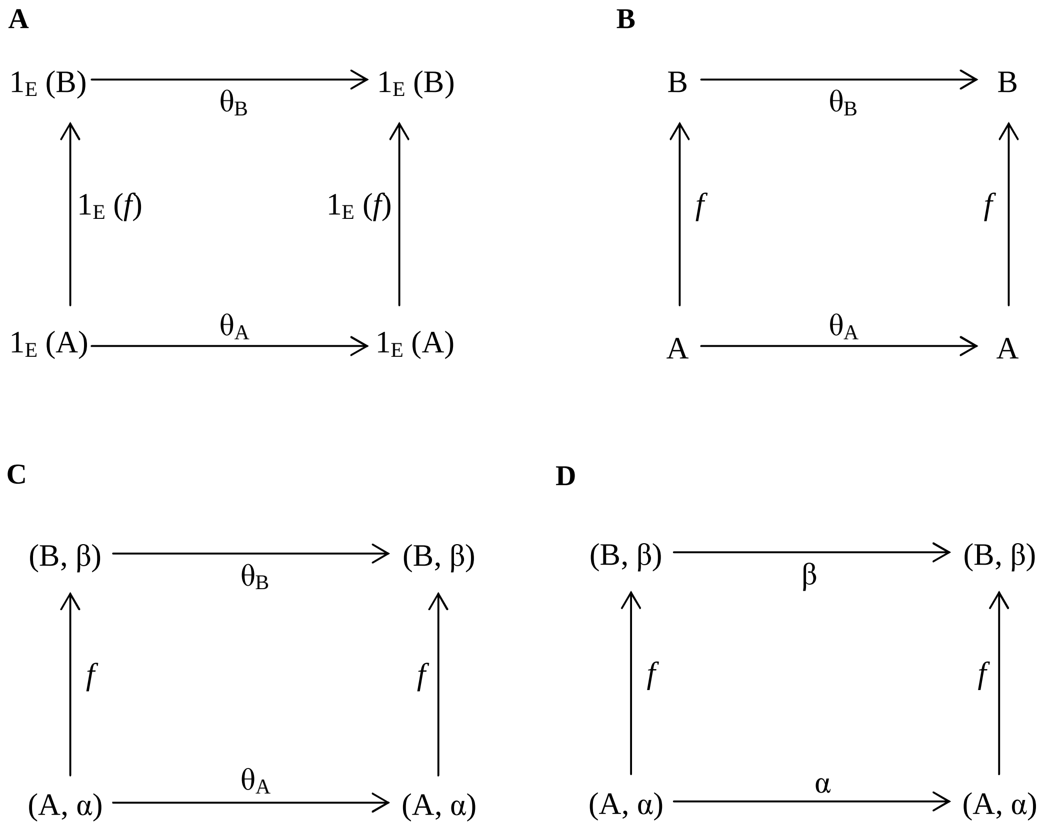
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Figure 6

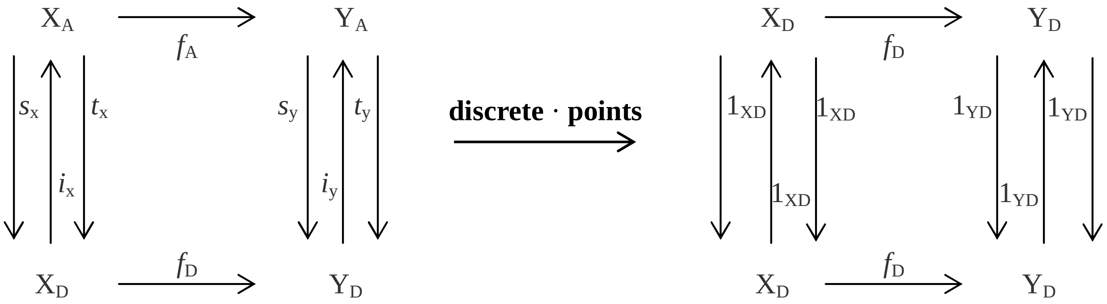
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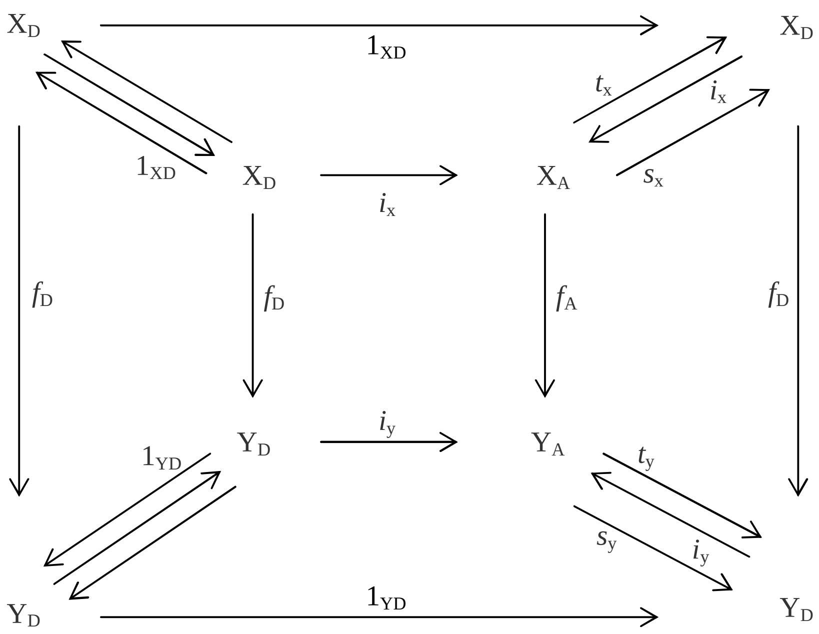
Figure 7



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