MORTAR CONTACT FORMULATION FOR HIERARCHICAL BASIS FUNCTIONS USING SMOOTH ACTIVE SET STRATEGY

***Ignatios Athanasiadis[1], Łukasz Kaczmarczyk[1], Zahur Ullah[2] and Chris J. Pearce[1]**

[1]Glasgow Computational Engineering Centre, School of Engineering, University of Glasgow, Glasgow
[2]School of Mechanical and Aerospace Engineering, Queen's University Belfast

*Ignatios.Athanasiadis@glasgow.ac.uk

Summary

This work focuses on 3D modelling of elastic two-body frictionless contact by means of the mortar method for small strains extended for hierarchical shape functions. Surfaces of two bodies discretised by tetrahedral elements are denoted as master and slave surfaces. When two triangular faces of tetrahedral elements are candidates for contact, with one face belonging to the master and the other to the slave surface, they are used to create a special prism element. These prisms are used as integration domains to solve the contact problem. For a given prism configuration, triangular faces can either be in contact or form a gap denoting an active or passive state, respectively. This state is determined by evaluation of the complementarity function proposed in [1], that is modified in the present work to yield a smooth Newton algorithm. Finally, results for sphere-to-sphere Hertz contact are compared to analytical solution for different orders of approximation.
Key Words: *Mortar Contact, Smooth Active Set, Hierarchical Basis Functions*

Introduction

Contact conditions are frequently observed in engineering applications. Even though contact has been an important research topic in computational mechanics for a long time, its modelling is still a challenge. Approaches that have been proposed are the so called node-to-node, node-to-segment and mortar contact, with the latter being the most promising method so far proposed. The majority of mortar contact related works involve usage of standard Lagrange shape functions or dual shape functions. In the present work, a mortar contact formulation using the active set strategy is presented for hierarchical shape functions for tetrahedral elements [2]. This paper is an extension of the work previously presented in [3]. Regularisation of non smoothness arising from the active set strategy formulation is proposed to avoid usage of semi-smooth Newton solver. This approach was chosen because it is well suited for integration within an already existing Arbitrary Lagrangian Eulerian fracture framework [4]. This future integration, will allow for investigation of influence of contact on crack propagation within nuclear graphite bricks.

Problem definition

The problem under consideration is schematically presented in Figure 1 where two bodies are potentially coming into contact. Current configuration of the two bodies is denoted by sets $\Omega^{(i)}$, where $i = 1, 2$. Furthermore, each body's surface, $\partial\Omega^{(i)}$, is divided into three sets, presented in Figure 1 with three different hatchings. The three sets are distinguished according to conditions applied and therefore there exist Dirichlet, Neumann and contact boundaries denoted by $\gamma_{\mathrm{u}}^{(i)}$, $\gamma_{\sigma}^{(i)}$ and $\gamma_{\mathrm{c}}^{(i)}$, respectively. Following assumptions presented in [1], the boundary sets are considered to be disjoined:

$$\partial\Omega^{(i)} = \gamma_{\mathrm{u}}^{(i)} \cup \gamma_{\sigma}^{(i)} \cup \gamma_{\mathrm{c}}^{(i)} \qquad \text{and} \qquad \gamma_{\mathrm{u}}^{(i)} \cap \gamma_{\sigma}^{(i)} = \gamma_{\sigma}^{(i)} \cap \gamma_{\mathrm{c}}^{(i)} = \gamma_{\mathrm{c}}^{(i)} \cap \gamma_{\mathrm{u}}^{(i)} = \emptyset \tag{1}$$

Moreover, the boundary value problem under consideration is described below

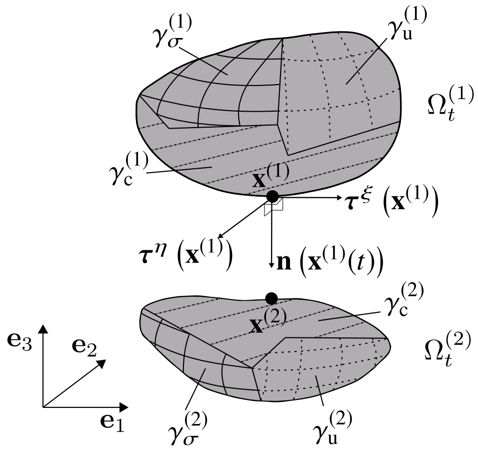

Figure 1: Schematic representation of 3D contact problem.

$$\mathrm{div}\boldsymbol{\sigma}^{(i)} = 0 \;\text{ in } \Omega^{(i)}, \quad \mathbf{u}^{(i)} = \bar{\mathbf{u}}^{(i)} \;\text{ on } \gamma_\mathrm{u}^{(i)} \quad \text{and} \quad \boldsymbol{\sigma}^{(i)}\mathbf{n}^{(i)} = \bar{\mathbf{t}}^{(i)} \;\text{ on } \gamma_\sigma^{(i)} \tag{2}$$

where, $\boldsymbol{\sigma}^{(i)}$ is the Cauchy stress tensor, $\mathbf{n}^{(i)}$ is unit vector normal to ${\gamma_\mathrm{u}}^{(i)}$ surfaces, $\bar{\mathbf{t}}^{(i)}$ is the vector of prescribed tractions on ${\gamma_\sigma}^{(i)}$ and $\mathbf{u}^{(i)}$ and $\bar{\mathbf{u}}^{(i)}$ are the unprescribed and prescribed displacement vectors, respectively. Vectors $\mathbf{u}^{(i)}$ and $\bar{\mathbf{u}}^{(i)}$ are evaluated as

$$\mathbf{u}^{(i)} = \mathbf{x}^{(i)} - \mathbf{X}^{(i)} \text{ on } \overline{\Omega^{(i)} \cap \gamma_\mathrm{u}^{(i)}} \quad \text{and} \quad \bar{\mathbf{u}}^{(i)} = \bar{\mathbf{x}}^{(i)} - \mathbf{X}^{(i)} \text{ on } \gamma_\mathrm{u}^{(i)} \tag{3}$$

where $\bar{\mathbf{x}}^{(i)}$ is the vector of prescribed current spatial positions on ${\gamma_\mathrm{c}}^{(i)}$ and $\mathbf{x}^{(i)}$ and $\mathbf{X}^{(i)}$ are the vectors of current and reference unprescribed spatial positions, respectively. Furthermore, gap between the two bodies is evaluated as

$$g(\mathbf{x}) = -\mathbf{n}(\mathbf{x}^{(1)}) \cdot \left[\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right] \quad \text{where} \quad \mathbf{n}(\mathbf{x}^{(1)}) = \boldsymbol{\tau}^\xi(\mathbf{x}^{(1)}) \times \boldsymbol{\tau}^\eta(\mathbf{x}^{(1)}) \tag{4}$$

where g is the scalar gap function and $\boldsymbol{\tau}^\xi(\mathbf{x}^{(1)})$ and $\boldsymbol{\tau}^\eta(\mathbf{x}^{(1)})$ are two tangent vectors to surface $\gamma_\mathrm{c}^{(1)}$ at $\mathbf{x}^{(1)}$. In addition, since contact is frictionless, only normal component, p_n, from contact tractions, $\mathbf{t}_\mathrm{c}$, over $\gamma_\mathrm{c}^{(1)}$ is taken into account and evaluated as

$$p_\mathrm{n} = \mathbf{t}_\mathrm{c} \cdot \mathbf{n}(\mathbf{x}^{(1)}) \tag{5}$$

The conditions that describe frictionless contact can be summerised by the Karun-Kuhn-Tucker (KKT) conditions as

$$g(\mathbf{x}) \geq 0, \qquad p_\mathrm{n} \leq 0, \qquad p_\mathrm{n} g(\mathbf{x}) = 0 \tag{6}$$

where the first inequality describes prohibition of penetration of the two bodies under consideration and the second one expresses development of normal tractions over the contact area. Moreover, the equality in (6) is a complementary argument that ensures gap closure when contact pressure is non-zero and zero pressure during gap opening.

Since it is computationally demanding to explicitly solve KKT conditions the three relationships in (6) can be captured by the alternative complementarity problem described by the complementarity function C as

$$C(\lambda, \mathbf{x}) = \lambda - \max(0, \lambda - c_\mathrm{n} g) = \frac{1}{2}\left(\lambda + c_\mathrm{n} g - |\lambda - c_\mathrm{n} g|\right), \quad c_\mathrm{n} > 0 \tag{7}$$

that was first presented in [1] for active set strategy that is well suited for semi-smooth Newton method. Here, λ is the so called Lagrange multiplier considered to be equal to p_{n} and c_{n} is a non-physical input parameter. In the present work, C function is regularised in order to avoid the primal dual active set strategy and usage of dual Lagrange multipliers. Regularisation is achieved by substituting the absolute (non-smooth) function with a strongly non-linear smooth function

$$\widetilde{C}(\lambda, \mathbf{x}) = \frac{1}{2}\left(\lambda + c_{\mathrm{n}} g - \frac{1}{r}|\lambda - c_{\mathrm{n}} g|^{r}\right) \tag{8}$$

where $\widetilde{C}$ is the regularised C function and r is non-physical regularisation parameter whose values could be chosen between 1 to 1.1.

Contact element formulation

The central objective of the proposed formulation is for it to be integrated with mesh partitioning schemes. Therefore, when two triangular faces of tetrahedral elements are candidates to be in contact, with one face belonging to the master and the other to the slave surface, they are used to create a special prism element. No integration is performed within the prism volume therefore contact prisms can overlap and can be arbitrarily distorted. This approach resolves the problem where the master and slave triangle lie between two different partitions. More details of the generation process of the prism elements can be found in [3].

The present section focusses on the description of the contact element solely, while virtual work related to work on the elastic bodies is omitted. Virtual work related to contact development and for complementarity function are presented below

$$\mathbf{r}_{\mathbf{x}} = \int_{\gamma_{\mathrm{c}}^{(1)}} \boldsymbol{\lambda} \delta g(\mathbf{x}) \mathrm{d}\gamma_{\mathrm{c}}^{(1)} = \int_{\gamma_{\mathrm{c}}^{(1)}} \boldsymbol{\lambda}(-\mathbf{n} \cdot (\delta \mathbf{x}^{(1)} - \delta \mathbf{x}^{(2)})) \mathrm{d}\gamma_{\mathrm{c}}^{(1)} \quad \text{and} \quad \mathbf{r}_{\boldsymbol{\lambda}} = \int_{\gamma_{\mathrm{c}}^{(1)}} \delta \boldsymbol{\lambda} \widetilde{C} \mathrm{d}\gamma_{\mathrm{c}}^{(1)} := 0 \tag{9}$$

Furthermore, the linearised system of equations is

$$\begin{bmatrix} \dfrac{\partial \mathbf{r}_{\mathbf{x}}}{\partial \mathbf{x}}^{(n)} & \dfrac{\partial \mathbf{r}_{\mathbf{x}}}{\partial \boldsymbol{\lambda}}^{(n)} \\ \dfrac{\partial \mathbf{r}_{\boldsymbol{\lambda}}}{\partial \mathbf{x}}^{(n)} & \dfrac{\partial \mathbf{r}_{\boldsymbol{\lambda}}}{\partial \boldsymbol{\lambda}}^{(n)} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^{(n+1)} \\ \Delta \boldsymbol{\lambda}^{(n+1)} \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_{\mathbf{x}}^{(n)} \\ -\mathbf{r}_{\boldsymbol{\lambda}}^{(n)} \end{bmatrix} \tag{10}$$

where n is the iteration number in the Newton algorithm within one step.

Results

Comparison of the model's results with the analytical solution for the Hertz problem for two spheres coming into contact is presented. The problem setup is schematically presented in Figure 2a) where only an eighth of each sphere is considered. Input parameters are: radius of the two spheres $R = 10$ [m], Young's modulus $E = 10$ [Pa], Poisson ratio $\nu = 0$, $c_{\mathrm{n}} = 10$ and $r = 1$. All planar surfaces of the two bodies are fixed in their perpendicular direction except for one where uniform normal displacements are applied incrementally (Figure 2a)). For each displacement increment, the total quarter surface force is calculated via summation of reaction forces of the nodes prescribed with non-zero displacements. Four analyses were run using the same mesh for increasing orders of approximation and having both fields of Lagrange multipliers and spatial positions to be equal. The quarter surface forces versus the uniform displacement increment curves resulting from the four analyses are compared in Figure 2b) to analytical curve for the given input according to the equations presented in [5]. It can be observed that for orders higher than 1^{st} results are very close to the analytical one and lie on top of each other.

Conclusions

A novel implementation of the mortar contact approach for hierarchical basis functions and regularisation of the complementarity function was presented. The model results for sphere-to-sphere Hertz problem matched well analytical solution for higher orders of approximation. The promising approach is a good candidate for simulating more challenging problems with spatially heterogeneous basis functions after further development.

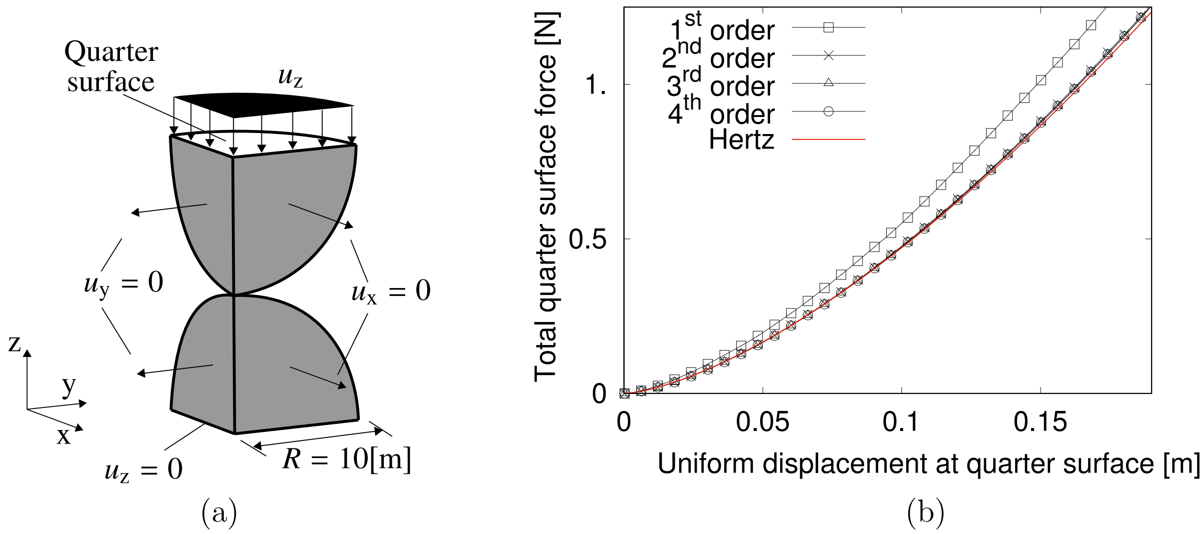

Figure 2: Sphere-to-sphere Hertz problem: (a) problem setup where only an eighth of each sphere is considered (b) comparison of model result with analytical curve for total quarter surface forces versus vertical displacements.

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