$$= \lim_{h \to 0} \frac{F_r\left(v\left(x\right) + h\left(a \cdot \nabla_x\right) v, x + ha\right) - F_r\left(v\left(x\right), x\right)}{h},$$
$$= \left(\left(\left(a \cdot \nabla_x\right) v\right) \cdot \nabla_v\right) F_r\left(v, x\right) + \left(a \cdot \nabla_x\right) F_r\left(v, x\right). \tag{63}$$

In eq. (63) one must be careful of the parenthesis since $\left(\left(a \cdot \nabla_x\right) v\right) \cdot \nabla_v$ is (must be) a scalar operator. The second term in eq. (63) is only non-zero if the coordinate connection is non-zero.

We now evaluate the vector operator, $\left(\nabla_x v\right) \cdot \nabla_v$, for a given coordinate system (using $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ and $\boldsymbol{e}^i = g^{ij} \boldsymbol{e}_j$) gives

$$\begin{aligned}
\left(\nabla_x v\right) \cdot \nabla_v &= \left(\boldsymbol{e}^i \partial_{x^i}\left(v^j \boldsymbol{e}_j\right)\right) \cdot \boldsymbol{e}^k \partial_{v^k}, \\
&= \boldsymbol{e}^i \left(\left(\partial_{x^i} v^j\right) \boldsymbol{e}_j + v^j \left(\partial_{x^i} \boldsymbol{e}_j\right)\right) \cdot \boldsymbol{e}^k \partial_{v^k}, \\
&= \boldsymbol{e}^i \left(\left(\partial_{x^i} v^j\right) \delta_j^k + v^j \left(\partial_{x^i} \boldsymbol{e}_j\right) \cdot g^{kl} \boldsymbol{e}_l\right) \partial_{v^k}, \\
&= \boldsymbol{e}^i \left(\left(\partial_{x^i} v^j\right) \partial_{v^j} + \frac{1}{2} v^j g^{kl} \left(\partial_{x^i} g_{jl}\right) \partial_{v^k}\right)
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
\nabla_x F_r\left(v\left(x\right), x\right) = &\left(\left(\partial_{x^i} v^j\right) \partial_{v^j} + \frac{1}{2} v^j g^{kl} \left(\partial_{x^i} g_{jl}\right) \partial_{v^k}\right) F_r^{i_1 \ldots i_r}\left(v\right) \boldsymbol{e}^i \left(\boldsymbol{e}_{i_1} \wedge \ldots \wedge \boldsymbol{e}_{i_1}\right) + \\
&F_r^{i_1 \ldots i_r}\left(v\right) \boldsymbol{e}^i \partial_{x^i} \left(\boldsymbol{e}_{i_1} \wedge \ldots \wedge \boldsymbol{e}_{i_1}\right).
\end{aligned} \tag{65}$$

1.9.4 Linear Differential Operators

First a note on partial derivative notation. We shall use the following notation for a partial derivative where the manifold coordinates are $x_1, \ldots, x_n$:

$$\frac{\partial^{j_1 + \cdots + j_n}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}} = \partial_{j_1 \ldots j_n}. \tag{66}$$

If $j_k = 0$ the partial derivative with respect to the k^{th} coordinate is not taken. If the $j_k = 0$ for all $1 \leq k \leq n$ then the partial derivative operator is the scalar one. If we consider a partial derivative where the x's are not in normal order such as

$$\frac{\partial^{j_1 + \cdots + j_n}}{\partial x_{i_1}^{j_1} \ldots \partial x_{i_n}^{j_n}},$$

and the i_k's are not in ascending order. The derivative can always be put in the form in eq (66) since the order of differentiation does not change the value of the partial derivative (for the smooth functions we are considering). Additionally, using our notation the product of two partial derivative operations is given by

$$\partial_{i_1 \ldots i_n} \partial_{j_1 \ldots j_n} = \partial_{i_1 + j_1, \ldots, i_n + j_n}. \tag{67}$$

A general general multivector linear differential operator is a linear combination of multivectors and partial derivative operators denoted by (in all of this section we will use the Einstein summation convention)

$$D \equiv D^{i_1 \ldots i_n} \partial_{i_1 \ldots i_n}. \tag{68}$$

Equation (68) is the normal form of the differential operator in that the partial derivative operators are written to the right of the multivector coefficients and do not operate upon the multivector coefficients. The operator of eq (68) can operate on mulitvector functions, returning a multivector function via the following definitions.

F as (Einstein summation convention)

$$D\circ F = D^{j_1\cdots j_n}\circ\partial_{j_1\cdots j_n}F, \tag{69}$$

or

$$F\circ D = \partial_{j_1\cdots j_n}F\circ D^{j_1\cdots j_n}, \tag{70}$$

where the $D^{j_1\cdots j_n}$ are multivector functions and $\circ$ is any of the multivector multiplicative operations.

Equations (69) and (70) are not the most general multivector linear differential operators, the most general would be

$$D\left(F\right) = D^{j_1\cdots j_n}\left(\partial_{j_1\cdots j_n}F\right), \tag{71}$$

where $D^{j_1\cdots j_n}\left(\right)$ are linear multivector functionals.

The definition of the sum of two differential operators is obvious since any multivector operator, $\circ$, is a bilinear operator $\left(\left(D_A+D_B\right)\circ F = D_A\circ F + D_B\circ F\right)$, the product of two differential operators D_A and D_B operating on a multivector function F is defined to be ($\circ_1$ and $\circ_2$ are any two multivector multiplicative operations)

$$\begin{aligned}
\left(D_A\circ_1 D_B\right)\circ_2 F &\equiv \left(D_A^{i_1\ldots i_n}\circ_1\partial_{i_1\ldots i_n}\left(D_B^{j_1\cdots j_n}\partial_{j_1\cdots j_n}\right)\right)\circ_2 F \\
&= \left(D_A^{i_1\ldots i_n}\circ_1\left(\left(\partial_{i_1\ldots i_n}D_B^{j_1\cdots j_n}\right)\partial_{j_1\cdots j_n} + D_B^{j_1\cdots j_n}\right)\partial_{i_1+j_1,\ldots,i_n+j_n}\right)\circ_2 F \\
&= \left(D_A^{i_1\ldots i_n}\circ_1\left(\partial_{i_1\ldots i_n}D_B^{j_1\cdots j_n}\right)\right)\circ_2\partial_{j_1\cdots j_n}F + \left(D_A^{i_1\ldots i_n}\circ_1 D_B^{j_1\cdots j_n}\right)\circ_2\partial_{i_1+j_1,\ldots,i_n+j_n}F,
\end{aligned}$$

where we have used the fact that the ∂ operator is a scalar operator and commutes with $\circ_1$ and $\circ_2$.

Thus for a pure operator product $D_A\circ D_B$ we have

$$D_A\circ D_B = \left(D_A^{i_1\ldots i_n}\circ\left(\partial_{i_1\ldots i_n}D_B^{j_1\cdots j_n}\right)\right)\partial_{j_1\cdots j_n} + \left(D_A^{i_1\ldots i_n}\circ_1 D_B^{j_1\cdots j_n}\right)\partial_{i_1+j_1,\ldots,i_n+j_n} \tag{72}$$

and the form of eq (72) is the same as eq(69). The basis of eq (72) is that the ∂ operator operates on all object to the right of it as products so that the product rule must be used in all differentiations. Since eq (72) puts the product of two differential operators in standard form we also evaluate $F\circ_2\left(D_A\circ_1 D_B\right)$.

A special case we must consider is $\nabla\circ F$ or $F\circ\nabla$. Should it return a multivector or a differential operator? We can make this situation unambiguous by defining an alternative geometric derivative operator $\bar{\nabla}$ so that $\nabla\circ F$ always immediately evaluates the derivatives of F and returns a multivector while $\bar{\nabla}\circ F$ defers evaluation of any derivatives and returns a differential operator. If the $D^{j_1\cdots j_n}$ are scalar functions these definitions agree with those used for the operator of quantum mechanics. For example consider the expectation value of the commutator of position and momentum:

$$\begin{aligned}
\int dx\;\psi^*\left[i\hbar\frac{\partial}{\partial x},x\right]\psi &= \int dx\;\psi^*\left(i\hbar\frac{\partial}{\partial x}x - xi\hbar\frac{\partial}{\partial x}\right)\psi \\
&= i\hbar\int dx\;\psi^*\left(\frac{\partial}{\partial x}\left(x\psi\right) - x\frac{\partial}{\partial x}\psi\right)
\end{aligned}$$

$$= i\hbar \int dx\ \psi^* \psi.$$

A general differential operator is built from repeated applications of the basic operator building blocks $(\bar{\nabla} \circ A)$, $(A \circ \bar{\nabla})$, $(\bar{\nabla} \circ \bar{\nabla})$, and $(A \pm \bar{\nabla})$. Both ∇ and $\bar{\nabla}$ are represented by the operator

$$\nabla = \bar{\nabla} = e^i \frac{\partial}{\partial x^i}, \tag{73}$$

but are flagged to produce the appropriate result.

For example if we wanted the commutator of the position and geometric derivative operating on a multivector function F we would write

$$\begin{aligned} \left[x, \bar{\nabla}\right] F &= \left(x\bar{\nabla} - \bar{\nabla}x\right) F \\ &= x\bar{\nabla}F - \dot{\bar{\nabla}}\dot{x}F - \dot{\bar{\nabla}}x\dot{F}. \end{aligned} \tag{74}$$

If we could use ∇ instead of $\bar{\nabla}$ we would get

$$\begin{aligned} \left[x, \nabla\right] F &= \left(x\nabla - \nabla x\right) F \\ &= x\nabla F - \left(\nabla x\right) F, \end{aligned} \tag{75}$$

but we cannot use ∇ unless we define (in general)

$$\nabla \circ A \equiv \bar{\nabla} \circ A, \tag{76}$$

but then we would have no way of writing the immediate evaluation of $\nabla \circ A$. Thus for the purpose of a computer algebraic representation of mulitvector differential operators we need both ∇ and $\bar{\nabla}$ since, at least in python, we cannot implement the dot notation for differentiation without writing a custom parser.

In the $\bar{\nabla}$ notation the directional derivative operator is $a \cdot \bar{\nabla}$, the Laplacian $\bar{\nabla} \cdot \bar{\nabla}$ and the expression for the Riemann tensor, R^i_{jkl}, is

$$\left(\bar{\nabla} \wedge \bar{\nabla}\right) \boldsymbol{e}^i = \frac{1}{2} R^i_{jkl} \left(\boldsymbol{e}^j \wedge \boldsymbol{e}^k\right) \boldsymbol{e}^l. \tag{77}$$

1.10 Manifolds and Submanifolds

A m-dimensional vector manifold[2], $\mathcal{M}$, is defined by a coordinate tuple (tuples are indicated by the vector accent "$\vec{\ }$")

$$\vec{x} = \left(x^1, \ldots, x^m\right), \tag{78}$$

and the differentiable mapping (U^m is an m-dimensional subset of $\Re^m$)

$$\boldsymbol{e}^{\mathcal{M}}\left(\vec{x}\right) : U^m \subseteq \Re^m \rightarrow \mathcal{V}, \tag{79}$$

where $\mathcal{V}$ is a vector space with an inner product[3] $(\cdot)$ and is of $\dim\left(\mathcal{V}\right) \geq m$.

[2] By the manifold embedding theorem any m-dimensional manifold is isomorphic to a m-dimensional vector manifold
[3] This product in not necessarily positive definite.