Chapter 10

The Conformal Model

Excerpted from *Linear and Geometric Algebra* by Alan Macdonald.
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A Jupyter notebook cm3 for 3D conformal model calculations is available at the book's webpage. It is also bundled with the $\mathcal{GA}$lgebra distribution.

10.1 The Geometric Algebra $\mathbb{G}^{r,s}$.

An orthonormal basis for the *indefinite* inner product space $\mathbb{R}^{r,s}$ has r $\mathbf{e}$'s with $\mathbf{e}\cdot\mathbf{e} = 1$ and s $\mathbf{e}$'s with $\mathbf{e}\cdot\mathbf{e} = -1$. And as in $\mathbb{R}^n$, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$.[1] According to *Sylvester's law of inertia*, r and s are independent of the orthonormal basis. The special case $\mathbb{R}^{n,0}$ is the $\mathbb{R}^n$ of earlier chapters.

Axioms I1-I3 in the Definition 4.9 of an inner product space are retained, but I4, "If $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{v}\cdot\mathbf{v} > 0$", is dropped. Set $r + s = n$. A basis is commonly written $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ with $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ for $i = 1, \ldots, r$ and $\mathbf{e}_i \cdot \mathbf{e}_i = -1$ for $i = r+1, \ldots, n$. Then for $\mathbf{u} = \sum_i u_i \mathbf{e}_i$ and $\mathbf{v} = \sum_j v_j \mathbf{e}_j$ (cf. Eq. (4.8)),

$$\mathbf{u}\cdot\mathbf{v} = u_1 v_1 + \cdots + u_r v_r - u_{r+1} v_{r+1} - \cdots - u_n v_n.$$

Exercise 10.1. Let $\mathbf{e}_+$ and $\mathbf{e}_-$ be orthogonal vectors in $\mathbb{R}^{r,s}$ with $\mathbf{e}_+ \cdot \mathbf{e}_+ = 1$ and $\mathbf{e}_- \cdot \mathbf{e}_- = -1$. Show that $\mathbf{e}_+ + \mathbf{e}_-$ is *null*: $(\mathbf{e}_+ + \mathbf{e}_-)\cdot(\mathbf{e}_+ + \mathbf{e}_-) = 0$.

The existence Theorem 6.1 for $\mathbb{G}^n$ extends to an existence theorem for the geometric algebra $\mathbb{G}^{r,s}$.[2] The inner and outer products of blades are defined as in $\mathbb{G}^n$: $A \cdot B = \langle AB \rangle_{k-j}$ and $A \wedge B = \langle AB \rangle_{j+k}$. (Blades of $\mathbb{G}^{r,s}$ will not be denoted in bold. We reserve bold for blades in $\mathbb{R}^{r,s}$.)

Most properties of $\mathbb{G}^n$ extend to $\mathbb{G}^{r,s}$, including the duality relations of Theorem 6.26 and the extended fundamental identity Theorem 6.28. Gram-Schmidt orthogonalization is an exception; it is not available in $\mathbb{G}^{r,s}$ (Problem 10.6.4).

Exercise 10.2. Theorem 7.2a is a test for membership in a subspace of $\mathbb{R}^n$: $x \in B \;\Leftrightarrow\; x \wedge B = 0$. ($B$ on the left is a set; on the right a multivector.) Its proof uses Gram-Schmidt orthogonalization. Prove the test for $\mathbb{R}^{r,s}$ without it.

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[1] Einstein's relativity theory uses $\mathbb{R}^{1,3}$. It is called *spacetime* (Cf. Problem 4.3.12).

[2] A. Macdonald, *An Elementary Construction of the Geometric Algebra*, Adv. Appl. Cliff. Alg. **12**, 1-6 (2002). An improved version is available at this book's web page.

10.2 The Conformal Model

The geometric algebra $\mathbb{G}^n$ is called the *standard model* of $\mathbb{R}^n$. The geometric algebra $\mathbb{G}^{n+1,1}$ is called the *conformal model* of $\mathbb{R}^n$.[3]

Extend $\mathbb{R}^n$ with two vectors e_+ and e_- orthogonal to $\mathbb{R}^n$, with $e_\pm^2 = \pm 1$. Then we have $\mathbb{R}^{n+1,1}$, and with it $\mathbb{G}^{n+1,1}$. More useful than $e_\pm$ are the vectors

$$o = \tfrac{1}{2}(e_- + e_+) \quad \text{and} \quad \infty = e_- - e_+.$$

Exercise 10.3. Show and memorize: $o^2 = \infty^2 = 0$ and $o \cdot \infty = -1$.

Definition 10.1 (Conformal point)**.** The conformal model represents the *point* $\mathbf{p} \in \mathbb{R}^n$ with the *vector*

$$p = o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2\infty \in \mathbb{R}^{n+1,1}. \tag{10.1}$$

The representation is *homogeneous*: nonzero scalar multiples ap also represent $\mathbf{p}$. The vector p in Eq. (10.1) is *normalized*: the coefficient of o is 1.

Exercise 10.4. a. Show that p is *null*: $p^2 = p \cdot p = 0$. Important!

b. More generally, show that

$$p \cdot q = -\tfrac{1}{2}(\mathbf{p} - \mathbf{q})^2. \tag{10.2}$$

Translations and rotations preserve $(\mathbf{p} - \mathbf{q})^2$, so preserve $p \cdot q$, so are orthogonal transformations on $\mathbb{G}^{n+1,1}$. Details are in Section 10.5.

Exercise 10.5. The *midplane* $\mathcal{M}$ of points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^3$ consists of points equidistant from $\mathbf{p}$ and $\mathbf{q}$. Show that $\mathbf{r} \in \mathcal{M} \;\Leftrightarrow\; r \cdot (p - q) = 0$.

Definition (Representations)**.** Let $\mathcal{G}$ be a geometric object in $\mathbb{R}^n$. We say that a blade G in $\mathbb{G}^{n+1,1}$ *directly represents* $\mathcal{G}$ if $\mathbf{x} \in \mathcal{G} \;\Leftrightarrow\; x \wedge G = 0$.

By duality, $\mathbf{x} \in \mathcal{G} \;\Leftrightarrow\; x \cdot G^* = 0$. We say that G^* *dually represents* $\mathcal{G}$.

Both representations are homogeneous, e.g., for $a \neq 0$, $x \wedge G = 0 \;\Leftrightarrow\; x \wedge aG = 0$.

Exercise 10.5 showed that the vector $p - q$ is a dual representation of $\mathcal{M}$.

Equation (10.1) assigns $p = o$ to $\mathbf{p} = \mathbf{0}$. So o represents the point at the origin. It is called the *origin*. Also from Eq. (10.1), $\lim_{|\mathbf{p}| \to \infty} p/(\tfrac{1}{2}\mathbf{p}^2) = \infty$ (the vector). Thus the choice of ∞ to denote this vector. It is called *infinity*.

Figure 10.1 shows an imperfect but helpful way to visualize this for $\mathbb{R}^2$. Set a unit sphere on $\mathbb{R}^2$ with its South Pole on the origin $\mathbf{0}$. This creates the 1-1 correspondence shown between points $\mathbf{p}$ in the plane and points p on the sphere, except the North Pole, labeled ∞. The vector ∞ represents $\boldsymbol{\infty}$, a single "point at infinity" appended to the plane. As $\mathbf{p}$ moves farther from $\mathbf{0}$, p gets closer to ∞.

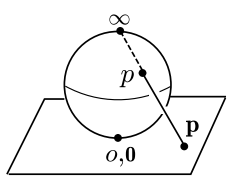

Fig. 10.1: $p \leftrightarrow \mathbf{p}$.

Vectors do double duty in $\mathbb{R}^n$ and in $\mathbb{G}^n$, representing both vectors and points (Figure 1.23). Null vectors p do single duty in the conformal model, representing points and only points. Other kinds of vectors represent other kinds of objects. For example, $p - q$ dually represents a midplane (Exercise 10.5).

[3] Much of this chapter was drawn from *Geometric Algebra for Computer Science (Revised Edition)* by L. Dorst, D. Fontijne, and S. Mann.

10.3 Dual Representations

This section and the next show that the conformal model represents several different kinds of geometric objects. Figure 10.2 lists them. Actually, they are only a small fraction of the total available (p. 196). All representations here are of $\mathbb{R}^3$ objects, the most important case, although much can be generalized.

Spheres. Equation (10.2) immediately gives an equation of the sphere with center $\mathbf{c}$ and radius ρ: $x \cdot c = -\frac{1}{2}\rho^2$. This is equivalent to $x \cdot (\,c - \frac{1}{2}\rho^2\infty) = 0$. Thus the *vector*

$$\sigma = c - \tfrac{1}{2}\rho^2\infty$$

is a dual representation of the *sphere*! *More succinctly*: σ is a *dual sphere.*

Exercise 10.6. Show that ρ can be extracted from σ: $\rho^2 = \sigma^2$.

Exercise 10.7. Let $p = o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2\infty$ represent a point and $\sigma = c - \frac{1}{2}\rho^2\infty$ dually represent a sphere. Show that $2p \cdot \sigma = \rho^2 - (\mathbf{p} - \mathbf{c})^2$. Thus $\mathbf{p}$ is inside, on, outside the sphere according as $p \cdot \sigma$ is $> 0,\ = 0,\ < 0$.

Rewrite σ in terms of the center $\mathbf{c}$ of the sphere and a point $\mathbf{p}$ on it:

$$\sigma = c - \tfrac{1}{2}\rho^2\infty = -(p \cdot \infty)\, c + (p \cdot c)\,\infty \stackrel{3}{=} p \cdot (c \wedge \infty). \tag{10.3}$$

Step (3) used Problem 9.6.6a. This is a second dual representation of a sphere.

Planes. The conformal model represents a unit *normal vector* $\mathbf{n}$ as $\pi = \mathbf{n} + d\infty$. (See Exercise 10.13 about normal vectors.) The normal vector is a dual representation of the plane orthogonal to $\mathbf{n}$ and at distance d from the origin. To see this, compute

$$x \cdot (\mathbf{n} + d\infty) = \left(o + \mathbf{x} + \tfrac{1}{2}\mathbf{x}^2\infty\right) \cdot (\mathbf{n} + d\infty) = -d + \mathbf{x} \cdot \mathbf{n}.$$

Setting this to zero gives the point-normal equation of the plane (Problem 4.1.3b).

Exercise 10.8. Extract d from $\mathbf{n} + d\infty$.

A given point $\mathbf{p}$ in the plane satisfies $p \cdot (\mathbf{n} + d\infty) = 0$, so $d = -(p \cdot \mathbf{n})/(p \cdot \infty)$. Substitute this into $\mathbf{n} + d\infty$ and multiply by the scalar[4] $-p \cdot \infty$ to obtain a dual representation of the plane through $\mathbf{p}$ and orthogonal to $\mathbf{n}$:

$$\pi = -(p \cdot \infty)\, \mathbf{n} + (p \cdot \mathbf{n})\, \infty \stackrel{3}{=} p \cdot (\mathbf{n}\infty). \tag{10.4}$$

Step (3) used Problem 9.6.6a. This is a second dual representation of a plane.

Circles. We seek a dual representation of the circle with center $\mathbf{c}$, with radius ρ, and is orthogonal to the unit vector $\mathbf{n}$. It is the intersection of the dual sphere $\sigma = c - \frac{1}{2}\rho^2\infty$ (Eq. (10.3)) and the dual plane $\pi = c \cdot (\mathbf{n}\infty)$ (Eq. (10.4)). From Theorem 10.3 the circle's dual representation is the bivector

$$c = \sigma \wedge \pi = (c - \tfrac{1}{2}\rho^2\infty) \wedge (c \cdot (\mathbf{n}\infty)).$$

Squaring the circle ☺ gives $c^2 = -\rho^2$.

Lines. Let π_1 and π_2 represent intersecting dual planes. Then the bivector $\lambda = \pi_1 \wedge \pi_2$ dually represents their line of intersection (Theorem 10.3 again).

[4]Remember, representations are homogeneous.

10.4 Direct Representations

We obtain direct representations of the geometric objects of the last section.

Lines. A direct representation of the line determined by points $\mathbf{p}$ and $\mathbf{q}$ is $L = p \wedge q \wedge \infty$. See Exercise 10.20. Thus a point $\mathbf{x}$ is on the line if and only if $x \wedge (p \wedge q \wedge \infty) = 0$.

Neither $\mathbf{p}$ nor $\mathbf{q}$ can be extracted from L. But the distance between them can: $L^2 = (\mathbf{p} - \mathbf{q})^2$.

If $\mathbf{x}, \mathbf{p}, \mathbf{q}$ are not collinear, then $x \wedge p \wedge q \wedge \infty \neq 0$. This leads to a measure of noncollinearity in numerical work.

Specify a line by a point $\mathbf{p}$ on it and a vector $\mathbf{d}$ parallel to it. Then $p \wedge \mathbf{d} \wedge \infty$ is a second direct representation of a line. See Problem 10.5.2 for $\mathbf{d} \wedge \infty$ here.

Circles. A direct representation of the circle determined by noncollinear points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is $C = p \wedge q \wedge r$. See Exercise 10.20. Its center is at $C\infty C$. Its radius ρ is given by $\rho^2 = -C^2/(C \wedge \infty)^2$. See Problem 10.6.1.

Intuitively, the lines $p \wedge q \wedge \infty$ above are "circles through infinity".

Planes. A direct representation of the plane determined by noncollinear points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ is $P = p \wedge q \wedge r \wedge \infty$.

The area A of the triangle is given by $A^2 = -P^2/4$.

Specify a plane by a point $\mathbf{p}$ on it and a bivector $\mathbf{D}$ parallel to it. Then $p \wedge \mathbf{D} \wedge \infty$ is a second direct representation of a plane. See Problem 10.6.2.

Spheres. A direct representation of the sphere determined by noncoplanar points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ is $S = p \wedge q \wedge r \wedge s$. Its radius ρ is given by $\rho^2 = S^2/(S \wedge \infty)^2$. Its center is at $S\infty S$ (unnormalized).

Intuitively, the planes $p \wedge q \wedge r \wedge \infty$ above are "spheres through infinity".

Vectors. The vectors of vector algebra represent several different kinds of vector-like objects in $\mathbb{R}^n$: direction vectors (Problem 10.5.2), normal vectors (Exercise 10.13), and tangent vectors (Problem 10.7.4). The conformal model represents these different kinds of geometric objects differently, as we will see.

Vector algebra vectors also represent points (Figure 1.23). The conformal model represents them as null vectors (Definition 10.1).

The various objects sometimes transform differently, for example, direction, normal, and null vectors under a translation by $\mathbf{a}$ (Problem 10.5.2). Yet the single linear transformation $\mathsf{T}_\mathbf{a}$ in the next section correctly translates them all.

Problems 10.4

10.4.1. a. Set $E = o \wedge \infty$. Show that $E^2 = 1$. *Hint*: Show that $o \wedge \infty = e_+ e_-$.

b. Show that $pE = (o - \frac{1}{2}\mathbf{p}^2\infty) + \mathbf{p}E$. This is called a *conformal split* of p.

c. Expand $\mathbf{p}E = \sum_i p_i \mathbf{e}_i E$. Show that the trivectors $\boldsymbol{\sigma}_i = \mathbf{e}_i E$ have the properties of an orthonormal vector basis for $\mathbb{R}^n$: $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i = 1$, and for $i \neq j$, $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = 0$. (Hence the use of bold for the $\boldsymbol{\sigma}_i$.)

10.4.2. Consider the plane through $\mathbf{e}_1, 2\mathbf{e}_2, 3\mathbf{e}_3$. Take the dual of its direct representation to find a normal vector and distance to the origin. Use $\mathcal{GA}$lgebra.

10.5 Transformations of Points

Exercise 10.9. a. Let $\mathbf{p} \in \mathbb{R}^n$. Show that $(\alpha o + \mathbf{p} + \gamma\infty)^2 = -2\alpha\gamma + \mathbf{p}^2$. *Hint*: Treat the $\alpha = 0$ and $\alpha \neq 0$ cases separately.

b. Show that null vectors in $\mathbb{R}^{n+1,1}$ are of the form βp ($\mathbf{p} \in \mathbb{R}^n$), $\gamma\infty$, or 0.

According to Part (b), null vectors, excluding 0, provide a (homogeneous) *algebraic* representation of $\overline{\mathbb{R}}^n \equiv \mathbb{R}^n \cup \{\boldsymbol{\infty}\}$. We will see that ∞ gives sensible results in calculations when thought of as representing $\boldsymbol{\infty}$, the point at infinity.

A *conformal transformation* of $\overline{\mathbb{R}}^n$ preserves scalar angles (not necessarily orientations) between intersecting curves. Examples include several fundamental geometric transformations: rotations, translations, reflections, dilations, inversion. Every conformal transformation is a (not unique) composition of these.

An orthogonal linear transformation O on $\mathbb{R}^{n+1,1}$ leaves inner products of vectors invariant (Exercise 9.20a). So $p \cdot p = 0 \;\Rightarrow\; \mathsf{O}(p) \cdot \mathsf{O}(p) = 0$, i.e., O represents a mapping from $\overline{\mathbb{R}}^n$ onto $\overline{\mathbb{R}}^n$.

The conformal model $\mathbb{G}^{n+1,n}$ represents
conformal transformations of $\overline{\mathbb{R}}^n$
with orthogonal transformations on $\mathbb{R}^{n+1,1}$.

We will show this for the five aforementioned conformal transformations. Theorem 10.2f and Problem 10.6.3 together show that the transformations preserve inner products of vectors, and so are indeed orthogonal.

Rotations. The rotation of a point $\mathbf{p}$ by angle $\mathbf{i}\theta$ is represented In $\mathbb{G}^n$ by $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{p}) = \mathrm{e}^{-\mathbf{i}\theta/2}\,\mathbf{p}\,\mathrm{e}^{\mathbf{i}\theta/2}$ (Section 7.2). In $\mathbb{G}^{n+1,1}$ the rotation is represented by

$$\mathsf{R}_{\mathbf{i}\theta}(p) = \mathrm{e}^{-\mathbf{i}\theta/2}\,p\,\mathrm{e}^{\mathbf{i}\theta/2} = \mathrm{e}^{-\mathbf{i}\theta/2}\left(o + \mathbf{p} + \tfrac{1}{2}\mathbf{p}^2\infty\right)\mathrm{e}^{\mathbf{i}\theta/2} = o + \mathsf{R}_{\mathbf{i}\theta}(\mathbf{p}) + \tfrac{1}{2}\mathsf{R}_{\mathbf{i}\theta}(\mathbf{p})^2\infty.$$

Exercise 10.10. Verify the just used $\mathsf{R}_{\mathbf{i}\theta}(o) = o$ and $\mathsf{R}_{\mathbf{i}\theta}(\infty) = \infty$.

Translations. The translation $\mathbf{p} \mapsto \mathbf{p} + \mathbf{a}$ in $\mathbb{R}^n$ is represented in the conformal model by

$$\mathsf{T}_{\mathbf{a}}(p) = \mathrm{e}^{-\mathbf{a}\infty/2}\,p\,\mathrm{e}^{\mathbf{a}\infty/2}.$$

Intuitively, this is a "rotation around infinity".

Translations in $\mathbb{R}^n$ are not linear: $(\mathbf{p} + \mathbf{q}) + \mathbf{a} \neq (\mathbf{p} + \mathbf{a}) + (\mathbf{q} + \mathbf{a})$. The conformal model *linearizes* them with the linear transformations $\mathsf{T}_{\mathbf{a}}$, a boon.

Exercise 10.11. Show that $\mathrm{e}^{\pm\mathbf{a}\infty/2} = 1 \pm \mathbf{a}\,\infty/2$ (exactly).

Exercise 10.12. Show: a. $\mathsf{T}_{\mathbf{a}}(o) = a$. b. $\mathsf{T}_{\mathbf{a}}(\infty) = \infty$. c. $\mathsf{T}_{\mathbf{a}}\mathsf{T}_{\mathbf{b}} = \mathsf{T}_{\mathbf{a}+\mathbf{b}}$.

Exercise 10.13. Show that a normal vector $\mathbf{n} + d\infty$ (Section 10.3) has no position; translating it in a direction $\mathbf{a}$ orthogonal to $\mathbf{n}$ leaves it invariant.

Reflections. Let $\mathbf{n}$ be a vector orthogonal to a hyperplane in $\mathbb{R}^n$. The standard model represents the reflection of point $\mathbf{p}$ in the hyperplane as $\mathsf{M}_\mathbf{n}(\mathbf{p}) = -\mathbf{n}\,\mathbf{p}\,\mathbf{n}^{-1}$ (Theorem 7.9). The conformal model represents it as

$$\mathsf{M}_\mathbf{n}(p) = -\mathbf{n}\,p\,\mathbf{n}^{-1}. \tag{10.5}$$

Exercise 10.14. Prove Eq. (10.5). *Hint*: First compute $\mathsf{M}_\mathbf{n}(o)$ and $\mathsf{M}_\mathbf{n}(\infty)$.

Dilations. The map $\mathbf{p} \mapsto \alpha\mathbf{p}$, $\alpha > 0$, is a *dilation* (by α). It is represented in the conformal model by

$$\mathsf{D}_\alpha(p) = \mathrm{e}^{E\ln\alpha/2}\,p\,\mathrm{e}^{-E\ln\alpha/2}.$$

To normalize the result, divide by the coefficient of o (which is α^{-1}).

Dilation can be thought of as a "rotation" in the E-plane. Recall: $\mathbf{i}^2 = -1$ and $\mathrm{e}^{\mathbf{i}\theta/2} = \cos(\theta/2) + \mathbf{i}\sin(\theta/2)$. In contrast, $E^2 = +1$ (Problem 10.4.1a) and $\mathrm{e}^{E\beta/2} = \cosh(\beta/2) + E\sinh(\beta/2)$, as a power series expansion reveals.

Inversion. The map $\mathbf{p} \mapsto \mathbf{p}^{-1} = \mathbf{p}/|\mathbf{p}|^2$ is called *inversion* (in the unit hypersphere). Points inside the sphere are mapped to the outside and vice-versa. Inversion in $\mathbb{R}^n$ is not linear $\big((\mathbf{p}+\mathbf{q})^{-1} \neq \mathbf{p}^{-1} + \mathbf{q}^{-1}\big)$, but is linearized in the conformal model:

$$\mathsf{I}(p) = -(o - \tfrac{1}{2}\infty)p(o - \tfrac{1}{2}\infty)^{-1}. \tag{10.6}$$

Note: $(o - \frac{1}{2}\infty)^{-1} = o - \frac{1}{2}\infty$. To normalize the result, divide by the coefficient of o (which is $\mathbf{p}^2$). Inversions reverse orientations.

Exercise 10.15. a. Which points in $\overline{\mathbb{R}}^n$ are represented by $\frac{1}{2}\infty$ and $2o$?
b. Show that $\mathsf{I}(o) = \frac{1}{2}\infty$. Intuitively, $\mathbf{0}^{-1} = \boldsymbol{\infty}$.
c. Show that $\mathsf{I}(\infty) = 2o$. Intuitively, $\boldsymbol{\infty}^{-1} = \mathbf{0}$.
d. Show that $\mathsf{I}(p)$ represents inversion.

Lines, planes, circles, and spheres map to the same under rotations, translations, reflections, and dilations. Inversion is the exception.

Under inversion, lines not through the origin map to circles through the origin and vice versa (Exercise 10.21). Planes not through the origin and spheres through origin are similarly related.

Problems 10.5

10.5.1. Show that inversion I is reflection in the hyperplane normal to e_+.

10.5.2 (Direction vectors)**.** A *direction vector* $\mathbf{d}$ has a direction, but not a position. They are like the oriented lengths of Section 1.1 – also called direction vectors there. The conformal model represents a direction vector $\mathbf{d}$ with $\mathbf{d} \wedge \infty$.

The line through a point $\mathbf{p}$ and parallel to a direction vector $\mathbf{d}$ is directly represented by $p \wedge (\mathbf{d} \wedge \infty)$ (Sections 10.4 and 10.6).

a. Show that $\mathbf{d}\wedge\infty$ has no position, i.e., is fixed under $\mathsf{T}_\mathbf{a}$: $\mathsf{T}_\mathbf{a}(\mathbf{d}\wedge\infty) = \mathbf{d}\wedge\infty$.
b. Show that direction vectors rotate as expected: $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{d}\wedge\infty) = \mathsf{R}_{\mathbf{i}\theta}(\mathbf{d})\wedge\infty$.

10.6 Covariance

The representations of rotations, translations, and dilations of points in Section 10.5 are of the form $\mathsf{O}(p) = VpV^{-1}$. Extend O to all $M \in \mathbb{G}^{n+1,1}$: $\mathsf{O}(M) = VMV^{-1}$. Then O is invertible: $V^{-1}(VMV^{-1})V = M$.

Theorem 10.2. Transformations of the form $\mathsf{O}(M) = VMV^{-1}$ preserve the entire algebraic structure of $\mathbb{G}^{n+1,1}$:

a. $\mathsf{O}(aM) = a\mathsf{O}(M)$.
b. $\mathsf{O}(M+N) = \mathsf{O}(M) + \mathsf{O}(N)$.
c. $\mathsf{O}(M \wedge N) = \mathsf{O}(M) \wedge \mathsf{O}(N)$.
d. B a blade $\Rightarrow \mathsf{O}(B)$ a blade (same grade).
e. $\mathsf{O}(MN) = \mathsf{O}(M)\,\mathsf{O}(N)$.
f. $\mathsf{O}(M \cdot N) = \mathsf{O}(M) \cdot \mathsf{O}(N)$.

Parts (a)-(c) tell us that O is an outermorphism of $\mathbb{G}^{n+1,1}$.

Exercise 10.16. Show that O preserves the inner product of vectors in $\mathbb{R}^{n+1,1}$. *Hint*: Recall cyclic reordering from Theorem 6.6.

Proof. a, b. These are obvious.

e. $\mathsf{O}(MN) = V(MN)V^{-1} = \big(VMV^{-1}\big)\big(VNV^{-1}\big) = \mathsf{O}(M)\mathsf{O}(N)$.

d. Let $B = b_1 \wedge \cdots \wedge b_r$, a blade of grade r. Then also $B = c_1 \cdots c_r$, a product of r orthogonal vectors.[5] From Part (e), $\mathsf{O}(B) = \mathsf{O}(c_1)\cdots\mathsf{O}(c_r)$. From Exercise 10.16, the $\mathsf{O}(c_i)$ are orthogonal, so $\mathsf{O}(B)$ is a blade of grade r.

c, f. See Exercise 10.17. □

Exercise 10.17. Prove Parts (c) and (f) of the theorem. *Hint*: Consider first blades $\mathbf{M}$ and $\mathbf{N}$. Use Part (d) and Definitions 6.12 and 6.13.

Let the blade $M \in \mathbb{G}^{n+1,1}$ represent a geometric object $\mathcal{M} \subseteq \mathbb{R}^n$. Let $\mathsf{O}(x) = VxV^{-1}$ represent a transformation of the *points* $\mathbf{x}$ of $\mathcal{M}$, as in the last section. Remarkably, we need not transform $\mathcal{M}$ pointwise: O transforms a geometric object as a *whole*!

Theorem (Covariance). *$\mathsf{O}(M)$ represents $\mathcal{M}$ transformed.*

Proof. Suppose that M directly represents $\mathcal{M}$. Then

$$\mathsf{O}(x) \in \mathsf{O}(M) \overset{1}{\Leftrightarrow} \mathsf{O}(x) \wedge \mathsf{O}(M) = 0 \overset{2}{\Leftrightarrow} \mathsf{O}(x \wedge M) = 0 \overset{3}{\Leftrightarrow} x \wedge M = 0 \overset{4}{\Leftrightarrow} x \in M.$$

Steps (1) and (4) use Exercise 10.2. Step (2) uses covariance. Step (3) uses the invertibility of O. □

Exercise 10.18. Give a similar argument when M dually represents $\mathcal{M}$.

[5] The *proof* of this for $\mathbb{G}^n$ (Theorem 6.19a), uses Gram-Schmidt orthogonalization, which is not available in $\mathbb{G}^{r,s}$. Nevertheless, the *result* remains true in $\mathbb{G}^{r,s}$ (C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press (2007), p. 88; `https://en.wikipedia.org/wiki/Geometric_algebra#Blades.2C_grades.2C_and_canonical_basis`). Problem 10.6.4 has an example.

As an example, translate by $\mathbf{a}$ the direct line $L = p \wedge q \wedge \infty$. According to the theorem, the result is $\mathsf{T}_{\mathbf{a}}(L)$. As a check, use covariance directly and $\mathsf{T}_{\mathbf{a}}(\infty) = \infty$: $\mathsf{T}_{\mathbf{a}}(L) = \mathsf{T}_{\mathbf{a}}(p) \wedge \mathsf{T}_{\mathbf{a}}(q) \wedge \infty$, which represents L translated.

The rest of this section illustrates the power of covariance.

Line through p in direction d. We show that the trivector $L = p \wedge \mathbf{d} \wedge \infty$ is a direct representation of the line through the point $\mathbf{p}$ and in the direction $\mathbf{d}$.

Start with $p = o$. We need to solve $x \wedge (o \wedge \mathbf{d} \wedge \infty) = 0$. The o and ∞ factors in the parentheses kill the o and ∞ terms in x, leaving $\mathbf{x} \wedge (o \wedge \mathbf{d} \wedge \infty)$. This is zero if and only if $\mathbf{x} \in \operatorname{span}\{o, \mathbf{d}, \infty\}$, i.e., if and only if $\mathbf{x}$ is a scalar multiple of $\mathbf{d}$. This establishes the result for $p = o$.

To finish, translate $o \wedge \mathbf{d} \wedge \infty$ by $\mathbf{p}$ to obtain L (Step (2) uses Problem 10.5.2):

$$\mathsf{T}_{\mathbf{p}}(o \wedge \mathbf{d} \wedge \infty) = \mathsf{T}_{\mathbf{p}}(o) \wedge \mathsf{T}_{\mathbf{p}}(\mathbf{d} \wedge \infty) \overset{2}{=} p \wedge \mathbf{d} \wedge \infty.$$

We have just seen an example of a common strategy:

Establish a representation of some well chosen geometric objects.
Extend the representation to transformations of the objects using covariance.

Exercise 10.19 (**Dual line**)**.** Let C represent a circle. Show that the bivector $\lambda = \infty \cdot C$ dually represents the line orthogonal to C and through its center.

Exercise 10.20 (**Direct lines and circles**)**.** Justify the direct representations $L = p \wedge q \wedge \infty$ of lines and $C = p \wedge q \wedge r$ of circles from Section 10.4. The justifications for planes and spheres are similar, but more complicated.

Here is another example of the coherence of the conformal model (without proof): $\mathrm{e}^{-L^*\theta/2}\, C\, \mathrm{e}^{L^*\theta/2}$ represents the rotation by angle θ of C about L.

Exercise 10.21. Show that under inversion lines not through the origin and circles through the origin map to each other.

Representations of reflections and inversions are of the form $p \mapsto -VpV^{-1}$. Define an extension $M \mapsto -VMV^{-1}$. It is not covariant (Problem 10.6.3). However, we can use the covariance of the VMV^{-1} part. Here is an example.

Reflect a line in a plane. The reflection of the point $\mathbf{p}$ in the plane through the origin with normal $\mathbf{n}$ is represented by $-\mathbf{n}\,\mathbf{p}\,\mathbf{n}^{-1}$ (Theorem 7.9).

The formula extends to the reflection of the line $L = o \wedge \mathbf{d} \wedge \infty$ in the plane. To see this, first use the covariance of $M \mapsto +\mathbf{n}M\mathbf{n}^{-1}$ (Theorem 10.2):

$$\mathbf{n}L\mathbf{n}^{-1} = (\mathbf{n}\, o\, \mathbf{n}^{-1}) \wedge (\mathbf{n}\,\mathbf{d}\,\mathbf{n}^{-1}) \wedge (\mathbf{n}\,\infty\,\mathbf{n}^{-1}) = (-o) \wedge (\mathbf{n}\,\mathbf{d}\,\mathbf{n}^{-1}) \wedge (-\infty).$$

Negate this: $-\mathbf{n}L\mathbf{n}^{-1} = o \wedge (-\mathbf{n}\,\mathbf{d}\,\mathbf{n}^{-1}) \wedge \infty$. This represents L reflected, since $-\mathbf{n}\,\mathbf{d}\,\mathbf{n}^{-1}$ represents $\mathbf{d}$ reflected.

Angle between lines. We compute the angle θ between intersecting lines. Consider first lines intersecting at the origin: $L_1 = o \wedge \mathbf{d}_1 \wedge \infty$, $L_2 = o \wedge \mathbf{d}_2 \wedge \infty$, with $|\mathbf{d}_1| = |\mathbf{d}_2| = 1$. Because of the simple form of L_1 and L_2, one can compute $L_1 \cdot L_2 = \mathbf{d}_1 \cdot \mathbf{d}_2 = \cos\theta$. Now translate the lines to an arbitrary position. By covariance, the left side with the new lines still gives $\cos\theta$.

Angle between circles. First note that the condition $|\mathbf{d}_1| = |\mathbf{d}_2| = 1$ for the lines above is equivalent to $\mathbf{d}_1^2 = \mathbf{d}_2^2 = 1$. Consequently, $L_1^2 = L_2^2 = 1$.

Let C be a direct circle determined by $\mathbf{p}, \mathbf{q}, \mathbf{r}$. Application of the results for circles and planes from Section 10.4 gives $C^2 = -\rho^2(C\wedge\infty)^2 = -\rho^2 P^2 = 4\rho^2 A^2$. Thus C can be normalized to square to 1: divide by $2\rho A$.

Now let C_1 and C_2 directly represent coplanar circles intersecting at two points, both normalized to $C^2 = 1$. Translate the circles so an intersection point is at the origin. Invert with I. This angle preserving conformal transformation maps the circles to straight lines (Exercise 10.21), and $C^2 = 1$ to $L^2 = 1$ (since $\mathsf{I}(1) = -1$). Thus the angle between the circles is given by the same formula as for the lines: $\cos\theta = C_1 \cdot C_2$.

Translate rotations. $\mathsf{R}_{\mathbf{i}\theta}(p) = \mathrm{e}^{-\mathbf{i}\theta/2}\, p\, \mathrm{e}^{\mathbf{i}\theta/2}$ represents a rotation around $\mathbf{0}$ by angle $\mathbf{i}\theta$ (Section 10.5). To rotate around $\mathbf{a}$ by angle $\mathbf{i}\theta$, translate by $-\mathbf{a}$, rotate by $\mathbf{i}\theta$ around $\mathbf{0}$, and translate back by $\mathbf{a}$:

$$\begin{aligned} p \mapsto (\mathsf{T}_\mathbf{a} \circ \mathsf{R}_{\mathbf{i}\theta} \circ \mathsf{T}_{-\mathbf{a}})(p) &= \mathrm{e}^{-\mathbf{a}\infty/2} \left(\mathrm{e}^{-\mathbf{i}\theta/2} (\mathrm{e}^{\mathbf{a}\infty/2}\, p\, \mathrm{e}^{-\mathbf{a}\infty/2})\, \mathrm{e}^{\mathbf{i}\theta/2} \right) \mathrm{e}^{\mathbf{a}\infty/2} \\ &= (\mathrm{e}^{-\mathbf{a}\infty/2}\, \mathrm{e}^{-\mathbf{i}\theta/2}\, \mathrm{e}^{\mathbf{a}\infty/2})\, p\, (\mathrm{e}^{-\mathbf{a}\infty/2}\, \mathrm{e}^{\mathbf{i}\theta/2}\, \mathrm{e}^{\mathbf{a}\infty/2}) = \mathsf{T}_\mathbf{a}(\mathrm{e}^{-\mathbf{i}\theta/2})\, p\, \mathsf{T}_\mathbf{a}(\mathrm{e}^{\mathbf{i}\theta/2}). \end{aligned}$$

So $\mathsf{T}_\mathbf{a}(\mathrm{e}^{-\mathbf{i}\theta/2})$ is the representative of the rotation about $\mathbf{a}$. It is the translation by $\mathbf{a}$ of the representative $\mathrm{e}^{-\mathbf{i}\theta/2}$ of the rotation about $\mathbf{0}$. Coherence!

Notice how easy it was to compose the rotation and translations. Try the composition in $\mathbb{G}^n$, where translation by $\mathbf{a}$ is $\mathbf{p} \mapsto \mathbf{p} + \mathbf{a}$. You won't like it. The problem is that in $\mathbb{G}^n$ translation is additive and rotation is multiplicative. Each is simple, but their combination is not.

Exercise 10.22. a. Show that $\mathsf{T}_\mathbf{a}(\mathrm{e}^{\mathbf{i}\theta/2}) = \mathrm{e}^{\mathsf{T}_\mathbf{a}(\mathbf{i}\theta)/2}$. Thus the rotation about $\mathbf{a}$ can also be written $\mathrm{e}^{-\mathsf{T}_\mathbf{a}(\mathbf{i}\theta)/2}\, p\, \mathrm{e}^{\mathsf{T}_\mathbf{a}(\mathbf{i}\theta)/2}$. Coherence again!

b. Show that $\mathsf{T}_\mathbf{a}(\mathbf{i})$ is a 2-blade with $(\mathsf{T}_\mathbf{a}(\mathbf{i}))^2 = -1$.

Problems 10.6

10.6.1 (Center, radius of a circle). Let $C = p \wedge q \wedge r$ represent a circle.

a. Show that its center is $C\infty C$.

b. Show that its radius ρ is given by $\rho^2 = -C^2/(C \wedge \infty)^2$.

10.6.2 (Plane through p in direction D). A *direction bivector* $\mathbf{D}$ has a direction but not a position. The conformal model represents the direction bivector with $\mathbf{D} \wedge \infty$, which is translation invariant (cf. Problem 10.5.2).

Show that $p \wedge (\mathbf{D} \wedge \infty)$ is a direct representation of the plane through $\mathbf{p}$ and parallel to $\mathbf{D}$. *Hint*: See "Line through $\mathbf{p}$ in direction $\mathbf{d}$" above.

10.6.3. a. Define $\mathsf{S}(M) = -VMV^{-1}$. Show that S does not preserve the geometric product. Thus Theorem 10.2 does not apply to it.

b. Show that S is an orthogonal transformation, i.e., preserves the inner product of vectors.

10.6.4. Show that Gram-Schmidt orthogonalization as used in Theorem 6.19 fails for $o \wedge \infty$. Nevertheless, the theorem is true for this blade: $o \wedge \infty = e_+ e_-$.

10.7 Intersections

Geometric objects in $\mathbb{R}^n$ *meet* in lower dimensional objects, their intersection. Examples: The meet of a line and plane intersecting in a point is that point. The meet of distinct intersecting planes is their line of intersection.

Theorem 10.3 (Representation of intersections)**.** Let A and B be blades in $\mathbb{G}^{n+1,1}$ directly representing geometric objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^n$. Let J be a blade representing $\operatorname{span}(A, B)$. Take duals with respect to it: $A^* = A/J$, etc.

Then $A^* \wedge B^*$ is a dual representation of the meet $\mathcal{A} \cap \mathcal{B}$.

Proof. (Gregory Grunberg.) Let $x \in J$. Steps (1) and (5) below use the definition of dual and direct representations (Section 10.2). Step (2) follows from Theorem 6.30a,[6] Step (3) from Theorem 6.26, and Step (4) from Theorem 6.29a:

$$\begin{aligned}
\mathbf{x} \in \mathcal{A} \;\;\&\;\; \mathbf{x} \in \mathcal{B} &\overset{1}{\Rightarrow} x \cdot A^* = 0 \;\;\&\;\; x \cdot B^* = 0 \\
&\overset{2}{\Rightarrow} x \cdot (B^* \wedge A^*) = 0 \;\;\&\;\; x \cdot (A^* \wedge B^*) = 0 \\
&\overset{3}{\Rightarrow} x \wedge (B^* \cdot A) = 0 \;\;\&\;\; x \wedge (A^* \cdot B) = 0 \\
&\overset{4}{\Rightarrow} x \wedge A = 0 \;\;\&\;\; x \wedge B = 0. \\
&\overset{5}{\Rightarrow} \mathbf{x} \in \mathcal{A} \;\;\&\;\; \mathbf{x} \in \mathcal{B}.
\end{aligned}$$

This chain of implications circles around on itself, so all of its statements are equivalent. In particular, for $x \in J$, $\mathbf{x} \in \mathcal{A}$ & $\mathbf{x} \in \mathcal{B} \Leftrightarrow x \cdot (A^* \wedge B^*) = 0$. ☐

To apply the theorem we need the blade J. It represents the *join* of the geometric objects $\mathcal{A}$ and $\mathcal{B}$. Examples: The join of distinct intersecting lines is the plane containing them. The join of a line and a point not on it is the plane containing them.

Unfortunately, there is no general geometric algebra formula expressing J in terms just of A and B. But there are efficient algorithms which compute it.

The join of distinct points $\mathbf{p}$ and $\mathbf{q}$ is a new kind of geometric object for us, a *point pair*, consisting of just the two points. It has a direct representation $p \wedge q$ (Problem 10.7.3). It can also be thought of as representing the 0-dimensional analog of 1-dimensional circles ($p \wedge q \wedge r$) and 2-dimensional spheres ($p \wedge q \wedge r \wedge s$).

Example. Let σ and π be dual representations of a sphere and plane intersecting in a circle. The join of the sphere and plane is $\mathbb{R}^3$, so the circle's dual representation in $\mathbb{G}^{3+1,1}$ is $\sigma \wedge \pi$.

Example. Let π_1 and π_2 be dual representations of distinct intersecting planes in $\mathbb{R}^3$. Their join is $\mathbb{R}^3$, so $\pi_1 \wedge \pi_2$ is the dual representation of their line of intersection.

There is an analog of this example in $\mathbb{R}^3$ vector algebra: the cross product of vectors normal to two planes is a vector along the intersection of the planes.

[6] For the application of the theorem here, we must be sure that $A^* \cap B^* = \{0\}$:
$y \in A^*$ & $y \in B^* \;\Rightarrow\; y \cdot A = 0$ & $y \cdot B = 0 \;\Rightarrow\; y \cdot \operatorname{span}(A, B) = 0 \;\Rightarrow\; y \cdot J = 0 \;\Rightarrow\; y = 0$.

Exercise 10.23. Let $\mathbf{n}$ and $\mathbf{n} + d\infty$ be dual representations of parallel planes.

a. Find a dual representation of their intersection.

b. Show: No p is in the dual representation, but ∞ is. That is, the intersection of the planes in $\overline{\mathbb{R}}^n$ is ∞. Intuitively, this makes sense.

Exercise 10.24. Use Theorem 10.3 to compute the intersection of the line containing the points $-\mathbf{e}_1$ and $\mathbf{e}_1$ with the line containing the points $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 - \mathbf{e}_2$. *Hint*: First determine J.

Object	Dual Representation	Direct Representation
Sphere	$\sigma = c - \frac{1}{2}\rho^2\infty$ $\sigma = p \cdot (c \wedge \infty)$	$S = p \wedge q \wedge r \wedge s$
Plane	$\pi = \mathbf{n} + d\infty$ $\pi = p \cdot (\mathbf{n}\infty)$	$P = p \wedge q \wedge r \wedge \infty$ $P = p \wedge \mathbf{D} \wedge \infty$
Circle	$c = \sigma \wedge \pi$	$C = p \wedge q \wedge r$
Line	$\lambda = \pi_1 \wedge \pi_2$ $\lambda = \infty \cdot C$	$L = p \wedge q \wedge \infty$ $L = p \wedge \mathbf{d} \wedge \infty$
Point pair		$p \wedge q$

Fig. 10.2: Representations of geometric objects in the conformal model.

Problems 10.7

10.7.1. Let σ_1 and σ_2 be dual spheres in $\mathbb{R}^3$ of radius ρ centered at $\pm\mathbf{e}_1$.

a. Compute $C = \sigma_1 \wedge \sigma_2$ from Theorem 10.3. If $\rho > 1$, then C is the dual representation of the intersection of the spheres, a circle.

b. Show that $C^2 = 4(1 - \rho^2)$. Thus the spheres intersect in a circle, a point, or not at all, according as $C^2 < 0, = 0, > 0$. By covariance this test works for all sphere pairs of equal radii.

10.7.2. Refer to the first example following Theorem 10.3. Take $\sigma = o - \frac{1}{2}\infty$ (unit sphere centered at o) and $\pi = \mathbf{e}_3 + d\infty$ (plane parallel to xy plane at distance d away). Compute $c = \sigma \wedge \pi$. Interpret c for different values of d.

10.7.3. Show that $p \wedge q$ represents the point pair $\{\mathbf{p}, \mathbf{q}\}$. *Hint*: By covariance it is sufficient to show this for two (carefully chosen) points. I used $\mathcal{GA}$lgebra.

10.7.4 (Tangent vectors)**.** Let $c - \frac{1}{2}\infty$, $\mathbf{c} = \pm\mathbf{e}_1$, dually represent two circles in $\mathbb{R}^2$ of radius 1. They intersect at the origin with common tangent vector $\mathbf{e}_2$.
a. Find a dual representation of the intersection of the circles. *Ans.* $o \wedge 2\mathbf{e}_1$.
b. Show that origin is the only point in the intersection, as expected.
c. The *dual tangent vector* $o \wedge 2\mathbf{e}_1$ represents what is common to the circles: the point $\mathbf{0}$ and the tangent vector $\mathbf{e}_2$, represented dually. Give reasons for:

$$\mathsf{T}_\mathbf{p}(o \wedge 2\mathbf{e}_1) \stackrel{1}{=} \mathsf{T}_\mathbf{p}(o) \wedge \mathsf{T}_\mathbf{p}(2\mathbf{e}_1) \stackrel{2}{=} p \wedge \big(p \cdot (2\mathbf{e}_1 \wedge \infty)\big).$$

For Step (2) use $\mathcal{GA}$lgebra. This (not $p \wedge 2\mathbf{e}_1$) is a dual tangent vector at $\mathbf{p}$.

Conformal Model: Final Words

The geometric objects and their representations in this chapter are only a fraction of those available. Pablo Colapinto has compiled two remarkable lists. One lists 25 different kinds of geometric objects representable in the conformal model. The other lists 200 constructions in total for them! [7]

Figure 10.3 shows a small sample of the 200, seven (of nine) ways to construct a dual line. The first item says that a dual line can be constructed as an inner product of ∞ and a direct circle (Exercise 10.19). (For $\rfloor$ see the footnote to Definition 6.12.)

Fig. 10.3: Operations that Construct a Dual Line

Compare this richness with vector algebra, where vectors are the only kind of geometric object.

We have two geometric algebras for Euclidean n-space: the standard model $\mathbb{G}^n$ and the conformal model $\mathbb{G}^{n+1,1}$. In both, blades represent geometric objects. In the standard model an outer product of three vectors represents an oriented volume. In the conformal model an outer product of three null vectors represents a circle. The semantics are different: oriented volumes are not circles. But the syntax is the same: the same rules of geometric algebra apply to both. Thus we need learn the rules only *once* to work with *both*.

It is even better than this. With suitable software it is possible to reap the benefits of the conformal model without knowing the rules.

The software must provide functions to (i) construct geometric objects (e.g., a sphere from a center and radius), (ii) transform geometric objects (e.g., translate them), and (iii) retrieve information about geometric objects (e.g. the center of a sphere or the intersection of two spheres).

A user can then think just in terms of geometric objects and transformations, with the conformal calculations hidden away in a magic "black box". Various software does this to various degrees, including the $\mathcal{GA}$lgebra notebook cm3.

It is natural to couple such software with graphing software. The dimension of the 3D conformal model $\mathbb{G}^{3+1,1}$ is $2^5 = 32$ (Problem 6.1.1). Surprisingly, efficient conformal model based graphics programs are only slightly less efficient than those using traditional methods. One estimate is 10%.

Surveying recent applications of geometric algebra to computer science and engineering shows that

The conformal model is where it is at!

[7]P. Colapinto, *VERSOR: Spatial Computing with Conformal Geometric Algebra*. See pp. 25 and 67 for the lists. Figure used with permission of Colapinto.