Order in the particle zoo

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Abstract
The standard model of particle physics classifies particles into elementary leptons and hadrons composed of quarks. There exists an alternate ordering principle based on a function Ψ that may be derived from the framework of the Einstein field equations, giving a convergent series of particle energies, to be quantized as a function of the fine-structure constant, α, with limits given by the energy values of the electron and the Higgs vacuum expectation value. The series expansion of the energy equation provides quantitative terms for Coulomb, strong and gravitational interaction. The value of α can be given numerically by the gamma functions of the integrals involved, extending the formalism to N-dimensions yields a single expression for the electroweak coupling constants.
The model can be expressed without use of free parameters.

1 Introduction

Particle zoo is the informal though fairly common nickname to describe what was formerly known as "elementary particles". The standard model of particle physics [1] divides these particles into leptons, considered to be fundamental "elementary particles" and the hadrons, composed of quarks. Well hidden in the data of particle energies lies another ordering principle based on an exponential function Ψ. The function Ψ was developed originally in a heuristic "ad hoc" approach to calculate particle energies, inspired by basic principles of quantum mechanics and the first part of this article will follow this line (chpt. 2). Applying Ψ as equivalent of a probability amplitude in the equations for calculating particle energy in a point charge and a photon expression will yield a convergent series of particle energies to be quantized as a function of the fine-structure constant, α [1]. α itself can be calculated directly from the basic form of the energy integrals for both symmetries, i.e. without the dimensioned values, and given in terms of the corresponding Γ-functions. The expansion of the incomplete Γ-function appearing in the point charge integral gives quantitative terms for Coulomb, strong and gravitational interaction. The latter provides a link between the electron and the Planck energy, allowing to identify the electron as ground state. The upper limit of the convergent energy series coincides with the Higgs vacuum expectation value. The relation with Planck terms allows to express the equations of the model "ab initio" as function of elementary charge, e, electric constant, ε, and gravitational constant G.
In the second part of this article (chpt. 3) it will be demonstrated that Ψ may be derived directly from the framework of the Einstein field equations (EFE). In particular an ansatz for a metric that is derived from solutions for the Kaluza scalar, Φ [4], requiring Ψ to be a function of the electromagnetic potential, A, in the approximation of this model the electric potential, $\varphi \sim e_c/(\varepsilon_c r)$ [2], reproduces the expression for point charge energy modified by Ψ that yields the results listed above - except for the absolute value of the energy scale.
This is due to the constant G/c_0^4 in the EFE having an order of magnitude that does not fit well to electromagnetic phenomena, a problem Kaluza met in his original work as well. The constant required to replace G/c_0^4 can be easily identified as $1/\varepsilon_c$, the electric constant in appropriate natural units. Retaining SI units for length, time and energy the electromagnetic constants may be defined as:

$$c_0^2 = (\varepsilon_c\ \mu_c)^{-1} \qquad (1)$$

with $\varepsilon_c = (2.998\text{E+8 [m}^2\text{/Jm]})^{-1} = (2.998\text{E+8})^{-1}$ [J/m]
$\mu_c = (2.998\text{E+8 [Jm/s}^2\text{]})^{-1} = (2.998\text{E+8})^{-1}$ [s²/Jm]

From the Coulomb term $b_0 = e^2/(4\pi\varepsilon_0) = e_c^2/(4\pi\varepsilon_c) = 2.307\text{E-28}$ [Jm] follows for the square of the elementary charge: $e_c^2 = 9.671\text{E-36}$ [J²]. In the following $e_c = 3.110\text{E-18}$ [J] and $e_c/\varepsilon_c = 9.323\text{E-10}$ [m] may be used as natural unit of energy and length. Using this unit system the necessary parameters of the model will be further reduced to e_c and ε_c.

1 The relation of the masses e, μ, π with α was noted in 1952 by Y.Nambu [2]. M.MacGregor calculated particle mass and constituent quark mass as *multiples* of α and related parameters [3].
2 Ψ = f(α, e/(εr)), e = elementary charge, ε = electric constant, r = radius;

The geometric approach for calculating the electromagnetic coupling constant, α, can be extended to different dimensions, yielding the three electroweak coupling constants for three point charges g, e and g' in 4, 3 and 2 dimensions, indicating at a possible coherent interpretation of the phenomena discussed here in 4D space / 5D space-time.

For both approaches used in chpt. 2 and 3 it might be helpful to use the following visualization: a photon with its intrinsic angular momentum interpreted as having its E-vector rotating around a central axis of propagation [3] will be transformed into an object that has the -still rotating- E-vector constantly oriented to a fixed point, the origin of the local coordinate system used, resulting in an SO(3) object with point charge properties [4]. The vectors E, B and C of the propagation velocity are supposed to be locally orthogonal and subject to the standard Maxwell equations, however, on the background of an appropriately curved space-time.

To focus on the more fundamental relationships some minor aspects of the model are exiled to an appendix, related topics to be marked as [A]. Typical accuracy of the calculations presented is in the order of 0.001-0.0001 [5]. QED corrections are not considered in this model.

The model presented here is far from being complete and occasionally requires minor assumptions, yet it provides a coherent, quantitative and parameter-free formalism, combines many particle phenomena [6] and may be based on the well established concepts of GTR.

2 Ad hoc approach

2.1 Energy terms

The model may essentially be based on a single assumption:

Particles can be described by using an appropriate exponential wave function, Ψ(r), that acts as a probability amplitude on an electromagnetic field.

An appropriate form of Ψ can be deduced from three boundary conditions:

1.) To be able to apply Ψ to a point charge Ψ(r = 0) = 0 is required.

2.) To ensure integrability an integration limit is needed.

3.) Ψ should be applicable regardless of the expression chosen to describe the electromagnetic object. In particular requiring a point charge and a photon representation of a localized electromagnetic field (particle) to have the same energy results in an exponent of 3 for r in the equation below (see 2.2).

Condition 1.) to 3.) are met by an expression (corresponding differential equation see [A1]):

$$\Psi_n(r) = \exp\left(-\left(\frac{\beta_n/2}{r^3} + \left[\left(\frac{\beta_n/2}{r^3}\right)^2 - 4\frac{\beta_n/2}{\sigma r^3}\right]^{0.5}\right)/2\right) \tag{2}$$

Up to the limit of the real solution, $r = r_n$, with

$$r_n = (\sigma\, \beta_n/8)^{1/3} \tag{3}$$

in all integrals over Ψ(r) given below equ. (4) may be used as approximation for (2):

$$\Psi_n(r \le r_n) \approx \exp\left(\frac{-\beta_n/2}{r^3}\right) \tag{4}$$

Phase will be neglected on this approximation level, properties of particles will be calculated by the integrals over $\Psi_n(r)^2$ [7] times some function of r which can be given by:

$$\int_0^{r_n} \Psi_n(r)^2 r^{-(m+1)} dr \approx \int_0^{r_n} \exp(-\beta_n/r_n^3) r^{-(m+1)} dr = \Gamma(m/3, \beta_n/r_n^3)\,\frac{\beta_n^{-m/3}}{3} = \int_{\beta_n/r_n^3}^{\infty} t^{\frac{m}{3}-1} e^{-t} dt\,\frac{\beta^{-m/3}}{3} \tag{5}$$

with m = {..-1;0;1;..}. The term $\Gamma(m/3, \beta/r_1^3)$) denotes the upper incomplete gamma function, given by the

3 Angular momentum J = 1, symmetry SO(2) as projected in propagation direction;

4 Neutral particles are supposed to exhibit nodes separating corresponding equal volume elements of reversed E-vector orientation and opposite polarity.

5 Including e.g. errors due to the numerical approximation of Γ-functions.

6 Additional particle properties discussed in [5];

7 Hence factor 2 in (2)ff

Euler integral of the second kind with $\beta_n/r_n^3 = 8/\sigma$ as lower integration limit [8]. For m ≥ 1 the complete gamma function $\Gamma_{m/3}$ is a sufficient approximation, for m ≤ 0 the integrals have to be integrated numerically.

Coefficient β_n may be given as partial product of a value for a ground state particle, β_{GS}, carrying a dimensional term, β_{dim} [m³], that will be demonstrated to have a particular useful expression using the cube of the natural unit for length as given in chpt.1 as [see A4]:

$$\beta_{dim} = \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 = 5.131\text{E-}30\ [\text{m}^3] \tag{6}$$

times particle specific dimensionless coefficients, α_n, of succeeding particles representing the ratio β_{n+1} / β_n:

$$\beta_n = \beta_{GS}\Pi^{n}_{k=1}\alpha_k = 2\sigma\alpha_{GS}\beta_{dim}\Pi^{n}_{k=1}\alpha_k = 2\sigma\alpha_{GS}\beta_{dim}\Pi_{\beta,n} \qquad n = \{1;2;..\} \tag{7}$$ [9]

Index n will indicate solutions of (2) and serve in the following as equivalent of a radial quantum number. For the angular terms of Ψ(r, ϑ, φ), to be indicated by index l, only rudimentary results exist, their contribution has to be incorporated in parameter σ (to be discussed in 2.4 - 2.6).

Particle energy is expected to be equally divided into electric and magnetic part, $W_n = 2W_{n,el} = 2W_{n,mag}$. To calculate energy the integral over the electrical field E(r) of a point charge is used, equ. (5) for m = 1 gives:

$$W_{pc,n} = 2\varepsilon_0\int_0^\infty E(r)^2\Psi_n(r)^2 d^3r = 2b_0\int_0^{r_n}\Psi_n(r)^2 r^{-2}dr = 2\,b_0\,\Gamma(1/3, \beta_n/r_n^3)\,\beta_n^{-1/3}/3 \approx 2\,b_0\,\Gamma_{1/3}\,\beta_n^{-1/3}/3 \tag{8}$$

Using equation (5) for m = -1 to calculate the Compton wavelength, λ_C, gives:

$$\lambda_{C,n} \approx \int_0^{\lambda_{C,n}}\Psi_n(r)^2 dr = \int_{\beta/\lambda^3_{C,n}}^{\infty} t^{-4/3}e^{-t}\,dt\ \ \beta_n^{1/3}/3 = \Gamma(-1/3, \beta_n/\lambda_{C,n}^3)\,\beta_n^{1/3}/3 \approx 36\,\pi^2\,|\Gamma_{-1/3}|\,\beta_n^{1/3}/3 \tag{9}$$ [10]

to be used in the expression for the energy of a photon, hc_0/λ_C:

$$W_{phot,n} = hc_0/\lambda_{C,n} = \frac{hc_0}{\int^{\lambda_{C,n}}\Psi_n(r)^2\,dr} \approx \frac{3hc_0}{36\pi^2|\Gamma_{-1/3}|\beta_n^{1/3}} \tag{10}$$

It should be noted that in both equations (8) and (10) the length $\beta_n^{1/3}$ is the only variable parameter. The dimensionless constants in the equations are π and the Γ-functions. In particular $|\Gamma_{-1/3}|$, as coefficient representing length, and the combination of these constants in form of coupling constants will be of importance in the following.

2.2 Fine-structure constant, α

The energy of a particle is assumed to be the same in both photon and point charge description. Equating (8) with (10) and rearranging to emphasize the relationship of α with the gamma functions ($\Gamma_{1/3} = 2.679$; $|\Gamma_{-1/3}| = 4.062$) gives as first approximation (note: h => ħ):

$$\frac{4\pi\Gamma_{1/3}|\Gamma_{-1/3}|}{0.998} = \frac{9hc_0}{18\pi b_0} = \frac{\hbar c_0}{b_0} = \alpha^{-1} \tag{11}$$

The agreement may be improved by using better approximations of the incomplete Γ-functions involved, see [A2].

In chapter 3.3, [A8], it will be demonstrated that this formalism may be extended to other than three spatial dimensions to give a general expression for electroweak coupling constants [11].

2.3 Quantization with powers of 1/3ⁿ over α

Inserting (7) in the product of the point charge and the photon expression of energy, (8) and (10), gives for the square of energy $W_n^2 = W_{pc,n} W_{phot,n}$:

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8 Euler integrals yield positive values, the absolute sign used for e.g. $|\Gamma_{-1/3}|$ is due to the sign convention of Γ-functions.

9 The product $\Pi_{\beta,n}$ includes all particle coefficients in the partial product for β_n except for the ground state particle (electron), related to the equivalent factor $\Pi_{W,n}$ in the energy expression (17) by $\Pi_{\beta,n} = \Pi_{W,n}^{-3}$ Factor 2 see note 7;

10 Factor ≈ 355 ≈ 36 π² may be calculated numerically from the Euler integral (5) for m = -1, using β_n of (32), (62) or from the integration limit according to chpt. 2.4 as discussed further in [A8].

11 As with all calculations in this work the calculation for coupling constants refers to a rest frame and thus corresponds to an IR limit. The geometric character of the “constants” implies that their values are subject to relativistic effects in other reference frames.

$$W_n^2 = 2b_0 hc_0 \frac{\int_{\lambda_{C,n}}^{r_n} \Psi_n(r)^2 r^{-2} dr}{\int^{\lambda_{C,n}} \Psi_n(r)^2 dr} \sim \frac{1}{\beta_n^{2/3}} \sim \frac{\alpha_1^{1/3}\alpha_2^{1/3}.....\alpha_n^{1/3}}{\alpha_1\alpha_2....\alpha_n} \qquad (12)$$

The last expression of (12) is obtained by expanding the product $\Pi_{\beta,n}^{-2/3}$ included in $\beta_n^{-2/3}$ of (7) with $\Pi_{\beta,n}^{1/3}$. The only non-trivial solution for W_n^2 where all intermediate particle coefficients cancel out and W_n becomes a function of coefficient α_1 only is given by a relation $\alpha_{n+1} = \alpha_n^{1/3}$:

$$W_n^2 \sim \frac{\alpha_1 \wedge (1/3^n)}{\alpha_1} \qquad n = \{1;2;..\} \text{ [12]} \qquad (13)$$

Including the other factors contained in (12) gives the square of (8) (term in square brackets cancels via (11):

$$W_n^2 = 2b_0 hc_0 \frac{\int_{\lambda_{C,n}}^{r_n} \Psi_n(r)^2 r^{-2} dr}{\int^{\lambda_{C,n}} \Psi_n(r)^2 dr} = \frac{4\pi b_0^2}{\alpha} \frac{\int_{\lambda_{C,n}}^{r_n} \Psi_n(r)^2 r^{-2} dr}{\int^{\lambda_{C,n}} \Psi_n(r)^2 dr} = \frac{4 b_0^2 \Gamma_{1/3}^2}{9[\alpha \mathbf{4\pi \Gamma_{1/3}|\Gamma_{-1/3}|}]\beta_n^{2/3}} \qquad (14)$$

According to (7), chpt 2.4, etc., β_n has to include additional Γ-, α-terms suggesting to test such a term as candidate for α_1. Identifying α_1 as $\alpha_1 = \alpha$ and comparing with experimental particle data shows that an expression for particle energies can be given using the muon as reference state, with (13) given as:

$$\left(\frac{\alpha \wedge (1/3^n)}{\alpha}\right)^{0.5} = \frac{\alpha \wedge (0.5/3^n)}{\alpha^{0.5}} = \Pi_{k=1}^{n} \alpha \wedge (-1/3^k) \qquad n = \{1;2;..\} \qquad (15)$$

and the corresponding term for particle energies as:

$$W_n = W_\mu \; \Pi_{k=1}^{n} \alpha \wedge (-1/3^k) \qquad n = \{1;2;..\} \qquad (16)$$

The partial product of (16) may be extended to include the electron by inserting *ad hoc* an additional factor ≈ 3/2 to represent an irregularity due to the energy ratio of e, μ, $W_\mu / W_e = 1.5088\ \alpha^{-1}$ (see 2.4, [A3]). In chpt. 2.8 it will be demonstrated that a fundamental relationship exists between the electron and the Planck energy, implying the electron to correspond to a ground state term. With W_e as ground state W_n would be given by (12)ff as:

$$W_n/W_e \approx \frac{3}{2}\frac{\alpha \wedge (1.5/3^n)}{\alpha^{1.5}} = \frac{3}{2}\Pi_{k=1}^{n} \alpha \wedge (-3/3^k) = \frac{3}{2}\; \Pi_{W,n} \qquad n = \{1;2;..\} \qquad (17)$$

for spherical symmetric states, see table 1. The electron coefficient in the series for β and energy would be given as:

$$\alpha_{\beta,e} \approx (3/2)^3\ \alpha^9 \quad \text{and} \quad \alpha_{W,e} \approx 2/3\ \alpha^{-3} \qquad (18)$$

2.4 Angular momentum, coefficients σ and α

A simple relation with angular momentum J for spherical symmetric states [13] will be given by applying a semi-classical approach using

$$J = r_2 x\, p(r_1) = r_2 W_n(r_1)/c_0 \qquad (19)$$

with $W_{kin,n}$=1/2 W_n , using term $2b_0$ of (8) as constant factor, integrating over a circular path of radius $|r_2| = |r_1|$ and setting r_n of (3), $8/\sigma_0$ according to (24) as integration limits. This will give:

$$|J| = \int_0^{r_n}\int_0^{2\pi} J_n(r)\, d\varphi\, dr = 4\pi \frac{b_0}{c_0} \int_0^{r_n} \Psi_n(r)^2 r^{-1} dr \qquad (20)$$

From (5) follows for m = 0:

$$\int_0^{r_n} \Psi_n(r)^2 r^{-1} dr = 1/3 \int_{8/\sigma_0}^{\infty} t^{-1} e^{-t} dt = \frac{\alpha^{-1}}{8\pi} \approx 5.45 \approx \Gamma_{1/3}|\Gamma_{-1/3}|/2 \text{ [14]} \qquad (21)$$

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12 For illustration purposes with n = 4: $\frac{\alpha^1 \alpha^{1/3} \alpha^{1/9} \boldsymbol{\alpha}^{\mathbf{1/27}}}{\boldsymbol{\alpha}^{\mathbf{3}} \alpha^1 \alpha^{1/3} \alpha^{1/9}} = \frac{\boldsymbol{\alpha}^{\mathbf{1/27}}}{\boldsymbol{\alpha}^{\mathbf{3}}}$

13 For coefficient σ indicated in the following by subscript 0, σ_0.

Inserting (21) in (20) gives:

$$|J| = \quad 4\pi \frac{b_0}{c_0} \frac{\alpha^{-1}}{8\pi} \quad = 1/2\ [\hbar] \tag{22}$$

Analyzing the components of σ_0, in addition to the mandatory term for length, $|\Gamma_{-1/3}|/3$, of the integral (5) for m = -1, r_n and σ_0 contain a factor $\approx 1.51\ \alpha^{-1}$, very close to the ratio $W_\mu/W_e = 206.8 = 1.5088\ \alpha^{-1}$. The exact value of 1.5133 for ≈1.51 has been chosen due to a geometrical interpretation of the terms in σ_0 [15]:

$$1.51\ \alpha^{-1} |\Gamma_{-1/3}|/3 \approx |\Gamma_{-1/3}|/\Gamma_{1/3}\ \ 4\pi |\Gamma_{-1/3}|\ \Gamma_{1/3}/0.998\ \ |\Gamma_{-1/3}|/3 \approx \frac{4\pi |\Gamma_{-1/3}|^3}{3} = (\sigma_0/8)^{1/3} \quad ^{16} \tag{23}$$

The various useful terms for σ_0 may be summed up as:

$$\sigma_0 = 8\ r_n^3/\beta_n = (1.5133\ \alpha^{-1}\ 2/3\ |\Gamma_{-1/3}|)^3 = 1.5133^3\ \sigma^* = 8\left(\frac{4\pi |\Gamma_{-1/3}|^3}{3}\right)^3 = 1.772E{+}8\ [-] \tag{24}$$

2.5 Upper limit of energy

According to the geometrical interpretation given in 2.4 non-spherical particles should exhibit lower values of σ (and r_n). The variable part in σ is given by the term $(1.5133\ \alpha^{-1})^3$ in (24), leaving the minimum for σ, defined by the Γ-term in the integral expression for r and the integers in the square bracket of equ.(2) to be:

$$\sigma_{min} = (2/3\ |\Gamma_{-1/3}|)^3 \tag{25}$$

The maximum angular contribution to W_{max} would be:

$$\Delta W_{max,\ angular} = 1.5133\ \alpha^{-1} \tag{26}$$

The limit of the partial product in (17) for a given l is $\alpha^{-1.5}$, the limit term of ≈ 3/2 by 1.5066 [A3], thus according to (17) and (26), the maximum energy will be $W_{max} = W_e\ 1.5066{*}1.5133\ \alpha^{-2.5} = 4.103E{-}8$ [J] (=1.041 Higgs vacuum expectation value, VEV = 246GeV = 3.941E-8 [J] [6]).

In the simple visualization sketched in the introduction the “rotating E-vector” might be interpreted to cover the whole angular range in the case of spherical symmetric states while an object with one angular node, as represented by the spherical harmonic Y_1^0 or an atomic p-orbital, might be interpreted as forming a double cone. Increasing the number of angular nodes would close the angle of the cone leaving in the angular limit case, l -> ∞, a state of minimal angular extension representing the original vector, however, extending in both directions from the origin and featuring parity p= -1. Considering only „half“ such a state, extending in one direction only and having p = +1, would feature an energy of 1.024 W_{Higgs}, the energy value of the Higgs boson. From (25) follows that such a particle includes a term $|\Gamma_{-1/3}|/3$, i.e. the characteristic coefficient representing length, in the denominator of the energy expression, to be referred to in chpt. 3.2.

2.6 Other non-spherical symmetric states

Except for the limit case of 2.5 angular solutions for particle states are not known yet and to extend the model to such states assumptions have to be made.

Assuming the angular part of Ψ to be related to spherical harmonics and exhibiting the corresponding nodes would give the analog of an atomic p-state for the 1st angular state, Y_1^0. With the additional assumption that $W_{n,l} \sim 1/r_{n,l} \sim 1/V_{n,l}^{1/3}$ (V = volume) is applicable for non-spherically symmetric states as well, this would give $W_1^0/W_0^0 = 3^{1/3} = 1.44$. A second partial product series of energies corresponding to these values (denoted y_1^0) approximately fits the data, see tab. 1.

A change in angular momentum has to be expected for a transition from spherical symmetric states, y_0^0, to y_1^0 which is actually observed with ΔJ = ± 1 except for the pair μ/π with Δ J = 1/2.

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14 The integral in (21) may be calculated numerically. However, obviously, to obtain J=1/2 the integral in $J = 4\pi\,\alpha\,\hbar \int \Psi^2 r^{-1} dr$, (20), *must* yield $\alpha_e^{-1}/8\pi \approx \Gamma_{1/3}|\Gamma_{-1/3}|/2$. Thus relation (21) may be conversely used to define σ_0 and r_n.

15 An additional reason is a 3rd power relationship between 1.5088 and 1.5133, (see [A3,4]), resulting in factor 1.5133 being also part of a minor term depending on the radial quantum number, n. Thus in the following β_n may be split into $\sigma^* = \sigma/1.5133^3 = 5.112E{+}7$ [-] and α(n)-terms containing factor 1.5133^3.

16 The term $4\pi |\Gamma_{-1/3}|^3/3$ is used for σ_0 in all calculations. In chpt. 3.3, [A8] it will be demonstrated that $\alpha_g = \alpha_{weak}$ can be calculated using an equivalent 4D term.

	n, l	$W_{n,Lit}$ [MeV]	α-coefficient (energy) equ (17)	α-coefficient in ß equ (7)	W_{calc}/ W_{Lit}	J	r_n [fm]
Planck	(-1,∞)	1.0 E+21*	$(2/3\ \alpha^{-3})^3\ 3/2\ \alpha^{-1}\ 2$ source term, relative to e !		0.9994 rel. to e !	-	-
e^{+-}	**0, 0**	**0.51**	**$2/3\ \alpha^{-3}$**	**$(3/2)^3\ \alpha^9$**	**1.0001**	**1/2**	**1412**
μ^{+-}	**1, 0**	**105.66**	**$\alpha^{-3}\alpha^{-1}$**	**$\alpha^9\alpha^3$**	**1.0001**	**1/2**	**6.83**
π^{+-}	1, 1	139.57	$\alpha^{-3}\alpha^{-1}\ 3^{1/3}$	$\alpha^9\alpha^3/3$	1.0919	0	4.74
K		495	see [A5]	see [A5]		0	
η^0	**2, 0**	**547.86**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}$**	**$\alpha^9\alpha^3\alpha^1$**	**0.9934**	**0**	**1.32**
ρ^0	2, 1	775.26	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1/3$	1.0124	1	0.92
ω^0	2, 1	782.65	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1/3$	1.0029	1	0.92
K*		894				1	
p^{+-}	**3, 0**	**938.27**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}$**	**1.0017**	**1/2**	**0.76**
n	**3, 0**	**939.57**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}$**	**1.0004**	**1/2**	**0.76**
η'		958	see [A5]	see [A5]		0	
Φ^0		1019	see [A5]	see [A5]		1	
Λ^0	**4, 0**	**1115.68**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}$**	**1.0107**	**1/2**	**0.63**
Σ^0	**5, 0**	**1192.62**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}\alpha^{-1/81}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}\alpha^{1/27}$**	**1.0047**	**1/2**	**0.61**
Δ	**∞, 0**	**1232.00**	**$\alpha^{-9/2}$**	**$\alpha^{27/2}$**	**1.0026**	**3/2**	**0.59**
Ξ		1318				1/2	
Σ^{*0}	3, 1	1383.70	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}/3$	0.9797	3/2	0.53
Ω^-	4, 1	1672.45	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}/3$	0.9725	3/2	0.45
N(1720)	5, 1	1720.00	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}\alpha^{-1/81})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}\alpha^{1/27}/3$	1.0047	3/2	0.43
tau^{+-}	∞, 1	1776.82	$(\alpha^{-9/2})\ 3^{1/3}$	$\alpha^{27/2}/3$	1.0025	1/2	0.40
Higgs	∞,∞ **	1.25 E+5	$(\alpha^{-9/2})\ 3/2\ \alpha^{-1}/2$	$(\alpha^{27/2})/(3/4\ \alpha^{-1})^3$	1.0230	0	0.006
VEV	∞,∞ **	2.46 E+5	$(\alpha^{-9/2})\ 3/2\ \alpha^{-1}$	$(\alpha^{27/2})/(3/2\ \alpha^{-1})^3$	1.04	0	0.003

Table 1: Particle energies for $\mathbf{y_0^0}$ **(bold)**, y_1^0 [17]; col. 2: radial, angular quantum number; col. 3: energy values of [7] except* (see (30)); col. 4: α-coefficient according to the energy terms of (17), *including (2/3) α^{-3} of electron*; col. 5: coefficients in β_n of (7); col. 6: W_{calc} calculated using the slightly more precise [A4 (61)f] in place of (17); ** see 2.5; Blanks in the table are discussed in [A5].

2.7 Expansion of the incomplete gamma function $\Gamma(1/3, \beta_n/r^3)$, strong interaction term

The series expansion of $\Gamma(1/3,\beta_n/r^3)$ in the equation for calculating particle energy (8) gives [8]:

$$\Gamma(1/3,\ \beta_n/(r^3)) \approx \Gamma_{1/3} - 3\left(\frac{\beta_n}{r^3}\right)^{1/3} + \frac{3}{4}\left(\frac{\beta_n}{r^3}\right)^{4/3} = \Gamma_{1/3} - 3\frac{\beta_n^{1/3}}{r} + \frac{3}{4}\frac{\beta_n^{4/3}}{r^4} \tag{27}$$

and for $W_n(r)$:

$$W_n(r) \approx W_n - 2b_0\frac{3\beta_n^{1/3}}{3\beta_n^{1/3} r} + 2b_0\frac{3}{4}\frac{\beta_n^{4/3}}{3\beta_n^{1/3} r^4} = W_n - \frac{2b_0}{r} + b_0\frac{\beta_n}{2r^4} \quad [18] \tag{28}$$

The 2nd term in (28) drops the particle specific factor β_n and gives twice [19] the electrostatic energy of two elementary charges at distance r. The 3rd term is an appropriate choice for the 0th order term of the differential equation [A1] as potential energy term. It is supposed to be responsible for the localized character of a particle state and may be identified with the "strong force" of the standard model.

According to this model it is suggestive to interpret strong interaction as evidenced in scattering events to be due to overlap of wave function Ψ depending on: 1) comparable size and energy of wave functions, 2)

17 up to Σ'^0 all resonance states given in [7] as **** included; Exponents of -9/2, 27/2 for Δ and tau are equal to the limit of the partial products in (7) and (17); r_n calculated with (3); 1.5133 approximated by 3/2;

18 Signs not adapted to conventional definition.

19 Due to adding up the electromagnetic contributions in (8): $W_n = 2W_{n,el} = 2W_{n,mag} = W_{n,el} + W_{n,mag}$

sufficient net overlap. Condition 1) prevents neutrino or electron to exhibit effective interaction with hadrons, condition 2) prevents the tauon which is at the end of the partial product series for y_1^0 and should exhibit a high, potentially infinite number of nodes, separating densely spaced volume elements of alternating wave function sign [20].

Overlap of wave functions might provide a possible description of nuclear bonding as well.

2.8 Gravitation

2.8.1 Planck scale

Expressing energy/mass in essentially electromagnetic terms suggests to test if mass interaction i.e. gravitational attraction can be derived from these terms as well. Assuming the expansion of the incomplete Γ-function for the integral over r^{-2}, $\Gamma(1/3,\beta_n/r^3)$ (27)f, to be an adequate starting point for gravitational attraction as well, implies that the Coulomb term b_0 will be part of the expression for F_G, i.e. the ratio between gravitational and Coulomb force, e.g. for the electron, $F_{G,e}/F_{C,e}$ = 2.41E-43, should be a completely separate, self-contained term.

This is equivalent to assume that gravitational interaction is a higher order effect with respect to electromagnetic interaction and as such should be of less or equal strength compared to the latter. This suggests to use the expression

$$b_0 = G\, m_{Pl}^2 = G\, W_{Pl}^2 / c_0^4 \qquad (29)$$

as definition for Planck terms, giving for the Planck energy, W_{Pl}:

$$W_{Pl} = c_0^2\, (b_0/G)^{0.5} = c_0^2\, (\alpha h c_0/G)^{0.5} = 1.671\text{ E+8 [J]} \qquad (30)$$

With definition (30) one may express a quantitative relationship for the ratio of W_e and W_{Pl} as:

$$1.0006\frac{W_e}{W_{Pl}} = \frac{\alpha_{W,e}^{-3}}{2\ \Delta W_{max,angular}} = 1.5133^2\alpha^{10}/2 = 4.903\text{ E-22} = \alpha_o \qquad (31)$$

i.e. the relation between the electrostatic part of $W_{e,\,elst} = W_e/2$ and the electrostatically defined W_{Pl} is given by $\alpha_{\beta,e} = \alpha_{W,e}^{-3}$, the cube of the electron coefficient for energy (18), i.e. an extension of relation (17) for spherical symmetric states, times the angular limit factor according to (26). In the next chapter a derivation will be given for this relation originating in the third term of the energy expansion (28).

With equ. (31) β_e of the electron can be approximated by a particularly simple expression:

$$\beta_e = \sigma{*}\alpha_0\ \beta_{\,dim} = \frac{\sigma_0{*}\alpha_0}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 = 1.286\text{E-43 [m}^3\text{]} \qquad (32)$$

Using (60) to express factor 1.5133 gives:

$$\left(\frac{W_e}{W_{Pl}}\right)^2 = \left(\frac{F_{G,e}}{F_{C,e}}\right)_{calc} \approx \left(\frac{1.5133^3\alpha^9}{1.5133\,\alpha^{-1}2}\right)^2 = \left(\frac{(4\pi)^2|\Gamma_{-1/3}|^4\alpha^{12}}{2}\right)^2 = 1.001^2\left(\frac{F_{G,e}}{F_{C,e}}\right)_{exp} = \frac{G\,W_e^2}{c_0^4\, b_0} = \alpha_0^2 \qquad (33)$$

Using (11) and (63) for calculating W_e would turn G into a coefficient based on electromagnetic constants:

$$G_{calc} \approx \frac{c_0^4}{4\pi\varepsilon_c}\left(\frac{1}{3\pi^{2/3}}\left(\frac{|\Gamma_{-1/3}|}{\Gamma_{1/3}}\right)^4\alpha^{12}\right)^2 \approx \frac{c_0^4}{4\pi\varepsilon_c}\ \frac{2}{3}\ \alpha^{24} = 1.0008\,G_{exp} \qquad (34)$$

2.8.2 Virtual superposition states

Within this model particles might interact via direct contact in place of boson-mediated interaction. The particles are not expected to exhibit a rigid radius. Within the limits of charge and energy conservation a superposition of many states might be conceivable, extending the particle in space with radius ~ r_n, $\lambda_{C,n}$ etc. appropriate for energy of each virtual particle state (VS) [21], providing a source of energy at a distance r_{VS} from the primary particle and in turn contributing to the stress-energy tensor responsible for curvature of space-time that manifests itself in gravitational attraction.

---

20 As for energy density ~ W_m/W_n^4: e/p ~ E-13, μ/p ~ 6E-4; μ/π ~ 1/3, i.e. in case of μ/π some measurable effect should be expected; different symmetry may play an additional role.

21 The superposition states considered here would not be virtual in a Heisenberg sense, the energy is provided by the primary particle.

Virtual states are not supposed to consist of analogs of e.g. spherical symmetric states covering the complete angular range of 4π but to be an instantaneous, short term extension of the E-vector thus requiring the angular limit factor of (26).

A long range effect of the 3rd, the strong interaction term, of (28) may be exerted via virtual particle states. To estimate such an effect in first approximation the following will be used:

- the 3rd term of the energy expansion equ. (28) with β according to (7), (32),
- the angular limit state of σ^*_{min} according to (24)f, $\sigma^*_{min} \approx 1$,
- $\beta_{dim} = (4\pi)^{-2} (e_c/\varepsilon_c)^3 \approx (\alpha^{-1} r_e)^3$, which might be considered to represent the cube of a natural unit of length, R.

For any VS at $r = \alpha^{-1}\ r_{VS} = \Pi_{\beta,VS}^{1/3} (\alpha^{-1} r_e)$, i.e. the radius of the VS in natural units, R_{VS}, equ. (35) will hold:

$$W_{VS}(r) \approx \frac{b_0 \beta_{VS}/2}{(\alpha^{-1}\ r_{VS})^4} \approx \frac{b_0 \alpha_0 \Pi_{\beta,VS} (\alpha^{-1} r_e)^3}{(\alpha^{-1} r_{VS})^3 (\alpha^{-1} r_{VS})} \approx \frac{b_0 \alpha_0 \Pi_{\beta,VS} (\alpha^{-1} r_e)^3}{(\Pi_{\beta,VS}^{1/3} \alpha^{-1} r_e)^3 (\alpha^{-1} r_{VS})} = \frac{b_0 \alpha_0}{(\alpha^{-1} r_{VS})} = \frac{b_0}{R_{VS}} \left(\frac{F_{G,e}}{F_{C,e}}\right)^{0.5} \quad [22] \tag{35}$$

Considering that the composition of the stress-energy tensor from virtual states is expected to be based on a much more complex mechanism requiring consideration of all possible virtual states at a particular point and appropriate averaging, (35) has to be a first approximation. The crucial factor that turns the r^{-4} dependence of the strong interaction term into r^{-1} of gravitational interaction is the proportionality of β_n to the cube of any characteristic particle length, r_n, $\lambda_{C,n}$ etc. which is valid for each particle state subject to the relations of this model.

Equ. (35) is a representation of the gravitational energy of the electron, terms for other particles may be obtained by inserting their energy values relative to the electron according to (17) in (35) which might be interpreted as the intensity/frequency of the emergence of virtual states being proportional to the energy of the primary particle.

As a consequence of (35) the highest possible particle energy value will be α_0^{-1}, i.e. the value of the Planck energy relative to the electron. This is the fundamental cause for equation (31) to relate W_e and W_{Pl} via an α-term [23] and define the electron as ground state and in turn corroborates the assumption used in the definition of equ. (29)f.

Such a VS-based model implies curvature of space-time to be in general identical to the presence of energy, and spatial coordinate and energy to be intertwined inextricably.

3 Relationship with the Einstein field equation and Kaluza theory

The quantitative relationship of a model for calculating particle energies with the phenomen of gravitational interaction suggests an origin of the equations of this model in the Einstein field equations of the general theory of relativity. The following will give a proof of concept for this assumption using the following steps:

1) demonstrating the relationship of the function Ψ with solutions for the Kaluza scalar, Φ, of a 5D metric,

2) using the solutions for Φ in a metric yielding equation (8) with β of type (32) or (61) as G_{00} element in the EFE,

3) giving additional support for a 5-dimensional ansatz by examining a possible relationship with elements of electroweak theory and the Higgs mechanism.

To get a correct absolute scale of particle energies will require to replace G/c_0^2 [m/kg] or G/c_0^4 [m/J] in the EFE by an appropriate substitution which has to be a term in the order of $\varepsilon_c/(4\pi)$, giving:

$$(8\pi) G/c_0^4 \quad => \quad \approx \ -\frac{2}{\varepsilon_c} \quad [24] \tag{36}$$

and an accordingly modified field equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{2}{\varepsilon_c} T_{\mu\nu} \tag{37}$$

22 The term for gravitational attraction, $F_{m,n;\,R}$ between two particles, m and n at a distance $r_{m,n}$, would be obtained by using $1/b_0$ as proportionality constant: $F_{m,n;R} \approx W_{VS(m,r)} W_{VS(n,r)}/b_0 \approx b_0\ \alpha_0^2\ \Pi_{W,m} \Pi_{W,n} R_{m,n}^{-2}$

23 i.e. in 1st approximation $\alpha_{\beta,e}$ in β has to be identical to $\alpha_{W,Pl}^{-1}$: $\alpha_{\beta,e} \approx \alpha_{W,e}^{-3} \approx \alpha_{W,Pl}^{-1} \approx \alpha_0$

24 Essentially this is equ. (34) with dropping α^{24}.

3.1 Kaluza theory

Kaluza theory is an extension of general relativity to 5D space-time with a metric given as [4], [9]:

$$g_{AB} = \begin{bmatrix} (g_{\alpha\beta} - \kappa^2 \Phi^2 A_\alpha A_\beta) & -\kappa \Phi^2 A_\alpha \\ -\kappa \Phi^2 A_\beta & -\Phi^2 \end{bmatrix} \quad {}^{25} \tag{38}$$

with κ being an electromagnetic coupling constant in the metric [26], A being the electromagnetic potential. In the context of the static approach of this model A will be assumed to be represented by the electric potential $\rho/r \sim e_c/(\varepsilon_c r)$.

For vacuum solutions, $R_{AB}=0$, Kaluza's ansatz yields the following differential equation for the scalar field Φ:

$$\nabla \Phi^\alpha = -\frac{1}{4}\kappa^2 \Phi^3 F^{\mu\nu} F_{\mu\nu} \tag{39}$$

To generalize the function Ψ of this model the following definition for Ψ_N [27] will be used:

$$\Psi_N(r) = \exp\left(-\left(\frac{\rho}{r}\right)^N\right) = e^\nu \qquad \rho \sim \left(\frac{e_c}{\varepsilon_c}\right) \qquad N = \{1, 2, 3, 4\} \tag{40}$$

Ψ_N will be part of an expression for Φ_N

$$\Phi_N \approx \left(\frac{\rho}{r}\right)^{N-1} e^{\nu/2} = \left(\frac{\rho}{r}\right)^{N-1} \exp\left(-\left(\frac{\rho}{r}\right)^N /2\right) \qquad \text{with} \quad \nu = -\left(\frac{\rho}{r}\right)^N \tag{41}$$

Function Φ_N yields solutions for (39), where the term of highest order of Φ'', $\sim \rho^{3N-1}/r^{3N+1}\, e^{\nu/2}$ may be interpreted to provide the terms for $A' \sim \rho/r^2$, see [A7] ($\kappa^2 \sim -c_0^2$):

$$\Phi_n'' \sim \left(\frac{\rho^{3N-1}}{r^{3N+1}}\right) e^{\nu/2} \sim \Phi_n^3 e^{-\nu} (A_0')^2 \approx \left[\left(\frac{\rho}{r}\right)^{N-1} e^{\nu/2}\right]^3 e^{-\nu} \left(\frac{\rho}{r^2}\right)^2 = \left(\frac{\rho}{r}\right)^{3N-3} e^{\nu/2} \left(\frac{\rho}{r^2}\right)^2 \tag{42}$$

Using terms of type Φ_N as ansatz in a 4D or 5D metric will produce various terms of (ρ^m/r^{m+2}) in G_{00} with m being some multiple of N. Volume integrals of these terms will in general produce results of type

$$W \sim \rho^m \int_0^{r_n} \exp\left(-\left(\frac{\rho^N}{r^N}\right)\right) r^{-(m+2)} d^3 r \sim \rho^m \int_0^{r_n} \exp\left(-\left(\frac{\rho^N}{r^N}\right)\right) r^{-m} dr \sim \rho^m \frac{\Gamma(1/N)}{N} \rho^{-(m-1)} = \frac{\Gamma(1/N)}{N}\rho \tag{43}$$

i.e. with using ε_c of (36)f and the more general version of ρ including a function α such as given in (15)ff, $\rho \sim \alpha(n)\, e_c/\varepsilon_c$ *any* term of kind (ρ^m/r^{m+2}) will produce a solution:

$$W_n \sim \varepsilon_c \alpha(n) \frac{e_c}{\varepsilon_c} \sim \alpha(n) e_c \tag{44}$$

thus providing a broad class of possible matrices to obtain an equation of type (8). The next chapter will discuss examples for this approach.

3.2 Example for Metric

The basic ansatz sketched in chpt 3.1 still leaves many possible approaches and requires some reduction of these. Some aspects to be considered for choosing the following examples include:

- a spherical symmetric coordinate system and metric will be used, with opposite sign of time and radial component, yet some additional freedom in angular components [28],
- an exact reproduction of (8) might require to differentiate between ρ in the exponent and the prefactor, resulting in terms $(\rho^*/r)^{N-1}\exp(-(\rho/r)^N)$, $\rho^* \neq \rho$,
- simplicity [29].

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25 Roman letters correspond to 5D, greek letters to 4D.

26 Kaluza's term $\kappa_G^2 = 16\pi G \varepsilon_c / c_0^2$ may be replaced by c_0 (using (36)): $\kappa_c^2 \approx \pm c_0^2$

27 N may be interpreted in the following as number of spatial dimensions.

28 According to the concept of the "rotating E-vector" the rotation of an object with extension in angular direction will result in some kind of self interaction increasing with r ->0 *unless* space(-time) is curved in such a way as to prevent that. This will be the case if the r^2-term in the angular coordinates is canceled, implying positive curvature and an expansion of curved space-time with r^2 at any given r, i.e. $R(r) \sim -1/r^2$, achievable e.g. through change of sign in g_{22}, g_{33}.

29 including dimensionality: a 5D solution should refer to a flat 5D space-time [9], thus a 4D metric might be sufficient as solution.

In [A6] the solution for G_{oo} of two examples of a metric of type

$$g_{\mu\nu} = \left[\left(\frac{\rho *}{r}\right)^2 \exp\left(-a\left(\frac{\rho}{r}\right)^3\right)\right]^{\boldsymbol{p}},\ -\left[\left(\frac{\rho *}{r}\right)^2 \exp\left(-b\left(\frac{\rho}{r}\right)^3\right)\right]^{\boldsymbol{p}},\ -/+\left[\left(\frac{\rho *}{r}\right)^2 \left(-c\left(\frac{\rho}{r}\right)^3\right)\right]^{\boldsymbol{q}} r^2, \quad -/+\left[\left(\frac{\rho *}{r}\right)^2 \left(-c\left(\frac{\rho}{r}\right)^3\right)\right]^{\boldsymbol{q}} r^2 \sin^2\theta \tag{45}$$

will be given in detail for p = 1, q = 0 and p = 2, q = 1 [30].

Coefficients ρ will be defined as $\rho^* = e_c/(4\pi\varepsilon_c)$, $\rho = \alpha(n)\,\rho^*$.

For both cases the Einstein tensor component G_{00} will be (with $v = -(\rho/r)^3$):

$$G_{00} = -/+\ \rho^{*2}/r^4\, e^{av} \tag{46}$$

and using equ. (37) will give (w = energy density):

$$-/+\frac{\rho^{*2}}{r^4} e^{av} \approx -2\frac{w}{\varepsilon_c} \quad => \quad \frac{\varepsilon_c \rho^{*2}}{2r^4} e^{av} \approx +/- w \tag{47}$$

The volume integral over (47)f gives the particle energy according to (a = 1):

$$W_n = \varepsilon_c \rho^{*2}/2 \int_0^{r_n} \frac{e^{av}}{r^4}\, d^3 r = \frac{b_0}{2}\frac{\Gamma_{1/3}}{3}\ \rho^{-1} \tag{48}$$

To recover (8), (32) for the electron, ρ in (48) has to be given by:

$$\rho^3 = \frac{\sigma^* \alpha_0}{8(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \tag{49}$$

i.e. a derivation from the EFE with $1/\varepsilon_c$ in place of $8\pi G/c_0^4$ reproduces the basic equations of chapter 2 with essentially the same set of coefficients [31].

3.3 Electroweak interaction and Higgs mechanism

This model originates from establishing a relation between a photon and rotating objects of SO(3), SU(2) symmetry i.e. it involves the symmetries of electroweak interaction. Symmetry SO(3) is directly related to the property "mass" via the requirement of a center of rotation, i.e. a rest frame, and considerations such as given in note 28 about the requirement of curvature to retain photon-like properties for particle objects.

The concept of chpt. 2.2 for calculating a 1st approximation for the fine-structure constant α may be extended directly to 4 and -with some additional assumptions- to 2 dimensions, based on the integral over the N-dimensional point charge term (S_N being the geometric factor for n-dimensional surface, in case of 3D: 4π):

$$\int_0^r \Psi_N(r)^2 r^{-2(N-1)}\ d^N r = S_N \int_0^r \Psi_N(r)^2 r^{-(N-1)} dr \sim S_N \int_0^r \Phi_N dr \tag{50}$$

multiplied by a complementary integral to yield a dimensionless constant. This results in (see [A8]):

$$\alpha_N^{-1} = \frac{(2\pi)^{\delta(N-2)}}{(2\pi)^{(N-2)}} \int_0^r \Psi_N(r)^2 r^{-(N-1)} dr \int_0^r \Psi_N(r)^2 r^{(N-3)} dr \tag{51}$$

with N = {2; 3; 4} or in terms of the Γ-functions:

$$\alpha_N^{-1} = S_n \frac{\Gamma_+(\Psi_N)\Gamma_-(\Psi_N)}{N^2\ arg(\Gamma(\Psi_N))^2} \tag{52}$$

with $\Gamma_{+/-}(\Psi_N)$ being the positive and negative Γ-functions attributed to the integrals over Ψ_N and $\arg(\Gamma(\Psi_N))$ being the argument of the Γ-functions attributed to Ψ_N [33], i.e. the three coupling constants of the electroweak charges g', e and g can be combined in a single function of spatial dimension only.

30 Terms with p = 2 correspond to Φ being squared in g_{44} of (38).

31 Equating with the photon term still has to hold. The simplest justification for using Ψ in (9) may be to consider it a convenient way of providing length, λ, in terms useful for this model, i.e. as function of $|\Gamma_{-1/3}|$, $\beta^{1/3}$.

33 I.e. in 4, 3 and 2D $\Gamma_{+/-}(\Psi_N)$ will be $\Gamma_{+/-1/2}$, $\Gamma_{+/-1/3}$ and $\Gamma(0, 8/\sigma_{2D}) = 7.872 \approx (2\pi^3)^{0.5}$ (numerical calculation); $\arg(\Gamma(\Psi_N))$ will be 1/2, 1/3, and for 2D *ad hoc* $\arg(\Gamma(0)) = 1$;

Dimension - space	coupling constant	Value of *inverse* of coupling constant, α_N^{-1}	
4D	α(g)	$2\pi^2 \Gamma_{+1/2} \lvert\Gamma_{-1/2}\rvert\ 4/16 = \pi^3 =$	31.006
2D	α(g')	$2\pi\ \Gamma(0, 8/\sigma_{2D})^2 /4 = \pi^4 =$	97.409
3D	α(e)	$4\pi\ \Gamma_{+1/3} \lvert\Gamma_{-1/3}\rvert\ 9/9 = 4\pi\ \Gamma_{+1/3} \lvert\Gamma_{-1/3}\rvert =$	137.036

Table 2: Values of electroweak coupling constants

The ratio of α_e and α_g represents the Weinberg angle, θ_W, and may be expressed as:

$$\sin^2\theta_W = \frac{\alpha_e}{\alpha_g} = \frac{\pi^2}{4\,\Gamma_{1/3}\lvert\Gamma_{-1/3}\rvert} = 0.2263 \tag{53}$$

(Experimental values: PDG [10]: $\sin^2\theta_W = 0.2312$, CODATA [11]: $\sin^2\theta_W = 0.2223$). The mass ratio of the W- and Z-bosons will be given by cos $\theta_{W,calc} = (m_W/m_Z)_{calc} = 0.8796 = 0.998\ (m_W/m_Z)_{exp}$ [12].

Equation (50) suggests a relationship of coupling constants with the Kaluza scalar, Φ. Together with the identification of a particle with the energy of the Higgs boson to represent a 1D object, to be characterized by $\lvert\Gamma_{-1/3}\rvert/3$, see 2.5, and a speculative mapping of all electroweak bosons relative to the expectation value of the Higgs field, $\langle\Phi\rangle_0$ = VEV/√2 = 246GeV/√2, one might tentatively merge coupling constants / their corresponding charges ~ $\alpha_N^{0.5}$, electroweak bosons and solutions for Kaluza's Φ in a 4D spatial scheme, see table 3:

	Point charge		Elements of electroweak / Higgs mechanism					
Dimension - space		Value relative to g	Electroweak bosons + VEV	W [GeV]	W_{exp} relative to VEV/√2	Γ-coefficient relative to VEV	VEV/√2 divided by Γ-coeff.	Kaluza coeff. N of (41) ,(50)
4D	g	1	VEV/√2	174.1				4
1D			Higgs boson	125.4	0.720	$\lvert\Gamma_{-1/3}\rvert/3$	128.6	1
2D	g'	**0.541**	Z^0	91.2	**0.524**	*$(\lvert\Gamma_{-1/3}\rvert/3)^2$*	95.0	2
3D	e	**0.476**	$W^{+/-}$	80.4	**0.462**	*$(\lvert\Gamma_{-1/3}\rvert)^2\ /(3\Gamma_{1/3})$*	84.8	3

Table 3: Comparison of values of coupling constant charges with electroweak energy scale and Φ_N. Values of electroweak charges in col. 3 are calculated from θ_W. Italic terms in col. 7 are a conjecture only [34].

4 Discussion

The authoritative theory to describe particles is the standard model of particle physics. However, the standard model has a major blind spot: it is not particularly efficient in calculating particle mass/energy [35] and gravitational phenomena are explicitly ignored which in turn restricts its applicability for addressing problems in cosmology.

A fundamental theory for mass of elementary particles should be expected to give some information about interaction of masses aka gravitation as well and a natural candidate for such a theory should be the general theory of relativity. The model presented here is a combination of electromagnetism and geometry of space-time as pioneered by T.Kaluza and it may be considered to be only a minor extension thereof, using an electromagnetic constant in place of a gravitational one in the field equations [36]. Such a tiny shift of interpretation of Kaluzas equations has however far reaching consequences. The applicability of the concepts of GTR in the subatomic range implies that effects conventionally attributed to strong force (cf. 2.7) and quantum mechanics are inherent in the formalism of GTR. Features of quantum mechanics that are covered by this model include quantization of energy, wave-character of particles and non-locality (cf. 2.8.2). Last not least the pivotal constant of quantum mechanics, Plancks constant, h, can be derived from the electromagnetic constants e_c, ε_c, and geometry of curved space, as expressed in α.

34 For $W^{+/-}$ $3/\Gamma_{1/3}$ is a first guess based on $\Gamma_{1/3}$ being the characteristic coefficient for energy and W ~1/r holds.

35 Lepton and quark masses are treated as parameters in the SM while the calculation of light hadron masses [13], [14], [15] with lattice QCD methods typically uses 2-3 quark masses, a coupling constant and a reference particle for the absolute energy scale, i.e. about 4-5 parameters, to calculate mass of ~9-12 particles with an accuracy of ~1%.

36 The original approach of Kaluza may suggest either of both constants and though he recognized the problems caused by using G and speculated about a possible replacement he obviously did not have a suitable candidate for this.
"..unter Preisgabe der etwas fragwürdigen Gravitationskonstante eine Versöhnung der widerstreitenden Größenordnungen.."
- ..reconciliation of the conflicting orders of magnitude by abandoning the somewhat questionable constant of gravitation..

The derivation of α from a photon and a point charge expression implies a rotation of electromagnetic fields and SO(3) symmetry for particles. This establishes a link to electroweak phenomena, further backed by the possibility to extend the derivation to 4D-space, giving the weak coupling constant, α_g, and the prominent position of the Higgs boson and vacuum expectation energy at the upper end of the energy series.

The results of this work may provide an opportunity for a quantitative approach to reexamine some phenomena in cosmology. In particular within this model the gravitational constant, G, depends on the speed of light which in turn depends on the gravitational field [16]. The resulting non-linear effect should decrease G and the product Gm with increasing mass, m, a trend that might contribute to effects attributed to dark matter. A variable G might affect other topics of cosmology as well, e.g. Friedmann equations [37].

The possibility to describe mass with a 5D Kaluza model has been studied extensively by P.Wesson and coworkers [9] discussing topics such as a photon-like character of particles in 5D, a possible relation to the Higgs mechanism and cosmological implications in depth. The more heuristic model presented here may complement their work and add some quantitative aspects to such a general model.

Conclusion

This article suggests a consistent and coherent model quantitatively connecting the concepts of general relativity with the properties of subatomic particles giving in particular the following results:

- a geometric expression for the electroweak coupling constants, in particular the fine-structure constant, α, as a coefficient defined by the product of the Γ- functions in the integrals over a function Ψ(r) related to photon and point charge symmetry, $4\pi\, \Gamma_{1/3}\, |\Gamma_{-1/3}| \approx \alpha^{-1}$,
- a quantization of energy levels with terms α^$(-1/3^n)$,
- electron and the Higgs vacuum expectation value energy as lower and upper limit of a convergent series for particle energy,
- a series expansion for particle energy, including terms for rest energy, electromagnetic interaction and a 3rd term which at short range yields effects associated with strong interaction, at long range gives a quantitative term for gravitational interaction.

The basic terms of the model may be derived directly from the framework of the Einstein field equations, using a metric derived from terms for the Kaluza scalar, and can be expressed without use of free parameters.

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37 Somewhat phenomenological: converting the spherical symmetric part of the ratio of electron to Planck energy of (31), $(3/2)^3\, \alpha^9$ = 1.98E-19, into an energy density, using the natural units of chpt. 1, $e_c/(e_c/\varepsilon_c)^3$= 3.84E+9[J/m^3] yields the value for critical density, $\rho_{c,calc}$ = 7.600E-10[J/m^3] = 1.005 $\rho_{c,exp}$ (7.640E-10 [J/m^3], [17]), i.e. ρ_c might represent the ratio of electromagnetism and gravitation, implying some of the "constants" involved to be time-dependent.

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Appendix

[A1] Differential equation

The approximation $\Psi(r<r_n)$ of equation (4) provides a solution to a differential equation of type

$$-\frac{r}{6}\frac{d^2\Psi(r)}{dr^2} + \frac{\beta_n/2}{2r^3}\frac{d\Psi(r)}{dr} - \frac{\beta_n/2}{r^4}\Psi(r) = 0 \tag{54}$$

which corresponds approximately to the limit l -> ∞ (σ -> ≈1) while has to be amended by σ in the denominator of the last term for the general case.
With the 3rd term in (28) used for potential energy, V:

$$V(r) = b_0\, \beta_{GS}/(2\, r^4) \tag{55}$$

and a corresponding expansion by $(\hbar c_0)^2/(\alpha^{-2} b_0^2)$ for the first term, the approximate differential equation for this model may be given as:

$$-\frac{(\hbar c_0)^2 r}{\alpha^{-2} b_0}\frac{d^2\Psi(r)}{dr^2} + r\,V(r)\frac{d\Psi(r)}{dr} - \frac{V(r)}{\sigma}\Psi(r) = 0 \tag{56}$$

Equations (54)ff give a satisfactory description for spherical symmetric states only. They may be considered to be a first approximation of the more complex terms resulting from a metric according to chapter 3.

[A2] Fine-structure constant

In (11) $\Gamma_{1/3}$ represents the limit of $\Gamma(1/3, \beta_n/r_n^3)$ for the lower bound of integration in the Euler integral approaching zero, β/r^3-> 0. Using the analog limit for $\Gamma(-1/3, \beta_n/\lambda_{C,n}^3)$ that may approximated by:

$$\Gamma(-1/N, \beta_n/r_x^3) = \int_{\beta_n/r_x^N}^{\infty} t^{-1-1/N} e^{-t} dt \quad \approx N\,(\beta_n/r_x^3)^{-1/3} \tag{57}$$

with N = 3, gives a more precise expression depending on $\lambda_{C,n}$ and β_n:

$$\frac{\Gamma_{1/3}\lambda_{C,n}}{3\pi\beta_n^{1/3}} = \alpha^{-1} \tag{58}$$

With (58) the precision for the calculation of α is identical to the precision for calculating particle energy with the respective β_n, e.g. with β_e of (62): α_{calc} = 1.0001 α.

[A3] Coefficient 1.51

Factor 1.5088 of the ratio W_μ/W_e is subject to a 3rd power relationship of the same kind as the α coefficients:

$$\left(\frac{1.5133}{1.5088}\right) = \left(\frac{1.5133}{1.5}\right)^{1/3} \tag{59}$$

indicating that the radial terms of $\Pi_{\beta,n}$ in β_n and the angular components of σ are not correctly separated yet or may not be separable even in the case of spherical symmetric states.
The limit of a corresponding partial product in the energy expression is given by 1.5133 Π_0^∞ (1.5/1.533)^1/3^k ≈ 1.5066.
The corresponding term in β will be: 1.5133^{-3} Π_0^n (1.533/1.5)^3/3^k, n={0;1;...}, for particles above the electron, see [A4]. The following relation holds:

$$1.5133 = 0.998\left|\Gamma_{-1/3}\right|/\Gamma_{1/3} = 4\pi\,\Gamma_{-1/3}^2\ \alpha \tag{60}$$

[A4] Particle parameter β

A more detailed expression for β than given in (32) will be attempted in the following.
The term (59) will be used within the particle specific factor (square brackets), thus coefficient 1.5133 of σ will be placed there, giving for the general term (i.e. excluding the electron):

$$\beta_n = \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \frac{2}{(2\pi)^3}\ 1.5133^{-3}\Pi_{k=0}^{n}\left[\alpha^3\left(\frac{1.5133}{1.5}\right)\right]\wedge\left(\frac{3}{3^k}\right) \qquad n = \{0;1;2;...\} \tag{61}$$

factor 1.5133^{-3} represents ≈3/2 for the ratio of W_μ/W_e, to be omitted in the term for the electron:

$$\beta_e = \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \boldsymbol{\frac{2}{(2\pi)^3}\left[\alpha^3\left(\frac{1.5133}{1.5}\right)\right]^3} \approx \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \boldsymbol{\alpha_0} \quad {}^{37} \tag{62}$$

the particle specific factor is given in square brackets (α_0 in bold). The other factors are due to
- factor 2: Ψ appearing squared in the integrals,
- factor $1/(2\pi)^3$: representing 2π of the integral limit in (20) ,
- factor 1.5133^{-3}: due to anomalous factor 2/3 in W_e/W_μ,
- $1/(4\pi)^2$: the power of 2 instead of the power of 3 as for the other components might be due to b_0 appearing squared in the expansion leading to (56) and its analog in the asymmetry of the ρ *, ρ components of chpt 3.2.

Using (62) W_e may be given as:

$$W_e = 2 b_0 \frac{\Gamma_{1/3}}{3}\left(\frac{9\pi^{5/3}\alpha}{|\Gamma_{-1/3}|}\left(\frac{\varepsilon_c}{e_c}\right)\left[\frac{\alpha^{-3}}{1.5133}\right]\right) = \frac{1.5\pi^{2/3}}{1.5133}\frac{\Gamma_{1/3}}{|\Gamma_{-1/3}|}\frac{e_c}{\alpha^2} = 1.0001\ W_{e,exp} \tag{63}$$

[A5] Additional particle states

Assignment of more particle states will not be obvious. The following gives some possible approaches.

[A5.1] Partial products

Additional partial product series will have to start with higher exponents n in α^$(-1/3^n)$ giving smaller differences in energy while density of experimentally detected states is high. There might be a tendency of particles to exhibit a lower mean lifetime (MLT), making experimental detection of particles difficult [38]. To determine the factor y_l^m requires an appropriate ansatz for the differential equation yet to be found.

One more partial product might be inferred from considering d-like-orbital equivalents with a factor of $5^{1/3}$ as energy ratio relative to η giving the start of an additional partial product series at $5^{1/3}$ W(η) = 937MeV = 0.98 W(η'), i.e. close to energy values of the first particles available as starting point, η', Φ^0. However, in general it is not expected that partial products can explain all values of particle energies.

[A5.2] Linear combinations

The first particle family that does not fit to the partial product series scheme are the kaons at ~ 495MeV. They might be considered to be linear combination states of π-states. The π-states of the y_1^0 series are assumed to exhibit one angular node, giving a charge distribution of +|+, -|- and +|-. A linear combination of two π-states would yield the basic symmetry properties of the 4 kaons as:

		+				-				-				+		
K^+	+		+	K^-	-		-	K_S^o	+		+	K_L^o	+		-	(+/- = charge)
		+				-				-				-		

providing two neutral kaons of different structure and parity, implying a decay with different parity and MLT values. For the charged Kaons, K^+, K^-, a configuration for wave function sign equal to the configuration for charge of K_S^o and K_L^o might be possible, giving two versions of P+ and P- parity of otherwise identical particles and corresponding decay modes not violating parity conservation.

[A6] Metric

(for both examples $v = -(\rho/r)^3$)

Example 1

$$g_{\mu\nu} = \left(\frac{\rho*}{r}\right)^2\exp\left(-a\left(\frac{\rho}{r}\right)^3\right),\ -\left(\frac{\rho*}{r}\right)^2\exp\left(-b\left(\frac{\rho}{r}\right)^3\right),\ -/+\ r^2,\ -/+\ r^2\sin^2\vartheta \tag{64}$$

$\Gamma_{01}^0 = \Gamma_{10}^0 = -1/r^1 + 3/2\ a\ \rho^3/r^4$ $\qquad \Gamma_{00}^1 = -1/r^1\ e^{(a-b)v} + 3/2\ a\ \rho^3/r^4 e^{(a-b)v}$

$\Gamma_{11}^1 = -1/r^1 + 3/2\ b\ \rho^3/r^4$

$\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = +1/r^1$ $\qquad \Gamma_{22}^1 = -/+\ r^3/\rho*^2\ e^{(c-b)v} = \Gamma_{33}^1/\sin^2\vartheta$

$\Gamma_{23}^3 = \Gamma_{32}^3 = \cot\vartheta$ $\qquad \Gamma_{33}^2 = -\sin\vartheta\cos\vartheta$

$R_{00} = e^{(a-b)v}\ [(-1/r^2 + 3\ (a-b)\ \rho^3/r^5 + 6\ a\ \rho^3/r^5 - 9/2\ a\ (a-b)\ \rho^6/r^8)\ |\ 2(\Gamma_{01}^0\ \Gamma_{00}^1) - \Gamma_{00}^1\ (\Gamma_{10}^0 + \Gamma_{11}^1 + 2\Gamma_{12}^2)]$
$= e^{(a-b)v}\ [(-1/r^2 + (9a - 3b)\ \rho^3/r^5 - 9/2a(a-b)\ \rho^6/r^8 - \Gamma_{00}^1\ (-\Gamma_{10}^0 + \Gamma_{11}^1 + 2\Gamma_{12}^2)]$
$= e^{(a-b)v}\ [(-1/r^2 + (9a - 3b)\ \rho^3/r^5 - 9/2a(a-b)\ \rho^6/r^8 + (+1/r^1 - 3/2\ a\ \rho^3/r^4)\ (+2/r^1)]$

---

37 Note: $2\ (2/3)^3/(2\pi)^3 \approx (1.5133\ \alpha^{-1}\ 2)^{-1}$, i.e. indicating a relation to the angular limit factor of chpt. 2.5.

38 Which might explain missing particles of higher n in the y_0^0 and y_1^0 series as well.

$= e^{(a-b)v}\ [(-1/r^2 + (9a -3b)\, \rho^3/r^5 - 9/2\, a(a-b)\, \rho^6/r^8 + 2/r^2 - 3a\rho^3/r^5]$

$R_{00} = e^{(a-b)v} [+1/r^2 + (6a -3b)\rho^3/r^5 - 9/2a(a-b)\rho^6/r^8]$

$R_{11} = [+ 1/r^2 - 6a\, \rho^3/r^5 + 1/r^2 - 6b\, \rho^3/r^5 - 2/r^2 - 1/r^2 + 6b\rho^3/r^5 + \Gamma_{10}{}^0\, \Gamma_{01}{}^0 + \Gamma_{11}{}^1\Gamma_{11}{}^1 + 2\Gamma_{12}{}^2\, \Gamma_{21}{}^2 - \Gamma_{11}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 + 2\Gamma_{12}{}^2)]$

$= [-1/r^2 - 6\, a\, \rho^3/r^5 + \Gamma_{10}{}^0\, \Gamma_{01}{}^0 + 2\Gamma_{12}{}^2\, \Gamma_{21}{}^2 - \Gamma_{11}{}^1\, (\Gamma_{10}{}^0 + 2\Gamma_{12}{}^2)]$

$= [-1/r^2 - 6a\, \rho^3/r^5 + 1/r^2 + 9/4\, a^2\, \rho^6/r^8 - 3\, a\, \rho^3/r^5 + 2/r^2 + (+ 1/r^1 - 3/2\, b\, \rho^3/r^4) (+1/r^1 + 3/2\, a\, \rho^3/r^4]$

$= [+2/r^2 - 9a\, \rho^3/r^5 + 9/4\, a^2\, \rho^6/r^8 +1/r^2 + 3/2b\rho^3/r^5 - 3/2b\rho^3/r^5 - 9/4\, ab\, \rho^6/r^8]$

$R_{11} = [+3/r^2 - (15/2a + 3/2b)\rho^3/r^5 + 9/4(+a^2 -ab)\rho^6/r^8]$

$R_{22} = - 1+ e^{(c-b)v} [(+/-3\, r^2/\rho*^2 +/- 3(c-b)\, \rho^3/(r\rho*^2) + 2[\Gamma_{21}{}^2\, \Gamma_{22}{}^1] - \Gamma_{22}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 + \Gamma_{12}{}^2 + \Gamma_{13}{}^3)]$

$= - 1+ e^{(c-b)v}\ [(+/-3\, r^2/\rho*^2 +/- 3(c-b)\, \rho^3/(r\rho*^2) - \Gamma_{22}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 - \Gamma_{12}{}^2 + \Gamma_{13}{}^3)]$

$= - 1+ e^{(c-b)v}\ [(+/-3\, r^2/\rho*^2 +/- 3(c-b)\, \rho^3/(r\rho*^2) +/- r^3/\rho*^2\, (-2/r^1 + 3/2(a + b)\rho^3/r^4)]$

$= - 1+ e^{(c-b)v}\ [(+/-3\, r^2/\rho*^2 +/- 3(c-b)\, \rho^3/(r\rho*^2) -/+ 2r^2/\rho*^2 +/- 3/2(a + b)\rho^3/(r\rho*^2)]$

$R_{22} = - 1+ e^{(c-b)v}\ [(+/- 1r^2/\rho*^2 +/- 3/2(+a -b + 2c)\rho^3/(r\rho*^2)]$

$g^{00}R_{00} = e^{-bv} [+1/\rho*^2 + (6a -3b)\rho^3/(r^3\rho*^2) - 9/2a(a-b)\rho^6/(r^6\rho*^2)]$

$g^{11}R_{11} = - e^{-bv} [+3/\rho*^2 - (15/2a + 3/2b)\rho^3/(r^3\rho*^2) + 9/4(+a^2 -ab)\rho^6/(r^6\rho*^2)]$

$g^{22}R_{22} + g^{33}R_{33} = +/- 2/r^2 -/+ e^{-bv} [(+/- 2/\rho*^2 +/- 3(+a -b + 2c)\rho^3/(r^3\rho*^2)]$

The two solutions for R with different sign of $R_{22,33}$ will be:

$R = +/- 2/r^2 + e^{-bv} [(- 4/\rho*^2 + (+21/2a + 3/2b - 6c)\, \rho^3/(r^3\rho*^2) - 9/4(+3a^2 - 3ab)\rho^6/(r^6\rho*^2)]$

G_{00} will be:

$G_{00} = e^{(a-b)v} [+1/r^2 + (6a -3b)\rho^3/r^5 - 9/8(4a^2 -4ab)\rho^6/r^8] -/+ \rho*^2/r^4\, e^{av} + e^{(a-b)v} [(+ 2/r^2 + (-21/4a - 3/4b + 3c)\, \rho^3/r^5 - 9/8(-3a^2 + 3ab)\rho^6/r^8] = -/+ \rho*^2/r^4\, e^{av} + e^{(a-b)v} [(+ 3/r^2 + (+3/4a - 15/4b + 3c)\, \rho^3/r^5 - 9/8(+a^2 - ab)\rho^6/r^8]$

giving a solution

$G_{00} = -/+ \rho*^2/r^4\, e^{av} + 3/r^2$

for $a = b = c$.

While higher orders of ρ^n-terms in G_{00} are in general easy to eliminate by appropriate choice of the factors in the exponents, a,b..., the lowest order term, i.e. in the metric of example 1: $\sim 1/r^2$, lacks these factors and needs a metric of the type of example 2 to be eliminated.

However, with the integral limits for the particles discussed here volume integrals over the second term will give negligible contributions to particle energy $< 10^{-6}$ and might still be considered a valid solution. It remains unclear if this term is an artefact or may have some actual significance.

Example 2

In the following an example for a metric without an $e^{(a-b)v}/r^2$ term will be given. The application of a $(\rho/r)^2$ term in the angular terms as well will cancel their r^2-dependence, implying the same effect as discussed in note 28.

$$g_{\mu\nu} = \left[\left[\left(\frac{\rho*}{r}\right)^2 \exp\left(-a\left(\frac{\rho}{r}\right)^3\right)\right]\right]^{\mathbf{2}},\ -\left[\left[\left(\frac{\rho*}{r}\right)^2 \exp\left(-b\left(\frac{\rho}{r}\right)^3\right)\right]\right]^{\mathbf{2}},\ -/+\boldsymbol{\rho*^2}\left(-c\left(\frac{\rho}{r}\right)^3\right),\ -/+\boldsymbol{\rho*^2}\left(-c\left(\frac{\rho}{r}\right)^3\right)\sin^2\theta$$

$\Gamma_{01}{}^0 = \Gamma_{10}{}^0 \quad = - 2/r^1 + 3\, a\, \rho^3/r^4 \qquad \Gamma_{00}{}^1 = - 2/r^1\, e^{(a-b)v} + 3\, a\, \rho^3/r^4 e^{(a-b)v}$

$\Gamma_{11}{}^1 \quad = - 2/r^1 + 3\, b\, \rho^3/r^4$

$\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = \Gamma_{13}{}^3 = \Gamma_{31}{}^3 \quad = \quad + 3/2\, c\, \rho^3/r^4 \qquad \Gamma_{22}{}^1 = -/+ 3/2\, c\, \rho^3/\rho*^2\, e^{(c-b)v} \quad = \Gamma_{33}{}^1/\sin^2\vartheta$

$\Gamma_{23}{}^3 = \Gamma_{32}{}^3 = \cot\vartheta \qquad \Gamma_{33}{}^2 = - \sin\vartheta \cos\vartheta$

$R_{00} = e^{(a-b)v} [(-2/r^2 + 6(a-b)\, \rho^3/r^5 + 12a\, \rho^3/r^5 - 9a\, (a-b)\, \rho^6/r^8) + 2(\Gamma_{01}{}^0\, \Gamma_{00}{}^1) - \Gamma_{00}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 + 2\Gamma_{12}{}^2)]$

$= e^{(a-b)v} [(-2/r^2 + 6(3a-b)\, \rho^3/r^5 - 9a\, (a-b)\, \rho^6/r^8) - \Gamma_{00}{}^1\, (-\Gamma_{10}{}^0 + \Gamma_{11}{}^1 + 2\Gamma_{12}{}^2)]$

$= e^{(a-b)v} [(-2/r^2 + 6(3a-b)\, \rho^3/r^5 - 9a\, (a-b)\, \rho^6/r^8) + (+ 2/r^1 - 3a\, \rho^3/r^4) (+ 3(-a + b + c)\rho^3/r^4)]$

$= e^{(a-b)v}\ [(-2/r^2 + 6(3a-b)\, \rho^3/r^5 - 9a\, (a-b)\, \rho^6/r^8) + 6(-a + b + c)\rho^3/r^5 - 9a(-a + b + c)\rho^6/r^8)]$

$R_{00} = e^{(a-b)v} [-2/r^2 + 6(+2a + c)\, \rho^3/r^5 - 9ac\rho^6/r^8)]$

$R_{11} = [+2/r^2 -12a\, \rho^3/r^5 +2/r^2 -12b\, \rho^3/r^5 -12\, c\, \rho^3/r^5 -2/r^2 +12b\rho^3/r^5 +\Gamma_{10}{}^0\, \Gamma_{01}{}^0 +\Gamma_{11}{}^1\Gamma_{11}{}^1 +2\Gamma_{12}{}^2\, \Gamma_{21}{}^2 - \Gamma_{11}{}^1\, (\Gamma_{10}{}^0 +\Gamma_{11}{}^1 + 2\Gamma_{12}{}^2)]$

$= [+2/r^2 - 12a\, \rho^3/r^5 - 12\, c\, \rho^3/r^5 + \Gamma_{10}{}^0\, \Gamma_{01}{}^0 + 2\Gamma_{12}{}^2\, \Gamma_{21}{}^2 - \Gamma_{11}{}^1\, (\Gamma_{10}{}^0 + 2\Gamma_{12}{}^2)]$

$= [+2/r^2 - 12(a + c)\, \rho^3/r^5 + 4/r^2 + 9a^2\, \rho^6/r^8 - 12a\, \rho^3/r^5 + 9/2\, c^2\, \rho^6/r^8 + (+2/r^1 - 3b\, \rho^3/r^4) (-2/r^1 + 3(+a + c)\, \rho^3/r^4)]$

$= [+6/r^2 - 12(2a + c)\, \rho^3/r^5 + 9/2(+2a^2 + c^2)\rho^6/r^8 -4/r^2 + 6b\rho^3/r^5 + 6(a + c)\, \rho^3/r^5 - 9b(a +c)\, \rho^6/r^8]$

$R_{11} = [+2/r^2 - 6(3a - b + c)\, \rho^3/r^5 + 9/2(+2a^2 + c^2 - 2ab - 2bc)\, \rho^6/r^8]$

$R_{22} = - 1+ e^{(c-b)v} [+/- 9/2\, c\, (c-b)\, \rho^6/(\rho*^2\, r^4) +2[\Gamma_{21}{}^2\, \Gamma_{22}{}^1] - \Gamma_{22}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 + \Gamma_{12}{}^2 + \Gamma_{13}{}^3)]$

$= - 1+ e^{(c-b)v}\ [+/- 9/2\, c\, (c-b)\, \rho^6/(\rho*^2\, r^4) - \Gamma_{22}{}^1\, (\Gamma_{10}{}^0 + \Gamma_{11}{}^1 - \Gamma_{12}{}^2 + \Gamma_{13}{}^3)]$

$= - 1+ e^{(c-b)v}\ [+/- 9/2\, c\, (c-b)\, \rho^6/(\rho*^2\, r^4) +/- 3/2\, c\, \rho^3/\rho*^2\, (-4/r^1 + 3(a + b)\rho^3/r^4)]$

$= - 1+ e^{(c-b)v}$ [+/− 9/2 c (c-b) $\rho^6/(\rho^{*2} r^4)$ −/+ 6 c $\rho^3/(\rho^{*2} r^1)$ +/− 9/2c(a + b)$\rho^6/(\rho^{*2} r^4)$)]
$R_{22} = - 1+ e^{(c-b)v}$ [−/+ 6 c$\rho^3/(\rho^{*2} r^1)$ +/− 9/2c(a + c)$\rho^6/(\rho^{*2} r^4)$]

$g^{00}R_{00} = e^{-bv}$ [-2r^2/ρ^{*4} + 6(+2a + c)$\rho^3/(r\rho^{*4})$ – 9ac $\rho^6/(r^4\rho^{*4})$)]
$g^{11}R_{11} = - e^{-bv}$ [+2r^2/ρ^{*4} - 6(3a - b + c)$\rho^3/(r\rho^{*4})$ + 9/2(+2a^2+ c^2 - 2ab - 2bc) $\rho^6/(r^4\rho^{*4})$)]
$g^{22}R_{22} + g^{33}R_{33}$ = +/−2e^{-cv}/ρ^{*2} −/+ e^{-bv} [−/+ 12 c $\rho^3/(r\rho^{*4})$ +/− 9c(a + c)$\rho^6/(r^4\rho^{*4})$)]

R= +/− 2e^{-cv}/ρ^{*2} + e^{-bv} [-4r^2/ρ^{*4} + 6(5a - b + 4c)$\rho^3/(r\rho^{*4})$ - 9/2(+2a^2 + 3c^2 - 2ab + 4ac - 2bc) $\rho^6/(r^4\rho^{*4})$)]

G_{00} will be:
$G_{00} = e^{(a-b)v}$ [-2/r^2 + 6(+2a + c) ρ^3/r^5 - 9acρ^6/r^8)] −/+ $\rho^{*2}/r^4\, e^{(a-c)v}$ + $e^{(a-b)v}$ [+2/r^2 - 3(5a -b + 4c) ρ^3/r^5 + 9/4(+2a^2 + 3c^2 - 2ab + 4ac - 2bc) ρ^6/r^8]
= −/+ $\rho^{*2}/r^4\, e^{(a-c)v}$ + $e^{(a-b)v}$ [3(-a +b -2c) ρ^3/r^5 + 9/4(+2a^2 + 3c^2 -2ab -2bc) ρ^6/r^8]
giving a solution
G_{00} = −/+ $\rho^{*2}/r^4\, e^{av}$
which will be particularly simple if choosing a = b, c = 0

[A7] Scalar potential Φ
The solutions for the scalar Φ depend on the complete metric used. As in [A6] the main problem to obtain $R_{44} = 0$ is to eliminate the terms of lowest order in ρ, which lack coefficients in their terms enabling an easy cancellation of them. As in [A6] solutions can be given by using a metric with squared terms, i.e. p = 2, for either g_{00} or g_{11} e.g.:

$$g_{\mu\nu} = \left[\left(\frac{\rho}{r}\right)^2 \exp\left(-a\left(\frac{\rho}{r}\right)^3\right)\right]^{\mathbf{2}},\ -\left(\frac{\rho}{r}\right)^2 \exp\left(-b\left(\frac{\rho}{r}\right)^3\right),\ -r^2\left(-c\left(\frac{\rho}{r}\right)^3\right),\ -r^2\left(-c\left(\frac{\rho}{r}\right)^3\right)\sin^2\theta, \\ -\left[\left(\frac{\rho}{r}\right)^2 \exp\left(-a\left(\frac{\rho}{r}\right)^3\right)\right]^{\mathbf{2}} \quad (65)$$

Using hyperspherical coordinates in a 5D metric with the line element

$$ds^2 = e^{av}dt^2 - e^{bv}dr^2 - r^2\left(d\psi^2 + \sin^2\psi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right) \quad (66)$$

gives a formal solution as well yet r would be supposed to represent the 5th coordinate.
Note: a general solution yielding (8) from a 5D metric may not require $R_{44} = 0$.

[A8] Coupling constant in 5D
3D case:
Equations (51)f have their origin in the integrals over Ψ_N to be recapped and examined in more depth for the 3D case: omitting the dimensioned constants in (8) and (10), α_e may be expressed directly via the integral over $1/r^2$ representing a point source in 3D times a complementary 2nd integral symmetric in the Γ-function to give a dimensionless term:

$$2\int_0^r \Psi_3(r)^2 r^{-2}\,dr \int_0^r \Psi_3(r)^2 dr = 2\left[\frac{\Gamma_{1/3}}{3}\right]\left[2\pi\, 2\pi\, 9\,\frac{\Gamma_{-1/3}}{3}\right] = 4\pi\,\Gamma_{1/3}\Gamma_{-1/3}\ \mathbf{2\pi} = \mathbf{2\pi}\ \alpha_e^{-1} \quad (67)$$

The term of 2*2π indicates that the volume integral over the square of $1/r^2$ is involved, as actually used in the derivation of (8)ff, $\int \Psi_3(r)^2 r^{-4}\, d^3r = \int \Psi_3(r)^2 r^{-4}\, \mathbf{4\pi} r^2\, dr$. One of the 2π terms originating from the second integral of equation (67) is required for turning h into ħ. Unless (67) is divided by 2π it would give a dimensionless constant $\alpha_e' = h\, c_0\, 4\pi\, \varepsilon/e^2$ and it is a matter of choice to include it in the dimensionless coupling constant [39].
The exact value of (67) depends on the integration limit of the second integral, i.e. the lower integration limit, r_{low}, of the corresponding Euler integral which can be expressed as 3D volume with $|\Gamma_{-1/3}|$ as radius (see 2.5):

$$r_{low} = \beta_n / \lambda_{C,n}^3 = 8/(3^{1.5}\sigma) = \left(3^{0.5}\frac{4\pi}{3}\ |\Gamma_{-1/3}|^3\right)^{-3} \quad (68)$$

in the limit r_{low} -> 0 to be multiplied by 1/arg(Γ(n)) = 3 according to equ. (57). The additional factor $3^{0.5}$ gives the ratio between r_n of equ. (3) and $\lambda_{C,n}$ as required in the expression for photon *energy*.
This limit yields the result of the second integral of (67) as: $\int\Psi_3(r)^2 dr = 3^{0.5} 4\pi|\Gamma_{-1/3}|^3 \approx 36\pi^2|\Gamma_{-1/3}|$.
The general N-dimensional version of (68) will be:

$$r_{low,N} = \left(3^{0.5\delta}\, V_N\ |\Gamma(-\mathrm{N})|^N\right)^{-N} \quad (69)$$

V_N is the coefficent for volume in N-D, coefficient $3^{0.5}$ will be omitted in 4D where coordinate r is considered to be directly related to energy via $r_n \sim 1/W_n$ and r might be directly identified with $\lambda_{C,n}$.

39 The term 2π may be traced back to the more detailed expression for β, equ. (61)f, including the cube of 2π.

4D case:
Using Ψ_4 according to the definition (40) and (69) for 4D:

$$r_{low} = \beta_n / r_{4,n}^4 = 8/\sigma_4 = \left(\frac{\pi^2}{2}\ |\Gamma_{-1/4}|^4\right)^{-4} \tag{70}$$

as integration limit the non-point charge integral in 4D will be given by:

$$\int_0^r \Psi_4(r)^2 r\, dr = \int_{8/\sigma_4}^{\infty} t^{-1.25} e^{-t}\, dt \approx 4(\pi^2/2\ |\Gamma_{-1/4}|^4) \approx 32\pi^4 |\Gamma_{-1/2}| \approx 1/11390 \quad ^{41} \tag{71}$$

The 4D equivalent of (67) will be:

$$\int_0^r \Psi_4(r)^2 r^{-3} dr \int_0^r \Psi_4(r)^2 r\, dr = \left[\frac{\Gamma_{1/2}}{4}\right]\left[2\pi^4 16 \frac{|\Gamma_{-1/2}|}{4}\right] = \frac{\pi^2}{2}\Gamma_{1/2}|\Gamma_{-1/2}|\ \mathbf{4\pi^2} = \pi^3 \mathbf{4\pi^2} = \alpha_g^{-1} \mathbf{4\pi^2} \tag{72}$$

The term $4\pi^2$ is the square of the 2π term in the last expression of (67) since the integrals in (72) refer to $\beta^{0.5}$ and thus to the square of energy and h, ħ.

While the integral $\int\Psi_3(r)^2 dr$ in 3D yields the wavelength of *one* photon, $\int\Psi_4(r)^2\, r\, dr$ may be considered as an integration over 1/W of *all* photons within the integration limits, giving a term $\int\Psi_4(\lambda)^2\, \lambda\, d\lambda \sim 1/W_n^2$.

2D case:
the 2D case is not as straightforward as the 4D case. The integral over the 1D point charge

$$\int_0^r \Psi_2(r)^2 r^{-1} dr = \Gamma(0, \beta_2/r_2^2)\ /\ 2 \tag{73}$$

features Γ(0, x) and with Γ(0, x) -> ∞ for x -> 0 the simple relation between integral limit and integral value according to (57) is not valid. Using nevertheless the 2D equivalent of the integration limit

$$r_{low} = \beta_n / \lambda_{C,n}^2 = 8/(3\sigma_2) = \left(3^{0.5}\pi\ |\Gamma_{-1/2}|^2\right)^{-2} \approx 1/4676 \tag{74}$$

and calculating $\Gamma(0, \beta_2/r_2^2)$ numerically gives $\int\Psi_2(r)^2\, r^{-1}\, dr \approx \Gamma(0, \beta_2/r_2^2)/2 = 7.872/2$. In the 2D case the complementary integral would be identical to the point charge integral, giving $(\int\Psi_2(r)^2\, r^{-1}\, dr)^2 \approx 2\pi^3/4$. This will give the expected value of $\alpha_{g'} \approx \pi^4$ *if multiplied by a factor 2π*. Unlike to the 3D, 4D case 2π will not appear in the denominator of the expression for α, since the 2D integrals yield dimensionless terms and refer to angular momentum rather than energy. Though the reason for the appearance of 2π in the nominator of the integral term is not obvious it is possible to include the 2D case in the unified expressions given by equations (50)f. [42]

[A9] Values used
π = 3.141592654
$\Gamma_{1/3}$ = 2.678938535
$|\Gamma_{-1/3}|$ = 4.062353818
α^{-1} = 137.035999084
c_0 = 2.99792458 [m/s]
e = 1.602176634 E-019 [C]
ε = 8.854187813 E-12 [F/m]
b_0 = 2.307077552 E-28 [Jm]
G = 6.67430 E-11 [$m^5/(Js^4)$]
$W_{e,\,exp}$ = 8.187105777 [J]
$\lambda_{C,e}$ = 2.426310239 E-12 [m]
e_c = 3.109751438 E-18 [J]
β_{dim} = 5.131205555 E-30 [m^3]
$\sigma = 8(4\pi\, |\Gamma_{-1/3}|^{3}/3)^3$ = 177155864 [-]
r_e = 1.413269970 E-12 [m]

41 Factor 2 representing electric and magnetic contributions in the 3D equations will be dropped in the 4D case.
42 Inserting a factor 2π in one of the two integrals $\int\Psi_2(r)^2\, r^{-1}$ dr would turn this integral into the volume integral over the square of $1/r^1$ in analogy to the derivation of the 3D term.