H. A. Lorentz on the equivalence of Huygens' construction and Fermat's principle

Edited by Gavin R. Putland *

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Annotated translation of H. A. Lorentz, *Abhandlungen über Theoretische Physik*, "vol. 1" (1907), chapter 14, secs. 12 & 13 (originally in French), and chapter 16, sec. 18 (originally in German), cited in De Witte, "Equivalence of Huygens' principle and Fermat's principle in ray geometry", *American Journal of Physics*, **27**: 293–301, 387 (1959).

Editor's preface

Adriaan J. de Witte, in footnote 2 of his 1959 paper "Equivalence of Huygens' principle and Fermat's principle in ray geometry" [1], cites two passages from "volume 1" (the only volume that ever appeared) of Lorentz's *Abhandlungen über Theoretische Physik* [5] of 1907, namely §§ 12 & 13 of chapter 14, and § 18 of chapter 16.

Chapter 14 was previously published in 1887 as "*De l'influence du mouvement de la terre sur les phénomènes lumineux*" [4]. This was itself a slightly edited French translation of the Dutch original [3], "*Over den invloed, dien de beweging der aarde op de lichtverschijnselen uitoefent*" ("On the influence of the earth's movement on the phenomena of light"), submitted and published in 1886. Chapter 16, with the German title "*Die Fortpflanzung von Wellen und Strahlen in einem beliebigen nicht absorbierenden Medium*" ("The propagation of waves and rays in an arbitrary nonabsorbent medium"), was dated 1906 and apparently not previously published.

De Witte acknowledges Lorentz's priority but adds: "The present argument, although in essence the same, is believed to be more cogent and more general." Indeed, the older part of Lorentz's treatment is totally immersed in aether theory, treats all media as homogeneous and (tacitly) isotropic, and allows for anisotropic effects solely through dragging of waves by the aether (which moves in the lab frame), and for inhomogeneous effects solely by admitting that the aether flow might be non-uniform. In this unnatural framework, the ray path as defined by Huygens' construction is shown to be the path of least time. The newer part, although purged of the aether, devotes much space to rectilinear propagation in homogeneous (not necessarily isotropic) media, and then only briefly generalizes the rectilinear path to the path of least time, referring to the earlier chapter for the connection with Huygens' construction. Fermat's principle is not named, and the path of interest is that of *least* time, not merely *stationary* time. De Witte, in contrast, gives a thoroughly modern treatment for general media, showing that Huygens' construction (which he calls Huygens' "principle") and Fermat's principle lead to the same differential equation of the ray path, and that in the latter case the converse is also true [1, p. 298].

De Witte undersells himself in that his treatment uses calculus of variations, whereas Lorentz's is geometric. But, for that very reason, Lorentz's approach is more accessible—or would be, if one could see through the discussion of moving aether. Even the discredited aether draws attention to a fact that De Witte passes over, namely that the equivalence of Huygens' construction and Fermat's principle holds in the presence of moving media; the equivalence depends on the *meaning* of the construction, and not on the reasons why the secondary waves propagate as they do. (As a bonus, in § 13 of ch. 14, Lorentz added his own proof that under Fresnel's aether-drag hypothesis, the ordinary laws of refraction and reflection are, to first order, insensitive to the aether wind.) Moreover, De Witte presents his proof for *two* dimensions and baldly asserts that it can be "extended without too much difficulty" to three.

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Hence it is appropriate to offer an English translation of Lorentz's original argument—the more so because of its historical interest, and because, as De Witte laments in the same footnote, "The matter seems to have escaped treatment in textbooks."

In the present translation, for consistency with chapter 16, page numbers and figure numbers for chapter 14 match the 1907 edition [5]. Otherwise the text of chapter 14 was initially taken from the 1887 French edition [4], which had by far the best OCR scan of the three sources, and then compared with the Dutch edition as seemed necessary. Footnotes are mine unless otherwise attributed. Editorial remarks are sometimes footnoted, and sometimes inlined in square brackets [thus].

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Chapter 14 (p. 341): On the influence of the earth's movement on the phenomena of light

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Section 12 (p. 357)

Huygens' principle allows us, in the same manner as in §4,[1] to investigate how the vibrations propagate from a plane wave or in any other case, in a space for which we admit the hypotheses of §8 [essentially Fresnel's aether-drag hypothesis, expressed in a reference frame fixed with respect to the lab, and without deciding whether opaque bodies obstruct the flow of aether]. We will suppose this space occupied by a homogeneous ponderable material, so that the entrainment coefficient κ [i.e., $1-k$, where k is Fresnel's drag coefficient] will have everywhere the same value.[2] For propagation in celestial space, or in the air if we want to neglect atmospheric refraction, it will suffice to set $\kappa = 1$.

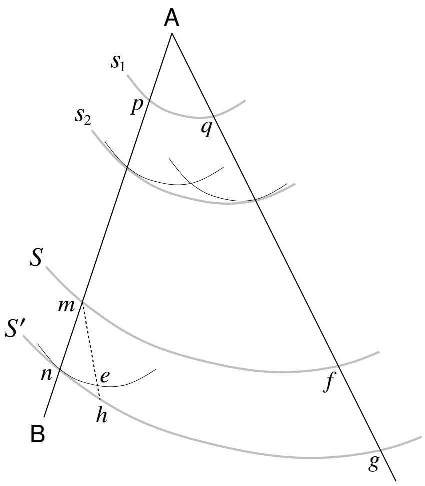

Fig. 25.

First consider a movement of light expanding from a center A (Fig. 25), either because some ponderable matter at A emits the light itself, or because at this point the aether and, if there is any, the ponderable matter receive vibrations from elsewhere. After an infinitesimal time, a disturbance from A

[1] §4 applies Huygens' *construction* to stellar aberration under Stokes's version of the aether-drag hypothesis [6].

[2] The entrainment coefficient is given by $\kappa = 1/n^2$, where n is the refractive index. Fresnel's drag coefficient k refers to the dragging of waves in aether by ponderable matter, in a reference frame fixed w.r.t. the aether, whereas the entrainment coefficient κ refers to the dragging of waves in ponderable matter by the aether, in a frame fixed w.r.t. that matter [5, pp. 355–6]. In all three sources (refs. [3]–[5]), it is hard to tell from the local font whether the symbol for the entrainment coefficient is κ or x. But the context suggests κ, and comparison between the cited passages in [5] confirms that x is a different character.

will have spread to the surface of an elementary wave s_1, like the one discussed in §10.[3] From points p, q, etc. of this wave, new elementary waves proceed during the next time element, and their envelope s_2 is the new position of the wave.[4] Continuing in this manner, we find all the successive positions of a wave which expands around A; let S and S' be any two of these positions, located at an infinitesimal distance from each other.

Each point B, wherever located, is eventually reached by the movement of light emanating from A, and we may ask how much time it takes.

[In Fig. 25, A and B (in sans-serif type) are points. In the equations, A and B (in *italic* type) are *speeds*.[5] In the frame of Fig. 26 (the lab frame), if $P'Q$ and PQ are the distances shown, then $A = P'Q/dt$ is the propagation speed in the given medium in the absence of motion of the aether, $B = PQ/dt$ is the propagation speed in the medium as affected by that motion, ρ is the speed of the aether in a vacuum, κ is the entrainment coefficient of the medium, and $\kappa\rho$ is the speed of the center of the elementary wavefront in the medium. In the time dt, the center travels the distance $PP' = \kappa\rho\, dt$.]

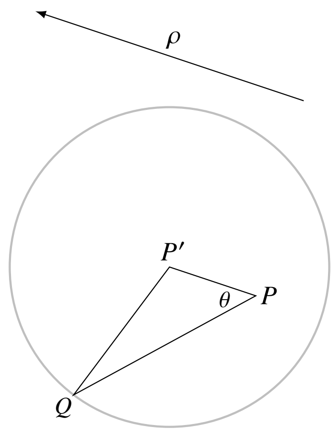

Fig. 26.

To determine this time, in Fig. 26, we again consider the elementary wave which forms around P in the time dt, as already proposed in Fig. 24.[6] At any point Q of this wave, the disturbance arrives in the same time as if it had propagated along the straight line PQ with the speed

$$B = \frac{PQ}{dt}\,.$$

Denoting by θ the angle that PQ makes with the direction of the speed ρ, i.e. with PP', we have [by the cosine rule]

$$P'Q^{\,2} \;=\; PQ^2 \,-\, 2\cdot PQ\cdot PP'\cdot\cos\theta \,+\, PP'^{\,2}$$

or, after division by dt^2,

$$A^2 = B^2 - 2B\kappa\rho\cos\theta + \kappa^2\rho^2.$$

[This can be written

$$A^2 = (B - \kappa\rho\cos\theta)^2 + \kappa^2\rho^2\sin^2\theta\,,$$

whence

[3] One whose center is moving in the lab frame, in accordance with the entrainment coefficient.

[4] In the redrawing of Fig. 25, all the elementary wavefronts are shown as having the same shape, for consistency with the homogeneity of the medium.

[5] The Dutch and French editions use italic type for both!

[6] Fig. 24 (not reproduced here) differs from Fig. 26 by lacking the point Q and the adjacent sides of the triangle. The label θ in Fig. 26 is an editorial addition.

$$\frac{B-\kappa\rho\cos\theta}{A} = \left(1-\frac{\kappa^2\rho^2}{A^2}\sin^2\theta\right)^{1/2}$$
$$\approx 1-\frac{\kappa^2\rho^2}{2A^2}\sin^2\theta \qquad \text{for small } \frac{\rho}{A}\,.$$

—*Editor*.]

Hence, if we limit ourselves to the first power of ρ,

$$B = A + \kappa\rho\cos\theta\,; \tag{5}$$

or if we want to take the second power into account,

$$B = A + \kappa\rho\cos\theta - \frac{\kappa^2\rho^2}{2A}\sin^2\theta\,. \tag{6}$$

In the first paragraphs that follow, only expression (5) will be used. The speed given by this depends on the direction of the element PQ, and moreover is different for elements in the same direction traced from different points in space, if at these points ρ does not have the same direction and the same magnitude.

Returning to Fig. 25, we should note that the points of two successive waves S and S' are linked pairwise so that one of these points, m, located on S, can be regarded as the center of vibration of the elementary wave which at the second point n is tangent to S'. These pairs, such as m and n, or f and g, we shall call *conjugate* points. For all the straight lines that join the conjugate points of S and S', the time needed for the light to traverse them with the speed B is the same,[7] and this time is that in which the wave is transported from S to S'. For any straight line mh, drawn between S and S', which on the contrary does not join two conjugate points, the time that the light would take to traverse it with speed B will be longer than that just mentioned. Indeed mh will cut the surface of the elementary wave emanating from m at some point e, inside S', and the time in question for me will already be the same as for mn.

Now suppose that a large number of lines connect [points] A and B [Fig. 25]. Among these will be one that will cut in conjugate points all the waves located between A and B, and the course of this line will require less time than the course of any other which does not entirely pass through conjugate points of successive waves. This line, traversed in a minimum time, I will call a *ray of light*; the time required for the passage of such a ray is that in which the waves expand from A to B.

The shape of the ray of light is easily deduced from what has just been said. Let ds be an element of one of the lines from A to B, and θ the angle that this element makes with the velocity ρ of the aether in its vicinity; the time needed to traverse this element will be

$$\frac{ds}{B} = \frac{ds}{A+\kappa\rho\cos\theta} = \frac{ds}{A} - \frac{\kappa\rho\cos\theta\, ds}{A^2}$$

[where the second equality is again a first-order approximation for small ρ], and the time required to traverse the entire path, whose length we call ℓ, will be

$$\int\frac{ds}{B} = \frac{\ell}{A} - \frac{\kappa}{A^2}\int\rho\cos\theta\, ds\,.$$

The factor $\rho\cos\theta$ is the speed of the aether in the direction of ds and therefore, since there exists an [aether] velocity potential φ,[8] can be represented by $\partial\varphi/\partial s$. It follows that the integral on the right has the value $\varphi_{\mathsf{B}}-\varphi_{\mathsf{A}}$, where the subscripts A and B distinguish the values of the velocity potential at points A and B.

In the expression thus obtained,

$$\frac{\ell}{A} - \frac{\kappa}{A^2}(\varphi_{\mathsf{B}}-\varphi_{\mathsf{A}})\,,$$

[7] The earlier Dutch edition notes, in parentheses, that B may be different for different lines [3, p.322].

[8] The velocity potential is discussed in previous sections; but only in the Dutch edition is this clause found here [3, p.323].

the last term is the same for all the lines from A to B. For the ray, the first term, and therefore ℓ, must accordingly become a minimum; the ray is therefore a straight line.

Stokes has already reached this result by another method for the case $\kappa = 1$. We get there again if, instead of assuming a single center A, we start with a wave S_1 of any shape. If S_2 is one of the later positions of this wave, and AB a line by which S_1, S_2, and all the intermediate positions are cut in conjugate points, then it is found again, by the same reasoning as above,[9] that this line will be traversed by the light[10] in a minimum time, implying again that it is straight.

Section 13 (p. 359)

Considerations similar to those of the previous section can be used to determine the change in direction that a light ray suffers when it passes from one medium to another. Let the surface V be the interface, of any shape, between two homogeneous ponderable materials, so that the propagation speed A and the entrainment coefficient κ have the uniform values A_1 and κ_1 in the first medium, and similarly the uniform values A_2 and κ_2 in the second medium. This general case includes that in which there is free aether [a vacuum] on one side of V.

Suppose that from any wave, of which the part that we have to consider is still entirely in the first medium, the luminous movement propagates towards the interface. Huygens' principle will again allow us to follow the progress of waves in infinitesimal steps, even after they have already partly penetrated into the second medium. In the latter case the wave is composed of two parts, which meet the interface along the same line, but which at each point of this line make between them a certain angle, and which will generally be of different shapes. These two parts will however be designated, in what follows, as a single wave. This wave, in the extent of it that we consider, can be cut by the interface along a single line that ends at the edges of the wave, or else along a curve that closes on itself, or finally according to two or more lines of either nature. The first case arises, for example, when a plane and limited wave falls obliquely on a plane surface; the second case, when such a surface is met by a spherical wave; finally, a cylindrical surface can be cut by a plane wave along two straight lines.

In any case, given an [initial] position S of the kinked wave, in order to deduce the position S' that it occupies after the time dt, we must construct two or, strictly speaking, three species of elementary waves. First, around the points of S which are already in the second medium, [there are] elementary waves similar to those discussed in § 10,[11] and for which we will use the values A_2 and κ_2 specific to the second medium. The envelope surface of these waves provides almost the entire the part of S' which is located in the second medium; only a narrow border, in the immediate vicinity of the boundary surface, is missing. Second, we have to construct elementary waves analogous to the previous ones, but with the values that A and κ have in the first medium, around all the points of S in this medium which are sufficiently far from the interface that the corresponding elementary waves fall short of it. The envelope of these waves is, up to a very small distance from V, the part of S' which is in the first medium.

There still remain the points of S which are so close to the interface that the [secondary] disturbances emanating from them cross it before the end of the time dt. Around these points we could construct a third group of elementary waves, but we do not need them to get to know the surface S'. Indeed, consideration of the elementary waves which are located entirely in the first or in the second medium leaves undetermined only an infinitesimally narrow band of S', near the interface, and we can fill this gap by extending each of the already known parts of S' by infinitesimal planes which connect to the direction of the surface already obtained.

Moreover, even when an elementary wave falls partly in the second medium, the part of this wave which is still located in the first medium will nevertheless have the form indicated in § 10. Therefore, in

[9] This clause is found only in the Dutch edition [3, p. 324].

[10] Instead of "by the light", the earlier Dutch edition says "with the speed B". But B may vary with direction. Perhaps, on second thought, this was considered a distraction because the problem of minimizing the time is reduced "by the same reasoning as above" to the trivial problem of minimizing ℓ.

[11] That is, elementary waves dragged by the aether, which is moving w.r.t. the lab frame.

the previous construction we can still use such waves on the only condition that their point of contact with the envelope surface falls within the first medium or on the boundary surface itself. Operating in this manner, one obtains in its entirety the part of S' located in the first medium.

The points of two successive waves are again in conjugate pairs, and by limiting ourselves to those pairs which are either both in the first medium or both in the second, we can say that all the straight lines which join two conjugate points, whether located in the first or in the second medium, are traversed in the same time with the speed B indicated in §12.

But if a straight line is drawn between two non-conjugate points of S and S', so that it is still entirely contained in the same medium, the traversal of this line[12] will take more time than the traversal of a straight line joining two conjugate points.

Let us imagine a line which, starting from a point A in the first medium, and even after its passage into the second medium, constantly joins conjugate points. Let B be the point where this "light ray" meets the interface, and C one of the points that it reaches beyond the interface. If we then draw between A and C some other line, which cuts the interface at (say) B′, where it may undergo a change of direction like the line ABC at point B, then the traversal of ABC will require less time than that of AB′C. To perceive this, we need only interpose between A and C an infinitude of waves,[13] of which one passes through B and one through B′, and notice that the elements of AB′C do not all join conjugate points of successive waves.

The light ray is therefore, of all the paths going from A to C, the one which[14] is traversed in the shortest time. It follows, according to the result of the preceding paragraph, that this ray must be composed of two straight lines, and B will be the position of the variable point B′ that minimizes the time needed to traverse the broken[15] line AB′C.

According to the formulas of the preceding section, the time needed to traverse AB′ is

$$\frac{\mathsf{AB'}}{A_1} - \frac{\kappa_1}{{A_1}^2}(\varphi_{\mathsf{B'}} - \varphi_{\mathsf{A}}), \tag{7}$$

and the time to traverse B′C is

$$\frac{\mathsf{B'C}}{A_2} - \frac{\kappa_2}{{A_2}^2}(\varphi_{\mathsf{C}} - \varphi_{\mathsf{B'}}); \tag{8}$$

and $\varphi_{\mathsf{B'}}$ has the same value in both expressions since, according to our hypothesis, the velocity potential is a continuous function.

The sum of (7) and (8) can be represented very simply, because of the value that we have accepted in §10 for the entrainment coefficient [namely $\kappa = 1/n^2$]. Indeed, denoting by n_1 and n_2 the absolute refractive indices of the two media, we have

$$\kappa_1 : \kappa_2 = {n_2}^2 : {n_1}^2,$$

and moreover we know that

$$A_1 : A_2 = n_2 : n_1 .$$

It follows [from these proportions] that

$$\frac{\kappa_1}{{A_1}^2} = \frac{\kappa_2}{{A_2}^2} .$$

I would add that the fraction $\frac{\kappa}{A^2}$ has the same value for *all* isotropic media.[16] If this is denoted by μ, the sum of (7) and (8) will be:

$$\frac{\mathsf{AB'}}{A_1} + \frac{\mathsf{B'C}}{A_2} - \mu(\varphi_{\mathsf{C}} - \varphi_{\mathsf{A}}) .$$

[12] The Dutch edition adds "with the speed B" [3, p. 326].

[13] This evidently means an infinite succession of positions of the same wave.

[14] The Dutch edition adds "with the speed B" [3, p. 327], although the same paragraph mentions a point B.

[15] French *brisée* = broken; Dutch *gebroken* = broken / refracted.

[16] This value is $1/c^2$, where c is the speed of light in vacuo.

Since the last term of this expression is independent of the location of B′, it is simply necessary that

$$\frac{\mathsf{AB'}}{A_1}+\frac{\mathsf{B'C}}{A_2} \tag{9}$$

is minimized when B′ occupies position B. But it follows from this that the lines AB and BC are located, with the normal to the interface at B, in the same plane, and that the sines of the angles which they make with this normal are between them in the ratio of A_1 to A_2 [the "Snell" or "Snell-Descartes" laws, in terms of phase velocity]. I can dispense with demonstrating this consequence here. Let us merely note that from expression (9), everything related to the movement of the aether with respect to the ponderable material has disappeared. Even when everything is at rest, the way by which the ray passes from one medium to another is determined by the condition that (9) is a minimum; however, it is known that in this case the laws of Snell apply.[17]

That these laws still subsist for the related rays when the aether is in motion, with respect to the ponderable matter, has been demonstrated in a general way, first by Stokes in his memoir on Fresnel's theory of aberration,[18] then by Veltmann. These scholars, however, took the Fresnel hypothesis as the starting point of their demonstration and their method is different from mine.

It is important to note that the result depends entirely on the value assigned to the entrainment coefficient. Indeed the given demonstration is in default as soon as $\varphi_{\mathsf{B'}}$ does not cancel in the sum of expressions (7) and (8). Now this [cancellation] takes place only if $\kappa_1/{A_1}^2 = \kappa_2/{A_2}^2$ — that is, only if for different media κ is inversely proportional to n^2, so that κ, which in free aether must to be equal to 1, has in any other medium the value $1/n^2$.

The reflection of light can be treated in the same way as refraction. There is the difference that the reflected waves intersect with the incident waves, but this circumstance does not change the reasoning. It will be easily perceived that the ordinary laws of reflection continue to apply to the related rays, and that to reach this conclusion one does not need any hypothesis on the entrainment coefficient. It will suffice to admit that in the same medium this coefficient always has the same value.

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Chapter 16 (p. 415): The propagation of waves and rays in an arbitrary nonabsorbent medium (1906)

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[**Editor's note:** The following theory, covered in the earlier sections of chapter 16, uses Huygens' construction and is needed for the first part of § 18, which deals with rectilinear propagation. In a general medium, which may be anisotropic, the *wave-slowness surface* (also called the *normal-slowness surface*) is the surface whose "distance" from the origin in any direction is the wave slowness (i.e., the reciprocal of the phase velocity) in that direction. If the Cartesian "components of normal slowness", as Hamilton called them [2], are denoted by α, β, γ, then the equation of the wave-slowness surface can be expressed in the form given by Lorentz (p. 417):

$$\varphi(\alpha,\beta,\gamma)=0\,, \tag{6}$$

where φ is some function (*not* to be confused with the aether velocity potential in chapter 14). In a homogeneous medium, if a plane wavefront passes through the origin at time 0, the time t at which it reaches a point at position (x, y, z) is the dot product of the wave slowness and the position vector; i.e.,

$$\alpha x+\beta y+\gamma z=t\,.$$

[17] The last clause is taken from the Dutch text [3, p.329]; the French refers to "the laws that I have just stated."

[18] Reference [7].

This is the equation of the wavefront at time t. Hence its equation at time 0 is

$$\alpha x + \beta y + \gamma z = 0\,, \tag{19}$$

and its equation after *unit time* is

$$\alpha x + \beta y + \gamma z = 1\,; \tag{20}$$

again the equation numbers are Lorentz's (p. 423). The wavefront that expands from a source at the origin in unit time is the *ray-velocity surface*—that is, the surface whose "distance" from the origin in any direction is the ray velocity in that direction. Lorentz conventionally called it the *wave surface* (*Wellenfläsche*). Hamilton evocatively called it the *unit wave*. By Huygens' construction, this surface is tangential to (i.e., it is the envelope of) all the plane wavefronts given by (20), corresponding to all the combinations of α, β, γ that satisfy (6). According to a theorem cited by Lorentz (p. 424), the direction from the origin to a general point of this surface—whose coordinates x, y, z are now the *components of the ray velocity*—is the direction of

$$\left(\frac{\partial\varphi}{\partial\alpha}, \frac{\partial\varphi}{\partial\beta}, \frac{\partial\varphi}{\partial\gamma}\right).$$

So (**§16**, p. 430) the components of the ray velocity can be written

$$x = \frac{1}{N}\frac{\partial\varphi}{\partial\alpha}\,, \quad y = \frac{1}{N}\frac{\partial\varphi}{\partial\beta}\,, \quad z = \frac{1}{N}\frac{\partial\varphi}{\partial\gamma}\,,$$

where N is a quantity to be determined. And it is determined by substituting for x, y, z in (20):

$$N = \alpha\frac{\partial\varphi}{\partial\alpha} + \beta\frac{\partial\varphi}{\partial\beta} + \gamma\frac{\partial\varphi}{\partial\gamma}\,. \tag{31}$$

In what follows, x, y, z revert to being ordinary (position) coordinates.]

Section 18 (p. 431)

We still have to show, what has already been said, that in a homogeneous medium the rays are always straight lines, whatever may be curvature of the wavefronts. For this purpose I note that for every given direction of the wave normal, we have a certain direction and a certain velocity u of the ray, and that we can (see §16) go from one wavefront σ to the neighboring one σ' by drawing, from every point P on σ, a line in the ray direction corresponding to the wave normal at P, and cutting from it an infinitesimal length $PP' = u\,dt$. The geometric location of the points P' is then the new wavefront.

If x, y, z are the coordinates of P, then according to the formulas given in §16 [for the components of the ray velocity], the coordinates of P' are determined by the equations

$$x' = x + \frac{1}{N}\frac{\partial\varphi}{\partial\alpha}\,dt\,, \quad y' = y + \frac{1}{N}\frac{\partial\varphi}{\partial\beta}\,dt\,, \quad z' = z + \frac{1}{N}\frac{\partial\varphi}{\partial\gamma}\,dt\,.$$

Now let Q be an arbitrary point of the surface σ that is infinitesimally close to P, and Q' the point where σ' is cut by the ray passing through Q. If the prefixed operator d denotes the changes that variables [other than t] undergo in the transition from P, P' to Q, Q', then, according to the preceding equations,

$$dx' = dx + d\!\left(\frac{1}{N}\frac{\partial\varphi}{\partial\alpha}\right)dt\,, \quad dy' = dy + d\!\left(\frac{1}{N}\frac{\partial\varphi}{\partial\beta}\right)dt\,, \quad dz' = dz + d\!\left(\frac{1}{N}\frac{\partial\varphi}{\partial\gamma}\right)dt\,.$$

To prove that the following element $P'P''$ of the ray has the same direction as PP' and that the ray is therefore a straight line, it suffices to show that the surface σ' at P' is parallel to the surface σ at P. To assure ourselves of this, we need only verify that for all possible displacements PQ of the surface σ, i.e. for all values of dx, dy, dz which satisfy the condition

$$\alpha\,dx + \beta\,dy + \gamma\,dz = 0\,,$$

we also have

$$\alpha\, dx' + \beta\, dy' + \gamma\, dz' = 0\,.$$

But this follows from the fact that the expression

$$\alpha\, d\left(\frac{1}{N}\,\frac{\partial\varphi}{\partial\alpha}\right) + \beta\, d\left(\frac{1}{N}\,\frac{\partial\varphi}{\partial\beta}\right) + \gamma\, d\left(\frac{1}{N}\,\frac{\partial\varphi}{\partial\gamma}\right),$$

which can also be written

$$d\left\{\frac{1}{N}\left[\alpha\,\frac{\partial\varphi}{\partial\alpha} + \beta\,\frac{\partial\varphi}{\partial\beta} + \gamma\,\frac{\partial\varphi}{\partial\gamma}\right]\right\} - \frac{1}{N}\left(\frac{\partial\varphi}{\partial\alpha}\,d\alpha + \frac{\partial\varphi}{\partial\beta}\,d\beta + \frac{\partial\varphi}{\partial\gamma}\,d\gamma\right),$$

is always zero; indeed, the first term of the latter expression vanishes because of equation (31), and the second because of equation (6).

An even simpler proof is given by a theorem, which follows directly from Huygens' construction,[19] and according to which, of all lines s connecting any two points A and B, the ray is the one for which the integral

$$\int \frac{ds}{u}$$

is minimized. From this theorem, in which u is to be understood as the speed of a ray coinciding with ds, it immediately follows that, as far as the course of a ray going from A to B is concerned, it does not matter whether A is a point of an already existing wavefront, or is itself a center of vibration. In this latter case, however, Huygens' construction leads to closed wavefronts that are similar to each other and have A as the center of similarity. The successive points of a ray have similar positions on these surfaces and therefore lie on a straight line emanating from A.

It is notable that in a homogeneous medium we can now directly deduce, for any given wavefront σ, the position σ' which it takes after a *finite* [non-infinitesimal] time τ. One need only draw a ray from each point of σ and cut a length $u\tau$ on it.

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Editor's acknowledgments

Deutsch habe ich fünfeinhalb Jahre gelernt und achtunddreißig Jahre lang vergessen. The less said about my French (let alone Dutch), the better. So this English edition makes heavy use of the online Google and Microsoft translators. My contribution is mostly editing and annotation, typesetting, redrawing of the figures (with minor enhancements as noted), and a pathological obsession with the subject-matter.

[19] [Footnote 1 in the original:] See chapter 14 of this collection, § 12.

References

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