Polar Diagram with respect to a Near Pole

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Abstract

Polar diagram of a set of points on the plane, the dual and applications of which has been introduced recently[1, 2]. In this paper we define the Near pole polar diagram and survey some properties of it. Then we present an optimal algorithm to obtain it and discuss the complexity of the algorithm. Also we introduce applications and future works.

Keywords: Polar Diagram, Near Pole Polar Diagram, Voronoi Diagram, Computational Geometry.

1 Introduction

The Voronoi diagram is one of the most fundamental concepts in Computational Geometry and its algorithms and applications have been studied extensively[6]. This concept has also been generalized in a variety of directions by replacing the Euclidean distance with other metrics such as L_p-distance, weighted distances [9], the geodesic distance[3], the power distance[8, 10], and a skew distance. However, some of them are difficult to compute. As the solution to many problems in computational geometry requires some kind of angle processing of the input, some other generalizations of Voronoi diagram based on angle have been studied in [2, 1]. Grima et al. propose a new locus approach for problems processing angles, the polar diagram. For any position q in the plane (represented by a point) the site with smallest polar angle, is the owner of the region where q lies in. Using this portion, it can be solved by an $O(n)$ search problem in an optimal $O(logn)$ location problem and so the polar diagram principle can be used in some important problems requiring angle processing in computational geometry. Grima et al. [2] proved that polar diagram, used as preprocessing, can be applied to many problems in computational geometry in order to speed up their processing times. Some of these applications are the convex hull, visibility problems, and Path Planning problems. Jarvis's March approach can be improved to become an optimal time process and visibility problems can take advantage of polar diagram principles as well. Also Sadeghi et al. introduced in [1] the dual of polar diagram and some properties and applications.

Polar Diagram is the plane partition with similar features to those of the Voronoi diagram. In fact, the polar diagram can be seen in the context of the generalized Voronoi diagram. The *Polar Angle* of the point p with respect to s_i, denoted as $ang_{s_i}(p)$, is the angle formed by the positive horizontal line of p and the straight line linking p and s_i.

Given a set S of n points in the plane, the locus of points having smaller positive polar angle with respect to $s_i \in S$ is called *Polar Region* of s_i. Thus,

$$\mathcal{P}_S(s_i) = \{(x,y) \in E^2 | ang_{S_i}(x,y) < ang_{S_j}(x,y); \forall j \neq i.$$

The plane is divided into different regions in such a way that if the point $(x,y) \in E^2$ lies into $\mathcal{P}_S(s_i)$, it is known that s_i is the first site found performing an angular scanning starting from (x,y). We can draw an analogy between this angular sweep and the behavior of a radar. Figure 1 depicts the polar diagram of a set of points in the plane and the final division constructed using the smallest polar angle criterion.

Although the polar diagram of n points in the plane is not a graph, we define its dual as well as a dual of graph. We said, two points (sites) are joined by the edge e^* in the dual of polar diagram if and only if their corresponding faces are separated by the edge e in polar diagram. So we may have some parallel edges or loops in the dual of a polar diagram. If we omit the loops and replace the parallel edges with one edge, then we will have another graph named Extracted Dual of polar diagram($\mathcal{EDPD}$) (Figure 2). There is an optimal algorithm to draw $\mathcal{EDPD}$ in [1].

In this paper, we define a new extension of Voronoi diagram and call it *Near Pole Polar Diagram ($\mathcal{NPPD}$)*. Then we present an optimal algorithm to find it and apply $\mathcal{NPPD}$ for some applications.

This paper is structured as follows: in Section 2 we introduce the problem Near Pole Polar Diagram and present an optimal algorithm to draw it. In Section 3 we present some applications and finally in Section 4 we state some feature works and open problems.

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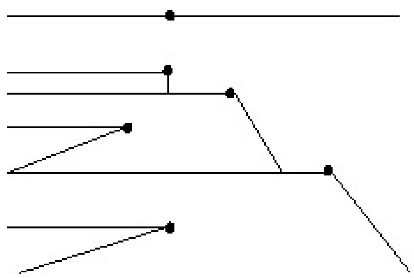

Figure 1: Polar diagram of 6 point sites.

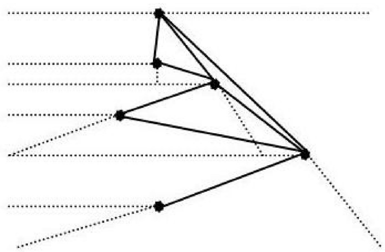

Figure 2: Extracted dual of polar diagram.

2 Polar diagram with respect to a near pole

In this section we introduce the Near pole polar diagram of a set of n point sites in the plane. Also we are going to present an optimal algorithm to draw it. Our approach is an incremental approach.

2.1 Near Pole Polar Diagram ($\mathcal{NPPD}$)

We introduce the Near pole polar diagram with similar features to those of the Polar diagram that can be seen in the context of the generalized Voronoi diagram [6]. As described in [1, 2], the pole mentioned lies on the left hand side of the plane at $-\infty$. In the polar diagram with respect to a near pole, it is assumed that the pole is located on the left hand side of the sites close to them. This allows us to find more applications for the problem. For example, the pole can be considered as the center of vision (eye) of a robot.

In addition to the given point sites, the point p in the plane is also given as a pole, and the partitioning of the plane will depend on the position of p. W.L.O.G assume that the pole p is located on the left-hand side of the sites. Figure 3 shows an example of $\mathcal{NPPD}$ for 7 points with respect to pole p.

In short, a Near pole polar diagram can be described as follows. Initially there is a radar at each of the point sites looking at the pole. They simultaneously start to rotate in counterclockwise direction and scan their periphery. The region in the plane observed by radar p_i before other radars will be called the region of p_i's $\mathcal{NPPD}$.

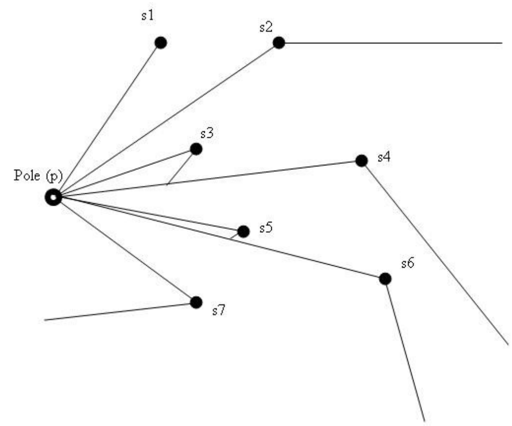

Figure 3: Near pole polar diagram of 7 points.

In Figure 4, we are given sites s_1 and s_2, pole p and point x in the plane. Since $\widehat{ps_1x} < \widehat{ps_2x}$, in the plane's partition, x will belong to the region of s_1. However this partitioning will produce disconnected regions with curved boundaries and makes the problem more complex and drawing the corresponding diagram more difficult. In this paper we have made an assumption which not only makes the problem simpler, but also increases its applications.

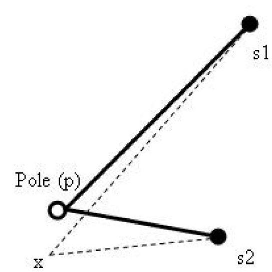

Figure 4: Our assumption: the line segment ps_2 blocks s_1's line of view.

2.2 Incremental approach

In this paper, we are going to present an incremental algorithm for drawing $\mathcal{NPPD}$, working in optimal time. In this paradigm, we first compute the tangent of the line segments $s_ip : i = 1, ..., n$ and sort them. Then a straight half line starting at pole p rotates in clockwise direction around the pole p and sweeps the plane. The $\mathcal{NPPD}$ region of s_i is built according to the following theorem.

Theorem 1 *Let S'_i denote the set of processed points when point s_i is reached, $S'_i = S'_{i-1} \cup s_i$. If $s_i \in$*

$\mathcal{NPPD}_S(s_k), s_k \in S'_{i-1}$ *then the near pole polar region of* s_i *is the angular sector defined by the half line from* s_i *to the pole* p *and the half line defined by* s_i *and* s_k*, which does not contain* s_k.

Proof. Consider point x within the region mentioned in the theorem. Since according to our initial assumption the line segment ps_i blocks other sites' (and specially s_k's) line of view, then the above mentioned region in $\mathcal{NPPD}$ belongs to s_i. Also as $\widehat{s_k p y} < \widehat{s_i p y}$, there can not exist any other points such as y within s_i's region (see Figure 5). □

Theorem 1 is the key to compute the $\mathcal{NPPD}$ using the Incremental method. Algorithm 1 describes the process: $S = \{s_0, s_1, ..., s_{n-1}\}$ is given. $TS = \{ts_0, ts_1, ..., ts_{n-1}\}$ is sorted from the largest member to the smallest one. The $\mathcal{NPPD}$ region of s_i ($\mathcal{NPPD}_S(s_i)$) is computed when $\mathcal{NPPD}_S(s_0)$, $\mathcal{NPPD}_S(s_1)$,...,$\mathcal{NPPD}_S(s_{i-1})$ have been already processed according to theorem 1.

In what follows an algorithm for drawing $\mathcal{NPPD}$ in the plane is presented which takes optimal $\theta(nlogn)$ time.

Algorithm 1
Input: A set $S = \{s_0, s_1, ..., s_{n-1}\}$ of n point sites and a point p as pole in E^2.
Output: $\mathcal{NPPD}(S, p)$.
Begin
  Step 1: Calculate the tangent of all lines $s_i p$ and make new set TS.
  Step 2: Sort TS by decreasing order obtaining $TS = \{ts_0, ts_1, ..., ts_{n-1}\}$
  Step 3: Let be $TS' := \{ts_0\}$
  Step 4: $TS := TS - \{ts_0\}$
  Step 5: While $\{TS \neq \emptyset\} Do$
    (a): Let ts_i be the maximum value of TS
    (b): Do $TS' := TS' \cup \{ts_i\}$ and $TS := TS - \{ts_i\}$
    (c): Construct $\mathcal{NPPD}_S(s_i)$ according to theorem 1.
    (d): Discard all edges inside $\mathcal{NPPD}_S(s_i)$
  Endwhile
End.

We can solve a sorting operation in $O(nlogn)$ time and theorem 1 ensures this time complexity after computing all near pole polar regions.

Assume that n numbers $x_i; i = 1, ..., n$ are given. We can find a function to map the numbers into interval $[-1, 1]$ and calculate n new numbers $x'_i; i = 1, ..., n$. Lets locate n point sites in the plane on $(1, x'_i); i = 1, ..., n$ coordination and the pole p on the $(0, 0)$ (Figure 6). Now using $\mathcal{NPPD}$ for these point sites with respect to the pole p we can sort the points $(1, x'_i); i = 1, ..., n$. So in this way we can sort

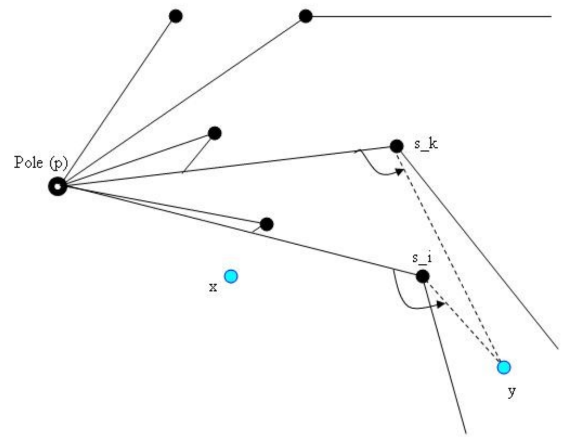

Figure 5: Incremental approach to drawing $\mathcal{NPPD}$.

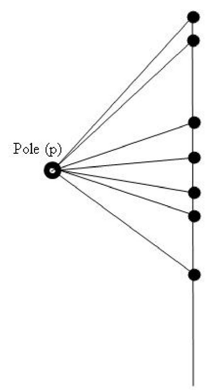

Figure 6: Contradiction: sorting n given numbers at a time less than $O(nlogn)$.

n given numbers at such time and this is a contradiction. Therefore this time complexity is a lower bound and we have the following:

Theorem 2 *The near pole polar diagram of a set of* n *points in the plane with respect to a given pole* p *must be computed in* $\theta(nlogn)$.

3 Some applications

In this section we are going to briefly address a number of applications for $\mathcal{NPPD}$. But before that, it is worth mentioning that as in [1], it is possible to define and solve the extracted dual of $\mathcal{NPPD}$. It can also be defined for other objects in the plane such as for line segments, convex polygons and circles.

Let S be a set of n disjoint line segments in the plane, and let p be a point not on any of the line segments of S. Using the $\mathcal{NPPD}$ of the line segments with respect to pole p and the extracted dual of it, we can find all line segments of S that p can see, that is, all line segments of S that contain some point q so that the open segment $\overline{pq}$ does not intersect any line segment of S.

In addition to the applications of $\mathcal{NPPD}$ to computer graphics, visibility and path planning problems, it is also possible to draw decorative patterns by assigning certain points in the plane and drawing $\mathcal{NPPD}$ in two directions (with point sites lying on the left-hand side or the right-hand side of the pole), with application in architecture. A sample of such patterns is shown in Figures 7, 8 in which the sites lie on some concentric circles.

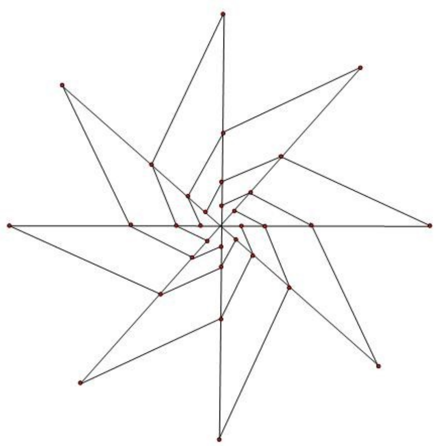

Figure 7: $\mathcal{NPPD}$ for some sites which lie on some concentric circles.

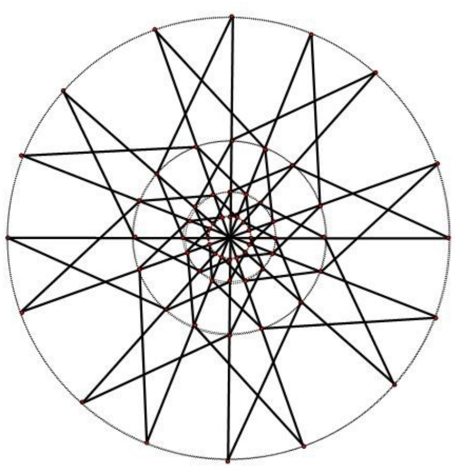

Figure 8: $\mathcal{NPPD}$ (in two directions) for some sites which lie on some concentric circles.

4 Conclusion and future works

We defined in this paper the Near pole polar diagram ($\mathcal{NPPD}$) and presented an optimal algorithm to find it. The algorithm runs in $\theta(nlogn)$ time for n given point sites and a given pole p in the plane. We also briefly introduced some applications of $\mathcal{NPPD}$.

Some problems related to robots' vision can be modeled using $\mathcal{NPPD}$ and its extracted dual. $\mathcal{NPPD}$ is also closely related to the collision detection problems, allowing one to employ methods similar to those in [4].

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