# Discretization of 2D FEM model

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## The Equations

We start with the momentum equations and the continuity equation:

$$\frac{\partial U}{\partial t} + gH\frac{\partial \zeta}{\partial x} + RU + X = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} + gH\frac{\partial \zeta}{\partial y} + RV + Y = 0 \tag{2}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{3}$$

where $x, y, t$ are the space coordinates and time, $U, V$ the transports in $x, y$ direction, $\zeta$ the water level, $R$ the friction parameter, $X, Y$ extra terms that cna be treated explicitly in the following discretization, $g$ the gravitational acceleration and $H$ the total water depth.

The transports $U, V$ can be obtained from the velocities by

$$U = \int u dz \qquad V = \int v dz \tag{4}$$

where $u, v$ are the current velocities.

The terms contained in $X, Y$ are the non-linear advective terms, the Coriolis terms, the wind stress and the lateral eddy friction. They may be written as

$$X = U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} - fV - \frac{\tau^x}{\rho_0} - A_H(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}) \tag{5}$$

$$Y = U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + fU - \frac{\tau^y}{\rho_0} - A_H(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}) \tag{6}$$

where $f$ is the Coriolis parameter, $\tau^x, \tau^y$ the wind stress, $\rho_0$ the reference density of water and $A_H$ the horizontal eddy viscosity.

# Discretization of the Momentum Equation

We now chose weighting parameters for the discretization. These parameters are $a_z$ for the transports in the continuity equation, $a_m$ for the pressure term in the momentum equations and $a_r$ for the friction term. Associated to these parameters are the parameters $\tilde{\alpha_z}, \tilde{\alpha_m}, \tilde{\alpha_r}$ that are defined as

$$\tilde{\alpha_z} = 1 - a_z \quad \tilde{\alpha_m} = 1 - a_m \quad \tilde{\alpha_r} = 1 - a_r. \tag{7}$$

All above parameters can take the values from 0 to 1 where 0 means an explicit treatment and 1 a complete implicit treatment.

Discretizing the $x$ momentum equation one obtains

$$\frac{U^{(1)} - U^{(0)}}{\Delta t} + gH[\alpha_m \frac{\partial \zeta^{(1)}}{\partial x} + \tilde{\alpha_m} \frac{\partial \zeta^{(0)}}{\partial x}] + R[\alpha_r U^{(1)} + \tilde{\alpha_r} U^{(0)}] + X = 0 \tag{8}$$

where the total depth $H$ and the extra terms $X$ are always taken at the old time step.

Solving for $U^{(1)}$ and introducing the new parameter

$$\delta = \frac{1}{1 + \Delta t R \alpha_r} \tag{9}$$

we obtain

$$U^{(1)} = \delta(1 - \Delta t R \tilde{\alpha_r})U - \Delta t \delta g H[\alpha_m \frac{\partial \zeta^{(1)}}{\partial x} + \tilde{\alpha_m} \frac{\partial \zeta^{(0)}}{\partial x}] - \Delta t \delta X. \tag{10}$$

Introducing two more auxiliary parameters

$$\gamma = \delta[1 - \Delta t R \tilde{\alpha_r}] \quad \beta = \Delta t \delta g H \tag{11}$$

we finally have for both momentum equations

$$U^{(1)} = \gamma U - \beta \alpha_m \frac{\partial \zeta^{(1)}}{\partial x} - \beta \tilde{\alpha_m} \frac{\partial \zeta^{(0)}}{\partial x} - \Delta t \delta X \tag{12}$$

$$V^{(1)} = \gamma V - \beta \alpha_m \frac{\partial \zeta^{(1)}}{\partial y} - \beta \tilde{\alpha_m} \frac{\partial \zeta^{(0)}}{\partial y} - \Delta t \delta Y \tag{13}$$

where the equation in $y$ direction has been obtained in a similar way obtained in a similar way as the one in $x$ direction.

# Spatial Integration of the Momentum Equation

We integrate the momentum equations over one element. For this we multiply every term with the constant weighting function $\Psi$ and integrate. Remember that $U, V$ are constant over an element, and $\zeta$ is varying linearly. The single terms give

$$\int \Psi U^{(1)} d\Omega = A_\Omega U^{(1)} \tag{14}$$

$$\int \Psi X d\Omega = \int X d\Omega \tag{15}$$

$$\int \frac{\partial \zeta^{(1)}}{\partial x} d\Omega = \int b_M \zeta_M^{(1)} d\Omega = A_\Omega b_M \zeta_M^{(1)} \tag{16}$$

$$\int \frac{\partial \zeta^{(1)}}{\partial y} d\Omega = \int c_M \zeta_M^{(1)} d\Omega = A_\Omega c_M \zeta_M^{(1)} \tag{17}$$

where $\Omega$ is the integration domain, $A_\Omega$ the area of the triangle and $b_M, c_M$ are the constant derivatives of the linear form functions $\Phi$

$$b_M = \frac{\partial \Phi}{\partial x} \quad c_M = \frac{\partial \Phi}{\partial y}. \tag{18}$$

We therefore obtain

$$U^{(1)} = \gamma U - \beta \alpha_m b_M \zeta_M^{(1)} - \beta \tilde{\alpha_m} b_M \zeta_M^{(0)} - \Delta t \delta \hat{X} \tag{19}$$

$$V^{(1)} = \gamma V - \beta \alpha_m c_M \zeta_M^{(1)} - \beta \tilde{\alpha_m} c_M \zeta_M^{(0)} - \Delta t \delta \hat{Y} \tag{20}$$

with

$$\hat{X} = \frac{1}{A_\Omega} \int X d\Omega \quad \hat{Y} = \frac{1}{A_\Omega} \int Y d\Omega. \tag{21}$$

# Integration of the Continuity Equations

We first integrate the continuity equation over one element and obtain

$$\int \Phi \frac{\partial \zeta^{(1)}}{\partial x} d\Omega = \int b_M \zeta_M^{(1)} d\Omega = A_\Omega b_M \zeta_M^{(1)} \tag{22}$$