

P versus NP

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Abstract

P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency. However, a precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. Another major complexity classes are L and NL. Whether $L = NL$ is another fundamental question that it is as important as it is unresolved. We demonstrate that every problem in NP could be NL-reduced to another problem in L.

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1 Introduction

In previous years there has been great interest in the verification or checking of computations [12]. Interactive proofs introduced by Goldwasser, Micali and Rackoff and Babai can be viewed as a model of the verification process [12]. Dwork and Stockmeyer and Condon have studied interactive proofs where the verifier is a space bounded computation instead of the original model where the verifier is a time bounded computation [12]. In addition, Blum and Kannan have studied another model where the goal is to check a computation based solely on the final answer [12]. More about probabilistic logarithmic space verifiers and the complexity class NP has been investigated on a technique of Lipton [12]. In this work, we show some results about the logarithmic space verifiers applied to the class NP .

The P versus NP problem is a major unsolved problem in computer science [5]. This is considered by many to be the most important open problem in the field [5]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US\$1,000,000 prize for the first correct solution [5]. The precise statement of the $P = NP$ problem was introduced in 1971 by Stephen Cook in a seminal paper [5]. In 2012, a poll of 151 researchers showed that 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore impossible to prove or disprove, 8 (5%) said either do not know or do not care or don't want the answer to be yes nor the problem to be resolved [9].

The $P = NP$ question is also singular in the number of approaches that researchers have brought to bear upon it over the years [7]. From the initial question in logic, the focus moved to complexity theory where early work used diagonalization and relativization techniques [7]. It was showed that these methods were perhaps inadequate to resolve P versus NP by demonstrating relativized worlds in which $P = NP$ and others in which $P \neq NP$ [4]. This shifted the focus to methods using circuit complexity and for a while this approach was deemed the one most likely to resolve the question [7]. Once again, a negative result showed that a class of techniques known as “Natural Proofs” that subsumed the above could not separate the classes NP and P , provided one-way functions exist [15]. There has been speculation that resolving the $P = NP$ question might be outside the domain of

mathematical techniques [7]. More precisely, the question might be independent of standard axioms of set theory [7]. Some results have showed that some relativized versions of the $P = NP$ question are independent of reasonable formalizations of set theory [10].

It is fully expected that $P \neq NP$ [14]. Indeed, if $P = NP$ then there are stunning practical consequences [14]. For that reason, $P = NP$ is considered as a very unlikely event [14]. Certainly, P versus NP is one of the greatest open problems in science and a correct solution for this incognita will have a great impact not only in computer science, but for many other fields as well [1]. Whether $P = NP$ or not is still a controversial and unsolved problem [1]. We show some results that could help us to prove this outstanding problem.

2 Theory and Methods

2.1 Preliminaries

In 1936, Turing developed his theoretical computational model [17]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [17]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [17]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [17].

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [3]. A Turing machine M has an associated input alphabet Σ [3]. For each string w in Σ^* there is a computation associated with M on input w [3]. We say that M accepts w if this computation terminates in the accepting state, that is $M(w) = \text{“yes”}$ [3]. Note that M fails to accept w either if this computation ends in the rejecting state, that is $M(w) = \text{“no”}$, or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when M outputs the string y on the input w) [3].

Another relevant advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet [6]. A complexity class is a set of problems, which are represented as a language, grouped by measures such as the running time, memory, etc [6]. The language accepted by a Turing machine M , denoted $L(M)$, has an associated alphabet Σ and is defined by:

$$L(M) = \{w \in \Sigma^* : M(w) = \text{“yes”}\}.$$

Moreover, $L(M)$ is decided by M , when $w \notin L(M)$ if and only if $M(w) = \text{“no”}$ [6]. We denote by $t_M(w)$ the number of steps in the computation of M on input w [3]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of M ; that is:

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

where Σ^n is the set of all strings over Σ of length n [3]. We say that M runs in polynomial time if there is a constant k such that for all n , $T_M(n) \leq n^k + k$ [3]. In other words, this means the language $L(M)$ can be decided by the Turing machine M in polynomial time. Therefore, P is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [6]. A verifier for a language L_1 is a deterministic Turing machine M , where:

$$L_1 = \{w : M(w, c) = \text{“yes” for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w , so a polynomial time verifier runs in polynomial time in the length of w [3]. A verifier uses additional information, represented by the symbol c , to verify that a string w is a member of L_1 . This information is called certificate. NP is the complexity class of languages defined by polynomial time verifiers [14].

A function $f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function if some deterministic Turing machine M , on every input w , halts in polynomial time with just $f(w)$ on its tape [17]. Let $\{0, 1\}^*$ be the infinite set of binary strings, we say that a language $L_1 \subseteq \{0, 1\}^*$ is polynomial time reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_p L_2$, if there is a polynomial time computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$:

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

An important complexity class is NP -complete [8]. A language $L_1 \subseteq \{0, 1\}^*$ is NP -complete if:

- $L_1 \in NP$, and
- $L' \leq_p L_1$ for every $L' \in NP$.

If L_1 is a language such that $L' \leq_p L_1$ for some $L' \in NP$ -complete, then L_1 is NP -hard [6]. Moreover, if $L_1 \in NP$, then $L_1 \in NP$ -complete [6]. A principal NP -complete problem is SAT [8]. An instance of SAT is a Boolean formula ϕ which is composed of:

1. Boolean variables: x_1, x_2, \dots, x_n ;
2. Boolean connectives: Any Boolean function with one or two inputs and one output, such as \wedge (AND), \vee (OR), \neg (NOT), \Rightarrow (implication), \Leftrightarrow (if and only if);
3. and parentheses.

A truth assignment for a Boolean formula ϕ is a set of values for the variables in ϕ . A satisfying truth assignment is a truth assignment that causes ϕ to be evaluated as true. A Boolean formula with a satisfying truth assignment is satisfiable. The problem SAT asks whether a given Boolean formula is satisfiable [8]. We define a CNF Boolean formula using the following terms:

A literal in a Boolean formula is an occurrence of a variable or its negation [6]. A Boolean formula is in conjunctive normal form, or CNF , if it is expressed as an AND of clauses, each of which is the OR of one or more literals [6]. A Boolean formula is in 3-conjunctive normal form or $3CNF$, if each clause has exactly three distinct literals [6].

For example, the Boolean formula:

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in $3CNF$. The first of its three clauses is $(x_1 \vee \neg x_1 \vee \neg x_2)$, which contains the three literals x_1 , $\neg x_1$, and $\neg x_2$. Another relevant NP -complete language is $3CNF$ satisfiability, or $3SAT$ [6]. In $3SAT$, it is asked whether a given Boolean formula ϕ in $3CNF$ is satisfiable.

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [17]. The work tapes may contain at most $O(\log n)$ symbols [17]. In computational complexity theory, L is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [14]. NL is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [14].

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and read/write work tapes [17]. The work tapes must contain at most

$O(\log n)$ symbols [17]. A logarithmic space transducer M computes a function $f : \Sigma^* \rightarrow \Sigma^*$, where $f(w)$ is the string remaining on the output tape after M halts when it is started with w on its input tape [17]. We call f a logarithmic space computable function [17]. We say that a language $L_1 \subseteq \{0, 1\}^*$ is logarithmic space reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_l L_2$, if there exists a logarithmic space computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$:

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

The logarithmic space reduction is used in the definition of the complete languages for the classes L and NL [14]. A Boolean formula is in 2-conjunctive normal form, or $2CNF$, if it is in CNF and each clause has exactly two distinct literals. There is a problem called $2SAT$, where we asked whether a given Boolean formula ϕ in $2CNF$ is satisfiable. $2SAT$ is complete for NL [14]. Another special case is the class of problems where each clause contains XOR (i.e. exclusive or) rather than (plain) OR operators. This is in P , since an $XOR SAT$ formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [13]. We denote the XOR function as \oplus . The $XOR 2SAT$ problem will be equivalent to $XOR SAT$, but the clauses in the formula have exactly two distinct literals. $XOR 2SAT$ is in L [2], [16].

2.2 Hypothesis

We can give a certificate-based definition for NL [3]. The certificate-based definition of NL assumes that a logarithmic space Turing machine has another separated read-only tape [3]. On each step of the machine, the machine's head on that tape can either stay in place or move to the right [3]. In particular, it cannot reread any bit to the left of where the head currently is [3]. For that reason this kind of special tape is called "read-once" [3].

► **Definition 1.** *A language L_1 is in NL if there exists a deterministic logarithmic space Turing machine M with an additional special read-once input tape polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x \in \{0, 1\}^*$:*

$$x \in L_1 \Leftrightarrow \exists u \in \{0, 1\}^{p(|x|)} \text{ such that } M(x, u) = \text{"yes"}$$

where by $M(x, u)$ we denote the computation of M where x is placed on its input tape and the certificate u is placed on its special read-once tape, and M uses at most $O(\log |x|)$ space on its read/write tapes for every input x where $|\dots|$ is the bit-length function [3]. M is called a logarithmic space verifier [3].

The two-way Turing machines may move their head on the input tape into two-way (left and right directions) while the one-way Turing machines are not allowed to move the head on the input tape to the left [11]. Hartmanis and Mahaney have investigated the classes $1L$ and $1NL$ of languages recognizable by deterministic one-way logarithmic space Turing machine and nondeterministic one-way logarithmic space Turing machine, respectively [11]. We state the following Hypothesis:

▷ **Hypothesis 2.** Given a nonempty language $L_1 \in L$, there is a language L_2 in NP -complete under logarithmic space reductions with a deterministic Turing machine M , where:

$$L_2 = \{w : M(w, u) = y, \exists u \text{ such that } y \in L_1\}$$

when M runs in one-way logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w . In this way, there is an NP -complete

language defined by a one-way logarithmic space verifier M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in L .

► **Theorem 3.** *If the Hypothesis 2 is true, then every problem in NP could be NL -reduced to another problem in L .*

Proof. We can simulate the computation $M(w, u) = y$ in the Hypothesis 2 by a nondeterministic logarithmic space Turing machine N such that $N(w) = y$, since we can read the certificate string u within the read-once tape by a work tape in a nondeterministic logarithmic space generation of symbols contained in u [14]. Certainly, we can simulate the reading of one symbol from the string u into the read-once tape just nondeterministically generating the same symbol in the work tapes using a logarithmic space [14]. We remove each symbol generated in the work tapes, when we try to generate the next symbol contiguous to the right on the string u . In this way, the generation will always be in logarithmic space. Moreover, we know that N is a nondeterministic one-way logarithmic space Turing machine.

For every language $L_3 \in NP$, then we can reduce the elements of the language L_3 to the elements of the language L_1 by a nondeterministic logarithmic space Turing machine M'' . In this way, there is a nondeterministic logarithmic space Turing machine $M''(x) = N(M'(x))$ which will nondeterministically output an element $y \in L_1$ when $x \in L_3$. The deterministic logarithmic space Turing machine M' will be the logarithmic space reduction of L_3 to L_2 , because of L_2 is in NP -complete under logarithmic space reductions. Actually, the nondeterministic logarithmic space reduction is possible, because of N is in one way. Indeed, it is not necessary to reset the computation of M' in the composition $N(M'(x))$ on the input x , because N never moves to the left the head on the input tape (that would be the output tape of M'). Since $L_1 \in L$, then we obtain that every problem in NP could be NL -reduced to another problem in L when the Hypothesis 2 is true. ◀

3 Results

We show a previous known NP -complete problem:

► **Definition 4. NAE 3SAT**

INSTANCE: A Boolean formula ϕ in 3CNF.

QUESTION: Is there a truth assignment for ϕ such that each clause has at least one true literal and at least one false literal?

REMARKS: $NAE\ 3SAT \in NP$ -complete [8].

We define a new problem:

► **Definition 5. MAXIMUM EXCLUSIVE-OR 2SAT**

INSTANCE: A positive integer K and a Boolean formula ϕ that is an instance of XOR 2SAT.

QUESTION: Is there a truth assignment in ϕ such that at most K clauses are unsatisfied?

REMARKS: We denote this problem as $MAX \oplus 2SAT$.

► **Theorem 6.** $MAX \oplus 2SAT \in NP$ -complete.

Proof. It is trivial to see $MAX \oplus 2SAT \in NP$ [14]. Given a Boolean formula ϕ in 3CNF with n variables and m clauses, we create three new variables a_{c_i} , b_{c_i} and d_{c_i} for each clause $c_i = (x \vee y \vee z)$ in ϕ , where x , y and z are literals, in the following formula:

$$P_i = (a_{c_i} \oplus b_{c_i}) \wedge (b_{c_i} \oplus d_{c_i}) \wedge (a_{c_i} \oplus d_{c_i}) \wedge (x \oplus a_{c_i}) \wedge (y \oplus b_{c_i}) \wedge (z \oplus d_{c_i}).$$

We can see P_i has at most one unsatisfied clause for some truth assignment if and only if at least one member of $\{x, y, z\}$ is true and at least one member of $\{x, y, z\}$ is false for the same truth assignment. Hence, we can create the Boolean formula ψ as the conjunction of the P_i formulas for every clause c_i in ϕ , such that $\psi = P_1 \wedge \dots \wedge P_m$. Finally, we obtain that:

$$\phi \in \text{NAE 3SAT} \text{ if and only if } (\psi, m) \in \text{MAX} \oplus \text{2SAT}.$$

Consequently, we prove $\text{NAE 3SAT} \leq_p \text{MAX} \oplus \text{2SAT}$ where we already know the language $\text{NAE 3SAT} \in \text{NP-complete}$ [8]. Moreover, this reduction is also a logarithmic space reduction [14]. To sum up, we show $\text{MAX} \oplus \text{2SAT} \in \text{NP-hard}$ and $\text{MAX} \oplus \text{2SAT} \in \text{NP}$ and thus, $\text{MAX} \oplus \text{2SAT} \in \text{NP-complete}$. ◀

Algorithm 1 Logarithmic space verifier

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1: /*A valid instance for MAX ⊕ 2SAT with its certificate*/
2: procedure VERIFIER((ψ, K), A)
3:   /*Initialize minimum and maximum values*/
4:   min ← 1
5:   max ← 0
6:   /*Iterate for the elements of the certificate array A*/
7:   for i ← 1 to K + 1 do
8:     if i = K + 1 then
9:       /*There exists a K + 1 element in the array*/
10:      if A[i] ≠ undefined then
11:        /*Reject the certificate*/
12:        return "no"
13:      end if
14:      /*m is the number of clauses in ψ*/
15:      max ← m + 1
16:    else if A[i] = undefined ∨ A[i] ≤ max ∨ A[i] < 1 ∨ A[i] > m then
17:      /*Reject the certificate*/
18:      return "no"
19:    else
20:      max ← A[i]
21:    end if
22:    /*Iterate for the clauses of the Boolean formula ψ*/
23:    for j ← min to max − 1 do
24:      /*Output the indexed jth clause in ψ*/
25:      output " ∧ cj"
26:    end for
27:    min ← max + 1
28:  end for
29: end procedure

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► **Theorem 7.** *There is a deterministic Turing machine M , where:*

$$\text{MAX} \oplus \text{2SAT} = \{w : M(w, u) = y, \exists u \text{ such that } y \in \text{XOR 2SAT}\}$$

when M runs in one-way logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w .

Proof. Given a valid instance (ψ, K) for $MAX \oplus 2SAT$ when ψ has m clauses, we can create a certificate array A which contains K different natural numbers in ascending order which represents the indexes of the clauses in ψ that we are going to remove from the instance. We read at once the elements of the array A and we reject whether this is not an appropriated certificate: That is when the numbers are not sorted in ascending order, or the array A does not contain exactly K elements, or the array A contains a number that is not between 1 and m . While we read the elements of the array A , we remove the clauses from the instance (ψ, K) for $MAX \oplus 2SAT$ just creating another instance ϕ for $XOR \ 2SAT$ where the Boolean formula ϕ does not contain the K different indexed clauses ψ represented by the numbers in A . Therefore, we obtain the array A would be valid according to the Theorem 7 when:

$$(\psi, K) \in MAX \oplus 2SAT \Leftrightarrow (\exists \text{ array } A \text{ such that } \phi \in XOR \ 2SAT).$$

Furthermore, we can make this verification in logarithmic space such that the array A is placed on the special read-once tape, because we read at once the elements in the array A and we assume the clauses in the input ψ are indexed from left to right. Hence, we only need to iterate from the elements of the array A to verify whether the array is an appropriated certificate and also remove the K different clauses from the Boolean formula ψ when we write the final clauses to the output. This logarithmic space verification will be the Algorithm 1. We assume whether a value does not exist in the array A into the cell of some position i when $A[i] = \text{undefined}$. In addition, we reject immediately when the following comparisons:

$$A[i] \leq \max \vee A[i] < 1 \vee A[i] > m$$

hold at least into one single binary digit. Note, in the loop j from min to $max - 1$, we do not output any clause when $max - 1 < min$. Certainly, the Algorithm 1 is in one-way, since this never moves the head on the input tape to the left. ◀

► **Theorem 8.** *Every problem in NP could be NL-reduced to another problem in L.*

Proof. Every *NP-complete* is logarithmic space reduced to $MAX \oplus 2SAT$. Certainly, every *NP* problem could be logarithmic space reduced to *SAT* by the Cook's Theorem algorithm [8]. In addition, the problem *SAT* could be logarithmic space reduced to *NAE 3SAT* [8]. Moreover, the problem *NAE 3SAT* could be logarithmic space reduced to $MAX \oplus 2SAT$ according to Theorem 6. Therefore, this is a direct consequence of Theorems 3, 6 and 7. ◀

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