

P versus NP

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Abstract

P versus NP is considered as one of the most important open problems in computer science. This consists in knowing the answer of the following question: Is P equal to NP? It was essentially mentioned in 1955 from a letter written by John Nash to the United States National Security Agency. However, a precise statement of the P versus NP problem was introduced independently by Stephen Cook and Leonid Levin. Since that date, all efforts to find a proof for this problem have failed. Another major complexity classes are L and NL. Whether $L = NL$ is another fundamental question that it is as important as it is unresolved. We demonstrate that every problem in NP could be NL-reduced to another problem in L. In addition, we show if L is equal to NL, then $L = NP$.

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1 Introduction

In previous years there has been great interest in the verification or checking of computations [13]. Interactive proofs introduced by Goldwasser, Micali and Rackoff and Babai can be viewed as a model of the verification process [13]. Dwork and Stockmeyer and Condon have studied interactive proofs where the verifier is a space bounded computation instead of the original model where the verifier is a time bounded computation [13]. In addition, Blum and Kannan have studied another model where the goal is to check a computation based solely on the final answer [13]. More about probabilistic logarithmic space verifiers and the complexity class NP has been investigated on a technique of Lipton [13]. In this work, we show some results about the logarithmic space verifiers applied to the class NP and logarithmic space disqualifiers applied to the class $coNP$.

The P versus NP problem is a major unsolved problem in computer science [5]. This is considered by many to be the most important open problem in the field [5]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US\$1,000,000 prize for the first correct solution [5]. The precise statement of the $P = NP$ problem was introduced in 1971 by Stephen Cook in a seminal paper [5]. In 2012, a poll of 151 researchers showed that 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore impossible to prove or disprove, 8 (5%) said either do not know or do not care or don't want the answer to be yes nor the problem to be resolved [9].

The $P = NP$ question is also singular in the number of approaches that researchers have brought to bear upon it over the years [7]. From the initial question in logic, the focus moved to complexity theory where early work used diagonalization and relativization techniques [7]. It was showed that these methods were perhaps inadequate to resolve P versus NP by demonstrating relativized worlds in which $P = NP$ and others in which $P \neq NP$ [4]. This shifted the focus to methods using circuit complexity and for a while this approach was deemed the one most likely to resolve the question [7]. Once again, a negative result showed that a class of techniques known as "Natural Proofs" that subsumed the above could not separate the classes NP and P , provided one-way functions exist [16]. There

has been speculation that resolving the $P = NP$ question might be outside the domain of mathematical techniques [7]. More precisely, the question might be independent of standard axioms of set theory [7]. Some results have showed that some relativized versions of the $P = NP$ question are independent of reasonable formalizations of set theory [10].

It is fully expected that $P \neq NP$ [15]. Indeed, if $P = NP$ then there are stunning practical consequences [15]. For that reason, $P = NP$ is considered as a very unlikely event [15]. Certainly, P versus NP is one of the greatest open problems in science and a correct solution for this incognita will have a great impact not only in computer science, but for many other fields as well [1]. Whether $P = NP$ or not is still a controversial and unsolved problem [1]. We show some results that could help us to prove this outstanding problem.

2 Theory and Methods

2.1 Preliminaries

In 1936, Turing developed his theoretical computational model [18]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [18]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [18]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [18].

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [3]. A Turing machine M has an associated input alphabet Σ [3]. For each string w in Σ^* there is a computation associated with M on input w [3]. We say that M accepts w if this computation terminates in the accepting state, that is $M(w) = \text{“yes”}$ [3]. Note that M fails to accept w either if this computation ends in the rejecting state, that is $M(w) = \text{“no”}$, or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when M outputs the string y on the input w) [3].

Another relevant advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet [6]. A complexity class is a set of problems, which are represented as a language, grouped by measures such as the running time, memory, etc [6]. The language accepted by a Turing machine M , denoted $L(M)$, has an associated alphabet Σ and is defined by:

$$L(M) = \{w \in \Sigma^* : M(w) = \text{“yes”}\}.$$

Moreover, $L(M)$ is decided by M , when $w \notin L(M)$ if and only if $M(w) = \text{“no”}$ [6]. We denote by $t_M(w)$ the number of steps in the computation of M on input w [3]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of M ; that is:

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

where Σ^n is the set of all strings over Σ of length n [3]. We say that M runs in polynomial time if there is a constant k such that for all n , $T_M(n) \leq n^k + k$ [3]. In other words, this means the language $L(M)$ can be decided by the Turing machine M in polynomial time. Therefore, P is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [6]. A verifier for a language L_1 is a deterministic Turing machine M , where:

$$L_1 = \{w : M(w, c) = \text{“yes” for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w , so a polynomial time verifier runs in polynomial time in the length of w [3]. A verifier uses additional information, represented by the symbol c , to verify that a string w is a member of L_1 . This information is called certificate. NP is the complexity class of languages defined by polynomial time verifiers [15]. If NP is the class of problems that have succinct certificates, then the complexity class $coNP$ must contain those problems that have succinct disqualifications [15]. That is, a “no” instance of a problem in $coNP$ possesses a short proof of its being a “no” instance [15].

► **Definition 1.** *We will extend the definition of succinct disqualification for an element $w \in L_2$ when $L_2 \in coNP$ as the polynomially bounded string c by w such that $M(w, c) = \text{“no”}$ and M is the polynomial time verifier of the complement of L_2 in NP . Note, this definition is different in relation to the exposed in Papadimitriou’s book [15].*

2.2 First Hypothesis

A function $f : \Sigma^* \rightarrow \Sigma^*$ is a polynomial time computable function if some deterministic Turing machine M , on every input w , halts in polynomial time with just $f(w)$ on its tape [18]. Let $\{0, 1\}^*$ be the infinite set of binary strings, we say that a language $L_1 \subseteq \{0, 1\}^*$ is polynomial time reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_p L_2$, if there is a polynomial time computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$:

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

An important complexity class is NP -complete [8]. A language $L_1 \subseteq \{0, 1\}^*$ is NP -complete if:

- $L_1 \in NP$, and
- $L' \leq_p L_1$ for every $L' \in NP$.

If L_1 is a language such that $L' \leq_p L_1$ for some $L' \in NP$ -complete, then L_1 is NP -hard [6]. Moreover, if $L_1 \in NP$, then $L_1 \in NP$ -complete [6]. A principal NP -complete problem is SAT [8]. An instance of SAT is a Boolean formula ϕ which is composed of:

1. Boolean variables: x_1, x_2, \dots, x_n ;
2. Boolean connectives: Any Boolean function with one or two inputs and one output, such as \wedge (AND), \vee (OR), \neg (NOT), \Rightarrow (implication), \Leftrightarrow (if and only if);
3. and parentheses.

A truth assignment for a Boolean formula ϕ is a set of values for the variables in ϕ . A satisfying truth assignment is a truth assignment that causes ϕ to be evaluated as true. A Boolean formula with a satisfying truth assignment is satisfiable. The problem SAT asks whether a given Boolean formula is satisfiable [8]. We define a CNF Boolean formula using the following terms:

A literal in a Boolean formula is an occurrence of a variable or its negation [6]. A Boolean formula is in conjunctive normal form, or CNF , if it is expressed as an AND of clauses, each of which is the OR of one or more literals [6]. A Boolean formula is in 3-conjunctive normal form or $3CNF$, if each clause has exactly three distinct literals [6].

For example, the Boolean formula:

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in $3CNF$. The first of its three clauses is $(x_1 \vee \neg x_1 \vee \neg x_2)$, which contains the three literals x_1 , $\neg x_1$, and $\neg x_2$. Another relevant NP -complete language is $3CNF$ satisfiability, or $3SAT$ [6]. In $3SAT$, it is asked whether a given Boolean formula ϕ in $3CNF$ is satisfiable.

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [18]. The work tapes may contain at most $O(\log n)$ symbols [18]. In computational complexity theory, L is the complexity class containing those decision problems that can be decided by a deterministic logarithmic space Turing machine [15]. NL is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [15].

A logarithmic space transducer is a Turing machine with a read-only input tape, a write-only output tape, and read/write work tapes [18]. The work tapes must contain at most $O(\log n)$ symbols [18]. A logarithmic space transducer M computes a function $f : \Sigma^* \rightarrow \Sigma^*$, where $f(w)$ is the string remaining on the output tape after M halts when it is started with w on its input tape [18]. We call f a logarithmic space computable function [18]. We say that a language $L_1 \subseteq \{0, 1\}^*$ is logarithmic space reducible to a language $L_2 \subseteq \{0, 1\}^*$, written $L_1 \leq_l L_2$, if there exists a logarithmic space computable function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for all $x \in \{0, 1\}^*$:

$$x \in L_1 \text{ if and only if } f(x) \in L_2.$$

The logarithmic space reduction is used in the definition of the complete languages for the classes L and NL [15]. A Boolean formula is in 2-conjunctive normal form, or $2CNF$, if it is in CNF and each clause has exactly two distinct literals. There is a problem called $2SAT$, where we asked whether a given Boolean formula ϕ in $2CNF$ is satisfiable. $2SAT$ is complete for NL [15]. Another special case is the class of problems where each clause contains XOR (i.e. exclusive or) rather than (plain) OR operators. This is in P , since an $XOR SAT$ formula can also be viewed as a system of linear equations mod 2, and can be solved in cubic time by Gaussian elimination [14]. We denote the XOR function as \oplus . The $XOR 2SAT$ problem will be equivalent to $XOR SAT$, but the clauses in the formula have exactly two distinct literals. $XOR 2SAT$ is in L [2], [17].

We can give a certificate-based definition for NL [3]. The certificate-based definition of NL assumes that a logarithmic space Turing machine has another separated read-only tape [3]. On each step of the machine, the machine's head on that tape can either stay in place or move to the right [3]. In particular, it cannot reread any bit to the left of where the head currently is [3]. For that reason this kind of special tape is called "read-once" [3]. Besides, in the certificate-based definition of NL , we assume the certificate string is appropriated for the instance [15]. For example, a truth assignment for a Boolean formula ϕ is appropriated for the instance when every possible variable in ϕ could be evaluated in that truth assignment string, but we cannot affirm the same for every possible binary string.

► **Definition 2.** A language L_1 is in NL if there exists a deterministic logarithmic space Turing machine M with an additional special read-once input tape polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x \in \{0, 1\}^*$:

$$x \in L_1 \Leftrightarrow \exists \text{ appropriated } u \in \{0, 1\}^{p(|x|)} \text{ such that } M(x, u) = \text{"yes"}$$

where by $M(x, u)$ we denote the computation of M where x is placed on its input tape and the certificate u is placed on its special read-once tape, and M uses at most $O(\log |x|)$ space on its read/write tapes for every input x where $|\dots|$ is the bit-length function [3]. M is called a logarithmic space verifier [3].

The two-way Turing machines may move their head on the input tape into two-way (left and right directions) while the one-way Turing machines are not allowed to move the head on the input tape to the left [12]. Hartmanis and Mahaney have investigated the classes $1L$ and $1NL$ of languages recognizable by deterministic one-way logarithmic space Turing machine and nondeterministic one-way logarithmic space Turing machine, respectively [11]. They have shown that $1L \neq 1NL$ (by looking at a uniform variant of the string non-equality problem from communication complexity theory) and have defined a natural complete problem for $1NL$ under deterministic one-way logarithmic space reductions [11]. Furthermore, they have proven that $1NL \subseteq L$ if and only if $L = NL$ [11].

We state the following Hypothesis:

▷ **Hypothesis 3.** Given a nonempty language $L_1 \in L$, there is a language L_2 in NP -complete under logarithmic space reductions with a deterministic Turing machine M , where:

$$L_2 = \{w : M(w, u) = y, \exists \text{ appropriated } u \text{ such that } y \in L_1\}$$

when M runs in one-way logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w . In this way, there is an NP -complete language defined by a one-way logarithmic space verifier M such that when the input is an element of the language with its certificate, then M outputs a string which belongs to a single language in L .

▶ **Theorem 4.** *If the Hypothesis 3 is true, then every problem in NP could be NL -reduced to another problem in L .*

Proof. We can simulate the computation $M(w, u) = y$ in the Hypothesis 3 by a nondeterministic logarithmic space Turing machine N such that $N(w) = y$, since we can read the certificate string u within the read-once tape by a work tape in a nondeterministic logarithmic space generation of symbols contained in u [15]. Certainly, we can simulate the reading of one symbol from the string u into the read-once tape just nondeterministically generating the same symbol in the work tapes using a logarithmic space [15]. We remove each symbol generated in the work tapes, when we try to generate the next symbol contiguous to the right on the string u . In this way, the generation will always be in logarithmic space. Moreover, we know that N is a nondeterministic one-way logarithmic space Turing machine.

For every language $L_3 \in NP$, then we can reduce the elements of the language L_3 to the elements of the language L_1 by a nondeterministic logarithmic space Turing machine M'' . In this way, there is a nondeterministic logarithmic space Turing machine $M''(x) = N(M'(x))$ which will output an element $y \in L_1$ when $x \in L_3$. The deterministic logarithmic space Turing machine M' will be the logarithmic space reduction of L_3 to L_2 , because of L_2 is in NP -complete under logarithmic space reductions. Actually, the nondeterministic logarithmic space reduction is possible, because of N is in one way. Indeed, it is not necessary to reset the computation of M' in the composition $N(M'(x))$ on the input x , because N never moves to the left the head on the input tape (that would be the output tape of M'). Since $L_1 \in L$, then we obtain that every problem in NP could be NL -reduced to another problem in L when the Hypothesis 3 is true. ◀

2.3 Second Hypothesis

An important complexity class is $coNP$ -complete [8]. A language $L_1 \subseteq \{0, 1\}^*$ is $coNP$ -complete if:

- $L_1 \in coNP$, and

■ $L' \leq_p L_1$ for every $L' \in \text{coNP}$.

If L_1 is a language such that $L' \leq_p L_1$ for some $L' \in \text{coNP-complete}$, then L_1 is *coNP-hard* [6]. Moreover, if $L_1 \in \text{coNP}$, then $L_1 \in \text{coNP-complete}$ [6]. A principal *coNP-complete* problem is *UNSAT* [8]. A Boolean formula without any satisfying truth assignment is unsatisfiable. The problem *UNSAT* asks whether a given Boolean formula is unsatisfiable [8].

coNL is the complexity class containing the languages such that their complements belong to *NL* [15]. We can give a disqualification-based definition for *coNL* [3]. The disqualification-based definition of *coNL* assumes that a logarithmic space Turing machine has another separated read-only tape, that is the same kind of special tape called “read-once” that we use in the certificate-based definition for *NL* [3]. Besides, in the disqualification-based definition of *coNL*, we assume the disqualification string is appropriated for the instance [15].

► **Definition 5.** *A language L_1 is in coNL if there exists a deterministic logarithmic space Turing machine M with an additional special read-once input tape polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x \in \{0, 1\}^*$:*

$$x \in L_1 \Leftrightarrow \forall \text{ appropriated } u \in \{0, 1\}^{p(|x|)} \text{ then } M(x, u) = \text{“yes”}$$

where by $M(x, u)$ we denote the computation of M where x is placed on its input tape and the disqualification u is placed on its special read-once tape, and M uses at most $O(\log |x|)$ space on its read/write tapes for every input x where $|\dots|$ is the bit-length function. M is called a *logarithmic space disqualifier*.

For example, there is a well-known *coNL* problem that states: Given a directed graph $G = (V, E)$ and two nodes $s, t \in V$, is there no possible path from s to t ? In that problem, an appropriated disqualification u is a sequence of nodes contained in V when s is the first node and t is the last one such that this sequence of nodes is not a path: There is at least a consecutive pair of nodes in the sequence where they are not connected by an edge.

We state the following Hypothesis:

▷ **Hypothesis 6.** Given a nonempty language $L_1 \in \text{1NL}$, there is a language L_2 in *coNP-complete* under logarithmic space reductions with a deterministic Turing machine M , where:

$$L_2 = \{w : M(w, u) = y, \forall \text{ appropriated } u \text{ such that } y \in L_1\}$$

when M runs in logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w . In this way, there is a *coNP-complete* language defined by a logarithmic space disqualifier M such that when the input is an element of the language with any of its appropriated disqualification, then M always outputs a string which belongs to a single language in *1NL*.

► **Theorem 7.** *When the Hypothesis 6 is true, therefore if $L = \text{NL}$, then $L = \text{NP}$.*

Proof. We can accept the elements of the language $L_1 \in \text{1NL}$ by a nondeterministic one-way logarithmic space Turing machine M' . In this way, there is a nondeterministic logarithmic space Turing machine $M''(w, u) = M'(M(w, u))$ which will accept when $w \in L_2$ for all the appropriated disqualification u , where u is placed on the special read-once tape of M'' .

The reason is because we can simulate the output string of $M(w, u)$ within a read-once tape and thus, we can compute in a nondeterministic logarithmic space the logarithmic

space composition using the same techniques of the logarithmic space composition reduction, but without any reset of the computation [15]. Certainly, we do not need to reset the computation of $M(w, u)$ for the reading at once of a symbol in the output string of $M(w, u)$ by the nondeterministic one-way logarithmic space Turing machine M' .

Consequently, M'' can be converted into a logarithmic space disqualifier for the language L_2 just assuming that $L = NL$, because of the nondeterministic logarithmic space Turing machine M'' could be simulated by a deterministic logarithmic space Turing machine. Therefore, L_2 is in $coNL$ and thus, $L_2 \in P$ due to $coNL \subseteq P$ [15]. If any single $coNP$ -complete problem can be solved in polynomial time, then $P = NP$ [15]. Since $L_2 \in P$ and $L_2 \in coNP$ -complete, then we obtain the complexity class P is equal to NP . Since $coNL = NL$ and the language L_2 is in $coNP$ -complete under logarithmic space reductions, then we obtain $L = NP$ under the assumption that $L = NL$ when the Hypothesis 6 is true. ◀ ◀

3 Results

3.1 First Hypothesis

We show a previous known NP -complete problem:

► **Definition 8. NAE 3SAT**

INSTANCE: A Boolean formula ϕ in 3CNF.

QUESTION: Is there a truth assignment for ϕ such that each clause has at least one true literal and at least one false literal?

REMARKS: $NAE\ 3SAT \in NP$ -complete [8].

We define a new problem:

► **Definition 9. MAXIMUM EXCLUSIVE-OR 2SAT**

INSTANCE: A positive integer K and a Boolean formula ϕ that is an instance of XOR 2SAT.

QUESTION: Is there a truth assignment in ϕ such that at most K clauses are unsatisfied?

REMARKS: We denote this problem as $MAX \oplus 2SAT$.

► **Theorem 10.** $MAX \oplus 2SAT \in NP$ -complete.

Proof. It is trivial to see $MAX \oplus 2SAT \in NP$ [15]. Given a Boolean formula ϕ in 3CNF with n variables and m clauses, we create three new variables a_{c_i} , b_{c_i} and d_{c_i} for each clause $c_i = (x \vee y \vee z)$ in ϕ , where x , y and z are literals, in the following formula:

$$P_i = (a_{c_i} \oplus b_{c_i}) \wedge (b_{c_i} \oplus d_{c_i}) \wedge (a_{c_i} \oplus d_{c_i}) \wedge (x \oplus a_{c_i}) \wedge (y \oplus b_{c_i}) \wedge (z \oplus d_{c_i}).$$

We can see P_i has at most one unsatisfied clause for some truth assignment if and only if at least one member of $\{x, y, z\}$ is true and at least one member of $\{x, y, z\}$ is false for the same truth assignment. Hence, we can create the Boolean formula ψ as the conjunction of the P_i formulas for every clause c_i in ϕ , such that $\psi = P_1 \wedge \dots \wedge P_m$. Finally, we obtain that:

$$\phi \in NAE\ 3SAT \text{ if and only if } (\psi, m) \in MAX \oplus 2SAT.$$

Consequently, we prove $NAE\ 3SAT \leq_p MAX \oplus 2SAT$ where we already know the language $NAE\ 3SAT \in NP$ -complete [8]. Moreover, this reduction is also a logarithmic space reduction [15]. To sum up, we show $MAX \oplus 2SAT \in NP$ -hard and $MAX \oplus 2SAT \in NP$ and thus, $MAX \oplus 2SAT \in NP$ -complete. ◀ ◀

► **Theorem 11.** *There is a deterministic Turing machine M , where:*

$$MAX \oplus 2SAT = \{w : M(w, u) = y, \exists \text{ appropriated } u \text{ such that } y \in XOR\ 2SAT\}$$

when M runs in one-way logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w .

Proof. Given a valid instance (ψ, K) for $MAX \oplus 2SAT$ when ψ has m clauses, we can create a certificate array A which contains K different natural numbers in ascending order which represents the indexes of the clauses in ψ that we are going to remove from the instance. We read at once the elements of the array A and we reject whether this is not an appropriated certificate: That is when the numbers are not sorted in ascending order, or the array A does not contain exactly K elements, or the array A contains a number that is not between 1 and m . While we read the elements of the array A , we remove the clauses from the instance (ψ, K) for $MAX \oplus 2SAT$ just creating another instance ϕ for $XOR\ 2SAT$ where the Boolean formula ϕ does not contain the K different indexed clauses ψ represented by the numbers in A . Therefore, we obtain the array A would be valid according to the Theorem 11 when:

$$(\psi, K) \in MAX \oplus 2SAT \Leftrightarrow (\exists \text{ appropriated array } A \text{ such that } \phi \in XOR\ 2SAT).$$

Furthermore, we can make this verification in logarithmic space such that the array A is placed on the special read-once tape, because we read at once the elements in the array A and we assume the clauses in the input ψ are indexed from left to right. Hence, we only need to iterate from the elements of the array A to verify whether the array is an appropriated certificate and also remove the K different clauses from the Boolean formula ψ when we write the final clauses to the output. This logarithmic space verification will be the Algorithm 1. We assume whether a value does not exist in the array A into the cell of some position i when $A[i] = \text{undefined}$. In addition, we reject immediately when the following comparisons:

$$A[i] \leq \max \vee A[i] < 1 \vee A[i] > m$$

hold at least into one single binary digit. Note, in the loop j from \min to $\max - 1$, we do not output any clause when $\max - 1 < \min$. Certainly, the Algorithm 1 is in one-way, since this never moves the head on the input to the left. ◀ ◀

3.2 Second Hypothesis

We show a previous known *coNP*-complete problem:

► **Definition 12.** *3UNSAT*

INSTANCE: A Boolean formula ϕ in 3CNF.

QUESTION: Is ϕ unsatisfiable?

REMARKS: $3UNSAT \in \text{coNP}$ -complete [8].

We define a new problem:

► **Definition 13.** *SUM ZERO*

INSTANCE: A collection of integers C such that $0 \notin C$ and every integer in C has the same bit-length of the number that represents the cardinality of C multiplied by 3 (we do not take into account the symbol minus in counting the bit-length of the negative integers).

QUESTION: Are there two elements $a, b \in C$, such that $a + b = 0$?

REMARKS: We denote this problem as $0SUM$.

Algorithm 1 Logarithmic space verifier

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1: /*A valid instance for  $MAX \oplus 2SAT$  with its certificate*/
2: procedure VERIFIER(( $\psi, K$ ),  $A$ )
3:   /*Initialize minimum and maximum values*/
4:    $min \leftarrow 1$ 
5:    $max \leftarrow 0$ 
6:   /*Iterate for the elements of the certificate array  $A$ */
7:   for  $i \leftarrow 1$  to  $K + 1$  do
8:     if  $i = K + 1$  then
9:       /*There exists a  $K + 1$  element in the array*/
10:      if  $A[i] \neq \text{undefined}$  then
11:        /*Reject the certificate*/
12:        return "no"
13:      end if
14:      /* $m$  is the number of clauses in  $\psi$ */
15:       $max \leftarrow m + 1$ 
16:      else if  $A[i] = \text{undefined} \vee A[i] \leq max \vee A[i] < 1 \vee A[i] > m$  then
17:        /*Reject the certificate*/
18:        return "no"
19:      else
20:         $max \leftarrow A[i]$ 
21:      end if
22:      /*Iterate for the clauses of the Boolean formula  $\psi$ */
23:      for  $j \leftarrow min$  to  $max - 1$  do
24:        /*Output the indexed  $j^{th}$  clause in  $\psi$ */
25:        output " $\wedge c_j$ "
26:      end for
27:       $min \leftarrow max + 1$ 
28:    end for
29: end procedure

```

► **Theorem 14.** $0SUM \in 1NL$.

Proof. Given a collection of integers C , we can read its elements from left to right, verify that every element is not equal to 0, check that every element in C has the same bit-length and count the amount of elements in C to finally multiply it by 3 and compare its bit-length with the single bit-length from the elements in C . In addition, we can nondeterministically pick two elements a and b from C and accept in case of $a + b = 0$ otherwise we reject. We can make all this computation in a nondeterministic one-way using logarithmic space. Certainly, the calculation and store of the bit-length of the elements in C could be done in logarithmic space since this is a unique value. On the one hand, we can count and store the number of elements that we read from the input and multiply it by 3 to finally compare its bit-length with the stored unique bit-length from the elements of the collection, since the cardinality of C multiplied by 3 could be stored in a binary number of bit-length that is logarithmic in relation to the encoded length of C . On the other hand, the two elements a and b that we pick from C have a logarithmic space in relation to the encoded length of C , because of every integer in C has the same bit-length of the number that represents the cardinality of C multiplied by 3. Indeed, we never need to read to the left on the input for the acceptance of the elements in $0SUM$ in a nondeterministic logarithmic space. ◀ ◀

► **Theorem 15.** *There is a deterministic Turing machine M , where:*

$$3UNSAT = \{w : M(w, u) = y, \forall \text{ appropriated } u \text{ such that } y \in 0SUM\}$$

when M runs in logarithmic space in the length of w , u is placed on the special read-once tape of M , and u is polynomially bounded by w .

Proof. Given a Boolean formula ϕ in $3CNF$ with n variables and m clauses, we can create a disqualification array A which contains m positive integers between 1 and 3 which represents the literals of the clauses in ϕ which appear from left to right. We read at once the elements of the array A and we reject whether this is not an appropriated disqualification: That is when the array A does not contain exactly m elements, or the array A contains a number that is not between 1 and 3. While we read the elements of the array A , we select from the clauses ϕ the literals such that these ones occupy the position that represents the number between 1 and 3, that is the first, second or third place within the clause from left to right. In this way, we output the selected literals that are represented by a positive or negative (in case of a negated variable) integer just creating another instance C for $0SUM$ where the collection C contains those integers which are the selected literals for each clause in ϕ . Therefore, we obtain that all the appropriated array A would be valid according to the Theorem 15 when:

$$\phi \in 3UNSAT \Leftrightarrow (\forall \text{ appropriated array } A \text{ such that } C \in 0SUM)$$

since we assume the positive and negated literals of some variable in the input ϕ correspond to a positive integer and its negative value, respectively. Furthermore, we can make this disqualification in logarithmic space such that the array A is placed on the special read-once tape, because we read at once the elements in the array A . Hence, we only need to iterate from the elements of the array A to verify whether the array is an appropriated disqualification and pick the m literals from the Boolean formula ϕ when we write the final integers that represent these literals to the output. This logarithmic space disqualification will be the Algorithm 2. We assume whether a value does not exist in the array A into the cell of some

position i when $A[i] = \text{undefined}$. In addition, we reject immediately when the following comparisons:

$$A[i] < 1 \vee A[i] > 3$$

hold at least into one single binary digit. Note, that every possible literal in ϕ could have a representation by an integer between $-3 \times m$ and $3 \times m$ with the exception of 0, where m is the cardinality of the collection C . In this way, we guarantee the output collection C is an appropriated instance of *OSUM* just filling with zeroes to the left the elements with bit-length lesser than $|3 \times m|$ where $|\dots|$ is the bit-length function. ◀ ◀

Algorithm 2 Logarithmic space disqualifier

```

1: /*A valid instance for 3UNSAT with its disqualification*/
2: procedure DISQUALIFIER( $\phi, A$ )
3:   /*Initialize an index*/
4:    $j \leftarrow 0$ 
5:   /* $m$  is the number of clauses in  $\phi$ */
6:   /*Iterate for the elements of the disqualification array  $A$ */
7:   for  $i \leftarrow 1$  to  $m + 1$  do
8:     if  $i = m + 1$  then
9:       /*There exists an  $m + 1$  element in the array*/
10:      if  $A[i] \neq \text{undefined}$  then
11:        /*Reject the disqualification*/
12:        return "no"
13:      end if
14:      /*Break the for loop*/
15:      break
16:    else if  $A[i] = \text{undefined} \vee A[i] < 1 \vee A[i] > 3$  then
17:      /*Reject the disqualification*/
18:      return "no"
19:    else
20:       $j \leftarrow A[i]$ 
21:    end if
22:    /*From the indexed  $i^{\text{th}}$  clause  $c_i = (x_j \vee y_k \vee z_r)$  in  $\phi$ */
23:    /*Where  $x, y$  and  $z$  are literals with local indexes  $\{j, k, r\} = \{1, 2, 3\}$  in  $c_i$ */
24:    /*Output the integer representation of the  $j^{\text{th}}$  literal, that is  $n(x_j)$ */
25:    /*Filled with zeroes to the left until a total of  $|3 \times m|$  bits including the literal*/
26:    /*But, the bit-length of the symbol minus is ignored in filling the negated literals*/
27:    output " ,  $n(x_j)$ "
28:  end for
29: end procedure

```

3.3 Consequences

► **Theorem 16.** Every problem in *NP* could be *NL*-reduced to another problem in *L*.

Proof. Every *NP*-complete is logarithmic space reduced to $MAX \oplus 2SAT$. Certainly, every *NP* problem could be logarithmic space reduced to *NAE 3SAT* by the Cook's Theorem

algorithm [8]. In addition, the problem $NAE\ 3SAT$ could be logarithmic space reduced to $MAX \oplus 2SAT$ according to Theorem 10. Therefore, this is a direct consequence of Theorems 4, 10 and 11. ◀ ▶

▶ **Theorem 17.** *If $L = NL$ then $L = NP$.*

Proof. Every *coNP*-complete is logarithmic space reduced to $3UNSAT$. Certainly, every *coNP* problem could be logarithmic space reduced to $3UNSAT$ by the Cook's Theorem algorithm [8]. Hence, this is a direct consequence of Theorems 7, 14 and 15. ◀ ▶

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