

FIBONACCI THRESHOLDING, SIGNAL REPRESENTATION AND MORPHOLOGICAL FILTERS

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ABSTRACT

This paper presents a new weighted thresholding concept for the set-theoretical representation of signals and design of new morphological filters. Such representation maps many operations of nonbinary signal and image processing into simple operations over the binary signals and images. The weighted thresholding is invariant relative to the morphological transforms, including the basic ones, erosion and dilation. The main idea of using the weighted thresholding is the right choice of special levels of thresholding for signal and image processing. Fibonacci thresholding is defined and decomposition of the median filter in terms of such thresholding is described.

1 INTRODUCTION

The thresholding is one of the simplest method used in several standard image segmentation techniques, which include the global thresholding, semi-thresholding, multilevel or variable thresholding [1, 2]. The goal of using the thresholding in image segmentation is to detect the regions of interest and remove all other regions from the consideration. Thresholding plays an important role in nonlinear filtering and is useful for set-theoretical representation of nonlinear morphological filters, which simplify the signal and image processing [3]-[9].

In the present paper, we consider a weighed thresholding and new decomposition of the multivalued signals and their medians by Fibonacci levels. For that, we introduce a concept of *multi-valued threshold images* that generalizes the threshold binary images and describe their properties.

2 REPRESENTATION OF SIGNALS

It is well-known that many signals can be represented either by one- or multi-dimensional functions or by combination of "elementary" functions. For $\alpha \in (-\infty, +\infty)$, *horizontal cross section*, or, a *threshold set* $T_\alpha(f)$ of f at the level α is the set of the points x of X such as $f(x) \geq \alpha$, and the binary signal, $f_\alpha(x)$, at that level is the characteristic function of $T_\alpha(f)$, i.e. $f_\alpha(x) = 1$ if $f(x) \geq \alpha$, and $f_\alpha(x) = 0$ otherwise.

We consider upper-semicontinuous functions (u.s.f), that is, functions which horizontal cross sections are closed. Every function f associates uniquely the family of the binary signals f_α , $\{f_\alpha; \alpha \in (-\infty, +\infty)\}$, and, one can reconstruct uniquely an u.s.c. function f by the horizontal cross sections using the following formula (the threshold decomposition of f):

$$f(x) = \sup\{\alpha; T_\alpha(f) \ni x\} = \sum_{\alpha} f_\alpha(x) \quad (1)$$

The binary images f_α are ordered by the relation \leq , i.e. $f_\alpha(x) \leq f_{\alpha+1}(x)$, for all x . The representation (1) with this constrain is the particular case of the general representation

$$f(x) = \sum_{\alpha} f'_\alpha(x) \quad (2)$$

by binary signals f'_α . Therefore, it is natural to raise the following problem: How do we choose the best, or *optimal*, representation in (2), when processing signals or images $f(x)$?

2.1 General threshold superposition

We consider a function $k(\alpha) \geq 1$ being non-decreasing (or, non-increasing) on the set of the non-negative integers $\alpha \geq 1$. Let $f(x)$ be a multi-level non-negative function which takes values of the interval $[0, m]$, $m > 1$. We will use the known notation $Z(m) = \{0, 1, 2, \dots, m\}$.

Definition 1 The threshold m -valued images of the function $f(x)$ at amplitude levels $0 \leq \alpha \leq m$ are respectively

$$\bar{f}_\alpha(x) = k(\alpha)f_\alpha(x) = \begin{cases} k(\alpha), & \text{if } f(x) \geq \alpha \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

The representation $f(x) \rightarrow (\bar{f}_1(x), \bar{f}_2(x), \dots, \bar{f}_m(x))$ is called a *k-weighted thresholding* of $f(x)$.

The function $f(x)$ can be reconstructed by the family of threshold m -valued images as

$$\begin{aligned} f(x) &= \sup\{\alpha; \bar{f}_\alpha(x) \neq 0\} \\ &= \sum_{\alpha=0}^m \bar{f}_\alpha(x)/k(\alpha) \quad x \in R^n \end{aligned} \quad (4)$$

which reminds the well known property of signal decomposition (expansion). The following function can be assigned uniquely to the function $f(x)$:

$$\mathcal{K}_f(x) = \sum_{\alpha=0}^m \bar{f}_\alpha(x), \quad x \in R^n. \quad (5)$$

The operator $\mathcal{K}: f(x) \rightarrow \mathcal{K}_f(x)$ keeps all straight (smooth) parts in the graph of f , increasing its peaks. $\mathcal{K}_f(x) \geq f(x)$ ($k(\alpha) \geq 1$), and the equality takes place in the case when the non-zero values $k(\alpha) \equiv 1$, i.e. when $m = 1$ and $\bar{f}_\alpha(x) = f_\alpha(x)$, for all α .

Example 1 Let $f(x) = \{0, 1, 4, 2, 0\}$. The maximal level of $f(x)$ is 4, and $f(x)$ can be represented by the following two forms, which we call to be a *nonoptimal* and *optimal* representation, respectively,

$$f(x) = \sum_1^4 f_\alpha(x) = \sum_1^3 \bar{f}'_\alpha(x)$$

$$\begin{aligned} \alpha = 1, f_1(x) &= \{0, 1, 1, 1, 0\} & f'_1(x) &= \{0, 1, 1, 1, 0\} \times 1 \\ \alpha = 2, f_2(x) &= \{0, 0, 1, 1, 0\} & f'_2(x) &= \{0, 0, 1, 1, 0\} \times 1 \\ \alpha = 3, f_3(x) &= \{0, 0, 1, 0, 0\} & f'_3(x) &= \{0, 0, 1, 0, 0\} \times 2 \\ \alpha = 4, f_4(x) &= \{0, 0, 1, 0, 0\} \end{aligned}$$

In the first presentation, $f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x)$, and, in the second presentation, the same expression takes place for the function $f'(x) = \{0, 1, 3, 2, 0\}$, i.e. $f(x) = \mathcal{K}_{f'}(x) = f'_1(x) + f'_2(x) + 2f'_3(x)$. This simple example illustrates the replacement of the four-level signal by the three-level signal, which can be written as $f'(x) = f_1(x) + f_2(x) + [f_3(x) + f_4(x)]/2$.

In the general, every n -valued function $f(x)$ can be presented as m -valued function $f'(x)$, by means of the threshold m -valued images, $m < n$,

$$f(x) = \sum_{\alpha=1}^m k(\alpha) f'_\alpha(x) \quad (6)$$

for a certain set of integer numbers $k(\alpha)$, $\alpha = 1, 2, \dots, m$. In the framework of the k -weighted thresholding, $f(x)$ is the function

$$f'(x) = \sum_{\alpha=1}^m f'_\alpha(x) \quad (7)$$

In the capacity of $k(\alpha)$, we can take the numbers of equal binary images of $f(x)$. Then, m is the number of different binary images of $f(x)$ and $k(1) + k(2) + \dots + k(m) = n$. We call the threshold transform $f(x) \rightarrow f'(x)$ a *canonical weighted representation* of $f(x)$. The canonical representation is unique and linear, i.e. if $f \rightarrow f'$, $g \rightarrow g'$, then $f + g \rightarrow (f' + g')'$.

It directly follows from equations (4) and (5), that the mapping $f(x) \rightarrow \mathcal{K}_f(x)$ is invertible and the inversion has a simple form even in the case the mapping is complex. It means, that in many cases one can replace the complicated nonlinear operations by means of the simple summing (in Boolean logic) of the binary signals.

Example 2 Let $k(\alpha)$ be the arithmetic progression, $k(\alpha) = 1, 2, 3, 4, \dots$. For the function $f(x) = \{\dots, 0, 0, 1, 2, 1, 4, 3, 5, 1, 1, 0, \dots\}$, we obtain that $\mathcal{K}_f(x) = \{\dots, 0, 0, 1, 3, 1, 10, 6, 15, 1, 1, 0, \dots\}$.

In the general, for a given function $f(x)$, the function $\mathcal{K}_f(x)$ is a sum of the arithmetical series and equals

$$\mathcal{K}_f(x) = f(x)(f(x) + 1)/2 \quad (8)$$

and the function $f(x)$ is defined from $\mathcal{K}_f(x)$ as

$$f(x) = (-1 + \sqrt{1 + 8\mathcal{K}_f(x)})/2 \quad (9)$$

We obtain the nonlinear (quadratic) transform of the signal and its simple inversion by the weighted thresholding:

$$f(x) \rightarrow \mathcal{K}_f(x) = \frac{f(x)(f(x) + 1)}{2} \rightarrow \sum_{\alpha=0}^m \bar{f}_\alpha(x)/k(\alpha)$$

$f(x)$ is determined by $\mathcal{K}_f(x)$ as the value satisfying the condition $\mathcal{K}_f(x) = 1 + 2 + \dots + f(x)$. Using the numbers of the arithmetic progression $k(\alpha)$, one can also calculate the square $g(x) = f^2(x)$. Indeed, owing to (8) $g(x) = 2\mathcal{K}_f(x) - f(x)$. If we take the coefficients $k(\alpha) = 2a - 1$, $a = 1, \dots, m$, we obtain $g(x) = \mathcal{K}_f(x) = f^2(x)$.

Example 3 Let the function $k(\alpha)$ be the Fibonacci series $k(\alpha) = 1, 1, 2, 3, 5, 8, 13, \dots$, ($\alpha = 1, 2, 3, 4, 5, 6, 7, \dots$). For the function $f(x)$ of Example 2, we have $\mathcal{K}_f(x) = \{\dots, 0, 0, 1, 2, 1, 7, 4, 12, 1, 1, 0, \dots\}$.

The function $\mathcal{K}_f(x)$ being a finite sum of the Fibonacci numbers can be written as

$$\mathcal{K}_f(x) = \sum_{\alpha=0}^{f(x)} k(\alpha) = k(f(x) + 2) - 1 \quad (10)$$

For each value $\mathcal{K}_f(x)$, we can calculate the number α such as $\mathcal{K}_f(x) + 1 = k(\alpha)$. The value $\alpha - 2$ must be the value of $f(x)$, and the following inversion holds

$$f(x) \rightarrow \mathcal{K}_f(x) \rightarrow \mathcal{K}_f(x) + 1 = k(\alpha) \rightarrow \alpha - 2 = f(x)$$

It is clear that, taking different types of coefficients $k(a)$ for the weighted thresholding, we obtain different operations the functions.

Example 4 Let p_a be nonnegative numbers such that $p_1 + p_2 + \dots + p_m = 1$, and the coefficients $k(a) = p_a$, $a = 1, \dots, m$. For instance, $p_1 = 0.4$, $p_2 = 0.2$, $p_3 = 0.2$, $p_4 = 0.1$, $p_5 = 0.1$, or, after multiplication by 10, $k(1) = 4$, $k(2) = 2$, $k(3) = 2$, $k(4) = 1$, $k(5) = 1$. For the function $f(x)$ of Example 2, we have $\mathcal{K}_f(x) = \{\dots, 0, 0, 4, 6, 4, 9, 8, 10, 4, 4, 0, \dots\}$.

Denoting the distribution function $F(1) = p_1$, $F(\alpha) = F(a - 1) + p_a$, $2 \leq a \leq m$, one can write $\mathcal{K}_f(x)$ as

$$\mathcal{K}_f(x) = \sum_{\alpha=0}^{f(x)} k(\alpha) = F(f(x)) \quad (11)$$

If p_a is the probability of the event $\{f(x) = a\}$, the weighted thresholding represents an operation of calculation the distribution of the random function $f(x)$.

3 WEIGHTED THRESHOLDING AND SEGMENTATION

The general method of using the weighted thresholding for processing the signal or image $f(x)$ can be described by the following algorithm.

Algorithm 1: Step 1. Calculate $\mathcal{K}_f(x)$. *Step 2.* Calculate futures (for instance, edges, if we are doing the edge detection, etc.) of $\mathcal{K}_f(x)$, by using the classical or new methods. *Step 3.* Find the corresponding futures in the original signal $f(x)$, inverting them from $\mathcal{K}_f(x)$.

The classical morphological approach for segmentation relates to gradients. We consider gradients which are based on morphological transforms, such as the dilation and erosion operations [3]. Let f be a nonnegative digital signal, let B be any subset of R^n , and let M be a morphological filter such as erosion $f \ominus B$, dilation $f \oplus B$, opening $f \circ B$, and closing, $f \bullet B$, or their linear combination. Then, we have the following

$$Mf = \sum_{\alpha=1}^m \frac{1}{k(\alpha)} M \bar{f}_\alpha \quad (12)$$

When the input gray-level signal is represented as a sum of its m -valued images, the M -filtered signal is represented by the linear combination of M -filtered m -valued images. In the $k(\alpha) \equiv 1$ case, the similar result has been described by Maragos [9], and in the general case the equality in (12) can be proved similarly.

Theorem 1 *Given n -valued function $f(x)$ and a structuring element B ,*

$$\begin{aligned} \mathcal{K}_f \ominus B &= \mathcal{K}_{f \ominus B} & \mathcal{K}_f \oplus B &= \mathcal{K}_{f \oplus B} \\ \mathcal{K}_f \circ B &= \mathcal{K}_{f \circ B} & \mathcal{K}_f \bullet B &= \mathcal{K}_{f \bullet B} \end{aligned} \quad (13)$$

The general image enhancement algorithm via the weighted thresholding and morphological operations can be described in the following way. *Algorithm 2: Step 1.* Choose and fix of $k(\alpha)$. *Step 2.* Calculate $\mathcal{K}_f(x)$. *Step 3.* Calculate erosion (or, dilation) of $\mathcal{K}_f(x)$. *Step 4.* Estimate the gradient. *Step 5.* Calculate the "enhancement" signal via inversion of $\mathcal{K}_f(x)$. The question arises how to choice the set of numbers $k(\alpha)$. Experimental results in image processing show the advantages of using the Fibonacci numbers for the weighted thresholding, when comparing with the arithmetical progression, as well with the canonical representation.

4 FIBONACCI THRESHOLDING

We describe the threshold m -valued images \bar{f}_a in the case, when the factor function $k(a)$ is the Fibonacci series. Let us take the Fibonacci numbers beginning with the second member, $\{u_n\}_{n>1} =$

$\{1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$, for which the recurrent formula $u_n = u_{n-1} + u_{n-2}$ ($u_0 = 1$) holds. For an integer $L \geq 1$, we denote by U_L the set of L first members of the Fibonacci series, i.e. $U_L = \{1, 2, 3, 5, \dots, u_L\}$. A given interval of integers $Z(m)$ can be represented by an union of the shifted sets U_L , i.e.

$$\{1, 2, \dots, m\} = \bigcup_{i \leq n} \{n_i + U_{L_i}\} \quad (14)$$

for certain integers $1 \leq n_i, L_i \leq m$, and $n \geq 1$.

Theorem 2 (Fibonacci representation) Any m -level signal can be represented by sum

$$f(x) = \sum_{i=1}^n u_{g_i(x)}$$

of signals $u_{g_i(x)}$, each of which is composed only by Fibonacci numbers.

So, for the 16-valued signal $f(x)$, it is enough to calculate its binary signals by the first six Fibonacci levels (which compose a signal $g_1(x)$) and three additional binary signals, for the difference $\Delta_1 f = f(x) - u_{g_1(x)}$. Each value of the function $u_{g_1(x)}$ is an element of the Fibonacci series. (As if the function f falls on the Fibonacci levels to compose f_1 .) It needs together 9 binary signals instead of 16 ones when using the traditional representation of $f(x)$ by means of the binary signals $\{f_a; a = 1, 2, \dots, 16\}$. We can continue the mentioned above reasoning and show that every finite-level signal is a sum of signals described by Fibonacci numbers.

Lemma 1 An m -level signal $f(x)$ can be represented as the sum of two signals

$$f(x) = u_{g_1(x)} + \Delta_1 f(x), \quad g_1 = \phi(f), \quad (15)$$

where the signal $\Delta_1 f$ is $(u_{r-3} - 1)$ -level signal, and the integer r is the minimal superior Fibonacci level of the signal $f(x)$, for which $r \geq f(x)$.

Using the reasoning given above for the signal $\Delta_1 f(x)$ instead of the initial one $f(x)$, we obtain the similar expansion

$$\Delta_1 f(x) = u_{g_2(x)} + \Delta_2 f(x), \quad g_2 = \phi(\Delta_1 f), \quad (16)$$

where the second difference $\Delta_2 f = \Delta_1(\Delta_1 f)$. Continue this procedure, we obtain the expansion of the function $f(x)$ in the form of the sum of signals

$$f(x) = u_{g_1(x)} + u_{g_2(x)} + \dots + u_{g_{n-1}(x)} + u_{\Delta_n f(x)}, \quad n \geq 1,$$

each of which is the sequence of Fibonacci series. Since the signal $\Delta_1 f$ is $(u_{r-3} - 1)$ -level, the signal $g_2(x)$ is $(r - 3)$ -level, and the second difference $\Delta_2 f$ is maximum $(u_{r-6} - 1)$ -level, etc. Defining the number n as $r = 3n + l$, $0 \leq l \leq 2$, we can state that the signal $f(x)$

can be represented as n signals $g_1(x), g_2(x), \dots, g_n(x)$, whose levels compose together the following number

$$\begin{aligned} L(m) = L_r &= r + (r-3) + (r-6) + \dots + (r-3n) \\ &= (r+l)(n+1)/2 \end{aligned}$$

The efficiency of the proposed signal representation by means of the corresponding Fibonacci signals g_1, g_2, \dots, g_n grows as the number m of levels becomes big. So, for the 2584-level signals, the efficiency equals $\epsilon_{17} = u_{17}/L_{17} = 46$. In the general, such value equals $\epsilon_r = u_r/L_r$, where u_r can be computed by means of the Buena formula.

5 FIBONACCI FILTERS

Suppose that: $W \subset R^n$, $n \geq 1$, is a finite set (window) and $W_x = \{w + x; w \in W\}$ is the translate window W at a point $x \in R^n$. We denote by *Med* a median filter by W , which maps the function $f(x)$ into such whose value at x is the median among the values $f(y)$, $y \in W_x$. Performance of the median filter over any finite-level signal is equivalent to the decomposition of the signal by binary signals and filtration them separately. Many nonlinear filters and stack filters have similar property. More general property of decomposition takes place for the mixed median filters which include the traditional median and order statistics [10].

The theorem given below shows the advances of using the decomposition of the median filter by Fibonacci levels. It allows to use only a few number of multi-valued signals (images) and shows a natural way to construct an extension of the median-type filters which can be defined by the formula $FILT f(x) = \bigvee_{i=1}^k (n_i + FILT_i u_{g_i(x)})$, where $FILT_i$ are filters defined on sequences consisting of Fibonacci numbers (Fibonacci signals). We shall call *Fibonacci filter* a filter defined by this kind of formula. In particular, when all $FILT_i = Med$, we have Fibonacci filter equal to median one.

Theorem 3 Every M -valued signal $f(x)$ can be represented as a finite union

$$f(x) = \bigvee_{i=1}^k (n_i + u_{g_i(x)})$$

such that the following decomposition is fulfilled

$$Med f(x) = \bigvee_{i=1}^k (n_i + Med u_{g_i(x)})$$

where n_i , $i = 1, \dots, k$, are integers, $k \leq r(M)/3$.

6 CONCLUSION

A new concept of weighted thresholding was introduced, which can be used for signal set-theoretical representation. The main advantage of this kind of representation is that it gives opportunity to implement the nonlinear

operations by weighted threshold operation and enhance a large number of geometrical features that are present in the original signals, manipulating with weights. The weighted thresholding was shown is invariant under the morphological transforms, including the basic ones, erosion and dilation, opening and closing. The particular case of the weighted thresholding, the Fibonacci thresholding, can be used for decomposition of the median and other filters.

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