Order in the particle zoo

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Abstract

The standard model of particle physics classifies particles into elementary leptons and hadrons composed of quarks. There exists an alternate ordering principle based on a function Ψ that may be derived from the framework of the Einstein field equations, giving a convergent series of particle energies, to be quantized as a function of the fine-structure constant, α, with limits given by the energy values of the electron and the Higgs vacuum expectation value. The series expansion of the energy equation provides quantitative terms for Coulomb, strong and gravitational interaction. The value of α can be given numerically by the gamma functions of the integrals involved, extending the formalism to N-dimensions yields a single expression for the electroweak coupling constants.

The model can be expressed without use of free parameters.

1 Introduction

Particle zoo is the informal though fairly common nickname to describe what was formerly known as "elementary particles". The standard model of particle physics [1] divides these particles into leptons, considered to be fundamental "elementary particles" and the hadrons, composed of quarks. Well hidden in the data of particle energies lies another ordering principle based on an exponential function Ψ [1]. Originally the function Ψ was developed in a heuristic "ad hoc" approach to calculate particle energies inspired by basic principles of quantum mechanics and the first part of this article will follow this line. This will yield a convergent series of particle energies to be quantized as a function of α [2]. α can be given numerically by the Γ-functions of the integrals for calculating particle energy in a point charge and a photon expression. The expansion of the incomplete Γ-function appearing in the point charge integral gives quantitative terms for Coulomb, strong and gravitational interaction. The latter provides a link between the electron and the Planck energy, allowing to identify the electron as ground state. The upper limit of the convergent energy series coincides with the Higgs vacuum expectation value. The relation with Planck terms allows to express the equations of the model "ab initio" as function of elementary charge, e, electric constant, ε, and gravitational constant G.

In the second part of this article it will be demonstrated that Ψ may be derived directly from the framework of the Einstein field equations (EFE). However, in 3 dimensional space an additional ad hoc term will be required. An extension of the approach to 5D space-time, i.e. 4D space, starting with a Kaluza-type ansatz, is currently in progress. Preliminary results show that the ad hoc term of 3D is related to solutions for the Kaluza scalar, Φ [4].

The geometric approach for calculating the electromagnetic coupling constant, α, can be extended to different dimensions, yielding three electroweak coupling constants for three point charges g, e and g' in 4, 3 and 2 dimensions. The solutions of the Kaluza scalar and the electroweak boson energies may be assigned to dimensions ≤ 4 as well suggesting a coherent classification in 4D space.

For both approaches it might be helpful to use the following visualization: a photon with its intrinsic angular momentum interpreted as having its E-vector rotating around a central axis of propagation [3] will be transformed to an object that has the -still rotating- E-vector constantly oriented to a fixed point, the origin of the local coordinate system used, resulting in a SO(3) object with point charge properties. The vectors E, B and V of the propagation velocity are supposed to be locally orthogonal and subject to the standard Maxwell equation, however, on the background of an appropriately curved space-time. Neutral particles are supposed to exhibit nodes separating corresponding equal volume elements of opposite polarity.

A particularly compact description of this model may be obtained by using a system of natural electromagnetic units, attributing the value of the speed of light, c_0, to the inverse value of electric and

1 Ψ = f(α, e/(εr)), where the electric potential, ~e/(εr), is supposed to be extendible to the electromagnetic potential A_a in a more general case (e = elementary charge, ε = electric constant, r = radius);

2 The relation of the masses e, μ, π with α was noted in 1952 by Y.Nambu [2]. M.MacGregor calculated particle mass and constituent quark mass as *multiples* of α and related parameters [3].

3 Symmetry SO(2) as projected in propagation direction;

magnetic constant, ε_c and μ_c, with units to be chosen to yield the elementary charge, e_c, in units of energy. Using SI units for length, time and energy this will result in:

$$c_0^2 = (\varepsilon_c \, \mu_c)^{-1} \tag{1}$$

with $\varepsilon_c = (2.998E+8 \, [m^2/Jm] \,)^{-1} = (2.998E+8)^{-1} \, [J/m]$
$\mu_c = (2.998E+8 \, [Jm/s^2] \,)^{-1} = (2.998E+8)^{-1} \, [s^2/Jm]$

From the Coulomb term $b_0 = e^2/(4\pi\varepsilon_0) = e_c^2 / (4\pi\varepsilon_c) = 2.307E\text{-}28$ [Jm] follows for the square of the elementary charge: $e_c^2 = 9.671E\text{-}36$ [J^2]. In the following $e_c = 3.110E\text{-}18$ [J] and $e_c/\varepsilon_c = 9.323E\text{-}10$ [m] may be used as natural unit of energy and length. Replacing the constant G/c_0^4 in the EFE by $1/\varepsilon_c$ will give a correct *absolute* scale for particle energies, see chpt. 3, and the necessary parameters of the model will be further reduced to e_c and ε_c.

To focus on the more fundamental relationships some minor aspects of the model are exiled to an appendix, related topics to be marked as [A]. The model may be used to calculate additional particle properties, see [5]. Typical accuracy of the calculations presented is in the order of 0.001 [4]. QED corrections are not considered in this model.

2 Ad hoc approach

2.1 Energy terms

The model may essentially be based on a single assumption:

Particles can be described by using an appropriate exponential wave function, Ψ(r), that acts as a probability amplitude on an electromagnetic field.

An appropriate form of Ψ can be deduced from three boundary conditions:

1.) To be able to apply Ψ to a point charge $\Psi(r = 0) = 0$ is required.

2.) To ensure integrability an integration limit is needed.

3.) Ψ should be applicable regardless of the expression chosen to describe the electromagnetic object. In particular requiring a point charge and a photon representation of a localized electromagnetic field (particle) to have the same energy results in an exponent of 3 for r in the equation below (see 2.2).

Condition 1.) to 3.) are met by an expression (corresponding differential equation see [A1]):

$$\Psi_n(r) = \exp\left(-\left(\frac{\beta_n/2}{r^3} + \left[\left(\frac{\beta_n/2}{r^3}\right)^2 - 4\,\frac{\beta_n/2}{\sigma\, r^3}\right]^{0.5}\right)/2\right) \tag{2}$$

Up to the limit of the real solution, $r = r_n$, with

$$r_n = (\sigma\, \beta_n/8)^{1/3} \tag{3}$$

in all integrals over Ψ(r) given below equ. (4) may be used as approximation for (2):

$$\Psi_n(r \le r_n) \approx \exp\left(\frac{-\beta_n/2}{r^3}\right) \tag{4}$$

Phase will be neglected on this approximation level, properties of particles will be calculated by the integrals over $\Psi_n(r)^2$ [5] times some function of r which can be given by:

$$\int_0^{r_n} \Psi_n(r)^2 r^{-(m+1)} dr \approx \int_0^{r_n} \exp(-\beta_n/r_n^3) r^{-(m+1)} dr = \Gamma(m/3, \beta_n/r_n^3)\,\frac{\beta_n^{-m/3}}{3} = \int_{\beta_n/r_n^3}^{\infty} t^{\frac{m}{3}-1} e^{-t}\, dt\,\frac{\beta^{-m/3}}{3} \tag{5}$$

with m = {..-1;0;1;..}. The term $\Gamma(m/3, \beta/r_l^3)$) denotes the upper incomplete gamma function, given by the Euler integral of the second kind with $\beta_n/r_n^3 = 8/\sigma$ as lower integration limit [6]. For $m \ge 1$ the complete gamma function $\Gamma_{m/3}$ is a sufficient approximation, for $m \le 0$ the integrals have to be integrated numerically.

Coefficient β_n may be given as partial product of a value for a ground state particle, β_{GS}, carrying a dimensional term, β_{dim} [m^3], that will be demonstrated to have a particular useful expression using the unit system defined in chpt. 1 as [see A3]:

4 Including e.g. errors due to the numerical approximation of Γ-functions;

5 Hence factor 2 in (2)ff

6 Euler integrals yield positive values, the absolute sign used for e.g. $|\Gamma_{-1/3}|$ is due to the sign convention of Γ-functions.

$$\beta_{dim} = \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 = 5.131\text{E-}30\ [\mathrm{m}^3] \quad (6)$$

times particle specific dimensionless coefficients, α_n, of succeeding particles representing the ratio β_{n+1} / β_n:

$$\beta_n = \beta_{GS}\Pi_{k=1}^{n}\alpha_k = 2\sigma\alpha_{GS}\beta_{dim}\Pi_{k=1}^{n}\alpha_k = 2\sigma\alpha_{GS}\beta_{dim}\Pi_{\beta,n} \qquad n = \{1;2;..\} \quad (7)$$ [7]

Index n will indicate solutions of (2) and serve in the following as equivalent of a radial quantum number. For the angular terms of $\Psi(r, \vartheta, \varphi)$, to be indicated by index l, only rudimentary results exist, their contribution has to be incorporated in parameter σ (to be discussed in 2.4 - 2.6).

Particle energy is expected to be equally divided into electric and magnetic part, $W_n = 2W_{n,el} = 2W_{n,mag}$. To calculate energy the integral over the electrical field E(r) of a point charge is used, equ. (5) for m = 1 gives:

$$W_{pc,n} = 2\varepsilon_0\int_0^\infty E(r)^2\Psi_n(r)^2 d^3r = 2b_0\int_0^{r_n}\Psi_n(r)^2 r^{-2}dr = 2\, b_0\, \Gamma(1/3, \beta_n/r_n^3)\, \beta_n^{-1/3}/3 \approx 2\, b_0\, \Gamma_{1/3}\, \beta_n^{-1/3}/3 \quad (8)$$

Using equation (5) for m = -1 to calculate the Compton wavelength, λ_C, gives:

$$\lambda_{C,n} \approx \int_0^{\lambda_{C,n}}\Psi_n(r)^2 dr = \int_{\beta/\lambda_{C,n}^3}^{\infty} t^{-4/3}e^{-t}\,dt\ \ \beta_n^{1/3}/3 = \Gamma(-1/3, \beta_n/\lambda_{C,n}^3)\, \beta_n^{1/3}/3 \approx 36\,\pi^2\, |\Gamma_{-1/3}|\, \beta_n^{1/3}/3 \quad (9)$$ [8]

to be used in in the expression for the energy of a photon, hc_0/λ_C:

$$W_{phot,n} = hc_0/\lambda_{C,n} = \frac{hc_0}{\int^{\lambda_{C,n}}\Psi_n(r)^2\,dr} \approx \frac{3hc_0}{36\pi^2|\Gamma_{-1/3}|\beta_n^{1/3}} \quad (10)$$

It should be noted that in both equations (8) and (10) the length $\beta_n^{1/3}$ is the only variable parameter. The dimensionless constants in the equations are π and the Γ-functions. In particular $|\Gamma_{-1/3}|$, as coefficient representing length, and the combination of these constants in form of coupling constants will be of importance in the following.

2.2 Fine-structure constant, α

The energy of a particle is assumed to be the same in both photon and point charge description. Equating (8) with (10) and rearranging to emphasize the relationship of α with the gamma functions ($\Gamma_{1/3} = 2.679$; $|\Gamma_{-1/3}| = 4.062$) gives as first approximation (note: h => ħ):

$$\frac{4\pi\Gamma_{1/3}|\Gamma_{-1/3}|}{0.998} = \frac{9hc_0}{18\pi b_0} = \frac{\hbar c_0}{b_0} = \alpha^{-1} \quad (11)$$

In (11) $\Gamma_{1/3}$ represents the limit of $\Gamma(1/3, \beta_n/r_n^3)$ for the lower bound of integration in the Euler integral approaching zero, $\beta/r^3 \to 0$. Using the analog limit for $\Gamma(-1/3, \beta_n/\lambda_{C,n}^3)$ that may approximated by:

$$\Gamma(-1/3, \beta_n/r_x^3) = \int_{\beta_n/r_x^3}^{\infty} t^{-4/3}e^{-t}dt \approx 3\,(\beta_n/r_x^3)^{-1/3} \quad (12)$$

gives a more precise expression depending on $\lambda_{C,n}$ and β_n:

$$\frac{\Gamma_{1/3}\lambda_{C,n}}{3\pi\beta_n^{1/3}} = \alpha^{-1} \quad (13)$$

With (13) the precision for the calculation of α is identical to the precision for calculating particle energy with the respective β_n, e.g. with β_e of (63): $\alpha_{calc} = 1.0001\ \alpha$.

In chapter 4 it will be demonstrated that this formalism may be extended to other than three spatial dimensions to give a general expression for electroweak coupling constants [9].

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7 $\Pi_{\beta,n}$ denoting the sum of all particle coefficients in the partial product for β_n except for the ground state particle (electron), related to the equivalent factor $\Pi_{W,n}$ in the energy expression (21) by $\Pi_{\beta,n} = \Pi_{W,n}^{-3}$ Factor 2 see note 5;

8 Factor ≈ 355 ≈ 36 π^2 may be calculated numerically from the Euler integral (5) for m = -1, using β_n of (19), (63) or from a fit of particle energy and angular momentum.

9 As with all calculations in this work the calculation for coupling constants refers to a rest frame and thus corresponds to an IR limit. The geometric character of the "constants" implies that their values are subject to relativistic effects in other reference frames.

2.3 Quantization with powers of $1/3^n$ over α

Inserting (7) in the product of the point charge and the photon expression of energy, (8) and (10), gives for $W_n^2 = W_{pc,n}\, W_{phot,n}$

$$W_n^2 = 2 b_0 hc_0 \frac{\int_{\lambda_{C,n}}^{r_n} \Psi_n(r)^2 r^{-2} dr}{\int \Psi_n(r)^2 dr} \sim \frac{1}{\beta_n^{2/3}} \sim \frac{\alpha_1^{1/3}\alpha_2^{1/3}.....\alpha_n^{1/3}}{\alpha_1\alpha_2....\alpha_n} \qquad (14)$$

The last expression of (14) is obtained by expanding the product $\Pi_{\beta,n}^{-2/3}$ included in $\beta_n^{-2/3}$ of (7) with $\Pi_{\beta,n}^{1/3}$. The only non-trivial solution for W_n^2 where all intermediate particle coefficients cancel out and W_n becomes a function of coefficient α_1 only is given by a relation $\alpha_{n+1} = \alpha_n^{1/3}$. Identifying α_1 as $\alpha_1 = \alpha_\mu = \alpha^3$ would give an expression using the muon as reference state:

$$\Pi_{k=1}^{n}\left(\frac{\alpha\wedge(3/3^k)}{\alpha\wedge(9/3^k)}\right) = \frac{\alpha\wedge(3/3^n)}{\alpha^3} \qquad n = \{1;2;..\}\ ^{10} \qquad (15)$$

and with its root

$$\left(\frac{\alpha\wedge(3/3^n)}{\alpha^3}\right)^{0.5} = \frac{\alpha\wedge(1.5/3^n)}{\alpha^{1.5}} = \Pi_{k=1}^{n}\alpha\wedge(-3/3^k) \qquad n = \{1;2;..\} \qquad (16)$$

the corresponding term for particle energies would be:

$$W_n = W_\mu\ \Pi_{k=2}^{n}\alpha\wedge(-3/3^k) \qquad n = \{2;3..\}\ ^{11} \qquad (17)$$

for spherical symmetry.

In an equation of type (17) no state is singled out in particular as a ground state in the equations. The partial product of (17) may be extended to include the electron by inserting *ad hoc* an additional factor ≈ 3/2 to represent an irregularity due to the energy ratio of e, μ, $W_\mu/W_e = 1.5088\ \alpha^{-1}$ (see 2.4, [A2]). In chpt. 2.8 it will be demonstrated that a fundamental relationship exists between the electron and the Planck energy [12], implying the electron to correspond to a ground state term. With the ratio of electron and Planck energy given as:

$$W_e/W_{Pl} = 4.903\text{ E-22} = \alpha_o\ , \qquad (18)$$

β_{GS} of the ground state, the electron, can be approximated in a particular simple expression:

$$\beta_{GS} = \beta_e = \sigma^*\alpha_0\ \beta_{dim} = \frac{\sigma_0^*\alpha_0}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 = 1.286\text{E-43 [m}^3] \quad ^{13} \qquad (19)$$

With W_e as ground state W_n would be given by (14) as [14] (n = {1;2;...}):

$$W_n = \frac{3}{2}\left(\frac{4\pi b_0^2}{\alpha}\ \frac{\int_{\lambda_{C,n}}^{r_n} \Psi_n(r)^2 r^{-2} dr}{\int \Psi_n(r)^2 dr}\right)^{0.5} = \frac{3}{2}\left(\frac{4 b_0^2\Gamma_{1/3}^2}{9[\alpha 4\pi\,\Gamma_{1/3}|\Gamma_{-1/3}|]\beta_n^{2/3}}\right)^{0.5} = \frac{3}{2}\left(\boldsymbol{\frac{2 b_0\Gamma_{1/3}(4\pi)^{2/3}}{3\ (\sigma^*\alpha_0)^{1/3}}\left(\frac{\varepsilon_c}{e_c}\right)}\right)\frac{\alpha\wedge(1.5/3^n)}{\alpha^{1.5}} \qquad (20)$$

and with adding a factor y_l^m for the contribution of non-spherical symmetric states [15] as:

$$W_n/W_e \approx 3/2(y_l^m)^{-1/3}\ \frac{\alpha\wedge(1.5/3^n)}{\alpha^{1.5}} = 3/2(y_l^m)^{-1/3}\Pi_{k=1}^{n}\alpha\wedge(-3/3^k) = 3/2(y_l^m)^{-1/3}\Pi_{W,n} \qquad n = \{1;2;..\}\ ^{16} \qquad (21)$$

see table 1.

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10 For illustration purposes with n = 4: $\frac{\alpha^1\alpha^{1/3}\alpha^{1/9}\boldsymbol{\alpha^{1/27}}}{\boldsymbol{\alpha^3}\alpha^1\alpha^{1/3}\alpha^{1/9}} = \frac{\boldsymbol{\alpha^{1/27}}}{\boldsymbol{\alpha^3}}$

11 Series starts with n=2 since n=1, i.e. α-coefficient of μ already included in W_μ.

12 as defined by (33);

13 In the following subscript 0 in σ_0 will refer to spherical symmetry, l = 0. $\sigma_0^* = \sigma_0/1.5133^3$, see 2.4, [A2,3].

14 factor 3/2 added ad hoc; expanding by $\Gamma_{1/3}$ and using (11) to eliminate term in square brackets; W_e given in bold; With equ. (33) for W_{Pl} this will give $W_e = 0.997\ W_{e,\,exp}$

15 $y_l^m = \int\int \Psi(\varphi,\vartheta)^2 \sin(\vartheta) d\varphi\, d\vartheta\ /\ 4\pi$ holds for l = 1; a general term is not known yet.

16 Implying a coefficient for the electron in W of $\alpha_{W,e} \approx 2/3\,\alpha^{-3}$ and in β_n of $\alpha_{\beta,e} \approx (3/2)^3\alpha^9$;

2.4 Angular momentum, coefficients σ and α

A simple relation with angular momentum J for spherical symmetric states will be given by applying a semi-classical approach using

$$J = r_2 x\, p(r_1) = r_2 W_n(r_1)/c_0 \tag{22}$$

with $W_{kin,n}$=1/2 W_n , using term $2b_0$ of (8) as constant factor, integrating over a circular path of radius $|r_2| = |r_1|$ and setting r_n of (3), $8/\sigma_0$ according to (27) as integration limits. This will give:

$$|J| = \int_0^{r_n}\int_0^{2\pi} J_n(r)\, d\varphi\, dr = 4\pi \frac{b_0}{c_0} \int_0^{r_n} \Psi_n(r)^2 r^{-1} dr \tag{23}$$

From (5) follows for m = 0:

$$\int_0^{r_n} \Psi_n(r)^2 r^{-1} dr = 1/3 \int_{8/\sigma}^{\infty} t^{-1} e^{-t} dt = \frac{\alpha^{-1}}{8\pi} \approx 5.45 \approx \Gamma_{1/3}|\Gamma_{-1/3}|/2 \tag{24}$$ [17]

Inserting (24) in (23) gives:

$$|J| = 4\pi \frac{b_0}{c_0} \frac{\alpha^{-1}}{8\pi} = 1/2\ [\hbar] \tag{25}$$

Analyzing the components of σ_0, in addition to the mandatory term for length of $|\Gamma_{-1/3}|\ \beta_n^{1/3}/3$ of the integrals (5) for m = -1, r_n and σ_0 contain a factor ≈1.51 α^{-1}, very close to the ratio W_μ/W_e = 206.8 = 1.5088 α^{-1}. The exact value of 1.5133 for ≈1.51 has been chosen due to a geometrical interpretation of the terms in σ_0 [18]:

$$1.51\ \alpha^{-1} |\Gamma_{-1/3}|/3 \approx |\Gamma_{-1/3}|/\Gamma_{1/3}\ 4\pi |\Gamma_{-1/3}|\ \Gamma_{1/3}/0.998\ |\Gamma_{-1/3}|/3 \approx \frac{4\pi|\Gamma_{-1/3}|^3}{3} = (\sigma_0/8)^{1/3} \tag{26}$$ [19]

The various useful terms for σ may be summed up as:

$$\sigma_0 = 8\, r_n^3/\beta_n = (1.5133\ \alpha^{-1}\ 2/3\ |\Gamma_{-1/3}|)^3 = 1.5133^3\ \sigma^* = 8\left(\frac{4\pi|\Gamma_{-1/3}|^3}{3}\right)^3 = 1.772\text{E+8}\ [-] \tag{27}$$

2.5 Upper limit of energy

According to the geometrical interpretation given in 2.4 non-spherical particles should exhibit lower values of σ (and r_n). The variable part in σ is given by the term $(1.5133\ \alpha^{-1})^3$, leaving the minimum for σ, defined by $(2/3|\Gamma_{-1/3}|)^3$, i.e. the term in the integral expression for r, and the integers in the square bracket of equ.(2) to be:

$$\sigma_{min} = (2/3\ |\Gamma_{-1/3}|)^3 \tag{28}$$

The maximum angular contribution to W_{max} would be :

$$\Delta W_{max,\ angular} = 1.5133\ \alpha^{-1} \tag{29}$$

The limit of the partial product in (21) for a given l is $\alpha^{-1.5}$, the limit term of ≈ 3/2 by 1.5066 [A2], thus according to (21) and (29), the maximum energy will be W_{max} = W_e 1.5066*1.5133 $\alpha^{-2.5}$ = 4.103E-8 [J] (=1.041 Higgs vacuum expectation value, VEV = 246GeV=3.941E-8 [J] [6]).

In the simple visualization sketched in the introduction the "rotating E-vector" might be interpreted to cover the whole angular range in the case of spherical symmetric states while an object with one angular node, as represented by the spherical harmonic Y_1^0 or an atomic p-orbital, might be interpreted as forming a double cone. Increasing the number of angular nodes would close the angle of the cone leaving in the angular limit case, l -> ∞, a state of minimal angular extension representing the original vector, however, extending in both directions from the origin and featuring parity p= -1. Considering only „half" such a state, extending in one direction only and having p = +1, would notably feature an energy of 1.024 W_{Higgs}, suggesting the energy value of the Higgs boson as possible high energy end for particle energy of (21). From (28) follows that such

17 The integral in (24) may be calculated numerically. However, obviously, to obtain J=1/2 the integral in J = 4π α ħ ∫$\Psi^2 r^{-1}$dr, (23), *must* yield $\alpha_e^{-1}/8\pi \approx \Gamma_{1/3}|\Gamma_{-1/3}|/2$.

18 An additional reason is a 3rd power relationship between 1.5088 and 1.5133, (see [A2,3]), resulting in factor 1.5133 being also part of a minor term depending on the radial quantum number, n. Thus in the following β_n may be split into σ* = σ/1.5133^3 = 5.112E+7 [-] and α(n)-terms containing factor 1.5133^3.

19 The term $4\pi|\Gamma_{-1/3}|^3/3$ is used for σ_0 in all calculations. In chpt. 4, [A7] it will be demonstrated that $\alpha_g = \alpha_{weak}$ can be calculated using an equivalent 4D term.

a particle includes a term $|\Gamma_{-1/3}|/3$, i.e. the characteristic coefficient representing length, in the denominator of the energy expression to be referred to in chpt. 4.2.

	n, l	$W_{n,Lit}$ [MeV]	α-coefficient (energy) equ (21)	α-coefficient in ß equ (7)	W_{calc}/W_{Lit}	J	r_n [fm]
Planck	(-1,∞)	1.0 E+21*	$(2/3\ \alpha^{-3})^3\ 3/2\ \alpha^{-1}\ 2$ source term, relative to e !		0.9994 rel. to e !	-	-
e^{+-}	**0, 0**	**0.51**	**$2/3\ \alpha^{-3}$**	$(3/2)^3$ **α^9**	**1.0001**	**1/2**	**1412**
μ^{+-}	**1, 0**	**105.66**	**$\alpha^{-3}\alpha^{-1}$**	**$\alpha^9\alpha^3$**	**1.0001**	**1/2**	**6.83**
π^{+-}	1, 1	139.57	$\alpha^{-3}\alpha^{-1}\ 3^{1/3}$	$\alpha^9\alpha^3/3$	1.0919	0	4.74
K		495	see [A5]	see [A5]		0	
η^0	**2, 0**	**547.86**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}$**	**$\alpha^9\alpha^3\alpha^1$**	**0.9934**	**0**	**1.32**
ρ^0	2, 1	775.26	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1/3$	1.0124	1	0.92
ω^0	2, 1	782.65	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1/3$	1.0029	1	0.92
K*		894				1	
p^{+-}	**3, 0**	**938.27**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}$**	**1.0017**	**1/2**	**0.76**
n	**3, 0**	**939.57**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}$**	**1.0004**	**1/2**	**0.76**
η'		958	see [A5]	see [A5]		0	
Φ^0		1019	see [A5]	see [A5]		1	
Λ^0	**4, 0**	**1115.68**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}$**	**1.0107**	**1/2**	**0.63**
Σ^0	**5, 0**	**1192.62**	**$\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}\alpha^{-1/81}$**	**$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}\alpha^{1/27}$**	**1.0047**	**1/2**	**0.61**
Δ	**∞, 0**	**1232.00**	**$\alpha^{-9/2}$**	**$\alpha^{27/2}$**	**1.0026**	**3/2**	**0.59**
Ξ		1318				1/2	
Σ^{*0}	3, 1	1383.70	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}/3$	0.9797	3/2	0.53
Ω^-	4, 1	1672.45	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}/3$	0.9725	3/2	0.45
N(1720)	5, 1	1720.00	$(\alpha^{-3}\alpha^{-1}\alpha^{-1/3}\alpha^{-1/9}\alpha^{-1/27}\alpha^{-1/81})\ 3^{1/3}$	$\alpha^9\alpha^3\alpha^1\alpha^{1/3}\alpha^{1/9}\alpha^{1/27}/3$	1.0047	3/2	0.43
tau^{+-}	∞, 1	1776.82	$(\alpha^{-9/2})\ 3^{1/3}$	$\alpha^{27/2}/3$	1.0025	1/2	0.40
Higgs	∞,∞ **	1.25 E+5	$(\alpha^{-9/2})\ 3/2\ \alpha^{-1}/2$	$(\alpha^{27/2})/(3/4\ \alpha^{-1})^3$	1.0230	0	0.006
VEV	∞,∞ **	2.46 E+5	$(\alpha^{-9/2})\ 3/2\ \alpha^{-1}$	$(\alpha^{27/2})/(3/2\ \alpha^{-1})^3$	1.04	0	0.003

Table 1: Particle energies for $\mathbf{y_0^0}$ **(bold)**, y_1^0 [20]; col. 2: radial, angular quantum number; col. 3: energy values of [7] except* (see (33)); col. 4: α-coefficient according to the energy terms of (21), *including (2/3) α^{-3} of electron*; col. 5: coefficients in β_n of (7); col. 6: W_{calc} calculated using the slightly more precise [A3 (62)f] in place of (21); ** see 2.5;
Blanks in the table are discussed in [A4].

2.6 Other non-spherical symmetric states

Except for the limit case of 2.5 angular solutions for particle states are not known yet and to extend the model to such states assumptions have to be made.

Assuming the angular part of Ψ to be related to spherical harmonics and exhibiting the corresponding nodes would give the analog of an atomic p-state for the 1st angular state, Y_1^0. With the additional assumption that $W_{n,l} \sim 1/r_{n,l} \sim 1/V_{n,l}^{1/3}$ (V = volume) is applicable for non-spherically symmetric states as well, this would give $W_1^0/W_0^0 = 3^{1/3} = 1.44$. A second partial product series of energies corresponding to these values approximately fits the data, see tab. 1.

A change in angular momentum has to be expected for a transition from Y_0^0 to Y_1^0 which is actually observed with $\Delta J = \pm 1$ except for the pair μ/π with $\Delta J = 1/2$.

20 up to Σ'^0 all resonance states given in [7] as **** included; Exponents of -9/2, 27/2 for Δ and tau are equal to the limit of the partial products in (7) and (21); r_n calculated with (3); 1.5133 approximated by 3/2;

2.7 Expansion of the incomplete gamma function $\Gamma(1/3, \beta_n/r^3)$

The series expansion of $\Gamma(1/3,\beta_n/r^3)$ in the equation for calculating particle energy (8) gives [8]:

$$\Gamma(1/3,\ \beta_n/(r^3)) \approx \Gamma_{1/3} - 3\left(\frac{\beta_n}{r^3}\right)^{1/3} + \frac{3}{4}\left(\frac{\beta_n}{r^3}\right)^{4/3} = \Gamma_{1/3} - 3\frac{\beta_n^{1/3}}{r} + \frac{3}{4}\frac{\beta_n^{4/3}}{r^4} \quad (30)$$

and for $W_n(r)$:

$$W_n(r) \approx W_n - 2b_0\frac{3\beta_n^{1/3}}{3\beta_n^{1/3}r} + 2b_0\frac{3}{4}\frac{\beta_n^{4/3}}{3\beta_n^{1/3}r^4} = W_n - \frac{2b_0}{r} + b_0\frac{\beta_n}{2r^4} \quad (31)$$ [21]

The 2nd term in (31) drops the particle specific factor β_n and gives twice [22] the electrostatic energy of two elementary charges at distance r. The 3rd term is an appropriate choice for the 0th order term of the differential equation [A1]. It is thus supposed to be responsible for the localized character of a particle state and may be identified with the "strong force" of the standard model.

2.8 Gravitation

2.8.1 Planck scale

Expressing energy/mass in essentially electromagnetic terms suggests to test if mass interaction i.e. gravitational attraction can be derived from the corresponding terms. Assuming the expansion of the incomplete Γ-function for the integral over r^{-2}, $\Gamma(1/3,\beta_n/r^3)$ (30)f, to be an adequate starting point for gravitational attraction as well, implies that the Coulomb term b_0 will be part of the expression for F_G, i.e. the ratio between gravitational and Coulomb force, e.g. for the electron, $F_{G,e}/F_{C,e}$ = 2.41E-43, should be a completely separate, self-contained term.

This is equivalent to assume that gravitational interaction is a higher order effect with respect to electromagnetic interaction and as such should be of less or equal strength compared to the latter. This suggests to use the expression

$$b_0 = G\, m_{Pl}^2 = G\, W_{Pl}^2/c_0^4 \quad (32)$$

as definition for Planck terms, giving for the Planck energy, W_{Pl}:

$$W_{Pl} = c_0^2\,(b_0/G)^{0.5} = c_0^2\,(\alpha h c_0/G)^{0.5} = 1.671\ \text{E+8 [J]} \quad (33)$$

Equation (33) allows to give the quantitative relationship used in (18) for the ratio of W_e and W_{Pl} as:

$$1.0006\frac{W_e}{W_{Pl}} = 1.5133^2\alpha^{10}/2 = 4.903\ \text{E-22} = \alpha_o \quad (34)$$

i.e. the relation between the electrostatic part of $W_{e,\,elst} = W_e/2$ and the electrostatically defined W_{Pl} is given by the cube of the electron coefficient for energy (see note 16) times the angular limit factor according to (29). In the next chapter a derivation will be given for this relation originating in the third term of the energy expansion (31).

Using (61) to express factor 1.5133 gives:

$$\left(\frac{W_e}{W_{Pl}}\right)^2 = \left(\frac{F_{G,e}}{F_{C,e}}\right)_{calc} \approx \left(\frac{1.5133^3\alpha^9}{1.5133\,\alpha^{-1}2}\right)^2 = \left(\frac{(4\pi)^2|\Gamma_{-1/3}|^4\alpha^{12}}{2}\right)^2 = 1.00075^2\left(\frac{F_{G,e}}{F_{C,e}}\right)_{\exp} = \frac{G W_e^2}{c_0^4 b_0} = \alpha_0^2 \quad (35)$$

Using (11) and (64) for calculating W_e would turn G into a coefficient based on electromagnetic constants:

$$G_{calc} \approx \frac{c_0^4}{4\pi\varepsilon_c}\left(\frac{1}{3\pi^{2/3}}\left(\frac{|\Gamma_{-1/3}|}{\Gamma_{1/3}}\right)^4\alpha^{12}\right)^2 \approx \frac{c_0^4}{4\pi\varepsilon_c}\,\frac{2}{3}\,\alpha^{24} = 1.0008\,G_{\exp} \quad (36)$$

2.8.2 Virtual superposition states

Within this model particles might interact via direct contact in place of boson-mediated interaction. The particles are not expected to exhibit a rigid radius. Within the limits of charge and energy conservation a superposition of many states might be conceivable, extending the particle in space with radius ~ r_n, $\lambda_{C,n}$ etc. appropriate for energy of each virtual particle state (VS) [23], providing a source of energy at a distance r_{VS}

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21 Signs not adapted to conventional definition.

22 Due to the assumption used in (8): $W_n = 2W_{n,el} = 2W_{n,mag} = W_{n,el} + W_{n,mag}$

23 The superposition states considered here would be not virtual in a Heisenberg sense, the energy is provided by the

from the primary particle and in turn contributing to the stress-energy tensor responsible for curvature of spacetime that manifests itself in gravitational attraction.

Virtual states are not supposed to consist of analogs of e.g. spherical symmetric states covering the complete angular range of 4π but to be an instantaneous, short term extension of the E-vector thus requiring the angular limit factor of (29).

A long range effect of the 3rd, the strong interaction term, of (31) may be exerted via virtual particle states. To estimate such an effect in first approximation the following will be used:

- the 3rd term of the energy expansion equ. (31) with β according to (7), (19),
- the angular limit state of σ^*_{min} according to (28), $\sigma^*_{min} \approx 1$,
- $\beta_{dim} = (4\pi)^{-2}\,(e_c/\varepsilon_c)^3 \approx (\alpha^{-1}\, r_e)^3$, which might be considered to represent the cube of a natural unit of length, R.

For any VS at $r = \alpha^{-1}\ r_{VS} = \Pi_{\beta,VS}^{1/3}\ (\alpha^{-1}\, r_e)$, i.e. the radius of the VS in natural units, R_{VS}, equ. (37) will hold:

$$W_{VS}(r) \approx \frac{b_0 \beta_{VS}/2}{(\alpha^{-1}\ r_{VS})^4} \approx \frac{b_0 \alpha_0 \Pi_{\beta,VS}(\alpha^{-1} r_e)^3}{(\alpha^{-1} r_{VS})^3 (\alpha^{-1} r_{VS})} \approx \frac{b_0 \alpha_0 \Pi_{\beta,VS}(\alpha^{-1} r_e)^3}{(\Pi_{\beta,VS}^{1/3} \alpha^{-1} r_e)^3 (\alpha^{-1} r_{VS})} = \frac{b_0 \alpha_0}{(\alpha^{-1} r_{VS})} = \frac{b_0}{R_{VS}} \left(\frac{F_{G,e}}{F_{C,e}}\right)^{0.5} \quad (37)$$

[24]

Considering that the composition of the stress-energy tensor from virtual states is expected to be based on a much more complex mechanism requiring consideration of all possible virtual states at a particular point and appropriate averaging, (37) has to be a first approximation. The crucial factor that turns the r^{-4} dependence of the strong interaction term into r^{-1} of gravitational interaction is the proportionality of β_n to the cube of any characteristic particle length, r_n, $\lambda_{C,n}$ etc. which is valid for each particle state subject to the relations of this model.

Equ. (37) is a representation of the gravitational energy of the electron, terms for other particles may be obtained by inserting their energy values relative to the electron according to (21) in (37) which might be interpreted as the intensity/frequency of the emergence of virtual states being proportional to the energy of the primary particle.

As a consequence of (37) the highest possible particle energy value will be α_0^{-1}, i.e. the value of the Planck energy relative to the electron. This is the fundamental cause for relation (34) and in turn corroborates the assumption used in the definition of equ. (32)f.

3 Derivation from the Einstein field equation

The quantitative relationship of the model for calculating particle energies with gravitational interaction via a mechanism that provides energy at a distance from a primary particle and thus a contribution to the stress-energy-tensor and curvature of space-time suggests to test if the equations of this model may be derived directly from the Einstein field equations.

The minute factor G/c_0^4 in the EFE is responsible for this equation not being particularly suited to attempt a calculation of particle energies based on this formalism. The interpretation of gravitation as a higher order effect with respect to electromagnetism suggests to replace G/c_0^2 [m/kg] or G/c_0^4 [m/J] by an equivalent electromagnetic term. A term of order $1/\varepsilon_c$ [m/J] may provide the appropriate units and the necessary order of magnitude, suggesting to use a substitution such as:

$$(8\pi) G/c_0^4 \quad => \quad \approx \frac{4\pi}{\varepsilon_c} \quad (38)$$

In the following the central concept will be the vizualisation of the “rotating E-vector” of the introduction.

The basic question will be: What kind of metric will yield an undisturbed photon propagation according to the Maxwell equations that manifests itself as a localized object in flat space-time ?

In a spherical coordinate system the rotation of an object with extension in angular direction will result in some kind of self interaction increasing with r ->0 *unless* space(-time) is curved in such a way as to prevent that. This will be the case if the r^2-term in the angular coordinates is canceled, implying positive curvature

primary particle.

24 The term for gravitational attraction, $F_{m,n;R}$ between two particles, m and n at a distance $r_{m,n}$, would be obtained by using $1/b_0$ as proportionality constant: $F_{m,n;R} \approx \frac{1}{b_0}\ W_{VS(m,r)} W_{VS(n,r)} \approx b_0\ \frac{\Pi_{W,m}\Pi_{W,n}}{R_{m,n}^2}\ \alpha_0^2$

and an expansion of curved space-time with r^2 at any given r, i.e. the Ricci scalar should be $R(r) \sim -1/r^2$.

The general approach will be to set all terms in an appropriately constructed Ricci scalar to be zero, except a $1/r^2$ component, thus obtaining a homogenous 2nd order differential equation.

In a simple 4-D metric of type $g_{\mu\nu} = (+1, -1, -r^2, -r^2 \sin^2\theta)$ [25] a factor of -1 arises in the Ricci components R_{22} and R_{33} due to the derivative of the term $\Gamma_{23}{}^3 = \Gamma_{32}{}^3 = \cot\theta$ with respect to θ, resulting in in a term $+1/r^2$ in the Ricci scalar. Changing the sign of $+1/r^2$ in R in such a 4D-metric can be formally achieved by changing the sign of $-r^2$ or the use of an imaginary value of θ in $g_{\mu\nu}$.

In the following this concept is illustrated as a formal, general approach, where the Ricci scalar will be required to be:

$$R = -2/r^2 \tag{39}$$

and an exponential ansatz will be used for $g_{00,11}$:

$g_{\mu\nu} = (+\exp(a\, v(r)), -\exp(b\, v(r)), +r^2, +r^2 \sin^2\theta)$

This will result in the following Ricci scalar (with the components belonging to ct, r and θ, φ still separated), (see [A5]):

$$R = \left(e^{-bv}\left[-a v'' - \frac{a^2 v'^2}{2} + \frac{ab\, v'^2}{2} - \frac{(a-b)v'}{r} \right]_{00,11} + e^{-bv}\left[\frac{(b-a)v'}{r} - \frac{2}{r^2} \right]_{22,33} \right) - 2/r^2 \tag{40}$$

To get $R = -2/r^2$ one has to set the term in curved brackets to zero.

The equation (40) refers to local coordinates and has to be solved for these or transformed to flat coordinates. The latter will be attempted by transforming the spherical object of a particle back into a photon of appropriate wavelength, assuming *ad hoc* that

1.) for r ->0 the angular coordinates have to reflect the expansion $\sim 1/r^2$, while

2.) the energy-space-time relation of a photon, i.e. $W_{ph} \sim 1/r$, $\sim 1/T$ reflects a contraction of space-time linear in coordinates ct, r .

A coefficient ρ [m] will be needed to obtain dimensionless terms . This gives:

$$R = \left(e^{-bv}\left[-a v'' - \frac{a^2 v'^2}{2} + \frac{ab\, v'^2}{2} - \frac{(a-b)v'}{r} \right]_{00,11} \frac{\boldsymbol{r}}{\boldsymbol{\rho}} + e^{-bv}\left[\frac{(b-a)v'}{r} - \frac{2}{r^2} \right]_{22,33} \frac{\boldsymbol{\rho}^2}{\boldsymbol{r}^2} \right) - \frac{2\boldsymbol{\rho}^2}{r^2 \boldsymbol{r}^2} \tag{41}$$

and

$$\left[-a v'' - \frac{a^2 v'^2}{2} + \frac{ab\, v'^2}{2} - \frac{(a-b)v'}{r} \right]_{00,11} \frac{\boldsymbol{r}}{\boldsymbol{\rho}} + \left[\frac{(b-a)v'}{r} - \frac{2}{r^2} \right]_{22,33} \frac{\boldsymbol{\rho}^2}{\boldsymbol{r}^2} = 0 \tag{42}$$

(In a 5D approach terms $(\rho/r)^N$ will appear in the Ricci tensor due to the ansatz for the Kaluza scalar, Φ, see chpt. 4.)

An equation of type (42) will in general feature solutions of type $\exp(v) = \exp(-x/r^3)$, which is a sufficient criterion to obtain equations (11), (15)ff i.e. the numerical expression for α and the quantization of particle energies. Setting e.g. a = b gives:

$$-a v'' \frac{r}{\rho} - \frac{2}{r^2}\frac{\rho^2}{r^2} = 0 \qquad \Rightarrow \qquad a v'' = -\frac{2\rho^3}{r^5} \tag{43}$$

and corresponds to equ. (4) if choosing an appropriate value for a. Using polar coordinates in flat space and setting a = b = 1/3 and $v = (-\rho/r)^3$ gives:

$$e^{v/3} = \Psi(r) = \exp\left(\frac{-\rho^3}{3r^3} \right) \qquad \Rightarrow \qquad \frac{-4\rho^3}{r^5} + \frac{2\rho^3}{r^5} = -\frac{2\rho^3}{r^5} \tag{44}$$

The Einstein tensor component G_{00} will be:

$$G_{00} = [-v''/(6) - v'/(3r)] + e^{v/3} \rho^2/r^4 = e^{v/3} \rho^2/r^4 \tag{45}$$

Equating with the component of the stress-energy tensor, $G_{00} = T_{00}$, and using the coefficient given in (38) will

25 coordinates t, r, θ, $\varphi = x_0, x_1, x_2, x_3$; only diagonal elements considered, μ=ν;

give (w = energy density):

$$e^{\nu/3}\frac{\rho^2}{r^4} \approx \frac{4\pi w}{\varepsilon_c} \quad => \quad \frac{\varepsilon_c e^{\nu/3}\rho^2}{4\pi r^4} \approx w \tag{46}$$

The volume integral over (46)f gives the particle energy according to

$$W_n = \frac{\varepsilon_c \ \rho^2}{4\pi}\int_0^{r_n}\frac{e^{\nu/3}}{r^4}d^3r = \frac{\Gamma_{1/3}}{3} \ 3^{1/3}\rho\,\varepsilon_c \tag{47}$$

To recover (8), (19) for the electron, ρ in (46)f has to be given by

$$\rho^3 = 8\frac{3}{\sigma^*\alpha_0(4\pi)}\left(\frac{e_c}{\varepsilon_c}\right)^3 \quad ^{26} \tag{48}$$

i.e. a derivation from the EFE with $4\pi/\varepsilon_c$ in place of $8\pi G/c_0^4$ reproduces the basic equation (4) with essentially the same set of coefficients as used in chapter 2.

4 Extension to 5 dimensions

Several aspects of this model hint at a possible 5 dimensional background. The ansatz of chpt. 3 may be improved in a 5D model based on the work of T. Kaluza [4] and extended by P.Wesson and collaborators [9]. Elements of the electroweak interaction and the Higgs mechanism may be interpreted geometrically as objects in 4D-space.

To extend this model to N spatial dimensions the following definition for Ψ_N will be used:

$$\Psi_N(r) = \exp(-\left(\frac{\rho}{r}\right)^N) \qquad \rho \sim \left(\frac{e_c}{\varepsilon_c}\right) \tag{49}$$

4.1 Kaluza theory

Several terms in Kaluza's work may be simplified using terms of this model. The electromagnetic coupling constant in the metric, κ, of Kaluza may be replaced by c_0 (using (38)) [27]:

$$\kappa_G = \left(\frac{16\pi G\varepsilon_c}{c_0^2}\right)^{0.5} \quad => \quad \kappa_c \approx c_0 \tag{50}$$

A major problem in Kaluza's model addressed by himself is that the mass to charge ratio of e.g. the electron results in values for the derivation of the 4th spatial coordinate dx_4/ds in an excessive order of magnitude, a problem which will be avoided if the substitution (50) is used [28].

For vacuum solutions, $R_{AB} = 0$, Kaluza's ansatz yields the following differential equation for a scalar field Φ:

$$\nabla\Phi^\alpha = -\frac{1}{4}\kappa^2\Phi^3 F^{\mu\nu}F_{\mu\nu} \tag{51}$$

(F = electromagnetic tensor) which has approximate solutions for r-> 0 of type $\Phi_n \sim \exp(-(\rho/r)^N/2)$, i.e. components of the functions $\Psi = f(\rho/r)$ used in this model might provide appropriate candidates for Φ:

$$\Phi_N \approx \left(\frac{\rho}{r}\right)^{N-1}e^{\nu/2} = \left(\frac{\rho}{r}\right)^{N-1}\exp(-\left(\frac{\rho}{r}\right)^N/2) \qquad \text{with} \quad \nu = -\left(\frac{\rho}{r}\right)^N \tag{52}$$

see [A6]. Inserting such solutions for Φ in a 5D metric will produce various terms of (ρ^N/r^{N+2}) in the Ricci tensor and the equations gain considerably in complexity.

4.2 Electroweak interaction and Higgs mechanism

This model originates from establishing a relation between a photon which represents SO(2), U(1) symmetry [29] and rotating objects -particles- of SO(3), SU(2) symmetry i.e. it involves the symmetries of electroweak

26 The term $\sigma^*\,\alpha_0$ has to appear in the denominator since ρ^2 appears in the nominator of equ (46), not affecting the validity of the equations of this model.

27 Essentially turning Kaluza's ansatz into an in 1st order electromagnetic one;

28 $dx_4/ds = e_c/(m_e\kappa_G) \approx 3E{+}29[\mathrm{m/s}] \gg c_0 \quad => \quad dx_4/ds = e_c/(m_e\kappa_c) \approx 1E{+}4[\mathrm{m/s}] \ll c_0$

29 U(1), the symmetry group for electromagnetism and its isomorphic rotation group SO(2) characterize a photon of

interaction. Symmetry SO(3) is directly related to the property “mass” via the reasoning in chpt. 3.

The concept of chpt. 2.2 for calculating the fine-structure constant α may be extended to other dimensions, N, based on the integral over the (square) of the N-D point charge term (S_N being the geometric factor for n-dimensional surface, in case of 3D: 4π):

$$\int_0^r \Psi_N(r)^2 r^{-2(N-1)}\ d^N r\ =\ S_N \int_0^r \Psi_N(r)^2 r^{-(N-1)} dr \tag{53}$$

multiplied by a complementary integral to yield a dimensionless constant. This results in (see [A7]):

$$\alpha_N^{-1}\ =\ \frac{(2\pi)^{\delta(2N-4)}}{(2\pi)^{(N-2)}} \int_0^r \Psi_N(r)^2 r^{-(N-1)} dr \int_0^r \Psi_N(r)^2 r^{(N-3)} dr \tag{54}$$

with N = {2; 3; 4} [30] or in terms of the Γ-functions:

$$\alpha_N^{-1}\ =\ S_n \frac{\Gamma_+(\Psi_N)\Gamma_-(\Psi_N)}{N^2\ \ arg(\Gamma(\Psi_N))^2} \tag{55}$$

with $\Gamma_{+/-}$ (Ψ_N) being the positive and negative Γ-functions attributed to Ψ_N and arg($\Gamma(\Psi_N)$) being the argument of the Γ-functions attributed to Ψ_N [31], i.e. the three coupling constants of the electroweak charges g', e and g can be combined in a single function of spatial dimension only.

Dimension - space	coupling constant	Value of *inverse* of coupling constant, α_N^{-1}	
4D	α(g)	$2\pi^2 \Gamma_{+1/2}\,\lvert\Gamma_{-1/2}\rvert\ 4/16\ =\ \pi^3\ =$	31.006
2D	α(g')	$2\pi\ \Gamma(0, 8/\sigma_{2D})^2\,/4\ = \pi^4\ \ =$	97.409
3D	α(e)	$4\pi\ \Gamma_{+1/3}\lvert\Gamma_{-1/3}\rvert\ 9/9 = 4\pi\ \Gamma_{+1/3}\lvert\Gamma_{-1/3}\rvert =$	137.036

Table 2: Values of electroweak coupling constants

The ratio of α_e and α_g represents the Weinberg angle, θ_W, and may be expressed as:

$$\sin^2\theta_W\ =\ \frac{\alpha_e}{\alpha_g}\ =\ \frac{\pi^2}{4\ \Gamma_{1/3}\lvert\Gamma_{-1/3}\rvert}\ =\ 0.2267 \tag{56}$$

(Experimental values: PDG [10]: $\sin^2\theta_W = 0.2312$, CODATA [11]: $\sin^2\theta_W = 0.2223$)
and $\cos\theta_W = m_W/m_Z = 0.8794 = 0.998\ \ (m_W/m_Z)_{exp}$ [12].

Comparing (53)f with equation (52) for the Kaluza scalar, Φ, demonstrates that except for a constant factor the terms are identical, $\Phi_N \sim \Psi_N\ r^{-(N-1)}$. Together with the identification of a particle with the energy of the Higgs boson to represent a 1D object, characterized by $\lvert\Gamma_{-1/3}\rvert/3$, see 2.5, and a speculative mapping of all electroweak bosons relative to the expectation value of the Higgs field, $\langle\Phi\rangle_0$ = VEV/√2 = 246GeV/√2, one might tentatively merge coupling constants / their corresponding charges ~ $\alpha_N^{0.5}$, electroweak bosons and solutions for Kaluza's Φ in a 4D spatial scheme, see table 3 [32]:

	Point charge		Elements of electroweak / Higgs mechanism					
Dimension - space		Value relative to g	Electroweak bosons + VEV	W [GeV]	W relative to VEV/√2	Γ-coefficient relative to VEV	VEV/√2 divided by Γ-coeff.	Kaluza coeff. N of (52)
4D	g	1	VEV/√2	174.1				4
1D			Higgs boson	125.4	0.720	$\lvert\Gamma_{-1/3}\rvert/3$	128.6	1
2D	g'	**0.541**	Z^0	91.2	**0.524**	$(\lvert\Gamma_{-1/3}\rvert/3)^2$	95.0	2
3D	e	**0.476**	$W^{+/-}$	80.4	**0.462**	$(\lvert\Gamma_{-1/3}\rvert)^2\ \ /(3\Gamma_{+1/3})$	84.8	3

Table 3: Comparison of values of coupling constant charges with electroweak energy scale and Φ_N

spin 1 if one considers the projection of the rotation of the E vector on the plane orthogonal to the axis of propagation.

30 Note: in the respective spherical coordinate systems r in 3D represents a conventional coordinate of type “length” while in 4D r is supposed to represent the inverse of energy, cf. [9].

31 I.e. in 4, 3 and 2D $\Gamma_{+/-}$ (Ψ_N) will be $\Gamma_{+/-1/2}$, $\Gamma_{+/-1/3}$ and $\Gamma(0, 8/\sigma_{2D}) = 7.872 = (2\pi^3)^{0.5}$ (numerical calculation); arg($\Gamma(\Psi_N)$) will be 1/2, 1/3, and for 2D ad hoc arg($\Gamma(0)$) = 1;
i.e. while 4 D is a straightforward extension of 3D, 2D fits in (54)f only with additional assumptions;

32 For $W^{+/-}$ $3/\Gamma_{1/3}$ is used as a first guess for the boson energy relation, since $\Gamma_{1/3}$ is the characteristic coefficient for energy and W ~1/r holds.

5 Discussion

The authoritative theory to describe particles is the standard model of particle physics (SM). However, the standard model has a major blind spot: it is not particularly efficient in calculating particle mass/energy and gravitational phenomena are explicitly not part of the SM which in turn restricts its applicability for addressing problems in cosmology.

Lepton and quark masses are treated as parameters in the SM while the calculation of light hadron masses [13], [14], [15] with lattice QCD methods typically uses 2-3 quark masses, a coupling constant and a reference particle for the absolute energy scale, i.e. about 4-5 parameters, to calculate mass of ~9-12 particles with an accuracy in the range of 1%. The model presented here achieves comparable results "ab initio" and includes both leptons and hadrons. The standard model distinguishes quite rigidly between both types, postulating that a set of physical objects characterized by an almost identical set of experimental observables is based on completely different physical principles. A major distinctive observable for both particle groups is assumed to be the strong force which is postulated to be zero for leptons, which per se is not verifiable beyond experimental accuracy.

According to this model it is suggestive to interpret strong interaction as evidenced in scattering events to be due to wave function overlap depending on [16]:

1) comparable size and energy of wave functions,

2) sufficient net overlap: If regions with same and opposite sign balance to give zero net overlap, no interaction occurs.

From condition 1) it is obvious that the wave functions of neutrino or electron can not be expected to exhibit effective interaction with hadrons [33]. In the case of the tauon the second rule is crucial. In this model the tauon is at the end of the partial product series for y_1^0 and should exhibit a high, potentially infinite number of nodes, separating densely spaced volume elements of alternating wave function sign prohibiting net overlap and effective interaction with hadrons of higher symmetry, such as the proton.

The completely different approach of this model compared to the QFT methods of the SM certainly produces more discrepancies than the lepton / hadron classification [34] and in this early stage it is impossible to anticipate if there will always be a more or less appropriate method to reconcile those. However, this model covers the "blind spot" of the SM quite thoroughly and might be a useful tool to complement it [35]: not only can mass/energy be calculated with a minimum of assumptions [36] and no free parameters but gravitational phenomena are an integral part of the model, i.e. it gives a coherent description for source and field. Moreover, these results seem to be founded within the established theory relating mass and gravitation: general relativity (GR), in particular its 5D Kaluza version, as evidenced by :

- the possibility to derive its basic equations - yielding quantized particle energies - from the Einstein field equation,
- obtaining absolute values for particle energies by replacing G/c_0^4 in the EFE by $1/\varepsilon_c$ = 2.998E+8[m/J],
- the electroweak coupling constants, in particular the fine-structure constant α, having a geometric interpretation related to curvature of space-time,
- the possibility to obtain a quantitative term for gravitational interaction from the expansion of the energy equation, implying curvature of space-time to be in general identical to (the presence of) energy, and spatial coordinate and energy to be intertwined inextricably,
- suggesting a close relationship of several mass/energy related phenomena - particle energy, elements of the Higgs mechanism, Planck energy - with GR.

Last not least GR is a general concept connecting geometry with energy and related phenomena and its applicability in the subatomic range would drastically underscore its universal validity.

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33 As for energy density ~ W_m/W_n^4 : e/p ~ E-13, μ/p ~ 6E-4; μ/π ~ 1/3, i.e. in case of μ/π some measurable effect should be expected; different symmetry may play an additional role.

34 Involving the three generation model, attributing a neutrino to each charged lepton, as well. This is not reflected in this model, which gives only some speculative information about neutrinos [5].

35 Other particle properties such as magnetic moments may be calculated with this model as well, see [5].

36 Ad hoc introduction of an exponential function Ψ, see 2.1 or those used for derivation of Ψ in chpt. 3.

Conclusion

This article suggests a consistent and coherent model quantitatively connecting the concepts of general relativity with the properties of subatomic particles giving in particular the following results:

- a geometric expression for the electroweak coupling constants, in particular the fine-structure constant, α, as a coefficient defined by the product of the Γ- functions in the integrals over Ψ(r) related to photon and point charge symmetry, $4\pi\ \Gamma_{+1/3}\ |\Gamma_{-1/3}| \approx \alpha^{-1}$,
- a quantization of energy levels with terms α^(-1/3^n)
- electron and the Higgs VEV energy as lower and upper limit of a convergent series for particle energy,
- additional information about particle properties e.g. the lepton character of the tauon,
- a series expansion for particle energy, including terms for rest energy, electromagnetic interaction and a 3rd term which at short range yields effects associated with strong interaction, at long range gives a quantitative term for gravitational interaction.

The basic terms of the model may be derived directly from the framework of the Einstein field equations and can be expressed without use of free parameters.

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Appendix

[A1] Differential equation

The approximation $\Psi(r<r_n)$ of equation (4) provides a solution to a differential equation of type

$$-\frac{r}{6}\frac{d^2\Psi(r)}{dr^2} + \frac{\beta_n/2}{2r^3}\frac{d\Psi(r)}{dr} - \frac{\beta_n/2}{r^4}\Psi(r) = 0 \tag{57}$$

which corresponds approximately to the limit l -> ∞ (σ -> ≈1) while has to be amended by σ in the denominator of the last term for the general case.

With the 3rd term in (31) used for potential energy, V:

$$V(r) = b_0\ \beta_{GS}/(2\ r^4) = b_0\ [\ \sigma^*\ \alpha_0\ (e_c/\varepsilon_c)^3\ /(4\pi)^2]\ /(2r^4) \tag{58}$$

and a corresponding expansion by $(\hbar c_0)^2\alpha^{-2}\ /b_0^2$ for the first term, the approximate differential equation for this model may be given as:

$$-\frac{(\hbar c_0)^2 r}{\alpha^{-2} b_0}\ \frac{d^2\Psi(r)}{dr^2} + r\,V(r)\ \frac{d\Psi(r)}{dr} - \frac{V(r)}{\sigma}\Psi(r) = 0 \tag{59}$$

Equations (57)ff give a satisfactory description for spherical symmetric states only.

[A2] Coefficient 1.51
Factor 1.5088 of the ratio W_μ/W_e is subject to a 3rd power relationship of the same kind as the α coefficients:

$$\left(\frac{1.5133}{1.5088}\right) = \left(\frac{1.5133}{1.5}\right)^{1/3} \tag{60}$$

indicating that the radial terms of $\Pi_{\beta,n}$ in β_n and the angular components of σ are not correctly separated yet or may not be separable even in the case of spherical symmetric states.
The limit of a corresponding partial product in the energy expression is given by 1.5133 Π_0^∞ (1.5/1.533)^1/3^k ≈ 1.5066.
The corresponding term in β will be: 1.5133^{-3} Π_0^n (1.533/1.5)^3/3^k , n={1;2;...}, for particles above the electron, see [A3].
The following relation holds:

$$1.5133 = 0.998 \left|\Gamma_{-1/3}\right| / \Gamma_{1/3} = 4\pi\, \Gamma^2_{-1/3}\ \alpha \tag{61}$$

[A3] Particle parameter β
A more detailed expression for β than given in (19) will be attempted in the following.
The term (60) will be used within the particle specific factor (square brackets), thus coefficient 1.5133 of σ will be placed there, giving for the general term (i.e. excluding the electron):

$$\beta_n = \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \frac{2}{(2\pi)^3}\ 1.5133^{-3}\Pi^{n}_{k=0}\left[\alpha^3\left(\frac{1.5133}{1.5}\right)\right]\wedge\left(\frac{3}{3^k}\right) \qquad n = \{0;1;2;...\} \tag{62}$$

factor 1.5133^{-3} represents ≈3/2 for the ratio of W_μ/W_e, to be omitted in the term for the electron:

$$\beta_e = \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \boldsymbol{\frac{2}{(2\pi)^3}\left[\alpha^3\left(\frac{1.5133}{1.5}\right)\right]^3} \approx \sigma * \frac{1}{(4\pi)^2}\left(\frac{e_c}{\varepsilon_c}\right)^3 \boldsymbol{\alpha_0} \tag{63}$$ [37]

the particle specific factor is given in square brackets (α_0 in bold). The other factors are due to
- factor 2: Ψ appearing squared in the integrals,
- factor $1/(2\pi)^3$: representing 2π of the integral limit in (23) ,
- factor 1.5133^{-3}: due to anomalous factor 2/3 in W_e/W_μ,
- $1/(4\pi)^2$: the power of 2 instead of the power of 3 as for the other components might be due to b_0 appearing squared in (59) and its analog in the asymmetry of the ρ/r components of (42)

Using (63) W_e may be given as:

$$W_e = 2b_0\frac{\Gamma_{+1/3}}{3}\left(\frac{9\pi^{5/3}\alpha}{\left|\Gamma_{-1/3}\right|}\left(\frac{\varepsilon_c}{e_c}\right)\left[\frac{\alpha^{-3}}{1.5133}\right]\right) = \frac{1.5\pi^{2/3}}{1.5133}\frac{\Gamma_{+1/3}}{\left|\Gamma_{-1/3}\right|}\frac{e_c}{\alpha^2} = 1.0001\ W_{e,exp} \tag{64}$$

[A4] Additional particle states
Assignment of more particle states will not be obvious. The following gives some possible approaches.

[A4.1] Partial products
Additional partial product series will have to start with higher exponents n in α^($-1/3^n$) giving smaller differences in energy while density of experimentally detected states is high. There might be a tendency of particles to exhibit a lower mean lifetime (MLT), making experimental detection of particles difficult [38]. To determine the factor y_l^m requires an appropriate ansatz for the differential equation yet to be found.
One more partial product might be inferred from considering d-like-orbital equivalents with a factor of $5^{1/3}$ as energy ratio relative to η giving the start of an additional partial product series at $5^{1/3}$ W(η) = 937MeV = 0.98 W(η'), i.e. close to energy values of the first particles available as starting point, η', Φ^0. However, in general it is not expected that partial products can explain all values of particle energies.

[A4.2] Linear combinations
The first particle family that does not fit to the partial product series scheme are the kaons at ~ 495MeV. They might be considered to be linear combination states of π-states. The π-states of the y_1^0 series are assumed to exhibit one angular node, giving a charge distribution of +|+, -|- and +|-. A linear combination of two π-states would yield the basic symmetry properties of the 4 kaons as:

		+				-				-				+	
K^+	+		+	K^-	-		-	K_S^o	+		+	K_L^o	+		-	(+/- = charge)
		+				-				-				-	

providing two neutral kaons of different structure and parity, implying a decay with different parity and MLT values. For the charged Kaons, K^+, K^-, a configuration for wave function sign equal to the configuration for charge of K_S^o and

37 Note: $2\,(2/3)^3/(2\pi)^3 \approx (1.5133\ \alpha^{-1}\ 2)^{-1}$, i.e. indicating a relation to the angular limit factor of chpt. 2.5.
38 Which might explain missing particles of higher n in the y_0^0 and y_1^0 series as well.

K_L^o might be possible, giving two versions of P+ and P- parity of otherwise identical particles and corresponding decay modes not violating parity conservation.

[A5] Metric

Coordinate variables: x0=t, x1=r, x2=θ, x3=φ

$g_{\mu\nu} = (+\exp(a\, v(r)),\ -\exp(b\, v(r)),\ +r^2,\ +r^2 \sin^2\theta)$

$g^{\mu\nu} = (+1/\exp(av(r)),\ -1/\exp(bv(r)),\ +1/r^2,\ +1/r^2 \sin^2\theta)$

$\Gamma_{01}^{0} = \Gamma_{10}^{0} = a\, v'/2$ $\quad$ $\Gamma_{00}^{1} = a\, v'\, e^{(a-b)v}/2$ $\quad$ $\Gamma_{11}^{1} = b\, v'/2$

$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \Gamma_{13}^{3} = \Gamma_{31}^{3} = 1/r$ $\quad$ $\Gamma_{22}^{1} = +r\, e^{-bv}$ $\quad$ $\Gamma_{33}^{1} = \Gamma_{22}^{1} \sin^2\theta$

$\Gamma_{23}^{3} = \Gamma_{32}^{3} = \cot\theta$ $\quad$ $\Gamma_{33}^{2} = -\sin\theta \cos\theta$

$R_{00} = e^{(a-b)v} [-a\, v''/2 - a^2 v'^2/4 + ab\, v'^2/4 - a\, v'/r]$ $\quad$ $R_{11} = [+a\, v''/2 + a^2 v'^2/4 - ab\, v'^2/4 - b\, v'/r]$

$R_{22} = e^{-bv} [(b-a)\, v' r/2 - 1] - 1$ $\quad$ $R_{33} = R_{22} \sin^2\theta$

$g^{00}R_{00} + g^{11}R_{11} = e^{-bv} [-a\, v'' - a^2 v'^2/2 + ab\, v'^2/2 - (a-b)\, v'/r]$

$g^{22}R_{22} + g^{33}R_{33} = e^{-bv} [(b-a)\, v'/r - 2/r^2] - 2/r^2$

[A6] Scalar potential Φ

Starting point for the following is the work of P.Wesson and J.Overduin [9], their equations are indicated in parenthesis (with their system of natural units used). The 5D metric is given as (6.68):

$$g_{AB} = \begin{bmatrix} (g_{\alpha\beta} - \Phi^2 A_\alpha A_\beta) & -\Phi^2 A_\alpha \\ -\Phi^2 A_\beta & -\Phi^2 \end{bmatrix} \tag{65}$$

(Roman letters correspond to 5D, greek letters to 4D, A_α = electromagnetic potential). Setting $R_{AB} = 0$ this results in a set of equations (6.75) describing Einstein- and Maxwell-like equations and a wave equation for the scalar potential, equ. (51). For the 4D part of the metric and the electromagnetic potential, A, (6.76) is used:

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - e^\mu r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{66}$$

$$A_\alpha = (A_0, 0, 0, 0) \tag{67}$$

With coefficients ν, λ, μ and A_0 depending only on r, equ. (51) gives (6.77):

$$\Phi'' + \left(\frac{\nu' - \lambda' + 2\mu'}{2} + \frac{2}{r}\right)\Phi' = \frac{1}{2}\Phi^3 e^{-\nu} (A_0')^2 \tag{68}$$

Using (52) the left side of equation (68) has approximate solutions for ν = λ = μ of type (52) where for r -> 0 equ. (68) will be dominated by the term of highest order, $\sim \rho^{3N-1}/r^{3N+1}\, e^{\nu/2}$:

$$\Phi_n'' \approx \left(\frac{\rho^{3N-1}}{r^{3N+1}}\right) e^{\nu/2} \approx \Phi_n^3 e^{-\nu} (A_0')^2 = \left[\left(\frac{\rho}{r}\right)^{N-1} e^{\nu/2}\right]^3 e^{-\nu} \left(\frac{\rho}{r^2}\right)^2 = \left(\frac{\rho}{r}\right)^{3N-3} e^{\nu/2} \left(\frac{\rho}{r^2}\right)^2 \tag{69}$$

[A7] Coupling constant in 5D

3D case:

Equations (54)f have their origin in the integrals over Ψ_N to be recapped and examined in more depth for the 3D case: α_e may be expressed directly via the volume integral over $1/r^2$ representing a point source in 3D times the corresponding integral symmetric in the Γ-function to give a dimensionless term:

$$2\int_0^r \Psi_3(r)^2 r^{-2}\, dr \int_0^r \Psi_3(r)^2\, dr = 2\left[\frac{\Gamma_{1/3}}{3}\right]\left[2\pi\, 2\pi\, 9 \frac{\Gamma_{-1/3}}{3}\right] = 4\pi\, \Gamma_{1/3} \Gamma_{-1/3}\ \mathbf{2\pi} = \mathbf{2\pi}\ \alpha_e^{-1} \tag{70}$$

The term of 2*2π indicates that the volume integral over the square of $1/r^2$ is involved, as actually used in the derivation of (8)ff, $\int \Psi_3(r)^2 r^{-4}\ d^3 r = \int \Psi_3(r)^2 r^{-4} \mathbf{4\pi} r^2\ dr$. One of the 2π terms originating from the second integral of equation (70) is required for turning h into ħ. Unless (70) is divided by 2π it would give a dimensionless constant $\alpha_e' = h\, c_0\, 4\pi\, \varepsilon/e^2$. The term 2π may be traced back to the more detailed expression for β, equ. (62)f, including the cube of 2π and it is a matter of choice to include it in the dimensionless coupling constant.

According to (12) the exact value of (70) depends on the integration limit of the second integral, i.e. the lower integration limit, r_{low}, of the corresponding Euler integral which can be expressed as 3D volume with $|\Gamma_{-1/3}|$ as radius:

$$r_{low} = \beta_n / \lambda_{C,n}^3 = 8/(3^{1.5}\sigma) = \left(3^{0.5} \frac{4\pi}{3}\ |\Gamma_{-1/3}|^3\right)^{-3} \qquad 39 \tag{71}$$

to be multiplied by 1/arg(Γ(n)) = 3. For this limit the result of the second integral of (70) is given by

$\int\Psi_{3(r)}{}^2 dr=3(3^{1.5}\sigma/8)^{1/3}= 3^{0.5}4\pi|\Gamma_{-1/3}|^3 \approx 36\pi^2|\Gamma_{-1/3}|$

4D case:
The results for 3D have an exact analogon in 4D:
Using Ψ_4 according to the definition (49) and an equivalent expression for (71) in 4D:

$$r_{low} = \beta_n / r_{4,n}^4 = 8/\sigma_4 = \left(\frac{\pi^2}{2}\ |\Gamma_{-1/4}|^4\right)^{-4} \quad [39] \tag{72}$$

as integration limit the non-point charge integral in 4D will be given by:

$$\int_0^r \Psi_4(r)^2 r\, dr = \int_{8/\sigma_4}^{\infty} t^{-1.25} e^{-t}\, dt \approx 4(\pi^2/2\ |\Gamma_{-1/4}|^4) \approx 32\pi^4 |\Gamma_{-1/2}| \approx 1/11390 \tag{73}$$

The 4D equivalent of (70) may be given as:

$$\int_0^r \Psi_4(r)^2 r^{-3} dr \int_0^r \Psi_4(r)^2 r\, dr = \left[\frac{\Gamma_{1/2}}{4}\right]\left[2\pi^4 16 \frac{|\Gamma_{-1/2}|}{4}\right] = \frac{\pi^2}{2}\Gamma_{1/2}|\Gamma_{-1/2}|\ \mathbf{4\pi^2} = \pi^3 \mathbf{4\pi^2} = \alpha_g^{-1}\mathbf{4\pi^2} \tag{74}$$

The term $4\pi^2$ is the square of the 2π term in the last expression of (70) since the integrals in (74) refer to $\beta^{0.5}$ and thus to the square of energy and h, ħ.

2D case:
the 2D case is not as straightforward as the 4D case. The integral over the 1D point charge

$$\int_0^r \Psi_2(r)^2 r^{-1} dr = \Gamma(0, \beta_2/r_2^2)\ /\ 2 \quad [39] \tag{75}$$

features $\Gamma(0, x)$, with $\Gamma(0, x) \to \infty$ for $x \to 0$ the simple relation between integral limit and integral value of 3D, 4D is not valid in this case. Using nevertheless the 2D equivalent of the integration limit

$$r_{low} = \beta_n / \lambda_{C,n}^2 = 8/(3\sigma_2) = \left(3^{0.5}\pi\ |\Gamma_{-1/2}|^2\right)^{-2} \approx 1/4676 \tag{76}$$

and calculating $\Gamma(0, \beta_2/r_2^2)$ numerically gives $\int\Psi_{2(r)}{}^2 r^{-1} dr \approx 7.872/2$ and $(\int\Psi_{2(r)}{}^2 r^{-1} dr)^2 \approx 2\pi^3/4$. This will give a value of $\alpha_{g'} \approx \pi^4$ *if multiplied by a factor 2π*. Unlike to the 3D, 4D case 2π will not appear in the denominator of the expression for α, since the 2D integrals yield dimensionless terms and refer to angular momentum rather than energy. Though the reason for the appearance of 2π in the nominator of the integral term is not obvious it is possible to include the 2D case in the unified expressions given by equations (53)f. [40]

[A8] Values used
π = 3.141592654
$\Gamma_{1/3}$ = 2.678938535
$|\Gamma_{-1/3}|$ = 4.062353818
α^{-1} = 137.035999084
c_0 = 2.99792458 [m/s]
e = 1.602176634 E-019 [C]
ε = 8.854187813 E-12 [F/m]
b_0 = 2.307077552 E-28 [Jm]
G = 6.67430 E-11 [$m^5/(Js^4)$]
$W_{e,\,exp}$ = 8.187105777 [J]
$\lambda_{C,e}$ = 2.426310239 E-12 [m]
e_c = 3.109751438 E-18 [J]
β_{dim} = 5.131205555 E-30 [m^3]
$\sigma = 8(4\pi\,|\Gamma_{-1/3}|^3/3)^3$ = 177155864 [-]
r_e = 1.413269970 E-12 [m]

39 Note: In (71), (76) λ_c is used in place of r_n, their relationship is given by $\lambda_c \approx 3^{0.5} = r_n$. In 4D the coordinate $r_{4,n}$ is already supposed to be related to energy: $r_{4,n} \sim 1/W$. Factor 2 representing electric and magnetic contributions in the 3D equations will be dropped in the 4D case.
40 Inserting a factor 2π in one of the two integrals $\int\Psi_{2(r)}{}^2 r^{-1}$ dr would turn this integral into the volume integral over the square of $1/r^1$ in analogy to the derivation of the 3D term.