

The Complexity of Goldbach's Conjecture

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Abstract

The Goldbach's conjecture has been described as the most difficult problem in the history of Mathematics. This conjecture states that every even integer greater than 2 can be written as the sum of two primes. The conjecture that all odd numbers greater than 7 are the sum of three odd primes is known today as the weak Goldbach conjecture. A major complexity classes are P and $NSPACE(S(n))$ for some $S(n)$. We prove if the complexity class P is equal to $NSPACE(S(n))$ for some $S(n) = o(\log n)$, then the weak Goldbach's conjecture is false. Since Harald Helfgott proved that the weak Goldbach's conjecture is true, then we obtain that P is not equal to $NSPACE(S(n))$ for all $S(n) = o(\log n)$.

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1 Introduction

Number theory is a branch of pure mathematics devoted primarily to the study of the integers and integer-valued functions [15]. Goldbach's conjecture is one of the most important and unsolved problems in number theory [8]. Nowadays, it is one of the open problems of Hilbert and Landau [8]. Goldbach's original conjecture, written on 7 June 1742 in a letter to Leonhard Euler, states: "... at least it seems that every number that is greater than 2 is the sum of three primes" [6]. This is known as the ternary Goldbach conjecture. We call a prime as a natural number that is greater than 1 and has exactly two divisors, 1 and the number itself [18]. However, the mathematician Christian Goldbach considered 1 as a prime number. Euler replied in a letter dated 30 June 1742 the following statement: "Every even integer greater than 2 can be written as the sum of two primes" [6]. This is known as the strong Goldbach conjecture.

Using Vinogradov's method [17], it has been showed that almost all even numbers can be written as the sum of two primes. In 1973, Chen showed that every sufficiently large even number can be written as the sum of some prime number and a semiprime [4]. The strong Goldbach conjecture implies the conjecture that all odd numbers greater than 7 are the sum of three odd primes, which is known today as the weak Goldbach conjecture [6]. In 2012 and 2013, Peruvian mathematician Harald Helfgott published a pair of papers claiming to improve major and minor arc estimates sufficiently to unconditionally prove the weak Goldbach conjecture [9], [10].

We define a non-regular language under the assumption that the weak Goldbach's conjecture is true. Indeed, the possible language can be easily proved that might be non-regular using the Pumping lemma for regular languages [13]. However, we prove this language is actually regular when the complexity class P is equal to $NSPACE(S(n))$ for some $S(n) = o(\log n)$. This result is based on the breakthrough approach that checking whether a number is prime can be decided in polynomial time by a deterministic Turing machine [1]. In this way, if $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$, then the weak Goldbach's conjecture is false. Since the weak Goldbach's conjecture is true, then we obtain that $P \neq NSPACE(S(n))$ for all $S(n) = o(\log n)$ [9], [10].

2 Background Theory

In 1936, Turing developed his theoretical computational model [16]. The deterministic and nondeterministic Turing machines have become in two of the most important definitions related to this theoretical model for computation [16]. A deterministic Turing machine has only one next action for each step defined in its program or transition function [16]. A nondeterministic Turing machine could contain more than one action defined for each step of its program, where this one is no longer a function, but a relation [16].

Let Σ be a finite alphabet with at least two elements, and let Σ^* be the set of finite strings over Σ [3]. A Turing machine M has an associated input alphabet Σ [3]. For each string w in Σ^* there is a computation associated with M on input w [3]. We say that M accepts w if this computation terminates in the accepting state, that is $M(w) = \text{"yes"}$ [3]. Note that M fails to accept w either if this computation ends in the rejecting state, that is $M(w) = \text{"no"}$, or if the computation fails to terminate, or the computation ends in the halting state with some output, that is $M(w) = y$ (when M outputs the string y on the input w) [3].

Another relevant advance in the last century has been the definition of a complexity class. A language over an alphabet is any set of strings made up of symbols from that alphabet [5]. A complexity class is a set of problems, which are represented as a language, grouped by measures such as the running time, memory, etc [5]. The language accepted by a Turing machine M , denoted $L(M)$, has an associated alphabet Σ and is defined by:

$$L(M) = \{w \in \Sigma^* : M(w) = \text{"yes"}\}.$$

Moreover, $L(M)$ is decided by M , when $w \notin L(M)$ if and only if $M(w) = \text{"no"}$ [5]. We denote by $t_M(w)$ the number of steps in the computation of M on input w [3]. For $n \in \mathbb{N}$ we denote by $T_M(n)$ the worst case run time of M ; that is:

$$T_M(n) = \max\{t_M(w) : w \in \Sigma^n\}$$

where Σ^n is the set of all strings over Σ of length n [3]. We say that M runs in polynomial time if there is a constant k such that for all n , $T_M(n) \leq n^k + k$ [3]. In other words, this means the language $L(M)$ can be decided by the Turing machine M in polynomial time. Therefore, P is the complexity class of languages that can be decided by deterministic Turing machines in polynomial time [5].

A logarithmic space Turing machine has a read-only input tape, a write-only output tape, and read/write work tapes [16]. The work tapes may contain at most $O(\log n)$ symbols [16]. In computational complexity theory, NL is the complexity class containing the decision problems that can be decided by a nondeterministic logarithmic space Turing machine [12].

We use o -notation to denote an upper bound that is not asymptotically tight. We formally define $o(g(n))$ ("little-oh of g of n ") as the set

$$o(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < c \times g(n) \text{ for all } n \geq n_0\}.$$

For example, $2 \times n = o(n^2)$, but $2 \times n^2 \neq o(n^2)$ [5].

In theoretical computer science and formal language theory, a regular language is a formal language that can be expressed using a regular expression [2]. The complexity class that contains all the regular languages is REG . The complexity class $NSPACE(f(n))$ is the set of decision problems that can be solved by a nondeterministic Turing machine M , using space $f(n)$, where n is the length of the input [11].

3 Results

► **Definition 1.** We define the Goldbach's language L_G as follows:

$$L_G = \{1^{2 \times n + 1} 0^p 0^q 0^r : n \in \mathbb{N} \wedge n \geq 4 \wedge p, q \text{ and } r \text{ are odd primes} \wedge 2 \times n + 1 = p + q + r\}.$$

► **Theorem 2.** If the weak Goldbach's conjecture is true, then the Goldbach's language L_G is non-regular.

Proof. If the weak Goldbach's conjecture is true, then the Goldbach's language L_G is equal to the another language L' defined as follows:

$$L' = \{1^{2 \times n + 1} 0^{2 \times n + 1} : n \in \mathbb{N} \wedge n \geq 4\}.$$

We can easily prove that L' is non-regular using the Pumping lemma for regular languages [13]. ◀

► **Definition 3.** We define the verification Goldbach's language L_{VG} as follows:

$$L_{VG} = \{(2 \times n + 1, p, q, r) : n \in \mathbb{N} \wedge n \geq 4 \wedge p, q \text{ and } r \text{ are odd primes} \wedge 2 \times n + 1 = p + q + r\}.$$

► **Theorem 4.** $L_{VG} \in P$.

Proof. This result is based on the breakthrough approach that checking whether a number is prime can be decided in polynomial time by a deterministic Turing machine [1]. Certainly, we can check in polynomial time whether p , q and r are odd primes and the other verifications can be easily done in polynomial time as well. ◀

► **Definition 5.** We define the Goldbach's language with separator L_{SG} as follows:

$$L_{SG} = \{0^{2 \times n + 1} \# 0^p \# 0^q \# 0^r : n \in \mathbb{N} \wedge n \geq 4 \wedge p, q \text{ and } r \text{ are odd primes} \wedge 2 \times n + 1 = p + q + r\}$$

where $\#$ is the blank symbol.

► **Lemma 6.** The Goldbach's language L_{SG} is the unary representation of the verification Goldbach's language L_{VG} .

Proof. This is trivially true under the definition of both languages. ◀

► **Theorem 7.** If $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$, then $L_G \in REG$.

Proof. If $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$, then we obtain that $L_{VG} \in NSPACE(S(n))$ because of Theorem 4. There is a nondeterministic Turing machine which decides L_{SG} that uses space that is smaller than $c \times \log \log n$ for all $c > 0$, because of L_{SG} is the unary version of L_{VG} due to Lemma 6 [7]. Certainly, the standard space translation between the unary and binary languages actually works for nondeterministic machines with small space [7]. This means that if some language belongs to $NSPACE(S(n))$, then the unary version of that language belongs to $NSPACE(S(\log n))$ [7]. In this way, we obtain that $L_{SG} \in REG$ because of $REG = NSPACE(o(\log \log n))$ [11]. In addition, we can reduce in a nondeterministic constant space the language L_G to L_{SG} just nondeterministically inserting the blank symbol $\#$ within two arbitrary positions between the 0's on the input. Moreover, this nondeterminism reduction inserts the blank symbol $\#$ between the 1's and 0's and converts the 1's to 0's from the original input of L_G just generating the final output to L_{SG} . Consequently, we prove $L_G \in REG$ under the assumption that $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$, since REG is also the complexity class of languages decided by nondeterministic Turing machines in constant space [14]. ◀

► **Theorem 8.** $P \neq NSPACE(S(n))$ for all $S(n) = o(\log n)$.

Proof. If the weak Goldbach's conjecture is true, then $L_G \notin REG$ as a consequence of Theorem 2. However, if $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$, then $L_G \in REG$ due to Theorem 7. In this way, the weak Goldbach's conjecture cannot be true under the assumption that $P = NSPACE(S(n))$ for some $S(n) = o(\log n)$. Since the weak Goldbach's conjecture is true, then we obtain that $P \neq NSPACE(S(n))$ for all $S(n) = o(\log n)$ [9], [10]. ◀

4 Conclusions

The NL versus P problem is one of the major unsolved problems in computer science [12]. This is considered by many to be one of the most important open problems in the field [12]. This work also proves under the assumption of $NL = P$ that $NSPACE(\log n) \neq NSPACE(S(n))$ for all $S(n) = o(\log n)$.

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