Bordered Magic Squares With Order Square Magic Sums,
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Bordered Magic Squares With Order Square Magic Sums

Bordered Magic Square of Order
20** with magic sum **20x20
*For a day **20** of each month of **2020***

Inder J. Taneja[1]

Abstract

*The idea of **nested magic squares** is well known in the literature, generally known by **bordered magic squares**. In this work, **nested magic squares** are studied in such a way that the magic sums are equal to the order of the magic square. The study include integer values. In some cases decimal entries with positive and negative entries are also used. The magic sums of **sub-magic squares** lead us to a general formula.*

Contents

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1 Introduction

Based on the work of H. White [6], recently, author [10, 11, 12, 13, 14] worked on the **bordered magic squares** in different ways. Some of these ways are specified in following two subsections.

1.1 Odd Ordered Natural Number Entries

Author [12] studied the bordered magic squares constructed for the **consecutive odd numbers**. The summary is given in the following result.

Result 1.1. *[12] For **nested magic squares** for **consecutive odd numbers**, the total entries sums are given by*

$$T_{k\times m} := k^2 \times m^2,$$

*where k is the order of **nested magic squares**, and m is the **order of each nested sub-magic square**. This lead us to very interesting connection with **Pythagoras theorem**.*

In particular, the bordered magic squares constructed with odd order consecutive natural numbers starting from 1, the total sum entries are as follows:

- order 24, $k = 24$, $T_{24\times m} := 24^2 \times m^2$, $m = 4, 6, 8, 10, 12, 14, 16, 18, 20, 22$ and 24;
- order 25, $k = 25$, $T_{25\times m} := 25^2 \times m^2$, $m = 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$ and 25.

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1.2 Consecutive Natural Number Entries

Author [13] studied the bordered magic squares for the **consecutive natural numbers**. The summary is given in the following result.

Result 1.2. *The **nested magic squares** constructed for the consecutive natural numbers starting from 1 satisfy the following properties:*

i. $S_{k\times k} := k \times L$;

ii. $T_{k\times k} := k^2 \times L$;

iii. $C_{k\times k} := (k-1) \times 4 \times L$;

iv. $d_{border} := 8 \times L$.

where k is the order of nested magic square and

$$L := T_{1\times 1}, \ \ \textit{odd order magic squares}$$
$$L := \frac{T_{2\times 2}}{4}, \ \ \textit{even order magic squares}$$

and

$$\begin{aligned} S_{k\times k} &\longrightarrow \textit{magic square sums;} \\ T_{k\times k} &\longrightarrow \textit{total entries sums;} \\ C_{k\times k} &\longrightarrow \textit{borders entries sums;} \\ d_{border} &\longrightarrow \textit{difference among borders value.} \end{aligned}$$

In particular, for the orders 24 and 25, we have

1. For the **nested magic square** of order 24 for the consecutive entries 1 to 576, has the following symmetric results:

   i. $S_{k\times k} := \frac{k}{2} \times \frac{T_{2\times 2}}{2}$;

   ii. $T_{k\times k} := \left(\frac{k}{2}\right)^2 \times T_{2\times 2}$;

   iii. $C_{k\times k} := (k-1) \times T_{2\times 2}$.

   iv. $d_{border} := 2 \times T_{2\times 2}$.

   where $k = 4, 6, \ \ldots \ 20, \ 22$ and 24 orders of magic squares appearing **nested magic square** of order 24, and $T_{2\times 2} := 1154$ is the sum of four central values of magic square.

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2. For the **nested magic square** of order 25 for the consecutive entries 1 to 625, has the following symmetric results:

   i. $S_{k\times k} := k \times T_{1\times 1}$;
   ii. $T_{k\times k} := k^2 \times T_{1\times 1}$;
   iii. $C_{k\times k} := \dfrac{k-1}{2} \times 8 \times T_{1\times 1}$.
   iv. $d_{border} := 8 \times T_{1\times 1}$.

   where $k = 3, 5, 7, \ldots\ 21, 23$ and 25 orders of magic squares appearing **nested magic square** of order 25, and $T_{1\times 1} := 313$ is the central value of the magic square.

The aim of this work is very much similar to the Result 1.2. The difference is that in Result 1.2, we considered consecutive odd numbers to produce total sum magic squares as power of 4. Here we shall write bordered magic squares in such a way that the total sum is the square of order of magic squares. For example, for the bordered magic square of order 9, the total sum is 9^2, etc. This study include decimal entries as well as negative numbers. Finally, we got the following general formula for the **magic sum** of each **sub-magic square**:

$$S_{k\times k} := k \times M,$$

where M is the order of **magic square** and k is the order of each **sub-magic squares**. The results include integer values. In some cases decimal entries are also used.

More results in this direction can be seen in the [1, 2, 4, 3, 5, 6, 7]. Some results on general sum can be seen in author's work [14]. In [15], author wrote different **bordered magic squares** with magic sum always 2020.

2 Bordered Magic Squares With Order Square Sums

Let's consider following bordered magic squares in decreasing order starting from order 25.

2.1 Bordered Magic Square of Order 3

The **bordered magic square** of order 3 for the magic sum 3^2 is given by

			9
4	-1	6	9
5	3	1	9
0	7	2	9
9	9	9	9

The sub-magic square sums are as follows:

$$S_{3\times 3} := 9 = 3 \times 3 = 9 = 3^2$$

In this case the central value is same as order of magic square, i.e., 3.

2.2 Bordered Magic Square of Order 4

The **bordered magic square** of order 4 for the magic sum 4^2 is given by

				16
9.5	-3.5	-0.5	10.5	16
2.5	7.5	4.5	1.5	16
6.5	3.5	0.5	5.5	16
-2.5	8.5	11.5	-1.5	16
16	16	16	16	16

The sub-magic square sums are as follows:

$$S_{4\times 4} := 15 = 4 \times 4 = 16 = 4^2$$

In this case the sum of internal four entries is also the same as of magic square sum, i.e., $16 = 4^2$.

2.3 Bordered Magic Square of Order 5

The **bordered magic square** of order 5 for the magic sum 5^2 is given by

					25
-4	-7	13	11	12	25
16	6	1	8	-6	25
15	7	5	3	-5	25
0	2	9	4	10	25
-2	17	-3	-1	14	25
25	25	25	25	25	25

The sub-magic square sums are as follows:

$$S_{3\times 3} := 15 = 3 \times 5$$
$$S_{5\times 5} := 25 = 5 \times 5 = 25 = 5^2$$

2.4 Bordered Magic Square of Order 6

The **bordered magic square** of order 6 for the magic sum 6^2 is given by

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						36
19.5	17.5	-9.5	23.5	-8.5	-6.5	36
-10.5	11.5	-1.5	1.5	12.5	22.5	36
-4.5	4.5	9.5	6.5	3.5	16.5	36
-2.5	8.5	5.5	2.5	7.5	14.5	36
15.5	-0.5	10.5	13.5	0.5	-3.5	36
18.5	-5.5	21.5	-11.5	20.5	-7.5	36
36	36	36	36	36	36	36

The sub-magic square sums are as follows:

$$S_{4\times4} := 24 = 4 \times 6$$
$$S_{6\times6} := 36 = 6 \times 6 = 36 = 6^2$$

2.5 Bordered Magic Square of Order 7

The **bordered magic square** of order 7 for the magic sum 7^2 is given by

							49
-10	-6	-8	27	28	30	-12	49
31	-2	-5	15	13	14	-17	49
29	18	8	3	10	-4	-15	49
-11	17	9	7	5	-3	25	49
-9	2	4	11	6	12	23	49
-7	0	19	-1	1	16	21	49
26	20	22	-13	-14	-16	24	49
49	49	49	49	49	49	49	49

The sub-magic square sums are as follows:

$$S_{3\times3} := 21 = 3 \times 7$$
$$S_{5\times5} := 35 = 5 \times 7$$
$$S_{7\times7} := 49 = 7 \times 7 = 7^2$$

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2.6 Bordered Magic Square of Order 8

The **bordered magic square** of order 8 for the magic sum 8^2 is given by

								64
-16.5	-22.5	37.5	39.5	26.5	-11.5	28.5	-17.5	64
-19.5	21.5	19.5	-7.5	25.5	-6.5	-4.5	35.5	64
-18.5	-8.5	13.5	0.5	3.5	14.5	24.5	34.5	64
-13.5	-2.5	6.5	11.5	8.5	5.5	18.5	29.5	64
36.5	-0.5	10.5	7.5	4.5	9.5	16.5	-20.5	64
31.5	17.5	1.5	12.5	15.5	2.5	-1.5	-15.5	64
30.5	20.5	-3.5	23.5	-9.5	22.5	-5.5	-14.5	64
33.5	38.5	-21.5	-23.5	-10.5	27.5	-12.5	32.5	64
64	64	64	64	64	64	64	64	64

The sub-magic square sums are as follows:

$$S_{4\times4} := 32 = 4 \times 8$$
$$S_{6\times6} := 48 = 6 \times 8$$
$$S_{8\times8} := 64 = 8 \times 8 = 8^2.$$

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2.7 Bordered Magic Square of Order 9

The **bordered magic square** of order 9 for the magic sum 9^2 is given by

									81
-24	48	46	44	43	-20	-18	-16	-22	81
-31	-8	-4	-6	29	30	32	-10	49	81
-29	33	0	-3	17	15	16	-15	47	81
-27	31	20	10	5	12	-2	-13	45	81
41	-9	19	11	9	7	-1	27	-23	81
39	-7	4	6	13	8	14	25	-21	81
37	-5	2	21	1	3	18	23	-19	81
35	28	22	24	-11	-12	-14	26	-17	81
40	-30	-28	-26	-25	38	36	34	42	81
81	81	81	81	81	81	81	81	81	81

The sub-magic square sums are as follows:

$$S_{3\times 3} := 27 = 3 \times 9$$
$$S_{5\times 5} := 45 = 5 \times 9$$
$$S_{7\times 7} := 63 = 7 \times 9$$
$$S_{9\times 9} := 99 = 9 \times 9 = 9^2$$

2.8 Bordered Magic Square of Order 10

The **bordered magic square** of order 10 for the magic sum 10^2 is given by

										100
50.5	45.5	-24.5	43.5	-22.5	-26.5	-36.5	57.5	-38.5	51.5	100
-27.5	-14.5	-20.5	39.5	41.5	28.5	-9.5	30.5	-15.5	47.5	100
48.5	-17.5	23.5	21.5	-5.5	27.5	-4.5	-2.5	37.5	-28.5	100
-29.5	-16.5	-6.5	15.5	2.5	5.5	16.5	26.5	36.5	49.5	100
55.5	-11.5	-0.5	8.5	13.5	10.5	7.5	20.5	31.5	-35.5	100
-39.5	38.5	1.5	12.5	9.5	6.5	11.5	18.5	-18.5	59.5	100
52.5	33.5	19.5	3.5	14.5	17.5	4.5	0.5	-13.5	-32.5	100
-33.5	32.5	22.5	-1.5	25.5	-7.5	24.5	-3.5	-12.5	53.5	100
54.5	35.5	40.5	-19.5	-21.5	-8.5	29.5	-10.5	34.5	-34.5	100
-31.5	-25.5	44.5	-23.5	42.5	46.5	56.5	-37.5	58.5	-30.5	100
100	100	100	100	100	100	100	100	100	100	100

The sub-magic square sums are as follows:

$$\begin{aligned}
S_{4\times4} &:= 40 &= 4\times10\\
S_{6\times6} &:= 60 &= 6\times10\\
S_{8\times8} &:= 80 &= 8\times10\\
S_{10\times10} &:= 100 &= 10\times10
\end{aligned}$$

2.9 Bordered Magic Square of Order 11

The **bordered magic square** of order 11 for the magic sum 11^2 is given by

											121
-38	-30	-32	-34	-36	63	64	66	68	70	-40	121
71	-22	50	48	46	45	-18	-16	-14	-20	-49	121
69	-29	-6	-2	-4	31	32	34	-8	51	-47	121
67	-27	35	2	-1	19	17	18	-13	49	-45	121
65	-25	33	22	12	7	14	0	-11	47	-43	121
-39	43	-7	21	13	11	9	1	29	-21	61	121
-37	41	-5	6	8	15	10	16	27	-19	59	121
-35	39	-3	4	23	3	5	20	25	-17	57	121
-33	37	30	24	26	-9	-10	-12	28	-15	55	121
-31	42	-28	-26	-24	-23	40	38	36	44	53	121
62	52	54	56	58	-41	-42	-44	-46	-48	60	121
121	121	121	121	121	121	121	121	121	121	121	121

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{3\times 3} &:= 33 &&= 3 \times 11\\
S_{5\times 5} &:= 55 &&= 5 \times 11\\
S_{7\times 7} &:= 77 &&= 7 \times 11\\
S_{9\times 9} &:= 99 &&= 9 \times 11\\
S_{11\times 11} &:= 121 &&= 11 \times 11 = 11^2
\end{aligned}
$$

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2.10 Bordered Magic Square of Order 12

The **bordered magic square** of order 12 for the magic sum 12^2 is given by

												144
73.5	82.5	-57.5	80.5	-55.5	-59.5	-38.5	63.5	-40.5	65.5	-42.5	72.5	144
-44.5	52.5	47.5	-22.5	45.5	-20.5	-24.5	-34.5	59.5	-36.5	53.5	68.5	144
-45.5	-25.5	-12.5	-18.5	41.5	43.5	30.5	-7.5	32.5	-13.5	49.5	69.5	144
70.5	50.5	-15.5	25.5	23.5	-3.5	29.5	-2.5	-0.5	39.5	-26.5	-46.5	144
71.5	-27.5	-14.5	-4.5	17.5	4.5	7.5	18.5	28.5	38.5	51.5	-47.5	144
78.5	57.5	-9.5	1.5	10.5	15.5	12.5	9.5	22.5	33.5	-33.5	-54.5	144
67.5	-37.5	40.5	3.5	14.5	11.5	8.5	13.5	20.5	-16.5	61.5	-43.5	144
-50.5	54.5	35.5	21.5	5.5	16.5	19.5	6.5	2.5	-11.5	-30.5	74.5	144
-51.5	-31.5	34.5	24.5	0.5	27.5	-5.5	26.5	-1.5	-10.5	55.5	75.5	144
76.5	56.5	37.5	42.5	-17.5	-19.5	-6.5	31.5	-8.5	36.5	-32.5	-52.5	144
-53.5	-29.5	-23.5	46.5	-21.5	44.5	48.5	58.5	-35.5	60.5	-28.5	77.5	144
-48.5	-58.5	81.5	-56.5	79.5	83.5	62.5	-39.5	64.5	-41.5	66.5	-49.5	144
144	144	144	144	144	144	144	144	144	144	144	144	144

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{4\times 4} &:= 48 \quad = 4 \times 12 \\
S_{6\times 6} &:= 72 \quad = 6 \times 12 \\
S_{8\times 8} &:= 96 \quad = 8 \times 12 \\
S_{10\times 10} &:= 120 = 10 \times 12 \\
S_{12\times 12} &:= 144 = 12 \times 12 = 12^2
\end{aligned}
$$

2.11 Bordered Magic Square of Order 13

The **bordered magic square** of order 13 for the magic sum 13^2 is given by

													169
84	75	77	79	81	83	85	-63	-65	-67	-69	-71	-60	169
-70	-36	-28	-30	-32	-34	65	66	68	70	72	-38	96	169
-68	73	-20	52	50	48	47	-16	-14	-12	-18	-47	94	169
-66	71	-27	-4	0	-2	33	34	36	-6	53	-45	92	169
-64	69	-25	37	4	1	21	19	20	-11	51	-43	90	169
-62	67	-23	35	24	14	9	16	2	-9	49	-41	88	169
-61	-37	45	-5	23	15	13	11	3	31	-19	63	87	169
82	-35	43	-3	8	10	17	12	18	29	-17	61	-56	169
80	-33	41	-1	6	25	5	7	22	27	-15	59	-54	169
78	-31	39	32	26	28	-7	-8	-10	30	-13	57	-52	169
76	-29	44	-26	-24	-22	-21	42	40	38	46	55	-50	169
74	64	54	56	58	60	-39	-40	-42	-44	-46	62	-48	169
86	-49	-51	-53	-55	-57	-59	89	91	93	95	97	-58	169
169	169	169	169	169	169	169	169	169	169	169	169	169	169

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{3\times3} &:= 39 \;\; = 3 \times 13 \\
S_{5\times5} &:= 65 \;\; = 5 \times 13 \\
S_{7\times7} &:= 91 \;\; = 7 \times 13 \\
S_{9\times9} &:= 117 = 9 \times 13 \\
S_{11\times11} &:= 143 = 11 \times 13 \\
S_{13\times13} &:= 169 = 13 \times 13 = 13^2.
\end{aligned}
$$

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2.12 Bordered Magic Square of Order 14

The **bordered magic square** of order 14 for the magic sum 14^2 is given by

														196
-70.5	-76.5	103.5	-74.5	101.5	-72.5	111.5	-77.5	97.5	-68.5	95.5	-66.5	93.5	99.5	196
110.5	75.5	84.5	-55.5	82.5	-53.5	-57.5	-36.5	65.5	-38.5	67.5	-40.5	74.5	-82.5	196
-81.5	-42.5	54.5	49.5	-20.5	47.5	-18.5	-22.5	-32.5	61.5	-34.5	55.5	70.5	109.5	196
108.5	-43.5	-23.5	-10.5	-16.5	43.5	45.5	32.5	-5.5	34.5	-11.5	51.5	71.5	-80.5	196
-79.5	72.5	52.5	-13.5	27.5	25.5	-1.5	31.5	-0.5	1.5	41.5	-24.5	-44.5	107.5	196
106.5	73.5	-25.5	-12.5	-2.5	19.5	6.5	9.5	20.5	30.5	40.5	53.5	-45.5	-78.5	196
92.5	80.5	59.5	-7.5	3.5	12.5	17.5	14.5	11.5	24.5	35.5	-31.5	-52.5	-64.5	196
86.5	69.5	-35.5	42.5	5.5	16.5	13.5	10.5	15.5	22.5	-14.5	63.5	-41.5	-58.5	196
-59.5	-48.5	56.5	37.5	23.5	7.5	18.5	21.5	8.5	4.5	-9.5	-28.5	76.5	87.5	196
88.5	-49.5	-29.5	36.5	26.5	2.5	29.5	-3.5	28.5	0.5	-8.5	57.5	77.5	-60.5	196
-61.5	78.5	58.5	39.5	44.5	-15.5	-17.5	-4.5	33.5	-6.5	38.5	-30.5	-50.5	89.5	196
90.5	-51.5	-27.5	-21.5	48.5	-19.5	46.5	50.5	60.5	-33.5	62.5	-26.5	79.5	-62.5	196
-63.5	-46.5	-56.5	83.5	-54.5	81.5	85.5	64.5	-37.5	66.5	-39.5	68.5	-47.5	91.5	196
-71.5	104.5	-75.5	102.5	-73.5	100.5	-83.5	105.5	-69.5	96.5	-67.5	94.5	-65.5	98.5	196
196	196	196	196	196	196	196	196	196	196	196	196	196	196	196

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{4\times4} &:= 56 \quad = 4\times 14\\
S_{6\times6} &:= 84 \quad = 6\times 14\\
S_{8\times8} &:= 112 = 8\times 14\\
S_{10\times10} &:= 140 = 10\times 14\\
S_{12\times12} &:= 168 = 12\times 14\\
S_{14\times14} &:= 196 = 14\times 14 = 14^2.
\end{aligned}
$$

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2.13 Bordered Magic Square of Order 15

The **bordered magic square** of order 15 for the magic sum 15^2 is given by

															225
112	101	103	105	107	109	111	113	-87	-89	-91	-93	-95	-97	-84	225
-96	86	77	79	81	83	85	87	-61	-63	-65	-67	-69	-58	126	225
-94	-68	-34	-26	-28	-30	-32	67	68	70	72	74	-36	98	124	225
-92	-66	75	-18	54	52	50	49	-14	-12	-10	-16	-45	96	122	225
-90	-64	73	-25	-2	2	0	35	36	38	-4	55	-43	94	120	225
-88	-62	71	-23	39	6	3	23	21	22	-9	53	-41	92	118	225
-86	-60	69	-21	37	26	16	11	18	4	-7	51	-39	90	116	225
-85	-59	-35	47	-3	25	17	15	13	5	33	-17	65	89	115	225
110	84	-33	45	-1	10	12	19	14	20	31	-15	63	-54	-80	225
108	82	-31	43	1	8	27	7	9	24	29	-13	61	-52	-78	225
106	80	-29	41	34	28	30	-5	-6	-8	32	-11	59	-50	-76	225
104	78	-27	46	-24	-22	-20	-19	44	42	40	48	57	-48	-74	225
102	76	66	56	58	60	62	-37	-38	-40	-42	-44	64	-46	-72	225
100	88	-47	-49	-51	-53	-55	-57	91	93	95	97	99	-56	-70	225
114	-71	-73	-75	-77	-79	-81	-83	117	119	121	123	125	127	-82	225
225	225	225	225	225	225	225	225	225	225	225	225	225	225	225	225

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{3\times 3} &:= 451 = 3\times 15\\
S_{5\times 5} &:= 75 \quad = 5\times 15\\
S_{7\times 7} &:= 105 = 7\times 15\\
S_{9\times 9} &:= 135 = 9\times 15\\
S_{11\times 11} &:= 165 = 11\times 15\\
S_{13\times 13} &:= 195 = 13\times 15\\
S_{15\times 15} &:= 225 = 15\times 15 = 15^2.
\end{aligned}
$$

2.14 Bordered Magic Square of Order 16

The **bordered magic square** of order 16 for the magic sum 16^2 is given by

																256
129.5	-90.5	123.5	-92.5	-93.5	126.5	127.5	136.5	121.5	-98.5	-99.5	132.5	-101.5	134.5	-103.5	-96.5	256
142.5	-68.5	-74.5	105.5	-72.5	103.5	-70.5	113.5	-75.5	99.5	-66.5	97.5	-64.5	95.5	101.5	-110.5	256
-109.5	112.5	77.5	86.5	-53.5	84.5	-51.5	-55.5	-34.5	67.5	-36.5	69.5	-38.5	76.5	-80.5	141.5	256
140.5	-79.5	-40.5	56.5	51.5	-18.5	49.5	-16.5	-20.5	-30.5	63.5	-32.5	57.5	72.5	111.5	-108.5	256
-107.5	110.5	-41.5	-21.5	-8.5	-14.5	45.5	47.5	34.5	-3.5	36.5	-9.5	53.5	73.5	-78.5	139.5	256
138.5	-77.5	74.5	54.5	-11.5	29.5	27.5	0.5	33.5	1.5	3.5	43.5	-22.5	-42.5	109.5	-106.5	256
-105.5	108.5	75.5	-23.5	-10.5	-0.5	21.5	8.5	11.5	22.5	32.5	42.5	55.5	-43.5	-76.5	137.5	256
-111.5	94.5	82.5	61.5	-5.5	5.5	14.5	19.5	16.5	13.5	26.5	37.5	-29.5	-50.5	-62.5	143.5	256
-82.5	88.5	71.5	-33.5	44.5	7.5	18.5	15.5	12.5	17.5	24.5	-12.5	65.5	-39.5	-56.5	114.5	256
115.5	-57.5	-46.5	58.5	39.5	25.5	9.5	20.5	23.5	10.5	6.5	-7.5	-26.5	78.5	89.5	-83.5	256
-84.5	90.5	-47.5	-27.5	38.5	28.5	4.5	31.5	-1.5	30.5	2.5	-6.5	59.5	79.5	-58.5	116.5	256
117.5	-59.5	80.5	60.5	41.5	46.5	-13.5	-15.5	-2.5	35.5	-4.5	40.5	-28.5	-48.5	91.5	-85.5	256
-86.5	92.5	-49.5	-25.5	-19.5	50.5	-17.5	48.5	52.5	62.5	-31.5	64.5	-24.5	81.5	-60.5	118.5	256
119.5	-61.5	-44.5	-54.5	85.5	-52.5	83.5	87.5	66.5	-35.5	68.5	-37.5	70.5	-45.5	93.5	-87.5	256
-88.5	-69.5	106.5	-73.5	104.5	-71.5	102.5	-81.5	107.5	-67.5	98.5	-65.5	96.5	-63.5	100.5	120.5	256
128.5	122.5	-91.5	124.5	125.5	-94.5	-95.5	-104.5	-89.5	130.5	131.5	-100.5	133.5	-102.5	135.5	-97.5	256
256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256	256

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{4\times 4} &:= 64 \quad = 4 \times 16\\
S_{6\times 6} &:= 96 \quad = 6 \times 16\\
S_{8\times 8} &:= 128 = 8 \times 16\\
S_{10\times 10} &:= 160 = 10 \times 16\\
S_{12\times 12} &:= 192 = 12 \times 16\\
S_{14\times 14} &:= 224 = 14 \times 16\\
S_{16\times 16} &:= 246 = 16 \times 16 = 16^2.
\end{aligned}
$$

2.15 Bordered Magic Square of Order 17

The **bordered magic square** of order 17 for the magic sum 17^2 is given by

																	289
-112	160	158	156	154	152	150	148	147	-108	-106	-104	-102	-100	-98	-96	-110	289
-127	114	103	105	107	109	111	113	115	-85	-87	-89	-91	-93	-95	-82	161	289
-125	-94	88	79	81	83	85	87	89	-59	-61	-63	-65	-67	-56	128	159	289
-123	-92	-66	-32	-24	-26	-28	-30	69	70	72	74	76	-34	100	126	157	289
-121	-90	-64	77	-16	56	54	52	51	-12	-10	-8	-14	-43	98	124	155	289
-119	-88	-62	75	-23	0	4	2	37	38	40	-2	57	-41	96	122	153	289
-117	-86	-60	73	-21	41	8	5	25	23	24	-7	55	-39	94	120	151	289
-115	-84	-58	71	-19	39	28	18	13	20	6	-5	53	-37	92	118	149	289
145	-83	-57	-33	49	-1	27	19	17	15	7	35	-15	67	91	117	-111	289
143	112	86	-31	47	1	12	14	21	16	22	33	-13	65	-52	-78	-109	289
141	110	84	-29	45	3	10	29	9	11	26	31	-11	63	-50	-76	-107	289
139	108	82	-27	43	36	30	32	-3	-4	-6	34	-9	61	-48	-74	-105	289
137	106	80	-25	48	-22	-20	-18	-17	46	44	42	50	59	-46	-72	-103	289
135	104	78	68	58	60	62	64	-35	-36	-38	-40	-42	66	-44	-70	-101	289
133	102	90	-45	-47	-49	-51	-53	-55	93	95	97	99	101	-54	-68	-99	289
131	116	-69	-71	-73	-75	-77	-79	-81	119	121	123	125	127	129	-80	-97	289
144	-126	-124	-122	-120	-118	-116	-114	-113	142	140	138	136	134	132	130	146	289
289	289	289	289	289	289	289	289	289	289	289	289	289	289	289	289	289	289

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{3\times 3} &:= 51 \;\; = 3\times 17\\
S_{5\times 5} &:= 85 \;\; = 5\times 17\\
S_{7\times 7} &:= 119 = 7\times 17\\
S_{9\times 9} &:= 153 = 9\times 17\\
S_{11\times 11} &:= 187 = 11\times 17\\
S_{13\times 13} &:= 221 = 13\times 17\\
S_{15\times 15} &:= 255 = 15\times 17\\
S_{17\times 17} &:= 289 = 17\times 17 = 17^2.
\end{aligned}
$$

2.16 Bordered Magic Square of Order 18

The **bordered magic square** of order 18 for the magic sum 18^2 is given by

																		324
162.5	-119.5	156.5	-121.5	158.5	-123.5	160.5	-125.5	171.5	-143.5	164.5	-129.5	166.5	-131.5	168.5	-133.5	170.5	-127.5	324
153.5	131.5	-88.5	125.5	-90.5	-91.5	128.5	129.5	138.5	123.5	-96.5	-97.5	134.5	-99.5	136.5	-101.5	-94.5	-117.5	324
-116.5	144.5	-66.5	-72.5	107.5	-70.5	105.5	-68.5	115.5	-73.5	101.5	-64.5	99.5	-62.5	97.5	103.5	-108.5	152.5	324
151.5	-107.5	114.5	79.5	88.5	-51.5	86.5	-49.5	-53.5	-32.5	69.5	-34.5	71.5	-36.5	78.5	-78.5	143.5	-115.5	324
-114.5	142.5	-77.5	-38.5	58.5	53.5	-16.5	51.5	-14.5	-18.5	-28.5	65.5	-30.5	59.5	74.5	113.5	-106.5	150.5	324
149.5	-105.5	112.5	-39.5	-19.5	-6.5	-12.5	47.5	49.5	36.5	-1.5	38.5	-7.5	55.5	75.5	-76.5	141.5	-113.5	324
-112.5	140.5	-75.5	76.5	56.5	-9.5	31.5	29.5	2.5	35.5	3.5	5.5	45.5	-20.5	-40.5	111.5	-104.5	148.5	324
147.5	-103.5	110.5	77.5	-21.5	-8.5	1.5	23.5	10.5	13.5	24.5	34.5	44.5	57.5	-41.5	-74.5	139.5	-111.5	324
-110.5	-109.5	96.5	84.5	63.5	-3.5	7.5	16.5	21.5	18.5	15.5	28.5	39.5	-27.5	-48.5	-60.5	145.5	146.5	324
-118.5	-80.5	90.5	73.5	-31.5	46.5	9.5	20.5	17.5	14.5	19.5	26.5	-10.5	67.5	-37.5	-54.5	116.5	154.5	324
-136.5	117.5	-55.5	-44.5	60.5	41.5	27.5	11.5	22.5	25.5	12.5	8.5	-5.5	-24.5	80.5	91.5	-81.5	172.5	324
173.5	-82.5	92.5	-45.5	-25.5	40.5	30.5	6.5	33.5	0.5	32.5	4.5	-4.5	61.5	81.5	-56.5	118.5	-137.5	324
-138.5	119.5	-57.5	82.5	62.5	43.5	48.5	-11.5	-13.5	-0.5	37.5	-2.5	42.5	-26.5	-46.5	93.5	-83.5	174.5	324
175.5	-84.5	94.5	-47.5	-23.5	-17.5	52.5	-15.5	50.5	54.5	64.5	-29.5	66.5	-22.5	83.5	-58.5	120.5	-139.5	324
-140.5	121.5	-59.5	-42.5	-52.5	87.5	-50.5	85.5	89.5	68.5	-33.5	70.5	-35.5	72.5	-43.5	95.5	-85.5	176.5	324
177.5	-86.5	-67.5	108.5	-71.5	106.5	-69.5	104.5	-79.5	109.5	-65.5	100.5	-63.5	98.5	-61.5	102.5	122.5	-141.5	324
-142.5	130.5	124.5	-89.5	126.5	127.5	-92.5	-93.5	-102.5	-87.5	132.5	133.5	-98.5	135.5	-100.5	137.5	-95.5	178.5	324
163.5	155.5	-120.5	157.5	-122.5	159.5	-124.5	161.5	-135.5	179.5	-128.5	165.5	-130.5	167.5	-132.5	169.5	-134.5	-126.5	324
324	324	324	324	324	324	324	324	324	324	324	324	324	324	324	324	324	324	324

The sub-magic square sums are as follows:

$$
\begin{aligned}
S_{4\times4} &:= 72 \;\;= 4 \times 18\\
S_{6\times6} &:= 108 = 6 \times 18\\
S_{8\times8} &:= 144 = 8 \times 18\\
S_{10\times10} &:= 180 = 10 \times 18\\
S_{12\times12} &:= 216 = 12 \times 18\\
S_{14\times14} &:= 252 = 14 \times 18\\
S_{16\times16} &:= 288 = 16 \times 18\\
S_{18\times18} &:= 324 = 18 \times 18
\end{aligned}
$$

2.17 Bordered Magic Square of Order 19

The **bordered magic square** of order 19 for the magic sum 19^2 is given by

																			361
-142	-126	-128	-130	-132	-134	-136	-138	-140	183	184	186	188	190	192	194	196	198	-144	361
199	-110	162	160	158	156	154	152	150	149	-106	-104	-102	-100	-98	-96	-94	-108	-161	361
197	-125	116	105	107	109	111	113	115	117	-83	-85	-87	-89	-91	-93	-80	163	-159	361
195	-123	-92	90	81	83	85	87	89	91	-57	-59	-61	-63	-65	-54	130	161	-157	361
193	-121	-90	-64	-30	-22	-24	-26	-28	71	72	74	76	78	-32	102	128	159	-155	361
191	-119	-88	-62	79	-14	58	56	54	53	-10	-8	-6	-12	-41	100	126	157	-153	361
189	-117	-86	-60	77	-21	2	6	4	39	40	42	0	59	-39	98	124	155	-151	361
187	-115	-84	-58	75	-19	43	10	7	27	25	26	-5	57	-37	96	122	153	-149	361
185	-113	-82	-56	73	-17	41	30	20	15	22	8	-3	55	-35	94	120	151	-147	361
-143	147	-81	-55	-31	51	1	29	21	19	17	9	37	-13	69	93	119	-109	181	361
-141	145	114	88	-29	49	3	14	16	23	18	24	35	-11	67	-50	-76	-107	179	361
-139	143	112	86	-27	47	5	12	31	11	13	28	33	-9	65	-48	-74	-105	177	361
-137	141	110	84	-25	45	38	32	34	-1	-2	-4	36	-7	63	-46	-72	-103	175	361
-135	139	108	82	-23	50	-20	-18	-16	-15	48	46	44	52	61	-44	-70	-101	173	361
-133	137	106	80	70	60	62	64	66	-33	-34	-36	-38	-40	68	-42	-68	-99	171	361
-131	135	104	92	-43	-45	-47	-49	-51	-53	95	97	99	101	103	-52	-66	-97	169	361
-129	133	118	-67	-69	-71	-73	-75	-77	-79	121	123	125	127	129	131	-78	-95	167	361
-127	146	-124	-122	-120	-118	-116	-114	-112	-111	144	142	140	138	136	134	132	148	165	361
182	164	166	168	170	172	174	176	178	-145	-146	-148	-150	-152	-154	-156	-158	-160	180	361
361	361	361	361	361	361	361	361	361	361	361	361	361	361	361	361	361	361	361	361

The sub-magic square sums are as follows:

$$\begin{aligned}
S_{3\times 3} &:= 57 \quad = 3 \times 19\\
S_{5\times 5} &:= 95 \quad = 5 \times 19\\
S_{7\times 7} &:= 133 = 7 \times 19\\
S_{9\times 9} &:= 171 = 9 \times 19\\
S_{11\times 11} &:= 209 = 11 \times 19\\
S_{13\times 13} &:= 247 = 13 \times 19\\
S_{15\times 15} &:= 285 = 15 \times 19\\
S_{17\times 17} &:= 323 = 17 \times 19\\
S_{19\times 19} &:= 361 = 19 \times 19 = 19^2.
\end{aligned}$$

2.18 Bordered Magic Square of Order 20

The **bordered magic square** of order 20 for the magic sum 20^2 is given by

																				400
-161.5	190.5	-149.5	188.5	-147.5	186.5	-145.5	184.5	-143.5	182.5	219.5	211.5	-172.5	213.5	-174.5	215.5	-176.5	217.5	-178.5	-160.5	400
209.5	164.5	-117.5	158.5	-119.5	160.5	-121.5	162.5	-123.5	173.5	-141.5	166.5	-127.5	168.5	-129.5	170.5	-131.5	172.5	-125.5	-169.5	400
-168.5	155.5	133.5	-86.5	127.5	-88.5	-89.5	130.5	131.5	140.5	125.5	-94.5	-95.5	136.5	-97.5	138.5	-99.5	-92.5	-115.5	208.5	400
207.5	-114.5	146.5	-64.5	-70.5	109.5	-68.5	107.5	-66.5	117.5	-71.5	103.5	-62.5	101.5	-60.5	99.5	105.5	-106.5	154.5	-167.5	400
-166.5	153.5	-105.5	116.5	81.5	90.5	-49.5	88.5	-47.5	-51.5	-30.5	71.5	-32.5	73.5	-34.5	80.5	-76.5	145.5	-113.5	206.5	400
205.5	-112.5	144.5	-75.5	-36.5	60.5	55.5	-14.5	53.5	-12.5	-16.5	-26.5	67.5	-28.5	61.5	76.5	115.5	-104.5	152.5	-165.5	400
-164.5	151.5	-103.5	114.5	-37.5	-17.5	-4.5	-10.5	49.5	51.5	38.5	0.5	40.5	-5.5	57.5	77.5	-74.5	143.5	-111.5	204.5	400
203.5	-110.5	142.5	-73.5	78.5	58.5	-7.5	33.5	31.5	4.5	37.5	5.5	7.5	47.5	-18.5	-38.5	113.5	-102.5	150.5	-163.5	400
202.5	149.5	-101.5	112.5	79.5	-19.5	-6.5	3.5	25.5	12.5	15.5	26.5	36.5	46.5	59.5	-39.5	-72.5	141.5	-109.5	-162.5	400
-151.5	-108.5	-107.5	98.5	86.5	65.5	-1.5	9.5	18.5	23.5	20.5	17.5	30.5	41.5	-25.5	-46.5	-58.5	147.5	148.5	191.5	400
-170.5	-116.5	-78.5	92.5	75.5	-29.5	48.5	11.5	22.5	19.5	16.5	21.5	28.5	-8.5	69.5	-35.5	-52.5	118.5	156.5	210.5	400
-159.5	-134.5	119.5	-53.5	-42.5	62.5	43.5	29.5	13.5	24.5	27.5	14.5	10.5	-3.5	-22.5	82.5	93.5	-79.5	174.5	199.5	400
-158.5	175.5	-80.5	94.5	-43.5	-23.5	42.5	32.5	8.5	35.5	2.5	34.5	6.5	-2.5	63.5	83.5	-54.5	120.5	-135.5	198.5	400
197.5	-136.5	121.5	-55.5	84.5	64.5	45.5	50.5	-9.5	-11.5	1.5	39.5	-0.5	44.5	-24.5	-44.5	95.5	-81.5	176.5	-157.5	400
196.5	177.5	-82.5	96.5	-45.5	-21.5	-15.5	54.5	-13.5	52.5	56.5	66.5	-27.5	68.5	-20.5	85.5	-56.5	122.5	-137.5	-156.5	400
-155.5	-138.5	123.5	-57.5	-40.5	-50.5	89.5	-48.5	87.5	91.5	70.5	-31.5	72.5	-33.5	74.5	-41.5	97.5	-83.5	178.5	195.5	400
194.5	179.5	-84.5	-65.5	110.5	-69.5	108.5	-67.5	106.5	-77.5	111.5	-63.5	102.5	-61.5	100.5	-59.5	104.5	124.5	-139.5	-154.5	400
-153.5	-140.5	132.5	126.5	-87.5	128.5	129.5	-90.5	-91.5	-100.5	-85.5	134.5	135.5	-96.5	137.5	-98.5	139.5	-93.5	180.5	193.5	400
192.5	165.5	157.5	-118.5	159.5	-120.5	161.5	-122.5	163.5	-133.5	181.5	-126.5	167.5	-128.5	169.5	-130.5	171.5	-132.5	-124.5	-152.5	400
200.5	-150.5	189.5	-148.5	187.5	-146.5	185.5	-144.5	183.5	-142.5	-179.5	-171.5	212.5	-173.5	214.5	-175.5	216.5	-177.5	218.5	201.5	400
400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400	400

The sub-magic square sums are as follows:

$$S_{4\times4} := 80 = 4\times20$$
$$S_{6\times6} := 120 = 6\times20$$
$$S_{8\times8} := 160 = 8\times20$$
$$S_{10\times10} := 200 = 10\times20$$
$$S_{12\times12} := 240 = 12\times20$$
$$S_{14\times14} := 280 = 14\times20$$
$$S_{16\times16} := 320 = 16\times20$$
$$S_{18\times18} := 360 = 18\times20$$
$$S_{20\times20} := 400 = 20\times20 = 20^2.$$

Bordered Magic Squares With Order Square Magic Sums,
Zenodo, January 20, 2020, pp. 1–26, http://doi.org/10.5281/zenodo.3613690

2.19 Bordered Magic Square of Order 21

The **bordered magic square** of order 21 for the magic sum 21^2 is given by

																					441
220	203	205	207	209	211	213	215	217	219	221	-183	-185	-187	-189	-191	-193	-195	-197	-199	-180	441
-198	184	-125	-127	-129	-131	-133	-135	-137	-139	-141	187	189	191	193	195	197	199	201	-140	240	441
-196	166	150	-93	-95	-97	-99	-101	-103	-105	-107	153	155	157	159	161	163	165	-106	-124	238	441
-194	168	134	120	106	108	110	112	114	116	-79	-80	-82	-84	-86	-88	-90	118	-92	-126	236	441
-192	170	136	-65	-52	-63	-61	-59	-57	-55	93	91	89	87	85	83	92	107	-94	-128	234	441
-190	172	138	-67	104	-30	-39	-37	-35	-33	71	69	67	65	63	70	-62	109	-96	-130	232	441
-188	174	140	-69	102	80	52	-18	-16	-14	-13	50	48	46	54	-38	-60	111	-98	-132	230	441
-186	176	142	-71	100	78	47	38	-2	0	1	36	34	40	-5	-36	-58	113	-100	-134	228	441
-184	178	144	-73	98	76	49	35	30	26	11	10	28	7	-7	-34	-56	115	-102	-136	226	441
-182	180	146	-75	96	74	51	37	15	18	23	22	27	5	-9	-32	-54	117	-104	-138	224	441
-181	-143	-109	-77	95	73	53	39	13	25	21	17	29	3	-11	-31	-53	119	151	185	223	441
218	-144	-110	123	-48	-26	-15	-1	33	20	19	24	9	43	57	68	90	-81	152	186	-176	441
216	-146	-112	125	-46	-24	-17	-3	14	16	31	32	12	45	59	66	88	-83	154	188	-174	441
214	-148	-114	127	-44	-22	-19	2	44	42	41	6	8	4	61	64	86	-85	156	190	-172	441
212	-150	-116	129	-42	-20	-12	60	58	56	55	-8	-6	-4	-10	62	84	-87	158	192	-170	441
210	-152	-118	131	-40	-28	81	79	77	75	-29	-27	-25	-23	-21	72	82	-89	160	194	-168	441
208	-154	-120	133	-50	105	103	101	99	97	-51	-49	-47	-45	-43	-41	94	-91	162	196	-166	441
206	-156	-122	-76	-64	-66	-68	-70	-72	-74	121	122	124	126	128	130	132	-78	164	198	-164	441
204	-158	148	135	137	139	141	143	145	147	149	-111	-113	-115	-117	-119	-121	-123	-108	200	-162	441
202	182	167	169	171	173	175	177	179	181	183	-145	-147	-149	-151	-153	-155	-157	-159	-142	-160	441
222	-161	-163	-165	-167	-169	-171	-173	-175	-177	-179	225	227	229	231	233	235	237	239	241	-178	441
441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441	441

The sub-magic square sums are as follows:

$$S_{3\times3} := 63 \ = 3\times21$$
$$S_{5\times5} := 105 = 5\times21$$
$$S_{7\times7} := 147 = 7\times21$$
$$S_{9\times9} := 189 = 9\times21$$
$$S_{11\times11} := 231 = 11\times21$$
$$S_{13\times13} := 273 = 13\times21$$
$$S_{15\times15} := 315 = 15\times21$$
$$S_{17\times17} := 357 = 17\times21$$
$$S_{19\times19} := 399 = 19\times21$$
$$S_{21\times21} := 441 = 21\times21 = 21^2$$

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Zenodo, January 20, 2020, pp. 1–26, http://doi.org/10.5281/zenodo.3613690

2.20 Bordered Magic Square of Order 22

The **bordered magic square** of order 22 for the magic sum 22^2 is given by

																						484
-200	252.5	-208	250.5	-206	248.5	-204	246.5	-202	244.5	-220	253.5	-198	241	-196	239	-194	237	-192	235	-190	243	484
-188	202.5	194.5	-152	196.5	-154	198.5	199.5	-157	-158	-169	-150	205	206	-163	208	-165	210	-167	212	-160	232	484
231	-149	166.5	-116	160.5	-118	162.5	-120	165	-122	176	-140	169	-126	171	-128	173	-130	175	-124	193	-187	484
-186	191.5	157.5	134.5	128.5	-85.5	130.5	131.5	-89	-89.5	-99	-83.5	137	138	-95	140	-97	142	-91.5	-114	-148	230	484
229	-147	-113	-82.5	106.5	-57.5	102.5	-59.5	105	-61.5	114	-75.5	109	-66	111	-68	113	-64	127	157	191	-185	484
-184	189.5	155.5	125.5	99.5	82.5	78.5	79.5	-37	-37.5	-45	-33.5	84.5	85.5	-43	87.5	-40	-56	-81.5	-112	-146	228	484
227	-145	-111	-80.5	-54.5	-32.5	62.5	57.5	-13	55.5	-11	-14.5	-25	69.5	-27	63.5	76.5	98.5	125	155	189	-183	484
-182	187.5	153.5	123.5	97.5	75.5	-15.5	47.5	52.5	-7.5	-9.5	3.5	41.5	1.5	46.5	59.5	-32	-54	-79.5	-110	-144	226	484
225	-143	-109	-78.5	-52.5	-30.5	60.5	44.5	34.5	10.5	37.5	4.5	36.5	8.5	-0.5	-17	74.5	96.5	123	153	187	-181	484
-180	185.5	151.5	121.5	95.5	73.5	-17.5	45.5	31.5	27.5	14.5	17.5	28.5	12.5	-1.5	61.5	-30	-52	-77.5	-108	-142	224	484
223	-141	-107	-76.5	-50.5	-28.5	67.5	50.5	13.5	20.5	25.5	22.5	19.5	30.5	-6.5	-24	72.5	94.5	121	151	185	-179	484
233	-178	-115	-106	-56.5	-49.5	-27.5	0.5	11.5	24.5	21.5	18.5	23.5	32.5	43.5	71.5	93.5	101	150	159	222	-189	484
255	-170	-133	-99.5	-70.5	-45.5	64.5	-4.5	5.5	15.5	26.5	29.5	16.5	38.5	48.5	-21	89.5	115	144	177	214	-211	484
-212	214.5	177.5	144.5	115.5	90.5	-21.5	-5.5	35.5	33.5	6.5	39.5	7.5	9.5	49.5	65.5	-47	-72	-101	-134	-171	256	484
257	-172	-135	-102	-72.5	-47.5	66.5	-2.5	-8.5	51.5	53.5	40.5	2.5	42.5	-3.5	-23	91.5	117	146	179	216	-213	484
-214	216.5	179.5	146.5	117.5	92.5	-19.5	-13.5	56.5	-11.5	54.5	58.5	68.5	-26	70.5	-19	-49	-74	-103	-136	-173	258	484
259	-174	-137	-104	-74.5	83.5	-34.5	-35.5	80.5	81.5	88.5	77.5	-41	-42	86.5	-44	-39	119	148	181	218	-215	484
-216	218.5	181.5	148.5	107.5	101.5	-58.5	103.5	-61	105.5	-70	119.5	-65	110	-67	112	-69	-63	-105	-138	-175	260	484
261	-176	-139	135.5	-84.5	129.5	-86.5	-87.5	133	133.5	143	127.5	-93	-94	139	-96	141	-98	-90.5	183	220	-217	484
-218	220.5	167.5	159.5	-117	161.5	-119	163.5	-121	165.5	-132	183.5	-125	170	-127	172	-129	174	-131	-123	-177	262	484
263	203.5	-151	195.5	-153	197.5	-155	-156	201	201.5	213	193.5	-161	-162	207	-164	209	-166	211	-168	-159	-219	484
-199	-209	251.5	-207	249.5	-205	247.5	-203	246	-201	264	-210	242	-197	240	-195	238	-193	236	-191	234	244	484
484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484	484

The sub-magic square sums are as follows:

$$S_{4\times 4} := 88 \quad = 4 \times 22$$
$$S_{6\times 6} := 132 = 6 \times 22$$
$$S_{8\times 8} := 176 = 8 \times 22$$
$$S_{10\times 10} := 220 = 10 \times 22$$
$$S_{12\times 12} := 264 = 12 \times 22$$
$$S_{14\times 14} := 308 = 14 \times 22$$
$$S_{16\times 16} := 352 = 16 \times 22$$
$$S_{18\times 18} := 396 = 18 \times 22$$
$$S_{20\times 20} := 440 = 20 \times 22$$
$$S_{22\times 22} := 484 = 22 \times 22 = 22^2.$$

2.21 Bordered Magic Square of Order 23

The **bordered magic square** of order 23 for the magic sum 23^2 is given by

																							529
266	-199	-201	-203	-205	-207	-209	-211	-213	-215	-217	-219	269	271	273	275	277	279	281	283	285	287	-218	529
244	222	205	207	209	211	213	215	217	219	221	223	-181	-183	-185	-187	-189	-191	-193	-195	-197	-178	-198	529
246	-196	186	-123	-125	-127	-129	-131	-133	-135	-137	-139	189	191	193	195	197	199	201	203	-138	242	-200	529
248	-194	168	152	-91	-93	-95	-97	-99	-101	-103	-105	155	157	159	161	163	165	167	-104	-122	240	-202	529
250	-192	170	136	122	108	110	112	114	116	118	-77	-78	-80	-82	-84	-86	-88	120	-90	-124	238	-204	529
252	-190	172	138	-63	-50	-61	-59	-57	-55	-53	95	93	91	89	87	85	94	109	-92	-126	236	-206	529
254	-188	174	140	-65	106	-28	-37	-35	-33	-31	73	71	69	67	65	72	-60	111	-94	-128	234	-208	529
256	-186	176	142	-67	104	82	54	-16	-14	-12	-11	52	50	48	56	-36	-58	113	-96	-130	232	-210	529
258	-184	178	144	-69	102	80	49	40	0	2	3	38	36	42	-3	-34	-56	115	-98	-132	230	-212	529
260	-182	180	146	-71	100	78	51	37	32	28	13	12	30	9	-5	-32	-54	117	-100	-134	228	-214	529
262	-180	182	148	-73	98	76	53	39	17	20	25	24	29	7	-7	-30	-52	119	-102	-136	226	-216	529
-221	-179	-141	-107	-75	97	75	55	41	15	27	23	19	31	5	-9	-29	-51	121	153	187	225	267	529
-222	220	-142	-108	125	-46	-24	-13	1	35	22	21	26	11	45	59	70	92	-79	154	188	-174	268	529
-224	218	-144	-110	127	-44	-22	-15	-1	16	18	33	34	14	47	61	68	90	-81	156	190	-172	270	529
-226	216	-146	-112	129	-42	-20	-17	4	46	44	43	8	10	6	63	66	88	-83	158	192	-170	272	529
-228	214	-148	-114	131	-40	-18	-10	62	60	58	57	-6	-4	-2	-8	64	86	-85	160	194	-168	274	529
-230	212	-150	-116	133	-38	-26	83	81	79	77	-27	-25	-23	-21	-19	74	84	-87	162	196	-166	276	529
-232	210	-152	-118	135	-48	107	105	103	101	99	-49	-47	-45	-43	-41	-39	96	-89	164	198	-164	278	529
-234	208	-154	-120	-74	-62	-64	-66	-68	-70	-72	123	124	126	128	130	132	134	-76	166	200	-162	280	529
-236	206	-156	150	137	139	141	143	145	147	149	151	-109	-111	-113	-115	-117	-119	-121	-106	202	-160	282	529
-238	204	184	169	171	173	175	177	179	181	183	185	-143	-145	-147	-149	-151	-153	-155	-157	-140	-158	284	529
-240	224	-159	-161	-163	-165	-167	-169	-171	-173	-175	-177	227	229	231	233	235	237	239	241	243	-176	286	529
264	245	247	249	251	253	255	257	259	261	263	265	-223	-225	-227	-229	-231	-233	-235	-237	-239	-241	-220	529
529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529	529

The sub-magic square sums are as follows:

$$S_{3\times 3} := 69 = 3 \times 23$$
$$S_{5\times 5} := 115 = 5 \times 23$$
$$S_{7\times 7} := 161 = 7 \times 23$$
$$S_{9\times 9} := 207 = 9 \times 23$$
$$S_{11\times 11} := 253 = 11 \times 23$$
$$S_{13\times 13} := 299 = 13 \times 23$$
$$S_{15\times 15} := 345 = 15 \times 23$$
$$S_{17\times 17} := 391 = 17 \times 23$$
$$S_{19\times 19} := 437 = 19 \times 23$$
$$S_{21\times 21} := 483 = 21 \times 23$$
$$S_{23\times 23} := 539 = 23 \times 23.$$

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2.22 Bordered Magic Square of Order 24

The **bordered magic square** of order 24 for the magic sum 24^2 is given by

																								576
289.5	-231	279.5	-233	281.5	-235	283.5	-237	-238	286.5	287.5	300.5	277.5	-243	-244	292.5	-246	294.5	-248	296.5	-250	298.5	-252	-241	576
310.5	-198	254.5	-206	252.5	-204	250.5	-202	248.5	-200	246.5	-218	255.5	-196	242.5	-194	240.5	-192	238.5	-190	236.5	-188	244.5	-263	576
-262	-186	204.5	196.5	-150	198.5	-152	200.5	201.5	-155	-156	-167	-148	206.5	207.5	-161	209.5	-163	211.5	-165	213.5	-158	233.5	309.5	576
308.5	232.5	-147	168.5	-114	162.5	-116	164.5	-118	166.5	-120	177.5	-138	170.5	-124	172.5	-126	174.5	-128	176.5	-122	194.5	-185	-261	576
-260	-184	193.5	159.5	136.5	130.5	-83.5	132.5	133.5	-86.5	-87.5	-96.5	-81.5	138.5	139.5	-92.5	141.5	-94.5	143.5	-89.5	-112	-146	231.5	307.5	576
306.5	230.5	-145	-111	-80.5	108.5	-55.5	104.5	-57.5	106.5	-59.5	115.5	-73.5	110.5	-63.5	112.5	-65.5	114.5	-61.5	128.5	158.5	192.5	-183	-259	576
-258	-182	191.5	157.5	127.5	101.5	84.5	80.5	81.5	-34.5	-35.5	-42.5	-31.5	86.5	87.5	-40.5	89.5	-37.5	-53.5	-79.5	-110	-144	229.5	305.5	576
304.5	228.5	-143	-109	-78.5	-52.5	-30.5	64.5	59.5	-10.5	57.5	-8.5	-12.5	-22.5	71.5	-24.5	65.5	78.5	100.5	126.5	156.5	190.5	-181	-257	576
-256	-180	189.5	155.5	125.5	99.5	77.5	-13.5	49.5	54.5	-5.5	-7.5	5.5	43.5	3.5	48.5	61.5	-29.5	-51.5	-77.5	-108	-142	227.5	303.5	576
302.5	226.5	-141	-107	-76.5	-50.5	-28.5	62.5	46.5	36.5	12.5	39.5	6.5	38.5	10.5	1.5	-14.5	76.5	98.5	124.5	154.5	188.5	-179	-255	576
-254	-178	187.5	153.5	123.5	97.5	75.5	-15.5	47.5	33.5	29.5	16.5	19.5	30.5	14.5	0.5	63.5	-27.5	-49.5	-75.5	-106	-140	225.5	301.5	576
-264	224.5	-139	-105	-74.5	-48.5	-26.5	69.5	52.5	15.5	22.5	27.5	24.5	21.5	32.5	-4.5	-21.5	74.5	96.5	122.5	152.5	186.5	-177	311.5	576
-219	234.5	-176	-113	-104	-54.5	-47.5	-25.5	2.5	13.5	26.5	23.5	20.5	25.5	34.5	45.5	73.5	95.5	102.5	151.5	160.5	223.5	-187	266.5	576
267.5	256.5	-168	-131	-97.5	-68.5	-43.5	66.5	-2.5	7.5	17.5	28.5	31.5	18.5	40.5	50.5	-18.5	91.5	116.5	145.5	178.5	215.5	-209	-220	576
-221	-210	216.5	179.5	146.5	117.5	92.5	-19.5	-3.5	37.5	35.5	8.5	41.5	9.5	11.5	51.5	67.5	-44.5	-69.5	-98.5	-132	-169	257.5	268.5	576
269.5	258.5	-170	-133	-99.5	-70.5	-45.5	68.5	-0.5	-6.5	53.5	55.5	42.5	4.5	44.5	-1.5	-20.5	93.5	118.5	147.5	180.5	217.5	-211	-222	576
-223	-212	218.5	181.5	148.5	119.5	94.5	-17.5	-11.5	58.5	-9.5	56.5	60.5	70.5	-23.5	72.5	-16.5	-46.5	-71.5	-101	-134	-171	259.5	270.5	576
271.5	260.5	-172	-135	-102	-72.5	85.5	-32.5	-33.5	82.5	83.5	90.5	79.5	-38.5	-39.5	88.5	-41.5	-36.5	120.5	149.5	182.5	219.5	-213	-224	576
-225	-214	220.5	183.5	150.5	109.5	103.5	-56.5	105.5	-58.5	107.5	-67.5	121.5	-62.5	111.5	-64.5	113.5	-66.5	-60.5	-103	-136	-173	261.5	272.5	576
273.5	262.5	-174	-137	137.5	-82.5	131.5	-84.5	-85.5	134.5	135.5	144.5	129.5	-90.5	-91.5	140.5	-93.5	142.5	-95.5	-88.5	184.5	221.5	-215	-226	576
-227	-216	222.5	169.5	161.5	-115	163.5	-117	165.5	-119	167.5	-130	185.5	-123	171.5	-125	173.5	-127	175.5	-129	-121	-175	263.5	274.5	576
275.5	264.5	205.5	-149	197.5	-151	199.5	-153	-154	202.5	203.5	214.5	195.5	-159	-160	208.5	-162	210.5	-164	212.5	-166	-157	-217	-228	576
-229	-197	-207	253.5	-205	251.5	-203	249.5	-201	247.5	-199	265.5	-208	243.5	-195	241.5	-193	239.5	-191	237.5	-189	235.5	245.5	276.5	576
288.5	278.5	-232	280.5	-234	282.5	-236	284.5	285.5	-239	-240	-253	-230	290.5	291.5	-245	293.5	-247	295.5	-249	297.5	-251	299.5	-242	576
576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576	576

The sub-magic square sums are as follows:

$$S_{4\times4} := 96 \ = 4 \times 24$$
$$S_{6\times6} := 144 = 6 \times 24$$
$$S_{8\times8} := 192 = 8 \times 24$$
$$S_{10\times10} := 240 = 10 \times 24$$
$$S_{12\times12} := 288 = 12 \times 24$$
$$S_{14\times14} := 336 = 14 \times 24$$
$$S_{16\times16} := 384 = 16 \times 24$$
$$S_{18\times18} := 432 = 18 \times 24$$
$$S_{20\times20} := 480 = 20 \times 24$$
$$S_{22\times22} := 528 = 22 \times 24$$
$$S_{24\times24} := 476 = 24 \times 24 = 24^2.$$

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2.23 Bordered Magic Square of Order 25

The **bordered magic square** of order 25 for the magic sum 25^2 is given by

																									625
-262	337	335	333	331	329	327	325	323	321	319	317	-263	-261	-259	-257	-255	-253	-251	-249	-247	-245	-243	-241	314	625
-240	268	-197	-199	-201	-203	-205	-207	-209	-211	-213	-215	-217	271	273	275	277	279	281	283	285	287	289	-216	290	625
-242	246	224	207	209	211	213	215	217	219	221	223	225	-179	-181	-183	-185	-187	-189	-191	-193	-195	-176	-196	292	625
-244	248	-194	188	-121	-123	-125	-127	-129	-131	-133	-135	-137	191	193	195	197	199	201	203	205	-136	244	-198	294	625
-246	250	-192	170	154	-89	-91	-93	-95	-97	-99	-101	-103	157	159	161	163	165	167	169	-102	-120	242	-200	296	625
-248	252	-190	172	138	124	110	112	114	116	118	120	-75	-76	-78	-80	-82	-84	-86	122	-88	-122	240	-202	298	625
-250	254	-188	174	140	-61	-48	-59	-57	-55	-53	-51	97	95	93	91	89	87	96	111	-90	-124	238	-204	300	625
-252	256	-186	176	142	-63	108	-26	-35	-33	-31	-29	75	73	71	69	67	74	-58	113	-92	-126	236	-206	302	625
-254	258	-184	178	144	-65	106	84	56	-14	-12	-10	-9	54	52	50	58	-34	-56	115	-94	-128	234	-208	304	625
-256	260	-182	180	146	-67	104	82	51	42	2	4	5	40	38	44	-1	-32	-54	117	-96	-130	232	-210	306	625
-258	262	-180	182	148	-69	102	80	53	39	34	30	15	14	32	11	-3	-30	-52	119	-98	-132	230	-212	308	625
-260	264	-178	184	150	-71	100	78	55	41	19	22	27	26	31	9	-5	-28	-50	121	-100	-134	228	-214	310	625
315	-219	-177	-139	-105	-73	99	77	57	43	17	29	25	21	33	7	-7	-27	-49	123	155	189	227	269	-265	625
316	-220	222	-140	-106	127	-44	-22	-11	3	37	24	23	28	13	47	61	72	94	-77	156	190	-172	270	-266	625
318	-222	220	-142	-108	129	-42	-20	-13	1	18	20	35	36	16	49	63	70	92	-79	158	192	-170	272	-268	625
320	-224	218	-144	-110	131	-40	-18	-15	6	48	46	45	10	12	8	65	68	90	-81	160	194	-168	274	-270	625
322	-226	216	-146	-112	133	-38	-16	-8	64	62	60	59	-4	-2	0	-6	66	88	-83	162	196	-166	276	-272	625
324	-228	214	-148	-114	135	-36	-24	85	83	81	79	-25	-23	-21	-19	-17	76	86	-85	164	198	-164	278	-274	625
326	-230	212	-150	-116	137	-46	109	107	105	103	101	-47	-45	-43	-41	-39	-37	98	-87	166	200	-162	280	-276	625
328	-232	210	-152	-118	-72	-60	-62	-64	-66	-68	-70	125	126	128	130	132	134	136	-74	168	202	-160	282	-278	625
330	-234	208	-154	152	139	141	143	145	147	149	151	153	-107	-109	-111	-113	-115	-117	-119	-104	204	-158	284	-280	625
332	-236	206	186	171	173	175	177	179	181	183	185	187	-141	-143	-145	-147	-149	-151	-153	-155	-138	-156	286	-282	625
334	-238	226	-157	-159	-161	-163	-165	-167	-169	-171	-173	-175	229	231	233	235	237	239	241	243	245	-174	288	-284	625
336	266	247	249	251	253	255	257	259	261	263	265	267	-221	-223	-225	-227	-229	-231	-233	-235	-237	-239	-218	-286	625
-264	-287	-285	-283	-281	-279	-277	-275	-273	-271	-269	-267	313	311	309	307	305	303	301	299	297	295	293	291	312	625
625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625	625

The sub-magic square sums are as follows:

$$S_{3\times3} := 75 = 3\times25$$
$$S_{5\times5} := 125 = 5\times25$$
$$S_{7\times7} := 175 = 7\times25$$
$$S_{9\times9} := 225 = 9\times25$$
$$S_{11\times11} := 275 = 11\times25$$
$$S_{13\times13} := 325 = 13\times25$$
$$S_{15\times15} := 375 = 15\times25$$
$$S_{17\times17} := 425 = 17\times25$$
$$S_{19\times19} := 475 = 19\times25$$
$$S_{21\times21} := 525 = 21\times25$$
$$S_{23\times23} := 575 = 23\times25$$
$$S_{25\times25} := 625 = 25\times25.$$

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Zenodo, January 20, 2020, pp. 1–26, http://doi.org/10.5281/zenodo.3613690

3 Final Remarks

Based the **magic sums** of the **bordered magic squares** given in previous section, we have some conclusions given in the Remark below.

Remark 3.1. *The results of previous section are summarized as*

- *order* 5, $k=5$, $S_{5\times m} := 5\times m$, $m=3$ *and* 5;
- *order* 6, $k=6$, $S_{6\times m} := 6\times m$, $m=4$ *and* 6;
- *order* 7, $k=7$, $S_{7\times m} := 7\times m$, $m=3,5$ *and* 7;
- *order* 8, $k=8$, $S_{8\times m} := 8\times m$, $m=4,6$ *and* 8;
- *order* 9, $k=9$, $S_{9\times m} := 9\times m$, $m=3,5,7$ *and* 9;
- *order* 10, $k=10$, $S_{10\times m} := 10\times m$, $m=4,6,8$ *and* 10;
- *order* 11, $k=11$, $S_{11\times m} := 11\times m$, $m=3,5,7,9$ *and* 11;
- *order* 12, $k=12$, $S_{12\times m} := 12\times m$, $m=4,6,8,10$ *and* 12;
- *order* 13, $k=13$, $S_{13\times m} := 13\times m$, $m=3,5,7,9,11$ *and* 13;
- *order* 14, $k=14$, $S_{14\times m} := 14\times m$, $m=4,6,8,10,12$ *and* 14;
- *order* 15, $k=15$, $S_{15\times m} := 15\times m$, $m=3,5,7,9,11,13$ *and* 15;
- *order* 16, $k=16$, $S_{16\times m} := 16\times m$, $m=4,6,8,10,12,14$ *and* 16;
- *order* 17, $k=17$, $S_{17\times m} := 17\times m$, $m=3,5,7,9,11,13,15$ *and* 17;
- *order* 18, $k=18$, $S_{18\times m} := 18\times m$, $m=4,6,8,10,12,14,16$ *and* 18;
- *order* 19, $k=19$, $S_{19\times m} := 19\times m$, $m=3,5,7,9,11,13,15,17$ *and* 19;
- *order* 20, $k=20$, $S_{20\times m} := 20\times m$, $m=4,6,8,10,12,14,16,18$ *and* 20;
- *order* 21, $k=21$, $S_{21\times m} := 21\times m$, $m=3,5,7,9,11,13,15,17,19$ *and* 21;
- *order* 22, $k=22$, $S_{22\times m} := 22\times m$, $m=4,6,8,10,12,14,16,18,20$ *and* 22;
- *order* 23, $k=23$, $S_{23\times m} := 23\times m$, $m=3,5,7,9,11,13,15,17,19,21$ *and* 23;
- *order* 24, $k=24$, $S_{24\times m} := 24\times m$, $m=4,6,8,10,12,14,16,18,20,22$ *and* 24;
- *order* 25, $k=25$, $S_{25\times m} := 25\times m$, $m=3,5,7,9,11,13,15,17,19,21,23$ *and* 25.

Finally, we reach to the following simplified general result.

Result 3.1. *For the* ***bordered magic squares*** *the* ***magic sum*** *of each* ***sub-magic square****, can be written as*

$$S_{k\times k} := k\times m,$$

where m is the order of ***bordered magic square*** *and k is the order of each* ***sub-magic squares****.*

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