Дослідження присвячено побудові математичної моделі, формулюванню крайових задач статики зернистого матеріалу стосовно до технологічних процесів сільськогосподарського виробництва. В якості робочого апарату побудови моделі зернистого матеріалу використовуються методи рівноважної термодинаміки. Сформульована основна термодинамічна рівність, що дозволяє отримати реологічне співвідношення, яке встановлює зв'язок між напруженнями і деформаціями зернистого матеріалу. У якості зернистого матеріалу обране сипке середовище з проявом ефекту Рейнольдса. Цей ефект має місце в разі малих деформацій і говорить про наявність залежності дилатації від девіатора тензора деформацій. На відміну від класичних методів, де розглядається модель суцільного середовища з недеформованими і гладкими частинками зерна, тут враховується як ефект Рейнольдса, так і наявність пружних деформацій. Отримане реологічне співвідношення дає залежність для тензора напружень від тензора деформацій відповідного співвідношенням лінійної теорії пружності.

У разі ізотермічного процесу деформування сформульована крайова задача статики зернового матеріалу в полі сил тяжіння. В роботі наведені постановка і рішення двох приватних завдань про рівновагу зернистого шару на горизонтальній площині: при відсутності поверхневих сил і при дії дотичних поверхневих сил на вільної поверхні.

Крайові задачі рівноваги зернистого матеріалу носять нелінійний характер, а отримане рішення представляє складний математичний апарат з залученням чисельних методів.

Отримані моделі статики суцільного середовища передує розгляду динамічних завдань, зокрема, вивчення стійкості рівноваги

Ключові слова: зернисті матеріали, рівноважна термодинаміка, ефект Рейнольдса, горизонтальний зерновий шар, крайова задача, граничні умови

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CONSTRUCTION OF MATHEMATICAL MODELS OF THE STATICS OF GRAIN MEDIA CONSIDERING THE REYNOLDS EFFECT

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1. Introduction

Granular materials (loose, grainy media are synonymous with the concept of granular) are widespread in nature and are used in various technological processes, for example, in agriculture. Grainy media have certain properties that distinguish them from «classic» materials – solid, liquid, and gaseous. This is due to that grainy media are composed of quite a large number of macroscopic particles (sand, grain, ore) whose dimensions significantly exceed the size of atoms (molecules). The interaction among particles occurs only when they collide or directly come into contact. This leads to that the environment does not perceive stretching efforts. In addition, the clashes are inelastic in character. A grainy material can manifest itself as a solid deformable body (a state of mechanical equilibrium), a viscoplastic or elastic body at minor deformations of the medium, or as a gas during its intense movement (a fast motion theory). The latter relates to the concept of pseudo-fluidization, which is caused either by the movement of a gas (a liquid) through pores between particles or by imposing high-frequency vibrations on the medium.

As regards agricultural technological processes, grain crop act as grain media. Therefore, it is important to study both the static and dynamic processes (outflow of grain from a bunker, the dynamics of a grain layer on a sieve, separation of pure grain and weed impurities). The considered material is a two-phase environment with a carrier in the form of a gas filling the space between grains, and a dispersed component – grain. The grain's density significantly exceeds the density of a gas. Therefore, the impact of gas on the state of such an environment can be neglected in general.

The phenomenon of loosening and compaction of granular media at displacement was noted for the first time by O. Reynolds in 1886. He termed the phenomenon dilatancy and explained it by repacking the particles of sand at displacement. The simplest environment in which dilatancy processes manifest themselves is a granular medium. It consists of separate solid particles, which, at a known density, form a certain packed arrangement in space. Dilatancy manifests itself at both elastic and plastic and viscous deformation, and, in all cases, it can be accompanied by both an increase in volume and its decrease. It is typical for most highly concentrated dispersed systems (mostly high-filled with a solid phase).

Constructing a model of a grainy environment taking into consideration the Reynolds effect and elastic properties employs a thermodynamic approach, based on the first and second laws of equilibrium thermodynamics. The characteristic kinetic property of a grainy environment at deformation, which distinguishes it from the elastic medium, is a dilatancy effect or the Reynolds effect. It implies that under shear deformations the particles are repackaged, which changes their volumetric concentration. Such an approach to modeling a grainy medium provides for the prospects of using the developed methods of nonequilibrium thermodynamics for a wider range of problems (the grain media dynamics).

The relevance of this study is due to the systematic use of equilibrium (and, subsequently, nonequilibrium) thermodynamics methods to derive equations of the balance of grainy materials mechanics, as well as to demonstrating the need to devise methods for solving appropriate boundary-value problems.

2. Literature review and problem statement

By its mechanical properties, grainy structures occupy an intermediate position between solid and liquid bodies. However, while the fundamental equations of the latter (the equations of elasticity theory, the Navier-Stokes equations, etc.) have long and firmly been established from the general principles of mechanics, the statement of such equations for a grainy medium motion is still an unresolved task. The grainy environment refers to a cohesive set of chaotically packaged particles that maintains contacts among neighbors when moving. At moderate deformations, the environment with a sufficiently dense initial packing is able to maintain equilibrium in a deformed state, like an elastic body. However, at large enough deformations, it loses stability and, like a liquid, demonstrates fluidity. The kinetic variables in this case are no longer the deformations, but the speed of the deformations.

Granular materials are widespread in nature and are used in various technological processes [1]. However, despite considerable efforts to develop the relevant theory, the successes in this field of mechanics are insignificant while the results are contradictory.

The difficulty of studying the behavior of granular materials is predetermined by their physical properties [2]. A granular material is a multi-phase environment, whose dispersed component is rather large solid particles, and the carrier environment is typically a gas (or a liquid) [3]. The effect of a carrier environment on dispersed particles is generally neglected [4].

Given that the particles of the environment are large and can acquire an intricate shape, the character of their interaction is reduced to a contact, impact, friction. Due to the absence of gravitational forces among the particles, there can be no stretching forces in the environment – normal stresses should be negative [5]. This leads to various effects in such media, in particular, their separation into fractions [6], which was experimentally confirmed by studies [7, 8].

The properties of a granular environment are significantly dependent on the state that this environment is in:

1) solid – particles are densely packed, which prevents them from executing significant relative displacements [11];

2) liquid – the relative movements of particles can be significant, but have weak fluctuations of the field of velocities [11]. In this case, the behavior of the environment is significantly influenced by the presence of a carrier environment – a liquid (a gas) [3];

3 gaseous – particles execute intense chaotic motion, similar to the movement of atoms (molecules) of gas at high enough temperatures. Such a state is called a «fast movement» state [12–14].

The second and third states are typically reduced to one and denote it a state of «pseudo-fluidization». This state can be entered by a granular environment by imposing small-scale rapidly oscillating external force fields (vibrations) on it. Technical implementation of the pseudo-fluidization of loose (granular) materials has received much attention in papers [10, 15, 16, 18].

The main issue is deriving a rheological ratio, which would define the relationship between a stress tensor and the deformation tensor. Such a relation should take into consideration the character of interaction among dispersed particles.

There are the following approaches to obtaining such ratios:

a) phenomenological, whereby mathematical ratios are established on the basis of the experiments [11]. It simulates a granular environment in the form of identical spherical balls of a small diameter. The dependence of stresses on an environment's density and a technique of initial arrangement of balls has been established. However, the cited work does not take into consideration the Reynolds effect in deforming the environment;

b) representation of a granular environment in the form of a finite set of particles of a certain shape (a system with the finite number of degrees of freedom). In this case, the behavior of an object is described by ordinary differential equations from analytical mechanics, which are usually solved numerically. Work [15] gives such a solution. The dynamic character of the circulation of gas and particulate matter, the distribution of their concentration within a working zone were investigated based on the rate of pseudo-fluidization and the average size of particles. However, the rheological ratio was not established either, which is a characteristic drawback of other works;

c) the use of statistical physics methods, whereby the type of a distribution function is established for a certain type of particle interaction, based on which the laws are formulated. These laws govern the behavior of a granular environment [18].

Our analysis of studies [11, 15, 19, 20] has also revealed a failure to account for the Reynolds effect.

This effect is considered in [21] with the construction of a rheological ratio, but only for the case of perfectly smooth, absolutely solid ball-shaped particles. In this case, the authors build an environment's equilibrium equations for the case of small deformations. However, solving specific applied equilibrium problems by such equations is difficult; they are not considered. In addition, resolving rheological ratios is based

on the variation principle of virtual works, but the potential energy of deformation is taken equal to zero, which does not make it possible to take into consideration elastic deformations in the model of an environment.

Thus, our analysis of known studies has identified an issue related to the absence of theoretical approaches to modeling the static of a grain granular environment taking into consideration the Reynolds effect.

3. The aim and objectives of the study

The aim of this study is to derive a closed system of equations and boundary conditions describing the static of a grain environment in the external force fields, taking into consideration the Reynolds effect.

To accomplish the aim, the following tasks have been set:

– to derive a principal thermodynamic equality for the chosen object of a continuous medium, the dependence of stress tensor on the deformation tensor in isothermal processes;

– to state the boundary-value problems on a grain medium's statics;

– to solve the simplest problems on a grain medium's equilibrium;

– to determine significant parameters for a grain medium's equilibrium process.

4. Mathematical methods for studying the equilibrium state of a grainy environment

A procedure for solving the problems on modeling the equilibrium state of a grainy environment implies the following algorithm: study the thermodynamics of a grainy material with the derivation of a rheological expression for its static; state a boundary-value problem and determine conditions for the course of the process; verify the devised procedure using applied problems on the equilibrium of a horizontal grainy layer along a solid horizontal plane in a two-dimensional statement.

4. 1. Thermodynamics of a grainy material

Introduce the following designations: x_1, x_2, x_3 – coordinates of the Cartesian system of coordinates; u_1, u_2, u_3 – vector of infinitesimal movements;

$$\varepsilon_{ik}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right),\quad \gamma_{ik}=\varepsilon_{ik}-\frac{1}{3}\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)\delta_{ik},$$

where δ_{ik} are the symbols by Kronecker; σ_{ik} is the stress tensor;

$$\tau_{ik}=\sigma_{ik}+p\delta_{ik},$$

where

$$p=-\frac{1}{3}\left(\sigma_{11}+\sigma_{22}+\sigma_{33}\right)$$

is the stress tensor deviator; ρ is the density of a grain material:

$$\rho=\rho_3\,\tau, \tag{1}$$

where τ is the volumetric grain density; ρ_3 is the true density of a grain's material.

The volumetric density τ depends on the way the grains are packed in a grainy environment [21, 22]. In the case of a random packing, we have $\tau=0.61\div 0.74$.

We shall consider infinitely small deformations, which corresponds to the linear theory of elasticity. In this case, one can consider that a body before and after the deformation occupies the same area of space. At the same time, a material's density ρ_0, prior to deformation, and ρ, after it, satisfies the following equation to an accuracy of second order small values:

$$\rho\left(1+\mathrm{d}iv\,\vec{u}\right)=\rho_0. \tag{2}$$

Let the grainy environment fill the area of space V, which is limited by surface $\Sigma=\Sigma_u+\Sigma_\sigma$ and which is exposed to external mass intensity forces g and surface forces p at border Σ of area V. Denote by U the displacements of surface points Σ_u. We consider the statics when a body under the influence of the specified force factors enters a certain strained deformed state of equilibrium. This state is described by the following equations:

$$\frac{\partial\sigma_{ki}}{\partial x_k}+\rho g_i=0 \tag{3}$$

and boundary conditions:

$$u_i=U_i\left(\Sigma_u\right), \tag{4}$$

$$n_k\sigma_{ki}=p_i\left(\Sigma_\sigma\right), \tag{5}$$

($i, k=1, 2, 3$). Over repeated indices, summation is from 1 to 3!

Set the body's points into virtual movements δu_i, which meet conditions [4]. Elementary work δA^e, performed by external bodies (external forces: mass g and surface p) to a given body in the specified virtual movements, is equal to:

$$\delta A^e=\int_V\rho g_i\delta u_i\,\mathrm{d}V+\oint_\Sigma p_i\delta u_i\mathrm{d}\Sigma.$$

Considering (3) to (5) and applying a Gauss-Ostrogradsky formula, this ratio is reduced to the form:

$$\delta A^e=\int_V\sigma_{ik}\delta\varepsilon_{ik}\,\mathrm{d}V,$$

which can be interpreted as the equality of elementary works by external bodies to the system and the work of the system to external bodies.

Thus, the magnitude of work of external forces to system δa^e, per unit of a body's mass, is equal to:

$$\delta a^e=\frac{1}{\rho}\sigma_{ik}\delta\varepsilon_{ik}, \tag{6}$$

where

$$\delta\varepsilon_{ik}=\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial x_k}+\frac{\partial\delta u_k}{\partial x_i}\right).$$

If one uses the representation of second-rank tensors in the form of amounts of ball tensors and deviators:

$$\sigma_{ik}=-p\delta_{ik}+\tau_{ik},\quad\left(p\equiv-\frac{1}{3}\sigma_{kk}\right),\quad\delta\varepsilon_{ik}=\frac{1}{3}\delta\vartheta\,\delta_{ik}+\delta\gamma_{ik},$$

ratio (6) can then be represented in the following form:

$$\delta a^e = -\frac{p}{\rho}\delta\vartheta + \frac{1}{\rho}\tau_{ik}\delta\gamma_{ik}. \quad (7)$$

Next, we proceed to considering the Reynolds effect. Resistance of granular materials when one part shifts against another is determined by the friction of sliding and rolling of moving particles, and in dense coarse-grained media – also by the so-called grip. The latter is part of the general resistance to shear, which in dense granular materials is necessary for some lifting and upward movement of particles, without which they, being wedged among neighboring particles, cannot move or roll over one another (dilatancy by O. Reynolds).

Studies into a dilatancy phenomenon and its effect on the character of deformation of granular media under load, on the example of soils, were reported in [23, 24]. The authors note that not taking into consideration the phenomenon of dilatancy leads to distortion of estimated indicators and schemes of soil deformations. It was established that the reliability of foundations and buildings in general is based on an in-depth representation of the mechanism of internal friction in soils, elucidation of the dilatancy properties, detailed research and studying the strength of loose soils, which serve as the base for structures.

Thus, using the Reynolds effect is justified in modeling the equilibrium state of grain granular media.

For the isotropic environment, we have the following ratio [21, 22]:

$$\vartheta = \vartheta\left(I_2\left(\gamma_{ik}\right)\right), \quad (8)$$

where $I_2(\gamma_{ik}, \gamma_{ik})=1/2\gamma_{ik}\gamma_{ik}$ is the second invariant of a stress tensor deviator. For the case of infinitesimal deformations, ratio (8) takes the form:

$$\vartheta = \mu\,\gamma_{ik}\,\gamma_{ki}, \quad (9)$$

where μ is the positive constant equal to $0.59\div 0.63$.

Variating the last ratio produces the following expression:

$$\delta\vartheta = 2\mu\gamma_{ik}\delta\gamma_{ki}.$$

Expression (7) can then be transformed to the form:

$$\delta a^e = \tau^*_{ik}\delta\gamma_{ki},$$

where

$$\tau^*_{ik} = \frac{1}{\rho}\left(\tau_{ik} - 2\mu\, p\,\gamma_{ik}\right). \quad (10)$$

Knowing the expression for elementary equilibrium work of external bodies to the body, we can record a principal thermodynamic equality, which is the result of the implementation of the first and second laws of thermodynamics:

$$\delta u = T\delta s + \tau^*_{ik}\delta\gamma_{ki}, \quad (11)$$

where $u=u(s, \gamma_{ik})$ is the internal energy of the system; T is temperature; s is entropy. It should be noted that the number of independent thermodynamic variables here is less by unity compared to the generally accepted basic thermodynamic equality.

Equation [11] can be rewritten in terms of the thermodynamic potential of free energy $f=f(T, \gamma_{ik})=u-Ts$:

$$\delta f = -s\delta T + \tilde{\tau}_{ik}\delta\gamma_{ki}, \quad (12)$$

hence, the expressions for caloric:

$$s = -\left(\frac{\partial f}{\partial T}\right)_{\gamma_{ik}}, \quad (13)$$

and thermal:

$$\tilde{\tau}_{ik} = \left(\frac{\partial f}{\partial \gamma_{ik}}\right)_{T}, \quad (14)$$

state equations.

Here, the thermodynamic potential f is a function of independent thermodynamic variables T, γ_{ik}. Consequently, there are dependences $s=s(T,\gamma_{ik})$, $\tau^*_{ik}=\tau^*_{ik}\left(T, \gamma_{ik}\right)$.

Following the procedure of linear thermodynamics, represent free energy in the form of a Taylor series in the vicinity of initial equilibrium state ($T=T_0$, $\gamma_{ik}=0$, $\sigma_{ik}=0$) and leave in the expansion those terms that do not exceed a second order:

$$f\left(T,\gamma_{ik}\right) = f\left(T_0,0\right) + \left(\frac{\partial f}{\partial T}\right)_{T_0,0}\left(T-T_0\right) + \left(\frac{\partial f}{\partial \gamma_{ik}}\right)_{T_0,0}\gamma_{ik} +$$
$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial T^2}\right)_{T_0,0}\left(T-T_0\right)^2 + \left(\frac{\partial^2 f}{\partial T\,\partial\gamma_{ik}}\right)_{T_0,0}\times$$
$$\times\left(T-T_0\right)\gamma_{ik} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial\gamma_{ik}\partial\gamma_{mn}}\right)_{T_0,0}\gamma_{ik}\gamma_{mn}. \quad (15)$$

Let the initial state be natural:

$$s \equiv \left(\frac{\partial f}{\partial T}\right)_{T_0,0} = 0,\ \sigma_{ik} \equiv \left(\frac{\partial f}{\partial \gamma_{ik}}\right)_{T_0,0} = 0.$$

Then the expression for free energy, determined with an accuracy to a constant, will be simplified. The following designations are introduced:

$$\theta = T - T_0,\ \left(\frac{\partial^2 f}{\partial T^2}\right)_{T_0,0} = -\frac{C_\gamma}{2T_0},\ \left(\frac{\partial^2 f}{\partial T\partial\gamma_{ik}}\right)_{T_0,0} =$$
$$= -b_{ik},\ \left(\frac{\partial^2 f}{\partial\gamma_{ik}\partial\gamma_{mn}}\right)_{T_0,0} = A_{ikmn},$$

where C_γ is the heat conductance in the absence of deformations; b, A_{ikmn} are constants. It follows from (15):

$$f\left(T,\gamma_{ik}\right) = -\frac{C_\gamma}{2T_0}\theta^2 - b_{ik}\theta\gamma_{ik} + \frac{1}{2}A_{ikmn}\gamma_{ik}\gamma_{mn}. \quad (16)$$

This expression makes it possible to derive rheological ratios for the case of an anisotropic environment. Hereafter, we shall consider isotropic media. For them, free energy will depend on invariants ϑ, $I_2(\gamma_{ik})$. A first invariant $I_1(\gamma_{ik})=\gamma_{11}+\gamma_{22}+\gamma_{33}$ is zero. Therefore, the second term in (16) for the case of an isotropic environment in the presence of the Reynolds effect is absent. The expression for free energy will take a simpler form:

$$f(T,\gamma_{ik})=-\frac{C_\gamma}{2T_0}\theta^2+\frac{\nu}{2\rho_0}\gamma_{ik}\gamma_{ki}, \quad (17)$$

where ν is the constant, ρ_0 is a material's density in the initial state; and at minor deformations we accept $\rho_0=\rho$.

It should be noted that the constants $C\gamma$, ν, present here, should be positive magnitudes, as it follows from a condition for the thermodynamic stability [25, 26]. Hereafter, we shall consider the equilibrium of a grainy environment at a constant temperature, that is an isothermal transition from the unloaded state to the stressed-deformed state. The necessary rheological ratio for a given case follows from (12), (14), (17) and from determining τ^*_{ik}:

$$\tau^*_{ik}=\left(\frac{\partial f}{\partial\gamma_{ik}}\right)=\frac{2\nu}{\rho}\gamma_{ik}, \quad \tau_{ik}=2(\nu+\mu p)\gamma_{ik}. \quad (18)$$

Returning to the stress tensor, we obtain a rheological ratio from (18):

$$\sigma_{ik}=-p\delta_{ik}+2(\nu+\mu p)\gamma_{ik}. \quad (19)$$

This rheological ratio (19) is the closing ratio in the system of equations of the statics of a grainy material. The difference between ratio (19) and those previously considered by other authors is that in the right-hand part there is an additional parameter ν, responsible for the presence of elastic deformations in a grainy environment.

Further verification of the resulting expressions implies stating a boundary-value problem and solving specific applied problems on the equilibrium of a horizontal grainy layer.

4. 2. A boundary-value problem on a grainy material's equilibrium

For the further research, we shall state a boundary-value problem on the equilibrium of a grainy material, which occupies region V, limited by the surface $\Sigma=\Sigma_u+\Sigma_\sigma$. If Σ_u is a solid unmovable wall, one can assign two types of conditions for it:

a) $u_i=0$;

b) $u_n=0, \quad n_k\sigma_{ki}u_i/|\vec{u}|=f_K n_k\sigma_{ki}n_i \quad (n_k\sigma_{ki}n_i>0), \quad u_n=0, \quad n_k\sigma_{ki}-n_k\sigma_{kj}n_j n_i-f_K n_k\sigma_{kj}n_j u_i/|\vec{u}|, \quad (n_k\sigma_{ki}n_i>0)$.

Here, n_i is the external single normal vector to Σ_u, f_K is the external ratio of dry friction. In the second case, it is believed that the boundary points of a body at deformation overcome the force of friction according to the law of dry friction by Coulomb. If a boundary Σ_σ divides two different media (in particular, it can separate a region, occupied by a grainy environment, from an empty space), then one assigns the «dynamic» boundary conditions on it:

$$\langle n_k\sigma_{ki}n_i\rangle=0, \langle n_k\sigma_{ki}\tau_i\rangle=0, \quad (20)$$

where τ_i is the single vector, tangent to Σ_σ; angular brackets in the expressions denote a jump of the corresponding magnitude in a transition over the surface Σ_σ.

In particular, if Σ_σ is exposed to external distributed forces $p=np_n+p_\tau$ (p_n, p_τ – normal and tangent components of these forces), boundary conditions (20) take the form:

$$n_k\sigma_{ki}n_i=p_n, \quad (21)$$

$$n_k\sigma_{ki}\tau_i^{(\alpha)}=p_m\tau_m^{(\alpha)}, \quad (22)$$

where $\tau_m^{(\alpha)}(\alpha=1,2)$ are the two noncollinear single vectors, tangent to Σ_σ. The stressed-deformed state of an environment is described by variables u_1,u_2,u_3,p that satisfy, in region V, static equations:

$$\frac{\partial\sigma_{ki}}{\partial x_k}+\rho g_i=0, \quad (23)$$

rheological ratios:

$$\sigma_{ik}=-p\delta_{ik}+2(\nu+\mu p)\varepsilon_{ik}, \quad \varepsilon_{ik}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right), \quad (24)$$

closing ratio:

$$\vartheta-\mu(\gamma_{ik}\gamma_{ki})=0 \left(\gamma_{ik}=\varepsilon_{ik}-\frac{1}{3}\vartheta\delta_{ik}\right). \quad (25)$$

For the resulting solution, the $p>0$ condition must be met.

Above is the relationship between the density of an environment and its volumetric density (1). In order not to disrupt the continuity of an environment, the volumetric density τ must accept limited values $\tau_{min}<\tau<\tau_{max}$. Therefore, when solving a problem, one needs to make sure that the density of a material is in the interval $\rho_3\tau_{min}<\rho<\rho_3\tau_{max}$. The law of mass preservation produces a ratio that makes it possible to define density as a function of deformations:

$$\rho(1+\vartheta)=\rho_0. \quad (26)$$

Thus, it is obvious that solving such problems is associated with significant mathematical difficulties, which testifies to the relevance of the devised methodology. Next, we shall consider two relatively simple applied problems on the equilibrium of a horizontal grainy layer along a solid horizontal plane in a two-dimensional statement, which is typical for the technological processes related to agricultural machines.

4. 3. Equilibrium of a two-dimensional horizontal grain layer in the field of gravity forces

Consider a layer of permanent thickness h, located on a horizontal wall, and is exposed to gravity force $g=e_3g$, where e_z is the Oz axis ort (Fig. 1). Consider a layer of permanent thickness h, located on a horizontal wall, and is exposed to gravity force $g=e_3g$, where e_z is the Oz axis ort (Fig. 1).

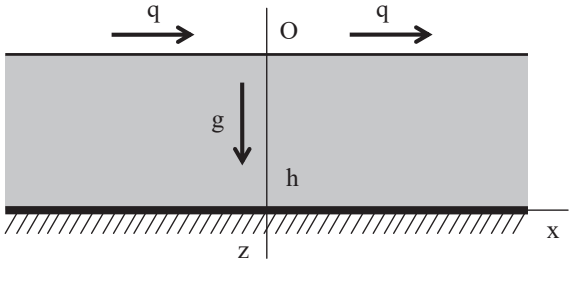

Fig. 1. Schematic of a horizontal grain layer: g – free fall acceleration; q – intensity of tangent stresses; h – layer's thickness

Let the field of displacements depend only on variable z and has a single non-zero component $u=e_z w(z)$. Then the strain tensor has one non-zero component $\varepsilon_{33}=w'$, where

(‘) means differentiation by variable z. The corresponding non-zero components of the deviator of strain tensor and a stress tensor are equal:

$$\gamma_{11}=-\frac{1}{3}w',\ \ \gamma_{22}=-\frac{1}{3}w',$$

$$\sigma_{11}=-p,\ \ \sigma_{22}=-p,\ \sigma_{33}=-p+2(\nu+\mu p)w',$$

$$\gamma_{ik}=\sigma_{ik}=0\,(i\neq k).$$

The first two equations in (23) suggest that pressure p does not depend on variables x, y, the third equilibrium equation takes the form of an ordinary differential equation:

$$-(1-2\mu w')p'+(\nu+2\mu p)w''=-\rho g. \tag{27}$$

Here, the function $\rho=\rho(z)$, in which a given dependence is determined from the law of mass preservation (26):

$$\rho=\frac{\rho_0}{1+\vartheta}\equiv\frac{\rho_0}{1+w'}. \tag{28}$$

A Reynolds condition is a closing equation:

$$w'-\frac{2}{3}\mu(w')^2=0. \tag{29}$$

The last equation produces two solutions:
a) $w'=0$;
b) $w'=3/(2\mu)$.
Hence, due to the boundary condition $w(h)=0$, we obtain:
a) $w=0$;
b) $w=3z/(2\mu)$.
Boundary condition (22) is met identically ($p_\tau=0$), and condition (21) takes the form $p(0)=0$. It follows from equation (27), (28), taking into consideration the boundary conditions at $z=h$:
– case a):

$$p=\rho_0 gz; \tag{30}$$

– case b):

$$p(z)=-\frac{\rho_0 g\mu}{3+2\mu}z. \tag{31}$$

Expression (31) produces negative values for pressure, which does not correspond to the physical state of a grainy environment. Ratio (30) corresponds to the distribution of pressure in a column of an ideal homogeneous environment.

4. 4. Equilibrium of a horizontal layer when a horizontal shear force acts on the free surface

A more interesting and more complex solution is produced by the problem, in which, in addition to gravity force, there is an evenly distributed tangent effort $q=e_x q$ acting on the free surface $(z=0)$:

$$\vec{u}=(u(z),0,w(z)), \tag{32}$$

dependent on a single variable z. Then the non-zero components of a strain tensor are equal to $\varepsilon_{13}=u'/2$, $\varepsilon_{33}=w'$, and the non-zero components of a stress tensor are equal to $\sigma_{11}=-p$, $\sigma_{33}=-p+2(\nu+\mu p)w'$.

The equilibrium equations take the form:

$$\frac{\partial\sigma_{31}}{\partial z}=0,\ \ \frac{\partial\sigma_{33}}{\partial z}+\rho g=0. \tag{33}$$

Consider that there is a friction force on the wall ($z=h$) that prevents the layer from moving:

$$u(h)=w(h)=0, \tag{34}$$

Conditions (21), (22) hold at the free surface ($z=0$):

$$\sigma_{31}\equiv(\nu+\mu p)u'=-q,\ (z=0), \tag{35}$$

$$\sigma_{33}\equiv-p+2(\nu+\mu p)w'=0,\ (z=0). \tag{36}$$

Note that there is the first integral derived from the first equation (33), which, taking into consideration boundary condition (35), can be written in the form:

$$[\nu+\mu p(z)]u'(z)=-q. \tag{37}$$

Let us consider this ratio as an equation for finding p:

$$p(z)=-\frac{1}{\mu}\left[\nu+\frac{q}{u'(z)}\right]. \tag{38}$$

The second ratio (33) considering (38) produces an ordinary differential equation relative to unknowns $u(z)$, $w(z)$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\mu}\left(\nu-\frac{q}{u'}\right)+2(\nu+\mu p)w'\right]+\frac{\rho_0 g}{1+w'}=0. \tag{39}$$

A Reynolds condition takes the form:

$$w'-2/3\,\mu w'^2-1/2\,\mu u'^2=0. \tag{40}$$

Differentiate (40) for z and we obtain:

$$-\mu u'u''+(1-4\mu w'/3)w''=0. \tag{41}$$

Excluding variables u', u'', using (40), (41), from ratio (39):

$$u'=\frac{\sqrt{6\mu w1'(3-2\mu w')}}{3\mu},\ \ u''=\frac{(4\mu w'-3)w''}{\sqrt{6\mu w'(3-2\mu w')}} \tag{42}$$

and solving it relative to w'', we obtain:

$$w''=\frac{2}{3}\frac{\rho_0 g\mu w'(2w'\mu-3)}{q(1+w')(2w'\mu+3)}\sqrt{6\frac{w'(3-2w'\mu)}{\mu}}. \tag{43}$$

Equation (43) can be reduced to a first-order equation by introducing a new variable:

$$W(z)=\frac{\mathrm{d}w(z)}{\mathrm{d}z}$$

and consider a Cauchy problem with an initial condition that follows from ratios (36), (38), (42):

$$W(0)=\frac{12q^2\mu^2+6\nu^2-\nu\sqrt{36\nu^2+96q^2\mu^2}}{8\left(3q^2\mu^2+\nu^2\right)\mu}. \quad (44)$$

After finding a solution to the Cauchy problem, we determine the displacement $u(z)$ by integrating the first ratio (42) and pressure according to (38). It should be noted that the variables u, w are determined with an accuracy to the constant component. This flaw is eliminated by the requirement that conditions $u(h)=w(h)=0$ should be met on a rigid wall ($z=h$).

5. Results of a numerical study of the equilibrium of a grainy layer lying on the horizontal plane

The calculation results are given in the form of charts in Fig. 2–5; their location is given in Table 1.

Table 1

Calculation results

h, m	0.01		0.02	
q, n/m^2	10	100	10	100
Figure	2	3	4	5

Each figure contains six graphic windows marked with letters a)–f) with four charts inside, which represent the following diagrams for different values of parameter $\nu=10$, 50, 100, 150 n/m^2 and value $\mu=0.65$:

a) distributions, along z, of the vertical movements of body points $w(z)$;

b) distributions, along z, of the horizontal movements of body points $u(z)$;

c) distribution, along z, of pressure $p(z)$;

d) distributions, along z, of tangent stresses $\sigma_{31}(z)$;

e) distributions, along z, of normal stresses $\sigma_{33}(z)$;

f) distributions, along z, of the medium's density $\rho=\rho(z)$.

Our analysis of dependences in Fig. 2–5, *a* has established that the normal movements are negative, they are directed towards the free surface ($z=0$) with their module increasing, which causes the layer to thicken. Our analysis of Fig. 2–5, *b* shows that the horizontal movements are directed in the same direction as the surface load q and increase as they approach the free surface.

Pressure in the layer increases with depth not according to a linear law (charts in all figures marked with letter *c*). Density of the environment increases with depth (Fig. 2–5, *f*). The dependences shown in Fig. 2–5, *e* indicate a slight dependence of tangential stresses and close values on their tangential stress, which is created on the free surface $\sigma_{31}\approx -q$. Normal stress turns to zero on the free surface, and in the remaining points of the layer is negative, which indicates the adequacy of the model built (Fig. 2–5, *e*).

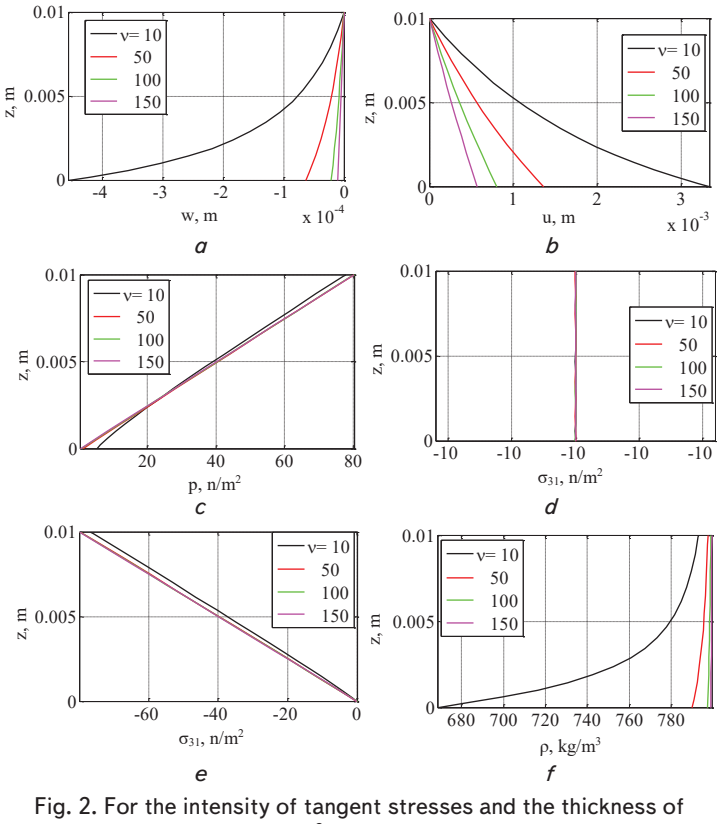

Fig. 2. For the intensity of tangent stresses and the thickness of a layer, equal to $q=10$ n/m^{-2}, $h=0.01$ m: *a* — vertical movements; *b* — horizontal movements; *c* — pressure, *d* — tangent stresses; *e* — normal stresses; *f* — environment's density

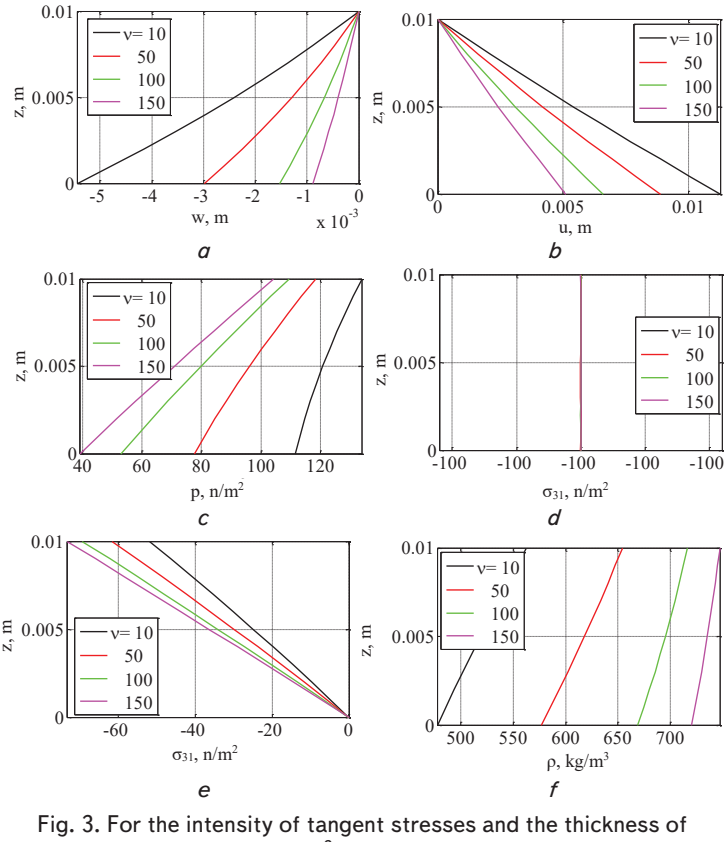

Fig. 3. For the intensity of tangent stresses and the thickness of a layer, equal to $q=100$ n/m^{-2}, $h=0.01$ m: *a* — vertical movements; *b* — horizontal movements; *c* — pressure, *d* — tangent stresses; *e* — normal stresses; *f* — environment's density

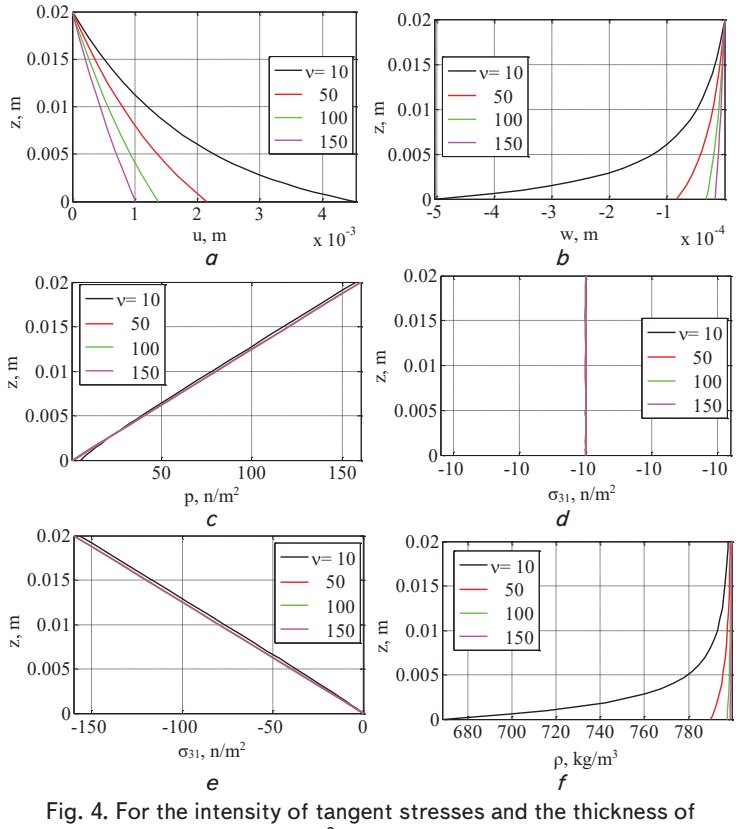

Fig. 4. For the intensity of tangent stresses and the thickness of a layer, equal to $q=10$ n/m^{-2}, $h=0.02$ m: *a* — vertical movements; *b* — horizontal movements; *c* — pressure, *d* — tangent stresses; *e* — normal stresses; *f* — environment's density

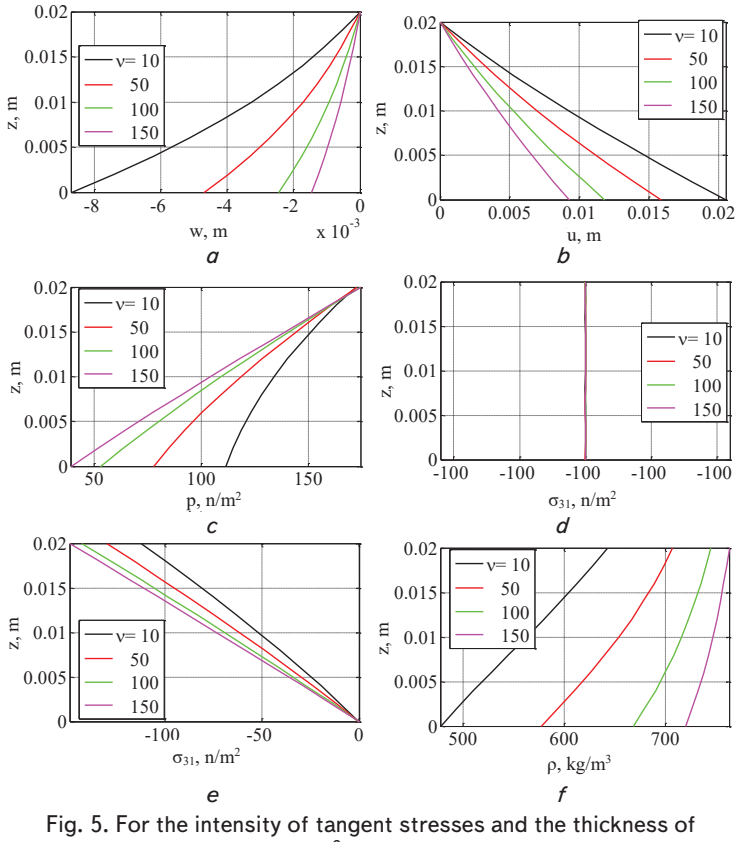

Fig. 5. For the intensity of tangent stresses and the thickness of a layer, equal to $q=100$ n/m^{-2}, $h=0.02$ m: *a* — vertical movements; *b* — horizontal movements; *c* — pressure; *d* — tangent stresses; *e* — normal stresses; *f* — environment's density

The effect of parameter ν (similar to the Lame coefficient in elasticity theory) is seen as follows. With an increase in ν:

– w grows;
– u falls;
– p falls;
– ρ grows;
– σ_{33} normal stress grows by module, while remaining negative.

The tangent stress σ_{31} acquires the value determined from the value of force q in accordance with the first integral in static equations (37).

6. Discussion of results from solving problems on the equilibrium of a horizontal grainy layer numerically

The results obtained, which relate to rheological dependence (19), are explained by taking into consideration the Reynolds effect at shear deformations of a grainy material. This ultimately leads to other static equations.

Special feature of the proposed method for stating the boundary-value problems on the statics of a grainy material is accounting for an additional limitation on the field of environment's deformations (9).

When stating the boundary-value problems on statics, characteristic are the limitations, which imply minor deformations of an environment.

The disadvantages of this work include the emergence of new phenomenological coefficients μ, ν. Identification of their natural values requires additional experiments.

Experimental studies into the gravitational flows of granular media are based on the use of various kinds of penetrating radiation (laser, X-ray, ultrasonic and microwave radiation), which makes it possible to obtain the integrated characteristics of a moving stream. Examples of implementing such methods include: continuous pulling, through the appropriate meter, of an examined material, made in the form of a tape [27]; using tomographic measurements, by γ-rays, of the binary mixture flow when it is unloaded from a bunker with a beam scanner [28]; passing, across a material's layer, a pulsed ultrasound beam or ultrasound tomography [29, 30].

Such methods make it possible to rather reliably control the integrated characteristics of the flow of a grainy medium. Determining the microstructural characteristics of some near-order of particle pairing, as well as defining phenomenological coefficients μ, ν, is related to significant technical difficulties, which require complex specialized equipment.

The implemented experimental-analytical method [31] for studying gravitational currents of grainy materials on a rough ramp using the modernized equation of the state of a grainy medium establishes the relationship between the structural, dynamic, and kinetic characteristics.

Experimental confirmation of the need to take into consideration the Reynolds effect in theoretical studies was given in [24]. With the help of designed devices for direct, oblique cutting of

soils, the authors investigated magnitudes of the limiting resistance of soil to a shear and the angle of dilatancy under the influence of applied tangential stresses. This has made it possible to establish the dependence of soil strength on the conditions of its destruction and properties.

The use of video recording with subsequent processing of photographic images in the experimental studies of statics and dynamics of different granular media has confirmed the complex non-linear behavior of particles at individual locations, and has proven the need to account for the dilatancy effect [32].

This gives grounds for the validity of the devised provisions and the use of the Reynolds effect for simulating the equilibrium state of grain media. Application of such procedures for experimental research [24, 32] would make it possible to perform the final verification of the devised theoretical provisions.

The current study could be advanced by applying a given procedure to the practical tasks related to the dynamics of grainy media.

7. Conclusions

1. The application of the devised theoretical procedure has made it possible to state the principal thermodynamic equality (the identity by Gibbs) and to derive a rheological ratio for a grainy material taking into consideration the Reynolds effect.

2. We have stated the boundary-value problem on the equilibrium of a grainy material taking into consideration the derived rheological ratio, the established conditions for the course of the examined process, taking into consideration the additional limitation on the field of environment's deformations.

3. Verification of the devised procedure has been carried out by solving the applied tasks on the equilibrium of a horizontal grainy layer in the field of gravity forces in the absence of tangential efforts on the free surface of the layer and in the presence of tangential stresses, distributed over a free surface, on a solid horizontal plane in a two-dimensional statement. The derived solutions to these problems are consistent with the heuristic concepts about an actual loading of grainy materials.

4. The implementation of the devised method has made it possible to obtain a degree of influence exerted by phenomenological coefficients μ, ν on the vertical and horizontal movements, pressure, tangential and normal stresses, the density of an environment, which define the dynamic character of the course of machines' technological processes.

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