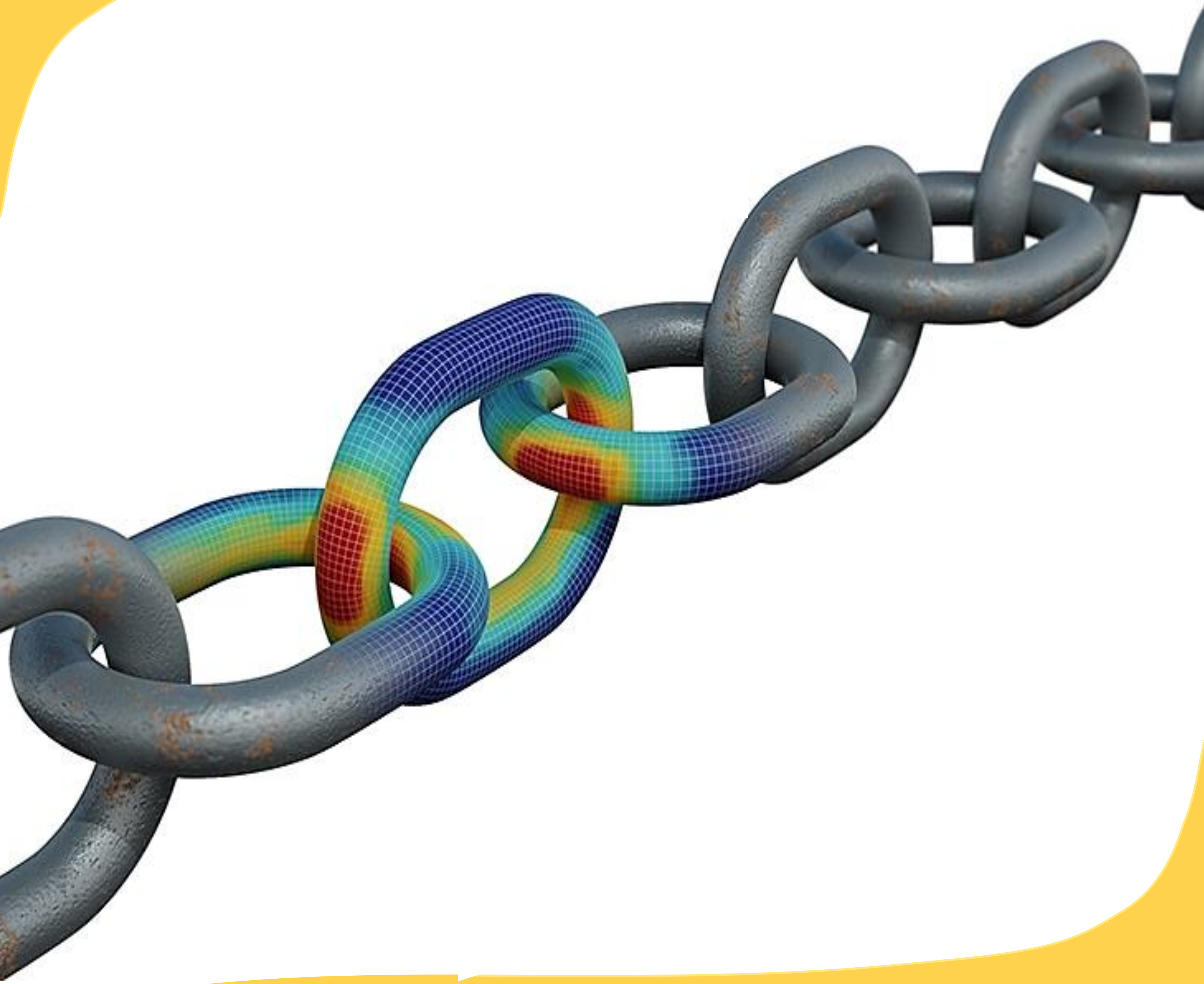
Introduction to Nonlinear Finite Element Analysis



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Introduction to Nonlinear Finite Element Analysis

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1.1.1 Introduction

Any vector in a two dimensional plane can be defined by a linear combination of two linear independent vectors. Independent vectors mean that they have different direction (not collinear), while space vector need a combination of 3 independent vectors such that they do not share the same plane (not coplanar). As shown schematically in Figure 1.1, vector $\boldsymbol{v}$ can be represented as follows:

$$\boldsymbol{v} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \qquad (2D\ case) \tag{1.1}$$

$$\boldsymbol{v} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \qquad (3D\ case) \tag{1.2}$$

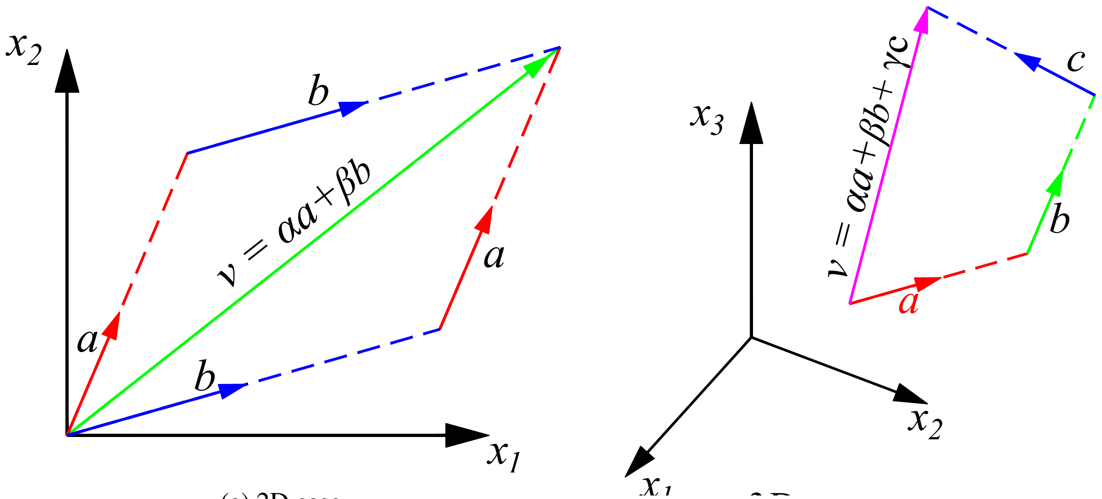
Note that bold small letters are used for vector while light letters are used for scalar values. Most vectors are introduced in terms of a combination of three *orthonormal basis vectors* (a set of three mutually orthogonal unit vectors). These basis vectors are defined as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$,and $\boldsymbol{e}_3$, and $\boldsymbol{x}_3$ coordinates axes forming what is so called *reference frame (coordinates system)* $\boldsymbol{I} = \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as shown in Figure 1.2, such that vector $\boldsymbol{v}$ can be defined as follows:

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_2 = \sum_{i=1}^{3} v_i\boldsymbol{e}_i \tag{1.3}$$

v_1, v_2, and v_3 are the components of vector $\boldsymbol{v}$ resolved in the reference frame $\boldsymbol{I}$. Also the components of vector $\boldsymbol{v}$ and basis vector $\boldsymbol{e}_i$ resolved in coordinate system $\boldsymbol{I}$ can be written in the matrix notation or column vector for $i = 1, 2, 3$ as follows:

$$[v]^{\boldsymbol{I}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad [\boldsymbol{e}_1]^{\boldsymbol{I}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad [\boldsymbol{e}_2]^{\boldsymbol{I}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \quad [\boldsymbol{e}_3]^{\boldsymbol{I}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \tag{1.4}$$

Superscript $\boldsymbol{I}$ indicates the frame of reference in which the components of vector $\boldsymbol{v}$ are resolved. For convenience $[\boldsymbol{V}]^{\boldsymbol{I}}$ can be written in this form $\boldsymbol{v}^{\boldsymbol{I}}$. Bear in mind that we can choose any suitable



(a) 2D case

(b) 3D case

Figure 1.1

coordinates system in which vector $\boldsymbol{v}$ can be resolved as indicated in Figure 1.3, such that the matrix components of vector $\boldsymbol{v}$ change with changing the coordinates system, while the vector itself remains at its same position in space, e.g. vector $\boldsymbol{v}$ can be resolved in two different bases $\boldsymbol{I}$ and $\boldsymbol{I}^*$ with different components given in the matrix notation as follows:

$$[\boldsymbol{v}]^{\boldsymbol{I}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \ [\boldsymbol{v}]^{\boldsymbol{I}^*} = \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix}, \boldsymbol{v}_i \ = \ \boldsymbol{v}_i^* \quad for \ i \ = \ 1, \ 2, \ 3 \tag{1.5}$$

Also we use a right-hand set of orthogonal axes as shown schematically in Figure 1.4. From above, we can conclude the vector properties as follows:

1. Commutative $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$
2. Distributive $\alpha(\boldsymbol{a}+\boldsymbol{b}) = \alpha\boldsymbol{a}+\alpha\boldsymbol{b}$
3. Associative under addition $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$
4. Vector length (magnitude) $|\boldsymbol{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$
5. Unit vector along vector $\boldsymbol{a}$ (vector direction) $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$, it is also called the vector direction as shown in Figure 1.5.
6. Identical vectors ($\boldsymbol{a}=\boldsymbol{b}$), if they share same length and direction illustrated in Figure 1.6.

Generally, vectors are considered *free vector*, if they are independent of a particular point of application, such that if two free vectors share the same magnitude and direction, they are identical as apparent in Figure 1.6, but in some cases, the location of application point is important for some vectors like force vector. Changing its location induces an additional moment. In this case, the vector is called *localized vector*.

1.1.2 Vector products

The first type of the vector product we are interested in to study is called *Scalar* (*dot/ inner*) *product*. Scalar product of vector ($\boldsymbol{a}$) and vector ($\boldsymbol{b}$) is defined by these two forms:

$$\boldsymbol{a}.\boldsymbol{b} = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.6}$$

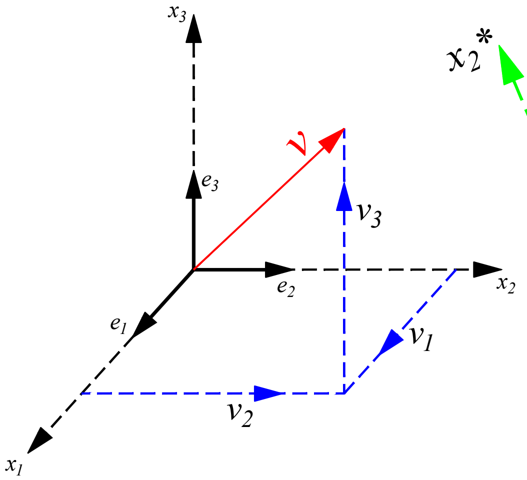


Figure 1.2

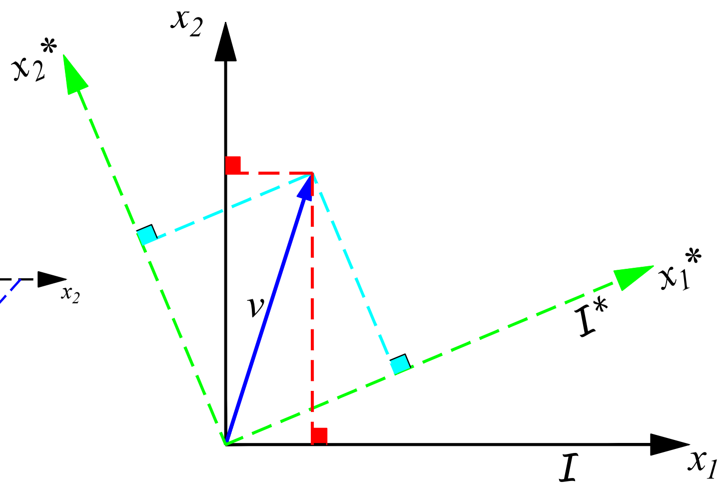


Figure 1.3

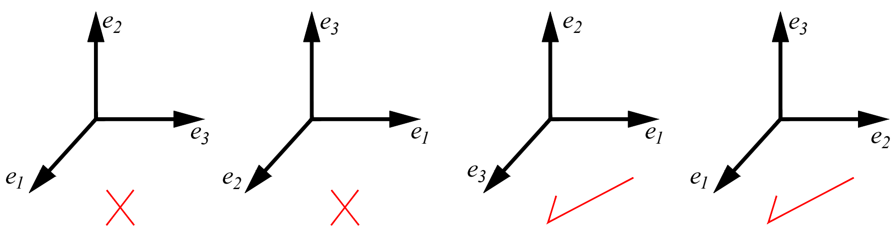


Figure 1.4

$$\boldsymbol{a}.\boldsymbol{b} = |a|\,|b|\cos(\theta) \tag{1.7}$$

The result of the dot product of two vector is a scalar value. Angle θ represents the angle bounded by the two vectors. Also, from expression above, the commutativity property achieves as follows:

$$\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a} \tag{1.8}$$

It has many applications like finding the projection of a some vector on another, angle between two vectors, and the projection of an area on a plane.

■ **Example 1.1** For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ defined as $\boldsymbol{a} = (3,4,5)$ and $\boldsymbol{b} = (1,0,1)$, calculate the following:

1. The projection of vector ($\boldsymbol{a}$) on vector ($\boldsymbol{b}$).
2. Angle between the two vectors.

Projection of vector $\boldsymbol{a}$ on vector $\boldsymbol{b}$ is defined as the dot product of vector ($\boldsymbol{a}$) and the unit vector along vector ($\boldsymbol{b}$) apparent in Figure 1.7.

$$\hat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{|\boldsymbol{b}|} = \frac{(1,0,1)}{\sqrt{1^2+1^2}} = \frac{(1,0,1)}{\sqrt{2}} \tag{1.9}$$

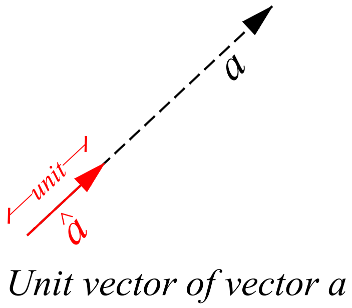


Figure 1.5

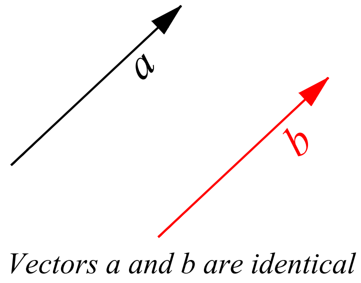


Figure 1.6

The projection will be:

$$(\boldsymbol{a}.\hat{\boldsymbol{b}}) = (3,4,5)\,.\,\frac{(1,0,1)}{\sqrt{2}} = (3*1+0+5*1)\,/\sqrt{2} = 4\sqrt{2} \tag{1.10}$$

Angle between the two vectors can be obtained from:

$$\boldsymbol{a}.\boldsymbol{b} = (1*3+1*5) = 8 = |\boldsymbol{a}|\ |\boldsymbol{b}|\cos(\theta) \tag{1.11}$$

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\cos(\theta) \tag{1.12}$$

$$|\boldsymbol{a}| = \sqrt{3^2+4^2+5^2} = 5\sqrt{2} \tag{1.13}$$

$$\cos(\theta) = 8/(5\sqrt{2}*\sqrt{2}) \tag{1.14}$$

$$\theta = 36.86^o \tag{1.15}$$

■

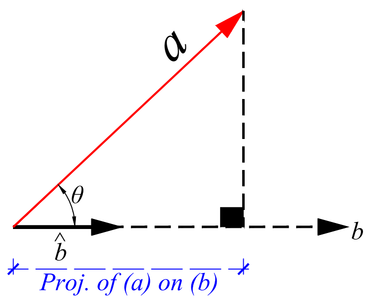


Figure 1.7

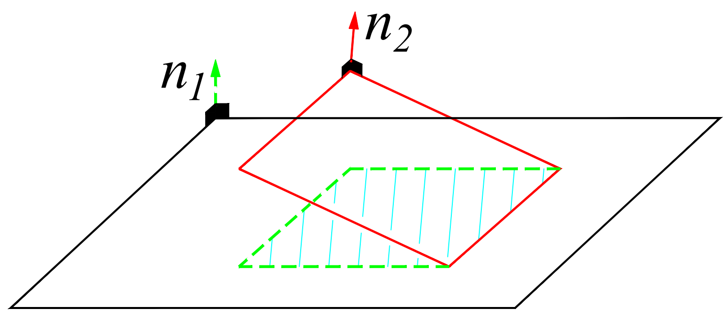


Figure 1.8

■ **Example 1.2** Plane with unit vector $\boldsymbol{n}_1 = (-2,\ 0,\ 1)/\sqrt{5}$ normal to it. Another plane with area $A_2 = 100m^2$ and normal direction $\boldsymbol{n}_2 = (-1,\ 1,\ 1)/\sqrt{3}$, calculate the projection of this area on plane $(\boldsymbol{n}_1)$.

Generally area vector is defined as a vector with magnitude equal to its area and a unit vector normal to its plane, such that the area vector is given by:

$$\boldsymbol{A}_2 = \boldsymbol{n}_2 A_2 \tag{1.16}$$

And, the projected area A_p shown in Figure 1.8 is defined as:

$$A_p = \boldsymbol{n}_1.\boldsymbol{A}_2 \ = \boldsymbol{n}_1.\boldsymbol{n}_2\ |area| = \ (-2*-1\ +\ 1*1)/\ \sqrt{15}*100 \ = \ 77.5\ m^2 \tag{1.17}$$

■ **Example 1.3** Calculate the work done by constant force $\boldsymbol{f} \ = \ (1,\ 5,\ 2)$ on an object after moving a vector distance $\boldsymbol{d} \ = \ (-2,\ 1,\ 1)$.

As schematically shown in Figure 1.9, the work done by force on an object moving distance d is equal to distance length times the force component in distance direction, and consequently, it follows:

$$work \ = \ \boldsymbol{f}.\boldsymbol{d} = \ (1*-2\ +\ 5*1\ +\ 1*1) \ = \ 4 \tag{1.18}$$

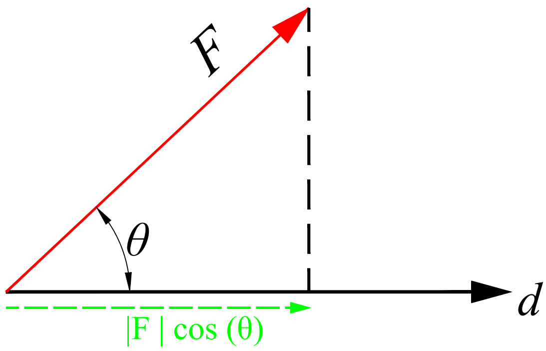


Figure 1.9

Also the components of vector $\boldsymbol{v}$ in Figure 1.3 can be conceived as the projection of the vectors on bases vector $\boldsymbol{e}_i$, such that vector $\boldsymbol{v}$ can be defined as follows:

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = (\boldsymbol{v}.\boldsymbol{e}_1)\,\boldsymbol{e}_1 + (\boldsymbol{v}.\boldsymbol{e}_3)\,\boldsymbol{e}_3 + (\boldsymbol{v}.\boldsymbol{e}_3)\,\boldsymbol{e}_3 = \sum_{i=}^{3}(\boldsymbol{v}.\boldsymbol{e}_i)\,\boldsymbol{e}_i \tag{1.19}$$

Note also that if $(\boldsymbol{a}.\boldsymbol{b} \ = \ 0)$, it means that either the magnitude of $\boldsymbol{a}$ or $\boldsymbol{b}$ is zero or vector ($\boldsymbol{a}$) is normal to vector ($\boldsymbol{b}$).

Another type of vectors product is called *cross (skew/ outer/ vector) product*. The cross product of vector ($\boldsymbol{a}$) and vector ($\boldsymbol{b}$) is given by:

$$\boldsymbol{c} \ = \ \boldsymbol{a}\times\boldsymbol{b} \tag{1.20}$$

With a magnitude $|\boldsymbol{c}| \ = \ |\boldsymbol{a}|\ |\boldsymbol{b}|\ sin\ \theta$ and a unit vector normal to vectors ($\boldsymbol{a}$) and ($\boldsymbol{b}$) formed by turning a right hand screw to bring ($\boldsymbol{a}$) to ($\boldsymbol{b}$) as schematically shown in the Figure 1.10. The expression used for calculating the cross product of vectors $\boldsymbol{a}$, and $\boldsymbol{b}$ is obtained from:

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= (a_2b_3-a_3b_2)\boldsymbol{e}_1+(a_3b_1-a_1b_3)\boldsymbol{e}_2+(a_1b_2-a_2b_1)\boldsymbol{e}_3\\ &= det\left(\begin{bmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}\right)\end{aligned} \tag{1.21}$$

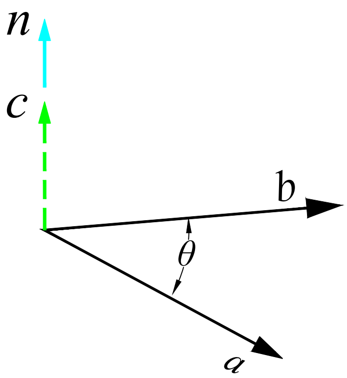


Figure 1.10

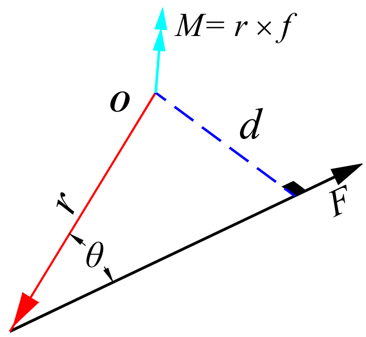


Figure 1.11

Where a_i, and b_i are components of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Symbol "det" indicates calculating the determinate of matrix. From above expression, cross product can achieve the distributive property, but it is not commutative as follows:

$$\begin{aligned}&\boldsymbol{a}\times\ (\boldsymbol{b}+\boldsymbol{c}) = \ \boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}\\ &\boldsymbol{a}\times\boldsymbol{b} = \ -\boldsymbol{b}\times\boldsymbol{a} \qquad \text{(commutative propery fails)}\end{aligned} \tag{1.22}$$

Note that last relation can be proven using right hand rule shown in Figure 1.10. As cross product of vector $\boldsymbol{b}$ and vector $\boldsymbol{a}$ results a vector identical to vector ($\boldsymbol{c} = \boldsymbol{a}\otimes\boldsymbol{b}$) in magnitude, but opposite in the direction. We also note that if cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ vanishes ($\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$), it means that either the magnitude of $\boldsymbol{a}$ or $\boldsymbol{b}$ is zero or vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel. Vector product includes many applications like evaluating the moment induced by some force about a particular point, area bounded by two vectors, velocity of an object attached to rigid body rotating about fixed axis, plane projection, etc. These applications are illustrated below as follows:

Example 1.4 — Moment $\boldsymbol{M}$ induced by force $\boldsymbol{F}$ about point O. As schematically shown in Figure 1.11, If force $\boldsymbol{F}$ passing through a particular point with position vector $\boldsymbol{r}$ and located at normal distance $|\boldsymbol{d}|$ from point O, the resulting moment $\boldsymbol{M}$ of force $\boldsymbol{F}$ about this point O will be obtained from:

$$|\boldsymbol{M}| = |\boldsymbol{F}|\,|\boldsymbol{d}| = |\boldsymbol{F}|\,|\boldsymbol{r}|\,\boldsymbol{sin\theta} \tag{1.23}$$

With direction normal to $\boldsymbol{r}$ and $\boldsymbol{F}$ so it follows that:

$$\boldsymbol{M} = \boldsymbol{r}\times\boldsymbol{F} \tag{1.24}$$

■ **Example 1.5 — Area bounded by two vectors.** As stated before in 1.2, area vector is defined as a vector with direction normal to its plane $\boldsymbol{n}$ and magnitude equal to the area. As shown in Figure 1.12, the magnitude of rectangular area formed by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ equals to:

$$\boldsymbol{c} = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\boldsymbol{sin\theta} \tag{1.25}$$

And consequently, area vector is obtained from:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \tag{1.26}$$

■

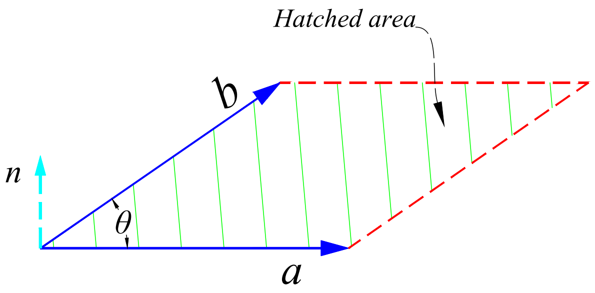


Figure 1.12

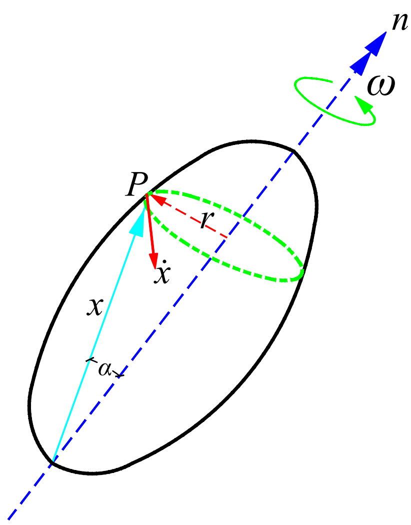


Figure 1.13

■ **Example 1.6 — Velocity of an object P attached to a rigid body rotating about fixed axis $\boldsymbol{n}$.** As shown schematically in Figure 1.13, time rate of rotation of a rigid body rotating about fixed axis is described by the angular velocity ($\boldsymbol{\omega}$) which is equivalent to 2π times number of cycles rotated in one second. It is also called spatial spin about axis $\boldsymbol{n}$. This rotation makes object P with position vector $\boldsymbol{x}$ to rotate in circle normal to axis $\boldsymbol{n}$. The object P has a velocity $\dot{\boldsymbol{x}}$ tangent to this circle in direction normal to vectors $\boldsymbol{x}$ and $\boldsymbol{n}$ with a magnitude equal to the angular velocity times the radius of the circle as follows:

$$|\dot{\boldsymbol{x}}| = |\boldsymbol{\omega}|\,|\boldsymbol{r}| = |\boldsymbol{\omega}|\,|\boldsymbol{x}|\,\boldsymbol{sin\alpha} \tag{1.27}$$

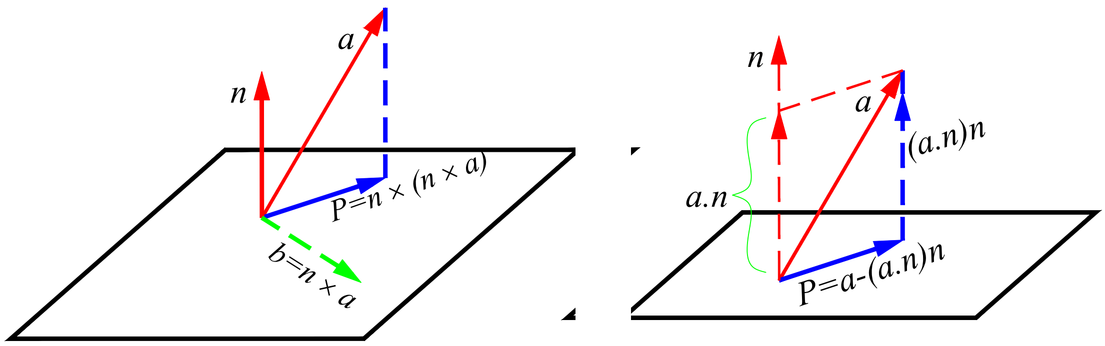
So that, the velocity vector is obtained from:

$$\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x} \tag{1.28}$$

Where $\boldsymbol{\omega}$ is spin vector in direction of $\boldsymbol{n}$ and vector dot $\dot{()}$ denotes the time rate of change of vector.

$$\boldsymbol{\omega} = |\boldsymbol{\omega}|\boldsymbol{n} \tag{1.29}$$

Note that position vector $\boldsymbol{x}$ is a line passing through fixed point located on axis of rotation and point P. ■



(a) (b)

Figure 1.14

■ **Example 1.7 — Perpendicular projection (plane projection).** Assume we need to evaluate the projection of vector $\boldsymbol{a}$ on a plane with unit vector $\boldsymbol{n}$ (axis normal to it) defined by vector $\boldsymbol{P}$ as indicated in Figure 1.14. There are two ways to evaluate it. As shown in Figure 1.14a, we can use an additional vector ($\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{n}$) with magnitude equal to the area bounded by vectors $\boldsymbol{a}$ and unit vector $\boldsymbol{n}$ as follows:

$$|\boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{n}|\,\boldsymbol{sin\theta} = |\boldsymbol{a}|\,\boldsymbol{sin\theta} \tag{1.30}$$

Where θ is the angle between vector $\boldsymbol{a}$ and unit vector $\boldsymbol{n}$. As $\boldsymbol{n}$ is a unit vector ($|\boldsymbol{n}| = 1$). From above equation the magnitude of the area is identical to the length of the projected vector $\boldsymbol{P}$ and we need to find its direction $\hat{\boldsymbol{P}}$ to fully describe this vector. The direction of vector $\boldsymbol{P}$ is normal to $\boldsymbol{n}$ and $\boldsymbol{b}$ obtained as follows:

$$\frac{\boldsymbol{n} \times \boldsymbol{b}}{|\boldsymbol{n} \times \boldsymbol{b}|} = \frac{\boldsymbol{n} \times (\boldsymbol{a} \times \boldsymbol{n})}{|\boldsymbol{n}|\ |\boldsymbol{b}|} = \frac{\boldsymbol{n} \times (\boldsymbol{a} \times \boldsymbol{n})}{|\boldsymbol{b}|} \tag{1.31}$$

As $\boldsymbol{n}$ is normal to vector $\boldsymbol{b}$, $|\boldsymbol{n} \times \boldsymbol{b}| = |\boldsymbol{n}|\ |\boldsymbol{b}|$, then vector $\boldsymbol{P}$ will be:

$$\boldsymbol{P} = |\boldsymbol{P}|\hat{\boldsymbol{P}} = \boldsymbol{n} \times (\boldsymbol{a} \times \boldsymbol{n}) \tag{1.32}$$

Also another way is schematically shown in Figure 1.14.b. Defining an additional vector $\boldsymbol{P}_1$ as a projection of vector $\boldsymbol{a}$ on a unit vector $\boldsymbol{n}$ which is equal to the dot product of vector $\boldsymbol{a}$ and $\boldsymbol{n}$ with direction parallel to unit vector $\boldsymbol{n}$ as follows:

$$\boldsymbol{P}_1 = (\boldsymbol{a}.\boldsymbol{n})\ \boldsymbol{n} \tag{1.33}$$

So subtracting vector $\boldsymbol{P}_1$ from vector a vector $\boldsymbol{P}$ yields the required vector $\boldsymbol{P}$ as follows:

$$\boldsymbol{P} = \boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{n})\boldsymbol{n} \tag{1.34}$$

Both methods are identical in results, so that we can conclude from these two methods that:

$$\boldsymbol{b}\times(\boldsymbol{a}\times\boldsymbol{c}) = (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c} \tag{1.35}$$

Last expression will be proven using index notation in subsection 1.1.3 Equation 1.66. ■

Scalar triple product Scalar triple product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is defined as $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}$. As illustrated in Figure 1.15, the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ defined by $(\boldsymbol{a}\times\boldsymbol{b})$, provides the area A of the rectangular bounded by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with direction $\boldsymbol{n}$ normal to them

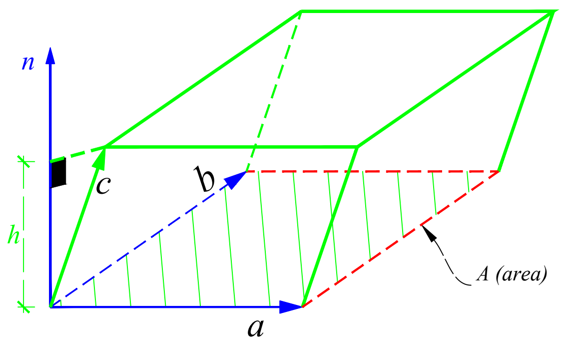


Figure 1.15

$$(\boldsymbol{a}\times\boldsymbol{b}) = A\,\boldsymbol{n} \tag{1.36}$$

And consequently, the scalar triple product of the $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}$ is obtained from:

$$(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = A\;(\boldsymbol{n}.\boldsymbol{c}) \tag{1.37}$$

But $(\boldsymbol{n}.\boldsymbol{c})$ defines the projection of vector $\boldsymbol{c}$ on direction $\boldsymbol{n}$which is identical to the height h of the parallelogram formed by three vector $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. And consequently, the Scalar triple product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ yields the volume V of parallelogram as follows:

$$(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = A\;(\boldsymbol{n}.\boldsymbol{c}) = A\,h = V \tag{1.38}$$

Where h and A are the height of parallelogram, and the magnitude of the area bounded by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

If $(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{0}$, it means that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ share the same plane (coplanar vectors). As the parallelogram volume is constant, the scalar triple product follows the following relations:

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} &= (\boldsymbol{b}\times\boldsymbol{c}).\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a}).\boldsymbol{b}\\ (\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c} &= -(\boldsymbol{a}\times\boldsymbol{c}).\boldsymbol{b}\end{aligned} \tag{1.39}$$

Vector triple product $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$

As schematically shown in Figure 1.16, after getting first $(\boldsymbol{a}\times\boldsymbol{b})$ as a vector normal to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, vector $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ will be normal to $(\boldsymbol{a}\times\boldsymbol{b})$ and $\boldsymbol{c}$ yielding a vector laying on the plane containing vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This product is evaluated as follows:

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a} \tag{1.40}$$

The above expression will be proven in details using index notation in the next section. It is easy to prove schematically that the vector triple product is not associative $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} \neq (\boldsymbol{a}\times\boldsymbol{c})\times\boldsymbol{b}$

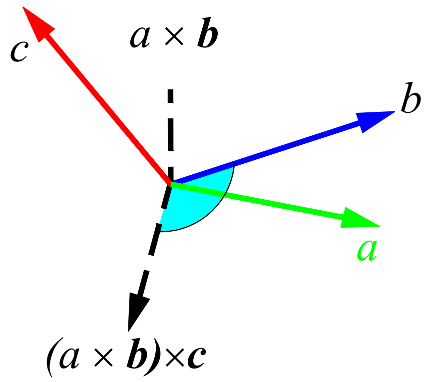


Figure 1.16

1.1.3 Index notation

The components of vector $\boldsymbol{v}$ in Equation 1.3 can be written using index notation by omitting the summation sign as follows:

$$\boldsymbol{v} \;=\; v_i\boldsymbol{e}_i, \;\; i \;=\; 1,\, 2,\, 3 \tag{1.41}$$

The repeated index (i) in v_i and $\boldsymbol{e}_i$ is called a summation or *dummy index*, so that the above expression can be expanded as follows:

$$v_i e_i = \sum_{i=1}^{3} v_i e_i = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \tag{1.42}$$

In the same manner, dot product can be represented as follows:

$$\boldsymbol{a}.\boldsymbol{b} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.43}$$

Another type of index we would like to address is *free index*. This index appears once in each term of the equation and translates this equation into three equations, so for:

$$\boldsymbol{a} = \alpha\boldsymbol{b} + \beta\boldsymbol{c} \tag{1.44}$$

It can be written in index notation as follows:

$$a_i = \alpha b_i + \beta c_i \tag{1.45}$$

Index i appears once in each term of the equation nd is considered free index which translate the above equation into three independent equations as follows:

$$\begin{aligned} a_1 &= \alpha b_1 + \beta c_1 \\ a_2 &= \alpha b_2 + \beta c_2 \\ a_3 &= \alpha b_3 + \beta c_3 \end{aligned} \tag{1.46}$$

Some equations include a combination of free indices and dummy indices, for example:

$$a_i = A_{ij} c_j \tag{1.47}$$

For dummy index (j), it yields that:

$$a_i = A_{i1}c_1 + A_{i2}c_2 + A_{i3}c_3 \tag{1.48}$$

While, for free index (i), it can be translated to three equations as follows:

$$a_1 = A_{11}c_1 + A_{12}c_2 + A_{13}c_3 \tag{1.49}$$
$$a_2 = A_{21}c_1 + A_{22}c_2 + A_{23}c_3 \tag{1.50}$$
$$a_3 = A_{31}c_1 + A_{32}c_2 + A_{33}c_3 \tag{1.51}$$

There are some rules to follow in using index notation:

1. Any index cannot appear more than twice.
2. The free index appears once in each term of the equation and dummy index appears twice in only one term of the equation

Example 1.8 Explain the validation of the following equations:

(a) $a_i = b_i c_j d_j e_j$

The expression is wrong as index j is repeated three times in one term.

(b) $f_j = a_i b_i c_j + \alpha\, m_j$
It is right as index j is used in each term of the equation as a free index, and dummy index i is used only in one term and it cn be translated to three equation (free index $j = 1,2,3$) as follows:
$f_1 = a_i b_i c_1 + \alpha\, m_1$
$f_2 = a_i b_i c_2 + \alpha\, m_2$
$f_3 = a_i b_i c_3 + \alpha\, m_3$
Where $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ for the dummy index j.

(c) $a_i = \alpha b_i + \beta c_j$
It is wrong as free indices i and j are not used in all terms of the equation.

(d) $f_j = a_i b_i c_j + \alpha\, d_i e_i m_j$
it is a wrong expression as dummy index i is repeated in more than one term.

3. Dummy index can be replaced by other index not used in the rest of the equation, e.g. $a_i b_i c_j$ and $a_k b_k c_j$ are identical, while the following expression is not:

$$a_i b_i c_j + d_k e_k m_j \neq a_i b_i c_j + d_i e_i m_j \tag{1.52}$$

The reason is that replacing dummy index k with index i used in other term in the equation is not allowed her.

4. We also have the freedom to flip between two scalar elements in one term of the equation as follows:

$$f_j = a_i b_i c_j = a_i c_j b_i \tag{1.53}$$

While flipping between vector elements is incorrect for most cases as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j \neq a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j \neq \boldsymbol{b} \times \boldsymbol{a} \tag{1.54}$$

as ($\boldsymbol{e}_i \times \boldsymbol{e}_j \neq \boldsymbol{e}_i \times \boldsymbol{e}_j$), while $b_j a_i = a_i b_j$
We shall introduce an operator called *Kroneckor delta* δ_{ij} defined as

$$\delta_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases} \tag{1.55}$$

It contains nine elements and it can be represented in index notation as a dot product of two bases vector $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ as follows:

$$\delta_{ij} = \boldsymbol{e}_i \,.\, \boldsymbol{e}_j \tag{1.56}$$

Where $\boldsymbol{e}_i$ represents three orthonormal bases, e.g.:

$$\boldsymbol{e}_1 \,.\, \boldsymbol{e}_1 = \boldsymbol{e}_2 \,.\, \boldsymbol{e}_2 = \boldsymbol{e}_3 \,.\, \boldsymbol{e}_3 = \mathbf{1}$$

$$\boldsymbol{e}_1 \,.\, \boldsymbol{e}_2 = \boldsymbol{e}_2 \,.\, \boldsymbol{e}_3 = \boldsymbol{e}_3 \,.\, \boldsymbol{e}_1 = \mathbf{0}$$

Also differentiating the components of some vector resolved in a particular basis of reference with each other yields this operator:

$$x_{i,j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij} \tag{1.57}$$

For i and $j = 1,2,3$, as x_i represents independent components of vector $\boldsymbol{x}$ e.g.:

$$\frac{\partial x_1}{\partial x_1} = \frac{\partial x_2}{\partial x_2} = \frac{\partial x_3}{\partial x_3} = 1$$

$$\frac{\partial x_1}{\partial x_2} = \frac{\partial x_2}{\partial x_3} = \frac{\partial x_3}{\partial x_1} = 0$$

Kroneckor delta can be used to contracts or flips indices as follows:

$$\delta_{ij} v_j = v_i \tag{1.58}$$

Which can be proven by expanding the above expression with dummy index as follows:

$$v_i = \ \delta_{i1} v_1 + \delta_{i2} v_2 + \delta_{i3} v_3$$

The free index i can translate the above equation into three equations as stated before to:

$$v_1 = \ \delta_{11} v_1 + \delta_{12} v_2 + \delta_{13} v_3 = v_1$$

$$v_2 = \ \delta_{21} v_1 + \delta_{22} v_2 + \delta_{23} v_3 = v_2$$

$$v_3 = \ \delta_{31} v_1 + \delta_{32} v_2 + \delta_{33} v_3 = v_3$$

That is why it also termed as a substitution operator.

Example 1.9

$$a_i a_j \delta_{ij} = a_i a_i = a_j a_j$$

$$\delta_{ij} \delta_{ik} = \delta_{kj}$$

$$A_{ij} \delta_{ij} = A_{ii}$$

In the same manner, dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be rewritten as follows:

$$\boldsymbol{a}.\boldsymbol{b} = \ (a_i e_i)\,.\,(b_j e_j) = a_i b_j\,(e_i.e_j) = a_i.b_j \delta_{ij} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Another operator we would like to introduce is called *Permutation symbol* ε_{ijk} which is given by:

$$\varepsilon_{ijk} = \begin{cases} 1 & for\ \ \varepsilon_{123},\ \varepsilon_{231},\ \varepsilon_{312} \\ -1 & for\ \ \varepsilon_{213},\ \varepsilon_{132},\ \varepsilon_{321} \\ 0 & for\ \ i=j\ or\ j=k\ or\ i=k \end{cases} \tag{1.59}$$

It is sometimes convenient to write the cross product of two vectors using permutation symbol as follows:

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk}\boldsymbol{e}_k \tag{1.60}$$

Where $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ are orthogonal bases. The above expression can be verified through the following examples:

■ Example 1.10

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \varepsilon_{12k}\boldsymbol{e}_k = \varepsilon_{121}\boldsymbol{e}_1 + \varepsilon_{122}\boldsymbol{e}_2 + \varepsilon_{123}\boldsymbol{e}_3 = \boldsymbol{e}_3$$

$$\boldsymbol{e}_1 \times \boldsymbol{e}_1 = \varepsilon_{11k}\boldsymbol{e}_k = \varepsilon_{111}\boldsymbol{e}_1 + \varepsilon_{112}\boldsymbol{e}_2 + \varepsilon_{113}\boldsymbol{e}_3 = \boldsymbol{0}$$

$$\boldsymbol{e}_1 \times \boldsymbol{e}_1 = \varepsilon_{11k}\boldsymbol{e}_k = \varepsilon_{111}\boldsymbol{e}_1 + \varepsilon_{112}\boldsymbol{e}_2 + \varepsilon_{113}\boldsymbol{e}_3 = \boldsymbol{0}$$

$$\boldsymbol{e}_2 \times \boldsymbol{e}_1 = \varepsilon_{21k}\boldsymbol{e}_k = \varepsilon_{211}\boldsymbol{e}_1 + \varepsilon_{212}\boldsymbol{e}_2 + \varepsilon_{213}\boldsymbol{e}_3 = -\boldsymbol{e}_3$$

■

In the same manner, the vector product $\boldsymbol{a}$ of two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ can be evaluated from:

$$\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c};\ \ a_k\boldsymbol{e}_k = (b_i\boldsymbol{e}_i \times c_j\boldsymbol{e}_j) = b_ic_j(\boldsymbol{e}_i \times \boldsymbol{e}_j) = \varepsilon_{ijk}b_ic_j\boldsymbol{e}_k \tag{1.61}$$

From which we can obtain

$$\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c} \leftrightarrow a_k = \varepsilon_{ijk}b_ic_j \tag{1.62}$$

From above we can conclude some rules as follows:

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} \tag{1.63a}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} \tag{1.63b}$$
$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.63c}$$

Also we can rewrite *vector triple product* in index notation as follows

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= \left(\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k\right) \times c_n\boldsymbol{e}_n \\ &= \varepsilon_{ijk}a_ib_jc_n\ (\boldsymbol{e}_k \times \boldsymbol{e}_n) \\ &= \varepsilon_{ijk}a_ib_jc_n\ \varepsilon_{knm}\boldsymbol{e}_m\end{aligned} \tag{1.64}$$

Using the above rules in equations Equation 1.63c yields:

$$\varepsilon_{ijk}\varepsilon_{knm} = \varepsilon_{kij}\varepsilon_{knm} = \delta_{in}\delta_{jm} - \delta_{im}\delta_{jn} \tag{1.65}$$

And substitute back in equation Equation 1.64 and remembering that the scalar elements can be flipped with each other results in:

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= a_ib_jc_n\left(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}\right)\boldsymbol{e}_m \\ &= \left(a_ic_ib_m - b_nc_na_m\right)\boldsymbol{e}_m \\ (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= (\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\,\boldsymbol{a}\end{aligned} \tag{1.66}$$

As ε_{ijk}, a_i, *and* b_j are scalar quantities, they can be flipped while vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}_n$ can not. The above expression is implemented in the previous sections without proof. Following the same above procedures, it can be easy to verify the following expression:

$$(\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{d}) - (\boldsymbol{a}.\boldsymbol{d})(\boldsymbol{b}.\boldsymbol{c}) \tag{1.67}$$

1.1.4 Matrix notation

Matrix $\boldsymbol{A}$ with coefficient element A_{ij} (i and $j = 1,2,3$) can be written as follow:

$$[\boldsymbol{A}] = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.68}$$

The diagonal elements include A_{11}, A_{22}, and A_{33}, while the remaining elements are called off-diagonal elements. Diagonal matrix is defined as a matrix with off-diagonal elements of zero value. Trace of matrix $\boldsymbol{A}$ ($Trace(\boldsymbol{A})$) is known as the sum of its diagonal elements $A_{11}+A_{22}+A_{33}$ termed in index notation as (A_{ii}) which can be defined using substitution operator δ_{ij} as follows:

$$Trace(\boldsymbol{A}) = A_{ij}\delta_{ij} = A_{ii} \tag{1.69}$$

Identity matrix $\mathbf{1}$ is a diagonal matrix with diagonal elements of unit value given by:

$$[\mathbf{1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.70}$$

Another operation we want to introduce is the product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ termed as $(A.B)$. Sometimes, dot product may be dropped for convenience. It can also be defined in index notation ($A_{ik}B_{kj}$) such that the element of the resulting matrix laying in i^{th} row and j^{th} column results form the dot product of i^{th} row of matrix $\boldsymbol{A}$ and j^{th} column of matrix $\boldsymbol{B}$.

Example 1.11 Let us assume matrix $\boldsymbol{C}$ is given by product of two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ defined as:

$$[\boldsymbol{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \qquad [\boldsymbol{B}] = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 6 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

If we need to evaluate, e.g. element C_{12}, it will be equal to the dot product of the first row of matrix $\boldsymbol{A}$ and the second column of matrix $\boldsymbol{B}$ as follows:

$$C_{12} = A_{1k}B_{k2} = (A_{11}, A_{12}, A_{13}).(B_{12}, B_{22}, B_{32}) \tag{1.71}$$

$$= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} = 1*2+2*6+3*-3 = 5 \tag{1.72}$$

In the same manner, matrix $\boldsymbol{C}$ will be:

$$[\boldsymbol{C}] = \begin{bmatrix} 11 & 5 & 3 \\ 5 & 0 & 2 \\ 4 & 5 & 1 \end{bmatrix}$$

While multiplying a matrix $\boldsymbol{A}$ with a vector $\boldsymbol{c}$ yields a vector as follows:

$$\boldsymbol{b} = \boldsymbol{A}.\boldsymbol{c} \quad or \quad \boldsymbol{b} = \boldsymbol{A}\boldsymbol{c} \;\; \text{dot product symbol is dropped for convenience} \tag{1.73}$$

And it can be written in index notation as follows:

$$b_i = A_{ij} c_j \tag{1.74}$$

the ith element of the resulting vector results form the dot product of ith row of matrix $\boldsymbol{A}$ and vector $\boldsymbol{c}$.

■ **Example 1.12** Let us assume vector $\boldsymbol{b}$ is given by product of matrix $\boldsymbol{A}$ and vector $\boldsymbol{c}$ as follows:

$$[\boldsymbol{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \qquad [\boldsymbol{c}] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{c} = \begin{bmatrix} (1,2,3).(1,2,0) \\ (0,1,2).(1,2,0) \\ (4,0,1).(1,2,0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

■

Also the above expression indicates that matrix A defines a linear mapping of vector c into vector a.

Note 1.1 From above, we can conclude the following properties of matrices:

1. Matrices do not commute under multiplication:

$$A.B \neq B.A \tag{1.75}$$

2. Associative property achieves as follows:

$$\boldsymbol{A}.(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}.\boldsymbol{B}+\boldsymbol{A}.\boldsymbol{C} \tag{1.76}$$

3. Multiplication with scalar means that each element of the matrix is multiplied by this scalar given by:

$$\boldsymbol{B} = \alpha \boldsymbol{A} \rightarrow B_{ij} = \alpha A_{ij} \tag{1.77}$$

■

The transpose of matrix $\boldsymbol{A}$ is termed as $\boldsymbol{A}^T$ which is obtained by swapping rows of the matrix $\boldsymbol{A}$ with its columns and defined in index notation as follows:

$$A^T_{ij} = A_{ji} \tag{1.78}$$

The transpose operation flipped the indices of the above matrix.

■ **Example 1.13** If we have matrix A equal to:

$$[\boldsymbol{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 4 & 0 & 1 \end{bmatrix}$$

Its transpose will be:

$$[\boldsymbol{A}]^T = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

A matrix $\boldsymbol{A}$ is considered a symmetric matrix, if it achieves the following condition

$$A = A^T \tag{1.79}$$

while skew-symmetric matrix follows this condition:

$$A = -A^T \tag{1.80}$$

Example 1.14 For example, matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ given by:

$$[\boldsymbol{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}, [\boldsymbol{B}] = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

These matrices are considered symmetric and skew-symmetric matrix, respectively.

We notice that skew-symmetric matrix includes zero value for diagonal elements and three different element with general form as follows:

$$[\boldsymbol{A}] = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{1.81}$$

Note that vector $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T$ is called the axial vector of the above skew-symmetric matrix $\boldsymbol{A}$ termed as:

$$w = axial\,(A) \tag{1.82}$$

While skew- symmetric matrix $\boldsymbol{A}$ can be written using tilde sign over the axial vector as follows:

$$A = \widetilde{\boldsymbol{w}} \tag{1.83}$$

From above property of skew-symmetric matrix, it can be easily proven that

$$\widetilde{\boldsymbol{w}}^T = -\widetilde{\boldsymbol{w}} \tag{1.84}$$

Matrix $\boldsymbol{A}$ is defined as a normal matrix, if it follows the following expression:

$$A.A^T \ = \ A^T.A \tag{1.85}$$

While matrix $\boldsymbol{A}$ is considered orthogonal matrix, if it follows this equation:

$$\boldsymbol{A}.\boldsymbol{A}^T \ = \ \boldsymbol{A}^T.\boldsymbol{A} = \boldsymbol{1} \tag{1.86}$$

Where $\boldsymbol{1}$ is identity matrix.

The transpose of matrix multiplication is obtained by reversing the order of multiplication with transpose operation for each element, e.g.:

$$\begin{aligned}(\boldsymbol{A}.\boldsymbol{B})^T &= \boldsymbol{B}^T.\boldsymbol{A}^T \\ \left(\boldsymbol{A}.\boldsymbol{B}^T.\boldsymbol{C}\right)^T &= \boldsymbol{C}^T.\boldsymbol{B}.\boldsymbol{A}^T\end{aligned} \tag{1.87}$$

We can also notice that $A^T.A$ and $A.A^T$ are symmetric matrix as:

$$\left(\boldsymbol{A}^T.\boldsymbol{A}\right)^T = \boldsymbol{A}^T.\boldsymbol{A} \tag{1.88}$$

Any matrix can be decomposed into two parts; symmetric part and skew- symmetric part given by

$$\begin{aligned}A &= S + W \\ S &= sym(A) = \left(A + A^T\right)/2 \\ W &= skew(A) = \left(A - A^T\right)/2\end{aligned} \tag{1.89}$$

The inverse of matrix $\boldsymbol{A}$ is defined as $\boldsymbol{A}^{-1}$, such that $\boldsymbol{A}.\boldsymbol{A}^{-1} = \mathbf{1}$. The transpose of inverse of matrix is equivalent to the inverse of its transpose as follows:

$$\left(\boldsymbol{A}^{-1}\right)^T = \boldsymbol{A}^{-T} \tag{1.90}$$

The determinate of matrix $\boldsymbol{A}$ is termed as $|\boldsymbol{A}|$ or $det(\boldsymbol{A})$ and defined as follows:

$$|\boldsymbol{A}| = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{1.91}$$

Example 1.15

$$[\boldsymbol{A}] = \begin{bmatrix} 2 & 2 & -1 \\ 5 & 6 & 2 \\ 4 & -3 & 1 \end{bmatrix} \tag{1.92}$$

$$= 2\,(6x1 + 2x3) - 2\,(5x1 - 2x4) - (-5x3 - 4x6) = 69 \tag{1.93}$$

With expression written above, the following results can be concluded:

$$\begin{aligned}|\boldsymbol{A}| &= \left(\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}\right).\boldsymbol{A}^{(3)} \\ det\ (\boldsymbol{A}.\boldsymbol{B}) &= det\ (\boldsymbol{A})\ det\ (\boldsymbol{B}) \\ det\ (\boldsymbol{A}^T) &= det\ (\boldsymbol{A})\end{aligned} \tag{1.94}$$

Where $\boldsymbol{A}^{(i)}$ represent the ith column of matrix $\boldsymbol{A}$. For any nonzero vector $\boldsymbol{v}$ ($|\boldsymbol{v}| \neq 0$), a positive definite matrix $\boldsymbol{A}$ is defined as:

$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} > \mathbf{0} \tag{1.95}$$

Which is important property for stiffness matrix of stable structures. While, for any nonzero vector $\boldsymbol{v}$ ($|\boldsymbol{v}| \neq 0$), semi-positive definite is defined as follows:

$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} \geq \mathbf{0} \tag{1.96}$$

Another operation called *Double dot product* of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, termed by $(\boldsymbol{A}:\boldsymbol{B})$ is defined as the trace of the dot product of one matrix and transpose of the other as follows:

$$\boldsymbol{A}:\boldsymbol{B} = Trace\left(\left(\boldsymbol{A}.\boldsymbol{B}^T\right)_{ij}\right) = Trace(A_{im}B^T_{mj}) = A_{im}B_{jm}\delta_{ij} = A_{im}B_{im} \tag{1.97}$$

Indices i and m are dummy indices as they are repeated twice which leads to the following expression for $\boldsymbol{A}:\boldsymbol{B}$

$$\begin{aligned}\boldsymbol{A}:\boldsymbol{B} = & A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} \\ & + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} \\ & + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}\end{aligned} \tag{1.98}$$

From above we can conclude the commutative property of the double dot product as follows:

$$\boldsymbol{A}:\boldsymbol{B} = \boldsymbol{B}:\boldsymbol{A} \tag{1.99}$$

■ **Example 1.16** Let's us evaluate double dot product $\boldsymbol{A}:\boldsymbol{B}$ of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined as follows:

$$[\boldsymbol{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \qquad [\boldsymbol{B}] = \begin{bmatrix} 2 & -2 & 0 \\ 5 & 6 & 0 \\ 7 & -3 & 1 \end{bmatrix}$$

$$\boldsymbol{A}:\boldsymbol{B} = 1\times 2 \ + \ 2\times -2 \ + \ 3\times 0 \ + \ 0\times 5 \ + \ 1\times 6 \ + 2\times 0 \ + \ 4\times 7 \ + \ 0\times -3 \ + \ 1\times 1 = 33$$

Or we can evaluate

$$\boldsymbol{A}:\boldsymbol{B} = Trace\ \left(\boldsymbol{A}.\boldsymbol{B}^T\right) = Trace\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 2 & 5 & 7 \\ -2 & 6 & -3 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= Trace\left(\begin{bmatrix} -2 & 17 & 4 \\ -2 & 6 & -1 \\ 8 & 20 & 29 \end{bmatrix}\right) = -2+6+29 = 33$$

■

For any symmetric matrix $\boldsymbol{A}$ and skew symmetric matrix $\boldsymbol{B}$, the relation below holds:

$$\boldsymbol{A}:\boldsymbol{B} = 0 \tag{1.100}$$

And consequently, for any matrix $\boldsymbol{B}$ and symmetric matrix $\boldsymbol{A}$ we get:

$$\boldsymbol{A}:\boldsymbol{B} = \boldsymbol{A}:\boldsymbol{sym}\,(\boldsymbol{B}) + \boldsymbol{A}:\boldsymbol{skew}\,(\boldsymbol{B}) = \boldsymbol{A}:\boldsymbol{sym}\,(\boldsymbol{B}) \tag{1.101}$$

Dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined using matrix operations as follows:

$$(\boldsymbol{a}.\boldsymbol{b}) = a_i b_i = [a]^T [b] \tag{1.102}$$

■ **Example 1.17** Let us calculate the dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ defined by:

$$[\boldsymbol{a}] = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad [\boldsymbol{b}] = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

$$[\boldsymbol{a}.\boldsymbol{b}] = [\boldsymbol{a}]^T [\boldsymbol{b}] = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = 2x1 + 1x5 + (-3x2) = 1$$

■

While the cross product ($\boldsymbol{a} \times \boldsymbol{b}$) takes this two forms:

$$\boldsymbol{a} \times \boldsymbol{b} = det\left(\begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right) \tag{1.103}$$

Which can be evaluated using skew-symmetric matrix $\tilde{\boldsymbol{a}}$ multiplied with vector $\boldsymbol{b}$ shown as follows:

$$[\boldsymbol{a} \times \boldsymbol{b}] = [\tilde{\boldsymbol{a}}\boldsymbol{b}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} (a_2b_3 - a_3b_2) \\ (a_3b_1 - a_1b_3) \\ (a_1b_2 - a_2b_1) \end{bmatrix} \tag{1.104}$$

In the same manner, we can prove the following:

$$\tilde{\boldsymbol{a}}\boldsymbol{b} = -\tilde{\boldsymbol{b}}\boldsymbol{a} \tag{1.105}$$

Note 1.2 We would like to mention some useful relations as follows:
Using Equation 1.66, we get

$$\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\,\boldsymbol{c} = \boldsymbol{a}^T\boldsymbol{c}\boldsymbol{b} - \boldsymbol{a}^T\boldsymbol{b}\boldsymbol{c} \tag{1.106}$$

Terms $\boldsymbol{a}^T\boldsymbol{c}$ or $\boldsymbol{a}.\boldsymbol{c}$ is considered as a scalar quantity, so it can be flipped with vector $\boldsymbol{b}$ as follows:

$$\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}\boldsymbol{c} = \boldsymbol{b}\boldsymbol{a}^T\boldsymbol{c} - \boldsymbol{a}^T\boldsymbol{b}\boldsymbol{c} = \left[\boldsymbol{b}\boldsymbol{a}^T - (\boldsymbol{a}^T\boldsymbol{b})\mathbf{1}\right]\boldsymbol{c} \tag{1.107}$$

And consequently, it follows:

$$\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}} = \boldsymbol{b}\boldsymbol{a}^T - (\boldsymbol{a}^T\boldsymbol{b})\mathbf{1} \tag{1.108}$$

Where $\mathbf{1}$ are identity matrix.
Another expression we would like to introduce is:

$$\widetilde{\tilde{\boldsymbol{a}}\boldsymbol{b}}\boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = -\tilde{\boldsymbol{c}}\tilde{\boldsymbol{a}}\boldsymbol{b} = \tilde{\boldsymbol{c}}\tilde{\boldsymbol{b}}\boldsymbol{a} \tag{1.109}$$

The last expression results from the fact that $\left(\tilde{\boldsymbol{a}}\boldsymbol{b} - \tilde{\boldsymbol{b}}\boldsymbol{a}\right)$. Using expression in Equation 1.108 results in:

$$\widetilde{\tilde{\boldsymbol{a}}\boldsymbol{b}} = \tilde{\boldsymbol{a}}\,\tilde{\boldsymbol{b}} - \tilde{\boldsymbol{b}}\,\tilde{\boldsymbol{a}} = \boldsymbol{b}\,\boldsymbol{a}^T - \boldsymbol{a}\boldsymbol{b}^T \tag{1.110}$$

For unit vector $\boldsymbol{n}$, we can conclude the following:

$$\begin{aligned} \widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} &= -\widetilde{\boldsymbol{n}} \\ \widetilde{\boldsymbol{n}}\boldsymbol{n} &= \boldsymbol{0} \end{aligned} \tag{1.111}$$

■

1.2 Tensor analysis

1.2.1 Introduction

Any physical quantity can be expressed using tensors. For examples, scalar value like temperature, length, etc. is considered as zeroth order tensor. Vector ($\boldsymbol{v}$) contains three elements and is represented by first order tensor ($3^1 = 3$), whereas second order tensor generally called tensor or dyad with nine elements ($3^2 = 9$) like stress tensor σ_{ij} and strain tensor ε_{ij}. There are higher order tensors like fourth order tensor C_{ijkl} with 81 elements which used in the constitutive relation between stress and strain $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$.

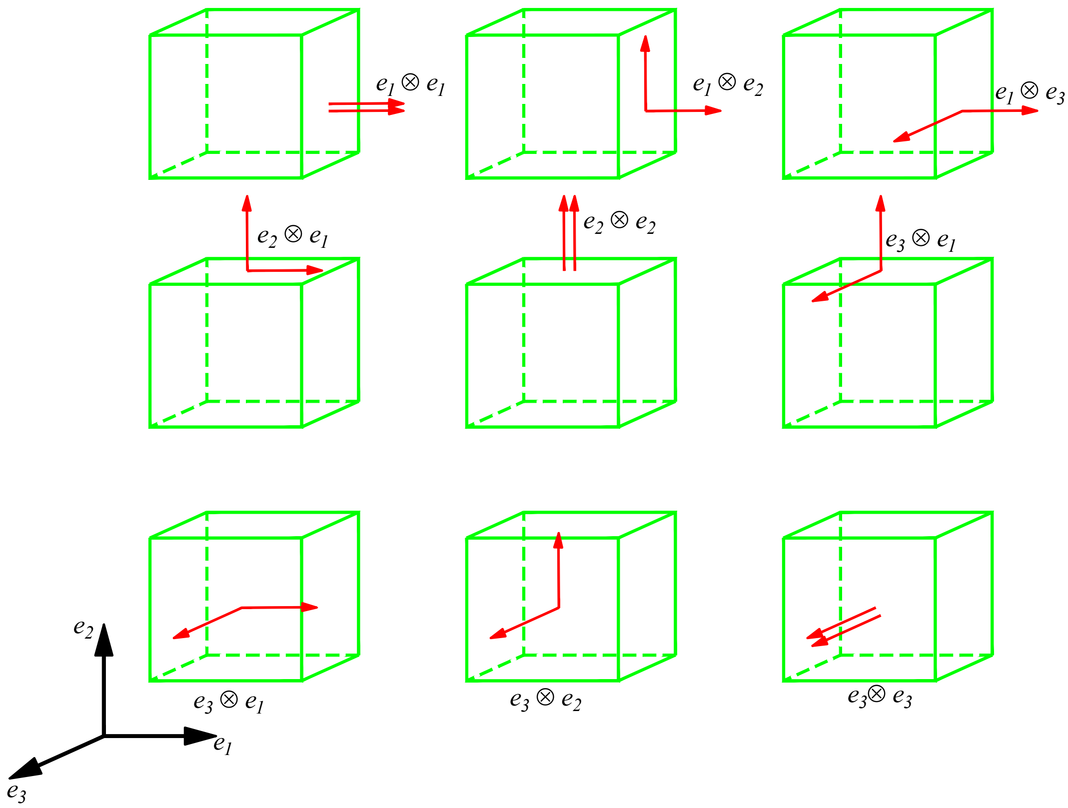


Figure 1.17

Dyad or 2^{nd} order tensor is defined by 2 vectors standing side by side and acting as a one unit. For example $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ represents a 2^{nd} order tensor as shown in Figure 1.17 where $\boldsymbol{e}_i$ is the basis i of the reference frame, such that any spatial tensor can be resolved in this reference frame as follows:

$$\boldsymbol{T} = \boldsymbol{T}_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.112}$$

Note that bold capital letter are used for tensors of second tensor. Tensor $\boldsymbol{T}$ also contains nine components by expanding the dummy indices i and j as follows:

$$\begin{aligned}\boldsymbol{T} &= \boldsymbol{T}_{11}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{T}_{12}\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + \boldsymbol{T}_{13}\boldsymbol{e}_1\otimes\boldsymbol{e}_3 \\ &+ \boldsymbol{T}_{21}\boldsymbol{e}_2\otimes\boldsymbol{e}_1 + \boldsymbol{T}_{22}\boldsymbol{e}_2\otimes\boldsymbol{e}_2 + \boldsymbol{T}_{23}\boldsymbol{e}_2\otimes\boldsymbol{e}_3 \\ &+ \boldsymbol{T}_{31}\boldsymbol{e}_3\otimes\boldsymbol{e}_1 + \boldsymbol{T}_{32}\boldsymbol{e}_3\otimes\boldsymbol{e}_2 + \boldsymbol{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3\end{aligned} \tag{1.113}$$

$\boldsymbol{T}_{ij}$ includes the nine components of second order tensor ($\boldsymbol{T}$) resolved in frame of reference $\boldsymbol{I}$, while $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ is defined as a dyadic product of two orthogonal bases (dyadic pair). Dyadic product $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ can be understood as a vector product of vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ with matrix representation $\boldsymbol{e}_i\boldsymbol{e}_j{}^T$, such that:

$$[\boldsymbol{e}_1\otimes\boldsymbol{e}_2] = \boldsymbol{e}_1\boldsymbol{e}_2{}^T = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$[\boldsymbol{e}_2\otimes\boldsymbol{e}_3] = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Each component $\boldsymbol{T}_{ij}$ is associated with dyadic pairs $\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ and second order tensor can be written in matrix form as follows:

$$[\boldsymbol{T}] = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \boldsymbol{T}_{13} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \boldsymbol{T}_{23} \\ \boldsymbol{T}_{31} & \boldsymbol{T}_{32} & \boldsymbol{T}_{33} \end{bmatrix} \tag{1.114}$$

Or using matrix composed of three vectors columns as follows:

$$[\boldsymbol{T}] = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 & \boldsymbol{T}_3 \end{bmatrix} \tag{1.115}$$

Where $\boldsymbol{T}_i$ is called tensor vectors defined by:

$$[\boldsymbol{T}_1] = \begin{bmatrix} \boldsymbol{T}_{11} \\ \boldsymbol{T}_{21} \\ \boldsymbol{T}_{31} \end{bmatrix}, \ [\boldsymbol{T}_2] = \begin{bmatrix} \boldsymbol{T}_{12} \\ \boldsymbol{T}_{22} \\ \boldsymbol{T}_{32} \end{bmatrix}, \ [\boldsymbol{T}_3] = \begin{bmatrix} \boldsymbol{T}_{13} \\ \boldsymbol{T}_{23} \\ \boldsymbol{T}_{33} \end{bmatrix}$$

And consequently, second order tensor can follow this definition:

$$\boldsymbol{T} = \boldsymbol{T}_1\otimes\boldsymbol{e}_1 + \boldsymbol{T}_2\otimes\boldsymbol{e}_2 + \boldsymbol{T}_3\otimes\boldsymbol{e}_3 \tag{1.116}$$

$$\boldsymbol{T} = \boldsymbol{T}_i\otimes\boldsymbol{e}_i \tag{1.117}$$

Where

$$\boldsymbol{T}_i = \boldsymbol{T}_{ji}\boldsymbol{e}_j \tag{1.118}$$

The transpose of tensor $\boldsymbol{T}$ can be understood as a mapping of coordinates basis into tensor vectors $\boldsymbol{T}_i$ for $(i = 1,2,3)$.

Identity tensor can be defined as:

$$\mathbf{1} = \delta_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{1.119}$$

This expression can be verified easily through expanding the tensor in matrix form to be:

$$\begin{aligned}
[\delta_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j] &= \delta_{11}[\boldsymbol{e}_1\otimes\boldsymbol{e}_1]+\delta_{12}[\boldsymbol{e}_1\otimes\boldsymbol{e}_2]+\delta_{13}[\boldsymbol{e}_1\otimes\boldsymbol{e}_3]\\
&+\delta_{21}[\boldsymbol{e}_2\otimes\boldsymbol{e}_1]+\delta_{22}[\boldsymbol{e}_2\otimes\boldsymbol{e}_2]+\delta_{23}[\boldsymbol{e}_2\otimes\boldsymbol{e}_3]\\
&+\delta_{31}[\boldsymbol{e}_3\otimes\boldsymbol{e}_1]+\delta_{32}[\boldsymbol{e}_3\otimes\boldsymbol{e}_2]+\delta_{33}[\boldsymbol{e}_3\otimes\boldsymbol{e}_3]\\
&=[\boldsymbol{e}_1\otimes\boldsymbol{e}_1]+[\boldsymbol{e}_2\otimes\boldsymbol{e}_2]+[\boldsymbol{e}_3\otimes\boldsymbol{e}_3]\\
&=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\mathbf{1}
\end{aligned} \tag{1.120}$$

Note 1.3 We also need to remark some of dyadic product operation in these following relations:

1. $\boldsymbol{u}\otimes\boldsymbol{v}\neq\boldsymbol{v}\otimes\boldsymbol{u}$ as $\boldsymbol{u}^T\boldsymbol{v}\neq\boldsymbol{v}\boldsymbol{u}^T$
2. $(\boldsymbol{u}\otimes\boldsymbol{v})^T=\boldsymbol{v}\otimes\boldsymbol{u}$
3. $\boldsymbol{u}\otimes(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\otimes\boldsymbol{v}+\boldsymbol{u}\otimes\boldsymbol{w}$
4. $(\boldsymbol{u}\otimes\boldsymbol{v}).\boldsymbol{w}=(\boldsymbol{v}.\boldsymbol{w})\boldsymbol{u}$

   As $(\boldsymbol{u}\otimes\boldsymbol{v}).\boldsymbol{w}=\boldsymbol{u}\boldsymbol{v}^T\boldsymbol{w}=\boldsymbol{v}^T\boldsymbol{w}\boldsymbol{u}=(\boldsymbol{v}.\boldsymbol{w})\boldsymbol{u}$ due to the fact that $\boldsymbol{v}^T\boldsymbol{w}$ is scalar and can be flipped with any element.
   In this equation,tensor $\boldsymbol{u}\otimes\boldsymbol{v}$ maps any vector to another in direction parallel to vector $\boldsymbol{u}$.
5. Using the same procedures, we can prove that:
   $(\boldsymbol{u}\otimes\boldsymbol{v}).\boldsymbol{A}=\boldsymbol{v}\otimes(\boldsymbol{A}^T\boldsymbol{u})$ where $\boldsymbol{A}$ is a second order tensor.
   As $(\boldsymbol{u}\otimes\boldsymbol{v}).\boldsymbol{A}=(\boldsymbol{u}\otimes\boldsymbol{v})^T\boldsymbol{A}=(\boldsymbol{u}\boldsymbol{v}^T)^T\boldsymbol{A}=\boldsymbol{v}\boldsymbol{u}^T\boldsymbol{A}=\boldsymbol{v}\left(\boldsymbol{u}^T\boldsymbol{A}\right)=\boldsymbol{v}\left(\boldsymbol{A}^T\boldsymbol{u}\right)^T=\boldsymbol{v}\otimes(\boldsymbol{A}^T\boldsymbol{u})$

The trace of dyadic product is defined as:

$Trace(\boldsymbol{u}\otimes\boldsymbol{v})=\boldsymbol{u}.\boldsymbol{v}$ *Double dot product of two tensors* $\boldsymbol{A}$ and $\boldsymbol{B}$ can be obtained from:

$$\boldsymbol{A}:\boldsymbol{B}=A_{ij}B_{ij}=trace(A^TB) \tag{1.121}$$

And consequently, double dot product satisfies the following relation:

$$(\boldsymbol{e}_i\otimes\boldsymbol{e}_j):(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=(\boldsymbol{e}_i.\boldsymbol{e}_k)(\boldsymbol{e}_j.\boldsymbol{e}_l)=\delta_{ik}\delta_{jl} \tag{1.122}$$

Such that $\boldsymbol{A}:\boldsymbol{B}$ can be rewritten in index notation as follows:

$$\begin{aligned}
&\boldsymbol{A}:\boldsymbol{B}=A_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j):B_{kl}(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=A_{ij}B_{kl}\delta_{ik}\delta_{jl}=A_{ij}B_{ij}\\
&\boldsymbol{a}\boldsymbol{b}^T:\boldsymbol{c}\boldsymbol{d}^T=(\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{d})
\end{aligned} \tag{1.123}$$

We also need to introduce the *Inner or dot product of two tensor* $\boldsymbol{A}$, $\boldsymbol{B}$ termed as $\boldsymbol{A}.\boldsymbol{B}$. Likewise matrix multiplication, it can be defined as:

$$(\boldsymbol{e}_i\otimes\boldsymbol{e}_j).(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)=\delta_{kj}(\boldsymbol{e}_i\otimes\boldsymbol{e}_l) \tag{1.124}$$

such that

$$\begin{aligned}
\boldsymbol{A}.\boldsymbol{B}&=A_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j).B_{kl}(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\
&=A_{ij}B_{kl}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j).(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\
&=A_{ij}B_{kl}\delta_{kj}(\boldsymbol{e}_i\otimes\boldsymbol{e}_l)\\
&=A_{ij}B_{jl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_l
\end{aligned} \tag{1.125}$$

It follows that:

$$(\boldsymbol{a}\otimes\boldsymbol{b}).(\boldsymbol{c}\otimes\boldsymbol{d}) = \boldsymbol{a}\boldsymbol{b}^T\boldsymbol{c}\boldsymbol{d}^T = \left(\boldsymbol{b}^T\boldsymbol{c}\right)\boldsymbol{a}\boldsymbol{d}^T = (\boldsymbol{b}.\boldsymbol{c})(\boldsymbol{a}\otimes\boldsymbol{d}) \tag{1.126}$$

For dot product of tensor and vector, it can follow:

$$\boldsymbol{e}_i.(\boldsymbol{e}_k\otimes\boldsymbol{e}_l) = \delta_{ik}\boldsymbol{e}_l \tag{1.127}$$

Which can be proven in matrix form as follows:

$$\boldsymbol{e}_2.(\boldsymbol{e}_2\otimes\boldsymbol{e}_3) = \delta_{22}\boldsymbol{e}_3 = \boldsymbol{e}_3$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{e}_3.(\boldsymbol{e}_2\otimes\boldsymbol{e}_3) = \delta_{32}\boldsymbol{e}_3 = \boldsymbol{0}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

And consequently, the relation $\boldsymbol{b} = \boldsymbol{A}.\boldsymbol{c}$ means in index notation that:

$$b_j = A_{ij}c_j \tag{1.128}$$

It can follow different expression as follows:

$$\boldsymbol{b} = \boldsymbol{A}.\boldsymbol{c} = \boldsymbol{c}.\boldsymbol{A}^T \tag{1.129}$$

Which can be proven using matrix or index notation as follows:

$$A_{ij}c_j = c_jA_{ij} = c_jA_{ji}{}^T \tag{1.130}$$

Likewise matrix multiplication, tensor multiplication does not follow the associative property:

$$\boldsymbol{A}.\boldsymbol{B} \neq \boldsymbol{B}.\boldsymbol{A} \tag{1.131}$$

The cross product of vector $\boldsymbol{a}$ and tensor $\boldsymbol{B}$ can follows this relation:

$$\boldsymbol{a}\times\boldsymbol{B} = \varepsilon_{ijk}a_iB_{jm}\boldsymbol{e}_k\otimes\boldsymbol{e}_m \tag{1.132}$$

So the above cross product is performed between vector $\boldsymbol{a}$ and each column of tensor $\boldsymbol{B}$ independently resulting a second order tensor, such that ith column of tensor of the resulting tensor is the cross product of vector $\boldsymbol{a}$ with ith column of tensor $\boldsymbol{B}$.

For second order tensors $\boldsymbol{B}$, $\boldsymbol{P}$ and vectors $\boldsymbol{a}$, $\boldsymbol{c}$, useful relations can be proven as follows:

$$\begin{aligned}\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{B}) &= c_m\boldsymbol{e}_m.(\boldsymbol{a}\times\boldsymbol{B})_{nk}\boldsymbol{e}_k\otimes\boldsymbol{e}_k \\ &= c_m(\boldsymbol{a}\times\boldsymbol{B})_{nk}\delta_{mn}\boldsymbol{e}_k \\ &= c_m(\boldsymbol{a}\times\boldsymbol{B})_{mk}\boldsymbol{e}_k \\ &= c_ma_iB_{mj}\varepsilon_{ijk}\boldsymbol{e}_k \\ &= a_iB_{jm}c_m\varepsilon_{ijk}\boldsymbol{e}_k \\ &= \boldsymbol{a}\times(\boldsymbol{B}.\boldsymbol{c})\end{aligned} \tag{1.133}$$

$$\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{B}) = \boldsymbol{a}\times(\boldsymbol{B}.\boldsymbol{c})$$

$$\boldsymbol{P}:(\boldsymbol{a}\times\boldsymbol{B}) = P_{mk}(\boldsymbol{a}\times\boldsymbol{B})_{mk} = P_{mk}a_iB_{mj}\varepsilon_{ijk} = a_iB_{mj}P_{mk}\varepsilon_{ijk} = a.(B_{mj}P_{mk}\varepsilon_{jkl}e_l) \tag{1.134}$$

For which vector $\{A\}^j$ represent the jth column of tensor $\boldsymbol{A}$ such that its elements will be $\{\boldsymbol{A}\}^j{}_i = A_{ij}$. From above expression it follows that

$$\boxed{\boldsymbol{P}:\boldsymbol{a}\times\boldsymbol{B} = \boldsymbol{a}.(\{\boldsymbol{B}\}^m\times\{\boldsymbol{P}\}^m)} \tag{1.135}$$

Where $\{\boldsymbol{B}\}^m\times\{\boldsymbol{P}\}^m = \sum_{\boldsymbol{m}=1}^{3}\{\boldsymbol{B}\}^m\times\{\boldsymbol{P}\}^m$

1.2.2 Eigen value analysis

For a matrix $\boldsymbol{A}$, a particular set of scalars λ and a set of vectors u can satisfy the following equation:

$$\boldsymbol{A}.\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{u} \tag{1.136}$$

The set of λ and $\boldsymbol{u}$ is called *Eigen values* and *Eigen vectors*, respectively. Rewriting the above equation as follows:

$$(\boldsymbol{A}-\boldsymbol{\lambda}\mathbf{1}).\boldsymbol{u} = \mathbf{0} \tag{1.137}$$

The above equation contains trivial solution $\boldsymbol{u} = \mathbf{0}$ and Non-trivial solution $\det(\boldsymbol{A}-\boldsymbol{\lambda}\mathbf{1}) = 0$. Non-trivial solution forms characteristic equation $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$ where I_1, I_2, I_3 are the invariants of matrix $\boldsymbol{A}$.

$$\begin{aligned}
I_1 &= trace(\boldsymbol{A})\\
I_2 &= \det\left(\begin{bmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix}\right) + \det\left(\begin{bmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{bmatrix}\right) + \det\left(\begin{bmatrix} a_{11} & a_{12}\\ a_{12} & a_{22}\end{bmatrix}\right)\\
I_3 &= \det(\boldsymbol{A})
\end{aligned} \tag{1.138}$$

Where a_{ij} are elements of matrix $\boldsymbol{A}$. Characteristic equation yields three roots for λ. One solution is always real where other two roots may be both real or may be complex and conjugate to each other. For each λ, we can solve homogeneous linear system of equations $(\boldsymbol{A}-\boldsymbol{\lambda}\mathbf{1}).\boldsymbol{u} = \mathbf{0}$ for Eigen vector u. The set of λ can form Eigen pairs; (λ_1, u_1), $(\lambda_2$, u$_2)$, and $(\lambda_3$, u$_3)$. If matrix $\boldsymbol{A}$ is symmetric, Eigen value analysis yields three real Eigen values, while symmetric and positive definite matrix results in three real positive Eigen values.

■ **Example 1.18** If $\boldsymbol{A}$ is defined as

$$A = \begin{bmatrix} 7 & 2 & -1\\ 2 & 3 & 4\\ -1 & 4 & 1\end{bmatrix}$$

Then

$$I_1 = trace(\boldsymbol{A}) = 11$$

$$I_1 = \det\left(\begin{bmatrix} 3 & 4\\ 4 & 1\end{bmatrix}\right) + \det\left(\begin{bmatrix} 7 & -1\\ -1 & 1\end{bmatrix}\right) + \det\left(\begin{bmatrix} 7 & 2\\ 2 & 3\end{bmatrix}\right) = -13+6+17 = 10$$

$$I_3 = \det(A) \;\; = -114$$

Characteristic equation

$$\lambda^3 - 11\lambda^2 + 10\lambda + 114 = 0$$

$$\lambda_1 = -2.5546,\ \lambda_2 = 7.9199,\ \lambda_3 = 5.6347$$

Or solving the following equation

$$\det(\boldsymbol{A} - \boldsymbol{\lambda 1}) = 0$$

$$\det(\boldsymbol{A} - \boldsymbol{\lambda 1}) = \det\left(\begin{bmatrix} 7-\boldsymbol{\lambda} & 2 & -1 \\ 2 & 3-\boldsymbol{\lambda} & 4 \\ -1 & 4 & 1-\boldsymbol{\lambda} \end{bmatrix}\right) = 0$$

Solving for Eigen vectors for $\lambda_1 = -2.5546$

$$\boldsymbol{0} = (\boldsymbol{A} - \boldsymbol{\lambda 1})\,.\boldsymbol{u} = \left(\begin{bmatrix} 7 & 2 & -1 \\ 2 & 3 & 4 \\ -1 & 4 & 1 \end{bmatrix} + 2.5546\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} 9.55 & 2 & -1 \\ 2 & 5.55 & 4 \\ -1 & 4 & 3.55 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 9.55u_1 + 2u_2 - u_3 \\ 2.0u_1 + 5.55u_2 + 4u_3 \\ -u_1 + 4u_2 + 3.55u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming $u_1 = 1$ and solving any two equations we get $u_2 = -2.97,\ \ u_3 = 3.62$

Normalizing vector $[u] = [u_1 \quad u_2 \quad u_3]^T = [1 \quad -2.97 \quad 3.62]^T$ to be a unit vector yielding:

$$\boldsymbol{u}^1 = \frac{[\boldsymbol{u}]}{|u|} = \begin{bmatrix} 0.209 & -0.62 & 0.756 \end{bmatrix}^T$$

Using the same above procedures, we get

$$For\ \lambda_2 = 7.9199,\ \ we\ get,\ \ u^2 = \begin{bmatrix} 0.45 & 0.626 & 0.637 \end{bmatrix}^T$$

$$For\ \lambda_3 = 5.6347,\ \ we\ get,\ \ u^3 = \begin{bmatrix} -0.868 & -0.474 & -0.148 \end{bmatrix}^T$$

Assuming matrix $P = \begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix}$, we can reach matrix P with three vector columns, each column is represented by u^1 as follows:

$$\boldsymbol{P} = \begin{bmatrix} 0.209 & -0.45 & -0.868 \\ -0.62 & 0.626 & -0.474 \\ 0.756 & 0.637 & -0.148 \end{bmatrix}$$

Note that $\boldsymbol{P}$ is an orthogonal matrix with $\left(\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{1}\right)$

Note also that for symmetric matrix $\boldsymbol{A}$ with Eigen vectors u^i, the following expression yields a diagonal matrix:

$$\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}^T[\boldsymbol{A}u^1, \boldsymbol{A}u^2, \boldsymbol{A}u^3] \ = \ \boldsymbol{P}^T[\lambda_1 u^1, \lambda_2 u^2, \lambda_3 u^3]$$

$$= \boldsymbol{P}^T\begin{bmatrix} u^1 & u^2 & u^3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \boldsymbol{P}^T\boldsymbol{P}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

■

1.2.3 Orthogonality of Eigen vectors for symmetric matrix A

If we have two pairs $(\boldsymbol{\lambda}_1, \boldsymbol{u}_1)$, $(\boldsymbol{\lambda}_2, \boldsymbol{u}_2)$ associated with the Eigen value analysis of symmetric matrix $\boldsymbol{A}$, orthogonality of Eigen vectors can be proven as follows:

$$\boldsymbol{A}.\boldsymbol{u}_i = \boldsymbol{\lambda}_i \boldsymbol{u}_i \boldsymbol{A}.\boldsymbol{u}_j = \boldsymbol{\lambda}_j \boldsymbol{u}_j \tag{1.139}$$

Pre-multiplying both above equation by $\boldsymbol{u}_j{}^T$, $\boldsymbol{u}_i{}^T$, respectively and subtracting both equation.

$$\boldsymbol{u}_j{}^T \boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{\lambda}_i \boldsymbol{u}_j{}^T \boldsymbol{u}_i \tag{1.140}$$
$$\boldsymbol{u}_i{}^T \boldsymbol{A}\boldsymbol{u}_j = \boldsymbol{\lambda}_j \boldsymbol{u}_i{}^T \boldsymbol{u}_j \tag{1.141}$$
$$(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j)\, \boldsymbol{u}_i{}^T \boldsymbol{u}_j = \boldsymbol{0} \tag{1.142}$$

As for any symmetric matrix A and any two vectors $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, the following identity can be achieved:

$$\boldsymbol{u}_j{}^T \boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{u}_i{}^T \boldsymbol{A}\boldsymbol{u}_j \tag{1.143}$$

Equation 1.142 leads to $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j$ or generally $\boldsymbol{u}_i{}^T \boldsymbol{u}_j = \boldsymbol{0}$ ($\boldsymbol{u}_i$ is normal to $\boldsymbol{u}_j$), so the Eigen vectors associated with different Eigen values are orthogonal to each other. Also this identity is proved in the previous example.

1.2.4 Spectral decomposition

Let us assume a known tensor $\boldsymbol{T}$ operating on another unknown tensor $\boldsymbol{L}$ using the following expression:

$$\boldsymbol{T} = Operator\,(\boldsymbol{L}) = O(\boldsymbol{L}) \tag{1.144}$$

Assuming a one-to-one mapping, the inverse of this operation yields:

$$\boldsymbol{L} = O^{-1}(\boldsymbol{T}) \tag{1.145}$$

Evaluation of the unknown tensor $\boldsymbol{L}$ requires following these procedures. First step is to transform tensor $\boldsymbol{T}$ into its principal coordinates, by finding its Eigen values and Eigen vectors, such that using the matrix notation, it can be written as follows:

$$\boldsymbol{T} = \boldsymbol{A} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \boldsymbol{A}^T \tag{1.146}$$

Which $\boldsymbol{A}$, λ_i are the Eigen vectors matrix and Eigen values of matrix $\boldsymbol{T}$. Tensor $\boldsymbol{L}$ can be evaluated by reverse the operation on the principle values of the tensor $\boldsymbol{T}$ such that tensor $\boldsymbol{L}$ will be defined as follows:

$$\boldsymbol{L} = \boldsymbol{A} \begin{bmatrix} O^{-1}(\lambda_1) & 0 & 0 \\ 0 & O^{-1}(\lambda_2) & 0 \\ 0 & 0 & O^{-1}(\lambda_3) \end{bmatrix} \boldsymbol{A}^T \tag{1.147}$$

■ **Example 1.19** Assume a known matrix $\boldsymbol{C}$ following this expression:

$$\begin{bmatrix} 1.2 & 0.3 & -0.2 \\ 0.3 & 1.3 & 0.4 \\ -0.2 & 0.4 & 1.4 \end{bmatrix} = \boldsymbol{C} = \boldsymbol{B}^2$$

And we need to evaluate matrix $\boldsymbol{B}$

$$\boldsymbol{C} = \boldsymbol{A} \lfloor \lambda_i \rceil \boldsymbol{A}^T$$

Calculating Eigen values, and Eigen vector of matrix $\boldsymbol{C}$

$$\lambda_i = (0.69,\ 1.45,\ 1.76)$$

$$[\boldsymbol{A}] = \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix}$$

$$O^{-1}(\lambda_i) = \sqrt{\lambda_i (U^2)}$$

$$\boldsymbol{B} = \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix} \begin{bmatrix} \sqrt{0.69} & 0 & 0 \\ 0 & \sqrt{1.45} & 0 \\ 0 & 0 & \sqrt{1.76} \end{bmatrix} \begin{bmatrix} -0.58 & 0.63 & -0.52 \\ 0.81 & 0.35 & -0.48 \\ -0.12 & -0.7 & -0.71 \end{bmatrix}$$

$$= \begin{bmatrix} 1.08 & 0.14 & -0.1 \\ 0.14 & 1.12 & 0.18 \\ -0.1 & 0.18 & 1.16 \end{bmatrix}$$

1.3 Vector calculus

Any function like temperature $T(\boldsymbol{x},t)$, velocity of fluid occupying some space $\boldsymbol{v}(\boldsymbol{x},t)$, or stress tensor distributed over a body $\boldsymbol{\sigma}(\boldsymbol{x},t)$ that, at any specific time t, varies with position $\boldsymbol{x}$ we need to understand its properties $\boldsymbol{x}$, is called a *field function*. Every position occupied with a particle has its own properties which probably change with time. Vector calculus studies variation of this field with position and time.

Differenting with time

Velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ is defined as the rate of change particles position with time at some position $\boldsymbol{x}$ at time t as follows:

$$\frac{d\boldsymbol{x}}{dt} = \frac{dx_i}{dt}\boldsymbol{e}_i = \frac{dx_1}{dt}\boldsymbol{e}_1 + \frac{dx_2}{dt}\boldsymbol{e}_2 + \frac{dx_3}{dt}\boldsymbol{e}_3 = \dot{\boldsymbol{x}}_i\boldsymbol{e}_i \tag{1.148}$$

Where $\boldsymbol{x}$ is the position vector and t indicates the time of recording the velocity. Similarly, acceleration can be evaluated as the time rate of change of velocity of particle yielding:

$$\frac{d\boldsymbol{v}}{dt} = \frac{d^2x_i}{dt^2}\boldsymbol{e}_i = \ddot{x}_i\boldsymbol{e}_i \tag{1.149}$$

From differentiation properties, we can conclude that:

$$\frac{d}{dt}(a.b) = \frac{d}{dt}(a).b + a.\frac{d}{dt}(b) \tag{1.150}$$

$$\frac{d}{dt}(a \times b) = \frac{d}{dt}(a) \times b + a \times \frac{d}{dt}(b) \tag{1.151}$$

$$\frac{d}{dt}(a \otimes b) = \frac{d}{dt}(a) \otimes b + a \otimes \frac{d}{dt}(b) \tag{1.152}$$

Differentiating with coordinates

Differentiation with coordinates is done using *Nabla operator* ∇ given by:

$$\nabla = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \tag{1.153}$$

While the matrix form is defined as:

$$[\nabla] = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}^T \tag{1.154}$$

Gradient of scalar field Φ with a continuous partial derivative is obtained from the following expression:

$$\nabla\Phi = \frac{\partial\Phi}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial\Phi}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial\Phi}{\partial x_3}\boldsymbol{e}_3 \tag{1.155}$$

Which can be written in the matrix form as follows:

$$[\nabla\Phi] = \begin{bmatrix} \frac{\partial\Phi}{\partial x_1} & \frac{\partial\Phi}{\partial x_2} & \frac{\partial\Phi}{\partial x_3} \end{bmatrix}^T \tag{1.156}$$

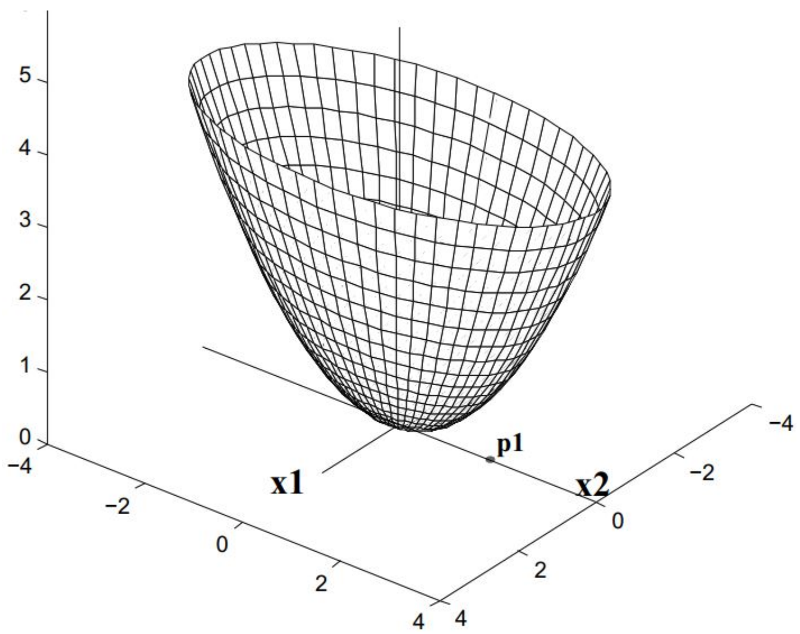


Figure 1.18: Scalar field function $f(x_1, x_2) = {x_1}^2 + 0.25{x_2}^2$

■ **Example 1.20** Calculate the gradient of field function $f(x_1, x_2) = {x_1}^2 + 0.25{x_2}^2$ shown in Figure 1.18 at points $p_1(x_1, x_2) = (0, 2)$

Gradient of the function $= \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 0.5x_2 \end{bmatrix}$

At point p_1. It means that functions $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 1$ and gradient will be (0,1), so moving to an adjacent point by only increasing x_1 by an infinitesimal amount, while x_2 is same, does not change the function $\left(\frac{\partial f}{\partial x_1} = 0\right)$. As in Figure 1.19, the increment in position x as indicated in the drawn arrow is in direction tangent to the contour lines which indicates no change in the function value, so only change in function can appear if we move in any direction except this tangent direction. Also maximum increase in function can be reached when moving normal to the contour line or in direction of the gradient $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (0, 1)$, while the value of the gradient $|\triangle f| = 1$ reflects the amount of increase in function with changing position (x_1, x_2).

Another derivative we would like to introduce is *directional derivative* of a scalar field in some direction $\boldsymbol{n}$ which is defined as $\nabla\Phi.\boldsymbol{n}$. For example, the directional derivative of the upper

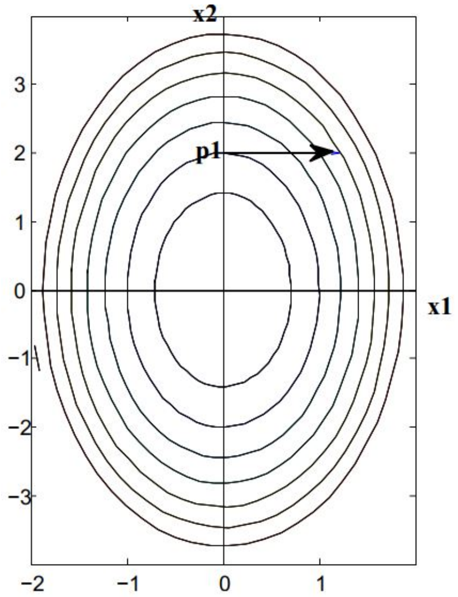


Figure 1.19: contour lines of the function projected on x_1 x_2 plane

function f in direction $\boldsymbol{n} = (1,\ 0)$ at point P_1 equals to $\nabla f.\boldsymbol{n} = (1,0).(0,1) = 0$ which means no change for the function in this direction, while if we evaluated it in the same direction of the gradient $\boldsymbol{n} = (0,\ 1)$, directional derivative yields $\nabla f.\boldsymbol{n} = (0,1).(0,1) = 1$ which provides the maximum change. Any other direction results smaller change or negative change, due to the fact that dot product of any two vectors is maximum if they share the same direction. ■

Gradient of vector $\boldsymbol{v}$ is the dyadic product of Nabla operator and vector field $\boldsymbol{v}$ which transforms the vector to a second order tensor. Generally gradient of a field increases the order of the field by one (gradient of a scalar is vector and the gradient of vector is second order tensor). This field should have a continuous partial derivative.

$$\nabla \boldsymbol{v} = \nabla \otimes \boldsymbol{v} = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = \frac{\partial v_j}{\partial x_i}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.157}$$

With matrix from

$$[\nabla \boldsymbol{v}]_{ij} = [\nabla \otimes \boldsymbol{v}]_{ij} = [\nabla]_i \otimes [\boldsymbol{v}]_j = \left[\frac{\partial v_j}{\partial x_i}\right] \tag{1.158}$$

$$[\nabla \boldsymbol{v}] = \begin{bmatrix} \frac{\partial \boldsymbol{v}_1}{\partial x_1} & \frac{\partial \boldsymbol{v}_2}{\partial x_1} & \frac{\partial \boldsymbol{v}_3}{\partial x_1} \\ \frac{\partial \boldsymbol{v}_1}{\partial x_2} & \frac{\partial \boldsymbol{v}_2}{\partial x_2} & \frac{\partial \boldsymbol{v}_3}{\partial x_2} \\ \frac{\partial \boldsymbol{v}_1}{\partial x_3} & \frac{\partial \boldsymbol{v}_2}{\partial x_3} & \frac{\partial \boldsymbol{v}_3}{\partial x_3} \end{bmatrix} \tag{1.159}$$

Gradient of 2^{nd} order tensor $\boldsymbol{A}$ forms 3^{rd} order tensor defined as follows:

$$\nabla \boldsymbol{A} = \nabla \otimes \boldsymbol{A} = \frac{\partial A_{jk}}{\partial x_i}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{1.160}$$

Divergence of a field tensor is the dot product of Nabla operator with the field tensor. For a vector field $\boldsymbol{v}$ and tensor field $\boldsymbol{A}$ with a continuous partial derivative, divergence of these fields is given by:

$$\nabla .\boldsymbol{v} = \frac{\partial}{\partial x_i}\boldsymbol{e}_i . v_j \boldsymbol{e}_j = \frac{\partial v_j}{\partial x_i}\boldsymbol{e}_i . \boldsymbol{e}_j = \frac{\partial v_j}{\partial x_i}\delta_{ij} = \frac{\partial v_1}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{1.161}$$

$$\nabla.\boldsymbol{A} = \frac{\partial}{\partial x_i}\boldsymbol{e}_i.A_{jk}\boldsymbol{e}_j \otimes \boldsymbol{e}_k = \frac{\partial A_{jk}}{\partial x_i}\boldsymbol{e}_i.\boldsymbol{e}_j \otimes \boldsymbol{e}_k = \frac{\partial A_{jk}}{\partial x_i}\delta_{ij}\boldsymbol{e}_k = \frac{\partial A_{ik}}{\partial x_i}\boldsymbol{e}_k \tag{1.162}$$

$\nabla.\boldsymbol{v}$ is a scalar value while $\nabla.\boldsymbol{A}$ is a vector field represented in matrix form as follows:

$$[\nabla.\boldsymbol{A}]_j = [\nabla]_j.[\boldsymbol{A}]_{ij} = \frac{\partial A_{ij}}{\partial x_i} \tag{1.163}$$

Rotation or curl of vector includes the cross product of the Nabla operator and the vector as follows:

$$\nabla \times \boldsymbol{v} = \frac{\partial}{\partial x_j}\boldsymbol{e}_j \times \boldsymbol{v}_k\boldsymbol{e}_k = \frac{\partial v_k}{\partial x_j}\boldsymbol{e}_j \times \boldsymbol{e}_k = \frac{\partial v_k}{\partial x_j}\boldsymbol{\varepsilon_{ijk}}\boldsymbol{e}_i \tag{1.164}$$

Curl of vector tells us about the spatial rate of rotation (ω) with magnitude defined as:

$$\omega = \frac{1}{2}\,|\nabla \times \boldsymbol{v}| \tag{1.165}$$

where $\boldsymbol{v}$ is the velocity vector field across the body studied.

■ **Example 1.21** Let us have a plate rotating about an axis x_3 with rate ω. The position of material points of the plate is changing as a function of time t according to the following:

$$x_1 = X_1\cos(\omega t) - X_2\sin(\omega t)$$

$$x_2 = X_1\sin(\omega t) + X_2\cos(\omega t)$$

$$x_3 = X_3$$

$$v = \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} -\omega X_1\sin(\omega t) - \omega X_2\cos(\omega t) \\ \omega X_1\cos(\omega t) - \omega X_2\sin(\omega t) \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega x_2 \\ \omega x_1 \\ 0 \end{bmatrix}$$

$$\nabla \times \boldsymbol{v} = (0, 0, 2\omega)$$

The spin have magnitude:

$$\frac{1}{2}\,|\nabla \times \boldsymbol{v}| = \frac{1}{2}|(0,0,2\omega)| = \omega$$

With direction $(0,0,1)$ and parallel to the axis of rotation. While $\nabla \times \boldsymbol{v}$ gives direction the axis of the rotations. ■

Laplacian of a scalar field function is the divergence of gradient of a function with a continuous second partial derivative termed as:

$$\nabla.\nabla\Phi = \nabla^2\Phi = \frac{\partial^2\Phi}{\partial {x_1}^2} + \frac{\partial^2\Phi}{\partial {x_2}^2} + \frac{\partial^2\Phi}{\partial {x_3}^2} \tag{1.166}$$

Laplacian of a vector field function is defined as:

$$\nabla.\nabla\boldsymbol{v} = \nabla^2\boldsymbol{v} = \frac{\partial^2 v_j}{\partial x_i \partial x_i}\boldsymbol{e}_j \tag{1.167}$$

Note 1.4 There are some useful expression we would like to address.
For scalar fields Φ and Ψ and vectors a and b, we note the following:

$$\nabla \times (\nabla \Phi) = \left(\frac{\partial}{\partial x_i}\boldsymbol{e}_i\right) \times \left(\frac{\partial \Phi}{\partial x_j}\boldsymbol{e}_j\right) = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}\varepsilon_{ijk}\boldsymbol{e}_k = \frac{\partial^2 \Phi}{\partial x_j \partial x_i}\varepsilon_{ijk}\boldsymbol{e}_k \tag{1.168}$$

As coordinate axes are linear independent so $\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\partial^2 \Phi}{\partial x_j \partial x_i}$. Using that $\varepsilon_{ijk} = -\varepsilon_{jik}$ so;

$$\nabla \times (\nabla \Phi) = -\frac{\partial^2 \Phi}{\partial x_j \partial x_i}\varepsilon_{jik}\boldsymbol{e}_k \tag{1.169}$$

As index notation can be flipped with each other so flipping index i with index j yields:

$$\nabla \times (\nabla \Phi) = -\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\varepsilon_{ijk}\boldsymbol{e}_k \tag{1.170}$$

Summing Equation 1.170 and Equation 1.168 leads to:

$$\boxed{\nabla \times (\nabla \Phi) = 0} \tag{1.171}$$

Also we can deduce the following relation

$$\begin{aligned}\nabla.(\nabla \Phi \times \nabla \Psi) &= \left(\frac{\partial}{\partial x_i}\boldsymbol{e}_i\right).\left(\frac{\partial \Phi}{\partial x_j}\boldsymbol{e}_j \times \frac{\partial \Psi}{\partial x_k}\boldsymbol{e}_k\right)\\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial \Phi}{\partial x_j}\frac{\partial \Psi}{\partial x_k}\right)\boldsymbol{e}_i.(\boldsymbol{e}_j \times \boldsymbol{e}_k)\\ &= \left(\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\frac{\partial \Psi}{\partial x_k} + \frac{\partial \Phi}{\partial x_j}\frac{\partial^2 \Psi}{\partial x_i \partial x_k}\right)\varepsilon_{ijk} = 0\end{aligned} \tag{1.172}$$

So we get

$$\boxed{\nabla.(\nabla \Phi \times \nabla \Psi) = 0} \tag{1.173}$$

In deriving the above expression, we used Equation 1.171 and the following identities:

$$\boldsymbol{e}_i.(\boldsymbol{e}_j \times \boldsymbol{e}_k) = \boldsymbol{e}_i.\left(\varepsilon_{jkm}\,\boldsymbol{e}_m\right) = \varepsilon_{jkm}\,\delta_{im} = \varepsilon_{jki} = \varepsilon_{ijk} \tag{1.174}$$

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\varepsilon_{ijk} = 0 \tag{1.175}$$

Another one we would like to introduce:

$$\begin{aligned}
\nabla \times (\nabla \times a) &= \left(\frac{\partial}{\partial x_i}\boldsymbol{e}_i\right) \times \left(\frac{\partial}{\partial x_j}\boldsymbol{e}_j \times a_k \boldsymbol{e}_k\right) \\
&= \left(\frac{\partial}{\partial x_i}\boldsymbol{e}_i\right) \times \left(\frac{\partial a_k}{\partial x_j}\varepsilon_{jkm}\boldsymbol{e}_m\right) \\
&= \frac{\partial^2 a_k}{\partial x_i \partial x_j}\varepsilon_{ijk}\boldsymbol{e}_i \times \boldsymbol{e}_m \\
&= \frac{\partial^2 a_k}{\partial x_i \partial x_j}\varepsilon_{ijk}\varepsilon_{imn}\boldsymbol{e}_n \\
&= \frac{\partial^2 a_k}{\partial x_i \partial x_j}\left(\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}\right)\boldsymbol{e}_n \\
&= \left(\frac{\partial^2 a_k}{\partial x_k \partial x_n} - \frac{\partial^2 a_n}{\partial x_i \partial x_i}\right)\boldsymbol{e}_n \\
&= \frac{\partial}{\partial x_n}\left(\frac{\partial a_k}{\partial x_k}\right)\boldsymbol{e}_n - \nabla^2 a \\
&= \nabla\left(\nabla . a\right) - \nabla^2 a
\end{aligned} \tag{1.176}$$

so we get:

$$\boxed{\nabla \times (\nabla \times a) = \nabla\left(\nabla . a\right) - \nabla^2 a} \tag{1.177}$$

Another one:

$$\begin{aligned}
\nabla . (a \times b) &= \frac{\partial}{\partial x_i}\boldsymbol{e}_i . \left(a_j a_k \varepsilon_{jkl}\boldsymbol{e}_l\right) \\
&= \left(\frac{\partial a_j}{\partial x_i}a_k + a_j \frac{\partial a_k}{\partial x_i}\right)\varepsilon_{jkl}\boldsymbol{e}_i . \boldsymbol{e}_l \\
&= \left(\frac{\partial a_j}{\partial x_i}a_k + a_j \frac{\partial a_k}{\partial x_i}\right)\varepsilon_{jkl}\delta_{il} \\
\nabla . (a \times b) &= \varepsilon_{ijk}\left(\frac{\partial a_j}{\partial x_i}a_k + a_j \frac{\partial a_k}{\partial x_i}\right) = \boldsymbol{b} . (\nabla \times \boldsymbol{a}) - \boldsymbol{a} . (\nabla \times \boldsymbol{b})
\end{aligned} \tag{1.178}$$

So we get

$$\boxed{\nabla . (a \times b) = \boldsymbol{b} . (\nabla \times \boldsymbol{a}) - \boldsymbol{a} . (\nabla \times \boldsymbol{b})} \tag{1.179}$$

We used the following expression in deriving the above equation:

$$\varepsilon_{jkl}\boldsymbol{e}_i . \boldsymbol{e}_l = \varepsilon_{jkl}\delta_{il} = \varepsilon_{jki} = \varepsilon_{ijk} \tag{1.180}$$

In the same manner:

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla . \boldsymbol{b})\boldsymbol{a} + \boldsymbol{b} . \nabla \boldsymbol{a} - [(\nabla . \boldsymbol{a})\boldsymbol{b} + \boldsymbol{a} . \nabla \boldsymbol{b}] \tag{1.181}$$

$$\boldsymbol{a} \times (\nabla \times \boldsymbol{b}) = \boldsymbol{a} . \left[\nabla \boldsymbol{a}^T - \nabla \boldsymbol{a}\right] \tag{1.182}$$

For the length of vector $\boldsymbol{x}$ defined as:

$$|\boldsymbol{x}| = \sqrt{|\boldsymbol{x}.\boldsymbol{x}|} = \sqrt{|x_i.x_i|} \tag{1.183}$$

The gradient of the length comes from:

$$\begin{aligned}
\nabla(|\boldsymbol{x}|) &= \frac{\partial}{\partial x_j}\boldsymbol{e}_j\sqrt{|x_i.x_i|} \\
&= \frac{\partial\left(\sqrt{|x_i.x_i|}\right)}{\partial x_j}\boldsymbol{e}_j \\
&= \frac{1}{2}\frac{2x_i}{\sqrt{|x_i.x_i|}}\frac{\partial x_i}{\partial x_j}\boldsymbol{e}_j \\
&= \frac{x_i}{\sqrt{|x_i.x_i|}}\delta_{ij}\boldsymbol{e}_j \\
&= \frac{x_j}{\sqrt{|x_i.x_i|}}\boldsymbol{e}_j = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}
\end{aligned} \tag{1.184}$$

■

1.3.1 Divergence or Gauss theorem

This theorem is used to solve mechanical and variational calculus problems, especially when integral is hard to evaluate in some forms and can be switched to other forms easier to handle. Divergence of a tensor $\boldsymbol{A}$ with a continuous partial derivative over some domain V (generally the body volume) can be converted into integral over the body boundary ∂V with an outward unit vector n normal to the boundary as in Figure 1.20. The general divergence theorem is defined as:

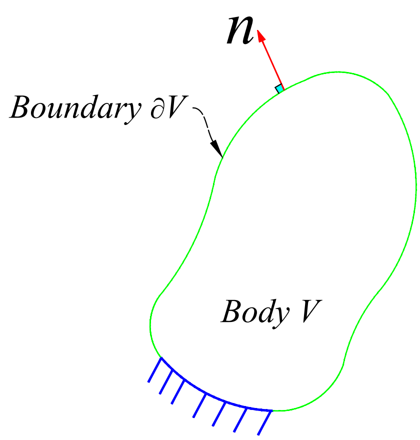


Figure 1.20

$$\int_V \nabla \odot \boldsymbol{A}\, dV = \int_{\partial V} \boldsymbol{n} \odot \boldsymbol{A}\, dS \tag{1.185}$$

Where $\odot$ is a general operator which can be dot, cross, or dyadic product as follows:

$$\int_V \nabla.\boldsymbol{A}\, dV = \int_{\partial V} \boldsymbol{n}.\boldsymbol{A}\, dS \tag{1.186}$$

$$\int_V \nabla\times\boldsymbol{A}\, dV = \int_{\partial V} \boldsymbol{n}\times\boldsymbol{A}\, dS \tag{1.187}$$

$$\int_V \nabla \otimes \boldsymbol{A}\, dV = \int_{\partial V} \boldsymbol{n} \otimes \boldsymbol{A}\, dS \tag{1.188}$$

From above, we can evaluate the integral over body volume using the properties of the outer parameter (surface) without need to dig into the body volume.
For a two dimensional analysis, integral over area a can be switched to integration over the area perimeter P as follows:

$$\int_V \nabla \odot \boldsymbol{A}\, da = \int_{\partial A} \boldsymbol{n} \odot \boldsymbol{A}\, dP \tag{1.189}$$

We will illustrate The following two examples to understand the divergence theorem as follows.

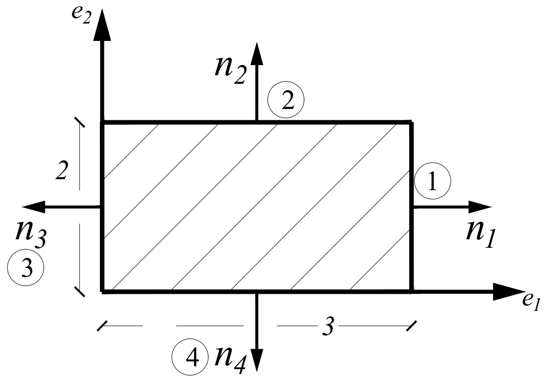


Figure 1.21

■ **Example 1.22 — Rectangular area.** If we need to evaluate the area of the shown rectangular in Figure 1.21. Area of the rectangular A is defined as follows:

$$A = \int_A dA = \int_A 1\ dA = \int_A \nabla . \boldsymbol{b} dA \tag{1.190}$$

Where $\boldsymbol{b}$ is any vector such that $\nabla . \boldsymbol{b} = 1$, e.g. assume $b = x_1 \boldsymbol{e}_1$. Using divergence theorem, area integral can be converted to line integral as follows:

$$A = \int_A \nabla . \boldsymbol{b} dA = \int_S \boldsymbol{n} . \boldsymbol{b} dS \tag{1.191}$$

Where $\boldsymbol{n}$ is the normal to the surface and S signifies the boundary of rectangular. We can divide the boundary of rectangular into four boundaries and the line integral can be defined over each boundary as follows:

$$\begin{array}{lll} \textit{Boundary } 1 & [\boldsymbol{n}] = (1,0) \rightarrow & \int_S (\boldsymbol{n}.\boldsymbol{b}|_{x_1=3}) dS = 3 \int_S dS = 3 * 2 = 6 \\ \textit{Boundary } 2 & [\boldsymbol{n}] = (0,1) & \int_S (\boldsymbol{n}.\boldsymbol{b}|_{x_2=2}) dS = 0 \\ \textit{Boundary } 3 & [\boldsymbol{n}] = (-1,0) & \int_S (\boldsymbol{n}.\boldsymbol{b}|_{x_1=0}) dS = 0 \\ \textit{Boundary } 4 & [\boldsymbol{n}] = (0,-1) & \int_S (\boldsymbol{n}.\boldsymbol{b}|_{x_1=0}) dS = 0 \end{array} \tag{1.192}$$

So the total integral is the sum over the four boundaries resulting the area:

$$A = \int_S \boldsymbol{n}.\boldsymbol{b} dS = 6 \tag{1.193}$$

■

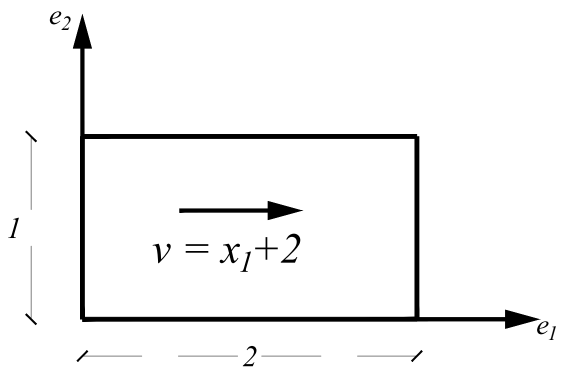


Figure 1.22

■ **Example 1.23 — Discharge of a rectangular body with a unit width.** If we have a fluid with velocity field $\boldsymbol{v} = (4x_1 + 2, 0)$, and it is required to find the discharge through rectangular shape shown in Figure 1.22 with unit width. Discharge Q is measured through the dot product of the velocity and the normal to the surface $\boldsymbol{n}$ as follows:

$$Q = width \times \int_A \boldsymbol{n}.\boldsymbol{v} dS = \int_A \nabla.\boldsymbol{v} dA = \int_A 4dA = 4*2*1 = 8 \tag{1.194}$$

■

Note 1.5 Useful relation

$$\nabla.\left(\boldsymbol{A}.\boldsymbol{v}\right) = \frac{\partial}{\partial x_i}\left(A_{ij}v_j\right) = \frac{\partial A_{ij}}{\partial x_i}.v_j + A_{ij}\frac{\partial v_j}{\partial x_i} = \left(\nabla.\boldsymbol{A}\right).\boldsymbol{v} + \boldsymbol{A} : \nabla\boldsymbol{v} \tag{1.195}$$

$$\int_{\partial V} \boldsymbol{n}.\left(\boldsymbol{A}.\boldsymbol{v}\right)\ dS = \int_V \nabla.\left(\boldsymbol{A}.\boldsymbol{v}\right)\ dV = \int_V \left(\left(\nabla.\boldsymbol{A}\right).\boldsymbol{v} + \boldsymbol{A} : \nabla\boldsymbol{v}\right) dV \tag{1.196}$$

But

$$\boldsymbol{n}.\left(\boldsymbol{A}.\boldsymbol{v}\right) = \boldsymbol{n}.\boldsymbol{A}.\boldsymbol{v} = \boldsymbol{v}.\boldsymbol{A}.\boldsymbol{n} = \boldsymbol{v}.(\boldsymbol{A}.\boldsymbol{n}) \tag{1.197}$$

We can deduce from above expressions and Equation 1.196 the following:

$$\boxed{\int_V \left(\nabla\boldsymbol{v} : \boldsymbol{A}\right) dV = \int_{\partial V} \boldsymbol{v}.\left(\boldsymbol{n}.\boldsymbol{A}\right) dS - \int_V \boldsymbol{v}.\left(\nabla.\boldsymbol{A}\right)\ dV} \tag{1.198}$$

The above derivation is called integration by part.

The term $\nabla.(\boldsymbol{v}\times\boldsymbol{A})$ can be defined as follows:

$$\begin{aligned}\nabla.(\boldsymbol{v}\times\boldsymbol{A}) &= \frac{\partial}{\partial x_n}e_n.\left(v_i A_{jm}\varepsilon_{ijk}e_k\otimes e_m\right)\\ &= \left(\frac{\partial v_i}{\partial x_n}A_{jm}+v_i\frac{\partial A_{jm}}{\partial x_n}\right)\varepsilon_{ijk}\delta_{nk}e_m\end{aligned} \tag{1.199}$$

But

$$\begin{aligned}\frac{\partial v_i}{\partial x_n}A_{jm}\varepsilon_{ijn}e_n\delta_{nm} &= \frac{\partial v_i}{\partial x_m}A_{jm}\varepsilon_{ijn}e_n\\ &= \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}^m\times\boldsymbol{A}^m\\ &= \nabla\boldsymbol{v}^m\times\boldsymbol{A}^m\end{aligned} \tag{1.200}$$

Where vector $\boldsymbol{A}^m$ represents the m^{th} column of matrix $\boldsymbol{A}$ with components $\boldsymbol{A}^m{}_k = A_{km}$ Where $\boldsymbol{B}^m\times\boldsymbol{P}^m = \sum_{m=1}^{3}\boldsymbol{B}^m\times\boldsymbol{P}^m$ as m is a dummy index. Also the second term in Equation 1.199 can be reduced to:

$$\begin{aligned}v_i\frac{\partial A_{jm}}{\partial x_n}\varepsilon_{ijk}\delta_{nk}e_m &= v_i\frac{\partial A_{nj}}{\partial x_n}\delta_{jm}\delta_{jn}\varepsilon_{ijk}\delta_{nk}\delta_{mk}e_k\\ &= (v\times\nabla.A)\,\delta_{jm}\delta_{jn}\delta_{nk}\delta_{mk}\\ &= \boldsymbol{v}\times(\nabla.\boldsymbol{A})\end{aligned} \tag{1.201}$$

So $\nabla.(\boldsymbol{v}\times\boldsymbol{A})$ in Equation 1.199 after using the divergence theorem will be:

$$\begin{aligned}\int_{\partial V}\boldsymbol{n}.(\boldsymbol{v}\times\boldsymbol{A})\,dS &= \int_V \nabla.(\boldsymbol{v}\times\boldsymbol{A})\;dV\\ &= \int_V(\boldsymbol{v}\times(\nabla.\boldsymbol{A})+\nabla\boldsymbol{v}^m\times\boldsymbol{A}^m)\,dV\end{aligned} \tag{1.202}$$

But

$$\int_V(\boldsymbol{v}\times(\nabla.\boldsymbol{A}))\,dV = \int_{\partial V}\boldsymbol{n}.(\boldsymbol{v}\times\boldsymbol{A})\;dS - \int_V \boldsymbol{v}\times(\nabla.\boldsymbol{A})\,dV \tag{1.203}$$

and

$$\boldsymbol{n}.(\boldsymbol{v}\times\boldsymbol{A}) = \boldsymbol{v}\times(\boldsymbol{A}.\boldsymbol{n}) \tag{1.204}$$

which yields:

$$\boxed{\int_V(\boldsymbol{v}\times(\nabla.\boldsymbol{A}))\,dV = \int_{\partial V}\boldsymbol{v}\times(\boldsymbol{A}.\boldsymbol{n})\;dS - \int_V \boldsymbol{v}\times(\nabla.\boldsymbol{A})\,dV} \tag{1.205}$$

The above expression can also be called integration by part. ■

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2. Finite Rotation and its Applications

2.1 Rotation in plane (rotation about origin)

2.1.1 Body rotation with fixed coordinate system

Lets us assume a body undergoing a counterclockwise rotation with angle θ in two dimensional plane about origin and referred to a fixed coordinate system with basis (triad) $\mathscr{B}$. If we assume that the solid line and dashed line are used for the body before and after rotation as shown in Figure 2.1 and the rotation θ is positive for rotating counter-clockwise (or using right-hand rule by upward pointing thumb normal to the paper plane in $\boldsymbol{e}_3$ direction), such that any vector attached to the body with position vector (X_1, X_2) is transformed after rotation to (x_1, x_2) given by:

$$\begin{aligned} x_1 &= X_1 \cos\theta - X_2 \sin\theta \\ x_2 &= X_1 \sin\theta + X_2 \cos\theta \end{aligned} \tag{2.1}$$

And it can be written in matrix form as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{2.2}$$

$$[\boldsymbol{x}]^B = [\boldsymbol{R}]^B [\boldsymbol{X}]^B \tag{2.3}$$

$[\boldsymbol{R}]^B$ is called the *rotation matrix* resolved in basis $\mathscr{B}$. If $[\boldsymbol{x}]^B$ and $[\boldsymbol{X}]^B$ are position vector after and before the rotation resolved in the same basis $\mathscr{B}$, the negative sign in $-\sin(\theta)$ in the above expressions comes from the fact that components of vector $\boldsymbol{X}$ in $\boldsymbol{e}_1$ direction is reduced with positive rotation.

The tensorial form of the transformation will be:

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{X} \tag{2.4}$$

Bear in mind that the coordinate system still the same after rotation. Also we need to note that the upper form implies that observer has the freedom to choose other coordinate system with other

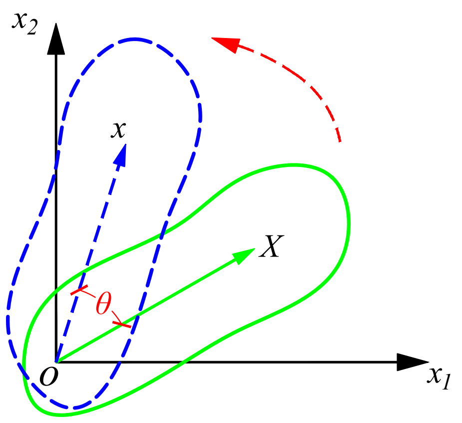


Figure 2.1

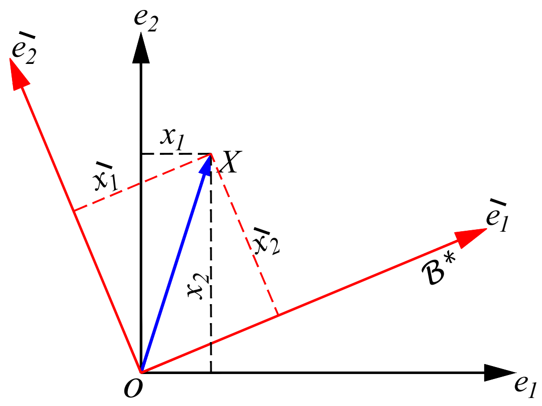


Figure 2.2

basis, e.g. $\grave{\boldsymbol{e}}_i$ such that the rotation $\boldsymbol{R}$ can be resolved in both bases; $\boldsymbol{e}_1$ and $\grave{\boldsymbol{e}}_i$ as follows:

$$\boldsymbol{x} = x_i \boldsymbol{e}_i = \grave{x}_i \grave{\boldsymbol{e}}_i \tag{2.5}$$

$$\boldsymbol{X} = X_i \boldsymbol{e}_i = \grave{X}_i \grave{\boldsymbol{e}}_i \tag{2.6}$$

$$\boldsymbol{R} = \boldsymbol{R}_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \grave{\boldsymbol{R}}_{ij} \grave{\boldsymbol{e}}_i \otimes \grave{\boldsymbol{e}}_j \tag{2.7}$$

Where x_i, and X_i are components of the vector after and before rotation and rotation matrix resolved in coordinate system with basis $\boldsymbol{e}_i$, while $\grave{x}_i$, and $\grave{X}_i$ are the components of the same vectors resolved in different basis $\grave{\boldsymbol{e}}_i$ as shown in Figure 2.2, for $i = 1, 2, 3$. $\boldsymbol{R}_{ij}$, and $\grave{\boldsymbol{R}}_{ij}$ represents the components of rotation tensor resolved in different bases and they are generally different to the same spatial tensor $\boldsymbol{R}$, but it can be proven that $\grave{\boldsymbol{R}}_{ij}$, and $\boldsymbol{R}_{ij}$ are identical in two dimensional plane rotation as it depends only on the rotation angle $\boldsymbol{\theta}$.

2.1.2 Body fixed in space referred to a rotated coordinate system.

Consider a body resolved in two coordinate system $\mathscr{B}$ and $\mathscr{B}^*$. As schematically shown in Figure 2.3 coordinate system $\mathscr{B}^*$ with dashed axes is obtained from applying a counterclockwise rotation by angle θ about origin O on coordinate system $\mathscr{B}$. Keep in mind that the body itself is fixed, while coordinate system undergoes rotation. If we have a vector attached to a body, it can be resolved in the both coordinate systems following these relations:

$$\begin{aligned} x_1{}^* &= x_1 \cos\theta + x_2 \sin\theta \\ x_2{}^* &= -x_1 \sin\theta + x_2 \cos\theta \end{aligned} \tag{2.8}$$

Where x_i, $x_i{}^*$ are components of the vector resolved in coordinate system with basis $\mathscr{B}$ and basis $\mathscr{B}^*$, respectively as shown in Figure 2.4 for $i = 1, 2, 3$.

$$\begin{bmatrix} x_1{}^* \\ x_2{}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.9}$$

$$[\boldsymbol{x}]^{B^*} = [\boldsymbol{Q}]^{B}_{B\to B^*} [\boldsymbol{x}]^{B} \;\; or \;\; [\boldsymbol{x}]^{B^*} = [\boldsymbol{Q}]^{B^*}_{B\to B^*} [\boldsymbol{x}]^{B} \tag{2.10}$$

$[\boldsymbol{Q}]^{B}_{B\to B^*}$, $[\boldsymbol{Q}]^{B^*}_{B\to B^*}$ are the transformation matrix from basis $\mathscr{B}$ to basis $\mathscr{B}^*$; as indicated in the subscript of $[\boldsymbol{Q}]$; resolved in basis $\mathscr{B}$ and basis $\mathscr{B}^*$, respectively (the superscript indicates the basis $[\boldsymbol{Q}]$ is resolved in). They are identical in two dimensional plane transformation. Subscript

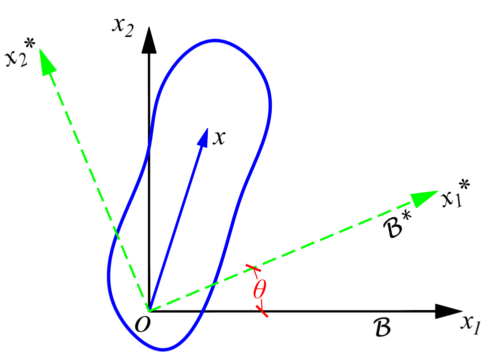


Figure 2.3

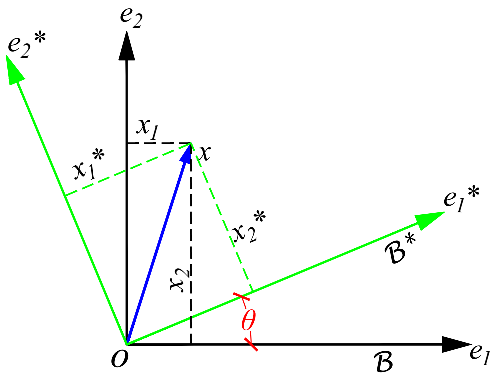


Figure 2.4

$(B \rightarrow B^*)$can be dropped for convenience. $[\boldsymbol{Q}]^B$ also called direction cosine matrix with elements expressed as:

$$Q_{ij} = \cos\left(\boldsymbol{e}_i^*, \boldsymbol{e}_j\right) \; = \boldsymbol{e}_i^*.\boldsymbol{e}_j \tag{2.11}$$

We emphasis again that the vector itself do not rotate and it is still the same spatial vector but described in a new coordinate system. We can also easily verify that rotation matrix and transformation matrix are orthogonal matrix carrying these relations:

$$det\left(\boldsymbol{R}\right) = \det\left(\boldsymbol{Q}\right) = 1 \tag{2.12}$$
$$\boldsymbol{R}\boldsymbol{R}^T \; = \boldsymbol{R}^T\boldsymbol{R} = 1 \; or \; \boldsymbol{R}^{-1} = \boldsymbol{R}^T \tag{2.13}$$
$$\boldsymbol{Q}\boldsymbol{Q}^T \; = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{1} \; or \; \boldsymbol{Q}^{-1} = \boldsymbol{Q}^T \tag{2.14}$$

we can generalize the transformation rule for higher order tensors. For example, second order tensor can be formed from a dyadic product of two arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and can be resolved in basis $\mathscr{B}$ as follows:

$$[\boldsymbol{A}]^{[\mathscr{B}]} = [\boldsymbol{u} \otimes \boldsymbol{v}]^{[\mathscr{B}]} = \left([\boldsymbol{u}][\boldsymbol{v}]^T\right)^{[\mathscr{B}]} = [\boldsymbol{u}]^{[\mathscr{B}]}\left([\boldsymbol{v}]^{[\mathscr{B}]}\right)^T \tag{2.15}$$

The components of this dyadic in another basis $\mathscr{B}^*$ could be determined as follows:

$$[\boldsymbol{A}]^{\boldsymbol{B}^*} = [\boldsymbol{u} \otimes \boldsymbol{v}]^{\boldsymbol{B}^*} = [\boldsymbol{u}]^{\boldsymbol{B}^*}\left([\boldsymbol{v}]^{\boldsymbol{B}^*}\right)^T = [\boldsymbol{Q}]^B[\boldsymbol{u}]^{[\mathscr{B}]}\left([\boldsymbol{Q}]^B[\boldsymbol{v}]^{[\mathscr{B}]}\right)^T = [\boldsymbol{Q}]^B\underline{[\boldsymbol{u}]^{[\mathscr{B}]}\left([\boldsymbol{v}]^{[\mathscr{B}]}\right)^T}\left([\boldsymbol{Q}]^B\right)^T \tag{2.16}$$

$$[\boldsymbol{A}]^{\boldsymbol{B}^*} = [\boldsymbol{Q}]\,[\boldsymbol{A}]^{[\mathscr{B}]}[\boldsymbol{Q}]^T \tag{2.17}$$

With index notation as follows:

$$A_{ij}^* = Q_{im}Q_{jn}A_{mn} \tag{2.18}$$

For example, assume a second order stress tensor $\boldsymbol{\sigma}$ at point P in two dimensional case as shown in Figure 2.5 and resolved in basis $\mathscr{B}$ as follows:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \tag{2.19}$$

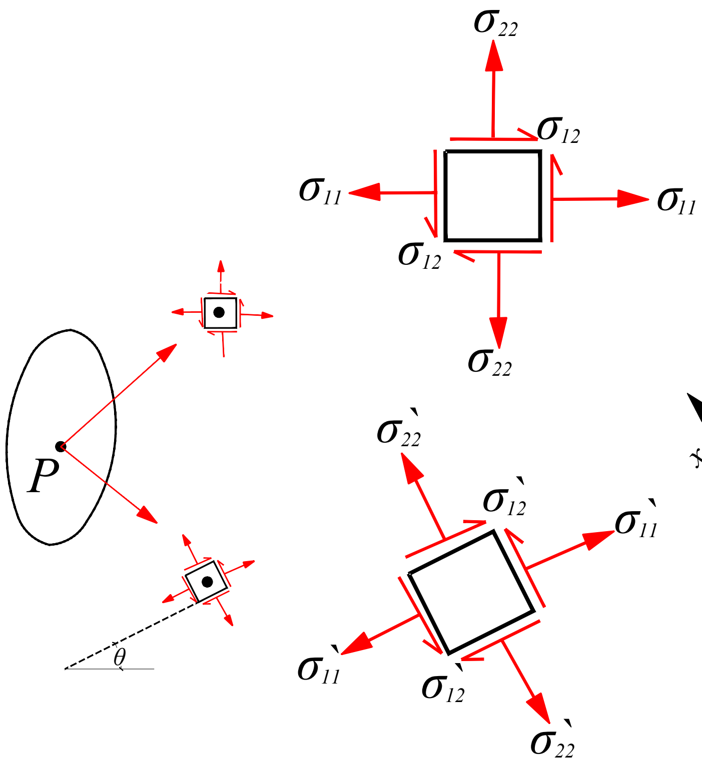


Figure 2.5

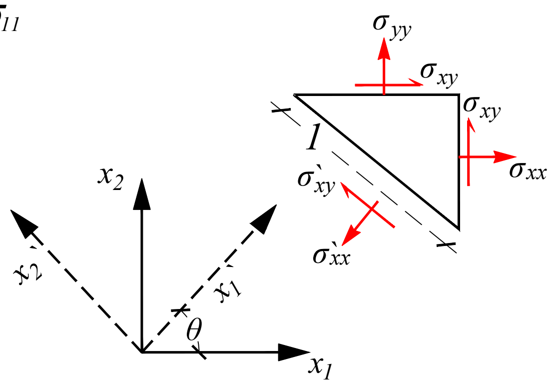


Figure 2.6

Resolving in other basis $\mathscr{B}'$ will follow this transformation relation:

$$[\boldsymbol{\sigma}]' = [\boldsymbol{Q}]\,[\boldsymbol{\sigma}]\,[\boldsymbol{Q}]^T \tag{2.20}$$

Or in index notation

$$\boldsymbol{\sigma'}_{ij} = \boldsymbol{Q_{im}Q_{jn}\sigma}_{ij} \tag{2.21}$$

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.22}$$

Which results in:

$$\begin{aligned} \sigma'_{11} &= \sigma_{11}\cos\theta^2 + \sigma_{22}\sin\theta^2 + 2\sigma_{12}\sin\theta\cos\theta \\ \sigma'_{22} &= \sigma_{11}\sin\theta^2 + \sigma_{22}\cos\theta^2 - 2\sigma_{12}\sin\theta\cos\theta \\ \sigma'_{12} &= (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \sigma_{12}\left(\cos\theta^2 - \sin\theta^2\right) \end{aligned} \tag{2.23}$$

The same results can be obtained using Mohr's circle or studying the equilibrium of a differential triangular element with thickness t and dimensions shown in Figure 2.6 by summing the force along x' coordinate as follows:.

$$\sigma'_{11} * 1 * t = \sigma_{11}\cos\theta * (\cos\theta * t) + \sigma_{22}\sin\theta * (\sin\theta * t) + \sigma_{12}\sin\theta * (\cos\theta * t) + \sigma_{12}\cos\theta * (\sin\theta * t) \tag{2.24}$$

Which leads to the same results of Equation 2.23.

For 4^{th} order tensor like the one used in constitutive relations can be resolved in two bases $\mathscr{B}^*$ and $\mathscr{B}'$ as follows:

$$\sigma_{mn} = C_{mnop}\,\varepsilon_{op} \tag{2.25}$$

$$\sigma'_{ij} = C'_{ijkl}\,\varepsilon'_{kl} \tag{2.26}$$

The transformation rule will be:

$$Q_{im}Q_{jn}\sigma_{mn} = C'_{ijkl}\,Q_{ko}Q_{lp}\varepsilon_{op} \tag{2.27}$$

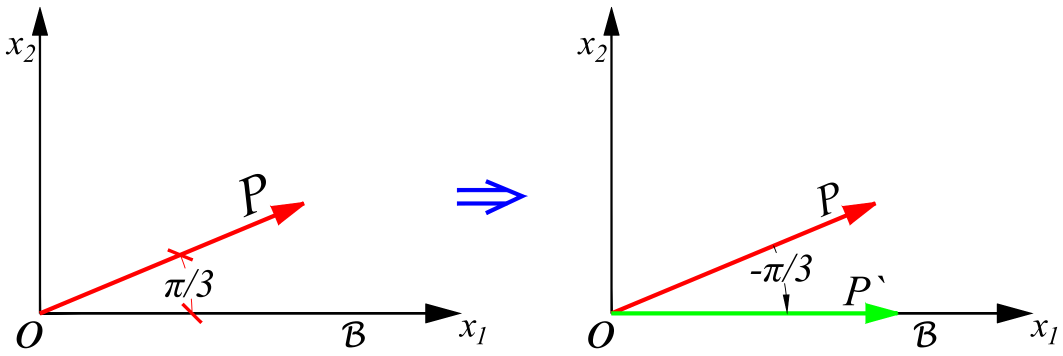
$$\sigma_{mn} = C'_{ijkl}\,Q_{im}{}^{T}Q_{jn}{}^{T}Q_{ko}Q_{lp}\varepsilon_{op} = Q_{mi}Q_{nj}Q_{ko}Q_{lp}C'_{ijkl}\varepsilon_{op} \tag{2.28}$$

$$C'_{ijkl} = Q_{im}Q_{jn}Q_{ok}Q_{pl}C_{mnop} \tag{2.29}$$

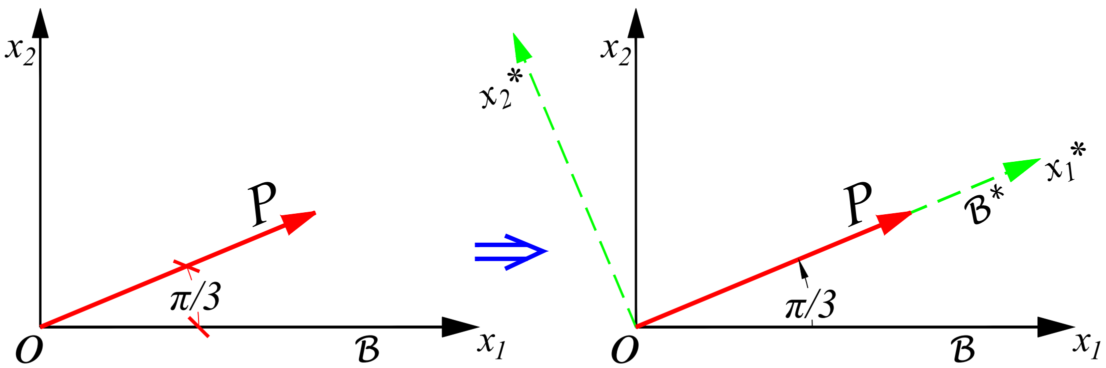
Also a two important role can be noticed. First, rotation matrix is transpose to transformation matrix for the same rotation angle, and second, rotation matrix for rotation angle θ is equivalent to transformation matrix for a rotation angle $-\theta$ as follows:

$$[\boldsymbol{Q}] = [\boldsymbol{R}]^{T} \tag{2.30}$$

$$[\boldsymbol{Q}(\theta)] = [\boldsymbol{R}(-\theta)] \tag{2.31}$$



(a)



(b)

Figure 2.7

■ **Example 2.1** A vector $\boldsymbol{P}$ in Figure 2.7a is originally oriented along direction $\left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$ in coordinate system $\mathscr{B}$. If the vector is subjected to a rotation by an angle $-\pi/3$, the new vector $\boldsymbol{P'}$ components in the same coordinate system $\mathscr{B}$ are $(1,0)$. While, in Figure 2.7b, another case involves rotating the coordinate system by angle $\pi/3$ to form new coordinate system $\mathscr{B}^*$, but

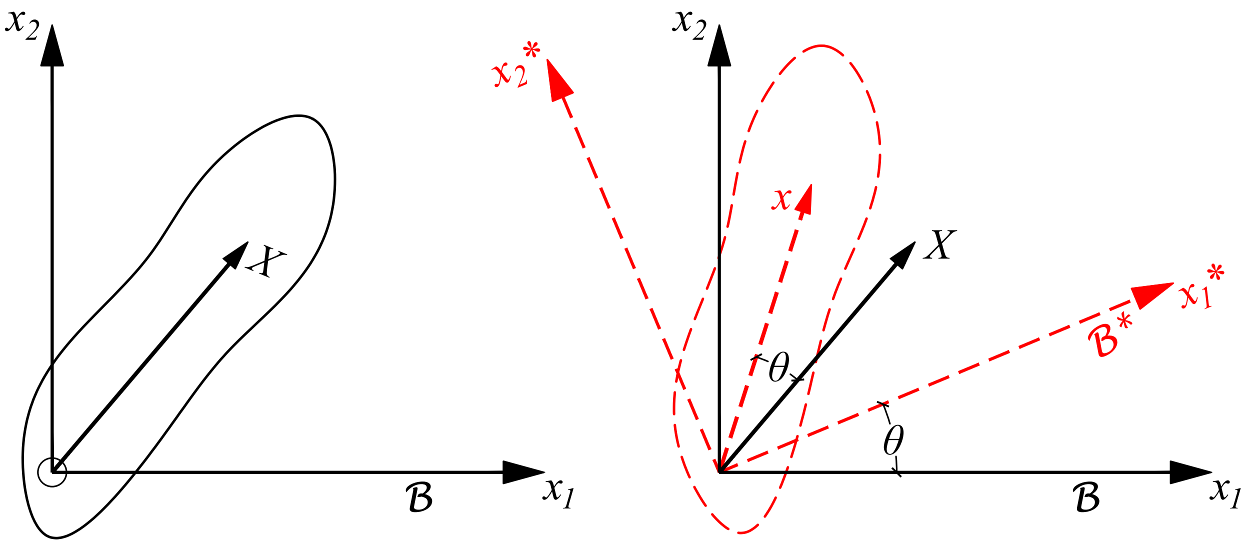


Figure 2.8

vector $\boldsymbol{P}$ stay still in its original position. The vector $\boldsymbol{P}$ resolved in the new coordinate system $\mathscr{B}^*$ will be $[\boldsymbol{P}]^{B^*} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ which results in the same components formed in the first case. Leading us to conclude that

$$[\boldsymbol{P'}]^B = \left[\boldsymbol{R}\left(-\frac{\pi}{3}\right)\right][\boldsymbol{P}]^B \tag{2.32}$$

$$[\boldsymbol{P}]^{B^*} = \left[\boldsymbol{Q}\left(\frac{\pi}{3}\right)\right][\boldsymbol{P}]^B \tag{2.33}$$

Both equations lead to same result which implies that $[\boldsymbol{Q}(\theta)] = [\boldsymbol{R}(-\theta)]$ ■

2.1.3 Rotation of the coordinate system and body together with same angle

In some cases, the coordinate system chosen may be attached to the body and rotates with it. This case is used when the body exhibits a large rotation while its internal deformations are infinitesimal. Observing these infinitesimal deformations required choosing a coordinate system attached to the body. This rotating or attached frame of reference is called co-rotated frame. As shown in Figure 2.8, a body with attached coordinate system to it is rotated counter clockwise by angle θ. By intuition, the new vector components resolved in the new coordinate system is identical to old vector components resolved in old coordinate system before rotation.

$$[\boldsymbol{x}]^{B^*} = [\boldsymbol{X}]^B \tag{2.34}$$

We also need to note this useful rule for rotation. Rotation preserves scalar quantities like vector length, projection of one vector on another, dot product of two vectors, and angle between two vectors. As shown in the Figure 2.9, angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ does not change after rotation by angle ϕ.

■ **Example 2.2** A scalar quantity like work W is defined as the dot product of the force $\boldsymbol{F}$ and displacement $\boldsymbol{d}$ as follows:

$$W = \boldsymbol{F}.\boldsymbol{d} = \boldsymbol{F}^T\boldsymbol{d} = (\boldsymbol{Q}^T\grave{\boldsymbol{F}})^T\boldsymbol{Q}^T\grave{\boldsymbol{d}} = \grave{\boldsymbol{F}}^T\boldsymbol{Q}\boldsymbol{Q}^T\grave{\boldsymbol{d}} = \grave{\boldsymbol{F}}^T\grave{\boldsymbol{d}} = \grave{\boldsymbol{F}}.\grave{\boldsymbol{d}} \tag{2.35}$$

So we conclude that the dot product of any two vector referred to two different coordinate systems are identical. ■

2.1.4 Compound rotation in two dimensions

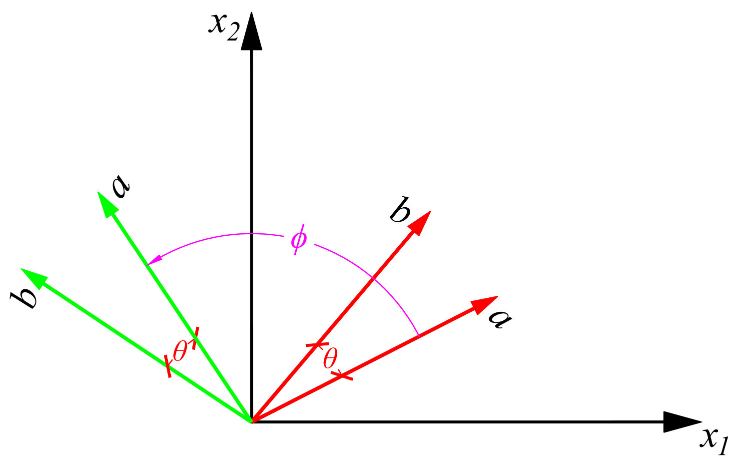


Figure 2.9

As shown in Figure 2.10a, if rotation $\boldsymbol{R}(\theta_1)$ is followed by rotation $\boldsymbol{R}(\theta_2)$, so the first rotation transforms vector $\boldsymbol{X}$ to vector $\boldsymbol{x}'$ and the second one rotates the vector $\boldsymbol{x}'$ to vector $\boldsymbol{x}$ as follows:

$$\boldsymbol{x}' = \boldsymbol{R}(\theta_1)\boldsymbol{X} \tag{2.36}$$
$$\boldsymbol{x} = \boldsymbol{R}(\theta_2)\boldsymbol{x}' \tag{2.37}$$

So the final vector $\boldsymbol{x}$ will be:

$$\boldsymbol{x} = \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1)\boldsymbol{X} = \boldsymbol{R}(\theta)\boldsymbol{X} \tag{2.38}$$

Where the equivalent rotation $\boldsymbol{R}(\theta)$ of two compound rotations $\boldsymbol{R}(\theta_1)$ and $\boldsymbol{R}(\theta_2)$ follows this relation:

$$\boldsymbol{R}(\theta) = \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1) \tag{2.39}$$

In two dimensional plane rotation, the equivalent rotation angle will be:

$$\theta = \theta_1 + \theta_2 \tag{2.40}$$

Also the sequence of rotation does not affect the final result as shown in Figure 2.10b, thus we can reach the same rotated vector if we started with angle of rotation θ_2 followed by rotation with angle θ_1.

$$\boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1) = \boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) \tag{2.41}$$

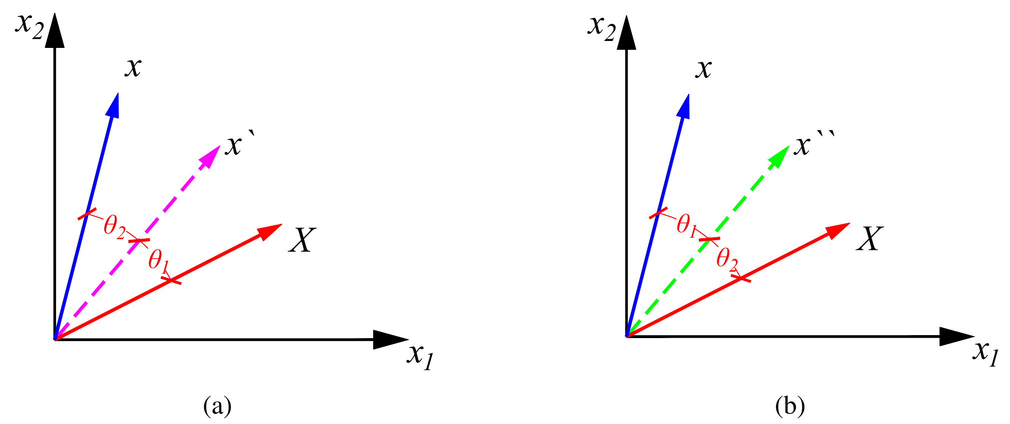


Figure 2.10

2.1.5 Rotation in three dimensions

Rotation in three dimensional space is defined by the angle and the axis of rotation. The rotation in two dimensional plane can be considered as a special case of rotation in which x_3 is the axis of rotation. Using Equation 2.2 and the fact that the position of any point laying on the axis of rotation (x_3) remain fixed after rotation, the point with initial coordinate $\boldsymbol{X}$ rotates to a new position $\boldsymbol{x}$ from this relation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.42}$$

Similarly rotation about x_1 axis follows this equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.43}$$

While rotation about x_2 axis comes from:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.44}$$

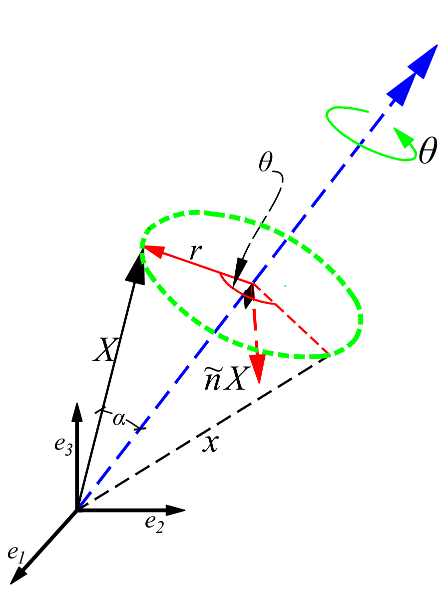
2.1.6 Rotation about any axis with unit vector $\boldsymbol{n}$

The rotation tensor can be paramterized using its intrinsic paramterization defined by $\boldsymbol{R}$ (orthogonal tensor with nine parameters and an element of Lie group called $SO(3)$, e.i. $\boldsymbol{R}^T R = \mathbf{1}, det(\boldsymbol{R}) = 1$) but group $SO(3)$ is non-linear space (manifold) and there will be some nonlinear issues when using it, so we can simplify the problenm using a vector-like parameterization so-called rotation vector. As in Figure 2.11a, assume a vector $\boldsymbol{X}$ rotated to a vector $\boldsymbol{x}$ (shown with dashed line) via a rotation of angle θ about axis with direction $\boldsymbol{n}$ through a circle normal to the axis of the rotation. The vector $\boldsymbol{X}$ makes angle α with axis of rotation. If we investigate the change in vector $\boldsymbol{X}$ via this circle as shown in Figure 2.11b, the vector after rotation increases in two directions $\boldsymbol{e}^{**}$ and $\boldsymbol{e}^{*}$ by length $|a|$ and $|a|$, respectively, as follows:

$$|\boldsymbol{r}| = |\boldsymbol{X}| \sin\alpha \tag{2.45}$$

$$|\boldsymbol{b}| = |\boldsymbol{r}| \sin\theta \tag{2.46}$$

$$|\boldsymbol{a}| = |\boldsymbol{r}| \, (1-\cos\theta) \tag{2.47}$$



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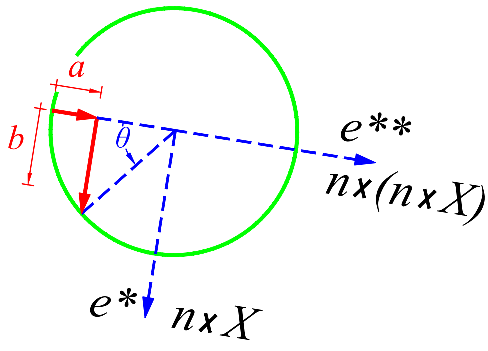


(b)

Figure 2.11

The direction of unit vectors $\boldsymbol{e}^*$ and $\boldsymbol{e}^{**}$

$$\boldsymbol{e}^* = \frac{\boldsymbol{n}\times\boldsymbol{X}}{|\boldsymbol{n}\times\boldsymbol{X}|} = \frac{\boldsymbol{n}\times\boldsymbol{X}}{|\boldsymbol{X}|\sin\alpha} = \frac{\boldsymbol{n}\times\boldsymbol{X}}{|r|} \tag{2.48}$$

$$\boldsymbol{e}^{**} = \frac{\boldsymbol{n}\times\boldsymbol{e}^*}{|\boldsymbol{n}\times\boldsymbol{e}^*|} = \frac{\boldsymbol{n}\times\boldsymbol{e}^*}{\sin(\pi/2)} = \boldsymbol{n}\times\boldsymbol{e}^* = \frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{X})}{|\boldsymbol{r}|} \tag{2.49}$$

The final vector $\boldsymbol{x}$ will be:

$$\boldsymbol{x} = \boldsymbol{X} + |\boldsymbol{b}|\,\boldsymbol{e}^* + |\boldsymbol{a}|\,\boldsymbol{e}^{**} \tag{2.50}$$

$$= \boldsymbol{X} + \sin\theta\,(\boldsymbol{n}\times\boldsymbol{X}) + (1-\cos\theta)\,(\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{X})) \tag{2.51}$$

$$= \boldsymbol{x} + \sin\theta\,\tilde{\boldsymbol{n}}\boldsymbol{X} + (1-\cos\theta)\,\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}\boldsymbol{X} \tag{2.52}$$

$$= (\mathbf{1} + \sin\theta\,\tilde{\boldsymbol{n}} + (1-\cos\theta)\,\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}\ )\boldsymbol{X} = \boldsymbol{R}\boldsymbol{X} \tag{2.53}$$

So the rotation tensor $\boldsymbol{R}$ is defined as:

$$\boldsymbol{R} = \mathbf{1} + \sin\theta\,\tilde{\boldsymbol{n}} + (1-\cos\theta)\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} \tag{2.54}$$

We need to note that the last term of the above equation $\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}$ is symmetric, while the middle term $\tilde{\boldsymbol{n}}$ is skew-symmetric. The last term is symmetric because

$$Skew\,(\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}}) = \frac{\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} - (\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}})^T}{2} = \mathbf{0} \tag{2.55}$$

$$(\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}})^T = \widetilde{\boldsymbol{n}}^T\widetilde{\boldsymbol{n}}^T = (-\widetilde{\boldsymbol{n}})\,(-\widetilde{\boldsymbol{n}}) = \widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} \tag{2.56}$$

This above Equation 2.54 is called Rodrigues' rotation formula. Another form we would like to introduce is exponential form of the rotation tensor as follows:

Using Taylor series

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \ldots, \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \ldots \tag{2.57}$$

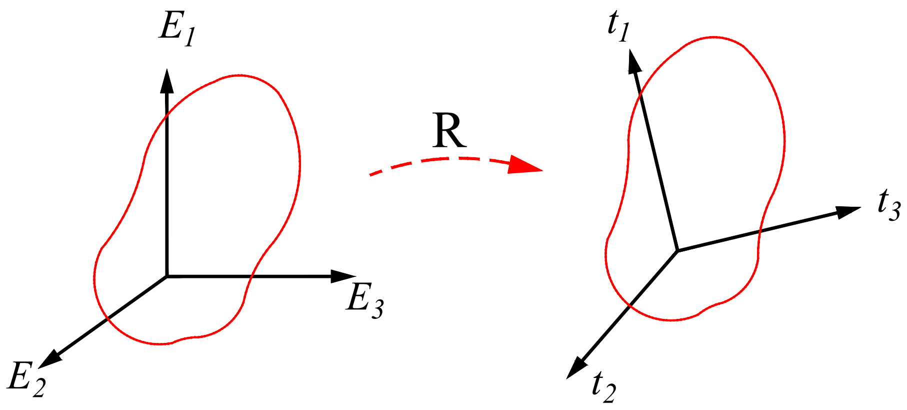


Figure 2.12

Also the skew-symmetric matrix with unit vector $\boldsymbol{n}$ as an axial vector follows this relation

$$\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} = -\widetilde{\boldsymbol{n}} \tag{2.58}$$

We can conclude that:

$$\boldsymbol{R} = 1 + \widetilde{\boldsymbol{\theta}} + \frac{\widetilde{\boldsymbol{\theta}}^2}{2!} + \frac{\widetilde{\boldsymbol{\theta}}^3}{3!} + \cdots = exp(\widetilde{\boldsymbol{\theta}}) \tag{2.59}$$

Where $\boldsymbol{\theta} = \theta\boldsymbol{n}$ is the rotational vector and θ is the magnitude of rotation, so the rotation $\boldsymbol{R}$ depends on three free independent parameters. There are other choices for parameterization like Euler angles, rotational pseudovector, quaternion, conformal rotation vector, Euler parameters, etc. Assume a rigid body rotation and we have two orthonormal frame; material (inertia) frame $(\mathscr{E} = \{\boldsymbol{E}_I\})$ and body-attached (moving) frame $(\mathscr{T} = \{\boldsymbol{t}_I\})$ as shown in Figure 2.12 such that a rotation operator $\boldsymbol{R}$ maps the material frame into the moving frame as follows:

$$\boldsymbol{t}_I = \boldsymbol{R}\boldsymbol{E}_I \tag{2.60}$$

We need to note that the material frame remains constant in the space at any time while the moving frame is attached to the body and change with time $\boldsymbol{t}_I(t)$, such that the moving frame is identical to the material frame at initial configuration $(t = 0)$. The rigid body rotation $\boldsymbol{R}$ can be interpreted as a rotation about axis $\boldsymbol{n}$ with angle θ. Resolving the above equation in material frame results in:

$$[\boldsymbol{t}_I]^{\mathscr{E}} = [\boldsymbol{R}]^{\mathscr{E}}[\boldsymbol{E}_I]^{\mathscr{E}} \tag{2.61}$$

With

$$[\boldsymbol{E}_1]^{\mathscr{E}} = \{1,0,0\}^T, \ [\boldsymbol{E}_2]^{\mathscr{E}} = \{0,1,0\}^T, \ [\boldsymbol{E}_3]^{\mathscr{E}} = \{0,0,1\}^T \tag{2.62}$$

So vector $[\boldsymbol{t}_i]^{\mathscr{E}}$ represent the i^{th} column of matrix $[\boldsymbol{R}]^{\mathscr{E}}$.

2.1.7 Recovering the axis and angle of rotation from rotation tensor

As stated before, the skew symmetric part of rotation tensor $\boldsymbol{R}$ is defined as:

$$skew(\boldsymbol{R}) = \frac{1}{2}(\boldsymbol{R} - \boldsymbol{R}^T) = \sin\theta\,\widetilde{\boldsymbol{n}} \tag{2.63}$$

The magnitude of the skew-symmetric part will be:

$$\sin\theta = |\, axial\,(skew\,(\boldsymbol{R}))\,| \tag{2.64}$$

Axial vector $\sin\theta\boldsymbol{n}$ of the skew-symmetric part define the direction of the rotation vector:

$$\boldsymbol{n} = \frac{axial\,(skew\,(\boldsymbol{R}))}{\sin\theta} \tag{2.65}$$

The range of angle θ is $]0-\pi[$. Note that the axis of rotation could be $-\boldsymbol{n}$ with corresponding angle $]\pi-2\pi[$. For example, rotating about axis $\boldsymbol{n}=(0,0,1)$ with angle $pi/3$ is equivalent to rotating about axis $-\boldsymbol{n}=(0,0,-1)$ with angle equal to $2\pi-\pi/3=5\pi/3$.

For relatively *small rotations*, we can neglect terms with order higher than second.

$$\boldsymbol{R} = \boldsymbol{1} + \widetilde{\boldsymbol{\theta}} + \frac{\widetilde{\boldsymbol{\theta}}^2}{2!} \tag{2.66}$$

For *infinitesimal small rotations*, neglecting higher order terms than first results in:

$$\boldsymbol{R} = \boldsymbol{1} + \widetilde{\boldsymbol{\theta}} \tag{2.67}$$

The infinitesimal rotation can be proven from Equation 4.575 as follows. Assume a vector $\boldsymbol{v}$ in a plane $\boldsymbol{x}_1 - \boldsymbol{x}_2$ directed with angle θ from x_1 axis. If the vector is subjected to an infinitesimal rotation $\triangle\theta$, it is transformed to vector $\boldsymbol{v}'$, such that if:

$$\boldsymbol{v} = |\boldsymbol{v}|\,(\cos\theta, \sin\theta) \tag{2.68}$$

The resulting vector will be:

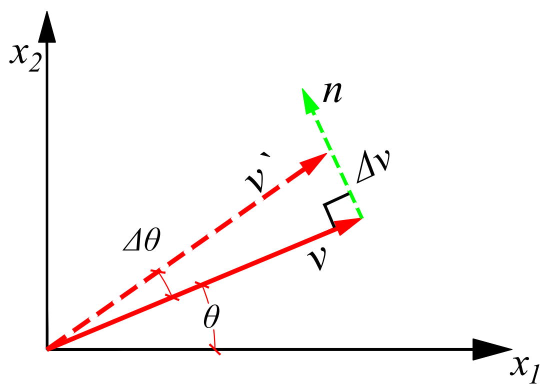


Figure 2.13

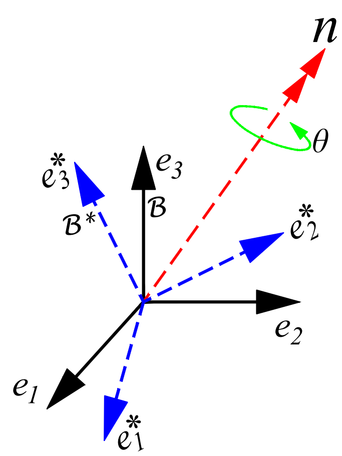


Figure 2.14

$$\boldsymbol{v}' = \boldsymbol{v} + \triangle\boldsymbol{v} = |\boldsymbol{v}|\,(\cos(\theta+\triangle\theta)\,, \sin(\theta+\triangle\theta)\,) \tag{2.69}$$

$$\boldsymbol{v}' = \boldsymbol{v} + \triangle\theta\,|\boldsymbol{v}|\;\boldsymbol{n} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\triangle\theta \\ \triangle\theta & 0 \end{bmatrix}\right)\boldsymbol{v} \tag{2.70}$$

Where the direction $\boldsymbol{n}=(-\sin\theta,\cos\theta)$ as $\boldsymbol{n}$ is orthogonal to vector $\boldsymbol{v}$ and axis of rotation.
In the same manner, we can conclude the general form for infinitesimal rotation about any arbitrary axis with rotational vector $\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3)$ as follows: First, we can deduce the vector $\boldsymbol{n}$ as follows:

$$\boldsymbol{n} = \frac{\boldsymbol{\theta}\times\boldsymbol{v}}{|\boldsymbol{\theta}\times\boldsymbol{v}|} = \frac{\boldsymbol{\theta}\times\boldsymbol{v}}{|\boldsymbol{v}|\sin\theta} \simeq \frac{\boldsymbol{\theta}\times\boldsymbol{v}}{|\boldsymbol{v}|\theta} \tag{2.71}$$

As for an infinitesimal angle θ, $\sin\theta \simeq \theta$, and using Equation 2.70

$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{\theta} \times \boldsymbol{v} = \left(\mathbf{1} + \widetilde{\boldsymbol{\theta}}\right) \boldsymbol{v} coc \tag{2.72}$$

The general form of rotation tensor for an infinitesimal rotation $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$:

$$\boldsymbol{R} = \mathbf{1} + \widetilde{\boldsymbol{\theta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \tag{2.73}$$

Generally any infinitesimal rotation $\Delta\boldsymbol{\theta}$ is also called *spin*.

Note 2.1 There are some useful properties we would like to introduce:

1. All the properties of rotation in three dimensional case is identical to those of the two dimensional case rotation except for dealing with compound rotations (see the next section).
2. Axis of rotation is not affected by rotation and remains fixed.
3. For the plane normal to the axis of rotation, any vector lying on this plane remains in the same plane after rotation.
4. Dot product of two vectors is preserved under rotation.
5. If we have two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ subjected to the same rotation $\boldsymbol{R}$ and the resulting vectors are $\boldsymbol{x}'_1$ and $\boldsymbol{x}'_2$.

$$\boldsymbol{x}'_1 = \boldsymbol{R}(\boldsymbol{\theta})\boldsymbol{x}_1, \quad \boldsymbol{x}'_2 = \boldsymbol{R}(\boldsymbol{\theta})\boldsymbol{x}_2 \tag{2.74}$$

6. The cross product of these two vectors before rotation $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and after rotation $(\boldsymbol{x}'_1, \boldsymbol{x}'_2)$ are related as follows:

$$\boldsymbol{x}'_1 \times \boldsymbol{x}'_2 = (\boldsymbol{R}(\boldsymbol{\theta})\boldsymbol{x}_1) \times (\boldsymbol{R}(\boldsymbol{\theta})\boldsymbol{x}_1) = \boldsymbol{R}(\boldsymbol{\theta})(\boldsymbol{x}_1 \times \boldsymbol{x}_2)^{a} \tag{2.75}$$

7. If a coordinate system with basis $\mathscr{B}1$ is subjected to a rotation $\boldsymbol{R}$ to form basis $\mathscr{B}2$, the rotation tensor resolved in both bases are identical. As, rotation formula in Equation 2.54 depends on the angle rotated and the axis of rotation as follows:

$$\boldsymbol{R}^{\mathscr{B}} = \boldsymbol{R}^{\mathscr{B}}(\theta, [\boldsymbol{n}]^{\mathscr{B}}) \tag{2.76}$$

As the axis of rotation $\boldsymbol{n}$ remains fixed after rotation as shown in Figure 2.14, so its components on both bases are identical $[\boldsymbol{n}]^{\mathscr{B}1} = [\boldsymbol{n}]^{\mathscr{B}2}$ and using Equation 2.54 results in:

$$[\boldsymbol{R}]^{\mathscr{B}1} = [\boldsymbol{R}]^{\mathscr{B}2} \tag{2.77}$$

■

[a] This expression can be proven by intition or from this relation $(Fa) \times (Fb) = det(F)F^{-T}(a \times b)$, where F is a linear mapping to vectors a and b

2.1.8 Non-commutative property of rotation

In Figure 2.15, a rectangular plate is subjected to rotation about $\boldsymbol{e}_1$ axis with angle $\pi/2$ then followed by a rotation about $\boldsymbol{e}_2$ axis with angle $\pi/2$ to finally reach to some configuration. While if we flipped the order of rotation starting with rotation about axis $\boldsymbol{e}_2$ followed by rotation about axis $\boldsymbol{e}_1$ using the same angles, we reach to another configuration, so we conclude that the sequence of rotations affects the final result of the compound rotation unlike the case of two dimensional rotation in subsection 2.1.4.

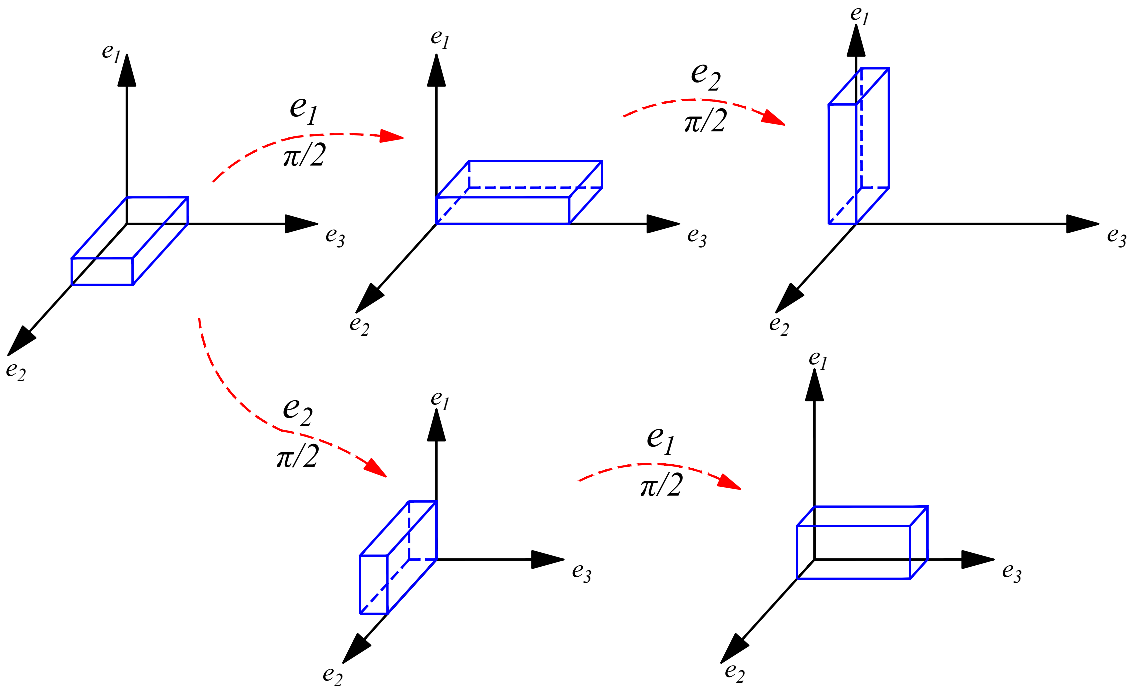


Figure 2.15

If rotation matrices $\boldsymbol{R}(\boldsymbol{\theta}_1)$, and $\boldsymbol{R}(\boldsymbol{\theta}_2)$, respectively, rotate a vector $\boldsymbol{v}_0$ to vector $\boldsymbol{v}_1$ and vector v_1 to vector $\boldsymbol{v}_2$ as follows:

$$\boldsymbol{v}_1 = \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{v}_0 \tag{2.78}$$

$$\boldsymbol{v}_2 = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{v}_1 = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{v}_0 \tag{2.79}$$

So the compound rotation tensor is:

$$\boldsymbol{R}(\boldsymbol{\theta}) = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1) \tag{2.80}$$

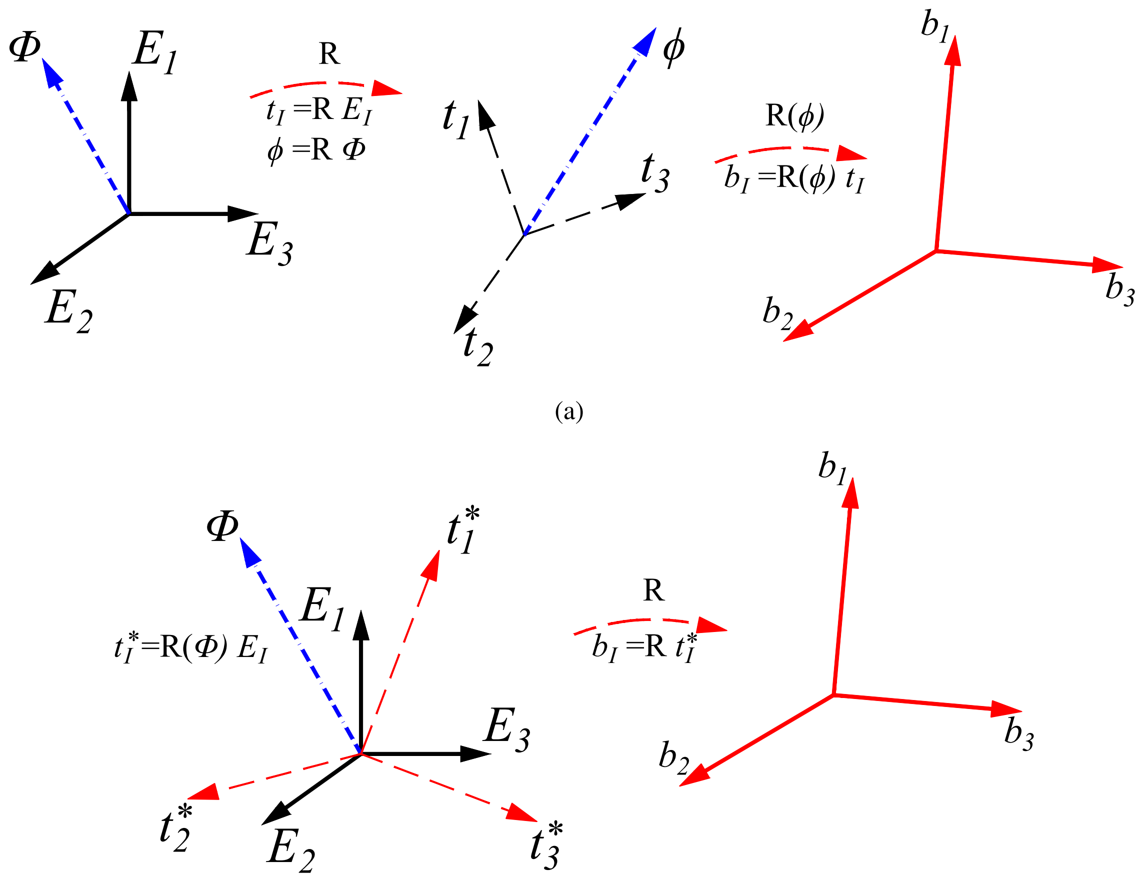
The resulting rotation $\boldsymbol{\theta}$ does not represent the algebraic vector sum of the two angles or ($\boldsymbol{\theta} \neq \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$) as confirmed from Figure 2.14. The above expression can be illustrated via Figure 2.21, in which basis $\mathscr{B}$ is transformed through rotation tensor $\boldsymbol{R}(\boldsymbol{\theta})$ to basis $\mathscr{B}^*$ then rotated through $(\triangle\boldsymbol{\phi})$ to reach finally to basis $\mathscr{B}^+$. This two subsequent rotations can be replaced with one equivalent rotation $\boldsymbol{\theta} + \triangle\boldsymbol{\theta}$ where $(\triangle\boldsymbol{\theta} \neq \triangle\boldsymbol{\phi})$. Also the not commutative property of $[\boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1) \neq \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{R}(\boldsymbol{\theta}_2)]$ argues the above discussion.

2.1.9 Compound Rotation

Consider a rotation operator $\boldsymbol{R}$ mapping from orthonormal frame $\boldsymbol{E}_I$ into another frame $\boldsymbol{t}_I$, then an incremental rotation is added which carries the rotation frame $\boldsymbol{t}_I$ to $\boldsymbol{b}_I$. There are two ways to apply thus rotation defined as follows:

- Through spatial rotation:
  In this case, the incremental rotation $\boldsymbol{\phi}$ is applied to moving frame $\boldsymbol{t}_I$ as shown in Figure 2.16a and the compound rotation is defined as:

$$\boldsymbol{b}_I = \boldsymbol{R}(\boldsymbol{\phi})\boldsymbol{R}\boldsymbol{E}_I \tag{2.81}$$



(a)

(b)

Figure 2.16

- Through material rotation:
  The incremental rotation $\boldsymbol{\Phi}$ is applied to the material frame $\boldsymbol{E}_I$ shown in Figure 2.16b and the resulting rotation is:

$$\boldsymbol{b}_I = \boldsymbol{R}\boldsymbol{R}(\boldsymbol{\Phi})\boldsymbol{E}_I \tag{2.82}$$

The updated compound rotation tensor is defined in the following two forms:

$$\boldsymbol{R}_{eq} = \boldsymbol{R}(\boldsymbol{\phi})\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}(\boldsymbol{\Phi}) \tag{2.83}$$

Where $\boldsymbol{\phi}$ ($\boldsymbol{\Phi}$) is the rotational vector corresponding to the incremental spatial (material) rotation. From above equation, they are related through the following:

$$\boldsymbol{R}(\boldsymbol{\Phi}) = \boldsymbol{R}^T\boldsymbol{R}(\boldsymbol{\phi})\boldsymbol{R} \rightarrow \boldsymbol{\Phi} = \boldsymbol{R}^T\boldsymbol{\phi} \leftrightarrow \boldsymbol{\phi} = \boldsymbol{R}\boldsymbol{\Phi} \tag{2.84}$$

The above equation can be interpreted through considering the rotation vector as a real vector attached to a rigid body like the moving frame $\boldsymbol{t}_I$ and subjected to rotation $\boldsymbol{R}$. As the angle between any two vectors subjected to the same rotation is preserved, we can imagine that rotating of frame $\boldsymbol{E}_i$ through rotation $\boldsymbol{R}$ followed by rotation $\boldsymbol{\phi}$ is equivalent to a rotation of the same frame with rotational vector $\boldsymbol{\Phi}$ followed by rotation $\boldsymbol{R}$.

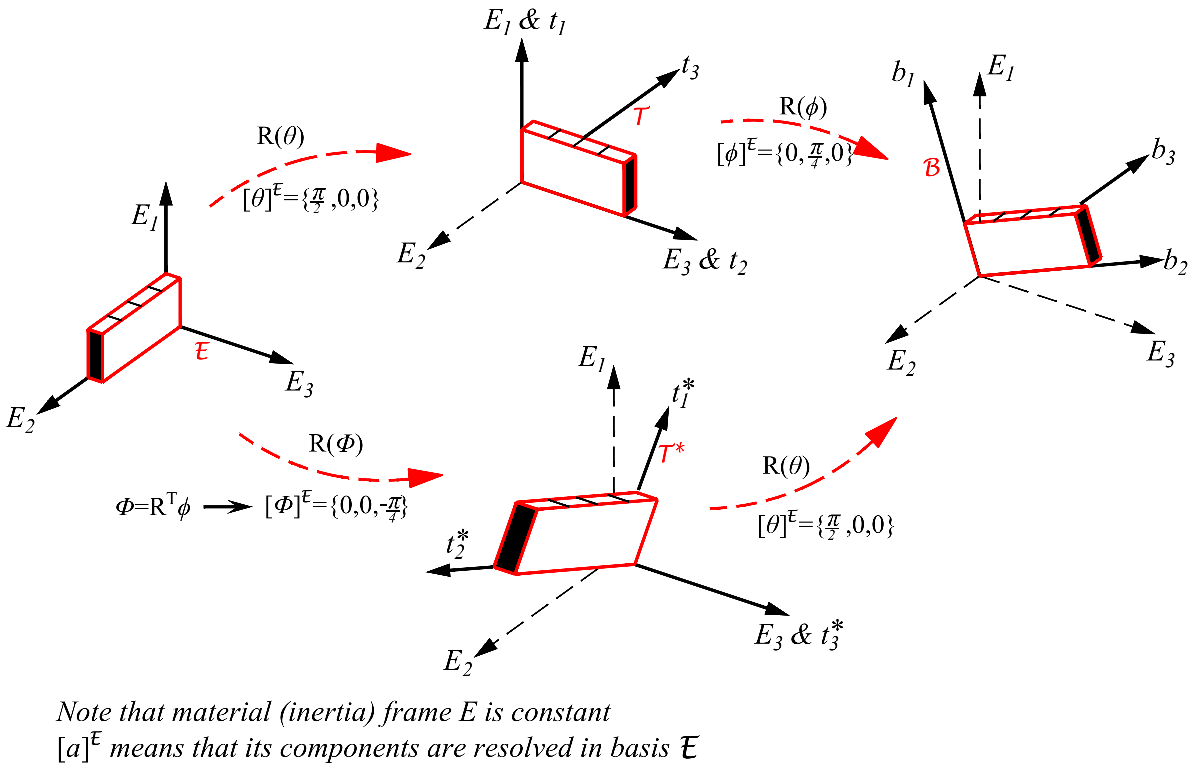


Figure 2.17

■ **Example 2.3** Assume a rigid body shown in Figure 2.17 subjected rotational vector resolved in basis E as $[\boldsymbol{\theta}]^E = \left\{\frac{\pi}{2},0,0\right\}^T$, then followed by an incremental rotation resolved in the same basis as $[\boldsymbol{\phi}]^E = \left\{0,\frac{\pi}{4},0\right\}^T$, resulting a body with attached frame $\mathscr{B}$ with bases resolved in frame of reference E as follows:

$$[\boldsymbol{b}_I]^{\mathscr{B}} = [\boldsymbol{R}(\boldsymbol{\phi})]^{\mathscr{B}} [\boldsymbol{R}]^{\mathscr{B}} [\boldsymbol{E}_I]^{\mathscr{B}} = \boldsymbol{R}([\boldsymbol{\phi}]^{\mathscr{B}}) [\boldsymbol{R}]^{\mathscr{B}} [\boldsymbol{E}_I]^{\mathscr{B}} \tag{2.85}$$

Also identical results can be reached from rotating about rotational vector $[\boldsymbol{\Phi}]^E = [\boldsymbol{R}^T\boldsymbol{\phi}]^E = \left\{0,0,-\frac{\pi}{4}\right\}^T$ followed by rotation about $\boldsymbol{\theta}$. Also, see the examples described in subsection 2.1.12.
■

2.1.10 Finite rotation followed by an infinitesimal rotation

This case is very common in nonlinear finite element analysis, as the solution is divided into small steps, each step includes number of increments with relatively small rotation. Updating rotations requires adding incremental rotations to the last converged step which is generally finite, such that if a vector subjected to a finite rotation $\boldsymbol{\theta}$ followed by an infinitesimal or linearized incremental spatial rotation $\triangle\boldsymbol{\phi}$ as shown in Figure 2.18, the compound rotation will be:

$$\boldsymbol{R}(\boldsymbol{\theta}+\triangle\boldsymbol{\theta}) = \boldsymbol{R}(\triangle\boldsymbol{\phi})\boldsymbol{R}(\boldsymbol{\theta}) \tag{2.86}$$

As the final rotation $\neq \boldsymbol{\theta}+\triangle\boldsymbol{\phi}$ but equal to $\boldsymbol{\theta}+\triangle\boldsymbol{\theta}$

$\triangle\boldsymbol{\phi}$, $\triangle\boldsymbol{\theta}$ are called non-additive and additive rotation vectors, respectively. From above expression:

$$\Delta\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\theta}+\triangle\boldsymbol{\theta}) - \boldsymbol{R}(\boldsymbol{\theta}) = \boldsymbol{R}(\triangle\boldsymbol{\phi})\boldsymbol{R}(\boldsymbol{\theta}) - \boldsymbol{R}(\boldsymbol{\theta}) = (\boldsymbol{R}(\triangle\boldsymbol{\phi})-\boldsymbol{1})\boldsymbol{R}(\boldsymbol{\theta}) \tag{2.87}$$

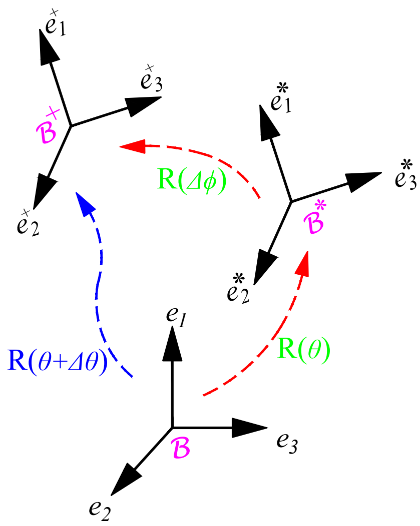


Figure 2.18

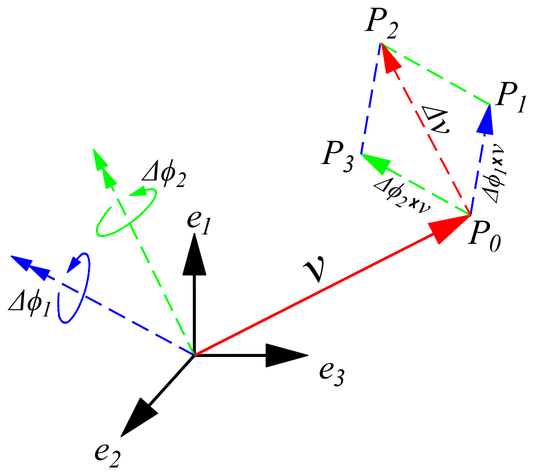


Figure 2.19

While for an infinitesimal rotation $\triangle\boldsymbol{\phi}$, rotation tensor will be $\boldsymbol{R}(\triangle\boldsymbol{\phi}) \approx \mathbf{1} + \widetilde{\triangle\boldsymbol{\phi}}$ and the above expression yields:

$$\Delta\boldsymbol{R} = \widetilde{\triangle\boldsymbol{\phi}}\boldsymbol{R}(\boldsymbol{\theta}) \tag{2.88}$$

which leads to:

$$\widetilde{\triangle\boldsymbol{\phi}} = \Delta\boldsymbol{R}(\boldsymbol{\theta}, \triangle\boldsymbol{\theta})\boldsymbol{R}(\boldsymbol{\theta})^T \tag{2.89}$$

For an infinitesimal rotation, $\triangle\boldsymbol{\phi}$ is also called spatial spin or angular variation.The above relation is equivalent to the following equation:

$$\triangle\boldsymbol{\phi} = \boldsymbol{T}(\boldsymbol{\theta})\triangle\boldsymbol{\theta} \leftrightarrow \triangle\boldsymbol{\theta} = \boldsymbol{T}(\boldsymbol{\theta})^{-1}\triangle\boldsymbol{\phi} \tag{2.90}$$

Where $\boldsymbol{T}(\boldsymbol{\theta})$ and $\boldsymbol{T}(\boldsymbol{\theta})^{-1}$ are defined as:

$$\boldsymbol{T}(\boldsymbol{\theta}) = \mathbf{1} + \frac{1-\cos\theta}{\theta^2}\widetilde{\boldsymbol{\theta}} + \frac{\theta - \sin\theta}{\theta^3}\widetilde{\boldsymbol{\theta}}\widetilde{\boldsymbol{\theta}} \tag{2.91}$$

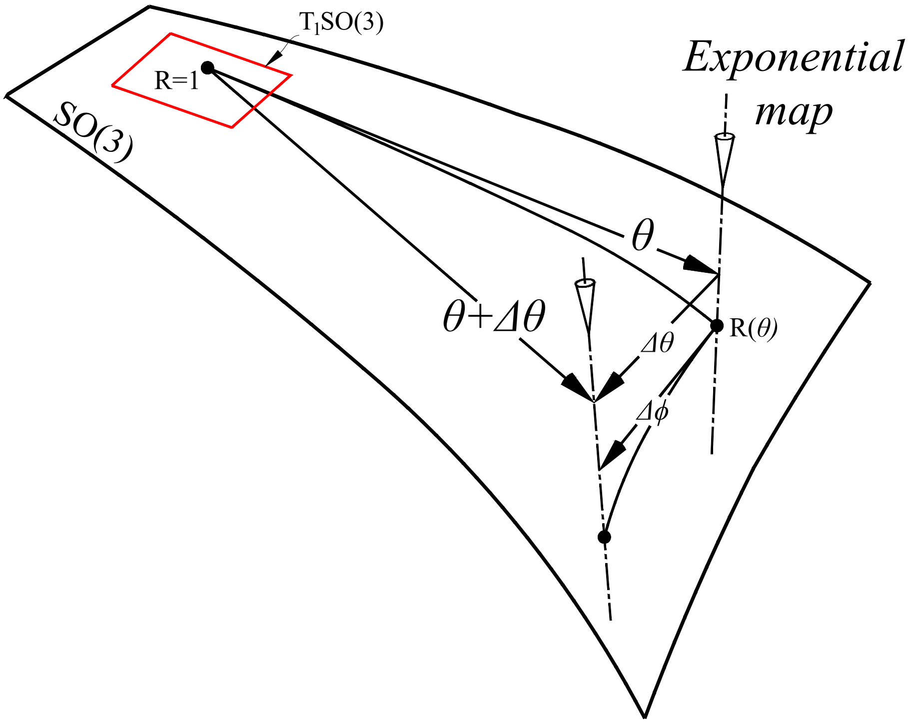
$$\boldsymbol{T}(\boldsymbol{\theta})^{-1} = \frac{\theta/2}{\tan(\theta/2)}\mathbf{1} + \left(1 - \frac{\theta/2}{\tan(\theta/2)}\right)\frac{\boldsymbol{\theta}\boldsymbol{\theta}^T}{\theta^2} + \frac{1}{2}\widetilde{\boldsymbol{\theta}} \tag{2.92}$$

The derivation of the above expressions is presented in Appendix 4.5.5. As stated before, rotation tensor $\boldsymbol{R}$ is an element of Lie group $SO(3)$. Rotation variation $\delta\boldsymbol{R}$ lies on the tangent space to $SO(3)$ at the current rotation $\boldsymbol{R}$ defined by $\boldsymbol{T_R}SO(3)$. Unlike non-linear manifold $SO(3)$, $\boldsymbol{T_R}SO(3)$ is a vector space as shown in Figure 2.20a and Figure 2.20b. At point with ($\boldsymbol{R} = \mathbf{1}$), the tangent space is defined as $\boldsymbol{T_1}SO(3)$, such that rotation vectors $\widetilde{\boldsymbol{\theta}}$, and $\delta\widetilde{\boldsymbol{\theta}}$ belong to the same vector space $\boldsymbol{T_1}SO(3)$ and the can added together as follows:

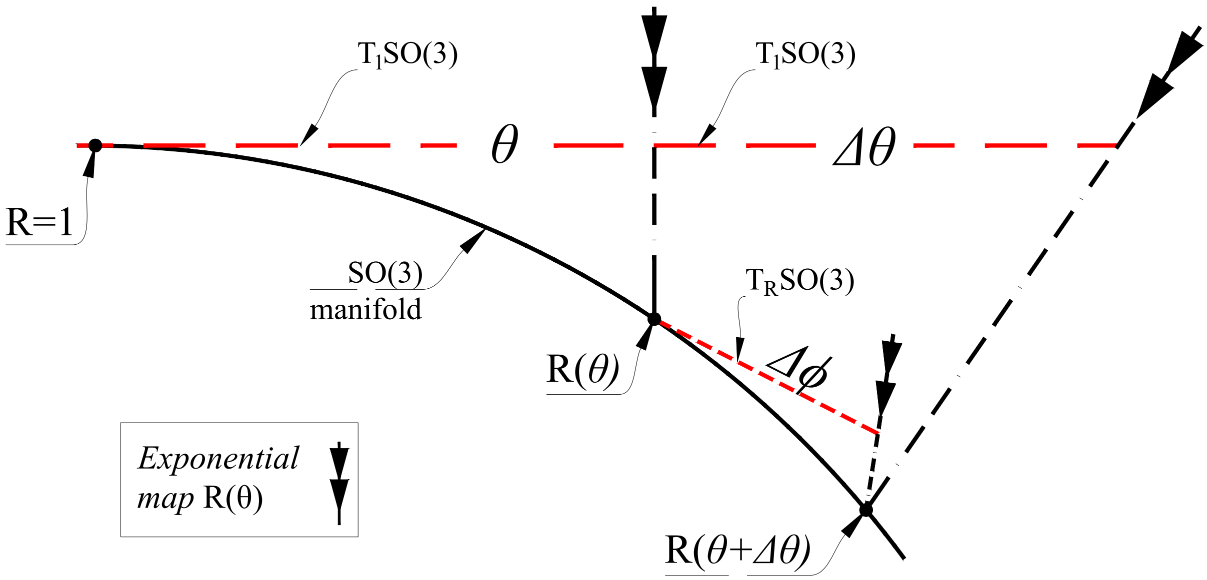
$$\widetilde{\boldsymbol{\theta}}, \widetilde{\delta\boldsymbol{\theta}} \in \boldsymbol{T_1}SO(3), \quad \widetilde{\boldsymbol{\theta}} + \widetilde{\delta\boldsymbol{\theta}} \in \boldsymbol{T_1}SO(3) \tag{2.93}$$

While variational rotation $\delta\boldsymbol{R}$ evaluated at rotation tensor $\boldsymbol{R}$ belongs to another tangent space $\boldsymbol{T_R}SO(3)$, such that it is defined as:

$$\delta\boldsymbol{R} = \widetilde{\delta\boldsymbol{\phi}}\boldsymbol{R} \in \boldsymbol{T_R}SO(3) \tag{2.94}$$



(a)



(b) cross section in $SO(3)$ manifold

Figure 2.20

As $\delta\boldsymbol{\phi}$ belong to a different vector space it can not be added to $\boldsymbol{\theta}$ and $\delta\boldsymbol{\theta}$, which leads us to use the mapping tensor $\boldsymbol{T}(\boldsymbol{\theta})$ in Equation 2.91 to relate the linearized rotation at $\boldsymbol{T_1}SO(3)$ defined as $\delta\boldsymbol{\theta}$ with the linearized rotation at $\boldsymbol{T_R}SO(3)$ defined as $\delta\boldsymbol{\phi}$ as stated in Equation 2.90.

When $|\boldsymbol{\theta}|$ approaches zero, $T(\boldsymbol{\theta})$ approaches identity matrix $\mathbf{1}$ and $\triangle\boldsymbol{\phi} = \triangle\boldsymbol{\theta}$, while for an infinitesimal rotation $\boldsymbol{\theta}_1[T(\boldsymbol{\theta}) \to 1]$, $\boldsymbol{T}$ can be approximated as follows:

$$\boldsymbol{T}(\boldsymbol{\theta}) \simeq \mathbf{1} + \frac{1}{2}\tilde{\boldsymbol{\theta}} \tag{2.95}$$

2.1.11 Adding two infinitesimal rotations or spin

As shown in Figure 2.19, imagine a rigid line $\boldsymbol{v}$ rotated about axis $\boldsymbol{n}_1$ with infinitesimal rotation $\triangle\boldsymbol{\phi}_1$ around point $\boldsymbol{O}$ moving the point $\boldsymbol{P}_0$ to point $\boldsymbol{P}_1$ by changing vector v as follows:

$$\triangle\boldsymbol{v}_1 = \triangle\boldsymbol{\phi}_1 \times \boldsymbol{v} \tag{2.96}$$

Then
around point $\boldsymbol{O}$ moving the point $\boldsymbol{P}_1$ to point $\boldsymbol{P}_2$ by changing vector $\boldsymbol{v}$ as follows:

$$\triangle\boldsymbol{v}_2 = \triangle\boldsymbol{\phi}_2 \times (\boldsymbol{v} + \triangle\boldsymbol{v}_1) = \triangle\boldsymbol{\phi}_2 \times \boldsymbol{v} + \triangle\boldsymbol{\phi}_2 \times (\triangle\boldsymbol{\phi}_1 \times \boldsymbol{v}) \simeq \triangle\boldsymbol{\phi}_2 \times \boldsymbol{v} \tag{2.97}$$

The last expression results form neglecting second order terms, so the resulting rotation $\triangle\boldsymbol{\phi}$ comes from:

$$\triangle\boldsymbol{v} = \triangle\boldsymbol{\phi} \times \boldsymbol{v} \tag{2.98}$$

$$\triangle\boldsymbol{v}_1 + \triangle\boldsymbol{v}_2 = \triangle\boldsymbol{\phi}_1 \times \boldsymbol{v} + \triangle\boldsymbol{\phi}_2 \times \boldsymbol{v} \tag{2.99}$$

$$= (\triangle\boldsymbol{\phi}_1 + \triangle\boldsymbol{\phi}_2) \times \boldsymbol{v} \tag{2.100}$$

Then the resulting infinitesimal compound rotation will be:

$$\triangle\boldsymbol{\phi} = \triangle\boldsymbol{\phi}_1 + \triangle\boldsymbol{\phi}_2 \tag{2.101}$$

This is called addition theorem. In this chapter, we generally use $\boldsymbol{\theta}$, and $\triangle\boldsymbol{\theta}$ for addition rotational vector, $\triangle\boldsymbol{\phi}$ for non-additive one following rotation $\boldsymbol{\theta}$.

■ **Example 2.4** For $[\boldsymbol{\theta}_1] = (\pi/3\ , 0, 0)$, $[\boldsymbol{\theta}_2] = (0, -\pi/3\ , 0)$, $[\triangle\boldsymbol{\phi}_1 =]\pi/100\,(1,1,0)$,and $[\triangle\boldsymbol{\phi}_2] = \pi/200\,(0,2,1)$, from formula in Equation 2.54.

$$[\boldsymbol{R}(\boldsymbol{\theta_1})] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\left(\frac{\pi}{3}\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$+ \left(1 - \cos\left(\frac{\pi}{3}\right)\right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.102}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{bmatrix}$$

Similarly, the second rotation tensor will be:

$$[\boldsymbol{R}(\boldsymbol{\theta_2})] = \begin{bmatrix} 0.5 & 0 & \frac{-\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{3}}{2} & 0 & 0.5 \end{bmatrix} \tag{2.103}$$

The resulting compound rotation will be:

$$\boldsymbol{R}(\boldsymbol{\theta}) = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1) = \begin{bmatrix} 0.5 & 0 & \frac{-\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{3}}{2} & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{bmatrix} \tag{2.104}$$

$$= \begin{bmatrix} 0.5 & -0.75 & \frac{-\sqrt{3}}{4} \\ 0 & 0.5 & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & 0.25 \end{bmatrix} \tag{2.105}$$

Using the procedures in subsection 2.1.7 to evaluate $\boldsymbol{\theta}$, we get

$$[\boldsymbol{\theta}]^T = \begin{bmatrix} 0.9463 & -0.9463 & 0.5463 \end{bmatrix} \tag{2.106}$$

We find that $\boldsymbol{\theta} \neq \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$. If we change the rotation sequence, the resulting compound rotation will be:

$$\boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{R}(\boldsymbol{\theta}_2) = \begin{bmatrix} 0.5 & 0 & \frac{-\sqrt{3}}{2} \\ -0.75 & 0.5 & \frac{-\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} & 0.25 \end{bmatrix} \tag{2.107}$$

We can conclude that:

$$\boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1) \; \neq \; \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{R}(\boldsymbol{\theta}_2) \tag{2.108}$$

and the sequence of rotation effect the final compound rotation.
The compound rotation formed by rotation $\boldsymbol{\theta}_1$ followed by infinitesimal rotation $\triangle\boldsymbol{\phi}_1$ will be:

$$\boldsymbol{R}(\boldsymbol{\theta}_1 + \triangle\boldsymbol{\theta}_1) = \boldsymbol{R}(\triangle\boldsymbol{\phi}_1)\boldsymbol{R}(\boldsymbol{\theta}_1) \tag{2.109}$$

The resulting additive rotation vector $\triangle\boldsymbol{\theta}_1$ will be:

$$[\triangle\boldsymbol{\theta}_1]^T = \begin{bmatrix} 0.0313 & 0.0286 & -0.0165 \end{bmatrix} \tag{2.110}$$

Or using Equation 2.91

$$\begin{aligned} [\boldsymbol{T}(\boldsymbol{\theta}_1)] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1-\cos\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ &+ \frac{\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.827 & -0.4775 \\ 0 & 0.4775 & 0.827 \end{bmatrix} \end{aligned} \tag{2.111}$$

The last solution is an approximate solution to the first one and can be used in the linearization of weak form of the finite element differential equation to evaluate the geometric stiffness matrix (predictor phase), while updating rotation after each increment can be done through the first one to ensure the exact results (corrector phase in which the accuracy of finite element depends on). We can get $\triangle\boldsymbol{\phi}_1$ from $\triangle\boldsymbol{\theta}_1$ as follows:

$$\Delta \boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\theta}_1 + \triangle\boldsymbol{\theta}_1) - \boldsymbol{R}(\boldsymbol{\theta}_1) \tag{2.112}$$

$$[\widetilde{\triangle\boldsymbol{\phi}_1}] = \Delta \boldsymbol{R}.\boldsymbol{R}(\boldsymbol{\theta}_1)^T = \begin{bmatrix} -0.0005 & 0.0005 & 0.0314 \\ 0.0005 & -0.0005 & -0.0314 \\ -0.0314 & 0.0314 & -0.001 \end{bmatrix} \tag{2.113}$$

Note that $\widetilde{\triangle\boldsymbol{\phi}_1}$ is totally skew-symmetric when the added rotation becomes infinitesimal, so we can consider the skew-symmetric part of the above equation to evaluate its axial vector $\triangle\boldsymbol{\phi}_1$ as follows:

$$[\triangle\boldsymbol{\phi}_1]^T = \begin{bmatrix} 0.0314 & -0.0314 & 0 \end{bmatrix} \tag{2.114}$$

Which is identical to $\triangle\boldsymbol{\phi}_1 = \pi/100(1,1,0)$ given in the start of the example.
For adding two infinitesimal rotations (spin) $\triangle\boldsymbol{\phi}_1$ and $\triangle\boldsymbol{\phi}_2$, we get:

$$\boldsymbol{R}(\triangle\boldsymbol{\theta}_{added} + \triangle\boldsymbol{\phi}_1) = \boldsymbol{R}(\triangle\boldsymbol{\phi}_2)\boldsymbol{R}(\triangle\boldsymbol{\phi}_1) = \begin{bmatrix} 0.9979 & -0.0142 & 0.0633 \\ 0.0162 & 0.9994 & -0.0307 \\ -0.0628 & 0.0316 & 0.99751 \end{bmatrix} \tag{2.115}$$

$$[\triangle\boldsymbol{\theta}_{added}] = \begin{bmatrix} -0.0002 \\ 0.0317 \\ 0.0152 \end{bmatrix} \tag{2.116}$$

$$\triangle\boldsymbol{\theta}_{added} \simeq \triangle\boldsymbol{\phi}_2 \tag{2.117}$$

Or using addition theorem:

$$\triangle\boldsymbol{\phi}_{final} = \triangle\boldsymbol{\phi}_1 + \triangle\boldsymbol{\phi}_2 \tag{2.118}$$

■

2.1.12 Manipulation with bases

As shown in Figure 2.21, assume a rotation tensor $\boldsymbol{R}(\boldsymbol{\theta}_1)$ that transforms bases $\mathscr{B} = [\boldsymbol{e}_1\ \boldsymbol{e}_2\ \boldsymbol{e}_3]$ to basis $\mathscr{B}^* = [\boldsymbol{e}_1^*\ \boldsymbol{e}_2^*\ \boldsymbol{e}_3^*]$, such that any axis of the resulting basis equals to:

$$\boldsymbol{e}_i^* = \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{e}_i \quad for\ i = 1,2,3 \tag{2.119}$$

Similarly, rotation tensor $\boldsymbol{R}(\boldsymbol{\theta}_2)$ brings basis $\mathscr{B}^*$ to $\mathscr{B}^+ = \left[\boldsymbol{e}_1^+\ \boldsymbol{e}_2^+\ \boldsymbol{e}_3^+\right]$ through $R(\boldsymbol{\theta}_2)$ as follows:

$$\boldsymbol{e}_i^+ = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{e}_i^* \tag{2.120}$$

The compound rotations will be:

$$\boldsymbol{R}(\boldsymbol{\theta}) = \boldsymbol{R}(\boldsymbol{\theta}_2)\boldsymbol{R}(\boldsymbol{\theta}_1) \tag{2.121}$$

Equation 2.119 can be resolved in any basis, e.g. it can be resolved in basis $\mathscr{B}$ as follows:

$$[\boldsymbol{e}_i^*]^{[\mathscr{B}]} = [\boldsymbol{R}(\boldsymbol{\theta}_1)]^{[\mathscr{B}]}[\boldsymbol{e}_i]^{[\mathscr{B}]} \tag{2.122}$$

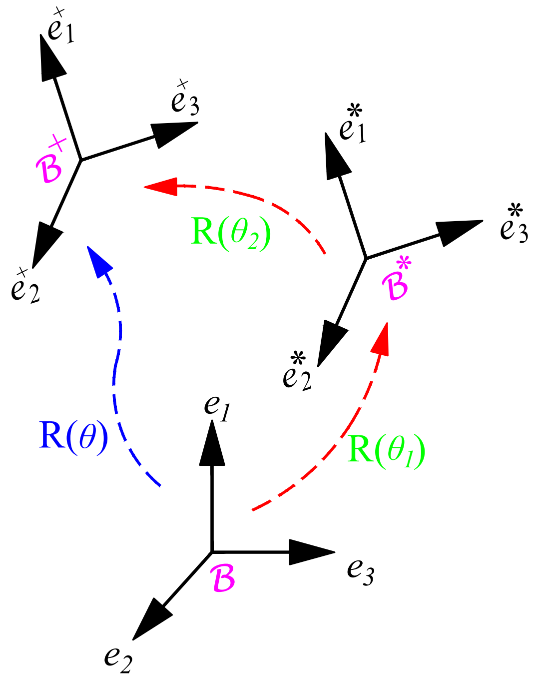


Figure 2.21

Where $\boldsymbol{A}^{[\mathscr{B}]}$ means that tensor $\boldsymbol{A}$ is resolved in basis $\mathscr{B}$. $[\boldsymbol{e}_i]^{[\mathscr{B}]}$ means that the basis $\boldsymbol{e}_i$ of frame $\mathscr{B}$ is resolved on itself which yields:

$$[\boldsymbol{e}_1]^{[\mathscr{B}]} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\boldsymbol{e}_2]^{[\mathscr{B}]} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\boldsymbol{e}_3]^{[\mathscr{B}]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.123}$$

From Equation 2.54, we get:

$$[\boldsymbol{R}(\boldsymbol{\theta}_1)]^{[\mathscr{B}]} = \mathbf{1} + \frac{\sin\theta}{\theta}\widetilde{\boldsymbol{\theta}_1^{[\mathscr{B}]}} + \frac{(1-\cos\theta)}{\theta^2}\widetilde{\boldsymbol{\theta}_1^{[\mathscr{B}]}}\widetilde{\boldsymbol{\theta}_1^{[\mathscr{B}]}} = \boldsymbol{R}\left(\boldsymbol{\theta}^{[\mathscr{B}]}\right) = \boldsymbol{R}^{[\mathscr{B}]}(\boldsymbol{\theta}) \tag{2.124}$$

The last equality is used for convenient. Also from Equation 2.122 and Equation 2.123, the rotation tensor $(\boldsymbol{\theta}_1)$ resolved in basis $\mathscr{B}$ will be:

$$[\boldsymbol{R}(\boldsymbol{\theta}_1)]^{[\mathscr{B}]} = \begin{bmatrix} [\boldsymbol{e}_1^*]^{[\mathscr{B}]} & [\boldsymbol{e}_2^*]^{[\mathscr{B}]} & [\boldsymbol{e}_3^*]^{[\mathscr{B}]} \end{bmatrix} \tag{2.125}$$

It means that a rotation tensor $\boldsymbol{R}$ rotating from basis $\mathscr{B}1$ to basis $\mathscr{B}2$ contains three columns, each one represents a unit vector in basis $\mathscr{B}2$ and resolved in basis $\mathscr{B}1$. Similarly, we can resolve the compound rotations in bases $\mathscr{B}$, $\mathscr{B}^*$ and $\mathscr{B}^+$ as follows:

$$\begin{array}{lll} \boldsymbol{R}\left(\boldsymbol{\theta}^{\mathscr{B}}\right) = \boldsymbol{R}\left(\boldsymbol{\theta}_2^{\mathscr{B}}\right)\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right) & or & \boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}) = \boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}_2)\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}_1) \\ \boldsymbol{R}(\boldsymbol{\theta}^*) = \boldsymbol{R}(\boldsymbol{\theta}_2^*)\boldsymbol{R}(\boldsymbol{\theta}_1^*) & or & \boldsymbol{R}^*(\boldsymbol{\theta}) = \boldsymbol{R}^*(\boldsymbol{\theta}_2)\boldsymbol{R}^*(\boldsymbol{\theta}_1) \\ \boldsymbol{R}(\boldsymbol{\theta}^+) = \boldsymbol{R}(\boldsymbol{\theta}_2^+)\boldsymbol{R}(\boldsymbol{\theta}_1^+) & or & \boldsymbol{R}^+(\boldsymbol{\theta}) = \boldsymbol{R}^+(\boldsymbol{\theta}_2)\boldsymbol{R}^+(\boldsymbol{\theta}_1) \end{array} \tag{2.126}$$

Where , $\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta})$, $\boldsymbol{R}^*(\boldsymbol{\theta})$ and $\boldsymbol{R}^+(\boldsymbol{\theta})$ are rotation matrices resolved in bases $\mathscr{B}$, $\mathscr{B}^*$ and $\mathscr{B}^+$ respectively. We also note that the rotation tensors $\boldsymbol{R}(\boldsymbol{\theta}_1)$ resolved in bases $\mathscr{B}$ and $\mathscr{B}^*$ are identical as the axis of rotation $\boldsymbol{\theta}_1$ remains the same after the rotation and its components in bases $\mathscr{B}$ and $\mathscr{B}^*$ are identical, so using Equation 2.124 results in

$$\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right) = \boldsymbol{R}(\boldsymbol{\theta}_1^*) \tag{2.127}$$

Similarly $\boldsymbol{\theta}_2$ when resolved in bases $\mathscr{B}^*$ and $\mathscr{B}^+$:

$$\boldsymbol{R}(\boldsymbol{\theta}_2^*) = \boldsymbol{R}(\boldsymbol{\theta}_2^+) \tag{2.128}$$

Also the components of rotation tensor resolved in different bases is related via:

$$\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}_2) = \boldsymbol{R}\left(\boldsymbol{\theta}_2^{\mathscr{B}}\right) = \boldsymbol{R}\left(\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)\boldsymbol{\theta}_2^*\right) = \boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)\boldsymbol{R}(\boldsymbol{\theta}_2^*)\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)^T \tag{2.129}$$

The last equality comes from Equation 2.124 and identity $\left(\widetilde{\boldsymbol{Ra}} = \boldsymbol{R}\tilde{\boldsymbol{a}}\boldsymbol{R}^T\right)$, so the compound rotation will be:

$$\boldsymbol{R}\left(\boldsymbol{\theta}^{\mathscr{B}}\right) = \boldsymbol{R}\left(\boldsymbol{\theta}_2^{\mathscr{B}}\right)\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right) \tag{2.130}$$

$$= \boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)\boldsymbol{R}(\boldsymbol{\theta}_2^*)\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)^T\boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right) \tag{2.131}$$

$$= \boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right)\boldsymbol{R}(\boldsymbol{\theta}_2^*) \Leftrightarrow \boldsymbol{R} = \boldsymbol{R}_1\boldsymbol{R}_2^* \tag{2.132}$$

$\boldsymbol{R}_1, \boldsymbol{R}_2$ are rotation tensor that brings basis $\mathscr{B}$ to basis $\mathscr{B}^*$ and basis $\mathscr{B}^*$ to $\mathscr{B}^+$, both resolved in basis $\mathscr{B}$, While $\boldsymbol{R}_2^*$ is the one that describes the rotation from basis $\mathscr{B}^*$ to basis $\mathscr{B}^+$ resolved in basis $\mathscr{B}^*$. We find that the sequence of rotation is reversed in Equation 2.132 compared to the sequence of rotation in Equation 2.80 and the order of multiplication depends on the basis which they are resolved in.

■ **Example 2.5** A basis $\mathscr{B}$ is subjected to rotation $\boldsymbol{\theta}$ resolved in basis $\mathscr{B}$ with $\boldsymbol{\theta}_1^{\mathscr{B}} = (0, 0, \pi/4)$ to form basis B^* and followed by $\boldsymbol{\theta}_2^{\mathscr{B}} = (0, \pi/2, 0)$ to form basis $\mathscr{B}^+$,

Rotation of basis $\mathscr{B}$ to basis $\mathscr{B}^*$ is shown in the Figure 2.22a through $\boldsymbol{\theta}_1^{\mathscr{B}}$ via R_1

$$\boldsymbol{R_1} = \boldsymbol{R}\left(\boldsymbol{\theta}_1^{\mathscr{B}}\right) = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.133}$$

Rotation of basis $\mathscr{B}^*$ to $\mathscr{B}^+$ as shown in Figure 2.22b through R_2

$$\boldsymbol{R_2^B} = \boldsymbol{R}\left(\boldsymbol{\theta}_2^{\mathscr{B}}\right) = \begin{bmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{bmatrix} \tag{2.134}$$

So the compound rotations will be:

$$\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}) = \boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}_2)\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}_1) = \begin{bmatrix} 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \tag{2.135}$$

The resulting rotation $\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta})$ resolved in basis $\mathscr{B}$ with bases $\boldsymbol{e}_1^+$ defined as:

$$\boldsymbol{R}^{\mathscr{B}}(\boldsymbol{\theta}) = [\ (\boldsymbol{e}_1^+)^{\mathscr{B}}\ (\boldsymbol{e}_2^+)^{\mathscr{B}}\ (\boldsymbol{e}_3^+)^{\mathscr{B}}] \tag{2.136}$$

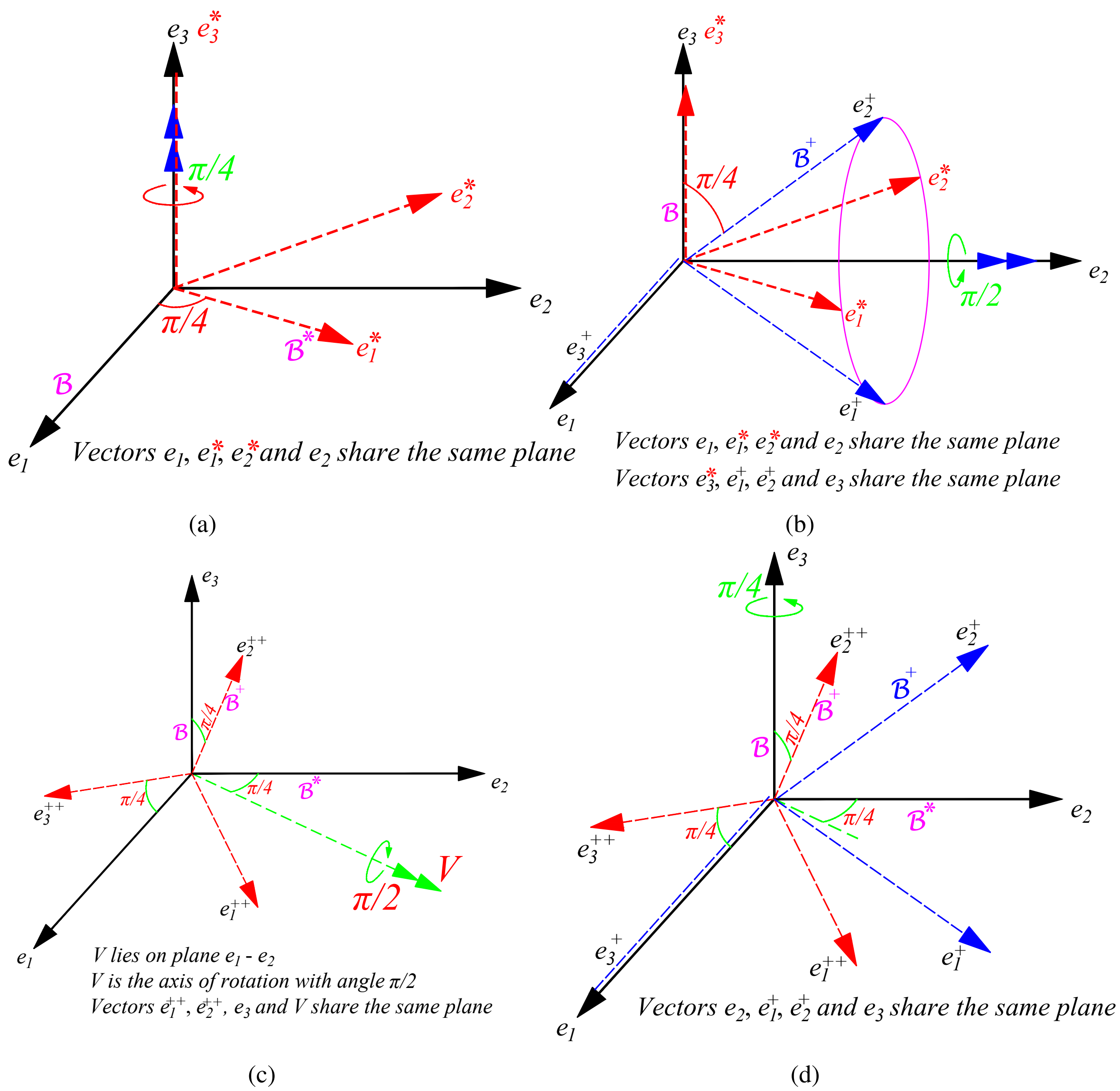


Figure 2.22

Where $\left(\boldsymbol{e}_i^{+}\right)^{\mathscr{B}}$ are components of $\boldsymbol{e}_i^{+}$ resolved in $\mathscr{B}$ for $i = 1,2,3$.

$$\left(\boldsymbol{e}_1^{+}\right)^{\mathscr{B}} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \left(\boldsymbol{e}_2^{+}\right)^{\mathscr{B}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \left(\boldsymbol{e}_3^{+}\right)^{\mathscr{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.137}$$

But if we calculate the components of $\boldsymbol{\theta}_2$ resolved in basis $\mathscr{B}^*$ as $(\boldsymbol{\theta}_2^*)$, it will be:

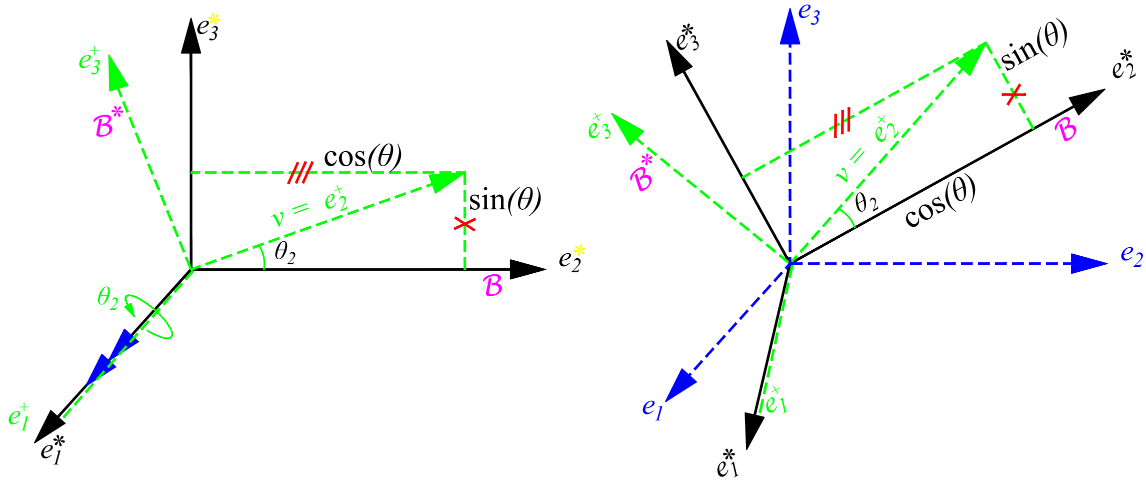
$$\boldsymbol{\theta}_2^* = \boldsymbol{R}_1^T \boldsymbol{\theta}_2^{\mathscr{B}} \tag{2.138}$$

Or from Figure 2.22d (resolving angle along axis $\boldsymbol{e}_2$ in basis $\mathscr{B}^*$), it follows:

$$\boldsymbol{\theta}_2^* = \frac{\pi}{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \tag{2.139}$$

Applying this rotation is shown in Figure 2.22c. Adding a rotation tensor $\boldsymbol{R}_1^T$ to the above rotation, $\boldsymbol{R} = \boldsymbol{R}_1 \boldsymbol{R}_2^*$ yields a identical results in Equation 2.135 as shown in Figure 2.22d.

From the last case, we reversed the rotation from $\mathscr{B}^+$ to $\mathscr{B}^{\oplus}$ through $\boldsymbol{R}_1^T$ then reverse the rotation through $\boldsymbol{R}_2^{*T}$ to transform finally to basis $\mathscr{B}$ as shown in Figure 2.22d. ■



(a) (b)

Figure 2.23

■ **Example 2.6** Imagine that we have a unit vector $\boldsymbol{v}$ come from the rotation of basis $\boldsymbol{e}_1^*$ about $\boldsymbol{e}_3^*$ via an angle $\boldsymbol{\theta}_2$. The components of the vector v resolved in basis $\mathscr{B}^*$ is $(\boldsymbol{v}_1^*, \boldsymbol{v}_2^*, \boldsymbol{v}_3^*) = (\cos(\boldsymbol{\theta}_2), \sin(\boldsymbol{\theta}_2), 0)$. If we track the bases $\boldsymbol{e}_i^*$ due to rotation $R(\boldsymbol{\theta}_2)$, we get new basis $\mathscr{B}^+$ with bases e_i^+ with components resolved in basis $\mathscr{B}^*$ as follows:

$$\boldsymbol{R}^*(\boldsymbol{\theta}_2) = [(\boldsymbol{e}_1^+)^* \ (\boldsymbol{e}_2^+)^* \ (\boldsymbol{e}_3^+)^*] \tag{2.140}$$

Where $(\boldsymbol{e}_i^+)^*$ are components of e_i^+ resolved in $\mathscr{B}^*$ for $i = 1,2,3$. (Note vector $\boldsymbol{v}$ and $\boldsymbol{e}_1^+$ are identical)

If the vector $\boldsymbol{v}$ and basis $\mathscr{B}^*$ are attached to rigid body, and this body is subjected to rotation $\boldsymbol{R}(\boldsymbol{\theta}_1)$, the vector $\boldsymbol{v}$ and basis $\mathscr{B}^*$ rotate also with the body. We find out the components of new vector $\boldsymbol{v}$ resolved in (projection on) basis $\mathscr{B}^*$ is still the same as old one and is not affected by $\boldsymbol{R}(\boldsymbol{\theta}_1)$ at all. Similar to vector $\boldsymbol{v}$, components of e_i^+ resolved in $\mathscr{B}^*$ and donated by $(\boldsymbol{e}_i^+)^*$ do not change with $\boldsymbol{R}_1$, so $\boldsymbol{R}^*(\boldsymbol{\theta}_2)$ is constant for any rotation $\boldsymbol{R}(\boldsymbol{\theta}_1)$ and the components of spatial abject or vector attached to a body referred to its local frame (basis $\mathscr{B}^*$ attached to this body) is called the material components. Studying material components is important, especially when a body is rotating with high speed ($\boldsymbol{R}_1$), while deformation (change in distance between any two point on it) is very small, so it is convenient to study this change relative to its local basis not global basis without affected by $\boldsymbol{R}_1$, the same case in our study. After rotation $(\boldsymbol{e}_i^+)^{\mathscr{B}}$ resolved in basis $\mathscr{B}$ is as follows:

$$\left(\boldsymbol{e}_i^{+}\right)^{\mathscr{B}} = \boldsymbol{R}(\boldsymbol{\theta}_1)\left(\boldsymbol{e}_i^{+}\right)^{*} \tag{2.141}$$

$$\left(\boldsymbol{e}_1^{+}\right)^{*} = \boldsymbol{R}^{*}(\boldsymbol{\theta}_2)\begin{bmatrix}1\\0\\0\end{bmatrix} = \boldsymbol{R}^{*}(\boldsymbol{\theta}_2)\left(\boldsymbol{e}_1^{\mathscr{B}}\right)^{\mathscr{B}} \tag{2.142}$$

$\left(\boldsymbol{e}_i^{\mathscr{B}}\right)^{\mathscr{B}}$ are components of $e_i^{\mathscr{B}}$resolved in $\mathscr{B}$, for $i = 1,\ 2,\ 3(components\ of\ basis\ on\ itself)$. Which equal to (1,0,0), (0,1,0), (0,0,1), respectively. So

$$\left(\boldsymbol{e}_i^{+}\right)^{\mathscr{B}} = \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{R}^{*}(\boldsymbol{\theta}_2)\left(\boldsymbol{e}_1^{\mathscr{B}}\right)^{\mathscr{B}} = \boldsymbol{R}(\boldsymbol{\theta})\left(\boldsymbol{e}_1^{\mathscr{B}}\right)^{\mathscr{B}} \Longleftrightarrow \boldsymbol{R}(\boldsymbol{\theta}) = \boldsymbol{R}(\boldsymbol{\theta}_1)\boldsymbol{R}^{*}(\boldsymbol{\theta}_2) \tag{2.143}$$

2.1.13 Angular velocity

From chapter 1, we concluded that the velocity of a point lying on object rotating with angular velocity ω about axis with unit vector $\boldsymbol{n}$ as shown in Figure 1.13 is defined as:

$$\dot{\boldsymbol{a}} = \tilde{\boldsymbol{\omega}}\boldsymbol{a} \tag{2.144}$$

Where $\boldsymbol{\omega} = \omega\boldsymbol{n}$, so the time derivative of vector with constant length equal to the cross product of angular velocity and vector itself.

Also vector $\boldsymbol{a}(t)$ can be formulated from rotation of vector $\boldsymbol{a}_0$ (constant with the time) through rotation $\boldsymbol{R}(t)$ which is a function of time:

$$\boldsymbol{a}(t) = \boldsymbol{R}(t)\boldsymbol{a}_0 \Leftrightarrow \boldsymbol{a}_0 = \boldsymbol{R}(t)^T\boldsymbol{a}(t) \tag{2.145}$$

$$\dot{\boldsymbol{a}}(t) = \dot{\boldsymbol{R}}(t)\,a_0 = \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\boldsymbol{a}(t) \tag{2.146}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T \tag{2.147}$$

As the angular velocity can be *imagined for constant axis of rotation as*

$$\boldsymbol{\omega} = \frac{\partial\boldsymbol{\phi}}{\partial t}\boldsymbol{n} = \frac{\triangle\boldsymbol{\phi}}{\triangle t}\boldsymbol{n} \tag{2.148}$$

So it is infinitesimal rotation rotated in infinitesimal time. There no vector its derivative is angular velocity due to the fact that:

$$\frac{d(\boldsymbol{\phi}\boldsymbol{n})}{dt} = \dot{\boldsymbol{\phi}}\boldsymbol{n} + \boldsymbol{\phi}\dot{\boldsymbol{n}} = \boldsymbol{\omega} + \boldsymbol{\phi}\dot{\boldsymbol{n}} \tag{2.149}$$

So angular velocity can be called the spin as it is similar to infinitesimal spin ($\widetilde{\triangle\boldsymbol{\phi}} = \Delta\boldsymbol{R}.\boldsymbol{R}(\boldsymbol{\theta})^T$)

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{2.150}$$

Following addition theorem $\triangle\boldsymbol{\phi} = \triangle\boldsymbol{\phi}_1 + \triangle\boldsymbol{\phi}_2$, adding two angular velocity follows:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \tag{2.151}$$

Also addition theorem can be proven as follow in Figure 2.24. Assume that $\boldsymbol{\omega}_1$ is spin that convert basis $\mathscr{B}$ (with basis $\boldsymbol{E}_i$) to basis $\mathscr{B}^*$ (with basis $\boldsymbol{e}_i$), and spin $\boldsymbol{\omega}_2$ convert basis $\mathscr{B}^*$ to basis $\mathscr{B}^+$(with basis $\boldsymbol{b}_i$), such that:

$$\boldsymbol{b}_i = \boldsymbol{R}_2^{\mathscr{B}}\boldsymbol{R}_1^{\mathscr{B}}\boldsymbol{E}_i \tag{2.152}$$

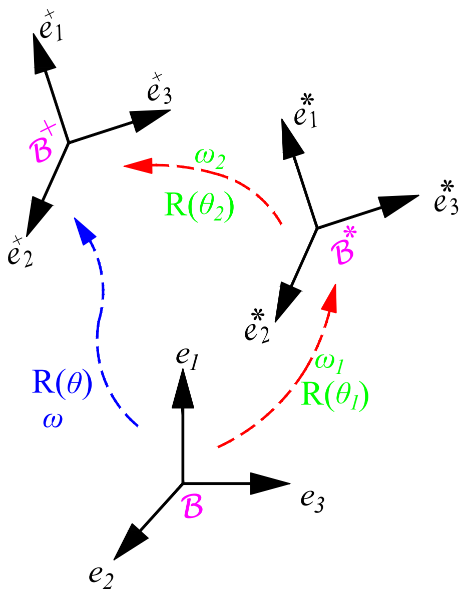


Figure 2.24

or

$$\boldsymbol{b}_i = \boldsymbol{R}_1^{\mathscr{B}} \boldsymbol{R}_2^{\mathscr{B}^*} \boldsymbol{E}_i = \boldsymbol{R}_1 \boldsymbol{R}_2^* \boldsymbol{E}_i \rightarrow \boldsymbol{E}_i = \boldsymbol{R}_2^{*T} \boldsymbol{R}_1{}^T \boldsymbol{b}_i \tag{2.153}$$

Then the time derivative of basis $\boldsymbol{b}_i$ will be:

$$\begin{aligned} \dot{\boldsymbol{b}}_i &= \left(\dot{\boldsymbol{R}}_1 \boldsymbol{R}_2^* + \boldsymbol{R}_1 \dot{\boldsymbol{R}}_2^*\right) \boldsymbol{E_i} \\ &= \left(\dot{\boldsymbol{R}}_1 \boldsymbol{R}_2^* + \boldsymbol{R}_1 \dot{\boldsymbol{R}}_2^*\right) \boldsymbol{R}_2^{*T} \boldsymbol{R}_1{}^T \boldsymbol{b}_i \\ &= \left(\dot{\boldsymbol{R}}_1 \boldsymbol{R}_1{}^T + \boldsymbol{R}_1 \dot{\boldsymbol{R}}_2^* \boldsymbol{R}_2^{*T} \boldsymbol{R}_1{}^T\right) \boldsymbol{b}_i \end{aligned} \tag{2.154}$$

Using the following expressions for angular velocity:

$$\tilde{\boldsymbol{\omega}}_1 = \dot{\boldsymbol{R}}_1 \boldsymbol{R}_1{}^T, \qquad \tilde{\boldsymbol{\omega}}_2^* = \dot{\overset{*}{\boldsymbol{R}_2}} \boldsymbol{R}_2^{*T} \tag{2.155}$$

where $\boldsymbol{\omega}_1$ is the angular velocity of basis $\mathscr{B}^*$ with respect to $\mathscr{B}$ resolved in basis $\mathscr{B}$. While $\boldsymbol{\omega}_2^*$ is angular velocity of basis $\mathscr{B}^+$ with respect to basis $\mathscr{B}^*$ and resolved in basis $\mathscr{B}^*$. Resolving $\boldsymbol{\omega}_2$ in basis $\mathscr{B}$ results in:

$$\tilde{\omega}_2 = \boldsymbol{R}_1 \tilde{\boldsymbol{\omega}}_2^* \boldsymbol{R}_1{}^T \tag{2.156}$$

Equation 2.154 will be:

$$\dot{\boldsymbol{b}}_i = \left(\tilde{\boldsymbol{\omega}}_1 + \boldsymbol{R}_1 \tilde{\boldsymbol{\omega}}_2^* \boldsymbol{R}_1{}^T\right) \boldsymbol{b}_i = \left(\tilde{\boldsymbol{\omega}}_1 + \tilde{\boldsymbol{\omega}}_2\right) \boldsymbol{b}_i = \tilde{\boldsymbol{\omega}} \boldsymbol{b}_i \tag{2.157}$$

And the equivalent angular velocity is:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \tag{2.158}$$

Note that

$$\dot{\boldsymbol{R}}_2 \boldsymbol{R}_2{}^T \neq \boldsymbol{\omega}_2 \tag{2.159}$$

As it is expressed in terms of basis ($\mathscr{B}$) different from the basis the angular velocity is supposed to be measured with respect to ($\mathscr{B}^*$), so it is not considered as an angular velocity as follows:

$$\dot{\boldsymbol{R}}_2\boldsymbol{R}_2{}^T = \frac{\partial(\boldsymbol{R}_1\boldsymbol{R}_2^*\boldsymbol{R}_1^T)}{\partial t}\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T \tag{2.160}$$

$$= \left(\dot{\boldsymbol{R}}_1\boldsymbol{R}_2^*\boldsymbol{R}_1^T + \boldsymbol{R}_1\dot{\boldsymbol{R}}_2^*\boldsymbol{R}_1^T + \boldsymbol{R}_1\boldsymbol{R}_2^*\dot{\boldsymbol{R}}_1^T\right)\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T \tag{2.161}$$

$$= \dot{\boldsymbol{R}}_1\boldsymbol{R}_1^T + \boldsymbol{R}_1\dot{\boldsymbol{R}}_2^*\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T + \boldsymbol{R}_1\boldsymbol{R}_2^*\dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T \tag{2.162}$$

$$= \widetilde{\boldsymbol{\omega}}_1 + \widetilde{\boldsymbol{\omega}}_2 - \boldsymbol{R}_2\widetilde{\boldsymbol{\omega}}_1\boldsymbol{R}_2^T \tag{2.163}$$

$$= \widetilde{\boldsymbol{\omega}}_2 + \left(\widetilde{\boldsymbol{1} - \boldsymbol{R}_2}\right)\widetilde{\boldsymbol{\omega}}_1 \tag{2.164}$$

As

$$\boldsymbol{R}_1\boldsymbol{R}_2^*\dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T = \boldsymbol{R}_1\boldsymbol{R}_2^*\boldsymbol{R}_1^T\boldsymbol{R}_1\dot{\boldsymbol{R}}_1^T\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T = \left(\boldsymbol{R}_1\boldsymbol{R}_2^*\boldsymbol{R}_1^T\right)\boldsymbol{R}_1\dot{\boldsymbol{R}}_1^T\left(\boldsymbol{R}_1\boldsymbol{R}_2^{*T}\boldsymbol{R}_1^T\right) \tag{2.165}$$

$$= -\boldsymbol{R}_2\widetilde{\boldsymbol{\omega}}_1\boldsymbol{R}_2^T \tag{2.166}$$

Where

$$\boldsymbol{R}_1\dot{\boldsymbol{R}}_1^T = \left(\dot{\boldsymbol{R}}_1\boldsymbol{R}_1{}^T\right)^T = \widetilde{\boldsymbol{\omega}}_1^T = -\widetilde{\boldsymbol{\omega}}_1 \tag{2.167}$$

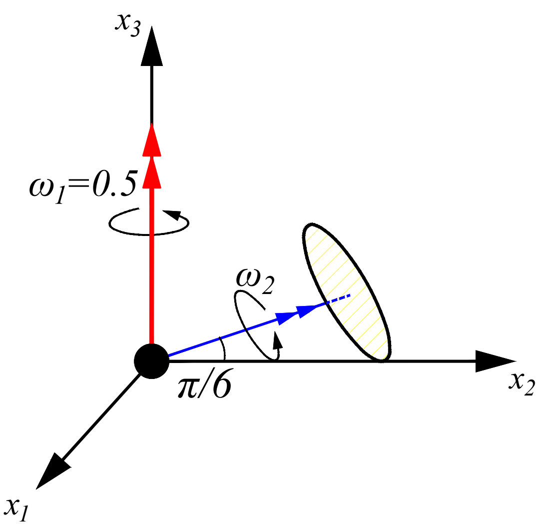


Figure 2.25

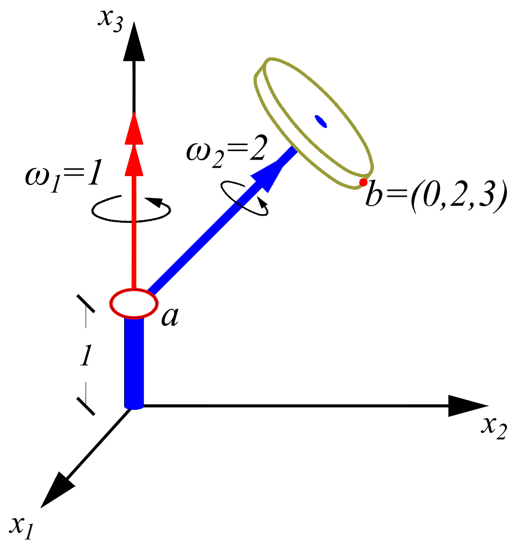


Figure 2.26

■ **Example 2.7** The plate rotates about the $\boldsymbol{x}_3$ axis at a constant rate $\omega_1 = 0.5\ rad/s$ without slipping on the horizontal plan pictured in Figure 2.25. Evaluate the ω_2.

Plate rotation is:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = -0.5\boldsymbol{e}_3 + \cos(\pi/6) * |\boldsymbol{\omega}_2| * \boldsymbol{e}_2 + \sin(\pi/6) * |\boldsymbol{\omega}_2| * \boldsymbol{e}_3 \tag{2.168}$$

As axis $\boldsymbol{x}_2$ represents the instantaneous axis of zero velocity, such that:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{e}_2 = 0 = \left(0.5 - \sin\left(\frac{\pi}{6}\right) * |\boldsymbol{\omega}_2|\right)\boldsymbol{e}_3 \rightarrow |\boldsymbol{\omega}_2| = 1 \tag{2.169}$$

From Equation 2.168, we get:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \frac{\sqrt{3}}{2}|\boldsymbol{\omega}_2|\boldsymbol{e}_2 = \frac{\sqrt{3}}{2}\boldsymbol{e}_2 \tag{2.170}$$

■

■ **Example 2.8** A disk attached to a shaft spinning with angular velocity $\boldsymbol{\omega}_2 = 2\ rad/s$ shown in Figure 2.26 attached through an internal hinge to another shaft rotating with angular velocity $\boldsymbol{\omega}_1 = 1\ rad/s$, calculate the velocity of point b.

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = (0, \sqrt{3}, 2) \tag{2.171}$$

The velocity of point b is defined through:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{2.172}$$

Where vector $\boldsymbol{r}$ is a position vector from point a to point b defined as $\boldsymbol{r} = (0, 2, 2)$, so the resulting velocity will be:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = (2\sqrt{3} - 4)\boldsymbol{e}_1 \tag{2.173}$$

■

2.2 Applications in structural analysis

2.2.1 Finite rotation of a rigid joint in framework

Assume a rigid joint connecting some structural members through rigid links with negligible length as shown in Figure 2.27. Each member i has its local axes formed through rotation transformation $\boldsymbol{R}^i$ of the global axes such that:

$$\boldsymbol{E}_I^i = \boldsymbol{R}^i \boldsymbol{e}_I \tag{2.174}$$

Where $\boldsymbol{E}_I^i$ is the local basis of element i in the direction I, while $\boldsymbol{e}_I$ represents the global axis. If the connecting joint is rotated through spatial rotation $\boldsymbol{\theta}$ resolved in global axes as:

$$[\boldsymbol{\theta}]_{\boldsymbol{e}_I} = [\ \theta_1 \quad \theta_2 \quad \theta_3\ ]^T \tag{2.175}$$

Because of the rigid links, this will result in a rotation of each element with rotation $\boldsymbol{\theta}^i$ resolved in the member local axes $(\overline{\boldsymbol{\theta}}^i)$ as follows:

$$\overline{\boldsymbol{\theta}}^i = [\boldsymbol{\theta}^i]_{\boldsymbol{E}_I} = \boldsymbol{R}^T[\boldsymbol{\theta}]_{\boldsymbol{e}_I} \tag{2.176}$$

This rotation leads to a motion of each material point on the member cross section, such that if the position of a point P relative to the beam centroid resolved in the member local bases is $[\boldsymbol{X}]_{\boldsymbol{t}_I} = [0, X_2, X_3]^T$ as shown in Figure 2.28, it will be $\boldsymbol{R}(\overline{\boldsymbol{\theta}}^i)X$, so the displacement $\boldsymbol{u}$ of the material point (X_2, X_3) resolved in the local axes of the member will be:

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = (\boldsymbol{R}(\overline{\boldsymbol{\theta}}^i) - 1)\boldsymbol{X} \tag{2.177}$$

For relatively small rotation and using Equation 2.66, the displacement will be:

$$\boldsymbol{u} = \left(\tilde{\overline{\boldsymbol{\theta}}}^i + \frac{1}{2}\tilde{\overline{\boldsymbol{\theta}}}^i\tilde{\overline{\boldsymbol{\theta}}}^i\right)\boldsymbol{X} \tag{2.178}$$

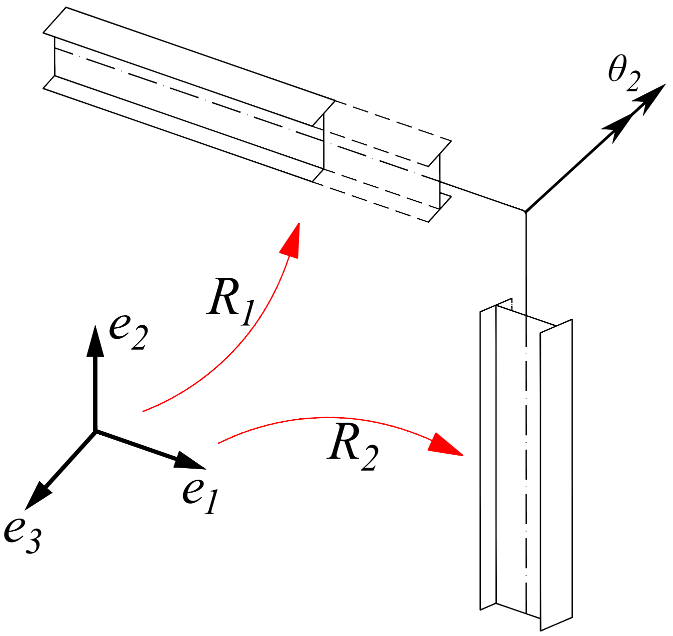


Figure 2.27

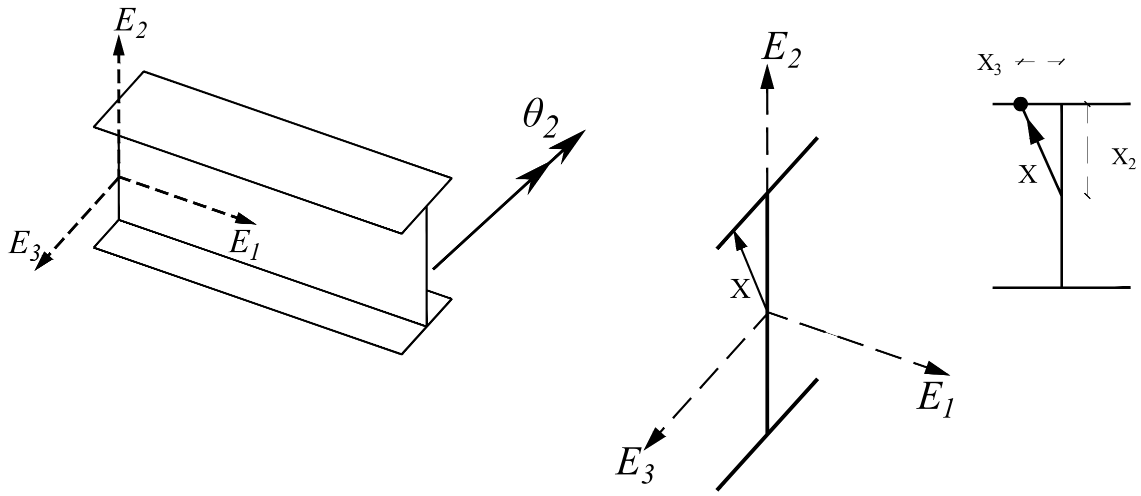


Figure 2.28

With $[\boldsymbol{X}]_{t_1} = [0, X_2, X_3]^T$ and $[\overline{\boldsymbol{\theta}}^i]_{t_1} = [\theta_x, \theta_y, \theta_z]$, the displacement components resolved in the member local axes are:

$$[\boldsymbol{u}]_{t_1} = \left(\begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -\left(\theta_y^2 + \theta_z^2\right) & (\theta_x\theta_y) & (\theta_x\theta_z) \\ (\theta_x\theta_y) & -\left(\theta_x^2 + \theta_z^2\right) & (\theta_y\theta_z) \\ (\theta_x\theta_z) & (\theta_y\theta_z) & -\left(\theta_x^2 + \theta_y^2\right) \end{bmatrix} \right) \begin{bmatrix} 0 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.179}$$

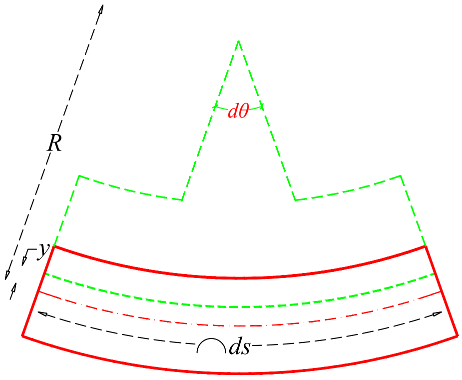


Figure 2.29

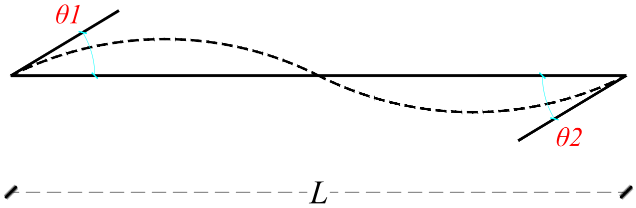


Figure 2.30

2.2.2 Curvature of two dimensional beams

For a two dimensional curve with a radius of curvature R, its curvature $\boldsymbol{\kappa}$ is defined as

$$\kappa = \frac{1}{R} = \frac{d\theta}{ds} \tag{2.180}$$

Where s is the arc length along the curve. Assume an Euler-Bernoulli in plane curved beams shown in Figure 2.29 with radius of curvature R, so any fiber located in distance y away from its center-line is stretched along the arc length by strain $\varepsilon_b(y)$ defined as:

$$\varepsilon_b\left(y\right) = \frac{length\ change}{original\ length} = \frac{\triangle\left(ds\right)}{ds} = \frac{-yd\theta}{ds} = -y\kappa \tag{2.181}$$

The above relation relates the strain induced in beam element with its curvature κ which ds is the undeformed or initial arc length. The beam is subjected to uniform axial strain across its cross section ε_a as shown in Figure 2.31 defined as

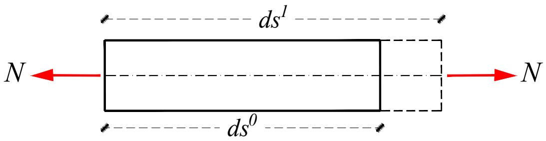


Figure 2.31

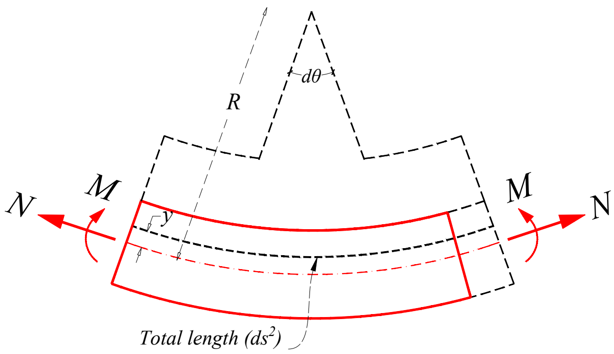


Figure 2.32

$$\varepsilon_a = \frac{ds^1 - ds^0}{ds^0} \tag{2.182}$$

Then followed by curvature in Figure 2.32 with total strain of:

$$\varepsilon = \frac{ds^2 - ds^0}{ds^0} = \frac{ds^1 - ds^0}{ds^0} + \frac{ds^2 - ds^1}{ds^0} = \varepsilon_a - y\frac{d\theta}{ds^0} = \varepsilon_a - y\boldsymbol{\kappa} \tag{2.183}$$

The third equality comes from:

$$ds^2 - ds^1 = (R-y)d\theta - Rd\theta = -yd\theta \tag{2.184}$$

As differential arc length (ds) is related to differential coordinates increment dx and dy through:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (y')^2} \tag{2.185}$$

($'$) means here differentiating with dx.

$$\theta = \frac{dy}{dx} \to d\theta = \frac{d^2y}{d^2x}dx = y''dx \tag{2.186}$$

$$\kappa = \frac{d\theta}{ds} = \frac{d\theta}{dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{y''}{\sqrt{1 + (y')^2}} \tag{2.187}$$

First order analysis assumes that the dominator of the upper equation equals to unity which results in:

$$\kappa \simeq y'' \tag{2.188}$$

■ **Example 2.9** Assume a beam shown in Figure 2.30 with length L directed along axis $\boldsymbol{e}_1$ with end rotations $\theta_1,\ \theta_2$. A smooth curve can be formed from the end boundary conditions:

$$y(0) = 0; y(L) = 0; y'(0) = \theta_1; y'(L) = \theta_2 \tag{2.189}$$

The curve will be a polynomial of third degree as follows:

$$y = ax^3 + bx^2 + cx + d \tag{2.190}$$

Solving for 4 unknowns a to d, we get the following:

$$y = \left(\frac{\theta_1 + \theta_2}{L^2}\right)x^3 - \left(\frac{2\theta_1 + \theta_2}{L}\right)x^2 + \theta_1 x \tag{2.191}$$

$$y' = 3\left(\frac{\theta_1 + \theta_2}{L^2}\right)x^2 - 2\left(\frac{2\theta_1 + \theta_2}{L}\right)x + \theta_1 \tag{2.192}$$

From Equation 2.188

$$\kappa(x) \simeq 6\left(\frac{\theta_1 + \theta_2}{L^2}\right)x - 2\left(\frac{2\theta_1 + \theta_2}{L}\right) = \left(-\frac{4}{L} + \frac{6x}{L^2}\right)\theta_1 + \left(-\frac{2}{L} + \frac{6x}{L^2}\right)\theta_2 \tag{2.193}$$

Curvature at beam mid point will be:

$$\kappa\left(\frac{L}{2}\right) = \left(\frac{\theta_1 - \theta_2}{L}\right) \tag{2.194}$$

Even if we assumed a constant curvature along the element, so the rotation y' (integration of curvature) would be a first-order polynomial as follow:

$$y' = ax + b \tag{2.195}$$

Applying only the rotational boundary conditions at ends in Equation 2.189:

$$y'(0) = \theta_1; y'(L) = \theta_2 \tag{2.196}$$

We conclude that

$$y' = \left(\frac{\theta_1 - \theta_2}{L}\right) x + \theta_1 \tag{2.197}$$

So the assumed constant curvature will be:

$$\kappa = y'' = \left(\frac{\theta_1 - \theta_2}{L}\right) \tag{2.198}$$

So for a constant curvature along the member, it can be evaluated from the changed in beam orientations at ends with beam length L as follows:

$$\kappa = \frac{\triangle\theta}{L} \tag{2.199}$$

For a three dimensional beam, the curvature will be:

$$\boldsymbol{\kappa} = \frac{\triangle\boldsymbol{\phi}}{L} \tag{2.200}$$

Or generally

$$\boldsymbol{\kappa} = \frac{d\boldsymbol{\phi}}{ds} \tag{2.201}$$

$\triangle\boldsymbol{\phi}$ is variation in non-additive rotation (see Equation 2.86). Also curvature $\boldsymbol{\kappa}_g$ in this case is vector. For more details, see subsection 2.2.4

2.2.3 Effect of beam bowing on axial strain

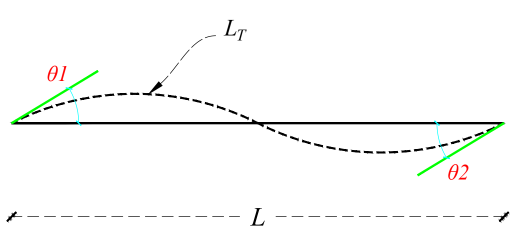


Figure 2.33

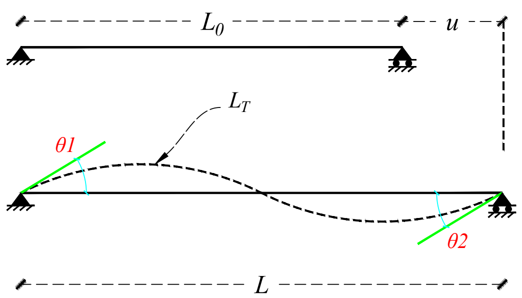


Figure 2.34

As shown in Figure 2.33, a straight beam is initially oriented along axis $\boldsymbol{e}_1$ and subjected to to ends rotation θ_1 and θ_2, the current beam length L_T compared to its projection on axis $\boldsymbol{e}_1$ is defined as:

$$L_t = \int ds = \int_0^L \sqrt{1+y'^2}\, dx \tag{2.202}$$

Using Equation 2.192, and solving the integration results in:

$$L_t = L\left(1+\frac{2\theta_1^2+2\theta_2^2-\theta_1\theta_2}{30}\right) \tag{2.203}$$

If a two dimensional beam with initial length L_0 shown in Figure 2.34 is subjected to axial displacement u, such that the axial strain ε_a will be:

$$\varepsilon_a = \frac{u}{L_0} \tag{2.204}$$

Then, its ends are subjected rotations θ_1 and θ_2. From Equation 2.203, the bowing created in the beam induces axial strain through beam elongation formed by end rotations as follows:

$$\begin{aligned} \varepsilon_a = \frac{\textit{change in beam length}}{\textit{original length}} &= \frac{L_t - L_0}{L_0} \\ &= \frac{u}{L_0} + \left(\frac{L_0+u}{L_0}\right)\frac{2\theta_1^2+2\theta_2^2-\theta_1\theta_2}{30} \\ &= \frac{u}{L_0} + \frac{2\theta_1^2+2\theta_2^2-\theta_1\theta_2}{30} \end{aligned} \tag{2.205}$$

For second order analysis, we can consider $\left(\frac{L_0+u}{L_0}\right)$ equals to unity. The total strain on the beam section due to axial strain and curvature will be:

$$\varepsilon(x,y) = \varepsilon_a + \varepsilon_b(x,y) = \varepsilon_a + \frac{2\theta_1^2+2\theta_2^2-\theta_1\theta_2}{30} - \left[\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\theta_1 + \left(-\frac{2}{L}+\frac{6x}{L^2}\right)\theta_2\right] y \tag{2.206}$$

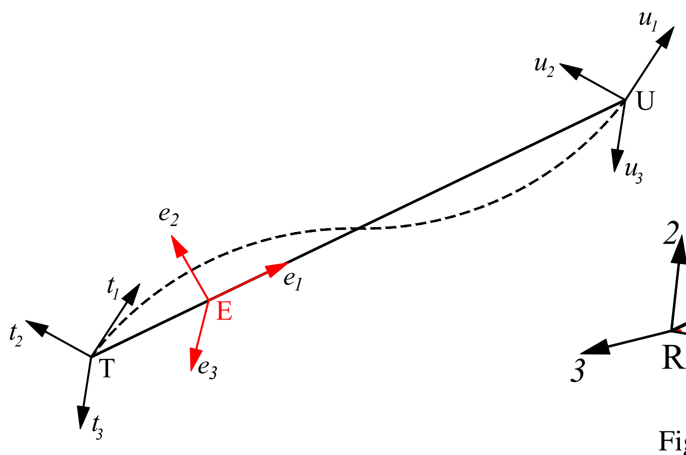


Figure 2.35

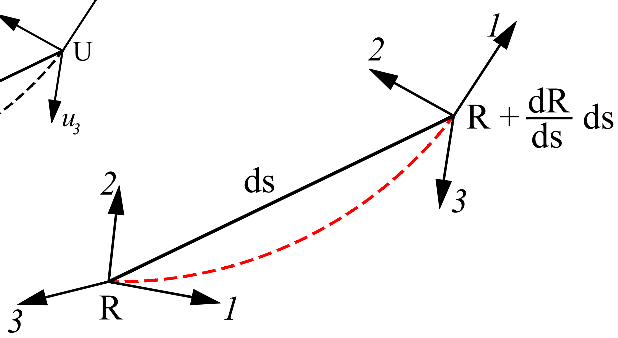


Figure 2.36

2.2.4 Curvature of three dimensional beams with small strain and large rotations

As shown in Figure 2.35, assume a three dimensional beam with two nodal triads (or nodal frame) at beam ends, $\boldsymbol{T}$ and $\boldsymbol{U}$ with axes $[t_1, t_2, t_3]$, $[u_1, u_2, u_3]$, respectively, so the first axis of each triad, $\boldsymbol{t}_1$

and $\boldsymbol{u}_1$, is directed along the beam tangent, while other two axes of each triad are directed along the principal axes of beam sections at ends. We can use another triad along the element $\boldsymbol{E}$ (generally with first axis linking two ends of the beam, while the other two axes are defined using many different procedures mentioned in subsection 2.2.8).
Assuming the relation between nodal triads as follows:

$$\boldsymbol{U} = \boldsymbol{R}(\triangle\boldsymbol{\phi})\,T \Leftrightarrow \boldsymbol{R}(\triangle\boldsymbol{\phi}) = \boldsymbol{U}\boldsymbol{T}^T \tag{2.207}$$

$\triangle\boldsymbol{\phi}$ is relatively small within beam element, so it can be approximated as follows:

$$\boldsymbol{U}\boldsymbol{T}^T = \boldsymbol{R} = \mathbf{1} + \widetilde{\triangle\boldsymbol{\phi}} + \frac{\widetilde{\triangle\boldsymbol{\phi}}^2}{2!} \tag{2.208}$$

The skew-symmetric part of the above rotation tensor will be:

$$skew\,(\boldsymbol{R}) = \frac{\boldsymbol{R} - \boldsymbol{R^T}}{2} = \widetilde{\triangle\boldsymbol{\phi}} = \frac{\boldsymbol{U}\boldsymbol{T}^T - \boldsymbol{U}^T\boldsymbol{T}}{2} \tag{2.209}$$

And from Equation 2.200, curvature will be:

$$\widetilde{\boldsymbol{\kappa}} = \frac{\boldsymbol{U}\boldsymbol{T}^T - \boldsymbol{U}^T\boldsymbol{T}}{2L} \tag{2.210}$$

The above formula can be resolved in any basis. To get the global curvature, it can be resolve in basis $I = [\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3]$ as follow

$$\widetilde{\boldsymbol{\kappa}}_g = \left[\widetilde{\boldsymbol{\kappa}}\right]^{[I]} = \left[\frac{\boldsymbol{U}\boldsymbol{T}^T - \boldsymbol{U}^T\boldsymbol{T}}{2L}\right]^{[I]} \tag{2.211}$$

While it can be resolved in local basis E to get the local curvature $\boldsymbol{\kappa}_l$ with axes $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ (or observed by triad $\boldsymbol{E}$)

$$\boldsymbol{\kappa}_l = [\boldsymbol{\kappa}]^E = \boldsymbol{E}^T\boldsymbol{\kappa}_g \tag{2.212}$$

Where $\boldsymbol{E}$ is the rotation tensor transforming from the global basis to the element local one $\boldsymbol{E}$. Also the local curvature can be written in this form:

$$\boldsymbol{\kappa}_l = \frac{\triangle\boldsymbol{\phi}_l = \triangle\boldsymbol{\phi}_{lb} - \triangle\boldsymbol{\phi}_{\boldsymbol{la}}}{L} \tag{2.213}$$

Where the local spin (non-additive) rotation related to global one via this relation:

$$\triangle\boldsymbol{\phi}_l = \boldsymbol{E}^T\triangle\boldsymbol{\phi}_g \Leftrightarrow \widetilde{\triangle\phi_l} = \boldsymbol{E}^T\widetilde{\triangle\boldsymbol{\phi}}_g\boldsymbol{E} \tag{2.214}$$

$\triangle\boldsymbol{\phi}_a$ is the rotation from $\boldsymbol{E}$ to $\boldsymbol{T}$ basis. Using formula in Equation 2.211 results in the global components of this rotation as follows:

$$\triangle\widetilde{\boldsymbol{\phi}_{ga}} = \left[\frac{\boldsymbol{T}\boldsymbol{E^T} - \boldsymbol{E}\boldsymbol{T^T}}{2}\right]^{[I]} \tag{2.215}$$

While the local components are:

$$\triangle\widetilde{\boldsymbol{\phi}}_{la} = \boldsymbol{E}^T\widetilde{\triangle\boldsymbol{\phi}_{ga}}\boldsymbol{E} = \left[\boldsymbol{E}^T\frac{\boldsymbol{T}\boldsymbol{E^T} - \boldsymbol{E}\boldsymbol{T^T}}{2}\boldsymbol{E}\right]^{[I]} = \left[\frac{\boldsymbol{E}^T\boldsymbol{T} - \boldsymbol{T^T}\boldsymbol{E}}{2}\right]^{[I]} \tag{2.216}$$

In the same manner, if $\triangle\boldsymbol{\phi_b}$ is the rotation from $\boldsymbol{E}$ to $\boldsymbol{U}$ basis, it follows that:

$$\left[\triangle\widetilde{\boldsymbol{\phi}}_{lb} = \frac{\boldsymbol{E}^T\boldsymbol{U}-\boldsymbol{U}^T\boldsymbol{E}}{2}\right]^{[I]} \tag{2.217}$$

So the variation in rotation between the two ends $\triangle\boldsymbol{\phi}_l = \triangle\boldsymbol{\phi}_{lb} - \triangle\boldsymbol{\phi_{la}}$ will be:

$$\triangle\widetilde{\boldsymbol{\phi}}_1 = \left[\frac{(\boldsymbol{E}^T\boldsymbol{U}-\boldsymbol{U}^T\boldsymbol{E})-(\boldsymbol{E}^T\boldsymbol{T}-\boldsymbol{T}^T\boldsymbol{E})}{2}\right]^{[I]} \simeq \left[\boldsymbol{E}^T\frac{\boldsymbol{U}\boldsymbol{T}^T-\boldsymbol{U}^T\boldsymbol{T}}{2}\boldsymbol{E}\right]^{[I]} \tag{2.218}$$

The last equality comes from the fact that $\boldsymbol{U}$ is close to $\boldsymbol{T}$ for small deformation inside the beam element, so it follows:

$$\boldsymbol{U}\boldsymbol{T}^T \simeq \boldsymbol{U}\boldsymbol{E}^T + \boldsymbol{E}\boldsymbol{T}^T \tag{2.219}$$

Using Equation 2.213, the local curvature will be:

$$\widetilde{\boldsymbol{\kappa}}_l = \left[\boldsymbol{E}^T\frac{\boldsymbol{U}\boldsymbol{T}^T-\boldsymbol{U}^T\boldsymbol{T}}{2L}\boldsymbol{E}\right]^{[I]} = \boldsymbol{E}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{E} \tag{2.220}$$

Which is identical to the findings in Equation 2.211, so the assumed formula in Equation 2.213 for local curvature is right.

2.2.5 Differential form of beam curvature

As shown in Figure 2.36, assume a differential beam ds with a nodal triad $\boldsymbol{R}$ changing along the arc length to $\boldsymbol{R}+\frac{d\boldsymbol{R}}{ds}ds$ at the other end, such that $\boldsymbol{T}$ and $\boldsymbol{U}$ in the previous section are replaced with $\boldsymbol{R}$ and $\boldsymbol{R}+\boldsymbol{R}'ds$, respectively, where R' is derivative of R with respect to arc length s.

$$\widetilde{\boldsymbol{\kappa}_g} = \frac{\boldsymbol{U}\boldsymbol{T}^T-\boldsymbol{U}^T\boldsymbol{T}}{2L} = \frac{1}{2}\left(\boldsymbol{R}'\boldsymbol{R}^T-\boldsymbol{R}\boldsymbol{R}'^T\right) = \boldsymbol{R}'\boldsymbol{R}^T \tag{2.221}$$

$$\widetilde{\boldsymbol{\kappa}}_l = \boldsymbol{E}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{E} = \boldsymbol{R}^T\boldsymbol{R}'\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}^T\boldsymbol{R}' \tag{2.222}$$

The second equality comes from $\boldsymbol{E}\simeq\boldsymbol{R}$

2.2.6 Effect of nodal spin on beam curvature

As shown in Figure 2.37, a beam with initial end rotation $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is subjected to spin at ends $\delta\boldsymbol{\phi}_1$ and $\delta\boldsymbol{\phi}_2$, such that the resulting nodal rotation at the ends will be:

$$\boldsymbol{R}(\boldsymbol{\theta}_1+\delta\boldsymbol{\theta}_1) = \boldsymbol{R}(\delta\boldsymbol{\phi}_1)\boldsymbol{R}(\boldsymbol{\theta}_1) = \boldsymbol{R}(\delta\boldsymbol{\phi}_1)\boldsymbol{R} \tag{2.223}$$

Assuming for an infinitesimal beam element of length ds that $\boldsymbol{R}(\boldsymbol{\theta}_1)=\boldsymbol{R}$ and $\boldsymbol{R}(\boldsymbol{\theta}_2)=\boldsymbol{R}+\frac{d\boldsymbol{R}}{ds}ds$

$$\boldsymbol{R}(\boldsymbol{\theta}_2+\delta\boldsymbol{\theta}_2) = \boldsymbol{R}(\delta\boldsymbol{\phi}_2)\boldsymbol{R}(\boldsymbol{\theta}_2) = \boldsymbol{R}(\delta\boldsymbol{\phi}_2)\left(\boldsymbol{R}+\frac{d\boldsymbol{R}}{ds}ds\right) \tag{2.224}$$

Before inducing the nodal spin, the initial global curvature is:

$$\widetilde{\boldsymbol{\kappa}_{go}} = \boldsymbol{R}'\boldsymbol{R}^T \tag{2.225}$$

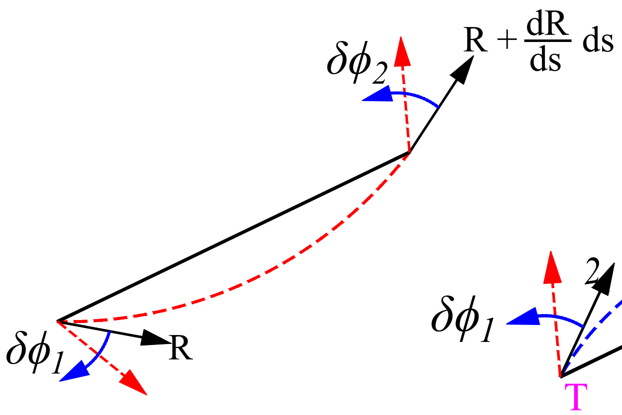


Figure 2.37

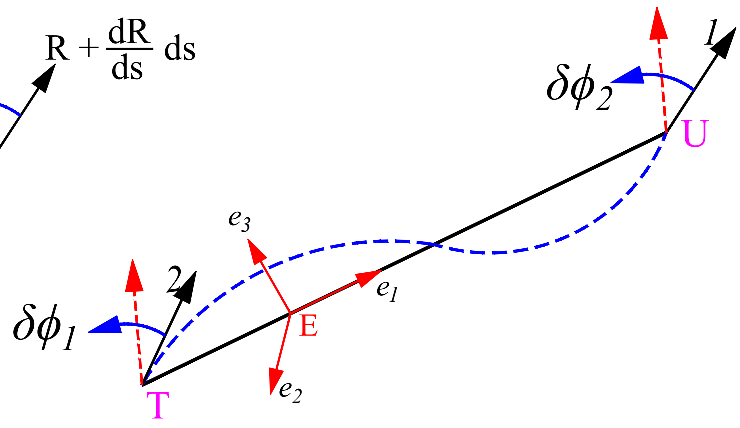


Figure 2.38

While the final global curvature will be:

$$
\begin{aligned}
\widetilde{\boldsymbol{\kappa}_{gf}} &= \frac{d\boldsymbol{R}_f}{ds}\boldsymbol{R}_f^T = \frac{\boldsymbol{R}(\boldsymbol{\theta}_2+\delta\boldsymbol{\theta}_2)-\boldsymbol{R}(\boldsymbol{\theta}_1+\delta\boldsymbol{\theta}_1)}{ds}\boldsymbol{R}^T\boldsymbol{R}(\delta\boldsymbol{\phi}_1)^T \\
&= \frac{\boldsymbol{R}(\delta\boldsymbol{\phi}_2)\left(\boldsymbol{R}+\frac{d\boldsymbol{R}}{ds}ds\right)-\boldsymbol{R}(\delta\boldsymbol{\phi}_1)\boldsymbol{R}}{ds}\boldsymbol{R}^T\boldsymbol{R}(\delta\boldsymbol{\phi}_1)^T \\
&= \frac{\boldsymbol{R}(\delta\boldsymbol{\phi}_2)\boldsymbol{R}(\delta\boldsymbol{\phi}_1)^T-1}{ds}+\boldsymbol{R}(\delta\boldsymbol{\phi}_2)\frac{d\boldsymbol{R}}{ds}\boldsymbol{R}^T\boldsymbol{R}(\delta\boldsymbol{\phi}_1)^T
\end{aligned}
\tag{2.226}
$$

Where subscripts f and o refer to the old and final state, respectively, while $\delta\boldsymbol{\phi}_1$, $\delta\boldsymbol{\phi}_2$ are infinitesimal change:

$$
\boldsymbol{R}(\delta\boldsymbol{\phi}) = \mathbf{1}+\widetilde{\delta\boldsymbol{\phi}}
\tag{2.227}
$$

$$
\begin{aligned}
\widetilde{\boldsymbol{\kappa}_{gf}} &= \frac{\left(\mathbf{1}+\widetilde{\delta\boldsymbol{\phi}_2}\right)\left(\mathbf{1}-\widetilde{\delta\boldsymbol{\phi}_1}\right)-\mathbf{1}}{ds}+\left(\mathbf{1}+\widetilde{\delta\boldsymbol{\phi}_2}\right)\frac{d\boldsymbol{R}}{ds}\boldsymbol{R}^T\left(\mathbf{1}-\widetilde{\delta\phi_1}\right) \\
&= \frac{\widetilde{\delta\boldsymbol{\phi}_2}-\widetilde{\delta\boldsymbol{\phi}_1}-\widetilde{\delta\boldsymbol{\phi}_2}\widetilde{\delta\boldsymbol{\phi}_1}}{ds}+\widetilde{\boldsymbol{\kappa}_{go}}+\widetilde{\delta\boldsymbol{\phi}_2}\widetilde{\boldsymbol{\kappa}_{go}}-\widetilde{\boldsymbol{\kappa}_{go}}\widetilde{\delta\boldsymbol{\phi}_1}-\widetilde{\delta\boldsymbol{\phi}_2}\widetilde{\boldsymbol{\kappa}_{go}}\widetilde{\delta\boldsymbol{\phi}_1}
\end{aligned}
\tag{2.228}
$$

Neglecting second order terms $(\delta\boldsymbol{\phi}_2\delta\boldsymbol{\phi}_1)$

$$
\widetilde{\boldsymbol{\kappa}_{gf}} = \frac{\widetilde{\delta\boldsymbol{\phi}_2}-\widetilde{\delta\boldsymbol{\phi}_1}}{ds}+\widetilde{\boldsymbol{\kappa}_{go}}+\widetilde{\delta\boldsymbol{\phi}_2}\widetilde{\boldsymbol{\kappa}_{go}}-\widetilde{\boldsymbol{\kappa}_{go}}\widetilde{\delta\boldsymbol{\phi}_1}
\tag{2.229}
$$

The infinitesimal change in global curvature due to end nodal spins $\delta\boldsymbol{\kappa}_g = \boldsymbol{\kappa}_{gf}-\boldsymbol{\kappa}_{go}$ is:

$$
\widetilde{\delta\boldsymbol{\kappa}_g} = \frac{\widetilde{\delta\boldsymbol{\phi}_2}-\widetilde{\delta\boldsymbol{\phi}_1}}{ds}+\widetilde{\delta\boldsymbol{\phi}_2}\widetilde{\boldsymbol{\kappa}_{go}}-\widetilde{\boldsymbol{\kappa}_{go}}\widetilde{\delta\boldsymbol{\phi}_1}
\tag{2.230}
$$

A similar expression to above can be deduced as follows:

$$
\widetilde{\boldsymbol{\kappa}_g} = \boldsymbol{R}'\boldsymbol{R}^T \Leftrightarrow \widetilde{\delta\boldsymbol{\kappa}_g} = \delta\boldsymbol{R}'\boldsymbol{R}^T+\boldsymbol{R}'\delta\boldsymbol{R}^T
\tag{2.231}
$$

Where $\delta\boldsymbol{R}$ and $\delta\boldsymbol{R}'$ are evaluated through:

$$\begin{aligned}\delta\boldsymbol{R} &= \boldsymbol{R}(\boldsymbol{\theta}+\delta\boldsymbol{\theta})-\boldsymbol{R}=\boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}-\boldsymbol{R}\\ &=\left(\mathbf{1}+\widetilde{\delta\boldsymbol{\phi}}\right)\boldsymbol{R}-\boldsymbol{R}=\widetilde{\delta\boldsymbol{\phi}}\boldsymbol{R}\rightarrow\delta\boldsymbol{R}^T=\boldsymbol{R}^T\delta\boldsymbol{\theta}^T=-\boldsymbol{R}^T\delta\boldsymbol{\theta}\end{aligned}\tag{2.232}$$

$$\delta\boldsymbol{R}'=\delta\left(\frac{d\boldsymbol{R}}{ds}\right)=\frac{d}{ds}(\delta\boldsymbol{R})=\frac{d}{ds}\left(\widetilde{\delta\boldsymbol{\phi}}\boldsymbol{R}\right)=\frac{d\widetilde{\delta\boldsymbol{\phi}}}{ds}\boldsymbol{R}+\widetilde{\delta\boldsymbol{\phi}}\left(\frac{d\boldsymbol{R}}{ds}\right)\tag{2.233}$$

Subtitling in Equation 2.231 and Equation 2.233 results in

$$\widetilde{\delta\boldsymbol{\kappa}_g}=\frac{d\widetilde{\delta\boldsymbol{\phi}}}{ds}+\widetilde{\delta\boldsymbol{\phi}}\widetilde{\boldsymbol{\kappa}_g}-\widetilde{\boldsymbol{\kappa}_g}\widetilde{\delta\boldsymbol{\phi}}\tag{2.234}$$

Which is identical to findings of Equation 2.230. if we assume that $\delta\boldsymbol{\phi}_1$, $\delta\boldsymbol{\phi}_2$ are very close to each other for an infinitesimal element

Using the identity that ($\widetilde{\boldsymbol{a}\boldsymbol{b}}=\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}-\tilde{\boldsymbol{b}}\tilde{\boldsymbol{a}}$), the infinitesimal change in curvature is related to induced nodal spin through the following expression:

$$\delta\boldsymbol{\kappa}_g=\frac{d\delta\boldsymbol{\phi}}{ds}+\widetilde{\delta\boldsymbol{\phi}}\boldsymbol{\kappa}_g\tag{2.235}$$

The corresponding infinitesimal change in beam curvature with respect to the local axes can be evaluated from the local curvature defined as:

$$\widetilde{\boldsymbol{\kappa}}_l=\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}}_g\boldsymbol{R}\tag{2.236}$$

Taking the variation results in:

$$\delta\widetilde{\boldsymbol{\kappa}}_l=\delta\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{R}+\boldsymbol{R}^T\widetilde{\delta\boldsymbol{\kappa}_g}\boldsymbol{R}+\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\delta\boldsymbol{R}\tag{2.237}$$

From Equation 2.234, Equation 2.232 and the identity $\left(\boldsymbol{R}\boldsymbol{R}^T=\mathbf{1}\rightarrow\boldsymbol{R}'\boldsymbol{R}^T+\boldsymbol{R}\boldsymbol{R}'^T=\mathbf{0}\ \&\ \delta\boldsymbol{R}\boldsymbol{R}^T+\boldsymbol{R}\delta\boldsymbol{R}^T=\mathbf{0}\right)$, we can deduce the following:

$$\begin{aligned}\delta\widetilde{\boldsymbol{\kappa}}_l &= \delta\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{R}+\boldsymbol{R}^T\left(\frac{d\widetilde{\delta\boldsymbol{\phi}}}{ds}+\widetilde{\delta\boldsymbol{\phi}}\widetilde{\boldsymbol{\kappa}_g}-\widetilde{\boldsymbol{\kappa}_g}\widetilde{\delta\boldsymbol{\phi}}\right)\boldsymbol{R}+\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\delta\boldsymbol{R}\\ &= \delta\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{R}+\boldsymbol{R}^T\left(\frac{d\widetilde{\delta\boldsymbol{\phi}}}{ds}\right)\boldsymbol{R}-\delta\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\boldsymbol{R}-\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\delta\boldsymbol{R}+\boldsymbol{R}^T\widetilde{\boldsymbol{\kappa}_g}\delta\boldsymbol{R}\\ &= \boldsymbol{R}^T\frac{d\widetilde{\delta\boldsymbol{\phi}}}{ds}\boldsymbol{R}\end{aligned}\tag{2.238}$$

Consequently, the infinitesimal change in curvature resolved in local axis (local curvature) will be:

$$\delta\boldsymbol{\kappa}_l=\boldsymbol{R}^T\frac{d\delta\boldsymbol{\phi}}{ds}\tag{2.239}$$

Equation 2.235 and Equation 2.239 can be used in formulating the geometric stiffness (predictor phase), but using them in updating curvature after each converged step results some computational errors, as they are formulated for an infinitesimal change in rotation (spin), while the incremental non-additive rotation at ends for each step is generally finite. Simo and Vu-Quoc proposed a method

to update the curvature as follows:
1. Evaluate the final rotation $\boldsymbol{R}_f$ and its derivative with respect to arc length s

$$\boldsymbol{R}_f = \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}_o \tag{2.240}$$

$$\boldsymbol{R}'_f = \boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}_o + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}'_o \tag{2.241}$$

2. Using substituting the above formulations into the global curvature expression $\widetilde{\boldsymbol{\kappa}_{gf}} = \frac{d\boldsymbol{R}_f}{ds}\boldsymbol{R}_f^T$ as follows:

$$\begin{aligned}\widetilde{\boldsymbol{\kappa}_{gf}} &= \left(\boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}_o + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}'_o\right)\boldsymbol{R}_o^T\boldsymbol{R}(\delta\boldsymbol{\phi})^T \\ &= \boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}(\delta\boldsymbol{\phi})^T + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}'_o\boldsymbol{R}_o^T\boldsymbol{R}(\delta\boldsymbol{\phi})^T \\ &= \widetilde{\boldsymbol{\kappa}_{g-add}} + \boldsymbol{R}(\delta\boldsymbol{\phi})\widetilde{\boldsymbol{\kappa}_{go}}\boldsymbol{R}(\delta\boldsymbol{\phi})^T\end{aligned} \tag{2.242}$$

Which results in:

$$\boldsymbol{\kappa}_{gf} = \boldsymbol{\kappa}_{g-add} + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{\kappa}_{go} \tag{2.243}$$

Where $\widetilde{\boldsymbol{\kappa}_{g-add}} = \boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}(\delta\boldsymbol{\phi})^T$ is evaluated from the incremental rotation induced in the current step/increment. Term $\boldsymbol{\kappa}_{g-add}$ can evaluated approximately for small values for $\delta\boldsymbol{\phi}$ through:

$$\boldsymbol{\kappa}_{g-add} = T(\delta\boldsymbol{\phi})\delta\boldsymbol{\phi}' \tag{2.244}$$

This expression is concluded from Equation 2.86 and Equation 2.90. Substituting Equation 2.2.6 into the above equation results in:

$$\boldsymbol{\kappa}_{gf} = \boldsymbol{T}(\delta\boldsymbol{\phi})\delta\boldsymbol{\phi}' + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{\kappa}_{go} \tag{2.245}$$

From above, we can deduce the following expression for beam global curvature:

$$\boxed{\Delta\boldsymbol{R} = \widetilde{\triangle\boldsymbol{\phi}}\boldsymbol{R}(\boldsymbol{\theta}) \Leftrightarrow \widetilde{\triangle\boldsymbol{\phi}} = \Delta\boldsymbol{R}.\boldsymbol{R}(\boldsymbol{\theta})^T \Leftrightarrow \triangle\boldsymbol{\phi} = T(\boldsymbol{\theta})\triangle\boldsymbol{\theta}} \tag{2.246}$$

Replacing the variation with time derivative results in:

$$\boxed{\dot{\boldsymbol{R}} = \widetilde{\dot{\boldsymbol{\phi}}}\boldsymbol{R}(\boldsymbol{\theta}) \Leftrightarrow \widetilde{\dot{\boldsymbol{\phi}}} = \dot{\boldsymbol{R}}.\boldsymbol{R}(\boldsymbol{\theta})^T \Leftrightarrow \dot{\boldsymbol{\phi}} = T(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}} \tag{2.247}$$

While differentiating with arc length s yields:

$$\boxed{\boldsymbol{R}' = \widetilde{\boldsymbol{\phi}}'\boldsymbol{R}(\boldsymbol{\theta}) \Leftrightarrow \widetilde{\boldsymbol{\phi}'} = \dot{\boldsymbol{R}}.\boldsymbol{R}(\boldsymbol{\theta})^T \Leftrightarrow \boldsymbol{\phi}' = T(\boldsymbol{\theta})\boldsymbol{\theta}'} \tag{2.248}$$

The beam curvature observed by beam element triad E (local curvature) will be:

$$\widetilde{\boldsymbol{\kappa}_{lf}} = \boldsymbol{R}_f^T\frac{d\boldsymbol{R}_f}{ds} \tag{2.249}$$

Substituting Equation 2.240 into the above equation yields:

$$\widetilde{\boldsymbol{\kappa}_{lf}} = \boldsymbol{R}_o^T\boldsymbol{R}(\delta\boldsymbol{\phi})^T\left(\boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}_o + \boldsymbol{R}(\delta\boldsymbol{\phi})\boldsymbol{R}'_o\right) \tag{2.250}$$

$$= \boldsymbol{R}_o^T\boldsymbol{R}(\delta\boldsymbol{\phi})^T\boldsymbol{R}'(\delta\boldsymbol{\phi})\boldsymbol{R}_o + \boldsymbol{R}_o^T\boldsymbol{R}'_o \tag{2.251}$$

Assuming ($\boldsymbol{\kappa}_{l-add} = \boldsymbol{R}(\delta\boldsymbol{\phi})^T\boldsymbol{R}'(\delta\boldsymbol{\phi})$) and using old local curvature ($\boldsymbol{\kappa}_{lo} = \boldsymbol{R}_o^T\boldsymbol{R}'_o$) results in:

$$\widetilde{\boldsymbol{\kappa}_{lf}} = \boldsymbol{R}_o^T\widetilde{\boldsymbol{\kappa}_{l-add}}\boldsymbol{R}_o + \widetilde{\boldsymbol{\kappa}_{lo}} \Leftrightarrow \boldsymbol{\kappa}_{lf} = \boldsymbol{R}_o^T\boldsymbol{\kappa}_{l-add} + \boldsymbol{\kappa}_{lo} \tag{2.252}$$

In the same manner, using $\boldsymbol{\kappa}_{l-add} = T(\delta\boldsymbol{\phi})^T \delta\boldsymbol{\phi}'$ approximation for small $\delta\boldsymbol{\phi}$, the final local curvature is evaluated from:

$$\boldsymbol{\kappa}_{lf} = \left(T\left(\delta\boldsymbol{\phi}\right)\boldsymbol{R}_o\right)^T \delta\boldsymbol{\phi}' + \boldsymbol{\kappa}_{lo} \tag{2.253}$$

Equation 2.245 and Equation 2.253 can be used to update the curvature after each converged step in nonlinear finite element analysis. Crisfield proposed an approximate update to the above equations using the fact that $\delta\phi$ is small during incremental step so $T\left(\delta\boldsymbol{\phi}\right)$ can be approximated using Equation 2.95 as follows:

$$\boldsymbol{T}\left(\delta\boldsymbol{\phi}\right) \simeq 1 + \frac{1}{2}\widetilde{\delta\boldsymbol{\phi}} \simeq \boldsymbol{R}\left(\frac{\delta\boldsymbol{\phi}}{2}\right) \tag{2.254}$$

Introducing a medium rotation tensor $\boldsymbol{R_m} = \boldsymbol{R}\left(\frac{\delta\phi}{2}\right)\boldsymbol{R}_o$ in Equation 2.253 results in another approximation for local curvature:

$$\boldsymbol{\kappa}_{lf} = {R_m}^T \delta\boldsymbol{\phi}' + \boldsymbol{\kappa}_{lo} \tag{2.255}$$

2.2.7 Methods of updating rotation and curvature in finite element analysis

There are two methods for updating rotation and curvature defined as follows:

1. The first method (updating on an iteration or incremental basis)
   - convergence at step **j** with following data:
     - Rotation and local curvature at Gauss points (g.p) $\overline{\boldsymbol{\kappa}}_{lj}$ - $\overline{\boldsymbol{\theta}}_j^{g.p.}$ and rotation at ends $\overline{\boldsymbol{\theta}}_j$ and the new unbalanced force vector $\boldsymbol{F}$.
     - initial: local curvature at the start of iteration phase $\boldsymbol{\kappa}_{l0} = \overline{\boldsymbol{\kappa}}_{lj}$ and rotation at Gauss points and at beam ends $\boldsymbol{\theta}_0^{g.p.} = \overline{\boldsymbol{\theta}}_j^{g.p.}$ & $\boldsymbol{\theta}_0 = \overline{\boldsymbol{\theta}}_j$ at ends - Start the iteration phase with $i = 0$
       - ■ iteration **i**
         - ★ solve $F = K\triangle$ to get $\triangle$ which includes incremental displacement and incremental spin at element nodes $\triangle\boldsymbol{\phi}$ then applying interpolations function to evaluate incremental spin and its derivative with respect to arc length s at Gauss points $\triangle\boldsymbol{\phi}^{g.p.}$ and $\triangle\boldsymbol{\phi'}^{g.p.}$.
         - ★ updating spin at ends and Gauss points:

$$\boldsymbol{R}(\boldsymbol{\theta}_{i+1}) = \boldsymbol{R}(\triangle\boldsymbol{\phi})\boldsymbol{R}(\boldsymbol{\theta}_i)$$

$$\boldsymbol{R}(\boldsymbol{\theta}_{i+1}^{g.p.}) = \boldsymbol{R}(\triangle\boldsymbol{\phi}^{\,g.p.})\boldsymbol{R}(\boldsymbol{\theta}_i^{g.p.})$$

         - ★ update the local curvature:

$$\boldsymbol{\kappa_{l(i+1)}} = \left(\boldsymbol{T}\left(\triangle\boldsymbol{\phi}_i^{g.p.}\right)\boldsymbol{R}\left(\boldsymbol{\theta}_i^{g.p.}\right)\right)^T \triangle\boldsymbol{\phi'}^{g.p.} + \boldsymbol{\kappa_{li}}$$

         - ★ Use the updated curvature to evaluate the unbalance vector force $\boldsymbol{F}$
         - ★ Stop iteration when the solution converge or the magnitude of unbalance force vector is less than the allowable or start new iteration with $i = i + 1$
     - New step curvature at Gauss points: $\overline{\boldsymbol{\kappa}}_{\boldsymbol{lj}} = \boldsymbol{\kappa_{li}}$
     - New rotations at beam ends and Gauss points:

$$\overline{\boldsymbol{\theta}}_j = \boldsymbol{\theta}_i$$

$$\overline{\boldsymbol{\theta}}_j^{g.p.} = \boldsymbol{\theta}_i^{g.p.}$$

   - Start a new step with $j = j+1$ and new external load.
2. The second method (updating on a step basis)
   - convergence at step **j** with following data:
     - Rotation and local curvature at Gauss points (g.p) $\overline{\boldsymbol{\theta}}_j^{g.p.}$ - $\overline{\boldsymbol{\kappa}}_{lj}$ and rotation at ends $\overline{\boldsymbol{\theta}}_j$ and the new unbalanced force vector $\boldsymbol{F}$.
     - initial: Null spin at ends and at Gauss points $\boldsymbol{\phi}_0^{inc} = 0$ & $\boldsymbol{\phi}_0^{inc\ g.p.} = 0$ - Start the iteration phase with $i = 0$
       - iteration **i**
         - ★ solve $F = K\triangle$ to get $\triangle$ which includes incremental displacement and incremental spin at element nodes $\triangle\boldsymbol{\phi}$ then applying interpolation functions to evaluate incremental spin at Gauss points $\triangle\boldsymbol{\phi}^{g.p.}$.
         - ★ updating spin at ends and Gauss points:

$$\boldsymbol{R}(\boldsymbol{\phi}_{i+1}^{inc}) = \boldsymbol{R}(\triangle\boldsymbol{\phi})\boldsymbol{R}(\boldsymbol{\phi}_i^{inc})$$

$$\boldsymbol{R}(\boldsymbol{\phi}_{i+1}^{inc\ g.p.}) = \boldsymbol{R}(\triangle\boldsymbol{\phi}^{\ g.p.})\boldsymbol{R}(\boldsymbol{\phi}_i^{inc\ g.p.})$$

         - ★ update local curvature:

$$\boldsymbol{\kappa}_{l(i+1)} = \left(\boldsymbol{T}\left(\boldsymbol{\phi}_i^{inc}\right)\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_j^{g.p.}\right)\right)^T \boldsymbol{\phi'}_{i+1}^{inc\ g.p.} + \overline{\boldsymbol{\kappa}}_{lj} \quad \text{where} \quad \boldsymbol{\phi'}_{i+1}^{inc\ g.p.} = \left.\frac{\partial\boldsymbol{\phi}_{i+1}^{inc\ g.p.}}{\partial s}\right|_{at g.p.}$$

         - ★ Use the updated curvature to evaluate the unbalance vector force $\boldsymbol{F}$
         - ★ Stop iteration when the solution converge or the magnitude of unbalance force vector is less than the allowable or start new iteration with $i = i+1$
     - New step curvature at Gauss points: $\overline{\boldsymbol{\kappa}}_{lj} = \boldsymbol{\kappa_{li}}$
     - New rotations at beam ends and Gauss points:

$$\boldsymbol{R}(\overline{\boldsymbol{\theta}}_{j+1}) = \boldsymbol{R}(\boldsymbol{\phi}_i^{inc})\boldsymbol{R}(\overline{\boldsymbol{\theta}}_j)$$

$$\boldsymbol{R}(\overline{\boldsymbol{\theta}}_{j+1}^{g.p.}) = \boldsymbol{R}(\boldsymbol{\phi}_i^{inc\ g.p.})\boldsymbol{R}(\overline{\boldsymbol{\theta}}_j^{g.p.})$$

     - Start a new step with $j = j+1$ and new external load.

We can use Equation 2.245 or Equation 2.255 instead of Equation 2.253 for updating curvature in both methods.

2.2.8 Beam element triad E with axes $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$

Calculating element triad $\boldsymbol{E}$ is essential step in co-rotational formulation for non-linear analysis and evaluating natural deformations which are responsible for inducing the internal stresses. There are various methods to evaluate this triad. However, these methods agree that the basis $\boldsymbol{e}_1$ of the element triad is pointed along the line connected beam ends. We will introduce three methods defined as follows:

According to Crisfield(6)

As shown in Figure 2.39, assume a medium triad $\boldsymbol{V}$ with axes $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ related to beam end triads $\boldsymbol{T}$ and $\boldsymbol{U}$ as follows:

$$\boldsymbol{U} = \boldsymbol{R}(\triangle\boldsymbol{\theta})\,T \tag{2.256}$$

$$\boldsymbol{V} = \boldsymbol{R}\left(\frac{\triangle\boldsymbol{\theta}}{2}\right)T \tag{2.257}$$

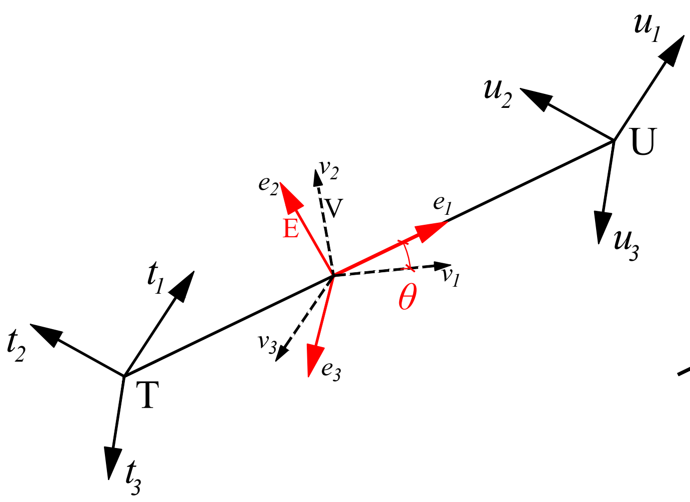


Figure 2.39

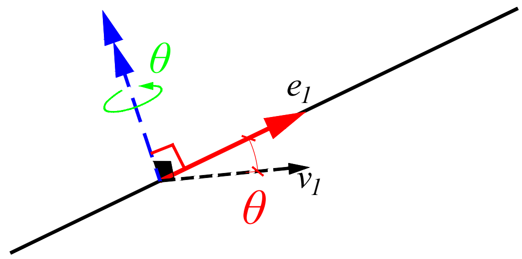


Figure 2.40

Triad $\boldsymbol{V}$ does not have to be identical to the element triad $\boldsymbol{E}$ as axis $\boldsymbol{v}_1$ is not necessary to be directed along the line connecting beam two ends, so we need to apply a rotation on triad $\boldsymbol{V}$ to transform axis $\boldsymbol{v}_1$ to axis $\boldsymbol{e}_1$ pointed to beam ends. There are an infinite number of rotation tensors to achieve this rotation, but we can choose the one with least angle of rotation. This transformation is achieved through rotating about axis $\boldsymbol{n}$ orthogonal to axes $\boldsymbol{v}_1$ and $\boldsymbol{e}_1$ as shown in Figure 2.40 with angle θ equal to the angle between these two axes as follows:

$$\cos(\theta) \;= v_1.e_1 \tag{2.258}$$

While the direction is defined as:

$$\boldsymbol{v}_1 \times \boldsymbol{e}_1 = \sin\theta\, \boldsymbol{n} \tag{2.259}$$

So the resulting rotation tensor will be:

$$\boldsymbol{R} = 1 + \sin\theta\, \widetilde{\boldsymbol{n}} + (1-\cos\theta)\, \widetilde{\boldsymbol{n}}\widetilde{\boldsymbol{n}} = 1 + \widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1} + \frac{(1-\cos\theta)}{\sin\theta^2}(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1}) \tag{2.260}$$

$$\boldsymbol{R} = 1 + \widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1} + \frac{1}{1+\cos(\theta)}(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1}) \tag{2.261}$$

Then the resulting axis $\boldsymbol{e}_2$ will be:

$$\boldsymbol{e}_2 = \boldsymbol{R}\boldsymbol{v}_2 = \boldsymbol{v}_2 + \left(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1}\right)\boldsymbol{v}_2 + \frac{1}{1+\cos(\theta)}(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})\boldsymbol{v}_2 \tag{2.262}$$

Using the following identity $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a}.\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\,\boldsymbol{a}$, we can conclude:

$$\left(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1}\right)\boldsymbol{v}_2 = (\boldsymbol{v}_1 \times \boldsymbol{e}_1) \times \boldsymbol{v}_2 = (\boldsymbol{v}_1.\boldsymbol{v}_2)\;\boldsymbol{e}_1 - (\boldsymbol{e}_1.\boldsymbol{v}_2)\;\boldsymbol{v}_1 \boldsymbol{=-}\;(\boldsymbol{e}_1.\boldsymbol{v}_2)\;\boldsymbol{v}_1 \tag{2.263}$$

$\boldsymbol{v}_1.\boldsymbol{v}_2 = \boldsymbol{0}$ as $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal to each others

$$\begin{aligned}
(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})\boldsymbol{v}_2 &= (\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})\;\;(\boldsymbol{v}_1 \times \boldsymbol{e}_1) \times \boldsymbol{v}_2 \\
&= -\,(\boldsymbol{e}_1.\boldsymbol{v}_2)\,(\widetilde{\boldsymbol{v}_1 \times \boldsymbol{e}_1})\;\boldsymbol{v}_1 \\
&= -\,(\boldsymbol{e}_1.\boldsymbol{v}_2)\,(\boldsymbol{v}_1 \times \boldsymbol{e}_1) \times \boldsymbol{v}_1
\end{aligned} \tag{2.264}$$

Using this identity $\widetilde{\tilde{a}b} = \tilde{a}\tilde{b} - \tilde{b}\tilde{a}$, it follows:

$$\widetilde{(\boldsymbol{v}_1 \times \boldsymbol{e}_1)}\widetilde{(\boldsymbol{v}_1 \times \boldsymbol{e}_1)}\boldsymbol{v}_2 = -(\boldsymbol{e}_1.\boldsymbol{v}_2)\left((\boldsymbol{v}_1.\boldsymbol{v}_1)\ \boldsymbol{e}_1 - (\boldsymbol{e}_1.\boldsymbol{v}_1)\ \boldsymbol{v}_1\right) = -(\boldsymbol{e}_1.\boldsymbol{v}_2)\left(\boldsymbol{e}_1 - \cos(\theta)\ \boldsymbol{v}_1\right) \quad (2.265)$$

If we assumed that $b_i = \boldsymbol{e}_1.\boldsymbol{v}_i$, for $i = 2,3$, we get:

$$\begin{aligned} \boldsymbol{e}_2 &= \boldsymbol{R}\boldsymbol{v}_2 \\ &= \boldsymbol{v}_2 - \boldsymbol{b}_2\boldsymbol{v}_1 + \frac{1}{1+\cos(\theta)}\left(\boldsymbol{b}_2\left(\cos(\theta)\,\boldsymbol{v}_1 - \boldsymbol{e}_1\right)\right) \\ &= \boldsymbol{v}_2 - \frac{\boldsymbol{b}_2}{1+\cos(\theta)}\left(\boldsymbol{e}_1 + \boldsymbol{v}_1\right) \end{aligned} \quad (2.266)$$

In the same manner:

$$\boldsymbol{e}_3 = \boldsymbol{v}_3 - \frac{\boldsymbol{b}_3}{1+\cos(\theta)}\left(\boldsymbol{e}_1 + \boldsymbol{v}_1\right) \quad (2.267)$$

According to Yang(12)

As shown in Figure 2.41a, the projections of beam axes $\boldsymbol{t}_{12}$, $\boldsymbol{t}_{13}$ and $\boldsymbol{t}_{22}$,$\boldsymbol{t}_{23}$ of the ends triads $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ on a plane orthogonal to $\boldsymbol{e}_1$ are defined using Equation 1.34 as follows:

$$\boldsymbol{p}_{ij} = \boldsymbol{t}_{ij} - (\boldsymbol{t}_{ij}.\boldsymbol{e}_1)\,\boldsymbol{e}_1 \quad (2.268)$$

These projections are pictured in Figure 2.41b with unit vector defined as:

$$\hat{\boldsymbol{p}}_{ij} = \frac{\boldsymbol{p}_{ij}}{|\boldsymbol{p}_{ij}|} \quad (2.269)$$

Then, as shown in Figure 2.41b, we will evaluate a medium vector $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ as follows:

$$\boldsymbol{p}_j = \hat{\boldsymbol{p}}_{1j} + \hat{\boldsymbol{p}}_{2j}, \quad j = 2,3 \quad (2.270)$$

In Figure 2.41c, we construct another two vectors $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ defined:

$$\hat{\boldsymbol{e}}_2^* = \frac{\boldsymbol{p}_2 + \boldsymbol{p}_3}{|\boldsymbol{p}_2 + \boldsymbol{p}_3|} \quad (2.271)$$

$$\hat{\boldsymbol{e}}_3^* = \frac{\boldsymbol{p}_3 - \boldsymbol{p}_2}{|\boldsymbol{p}_3 - \boldsymbol{p}_2|} \quad (2.272)$$

Then rotating axes $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ by an angle $-\pi/4$ about $\boldsymbol{e}_1$, such that the final element triad will be:

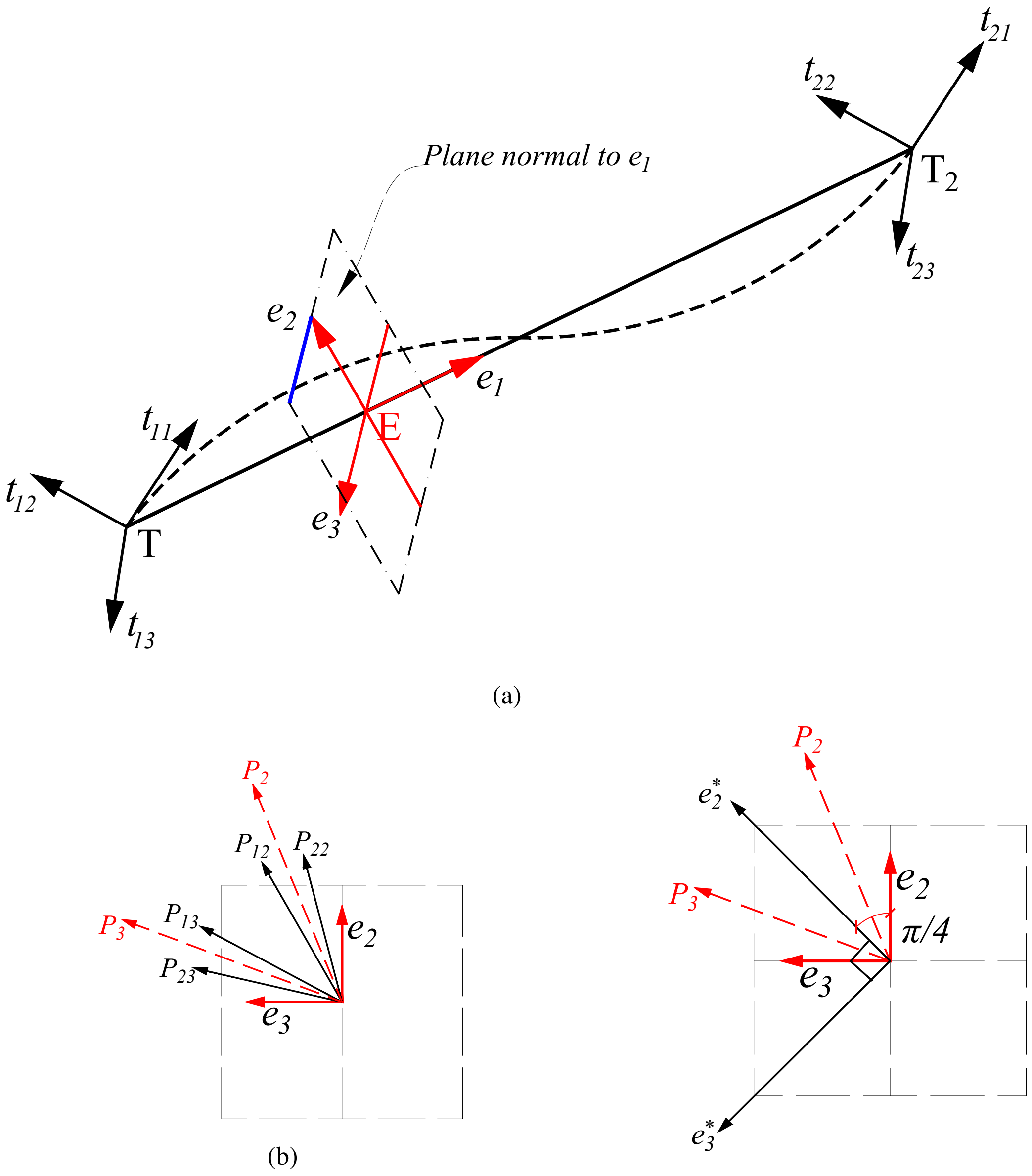
$$\boldsymbol{e}_2 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_2^* - \hat{\boldsymbol{e}}_3^*) \quad (2.273)$$

$$\boldsymbol{e}_3 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_2^* + \hat{\boldsymbol{e}}_3^*) \quad (2.274)$$

According to Battini(3)

Assume a straight beam shown in Figure 2.42 with initial triad $\boldsymbol{E}^0$ for the beam element and $\boldsymbol{T}_1^0$, $\boldsymbol{T}_2^0$ for the two ends. We can see that the three triad are identical for an initially straight beam and have equal transformation tensor as follows:

$$\boldsymbol{E}^0 = \boldsymbol{T}_1^0 = \boldsymbol{T}_2^0 = \boldsymbol{R}_0 \quad (2.275)$$



(a)

(b)

(c)

with axes defining the above rotation tensor as follows:

$$\boldsymbol{E}^0 = \left[\boldsymbol{e}_1^0, \boldsymbol{e}_2^0, \boldsymbol{e}_3^0\right], \quad \boldsymbol{T}_1^0 = \left[\boldsymbol{t}_{11}^0, \boldsymbol{t}_{12}^0, \boldsymbol{t}_{13}^0\right], \quad \boldsymbol{T}_2^0 = \left[t_{21}^0, \boldsymbol{t}_{22}^0, \boldsymbol{t}_{23}^0\right] \tag{2.276}$$

If the nodal triads at ends are rotated via $\boldsymbol{R}_{g1}$ and $\boldsymbol{R}_{g2}$, the final triads of the beam ends T_1, T_2 will be:

$$[\boldsymbol{T}_1] = [\boldsymbol{t}_{11}, \boldsymbol{t}_{12}, \boldsymbol{t}_{13}] = \boldsymbol{R}_{g1}\boldsymbol{R}_0, \quad \boldsymbol{T}_2 = [\boldsymbol{t}_{21}, \boldsymbol{t}_{22}, \boldsymbol{t}_{23}] = \boldsymbol{R}_{g2}\boldsymbol{R}_0 \tag{2.277}$$

where $\boldsymbol{t}_{ij}$ represents the j^{th} axis of nodal end i. The second axis of each nodal triads $\boldsymbol{t}_{i2}$ can be evaluated through:

$$\boldsymbol{t}_{12} = \boldsymbol{R}_{g1}\boldsymbol{t}_{12}^0 = \boldsymbol{R}_{g1}\boldsymbol{R}_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{t}_{22} = \boldsymbol{R}_{g2}\boldsymbol{t}_{22}^0 = \boldsymbol{R}_{g2}\boldsymbol{R}_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.278}$$

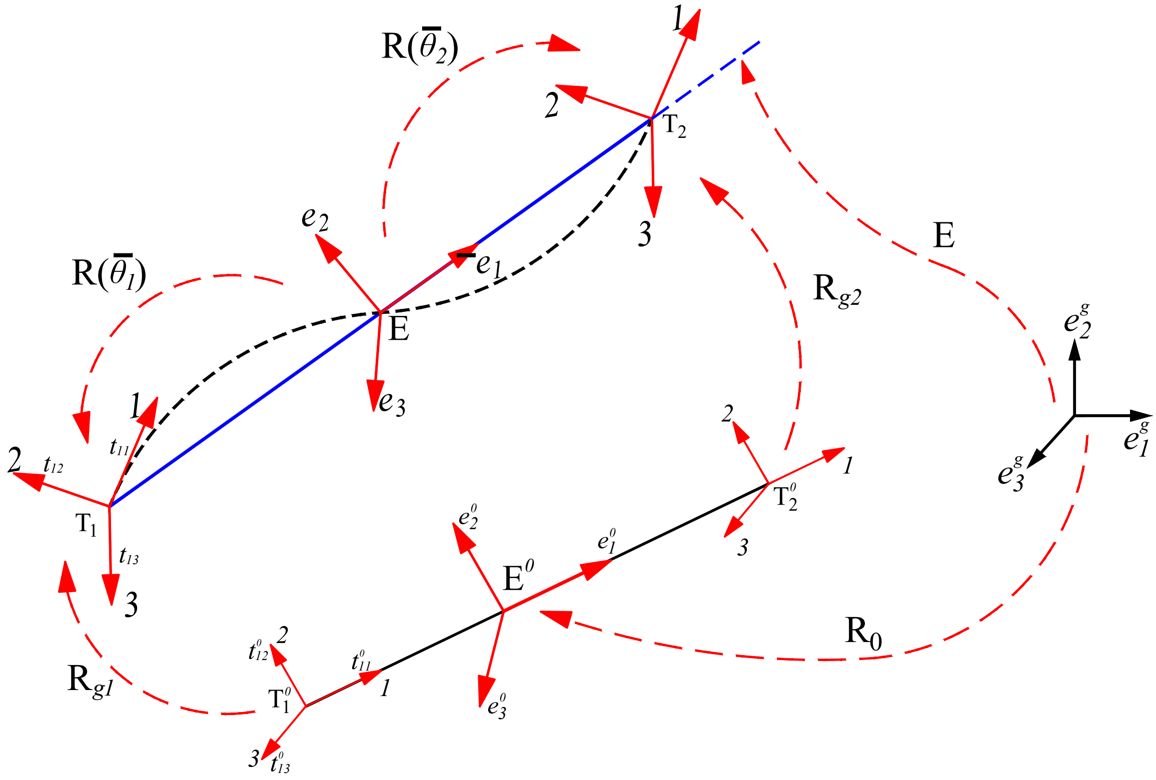


Figure 2.42: Rotation tensor $\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)$ with $i = 1,2$ defines the rotation of basis $\boldsymbol{E}$ to basis $\boldsymbol{T}_i$ ($t_{ij} = \boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)\boldsymbol{e}_j$) with $j = 1,2,3$ and defines the natural rotation deformation which is responsible for internal stresses.

The last equality in the above equation ($\boldsymbol{t}_{22}^0 = \boldsymbol{R}_0 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$) comes from the fact that $\boldsymbol{t}_{22}^0$ represents the second column of the rotation tensor $\boldsymbol{R}_0$. Defining a new vector $\boldsymbol{t}_2$ by taking the average of $\boldsymbol{t}_{12}$ and $\boldsymbol{t}_{22}$ as follows:

$$\boldsymbol{t}_2 = \frac{\boldsymbol{t}_{12} + \boldsymbol{t}_{22}}{2} \tag{2.279}$$

Vector $\boldsymbol{e}_1$ can be defined from position of beam ends, but generally $\boldsymbol{t}_2$ is not necessary pointed normal to $\boldsymbol{e}_1$. However, we can create basis $\boldsymbol{e}_2$, such that it share the same plane with basis $\boldsymbol{e}_1$ and vector $\boldsymbol{t}_2$ as shown in Figure 2.43. In this case basis $\boldsymbol{e}_3$ is orthogonal to this plane with direction defined as follows:

$$\boldsymbol{e}_3 = \frac{\boldsymbol{e}_1 \times \boldsymbol{t}_2}{|\boldsymbol{e}_1 \times \boldsymbol{t}_2|} \tag{2.280}$$

Then basis vector $\boldsymbol{e}_2$ will be:

$$\boldsymbol{e}_2 = \boldsymbol{e}_3 \times \boldsymbol{e}_1 \tag{2.281}$$

The formulated element basis $\boldsymbol{E} = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ will be evaluated.

2.3 Natural deformations

Evaluating the local (natural deformation) that is responsible for internal stresses requires removing any rigid body motion (displacement or rotation) from the beam nodal displacements. As shown in

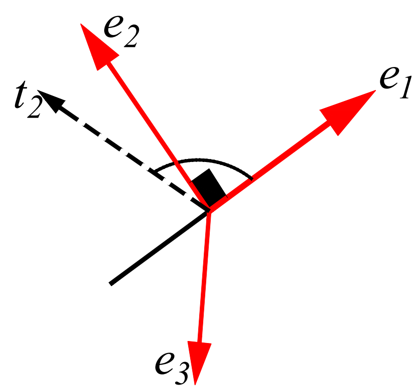


Figure 2.43

Figure 2.44, axial displacement induced in the element can be evaluated through comparing the beam length before and after deformation.As shown in Figure 2.42, after defining the element triad using one of the above three methods, we can evaluate the rotation tensors $\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)$ that transform element triad $\boldsymbol{E}$ to nodal triads $\boldsymbol{T}_i$ at beam ends resolved in global (local or element) basis $\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)\right]^{\boldsymbol{I}}$ ($\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)\right]^{\boldsymbol{E}}$) as follows:

$$\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_1\right)\right]^{\boldsymbol{I}} = \boldsymbol{T}_1\boldsymbol{E}^T = \boldsymbol{R}_{g1}\boldsymbol{R}_0\boldsymbol{E}^T \tag{2.282}$$

In the same manner:

$$\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_2\right)\right]^{\boldsymbol{I}} = \boldsymbol{R}_{g2}\boldsymbol{R}_0\boldsymbol{E}^T \tag{2.283}$$

Where frame of reference $\boldsymbol{I}$ is formed by inertia basis $\boldsymbol{e}_i^g$ shown in Figure 2.42, while the local (element) components will be:

$$\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_1\right)\right]_{\boldsymbol{E}} = \boldsymbol{E}^T\left(\boldsymbol{R}_{g1}\boldsymbol{R}_0\boldsymbol{E}^T\right)\boldsymbol{E} = \boldsymbol{E}^T\boldsymbol{R}_{g1}\boldsymbol{R}_0 \tag{2.284}$$

$$\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_2\right)\right]_{\boldsymbol{E}} = \boldsymbol{E}^T\boldsymbol{R}_{g2}\boldsymbol{R}_0 \tag{2.285}$$

Where frame of reference $\boldsymbol{E}$ is formed by element attached basis $\boldsymbol{e}_i$ shown in Figure 2.42. Generally, local end rotations $\left[\boldsymbol{R}\left(\overline{\boldsymbol{\theta}}_i\right)\right]_{\boldsymbol{E}}$ are directly responsible for beam bending stresses.

2.3.1 Variation in natural deformations

In this section, our goal is to evaluate the variation in natural deformations $\delta\boldsymbol{d}_l$ due to variation in global displacements at element nodes $\delta\boldsymbol{d}_g$ through the following equation:

$$[\delta\boldsymbol{d}_l] = \boldsymbol{B}[\delta\boldsymbol{d}_g] \tag{2.286}$$

Where $[\delta\boldsymbol{d}_l]$ and $[\delta\boldsymbol{d}_g]$ are the local natural deformation and global displacements variation in beam element, respectively. This process is done through using so-called linearization. This step is essential in deducing the geometric stiffness matrix in co-rotational formulation of beam element. In the next two subsections, we will illustrate how to evaluate $\boldsymbol{B}$ matrix for two and three dimensional beams.

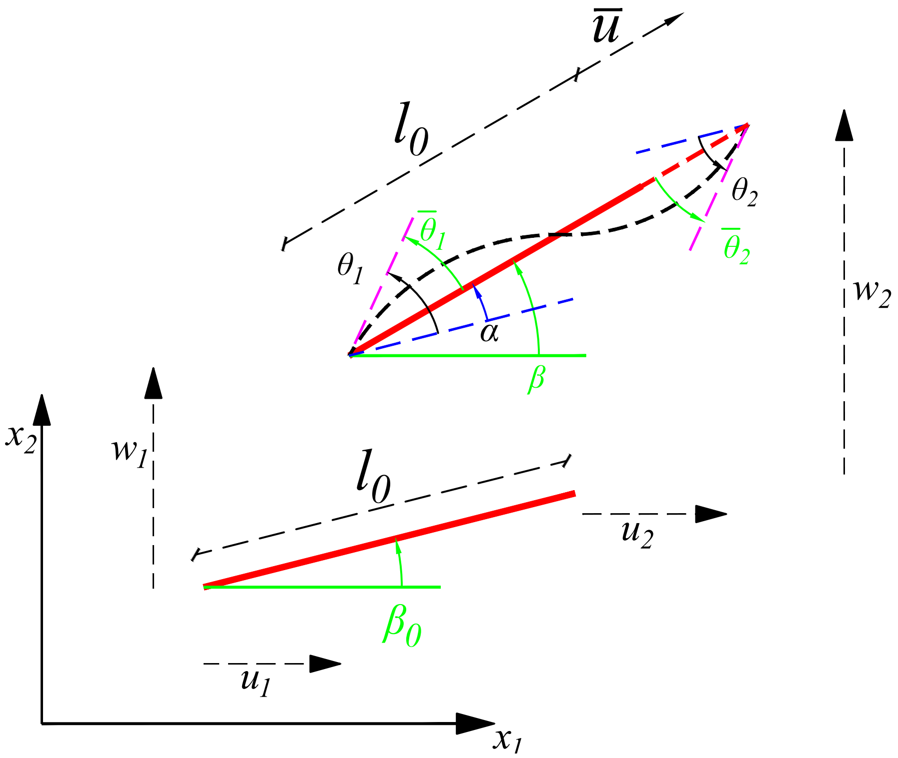


Figure 2.44: Natural deformation include axial displacement $\overline{u}$ and local end rotations $\overline{\theta}_1$, $\overline{\theta}_2$

Two dimensional beam

As shown in Figure 2.44, the deformed beam possess local (natural) deformations $\boldsymbol{d}_l$, and global displacement $\boldsymbol{d}_g$ defined as follows:

$$\boldsymbol{d}_g = \begin{bmatrix} u_1 & w_1 & \theta_1 & u_1 & w_1 & \theta_2 \end{bmatrix}^T \tag{2.287}$$

$$\boldsymbol{d}_l = \begin{bmatrix} \overline{u} & \overline{\theta_1} & \overline{\theta_2} \end{bmatrix}^T \tag{2.288}$$

Relation between $\boldsymbol{d}_l$ and $\boldsymbol{d}_g$ can be defined as follows:
The change in beam length comes from:

$$\overline{u} = l_n - l_0 \tag{2.289}$$

And the local natural rotation is defined as:

$$\overline{\theta_i} = \theta_i - \alpha \; for \; i = 1,\ 2 \tag{2.290}$$

$$\alpha = \beta - \beta_0 \tag{2.291}$$

Where

$$l_0 = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \tag{2.292}$$

$$l_n = \sqrt{(x_2 + u_2 - x_1 - u_1)^2 + (z_2 + w_2 - z_1 - w_1)^2} = \sqrt{\triangle x^2 + \triangle z^2} \tag{2.293}$$

Where $\triangle x = x_2 + u_2 - x_1 - u_1$ and $\triangle z = z_2 + w_2 - z_1 - w_1$.
Assuming the following:

$$s = \sin\beta = \frac{\triangle z}{l_n}, \;\; c = \cos\beta = \frac{\triangle x}{l_n}, \;\; s_0 = \sin\beta_o = \frac{\triangle z_0}{l_0}, \;\; c_0 = \cos\beta_0 = \frac{\triangle x_0}{l_o} \tag{2.294}$$

We get:

$$\sin\alpha = \sin(\beta - \beta_0) = \sin(\beta)\cos(\beta_0) - \cos(\beta)\sin(\beta_0) = s\, c_0 - c\, s_0 \tag{2.295}$$

In the same manner

$$\cos\alpha = c\, c_0 - c\, s_0 \tag{2.296}$$

Relation between the variation or increment in local deformation$\boldsymbol{\delta d}_l$ and global displacement $\boldsymbol{\delta d}_g$ is defined as:

$$\delta d_g = \begin{bmatrix} \delta u_1 & \delta w_1 & \delta\theta_1 & \delta u_1 & \delta w_1 & \delta\theta_2 \end{bmatrix}^T \tag{2.297}$$

$$\boldsymbol{\delta d}_l = \begin{bmatrix} \overline{\delta u} & \overline{\delta\theta_1} & \overline{\delta\theta_2} \end{bmatrix}^T \tag{2.298}$$

From Equation 2.292 and Equation 2.293, we can evaluate the variation in the axial displacement $\overline{u}$ as follows:

$$\overline{\delta u} = \delta l_n - \delta l_0 = \delta l_n \tag{2.299}$$

The variation in the initial length δl_0 is null.
As the change in beam length or axial deformation depends only on ends displacement, it follows

$$\delta l_n = \frac{\partial l_n}{\partial u_1}\delta u_1 + \frac{\partial l_n}{\partial u_2}\delta u_2 + \frac{\partial l_n}{\partial w_1}\delta w_1 + \frac{\partial l_n}{\partial w_2}\delta w_2 \tag{2.300}$$

Where:

$$\frac{\partial l_n}{\partial u_1} = \frac{(x_2 + u_2 - x_1 - u_1) * -1}{\sqrt{(x_2 + u_2 - x_1 - u_1)^2 + (z_2 + w_2 - z_1 - w_1)^2}} = \frac{-\triangle x}{l_n} = -c \tag{2.301}$$

In the same manner

$$\frac{\partial l_n}{\partial u_2} = \frac{\triangle x}{l_n} = c \tag{2.302}$$

$$\frac{\partial l_n}{\partial w_1} = \frac{-\triangle z}{l_n} = -s \tag{2.303}$$

$$\frac{\partial l_n}{\partial w_2} = \frac{\triangle z}{l_n} = s \tag{2.304}$$

So the resulting variation in beam length will be:

$$\delta l_n = -c\;\delta u_1 + c\;\delta u_2 - s\;\delta w_1 + s\;\delta w_2 = c\;\delta\left(\triangle x\right) + s\delta\left(\triangle z\right) \tag{2.305}$$

Where $\delta\left(\triangle x\right) = \delta u_2 - \delta u_1$, $\delta\left(\triangle z\right) = \delta w_2 - \delta w_1$, so the variation in beam length or axial deformation will be related to the variation in the global displacement $\boldsymbol{\delta d}_g$ as follows:

$$\overline{\delta u} = \delta l_n = \begin{bmatrix} -c & -s & 0 & c & s & 0 \end{bmatrix} \boldsymbol{\delta d}_g \tag{2.306}$$

Also we need to evaluate the variation in beam orientation $\delta\beta$. We find that it is related to the increments $\delta(\triangle x)$ and $\delta(\triangle z)$ by differentiating equation $\sin\beta = \frac{\triangle z}{l_n}$ as follows:

$$\delta\left(\sin\beta = \frac{\triangle z}{l_n}\right) \to \cos(\beta)\ \delta\beta = \frac{\delta(\triangle z)}{l_n} - \frac{\triangle z}{l_n^2}\delta l_n \tag{2.307}$$

$$= \frac{1}{l_n}\left(\delta(\triangle z) - s\ [c\ \delta(\triangle x) + s\delta(\triangle z)]\right) \tag{2.308}$$

$$= \frac{1}{l_n}\left(\delta(\triangle z)\left(1-s^2\right) - sc\ \delta(\triangle x)\right) \tag{2.309}$$

$$= \frac{1}{l_n}\left(c^2\delta(\triangle z) - sc\ \delta(\triangle x)\right) \tag{2.310}$$

The spin of the beam element orientation $\delta\beta$ will be:

$$\delta\beta = \frac{1}{l_n}\left(c\ \delta(\triangle z) - s\ \delta(\triangle x)\right) \tag{2.311}$$

$$= \frac{1}{l_n}\begin{bmatrix} s & -c & 0 & -s & c & 0 \end{bmatrix}\ \delta\boldsymbol{d}_g \tag{2.312}$$

From above equation, the infinitesimal change in beam orientation $\delta\beta$ is related directly to the variation in position of nodal coordinates $\delta(\triangle x) = \delta u_2 - \delta u_1$, $\delta(\triangle z) = \delta w_2 - \delta w_1$, while the variation in local rotations at ends results from:

$$\overline{\delta\theta_i} = \delta\theta_i - \delta\alpha = \delta\theta_i - (\delta\beta - \delta\beta_0) = \delta\theta_i - \delta\beta \qquad for\ i = 1,\ 2 \tag{2.313}$$

$$\overline{\delta\theta_1} = \delta\theta_1 - \delta\beta \tag{2.314}$$

$$= \frac{1}{l_n}\begin{bmatrix} -s & c & 1 & s & -c & 0 \end{bmatrix}\ \delta\boldsymbol{d}_g \tag{2.315}$$

$$\tag{2.316}$$

Similarly

$$\overline{\delta\theta_2} = \delta\theta_2 - \delta\beta = \frac{1}{l_n}\begin{bmatrix} -s & c & 0 & s & -c & 1 \end{bmatrix}\delta\boldsymbol{d}_g \tag{2.317}$$

So the relation between variation in local deformations and global displacements will be:

$$\delta\boldsymbol{d}_l = B\ \delta\boldsymbol{d}_g \tag{2.318}$$

Where matrix $\boldsymbol{B}$ is defined as:

$$\boldsymbol{B} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/l_n & c/l_n & 1 & s/l_n & -c/l_n & 0 \\ -s/l_n & c/l_n & 0 & s/l_n & -c/l_n & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} - \boldsymbol{b}_2 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \boldsymbol{b}_2 \end{bmatrix} \tag{2.319}$$

Where

$$\begin{aligned} \boldsymbol{b}_1 &= \begin{bmatrix} -c & -s & 0 & c & s & 0 \end{bmatrix}/l_n \\ \boldsymbol{b}_2 &= \begin{bmatrix} s & -c & 0 & -s & c & 0 \end{bmatrix}/l_n \\ \delta\beta &= \boldsymbol{b}_2\ \delta\boldsymbol{d}_g \\ \delta l_n &= \boldsymbol{b}_1\delta\boldsymbol{d}_g \end{aligned} \tag{2.320}$$

Three dimensional beam

Relation between $\boldsymbol{d}_l$ and $\boldsymbol{d}_g$ is defined as:

$$\boldsymbol{d}_g = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{\theta}_1 & \boldsymbol{d}_2 & \boldsymbol{\theta}_2 \end{bmatrix}^T \ with\ size\ 12\times 1 \tag{2.321}$$

Where the components of global displacement is defined as:

$$\boldsymbol{d}_1 = \begin{bmatrix} u_1 & v_1 & w_1 \end{bmatrix} \tag{2.322}$$
$$\boldsymbol{d}_2 = \begin{bmatrix} u_2 & v_2 & w_2 \end{bmatrix} \tag{2.323}$$
$$\boldsymbol{\theta}_1 = \begin{bmatrix} \theta_1^1 & \theta_2^1 & \theta_3^1 \end{bmatrix} \tag{2.324}$$
$$\boldsymbol{\theta}_2 = \begin{bmatrix} \theta_1^2 & \theta_2^2 & \theta_3^2 \end{bmatrix} \tag{2.325}$$

while the local (natural) deformation is:

$$\boldsymbol{d}_l = \begin{bmatrix} \overline{u} & \overline{\boldsymbol{\theta}_1} & \overline{\boldsymbol{\theta}_2} \end{bmatrix}^T \ with\ size\ 7\times 1 \tag{2.326}$$

The local axial deformation expresses the beam change in length as follows:

$$\overline{u} = l_n - l_0 \tag{2.327}$$

Where the initial and final length are defined as:

$$l_0 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{2.328}$$

$$l_n = \sqrt{(x_2+u_2-x_1-u_1)^2+(y_2+v_2-y_1-v_1)^2+(z_2+w_2-z_1-w_1)^2} = \sqrt{\triangle x^2+\triangle y^2+\triangle z^2} \tag{2.329}$$

Where $\triangle x = x_2+u_2-x_1-u_1$, $\triangle y = y_2+v_2-y_1-v_1$ and $\triangle z = z_2+w_2-z_1-w_1$. While the local rotation angles, $\overline{\boldsymbol{\theta}_i}$ observed from element triad $\boldsymbol{E}$ are defined from Equation 2.284 as follows:

$$\boldsymbol{R}\left(\overline{\boldsymbol{\theta}_i}\right) = \boldsymbol{E}^T\boldsymbol{R}_{gi}R_0 \leftrightarrow \boldsymbol{R}_{gi} = \boldsymbol{R}(\boldsymbol{\theta}_i) \quad for\ i=1,\ 2 \tag{2.330}$$

Assume a unit vector $\boldsymbol{e}_1$ along element axis with components resolved in the global frame of reference $\boldsymbol{I}$ with basis $\boldsymbol{e}_i^g$ shown in Figure 2.42 as $[\boldsymbol{e}_1]^{\boldsymbol{I}} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T$ which represents the first column of element rotation tensor $\boldsymbol{E}$ resolved in the global frame as follows:

$$[\boldsymbol{e}_1]^{\boldsymbol{I}} = \boldsymbol{E}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.331}$$

In the same manner as in Equation 2.305

$$\delta l_n = r_1\ \delta(\triangle x) + r_2\delta(\triangle y) + r_3\delta(\triangle z) = [\boldsymbol{e}_1]^T \begin{bmatrix} \delta(\triangle x) \\ \delta(\triangle y) \\ \delta(\triangle z) \end{bmatrix} \tag{2.332}$$

$$\overline{\delta u} = \begin{bmatrix} -\boldsymbol{e}_1^T & \boldsymbol{0}_{1\times 3} & \boldsymbol{e}_1^T & \boldsymbol{0}_{1\times 3} \end{bmatrix} [\delta \boldsymbol{d}_g]^{\boldsymbol{I}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{E}_4^T [\delta \boldsymbol{d}_g]^{\boldsymbol{I}}$$

(2.333)

$$\overline{\delta u} = \begin{bmatrix} -\mathbf{1}_{1\times 6} & \mathbf{1}_{1\times 6} \end{bmatrix} E_4^T \left[\delta \boldsymbol{d}_g\right]^{\boldsymbol{I}} = \boldsymbol{r} E_4^T \left[\delta \boldsymbol{d}_g\right]^{\boldsymbol{I}} \tag{2.334}$$

Where $\delta(\triangle x) = \delta u_2 - \delta u_1$, $\delta(\triangle y) = \delta v_2 - \delta v_1$, $\delta(\triangle z) = \delta w_2 - \delta w_1$, $\boldsymbol{r} = \begin{bmatrix} -\mathbf{1}_{1\times 6} & \mathbf{1}_{1\times 6} \end{bmatrix}$, $\mathbf{1}_{1\times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and $[\boldsymbol{E}_4]_I = \begin{bmatrix} \boldsymbol{E} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{E} \end{bmatrix}^{\boldsymbol{I}}$ *with size* 12×12

If the beam is displaced $\delta d_l^{12}, \delta d_l^{22}$ (displacement in the direction of current element axis e_2), the element triad $\boldsymbol{E}$ exhibits a spin rotation via a rotation about axis e_3 by angle $= \frac{\delta d_l^{22} - \delta d_l^{12}}{l_n}$, so the spin vector of the element will be:

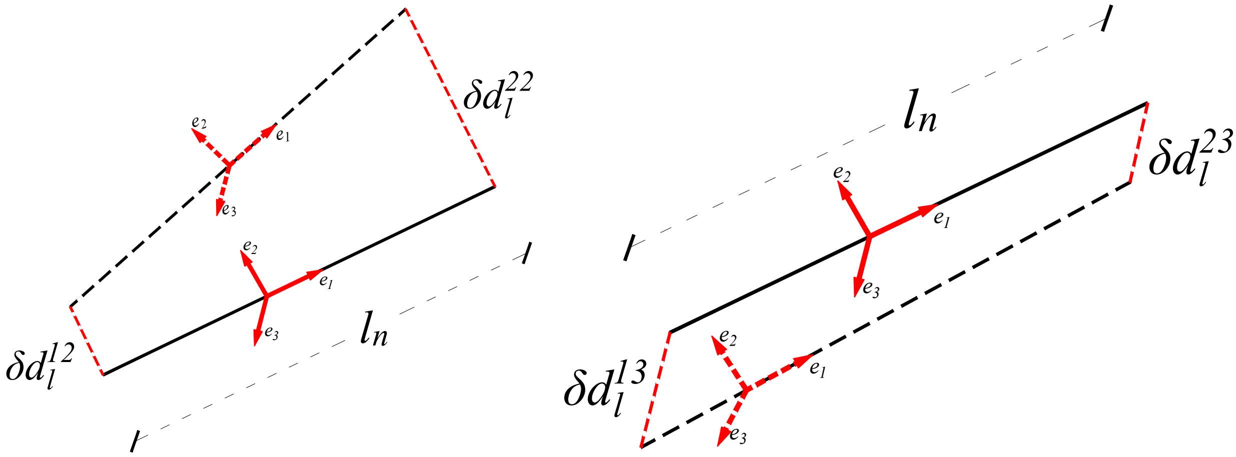


Figure 2.45: The displacement shown ar parallel to the element triads $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$

$$[\delta\phi_e^{r_3}]^{\boldsymbol{E}} = \left[0, 0, \frac{\delta d_l^{22} - \delta d_l^{12}}{l_n}\right]^T \tag{2.335}$$

$[\delta\phi_e]^{\boldsymbol{E}}$ is the spin of element resolved in basis $\boldsymbol{E}$. In the same way, if the displacement in $\boldsymbol{e}_3$ direction through $\delta d_l^{13}, \delta d_l^{23}$, the spin will be:

$$[\delta\phi_e^{r_2}]^{\boldsymbol{E}} = \left[0, \frac{\delta d_l^{13} - \delta d_l^{23}}{l_n}, 0\right]^T \tag{2.336}$$

For local nodal spin $\delta\theta_l^{11}, \delta\theta_l^{21}$, about axis $\boldsymbol{e}_1$ contributes greatly to element spin around axis e_1.

$$[\delta\phi_e^{r_1}]^{\boldsymbol{E}} = \left[\frac{\delta\phi_l^{11} + \delta\phi_l^{21}}{2}, 0, 0\right]^T \tag{2.337}$$

Using addition theorem for spin $[\delta\phi_e]^{\boldsymbol{E}} = \sum_{i=1}^{3} [\delta\phi_e^{r_i}]^{\boldsymbol{E}}$

$$[\delta\boldsymbol{\phi}_e]^{\boldsymbol{E}} = \begin{bmatrix} \frac{\delta\phi_l^{11} + \delta\phi_l^{21}}{2} \\ \frac{\delta d_l^{13} - \delta d_l^{23}}{l_n} \\ \frac{\delta d_l^{22} - \delta d_l^{12}}{l_n} \end{bmatrix} \tag{2.338}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/L_n & 0 & 0 & 0 & 0 & 0 & -1/L_n & 0 & 0 & 0 \\ 0 & -1/L_n & 0 & 0 & 0 & 0 & 0 & 1/L_n & 0 & 0 & 0 & 0 \end{bmatrix} [\delta \boldsymbol{d}_g]^{\boldsymbol{E}} \tag{2.339}$$

Where $[\delta \boldsymbol{d}_g]^{\boldsymbol{E}}$ defines the global nodal displacement but resolved im the element triad $\boldsymbol{E}$. From above, we can define the following expression:

$$[\delta \boldsymbol{\phi}_e]^{\boldsymbol{E}} = \boldsymbol{A}[\delta \boldsymbol{d}_g]^{\boldsymbol{E}} \tag{2.340}$$

Where $[\delta \boldsymbol{d}_g]^{\boldsymbol{E}}$ is defined as follows:

$$[\delta \boldsymbol{d}_g]^{\boldsymbol{E}} = \begin{bmatrix} \delta \boldsymbol{d}_1 & \delta \boldsymbol{\phi}_1 & \delta \boldsymbol{d}_2 & \delta \boldsymbol{\phi}_2 \end{bmatrix}^{\boldsymbol{E}} \tag{2.341}$$

$$= \begin{bmatrix} \delta d_l^{11} & \delta d_l^{12} & \delta d_l^{13} & \delta \phi_l^{11} & \delta \phi_l^{12} & \delta \phi_l^{13} & \delta d_l^{21} & \delta d_l^{22} & \delta d_l^{23} & \delta \phi_l^{21} & \delta \phi_l^{22} & \delta \phi_l^{23} \end{bmatrix} \tag{2.342}$$

Where δd_l^{ij} defines the displacement of beam end i in direction j parallel to element basis e_j as shown in Figure 2.45a and Figure 2.45b, while rotation vector with components resolved in the element basis $(\delta \phi_l^{i1}, \delta \phi_l^{i2}, \delta \phi_l^{i3})$ defines the end i orientation. Term $\boldsymbol{A}$ is equal to:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/L_n & 0 & 0 & 0 & 0 & 0 & -1/L_n & 0 & 0 & 0 \\ 0 & -1/L_n & 0 & 0 & 0 & 0 & 0 & 1/L_n & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.343}$$

But the components of the global displacement resolved in element frame of reference $\boldsymbol{E}$ ($[\delta \boldsymbol{d}_g]^{\boldsymbol{E}}$) are related to these resolved in the global frame $\boldsymbol{I}$ ($[\delta \boldsymbol{d}_g]^{\boldsymbol{I}}$) through the transformation rule defined as follows:

$$[\delta \boldsymbol{d}_g]^{\boldsymbol{E}} = \boldsymbol{E}_4^T [\delta \boldsymbol{d}_g]^{\boldsymbol{I}} \tag{2.344}$$

Using addition theorem, the nodal spin measured from the element triad is equal to the nodal spin measured from the global triad minus element triad spin measured from the global triad as follows:

$$\delta \overline{\boldsymbol{\phi}_i} = \delta \boldsymbol{\phi}_i - \delta \boldsymbol{\phi}_e \tag{2.345}$$

This spin can be resolved in any basis, such that if we choose the local element basis $\boldsymbol{E}$, the spin of the first nodal beam measured from element triad $\delta \overline{\boldsymbol{\phi}_1}$ is defined as:

$$[\delta \overline{\boldsymbol{\phi}_i}]^{\boldsymbol{E}} = [\delta \boldsymbol{\phi}_i]^{\boldsymbol{E}} - [\delta \boldsymbol{\phi}_e]^{\boldsymbol{E}} \tag{2.346}$$

$$= \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{1}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} [\delta \boldsymbol{d}_g]^{\boldsymbol{E}} - \boldsymbol{A}[\delta \boldsymbol{d}_g]^{\boldsymbol{E}} \tag{2.347}$$

$$= \boldsymbol{P}_1 [\delta \boldsymbol{d}_g]^{\boldsymbol{E}} \tag{2.348}$$

$$= \boldsymbol{P}_1 \boldsymbol{E}_4^T [\delta \boldsymbol{d}_g]^{\boldsymbol{I}} \tag{2.349}$$

Where $\boldsymbol{P}_1$ matrix is defined as:

$$\boldsymbol{P}_1 = \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{1}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} - \boldsymbol{A} \tag{2.350}$$

and $[\delta \boldsymbol{\phi}_{\boldsymbol{e}}]^{\boldsymbol{E}}$ can be defined as:

$$[\delta \boldsymbol{\phi}_{\boldsymbol{e}}]^{\boldsymbol{E}} = \boldsymbol{A}[\delta \boldsymbol{d}_g]^{\boldsymbol{E}} = \boldsymbol{A}\boldsymbol{E}_4^T [\delta \boldsymbol{d}_g]^{\boldsymbol{I}} \tag{2.351}$$

Where

$$\mathbf{1}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{0}_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.352}$$

In the same manner:

$$\left[\delta\overline{\boldsymbol{\phi}_2}\right]^{\boldsymbol{E}} = P_2[\delta\boldsymbol{d}_g]^{\boldsymbol{E}} = \boldsymbol{P}_2\boldsymbol{E}_4^T\,[\delta\boldsymbol{d}_g]^{\boldsymbol{I}} \tag{2.353}$$

With $\boldsymbol{P}_2$ defined as:

$$\boldsymbol{P}_2 = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{1}_{3\times3} \end{bmatrix} - \boldsymbol{A} \tag{2.354}$$

We get from above that

$$\boldsymbol{B} = \boldsymbol{P}\boldsymbol{E}_4^T \tag{2.355}$$

with

$$[\delta\boldsymbol{d}_l] = \begin{bmatrix} \delta\overline{u} & \overline{\delta\boldsymbol{\phi}_1} & \overline{\delta\boldsymbol{\phi}_2} \end{bmatrix}^T \; \textit{with size } 7\times1 \tag{2.356}$$

$$[\delta\boldsymbol{d}_g] = \begin{bmatrix} \delta\boldsymbol{d}_1 & \delta\boldsymbol{\phi}_1 & \delta\boldsymbol{d}_2 & \delta\boldsymbol{\phi}_2 \end{bmatrix} \; \textit{with size } 12\times1 \tag{2.357}$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{bmatrix} \tag{2.358}$$

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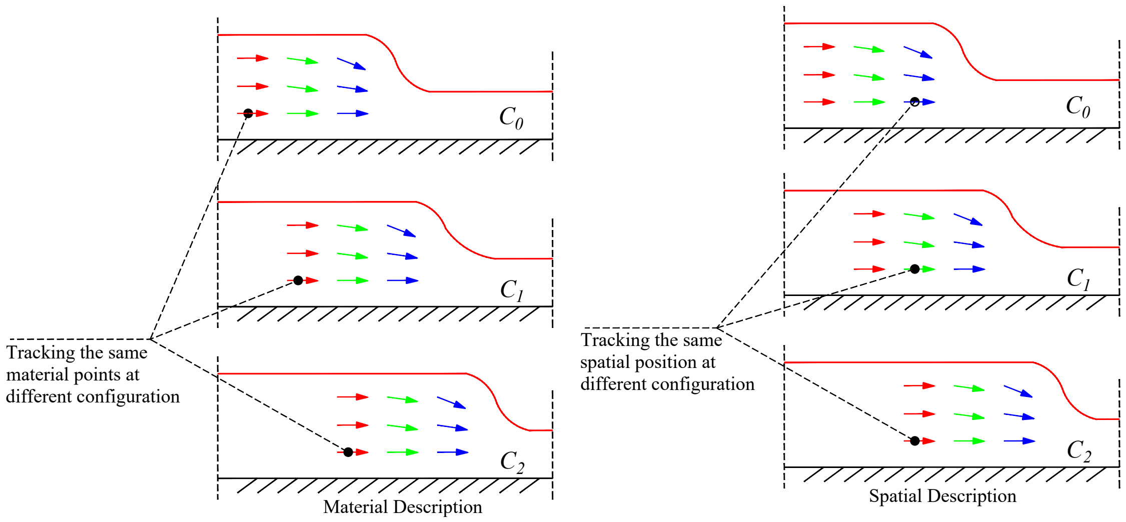
3. Introduction in Continuum Mechanics

3.1 Description of motion

Material can be described using two scale; microscopic and macroscopic scale. Microscopic scale considers that the material is discontinuous and takes into account the gap between the particles and the sliding of particles relative to each other. Continuum mechanics study the material at macroscopic level in which it is assumed that the material is continuous with no gaps, and the body completely fills the space. Also it studies the macroscopic geometric change undergone on the body under external loadings or kinematics of the body. This loading yields a geometric change and internal stresses, forcing the body to occupy continuous sequences of geometric regions. Body motion includes two types of motion; deformation and rigid body motion. Rigid body motion neither changes body shape nor contributes to internal stresses, while the deformation (change in the distance between any two particles attached on the body) is responsible for stresses. First we shall introduce some definition used commonly in continuum mechanics like configuration, material and spatial descriptions, then we will move to deformation gradient and how to separate rigid body motion out of the body motion. After that we will give different measures of strains and stresses followed by introducing an objective stress rate for nonlinear finite element.

Any continuum medium is formed by an infinite number of particles, each one occupies a particular position in space during its movement with time. Every particle attached to the body, we are interested in, is called material point, while any position in space, constant with time, is called spatial point. As a result, the location of material points changes with body motion, whereas spatial points have fixed position in space. As shown in Figure 3.1a, if we focus on a particle moving in a river, we find that it occupies different spatial positions with time, but if we are observing a particular position as shown in Figure 3.1b, we will record many material particles passing this spatial point ith time.

Also we need to introduce another definition called configuration C_t at time t which is defined as a set of positions occupied by particles of the body or the region occupied by the body in space at this time. As illustrated in Figure 3.2, anybody has a different configuration each time. The initial configuration C_0 at time $(t = 0)$ is called reference or known configuration. While the current or deformed configuration C_t defines the region occupied by the body at the current time t. As



(a) (b)

Figure 3.1

schematically shown in Figure 3.3, the position vector of a particular particle at the reference configuration is $\boldsymbol{X}$ with components $[X_1, X_2, X_3]$ referred to the spatial frame. This initial position $\boldsymbol{X}$ is called the material coordinates of the particle of label $\boldsymbol{X}$ which is a fixed property for the particle and does not change with time. The position of material points of label $\boldsymbol{X}$ in the current configuration at time t is called spatial position $\boldsymbol{x}$ with components $[x_1, x_2, x_3]$ referred to the spatial frame, such that it will be a function of material position of particle label $\boldsymbol{X}$ and time t as follows:

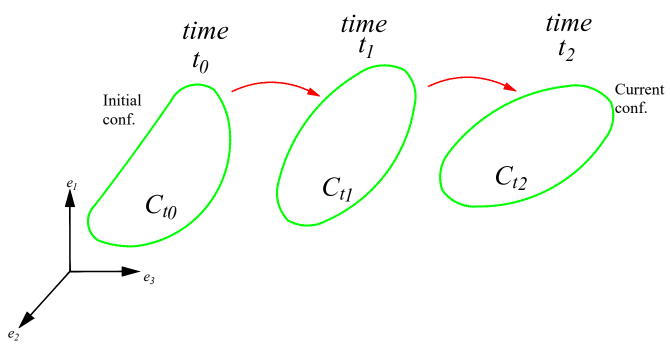


Figure 3.2

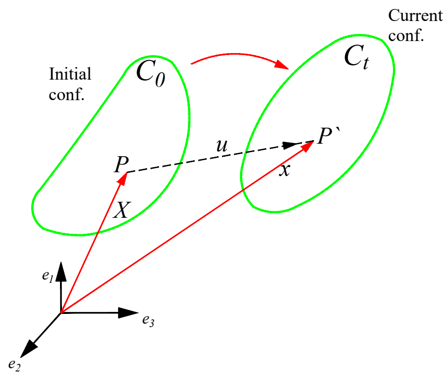
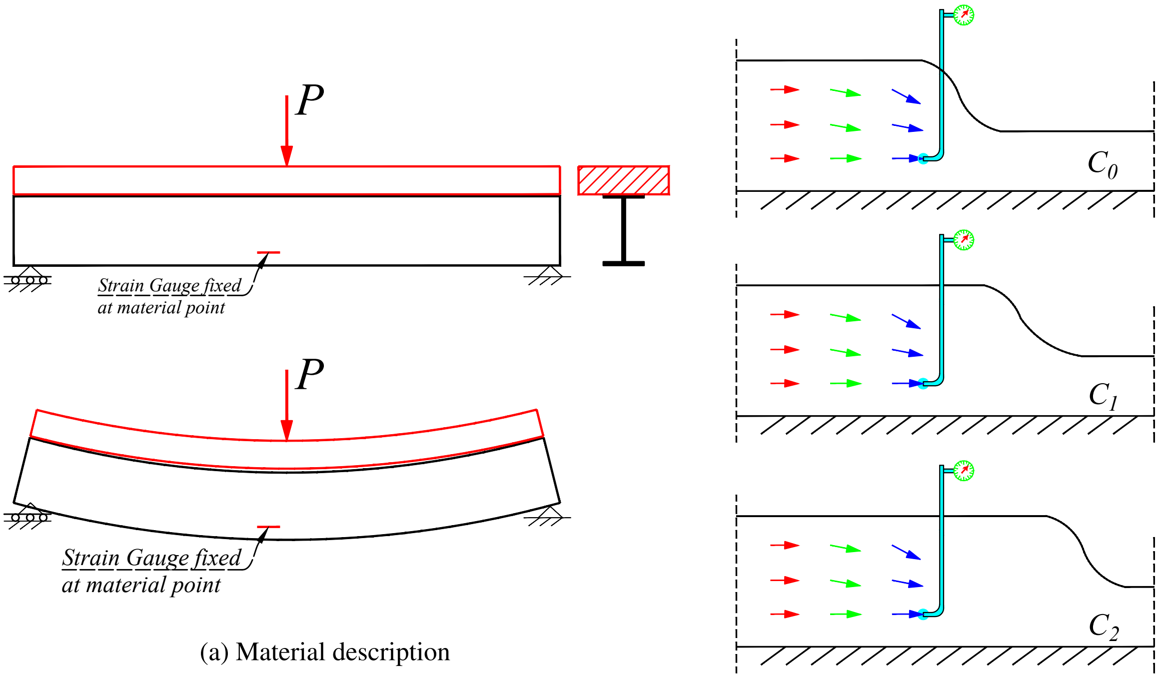


Figure 3.3

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u} \tag{3.1}$$

The above equation is called the canonical form of the equation of motion. $\boldsymbol{x}(\boldsymbol{X}, t)$ defines the current position of a particle point at time t with initial position $\boldsymbol{X}$, while $\boldsymbol{u}$ refers to the displacement displaced by the material point X from the initial configuration to the current one.

The mechanical properties of the bodies are defined using two descriptions, material and spatial description. If we are concerned with properties of a particle moving with time, we shall use the material or Lagrangian description, but if we study the properties of particles passing particular position in space, we can use the spatial or Eulerian description. For example shown in Figure 3.4a when testing a composite beam, we attach strain gauges at some points and record strain readings with loading. In this case, the description used in tracking the properties of these material points with time is Lagrangian description which is more suitable for studying solids, while an example of Eulerian description is installing velocity readers (velocity-meter) in some fixed positions in fluid channel to record its velocity with time as shown in Figure 3.4b. It is hard to track the motion of fluid particles as the case of Lagrangian description, so the better choice for fluid description is to implement Eulerian description. The general Lagrangian description for property Φ is defined as:



(a) Material description

(b) Spatial description

Figure 3.4

$$\Phi = \Phi(\boldsymbol{X}, t) \tag{3.2}$$

Which $\Phi(\boldsymbol{X}, t)$ is a function of the initial position of $\boldsymbol{X}$ and the current time t e.g. the Lagrangian description of position vector $\boldsymbol{x}$ and strain $\boldsymbol{\varepsilon}$ of a material point at time t with initial position $\boldsymbol{X}$ is given by:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{X}, t) \tag{3.3}$$

Whereas the general Eulerian description is defined as:

$$\Phi = \Phi(\boldsymbol{x}, t) \tag{3.4}$$

Which $\Phi(\boldsymbol{x}, t)$ is a function of the spatial position $\boldsymbol{x}$ recorded at it the property Φ and the time of recording t. For example, the Eulerian description of particle velocity at spatial position $\boldsymbol{x}$ and time t is given by:

$$v = v(x, t) \tag{3.5}$$

3.1.1 Time derivative

Time derivative of a property with a material description is defined as a time rate of change of a particular property as follows:

$$\frac{d\Phi(\boldsymbol{X},t)}{dt} = \frac{\partial\Phi(\boldsymbol{X},t)}{\partial t} \tag{3.6}$$

In the above expression, we equalize the total derivative $\frac{d\Phi(\boldsymbol{X},t)}{dt}$ and partial derivative $\frac{\partial\Phi(\boldsymbol{X},t)}{\partial t}$ of the property Φ as the time derivative of property Φ tracks the same particle of label $\boldsymbol{X}$, so it depends only on time, whereas the total time derivative of a property descried using a spatial description is given by:

$$\frac{d\Phi(\boldsymbol{x},t)}{dt} = \underbrace{\frac{\partial\Phi(\boldsymbol{x},t)}{\partial t}}_{Local\ derivative} + \underbrace{\frac{\partial\Phi(\boldsymbol{x},t)}{\partial \boldsymbol{x}}.\frac{\partial \boldsymbol{x}}{\partial t}}_{Convective\ derivative} \tag{3.7}$$

As the total derivative tracks the change in particle property with time, it includes two parts for spatial description; local derivative $\frac{\partial\Phi(\boldsymbol{x},t)}{\partial t}$ defined as the rate of change of the property measured at a fixed spatial position with time, and convective derivative, which compensates for the effect of particles motion at this fixed position. The convective derivative part is defined as follows:

$$\frac{\partial\Phi(\boldsymbol{x},t)}{\partial x_i}.\frac{\partial x_i}{\partial t} = \frac{\partial\Phi}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial\Phi}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial\Phi}{\partial x_3}\frac{\partial x_3}{\partial t} \tag{3.8}$$

$$\frac{\partial\Phi(\boldsymbol{x},t)}{\partial \boldsymbol{x}}.\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{\nabla}\Phi.\boldsymbol{v} \tag{3.9}$$

Where $\boldsymbol{v}$ defines the velocity of the particle passing the spatial position $\boldsymbol{x}$ and $\boldsymbol{\nabla}\Phi$ is the gradient of Φ. The above expression of time derivative does not need the current position function $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X},t)$ but the velocity of the particle and gradient of the property $\boldsymbol{\nabla}\Phi$ at particular position $\boldsymbol{x}$.

■ **Example 3.1** Let us consider a steady flow through tapered pipe shown in Figure 3.5, and we want to evaluate the time derivative of particles velocity with spatial description $\boldsymbol{v}(\boldsymbol{x},t)$. As the discharge for the steady flow is constant, the velocity recorded at any spot shall be constant with time, but if we track a particle velocity through its motion in the pipe, it increases with time due to pipe contraction. Applying the above expression, we find that the local derivative vanishes as the velocity do not change for the same spatial point for steady flow, while the convective part results in $(\boldsymbol{\nabla}\Phi.\boldsymbol{v} = \boldsymbol{\nabla}\boldsymbol{v}.\boldsymbol{v})$ which makes up for the increasing velocity of the particle with time. ■

Also we will states two definitions for volume, material volume and spatial (control) volume. The material volume generally expresses the volume of the body occupying series of configuration. The material volume has a constant mass and a varied shape or space occupation with time, while control volume has a constant shape and position with time, so the particles is expected to move in and out of it.

3.2 Deformation gradient

Let us assume a body shown in Figure 3.6 with undeformed configuration C_0 is gradually displaced to the current configuration C_t under the application of external loads body. Through this displacement, the body undergoes two different types of motion; stretch (deformation) and rigid body motion. In rigid body motion, the distance between any two particles does not change, such that

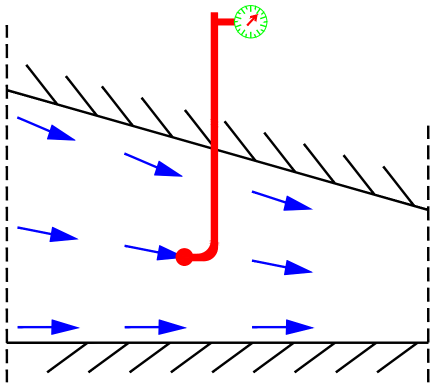


Figure 3.5: Spatial description

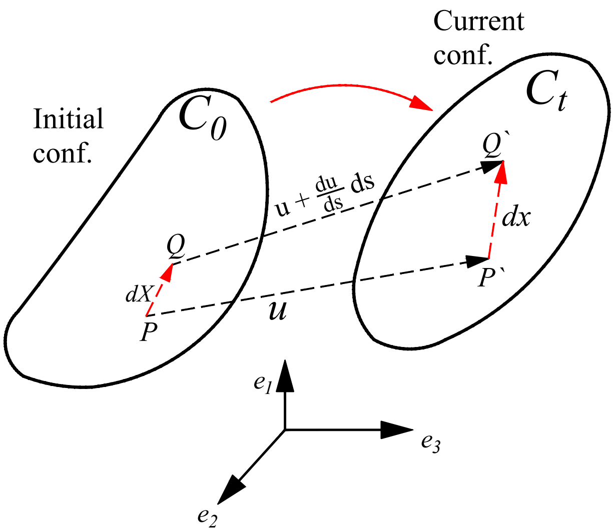


Figure 3.6: Material description

all the material particles undergo the same linear and angular displacement. Assume two arbitrary particles, P and Q embedded in the body, infinitesimally close to each other and spaced by vector $\boldsymbol{dX}$ in the undeformed configuration. After deformation, line PQ translates to line $P`Q`$, such that point P with material position $\boldsymbol{X}$ relative to global axes is translated through displacement $\boldsymbol{u}$ to point $P`$ with new position vector $\boldsymbol{x}$ defined as follows:

$$\boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X},t) \tag{3.10}$$

An infinitesimal vector $\boldsymbol{dX}$ is transformed to its deformed state $\boldsymbol{dx}$ through what is called the deformation gradient $\boldsymbol{F}$ such that the components of the new deformed vector $\boldsymbol{dx}$ can be evaluated through:

$$\begin{aligned}
dx_1 &= \frac{\partial x_1}{\partial X_1}dX_1 + \frac{\partial x_1}{\partial X_2}dX_2 + \frac{\partial x_1}{\partial X_3}dX_3 \\
dx_2 &= \frac{\partial x_2}{\partial X_1}dX_1 + \frac{\partial x_2}{\partial X_2}dX_2 + \frac{\partial x_2}{\partial X_3}dX_3 \\
dx_3 &= \frac{\partial x_3}{\partial X_1}dX_1 + \frac{\partial x_3}{\partial X_2}dX_2 + \frac{\partial x_3}{\partial X_3}dX_3
\end{aligned} \tag{3.11}$$

Where dx_i and dX_i are the components of vector $\boldsymbol{dx}$ and $\boldsymbol{dX}$ for $i = 1,2,3$. Writing these components in matrix form yields:

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} \rightarrow \boldsymbol{dx} = \boldsymbol{F}\boldsymbol{dX} \tag{3.12}$$

Deformation gradient $\boldsymbol{F}$ provides a mapping from the reference configuration C_0 to the current configuration C_t, so it can be written in this form $({}^t_0\boldsymbol{F})$. Also it provides a complete description of the displacement (excluding translations) which includes deformation and rigid body rotation. Using Equation 3.10, deformation gradient takes many forms as follows:

$$\boldsymbol{F} = \nabla_0 \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \mathbf{1} + \nabla_o \boldsymbol{u} = \mathbf{1} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \tag{3.13}$$

which Nabla operator $\boldsymbol{\nabla}_o = \frac{\partial}{\partial \boldsymbol{X}}$ operates on the initial configuration and $\boldsymbol{\nabla}_o \boldsymbol{u}$ is the displacement gradient. For infinitesimal vectors $d\boldsymbol{X}$ and $d\boldsymbol{x}$ with components defined respectively with respect to the initial or material $\boldsymbol{E}_I$ and final or spatial frame of reference $\boldsymbol{e}_i$ as $d\boldsymbol{X} = dX_I \boldsymbol{E}_I$ and $d\boldsymbol{x} = dx_i \boldsymbol{e}_i$, the index notation of the above equation will be:

$$\boldsymbol{F} = F_{iJ} \boldsymbol{e}_i \otimes \boldsymbol{E}_J \tag{3.14}$$

So the deformation gradient is called a two-point tensor as it maps between two different configurations, each one defined with respect to a particular frame of reference. The components of deformation gradient will be as follows:

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = x_{i,J} = u_{i,J} + \delta_{iJ} \tag{3.15}$$

Where $u_{i,J}$ is defined as $\frac{\partial u_i}{\partial X_J}$. While the inverse of deformation gradient is defined as:

$$\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \tag{3.16}$$

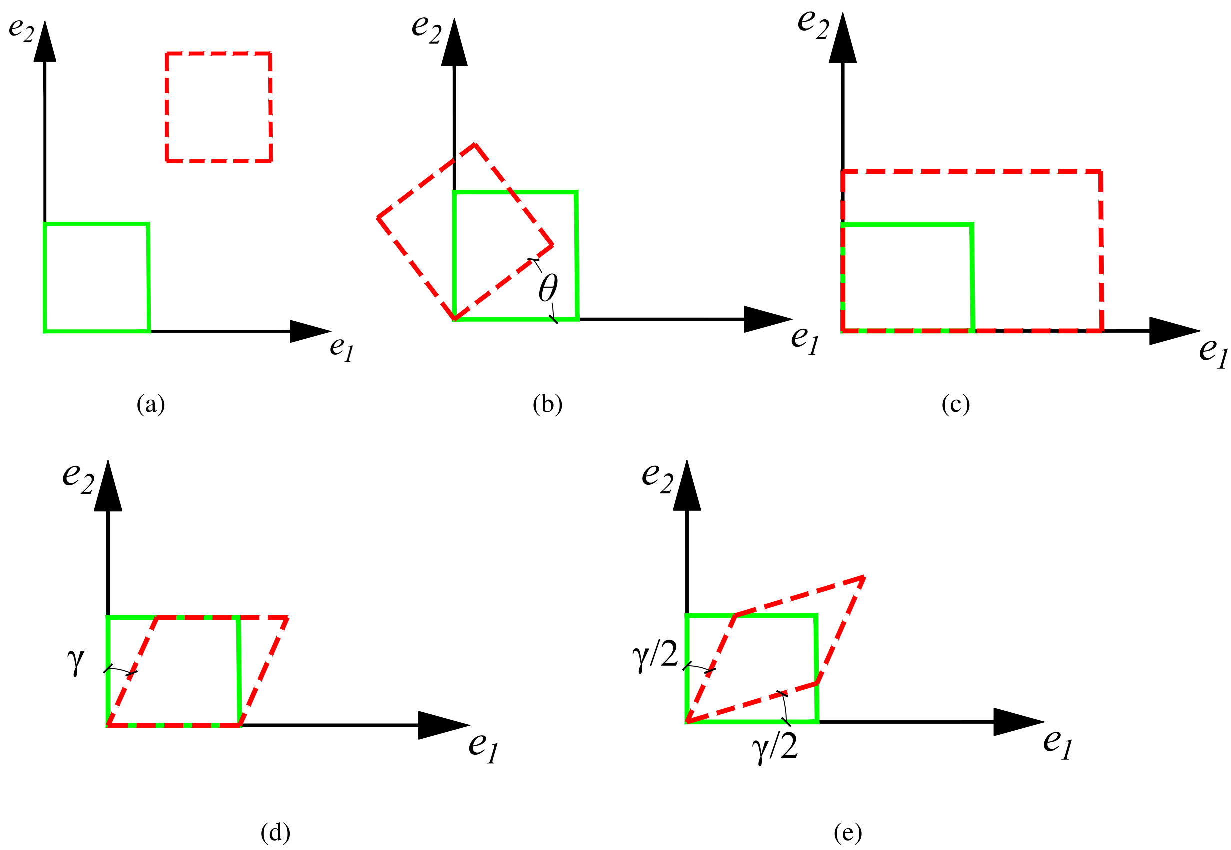


Figure 3.7

■ **Example 3.2** *Rigid body translation* shown in Figure 3.7a, the deformation gradient $\boldsymbol{F}$ will be:

$$\boldsymbol{F} = \mathbf{1}$$

Finite rotation shown in Figure 3.7b

$$[\boldsymbol{F}] = [\boldsymbol{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where $\boldsymbol{R}$ is a rotation matrix.
Pure stretching in Figure 3.7c, the deformation gradient is evaluated as follows:

$$x = 2X, \quad y = 1.5Y \quad \rightarrow \quad [\boldsymbol{F}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.17}$$

Shear with rotation in Figure 3.7d, it follows from the figure that two dimensional deformation gradient will be:

$$x = X + \gamma Y, \quad y = Y \quad \rightarrow \quad [\boldsymbol{F}] = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \tag{3.18}$$

Pure shear in Figure 3.7e, it follows that:

$$x = X + \frac{\gamma}{2}Y, \quad y = \frac{\gamma}{2}X + Y \quad \rightarrow \quad [\boldsymbol{F}] = \begin{bmatrix} 1 & 0.5\gamma \\ 0.5\gamma & 1 \end{bmatrix} \tag{3.19}$$

The un-symmetry of deformation gradient indicates that body motion contains rigid body rotation as shown in Figure 3.7b and Figure 3.7d. Off-diagonal elements in deformation gradient matrix reflect the existence of shear deformation in Figure 3.7d and Figure 3.7e which result from change of the angle between two perpendicular planes initially oriented along material frame $\boldsymbol{E}_I$. ■

3.2.1 Volume and area change

Assume an infinitesimal cubic with dimension shown in Figure 3.8 subjected to deformation gradient $\boldsymbol{F}$. Assuming the following expressions:

$$\boldsymbol{H} = \begin{bmatrix} dX_1^1 & dX_1^2 & dX_1^3 \\ dX_2^1 & dX_2^2 & dX_2^3 \\ dX_3^1 & dX_3^2 & dX_3^3 \end{bmatrix} \boldsymbol{h} = \begin{bmatrix} dx_1^1 & dx_1^2 & dx_1^3 \\ dx_2^1 & dx_2^2 & dx_2^3 \\ dx_3^1 & dx_3^2 & dx_3^3 \end{bmatrix} \tag{3.20}$$

Where dx_i^j are components of vector $d\boldsymbol{x}_i$ resolved in the global bases $\boldsymbol{e}_j$. These above matrices are related through deformation gradient as follows:

$$\boldsymbol{h} = \boldsymbol{H}\boldsymbol{F}^T \rightarrow det(\boldsymbol{h}) = det(\boldsymbol{H}\boldsymbol{F}^T) = det(\boldsymbol{H})det(\boldsymbol{F}) \tag{3.21}$$

Evaluating the volume of the cube before and after deformation dV_0, dV_1 as follows:

$$dV = d\boldsymbol{x}_1.(d\boldsymbol{x}_2 \times d\boldsymbol{x}_3) = det\left(\begin{bmatrix} dx_1^1 & dx_1^2 & dx_1^3 \\ dx_2^1 & dx_2^2 & dx_2^3 \\ dx_3^1 & dx_3^2 & dx_3^3 \end{bmatrix}\right)$$
$$= det(\boldsymbol{h}) = det(\boldsymbol{F})det(\boldsymbol{H}) = det(\boldsymbol{F})\,[d\boldsymbol{X}_1.(d\boldsymbol{X}_2 \times d\boldsymbol{X}_3)] = JdV \tag{3.22}$$

Where J is the determinant of the deformation gradient. Some formulation can be proved as follows:

$$d\boldsymbol{x}_1.(d\boldsymbol{x}_2 \times d\boldsymbol{x}_3) = d\boldsymbol{X}_1^T\boldsymbol{F}^T(\boldsymbol{F}d\boldsymbol{X}_2 \times \boldsymbol{F}d\boldsymbol{X}_3)^1 =^T \boldsymbol{F}^T\,J\boldsymbol{F}^{-T}\,(d\boldsymbol{X}_2 \times d\boldsymbol{X}_3) = J\,(d\boldsymbol{X}_1.(d\boldsymbol{X}_2 \times d\boldsymbol{X}_3))$$

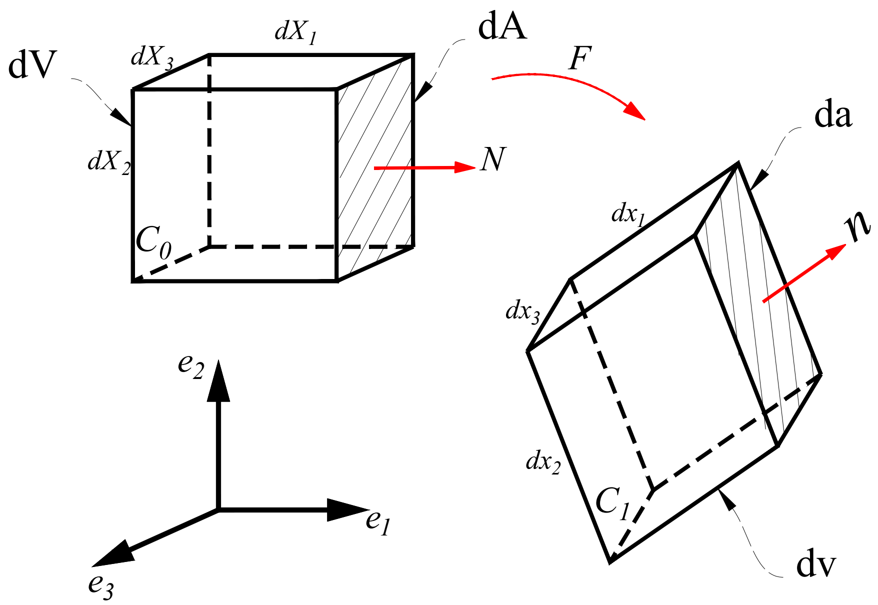


Figure 3.8

$$ \tag{3.23} $$

Which is identical to the first expression. Also infinitesimal areas before and after deformation are related as follows

$$
\begin{aligned}
dv = JdV \rightarrow d\boldsymbol{x_1}.d\boldsymbol{a} &= Jd\boldsymbol{X_1}.d\boldsymbol{A} \\
d\boldsymbol{X_1}^T\boldsymbol{F}^T d\boldsymbol{a} &= Jd\boldsymbol{X_1}^T d\boldsymbol{A} \\
d\boldsymbol{a}.d\boldsymbol{X_1} &= \left(J\boldsymbol{F}^{-T}d\boldsymbol{A}\right).d\boldsymbol{X_1} \\
d\boldsymbol{a} &= J\boldsymbol{F}^{-T}d\boldsymbol{A} \\
\boldsymbol{n}da &= J\boldsymbol{F}^{-T}\boldsymbol{N}dA
\end{aligned} \tag{3.24}
$$

Where $\boldsymbol{N}, \boldsymbol{n}$ are unit vectors normal to the areas $d\boldsymbol{A}, d\boldsymbol{a}$, respectively. This formula is called Nanson's formula.

3.2.2 Polar decomposition

As stated before, the stretch is responsible for stresses, while rigid body rotation is not, such that if we need to measure the stresses, we shall first remove rigid body rotation part out of the deformation gradient to keep only the part responsible for stresses. As schematically shown in Figure 3.9, a body is subjected to pure deformation, such that an infinitesimal line $d\boldsymbol{X}$ transforms to $d\boldsymbol{x}_1$ through what is called stretch tensor $\boldsymbol{U}$ and then the body is subjected to a rotation tensor $\boldsymbol{R}$ to yield finally $d\boldsymbol{x}$ defined as follows:

$$ d\boldsymbol{x}_1 = \boldsymbol{U}d\boldsymbol{X} \rightarrow d\boldsymbol{x} = \boldsymbol{R}d\boldsymbol{x}_1 = \boldsymbol{RU}d\boldsymbol{X} \tag{3.25} $$

So the final deformation gradient will be defined as:

$$ \boldsymbol{F} = \boldsymbol{RU} \tag{3.26} $$

As rotation tensor $\boldsymbol{R}$ does not contribute in body stress, the stretch tensor $\boldsymbol{U}$ is a symmetric tensor and responsible for the deformation and can be considered as a strain measure to evaluate body stresses. Stretch tensor can be evaluated as follows:

$$ \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{RU} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2 \tag{3.27} $$

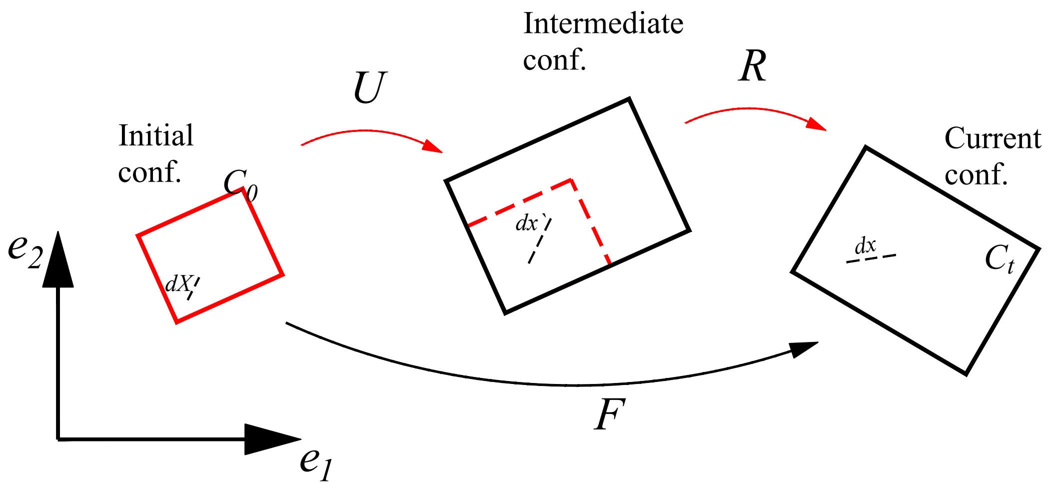


Figure 3.9

For example shown in Figure 3.10, if we have a rectangular block undergoing a pure stretch in $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions, then followed by a rotation with angle $\pi/3$, the stretch and rotation tensors can be given by:

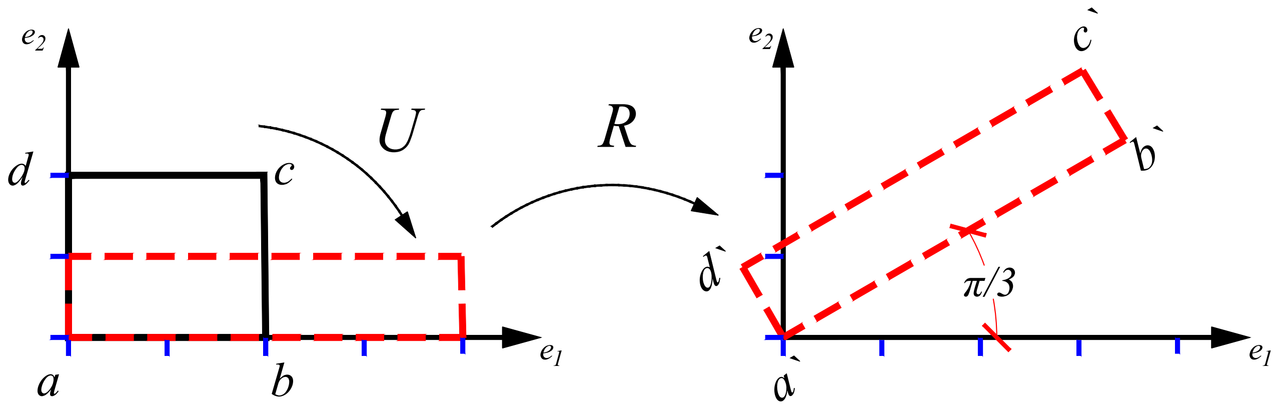


Figure 3.10

$$[\boldsymbol{U}] = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad [\boldsymbol{R}] = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \tag{3.28}$$

So the resulting deformation gradient will be:

$$[\boldsymbol{F}] = [\boldsymbol{RU}] = \begin{bmatrix} 2C & -0.5S \\ 2S & 0.5C \end{bmatrix}, \; where\ S = \sin\pi/3,\ C = \cos\pi/3 \tag{3.29}$$

We can evaluate the deformation gradient in a different way by tracking the coordinates of the new rectangular block points after deformation and comparing them with its initial positions as follows:

If the coordinate of points b and d are $(X,0)$ and $(0,Y)$, respectively, before deformation and reached to $b\grave{} = (2CX, 2SX)$ and $d\grave{} = (-0.5SY, 0.5CY)$, any general point like point c with coordinates (X,Y) transforms to point $c\grave{}$ as follows:

$$c\grave{} = b\grave{} + d\grave{} = (2CX - 0.5SY, 2SX + 0.5CY) \; or\ x = 2CX - 0.5SY \; and \;\; y = 2SX + 0.5CY \tag{3.30}$$

$$[F]_{iJ} = \left[\frac{\partial x_i}{\partial X_J}\right] = \begin{bmatrix} 2C & -0.5S \\ 2S & 0.5C \end{bmatrix} \tag{3.31}$$

Deformation gradient can be evaluated for two dimensional cases, whereas general three dimensional case needs some effort to perform polar decomposition in extracting stretch tensor $\boldsymbol{U}$ from deformation gradient $\boldsymbol{F}$.

■ **Example 3.3** Assume the deformation gradient $\boldsymbol{F}$ as follows:

$$[\boldsymbol{F}] = \begin{bmatrix} 0.415 & -0.894 & -0.208 \\ 1.009 & 0.684 & 0.004 \\ -0.1 & 0.18 & 1.165 \end{bmatrix} \tag{3.32}$$

we can evaluate $\boldsymbol{F}^T\boldsymbol{F}$ as follows

$$[\boldsymbol{F}^T\boldsymbol{F}] = [\boldsymbol{U}]^2 = \begin{bmatrix} 1.2 & 0.3 & -0.2 \\ 0.3 & 1.3 & 0.4 \\ -0.2 & 0.4 & 1.4 \end{bmatrix} \tag{3.33}$$

We can extract $\boldsymbol{U}$ from $\boldsymbol{U}^2$ through spectral decomposition as follows:

$$[\boldsymbol{F}^T\boldsymbol{F}] = \boldsymbol{A}\boldsymbol{\lambda}\boldsymbol{A}^T \tag{3.34}$$

Which $\boldsymbol{A}$, $\boldsymbol{\lambda}_i$ are the Eigen vectors matrix and Eigen values of matrix F^TF evaluated as follows:

$$[\boldsymbol{\lambda}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.69 & 0 & 0 \\ 0 & 1.45 & 0 \\ 0 & 0 & 1.76 \end{bmatrix} \tag{3.35}$$

$$[\boldsymbol{A}] = \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix} \tag{3.36}$$

So $\boldsymbol{U}$ is defined $[\boldsymbol{U}] = [\boldsymbol{A}] \left\lfloor \lambda_i^{0.5} \right\rceil [\boldsymbol{A}]^T$ as follows:

$$[\boldsymbol{U}] = \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix} \begin{bmatrix} \sqrt{0.69} & 0 & 0 \\ 0 & \sqrt{1.45} & 0 \\ 0 & 0 & \sqrt{1.76} \end{bmatrix} \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix}$$
$$= \begin{bmatrix} 1.08 & 0.14 & -0.1 \\ 0.14 & 1.12 & 0.18 \\ -0.1 & 0.18 & 1.16 \end{bmatrix} \tag{3.37}$$

Then the rotation matrix $\boldsymbol{R}$ will be:

$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1} \tag{3.38}$$

$$[\boldsymbol{U}]^{-1} = [\boldsymbol{A}] \left\lfloor \frac{1}{\lambda_i} \right\rceil [\boldsymbol{A}]^T \tag{3.39}$$

$$[\boldsymbol{U}]^{-1} = \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix} \begin{bmatrix} 1/0.83 & 0 & 0 \\ 0 & 1/1.2 & 0 \\ 0 & 0 & 1/1.33 \end{bmatrix} \begin{bmatrix} -0.58 & 0.81 & -0.12 \\ 0.63 & 0.35 & -0.7 \\ -0.52 & -0.48 & -0.71 \end{bmatrix}$$
$$= \begin{bmatrix} 0.954 & -0.141 & 0.105 \\ -0.141 & 0.94 & -0.159 \\ 0.105 & -0.159 & 0.892 \end{bmatrix} \tag{3.40}$$

$$[\boldsymbol{R}] = [\boldsymbol{F}][\boldsymbol{U}]^{-1} = \begin{bmatrix} 0.415 & -0.894 & -0.208 \\ 1.009 & 0.684 & 0.004 \\ -0.1 & 0.18 & 1.165 \end{bmatrix} \begin{bmatrix} 0.954 & -0.141 & 0.105 \\ -0.141 & 0.94 & -0.159 \\ 0.105 & -0.159 & 0.892 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.41}$$

From above calculation, using stretch tensor $\boldsymbol{U}$ as a strain measure can be tedious and time-wasting, so we will mention another strain measures in the following section.

3.2.3 Strain measure

As stated before, deformation gradient cannot be used as a strain measure as it includes rigid body rotation, while stretch tensor $\boldsymbol{U}$ can be used as a strain measure, but it requires some effort to extract. However, we can measure the strain from the change in the length between two infinitesimally-spaced points. Let us assume infinitesimal line of length ds in the deformation configuration with initial length dS at the reference configuration. The length square of a vector can be evaluated from the dot product of the vector with itself as follows:

$$ds^2 = d\boldsymbol{x}.d\boldsymbol{x} = d\boldsymbol{x}^T d\boldsymbol{x} = (\boldsymbol{F}d\boldsymbol{X})^T \boldsymbol{F}d\boldsymbol{X} = d\boldsymbol{X}^T \left(\boldsymbol{F}^T\boldsymbol{F}\right) d\boldsymbol{X} = d\boldsymbol{X}^T\boldsymbol{C}d\boldsymbol{X} \tag{3.42}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2 \tag{3.43}$$

Where $\boldsymbol{C}$ is called left Cauchy-Green tensor. It depends on the stretch tensor $\boldsymbol{U}$, and consequently excludes rigid body rotation from body motion and can be used as a strain measure. However, it yields identity matrix $\mathbf{1}$ when ds and dS are identical (no strain case), so the appropriate strain measure can be evaluated from the length change defined as follows:

$$ds^2 - dS^2 = d\boldsymbol{x}.d\boldsymbol{x} - d\boldsymbol{X}.d\boldsymbol{X} = d\boldsymbol{X}^T\boldsymbol{C}d\boldsymbol{X} - d\boldsymbol{X}^T d\boldsymbol{X} = d\boldsymbol{X}^T(\boldsymbol{C}-\mathbf{1})\,d\boldsymbol{X} \tag{3.44}$$
$$= 2d\boldsymbol{X}^T\boldsymbol{E}d\boldsymbol{X} = 2d\boldsymbol{X}.\boldsymbol{E}.d\boldsymbol{X} \tag{3.45}$$
$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{1}\right) \tag{3.46}$$

Where $\boldsymbol{E}$ is a symmetric tensor called Green-Lagrange strain. It can be evaluated in index notation as follows:

$$\boldsymbol{E} = E_{IJ}\boldsymbol{E}_I \otimes \boldsymbol{E}_J \quad where \quad E_{IJ} = \frac{1}{2}\left(F_{mI}F_{mJ} - \delta_{IJ}\right) \tag{3.47}$$

Where $\boldsymbol{E}_I$ represent vector bases of the material frame at the initial configuration for $I = 1,2,3$. Using Equation 3.15 yields:

$$E_{ij} = \frac{1}{2}\left((\delta_{ki} + u_{k,i})\left(\delta_{kj} + u_{k,j}\right) - \delta_{ij}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{3.48}$$

Or in tensor notation:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{\nabla}_o \boldsymbol{u} + \boldsymbol{\nabla}_o \boldsymbol{u}^T + \boldsymbol{\nabla}_o \boldsymbol{u}^T \boldsymbol{\nabla}_o \boldsymbol{u}\right) \tag{3.49}$$

With components defined as:

$$\begin{aligned}
E_{11} &= u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2\right) \\
E_{22} &= u_{2,2} + \frac{1}{2}\left(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2\right) \\
E_{33} &= u_{3,3} + \frac{1}{2}\left(u_{1,3}^2 + u_{2,3}^2 + u_{3,3}^2\right) \\
E_{12} &= \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) + \frac{1}{2}\left(u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}\right) = E_{21} \\
E_{13} &= \frac{1}{2}\left(u_{1,3} + u_{3,1}\right) + \frac{1}{2}\left(u_{1,1}u_{1,3} + u_{2,1}u_{2,3} + u_{3,1}u_{3,3}\right) = E_{31} \\
E_{23} &= \frac{1}{2}\left(u_{2,3} + u_{3,2}\right) + \frac{1}{2}\left(u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3}\right) = E_{32}
\end{aligned} \tag{3.50}$$

Where $u_{i,j} = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{X}_j}$.

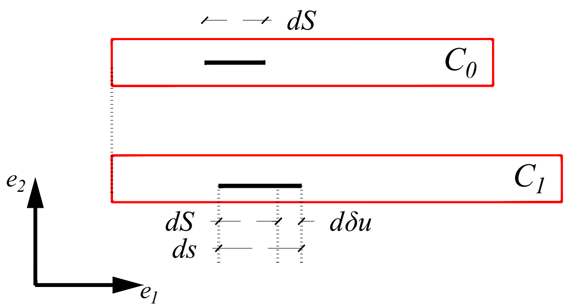


Figure 3.11

■ **Example 3.4** Lets assume an infinitesimal line attached to a bar and directed along its longitudinal as shown in Figure 3.11. The bar is stretched, such that the initial and final length of the line are dS and ds, respectively, with a change in its length of value ($d\delta u = ds - dS$), so the axial Green-Lagrange strain E_{11} using Equation 3.44 will be obtained from:

$$E_{11} = \frac{ds^2 - dS^2}{2dS^2} = \frac{(dS + d\delta u)^2 - dS^2}{2dS^2} = \frac{2\,(d\delta u)\,dS + d\delta u^2}{2dS^2} \tag{3.51}$$

Neglecting second order terms in above expression yields:

$$E_{11} \simeq \frac{d\delta u}{dS} \tag{3.52}$$

Which is similar to strain evaluate using small strain theory, so using half used in Equation 3.46 is necessary to define a physical meaning for Green-Lagrange strain. We also need to note that,

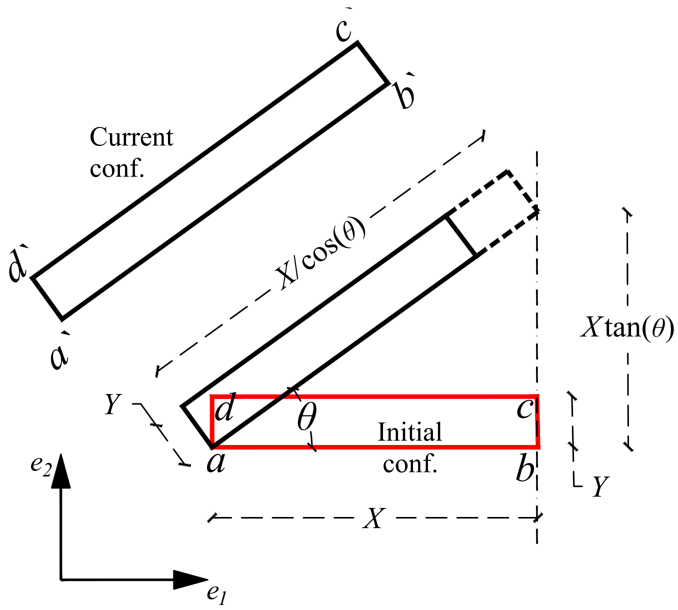


Figure 3.12

for a body undergoing small strains and large rotations, Green-Lagrange strain is very similar to stretch tensor minus identity matrix $\boldsymbol{E} \simeq \boldsymbol{U} - \mathbf{1}$. ■

■ **Example 3.5** Assume a rectangular body shown in Figure 3.12 undergoing only a finite rotation by rotating counter-wise an angle θ about axis x_3 such that the deformation gradient will be given by:

$$[\boldsymbol{F}] = [\boldsymbol{RU}] = [\boldsymbol{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.53}$$

We can conclude that Green-Lagrange strain vanishes for rigid body rotation as follows:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{1}\right) = \frac{1}{2}\left(\boldsymbol{R}^T\boldsymbol{R} - \mathbf{1}\right) = \mathbf{0} \tag{3.54}$$

■

■ **Example 3.6** Let us assume that this rectangular body is subjected uniaxial strain after rigid body rotation, such that the final configuration is Coordinate of points b, d, c before deformation will be $(X,0)$, $(0,Y)$ and (X,Y), respectively, and reached to following points:

$$\grave{b} = (1,T)X, \grave{d} = (-S,C)Y, \grave{c} = \grave{b} + \grave{d} = (X - SY, TX + CY). \tag{3.55}$$

where $T = \tan(\theta)$, $C = \cos(\theta)$ and $S = \sin(\theta)$, so the deformation gradient $\boldsymbol{F}$ stretch tensor $\boldsymbol{U}$

and rotation tensor $\boldsymbol{R}$ will be given by:

$$[\boldsymbol{F}]=\begin{bmatrix}1 & -S & 0\\ T & C & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.56}$$

$$[\boldsymbol{F}^T\boldsymbol{F}]=\begin{bmatrix}1 & T & 0\\ -S & C & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -S & 0\\ T & C & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{C^2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=[\boldsymbol{U}]^2 \tag{3.57}$$

$$[\boldsymbol{U}]=\begin{bmatrix}1/C & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.58}$$

$$[\boldsymbol{R}]=[\boldsymbol{F}][\boldsymbol{U}]^{-1}=\begin{bmatrix}1 & -S & 0\\ T & C & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}C & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}C & -S & 0\\ S & C & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.59}$$

We conclude that the body is rotated through by angle θ about origin, then subjected to a stretch through uniaxial strain of amount $1/\cos(\theta)$. ■

3.2.4 Infinitesimal strain tensor

For small displacement gradient $\nabla_o\boldsymbol{u}$, the strain tensor can be approximated by neglecting second order terms and assuming that the final configuration is very close to the initial one, such that the gradient operating on the initial and final configuration can be identical $(\nabla_o\boldsymbol{u}=\nabla\boldsymbol{u})$, so the resulting strain will be obtained from:

$$\boldsymbol{\varepsilon}=\frac{1}{2}\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right)\quad\rightarrow\quad\boldsymbol{\varepsilon}_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{3.60}$$

Where $\boldsymbol{\varepsilon}$ is a symmetric tensor called an infinitesimal strain. This strain measure can not be used for a body undergoing a finite rotation or it will introduce large errors for strain results. Engineering strain vector $\boldsymbol{\varepsilon_e}$ is identical to infinitesimal strain tensor, but its shear components are twice the shear components of the infinitesimal strain tensor as follows:

$$\boldsymbol{\varepsilon_e}=\left\{\ \varepsilon_{11}\quad\varepsilon_{22}\quad\varepsilon_{33}\quad\gamma_{12}\quad\gamma_{13}\quad\gamma_{23}\ \right\}^T\rightarrow\boldsymbol{\gamma}_{ij}=2\varepsilon_{ij}\ for\ i\neq j \tag{3.61}$$

3.2.5 Velocity gradient, rate of deformation and spin

Assume a velocity field $\boldsymbol{v}(\boldsymbol{x})$ shown in Figure 3.13, such that the change in velocity $d\boldsymbol{v}$ between two particles of the body infinitesimally-spaced by spatial vector $d\boldsymbol{x}$ measured in the deformed configuration is evaluated through:

$$d\boldsymbol{v}=\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{X}}d\boldsymbol{x}=\boldsymbol{L}d\boldsymbol{x} \tag{3.62}$$

where $\boldsymbol{L}$ is called the velocity gradient that describes the spatial rate of change of the velocity field. It can be written in index notations as follows:

$$dv_i=\frac{\partial v_i}{\partial x_j}dx_j=L_{ij}dx_j \tag{3.63}$$

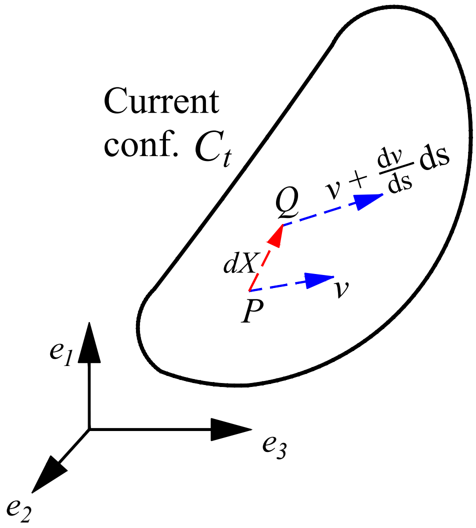


Figure 3.13

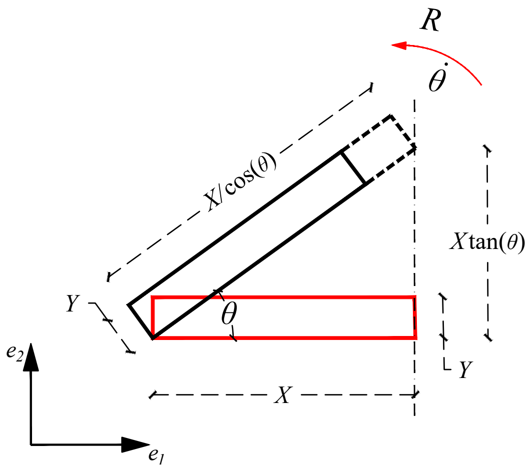


Figure 3.14

But the time rate of change of deformation gradient can be defined as:

$$\dot{\boldsymbol{F}} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right) = \frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial \boldsymbol{x}}{\partial t}\right) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{L}\boldsymbol{F} \rightarrow \boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \tag{3.64}$$

From above equation, the velocity gradient maps deformation gradient onto rate of change of deformation gradient. Generally the rate of change of deformation is implemented for nonlinear analysis, in which it uses incremental process or time rate of change. Velocity gradient can be decomposed into two parts; symmetric part called the rate of deformation tensor $\boldsymbol{D}$ and anti-symmetric part called spin or vorticity tensor $\boldsymbol{W}$ defined as follows:

$$\begin{gathered} \boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W} \\ \boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right), \;\; \boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right) \end{gathered} \tag{3.65}$$

Also from polar decomposition expression in ($\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$), time rate of change of deformation gradient will be:

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{R}}\boldsymbol{U} + \boldsymbol{R}\dot{\boldsymbol{U}} \tag{3.66}$$

And consequently, the velocity gradient and vorticity tensors $\boldsymbol{W}$ can be evaluated as follows:

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \left(\dot{\boldsymbol{R}}\boldsymbol{U} + \boldsymbol{R}\dot{\boldsymbol{U}}\right)\boldsymbol{U}^{-1}\boldsymbol{R}^T = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T \tag{3.67}$$

$$\begin{aligned} \boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right) &= \frac{1}{2}\left(\dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T - \boldsymbol{R}\dot{\boldsymbol{R}}^T - \boldsymbol{R}\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\right)^T\boldsymbol{R}^T\right) \\ &= \dot{\boldsymbol{R}}\boldsymbol{R}^T + \frac{1}{2}\boldsymbol{R}\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - \left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\right)^T\right)\boldsymbol{R}^T \end{aligned} \tag{3.68}$$

As the rotation tensor is orthogonal $\left(\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{1}\right)$, we can derive that:

$$\boldsymbol{R}\dot{\boldsymbol{R}}^T = -\dot{\boldsymbol{R}}\boldsymbol{R}^T \leftrightarrow \boldsymbol{\Omega} = \boldsymbol{R}\dot{\boldsymbol{R}}^T = \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{v} \tag{3.69}$$

Where $\boldsymbol{\Omega}$ is the angular velocity tensor, which depend on rigid body rotation and its time rate of change. From above, we can express vorticity tensor $\boldsymbol{W}$ as follows:

$$\boldsymbol{W} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{R}\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - \left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\right)^T\right)\boldsymbol{R}^T \tag{3.70}$$

Generally term $\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - (\dot{\boldsymbol{U}}\boldsymbol{U}^{-1})^T\right)$ has a negligible value and vorticity and angular velocity tensor can be considered approximately equal ($\boldsymbol{W} \simeq \boldsymbol{\Omega}$). We can also express the relation between time rate of change of Green-Lagrange strain tensor and rate of deformation tensor as follows:

$$\dot{\boldsymbol{E}} = \frac{1}{2}\left[\boldsymbol{F}^T\dot{\boldsymbol{F}} + \dot{\boldsymbol{F}}^T\boldsymbol{F}\right] = \frac{1}{2}\left[\boldsymbol{F}^T\boldsymbol{L}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{L}^T\boldsymbol{F}\right] = \frac{1}{2}\boldsymbol{F}^T\left[\boldsymbol{L} + \boldsymbol{L}^T\right]\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F} \tag{3.71}$$

$$\boldsymbol{D} = \boldsymbol{F}^{-T}\dot{\boldsymbol{E}}\boldsymbol{F}^{-1} \tag{3.72}$$

In some textbooks, rate change $\dot{\boldsymbol{E}}$ is defined as a push back to rate of deformation tensor $\boldsymbol{D}$ while $\boldsymbol{D}$ is considered as a push forward to $\dot{\boldsymbol{E}}$. Also using polar decomposition expression ($\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$), time rate of change of Green-Lagrange strain tensor $\dot{\boldsymbol{E}}$ is obtained from:

$$\begin{aligned}\dot{\boldsymbol{E}} &= \frac{1}{2}\left[\boldsymbol{F}^T\dot{\boldsymbol{F}} + \dot{\boldsymbol{F}}^T\boldsymbol{F}\right] = \frac{1}{2}\left[\boldsymbol{U}\boldsymbol{R}^T\left(\dot{\boldsymbol{R}}\boldsymbol{U} + \boldsymbol{R}\dot{\boldsymbol{U}}\right) + \left(\boldsymbol{U}\dot{\boldsymbol{R}}^T + \dot{\boldsymbol{U}}^T\boldsymbol{R}\right)\boldsymbol{R}\boldsymbol{U}\right] \\ &= \frac{1}{2}\left[\boldsymbol{U}\underline{\boldsymbol{R}^T\dot{\boldsymbol{R}}}\boldsymbol{U} + \boldsymbol{U}\dot{\boldsymbol{U}} + \boldsymbol{U}\underline{\dot{\boldsymbol{R}}^T\boldsymbol{R}}\boldsymbol{U} + \left(\boldsymbol{U}\dot{\boldsymbol{U}}\right)^T\right]\end{aligned} \tag{3.73}$$

As the underlined terms cancel each other, the final expression of $\dot{\boldsymbol{E}}$ will be:

$$\dot{\boldsymbol{E}} = \frac{1}{2}\left[\boldsymbol{U}\dot{\boldsymbol{U}} + \left(\boldsymbol{U}\dot{\boldsymbol{U}}\right)^T\right] = sym(\boldsymbol{U}\dot{\boldsymbol{U}}) \tag{3.74}$$

■ **Example 3.7** Lets assume a rectangular body shown in Figure 3.14, stretching and rotating with constant angular velocity $\dot{\theta}$ such that the time rate of change of current stretch and rotation tensor can be obtained using Equation 3.58 and Equation 3.6 as follows:

$$[\dot{\boldsymbol{U}}] = \dot{\theta}\begin{bmatrix} S/C^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\dot{\boldsymbol{R}}] = \dot{\theta}\begin{bmatrix} -S & -C & 0 \\ C & -S & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.75}$$

where $C = \cos(\theta)$, $S = \sin(\theta)$.

$$[\dot{\boldsymbol{U}}][\boldsymbol{U}]^{-1} = \dot{\theta}\begin{bmatrix} S/C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.76}$$

$$[\boldsymbol{L}] = [\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}] = [\dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T] = \dot{\theta}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dot{\theta}\begin{bmatrix} SC & S^2 & 0 \\ S^2 & S^3/C & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.77}$$

$$[\boldsymbol{D}] = [\boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T] = \dot{\theta}\begin{bmatrix} SC & S^2 & 0 \\ S^2 & S^3/C & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.78}$$

Also rate of deformation tensor $\boldsymbol{D}$ is known as a push forward to tensor $\dot{U}U^{-1}$, the verticity tensor will be:

$$[\boldsymbol{W}] = \dot{\theta}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \widetilde{\boldsymbol{\omega}} \leftrightarrow \boldsymbol{\omega} = \left[0, 0, \dot{\theta}\right]^T \tag{3.79}$$

From above example, we find $\boldsymbol{D}$ and $\boldsymbol{W}$ are identical and another expression for rate of deformation tensor $\boldsymbol{D}$ is approximated as follows:

$$\boldsymbol{D} = \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T \tag{3.80}$$

■

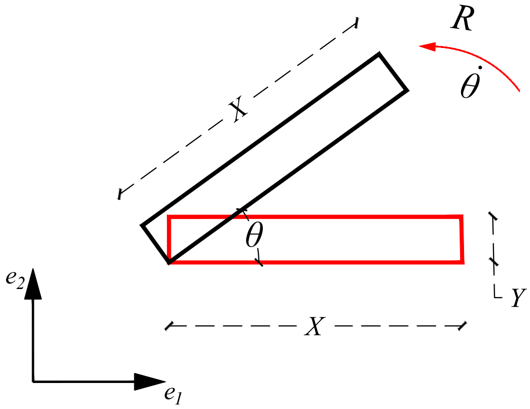


Figure 3.15

■ **Example 3.8** If a rectangular body shown in Figure 3.15 is rotating with angular velocity $\dot{\theta}$ without axial strain, the deformation gradient and rate of deformation tensors at any configuration orientated at angle θ are given by:

$$[F] = [\boldsymbol{RU}] = [\boldsymbol{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.81}$$

$$\boldsymbol{D} = \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}\boldsymbol{R}^T = \boldsymbol{0} \tag{3.82}$$

From the last equality in the above equation, we can use rate of deformation tensor $\boldsymbol{D}$ in nonlinear geometric analysis as it depends on the time rate of change of stretch tensor $\dot{\boldsymbol{U}}$ and vanishes for rigid body rotation. ■

3.3 Introduction to stress analysis

As schematically shown in Figure 3.16, Let us assume a bar with rectangular section of area A subjected to axial load P, such that the stress distribution σ induced on a cut plane normal to the cross section is defined as follows:

$$\sigma = \frac{P}{A} \tag{3.83}$$

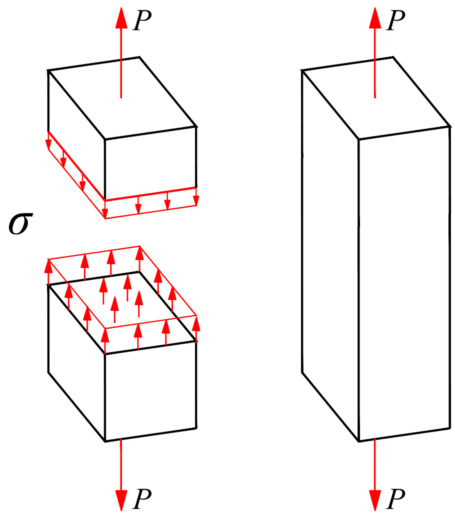


Figure 3.16

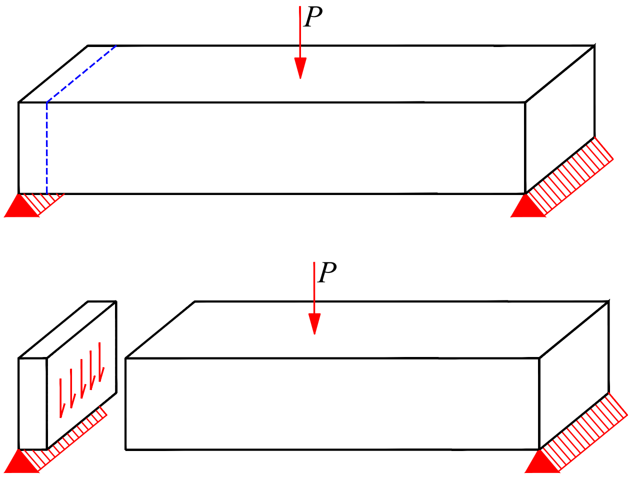


Figure 3.17

As the force is normal to the cut section, the stresses induced are normal stresses, while shear stresses are tangent to the section cut as the case of two hinged beam with normal section cut near the support, as shown in Figure 3.17. Complexity arises if we choose another cut plane with normal axis different from the force vector direction. For example, if the cut plane is oriented at angle θ relative to the plane normal to force vector, as shown in Figure 3.18, the new cut plane has surface area equal to $A/\cos(\theta)$. From equilibrium, Force normal to the cut plane equals to $P\cos(\theta)$ resulting normal stresses σ_n given by:

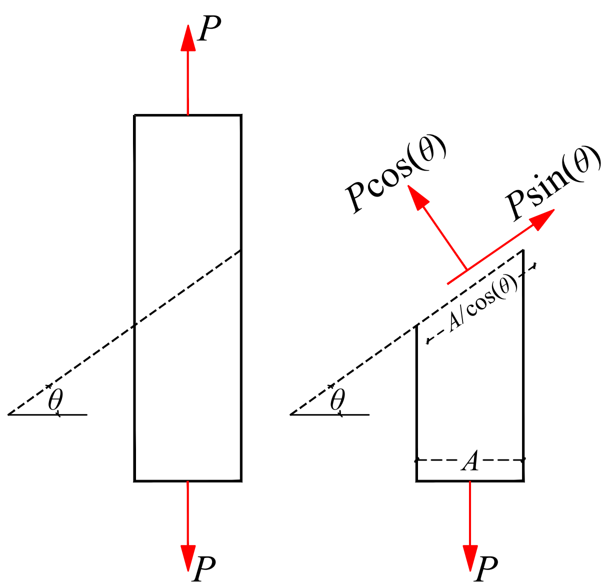


Figure 3.18

$$\sigma_n = (P\cos(\theta))/(A/\cos(\theta)) = \frac{P}{A}\cos(\theta)^2 \tag{3.84}$$

While force tangent to the surface equals to $P\sin(\theta)$ resulting shear stress τ_n obtained from:

$$\tau_n = \frac{P}{A}\sin(\theta)\cos(\theta) \tag{3.85}$$

These results are identical to the findings of Mohr's circle. Also using axes transformation form axes x_i to x_ì, shown in Figure 3.19, leads to the followings:

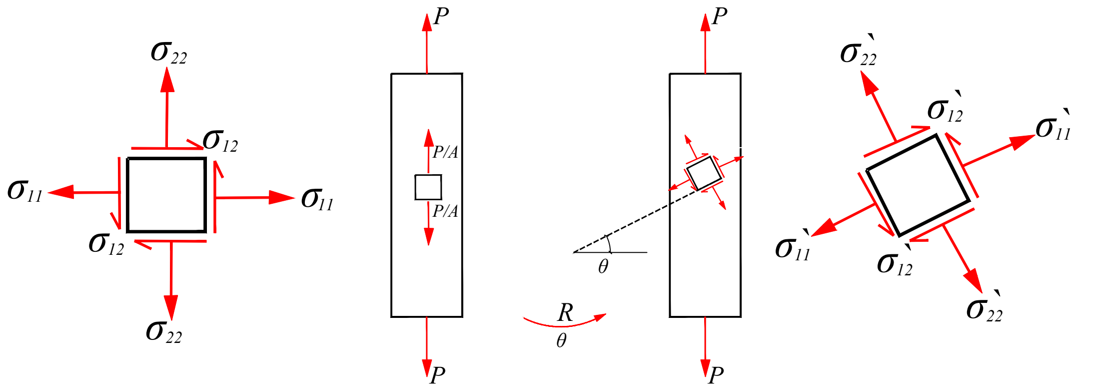


Figure 3.19

$$\grave{\boldsymbol{\sigma}} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T \tag{3.86}$$

With:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}, \quad [\boldsymbol{Q}] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3.87}$$

The transformed stress tensor will be:

$$[\grave{\boldsymbol{\sigma}}] = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} \sigma\sin(\theta)^{2} & \sigma\sin(\theta)\cos(\theta) \\ \sigma\sin(\theta)\cos(\theta) & \sigma\cos(\theta)^{2} \end{bmatrix} \tag{3.88}$$

3.3.1 Stress vector

Let's assume a body subjected to external forces (body or surface forces) shown in Figure 3.20, and a cut plane with normal direction n is used to divide the body into two parts. Focusing on an infinitesimal area located on the cut plane $\triangle A$ it will be subjected to small force vector $\triangle \boldsymbol{F}$ such that the stress vector or surface traction acting on this area will be:

$$\boldsymbol{t}^{(\boldsymbol{n})} = \left(\frac{\triangle \boldsymbol{F}}{\triangle A}\right)_{\triangle A \to 0} \tag{3.89}$$

Superscript $(\boldsymbol{n})$ means that the stress vector is associated with plane $\boldsymbol{n}$. Stress vector has two components; normal stress σ normal to the section cut, and shear stress τ tangent to the cut section. If we change the orientation of the cut section, it will result in different stress vector as concluded from the previous example. Also, at any point, there is an infinite number of section planes at this point, such that each one has its own stress vector, but tracking the stress vectors associated with three perpendicular or independent planes is enough to define the stress state at that point. These three planes with three different components of stress vector associated with each plane can be combined together in what is called dyadic or second order stress tensor of nine elements shown on rectangular block as shown in Figure 3.21 and expressed as follow:

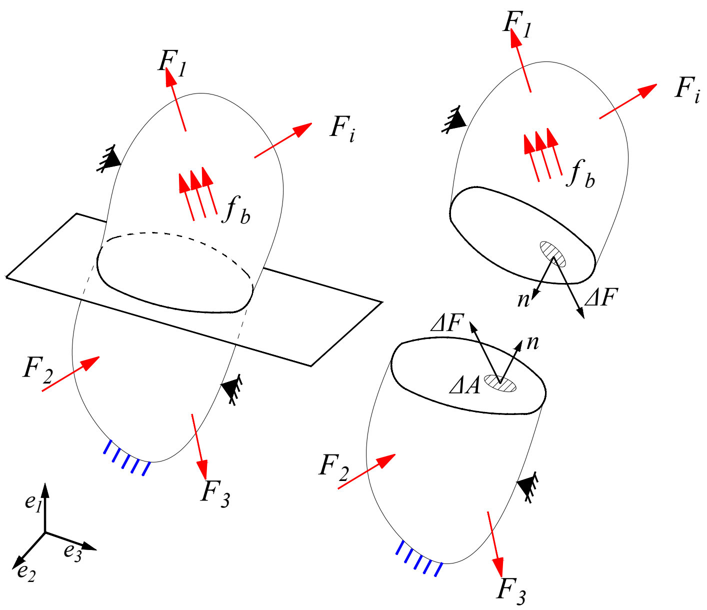


Figure 3.20

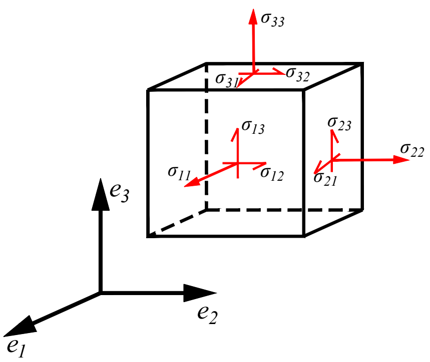


Figure 3.21

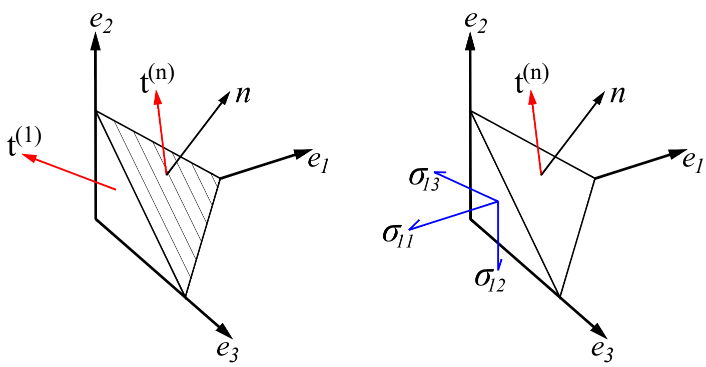


Figure 3.22

$$\boldsymbol{\sigma} = \ \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.90}$$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.91}$$

We shall exhibit here how to extract stress vector $\boldsymbol{t}^{(\boldsymbol{n})}$ associated with plane $\boldsymbol{n}$ from stress tensor $\boldsymbol{\sigma}$. Assume a rectangular block shown in Figure 3.22, with plane cut with normal $\boldsymbol{n}$ with area equal to A and surface traction $\boldsymbol{t}^{(n)}$, while the traction force associated with plane normal to axis $\boldsymbol{e}_i$ can be defined as $\boldsymbol{t}^{(i)}$, for $i = 1,\ 2,\ 3$ defined as :

$$\begin{aligned} \boldsymbol{t}^{(1)} &= \{\ \sigma_{11} \quad \sigma_{12} \quad \sigma_{13}\ \}^T \\ \boldsymbol{t}^{(2)} &= \{\ \sigma_{21} \quad \sigma_{22} \quad \sigma_{23}\ \}^T \\ \boldsymbol{t}^{(3)} &= \{\ \sigma_{31} \quad \sigma_{32} \quad \sigma_{33}\ \}^T \end{aligned} \tag{3.92}$$

Figure 3.22 shows the components of stress vector $\boldsymbol{t}^{(1)}$. We can evaluate the area of each side $A^{(i)}$ normal to axis x_i through the projection of area A on each side as follows:

$$A^{(i)} = (\boldsymbol{n})^{(i)}.A\boldsymbol{n} = \left((\boldsymbol{n})^{(i)}.\boldsymbol{n}\right)A^2 \tag{3.93}$$

The unit vectors normal to each surface shown in Figure 3.22 and resolved in the global frame is given by:

$$\boldsymbol{n} = \left\{ \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \right\}^T, \;\; \boldsymbol{n}^{(1)} = \left\{ \begin{array}{ccc} 1 & 0 & 0 \end{array} \right\}^T, \;\; (\boldsymbol{n})^{(2)} = \left\{ \begin{array}{ccc} 0 & 1 & 0 \end{array} \right\}^T, \;\; (\boldsymbol{n})^{(3)} = \left\{ \begin{array}{ccc} 0 & 0 & 1 \end{array} \right\}^T \tag{3.94}$$

And consequently,

$$A^{(1)} = An_1, \;\; A^{(2)} = An_2, \;\; A^{(3)} = An_3 \tag{3.95}$$

Applying equilibrium over the this part of rectangular block in Figure 3.22 results in:

$$\boldsymbol{t}^{(\boldsymbol{n})}A = \boldsymbol{t}^{(1)}An_1 + \boldsymbol{t}^{(2)}An_2 + \boldsymbol{t}^{(3)}An_3 \tag{3.96}$$

Dividing by the area A yields:

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{t}^{(1)}n_1 + \boldsymbol{t}^{(2)}n_2 + \boldsymbol{t}^{(3)}n_3 = \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{array} \right\} n_1 + \left\{ \begin{array}{c} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{array} \right\} n_2 + \left\{ \begin{array}{c} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{array} \right\} n_3 \tag{3.97}$$

$$= \left[\begin{array}{ccc} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{array} \right] \left\{ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \right\} \tag{3.98}$$

The above matrix form can be rewritten in tensor or index notation as follows:

$$\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{\sigma}^T\boldsymbol{n} = \boldsymbol{n}.\boldsymbol{\sigma} = \boldsymbol{n}^T\boldsymbol{\sigma} \tag{3.99}$$

The above equation is called Cauchy formula. The components of stress vector $\boldsymbol{t}^{(\boldsymbol{n})} = t_i\boldsymbol{e}_i$ on plane $\boldsymbol{n} = n_i\boldsymbol{e}_i$ are defined in index notation from above equation as follows:

$$t_i = \sigma_{ji}n_j \tag{3.100}$$

3.3.2 Conservation of linear and angular momentum

Conservation of linear momentum or Newton's second law of motion states that the time rate of change of linear momentum ($m\boldsymbol{v}$) of a particle of mass m and velocity $\boldsymbol{v}$ equals to the net force $\sum\boldsymbol{F}$ exerted on this particle as follows:

$$\frac{d}{dt}(m\boldsymbol{v}) = \sum\boldsymbol{F} \tag{3.101}$$

If its mass is constant with time, the above expression reduces to:

$$m\frac{d}{dt}(\boldsymbol{v}) = m\frac{\partial^2\boldsymbol{x}}{\partial t^2} = m\boldsymbol{a} = \sum\boldsymbol{F} \tag{3.102}$$

Where $\boldsymbol{a}$ and $\boldsymbol{x}$ are particle acceleration and position. Generally the forces are divided into two parts; internal and external forces. The internal forces result from stresses induced in the cut plane, while external forces include body forces and surface forces. Body forces act on mass distribution like inertia, gravity, electromagnetic forces and are generally measured per unit mass, so if the body force per unit mass is $\boldsymbol{f}_b$, the total body force $\boldsymbol{F}_b$ will be obtained from:

$$\boldsymbol{F}_b = \int_V \rho \boldsymbol{f}_b dV \tag{3.103}$$

And consequently, the inertia force $\boldsymbol{F}_I$ is given by:

$$\boldsymbol{F}_I = \int_V \rho \frac{\partial^2 \boldsymbol{x}}{\partial t^2} dV \tag{3.104}$$

While the surface traction $\boldsymbol{t}^{(\boldsymbol{n})}$ includes the forces acting on the boundary surface of the body and measured per unit area with normal vector $\boldsymbol{n}$, e.g. contact forces, such that the total surface body $\boldsymbol{F}_s$ can be evaluated through integrating surface traction over the area as follows:

$$\boldsymbol{F}_s = \int_S \boldsymbol{t}^{(\boldsymbol{n})} dA = \int_S \boldsymbol{n}.\boldsymbol{\sigma} dA \tag{3.105}$$

From divergence theorem, the above expression can be rewritten in this form:

$$\boldsymbol{F}_s = \int_V \nabla.\boldsymbol{\sigma} dV \tag{3.106}$$

Substituting the above relations into Equation 3.102 results in:

$$\boldsymbol{F}_b + \boldsymbol{F}_s = \boldsymbol{F}_I \to \int_V \rho \boldsymbol{f}_b dV + \int_V \nabla.\boldsymbol{\sigma} dV = \int_V \rho \frac{\partial \boldsymbol{v}}{\partial t} dV \tag{3.107}$$

And consequently, we reach to the equilibrium equation of motion as follows:

$$\nabla.\boldsymbol{\sigma} + \rho \boldsymbol{f}_b = \rho \frac{d\boldsymbol{v}}{dt} = \rho \frac{\partial^2 \boldsymbol{x}}{\partial t^2} = \rho \boldsymbol{a} \tag{3.108}$$

It can also be expressed in tensor notation as follows:

$$\frac{\partial \sigma_{ji}}{\partial n_j} + \rho f_{bi} = \rho \frac{\partial^2 x_i}{\partial t^2} = \rho a_i \tag{3.109}$$

On the other hand, conservation of angular momentum states that the time rate of change of the total angular momentum of a body equal to vector sum of the moments of external forces acting on this body. This principle leads to the symmetry of the stress tensor as follows:

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{13} = \sigma_{31}, \quad \sigma_{23} = \sigma_{32} \tag{3.110}$$

3.3.3 Work and power

Change in work dW done by a force $\boldsymbol{F}$ on some particle equals to the dot product of the force vector and displacement change $d\boldsymbol{x}$ as follows:

$$dW = \boldsymbol{F}.d\boldsymbol{x} \tag{3.111}$$

Such that the total work done through the particle path c will be:

$$W = \int_c \boldsymbol{F}.d\boldsymbol{x} \tag{3.112}$$

while power p is the time derivative of the work W defined as follows:

$$p = \frac{dW}{dt} = \boldsymbol{F}.\frac{d\boldsymbol{x}}{dt} = \boldsymbol{F}.\boldsymbol{v} \tag{3.113}$$

From above expression, the power can be defined as the dot product of the force vector with velocity vector $\boldsymbol{v}$. The power generated by the external forces includes the contribution of the body and surface forces as follows:

$$P = (\boldsymbol{F}_b + \boldsymbol{F}_s).\boldsymbol{v} = \int_s \boldsymbol{T}.\boldsymbol{v}ds + \int_V \rho \boldsymbol{f}_b.\boldsymbol{v}dV = \int_s \boldsymbol{n}.(\boldsymbol{\sigma}.\boldsymbol{v})\,ds + \int_V \rho \boldsymbol{f}_b.\boldsymbol{v}dV \tag{3.114}$$

The velocity $\boldsymbol{v}$ here is considered as a velocity field as it can be varied over the body volume. Using divergence theorem on the first term of the right hand side in the above expression yields:

$$p = \int_V \nabla.(\boldsymbol{\sigma}.\boldsymbol{v})\,dV + \int_V \rho \boldsymbol{F}.\boldsymbol{v}dV \tag{3.115}$$

$$\nabla.(\boldsymbol{\sigma}.\boldsymbol{v}) = \frac{\partial}{\partial x_i}(\sigma_{ij}v_j) = \frac{\partial \sigma_{ij}}{\partial x_i}v_j + \sigma_{ij}\frac{\partial v_j}{\partial x_i} = (\nabla.\boldsymbol{\sigma}).\boldsymbol{v} + \boldsymbol{\sigma} : \boldsymbol{L}^T \tag{3.116}$$

As stress tensor $\boldsymbol{\sigma}$ is a symmetric matrix, we can conclude using Equation 1.100:

$$\boldsymbol{\sigma} : \boldsymbol{L}^T = \boldsymbol{\sigma} : \boldsymbol{L} = \boldsymbol{\sigma} : sym(\boldsymbol{L}) = \boldsymbol{\sigma} : \boldsymbol{D} \tag{3.117}$$

such that power will be given by:

$$p = \int_V (\nabla.\boldsymbol{\sigma} + \rho \boldsymbol{f}_b).\boldsymbol{v}dV + \int_V \boldsymbol{\sigma} : \boldsymbol{D}dV \tag{3.118}$$

From equilibrium Equation 3.108, it follows:

$$P = \int_V \rho \boldsymbol{a}.\boldsymbol{v}dV + \int_V \boldsymbol{\sigma} : \boldsymbol{D}dV \tag{3.119}$$

$$\int_V \rho \boldsymbol{a}.\boldsymbol{v}dV = \frac{d}{dt}\left(\frac{1}{2}\int_V \rho \boldsymbol{v}.\boldsymbol{v}dV\right) = \frac{d}{dt}(K.E) \tag{3.120}$$

$$= \frac{d}{dt}(K.E) + \int_V \boldsymbol{\sigma} : \boldsymbol{D}dV \tag{3.121}$$

From above equation, the external power is converted into two parts; time rate of change of kinetic energy $K.E$ associated with body motion and time rate of change of strain energy associated with deformation. Cauchy stress tensor and rate of deformation strain rate $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ are called energetically conjugate pairs of stresses and strain rates. There are other energetically conjugate pairs other than Cauchy stress and deformation strain rate. For example, if we need to evaluate the stress measure conjugate to time rate of change of deformation gradient $\dot{\boldsymbol{F}}$, we need to convert the power part associated with deformation as follows:

$$\begin{aligned}\int_V \boldsymbol{\sigma} : \boldsymbol{D}dV = \int_V \boldsymbol{\sigma} : \boldsymbol{L}dV = \int_V \boldsymbol{\sigma} : (\dot{\boldsymbol{F}}\boldsymbol{F}^{-1})dV \\ = \int_V \sigma_{ij}\dot{F}_{im}F_{mj}^{-1}dV = \int_V \sigma_{ij}F_{jm}^{-T}\dot{F}_{im}dV \\ = \int_V \left(\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right) : \dot{\boldsymbol{F}}dV = \int_V \left(\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}\right) : \dot{\boldsymbol{F}}dV\end{aligned} \tag{3.122}$$

So $\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}$ is conjugate to the time rate of change of deformation gradient $\dot{\boldsymbol{F}}$ and integrated over the current volume V. Using $dV = JdV_0$, where dV, dV_0 are the volume of a differential body in the final and initial configurations, respectively, we can convert the current volume integration into integration over the initial volume as follows:

$$\int_V \boldsymbol{\sigma} : \boldsymbol{D} dV = \int_V J\boldsymbol{\sigma}^T\boldsymbol{F}^{-T} : \dot{\boldsymbol{F}} dV_0 = \int_{V_0} \boldsymbol{P} : \dot{\boldsymbol{F}} dV_0 \tag{3.123}$$

Where $\boldsymbol{P} = J\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}$ is called first Piola Kirchhoff stress tensor, such that $\boldsymbol{P}$ and $\dot{\boldsymbol{F}}$ are considered energetically conjugate pairs. Cauchy stress can be evaluated from the following:

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{P}^T \tag{3.124}$$

From the above relation, it seems that $\boldsymbol{P}$ is unsymmetric tensor. However, the symmetry of Cauchy stress $\boldsymbol{\sigma}$ leads to this expression:

$$\boldsymbol{F}\boldsymbol{P}^T = \boldsymbol{P}\boldsymbol{F}^T \tag{3.125}$$

Also we can search for another stress measure conjugate to time rate of change of Green Lagrange tensor using Equation 3.71 as follows:

$$\begin{aligned}
&\int_V \boldsymbol{\sigma} : \boldsymbol{D} dV = \int_V \boldsymbol{\sigma} : \left(\boldsymbol{F}^{-T}\dot{\boldsymbol{E}}\boldsymbol{F}^{-1}\right) dV \\
&= \int_V \sigma_{ij} F_{im}^{-T} \dot{E}_{mn} F_{nj}^{-1} dV = \int_V F_{mi}^{-1} \sigma_{ij} F_{jn}^{-T} \dot{E}_{mn} dV \\
&= \int_{V_0} J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} : \dot{\boldsymbol{E}} dV_0 = \int_{V_0} J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T\boldsymbol{F}^{-T} : \dot{\boldsymbol{E}} dV_0 = \int_{V_0} \boldsymbol{S} : \dot{\boldsymbol{E}} dV_0
\end{aligned} \tag{3.126}$$

Where $\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}$ is called second Piola Kirchhoff stress tensor, such that $\boldsymbol{S}$ and $\dot{\boldsymbol{E}}$ are considered energetically conjugate pairs. Also it is easily to verify that $\boldsymbol{S}$ is a symmetric tensor. Also it is considered as a push back of Cauchy stress from the current configuration C_t to the initial configuration C_0 which takes sometimes this form ${}_0^t\boldsymbol{S}$. Also the above expressions can be rewritten in variational rate using virtual work principle[3] as follows:

$$\int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dV = \int_{V_0} \boldsymbol{P} : \delta\boldsymbol{F} dV_0 = \int_{V_0} \boldsymbol{S} : \delta\boldsymbol{E} dV_0 \tag{3.127}$$

Using Equation 3.71, Equation 3.64, and Equation 3.65 results in:

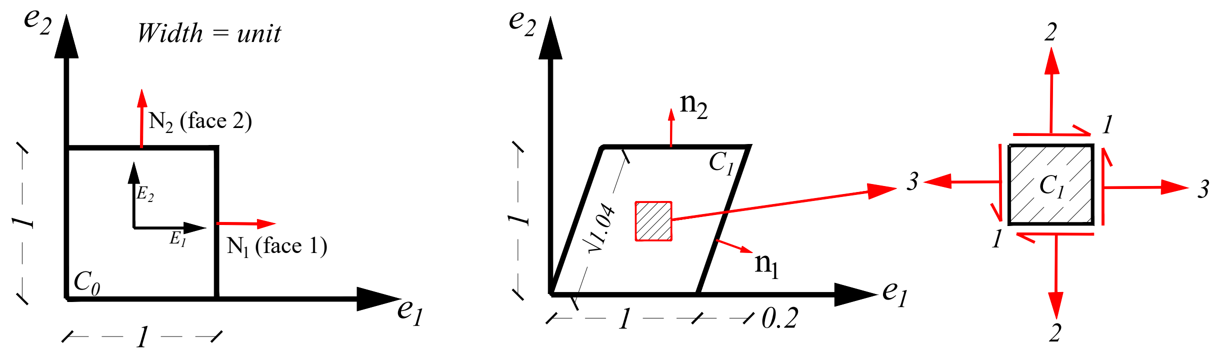
$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}\left(\delta\boldsymbol{F}\boldsymbol{F}^{-1} + \left(\delta\boldsymbol{F}\boldsymbol{F}^{-1}\right)^T\right), \quad \delta\boldsymbol{E} = \boldsymbol{F}^T\delta\boldsymbol{\varepsilon}\boldsymbol{F} \tag{3.128}$$

3.3.4 The physical meaning of the first and second Piola Kirchhoff stress tensor

■ **Example 3.9** Assume a four-node element with undeformed configuration C_0 and subjected to deformation to reach configuration C_1 shown in Figure 3.23. The stress tensor resolved in the inertia basis $\boldsymbol{e}_i$ is:

$$[\boldsymbol{\sigma}]_{[\boldsymbol{e}_i \otimes \boldsymbol{e}_j]} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.129}$$

[3] see chapter 4



(a) Initial unstressed configuration C_0 with section normal $\boldsymbol{N}^{(1)}$ and $\boldsymbol{N}^{(2)}$ and material frame of reference $\boldsymbol{E}$

(b) Final stressed configuration C_1 with section normal $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$. Cauchy stress state is shown for an infinitesimal element

Figure 3.23: Configurations C_0 and C_1

From the above figure, deformation gradient is defined as

$$[\boldsymbol{F}]_{[\boldsymbol{e}_i \otimes \boldsymbol{E}_I]} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}; \quad \text{with} \quad J = 1; \quad \boldsymbol{F}^{-1} = \begin{bmatrix} 1 & -0.2 \\ 0 & 1 \end{bmatrix} \tag{3.130}$$

First and second Piola Kirchhoff stresses will be:

$$[\boldsymbol{P}]_{[\boldsymbol{e}_i \otimes \boldsymbol{E}_I]} = \begin{bmatrix} 2.8 & 1 \\ 0.6 & 2 \end{bmatrix}; \quad [\boldsymbol{S}]_{[\boldsymbol{E}_i \otimes \boldsymbol{E}_I]} = \begin{bmatrix} 2.68 & 0.6 \\ 0.6 & 2 \end{bmatrix} \tag{3.131}$$

Kirchhoff ■

First Piola Kirchhoff stress means that plane $\boldsymbol{n}^{(1)}$ has force $\boldsymbol{P}_1 = (2.8, 0.6)$ on face 1 with initial normal $\boldsymbol{N}^{(1)} = [1, 0]$ and initial area $|\boldsymbol{A}_1| = 1$ and current area $|\boldsymbol{a}_1| = \sqrt{1.04}$

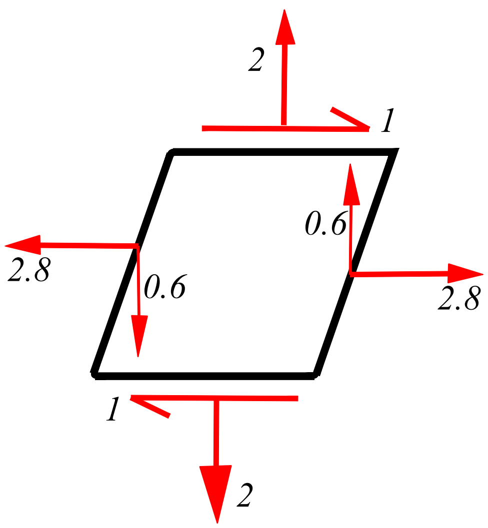


Figure 3.24: Force distribution $\boldsymbol{F}$ on the deformed surfaces deduced from firts Piola Kirchhoff stress $\boldsymbol{P}$ and the initial area $\boldsymbol{A}$ ($\boldsymbol{F} = \boldsymbol{P}.\boldsymbol{A}$)

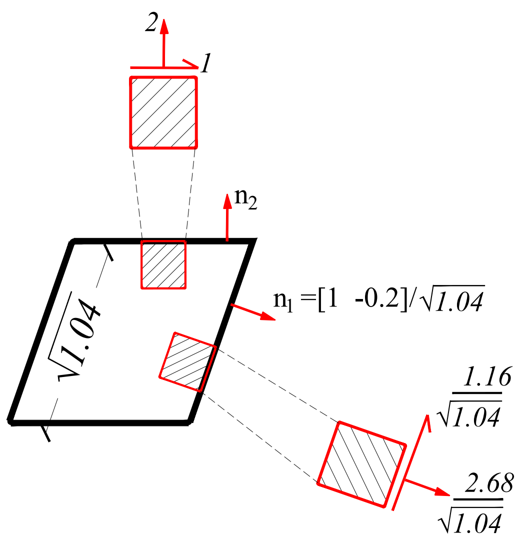


Figure 3.25: Stress distribution on the deformed surfaces after resolving the forces in surface normal and tangent direction

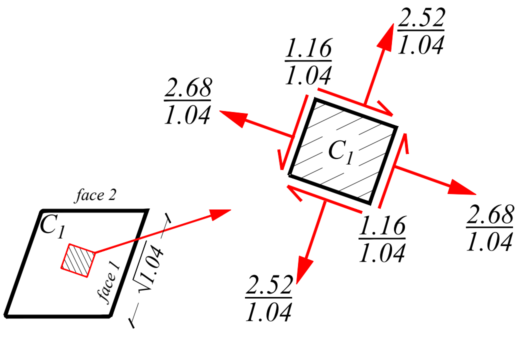


Figure 3.26: Cauchy stress state transformed in the direction of $\boldsymbol{n}^1$ and its normal $\boldsymbol{n}^2$

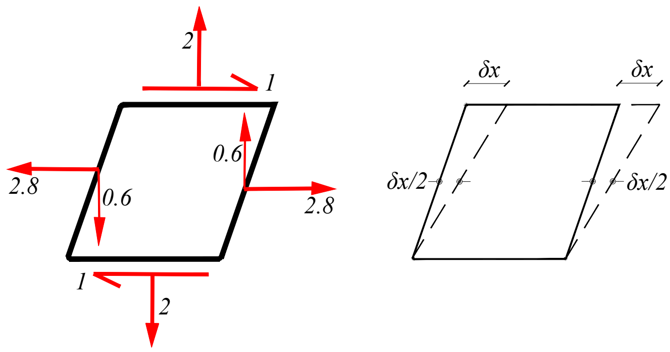


Figure 3.27: Applying virtual displacement δx

■ **Example 3.10 — Equilibrium study.** In the above example, the $i^t h$ column in $\boldsymbol{P}$ represents the force applied on the material surface with current normal $\boldsymbol{n}_i$ and initial normal $\boldsymbol{N}_i$ with unit initial area $(dA_i^{(1)} = 1)$ as pictured in Figure 3.24. It is denoted by stress vector $\boldsymbol{P}_i$ defined as follows:

$$\boldsymbol{P} = \boldsymbol{P}_I \otimes \boldsymbol{E}_I \rightarrow \boldsymbol{P}_I = \boldsymbol{P}\boldsymbol{E}_I \tag{3.132}$$

$$\begin{aligned} &\boldsymbol{P} = \boldsymbol{P}_{iI}\boldsymbol{e}_i \otimes \boldsymbol{E}_I \rightarrow \boldsymbol{P}_I = \boldsymbol{P}_{iI}\boldsymbol{e}_i \\ &\qquad \boldsymbol{P}_1 = 2.8\boldsymbol{e}_1 + 0.6\boldsymbol{e}_2 \\ &\qquad \boldsymbol{P}_2 = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 \end{aligned} \tag{3.133}$$

The resulting force on plane $\boldsymbol{n}_1$ will be $\boldsymbol{F}_1 = \boldsymbol{P}_1\boldsymbol{A}_1 = \boldsymbol{P}_1\boldsymbol{N}_1 dA_1 = (2.8\boldsymbol{e}_1 + 0.6\boldsymbol{e}_2) * 1 = 2.8\boldsymbol{e}_1 + 0.6\boldsymbol{e}_2$ as shown in Figure 3.24, while the corresponding force to plane $\boldsymbol{n}_2$ is $\boldsymbol{F}_2 = \boldsymbol{e}_1 + 2\boldsymbol{e}_2$. We can get the deformed area using Nanson's formula $\boldsymbol{n}_1 da_1 = J\boldsymbol{F}^{-T}\boldsymbol{N}_1 dA_1 = \begin{bmatrix} 1 & -0.2 \end{bmatrix}^T$ with unit vector $\frac{\begin{bmatrix} 1 & -0.2 \end{bmatrix}^T}{\sqrt{1.04}}$ and area magnitude $da_1 = \sqrt{1.04}$ as shown in Figure 3.25, such that the Cauchy stress vector on this plane is defined as

$$\boldsymbol{\sigma}^{(\boldsymbol{n}_1)} = \boldsymbol{n}_1.\boldsymbol{\sigma} = \frac{\begin{bmatrix} 1 & -0.2 \end{bmatrix}}{\sqrt{1.04}} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\begin{bmatrix} 2.8 & 0.6 \end{bmatrix}}{\sqrt{1.04}} \tag{3.134}$$

$$\boldsymbol{\sigma}^{(\boldsymbol{n}_2)} = \boldsymbol{n}_2.\boldsymbol{\sigma} = \begin{bmatrix} 1 & 2 \end{bmatrix} \tag{3.135}$$

The stress distribution is shown in Figure 3.25. The resulting forces $\boldsymbol{F}_1 = \boldsymbol{\sigma}^{(\boldsymbol{n}_1)} da_1 = \begin{bmatrix} 2.8 & 0.6 \end{bmatrix}$, $\boldsymbol{F}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}$, which is identical to the first Piola Kirchhoff resultant force mentioned in the previous paragraph. Also the same results can be obtained from Cauchy stresses $\overline{\boldsymbol{\sigma}}$ on plane $\boldsymbol{n}_1$ can be defined using transformation rule

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{R}^T\boldsymbol{\sigma}\boldsymbol{R}, \quad \text{with } \boldsymbol{R} = \frac{1}{\sqrt{1.04}} \begin{bmatrix} 1 & 0.2 \\ -0.2 & 1 \end{bmatrix} \tag{3.136}$$

It follows as shown in Figure 3.25 and Figure 3.26 that

$$\overline{\boldsymbol{\sigma}} = \frac{1}{1.04}\begin{bmatrix} 2.68 & 1.16 \\ 1.16 & 2.52 \end{bmatrix} \tag{3.137}$$

Evaluating the components of the resultant force over face 1 shown in Figure 3.26 results in:

$$\boldsymbol{F}_{11} = \left(\frac{2.68}{1.04}\frac{1}{\sqrt{1.04}} + \frac{1.16}{1.04}\frac{0.2}{\sqrt{1.04}}\right)a_1 = \left(\frac{2.68}{1.04}\frac{1}{\sqrt{1.04}} + \frac{1.16}{1.04}\frac{0.2}{\sqrt{1.04}}\right) * \sqrt{1.04} = 2.8 \tag{3.138}$$

As the surface $\boldsymbol{n}_1$ has area $\sqrt{1.04}$. This is the component of resultant force on face 1 in $\boldsymbol{e}_1$ direction ($\boldsymbol{F}_{11}$), while, in $\boldsymbol{e}_2$ direction, it will be:

$$\boldsymbol{F}_2 = \left(-\frac{2.68}{1.04}\frac{0.2}{\sqrt{1.04}} + \frac{1.16}{1.04}\frac{1}{\sqrt{1.04}}\right)a_1 = \left(-\frac{2.68}{1.04}\frac{0.2}{\sqrt{1.04}} + \frac{1.16}{1.04}\frac{1}{\sqrt{1.04}}\right) * \sqrt{1.04} = 0.6 \tag{3.139}$$

■ **Example 3.11 — Virtual work.** We can also prove Equation 3.127 as follows. Assume a virtual displacement δx shown in the Figure 3.27 applied over the deformed configuration C_1, such that the resulting deformation gradient and its variation will be:

$$\boldsymbol{F}_{new} = \begin{bmatrix} 1 & 0.2+\delta x \\ 0 & 1 \end{bmatrix} \tag{3.140}$$

$$\delta \boldsymbol{F} = \boldsymbol{F}_{new} - \boldsymbol{F} = \begin{bmatrix} 0 & \delta x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\delta x \tag{3.141}$$

Also the variation in infinitesimal strain and variation in Green-Lagrange strain using Equation 3.128 will be:

$$\delta \boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}\delta x \tag{3.142}$$

$$\delta \boldsymbol{E} = \boldsymbol{F}^T \delta\boldsymbol{\varepsilon}\boldsymbol{F} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.2 \end{bmatrix}\delta x \tag{3.143}$$

Such that the resulting virtual work in terms of different stress measures using Equation 3.127

$$\delta W = \int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} dv = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} : \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}\delta x = \delta x \tag{3.144}$$

$$\delta W = \int_{V_0} \boldsymbol{P} : \delta\boldsymbol{F} dV = \begin{bmatrix} 2.8 & 1 \\ 0.6 & 2 \end{bmatrix} : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\delta x = \delta x \tag{3.145}$$

$$\delta W = \int_{V_0} \boldsymbol{S} : \delta\boldsymbol{E} dV = \begin{bmatrix} 2.68 & 0.6 \\ 0.6 & 2 \end{bmatrix} : \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.2 \end{bmatrix}\delta x \tag{3.146}$$

$$= (0.6 * 0.5 + 0.6 * 0.5 + 2 * 0.2)\delta x = \delta x \tag{3.147}$$

Where volume before and after deformation is equal 1 ($v = V = 1$). Also the same result can be obtained if we use Figure 3.27 to evaluate the virtual work exerted by first Piola Kirchhoff stress vectors $\boldsymbol{P}_1 = (2.8\boldsymbol{e}_1 + 0.6\boldsymbol{e}_2)$ and $\boldsymbol{P}_1 = (1\boldsymbol{e}_1 + 2\boldsymbol{e}_2)$ as follows:
The virtual work done by these forces $=2.8 * \left(\frac{\delta x}{2}\right) - 2.8 * \left(\frac{\delta x}{2}\right) + 1 * \delta x = \delta x$.
Which gives the same findings of the above equations ■

3.3.5 Geometrically exact beam theory

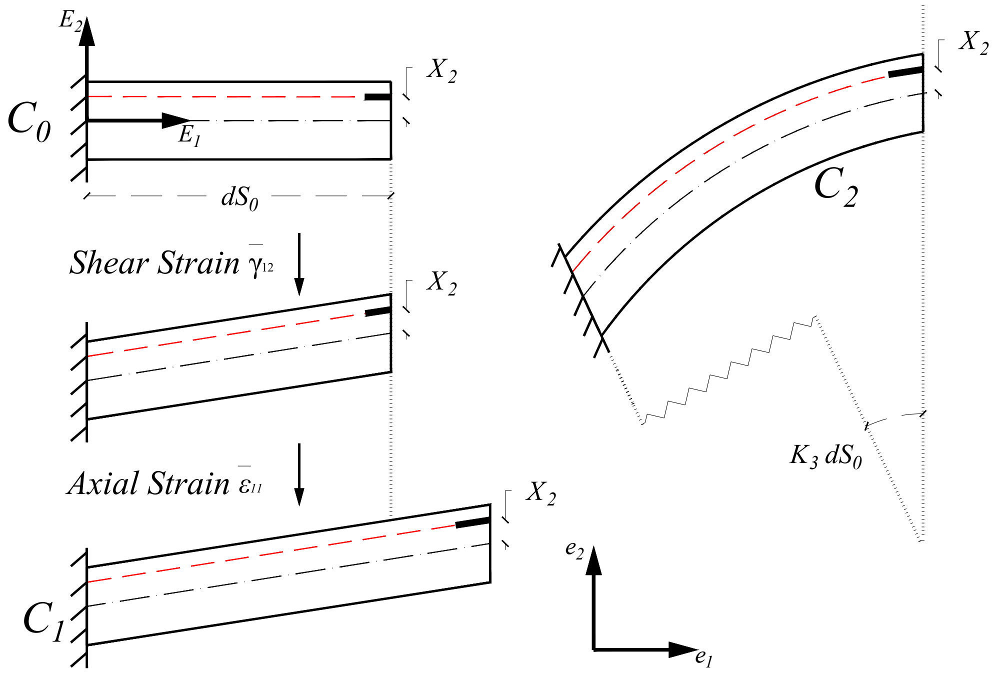


Figure 3.28

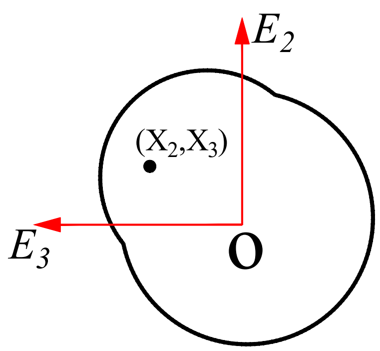


Figure 3.29: Position of point $\boldsymbol{X}$ relative to the material triad $\boldsymbol{E}$ at configuration C_1 in Figure 3.28

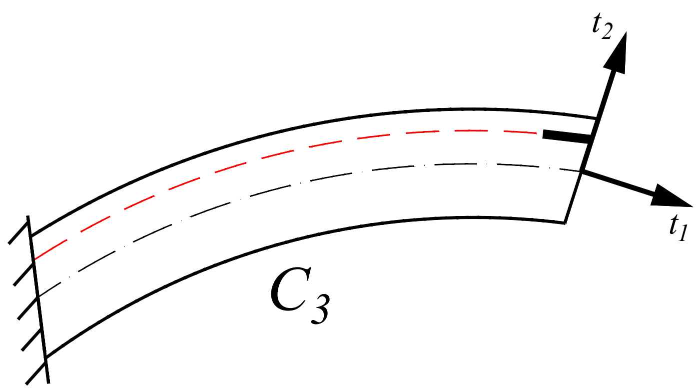


Figure 3.30: Applying rigid body rotation $\boldsymbol{R}$ on configuration C_2 in Figure 3.28

Assume a Timoshenko beam (rigid cross section assumption) shown in Figure 3.28 with an undeformed infinitesimal arc length dS_0 and material basis[4] $\boldsymbol{E}_I$ subjected to shear strain $\bar{\gamma}_{12}$ and

[4] The material basis $\boldsymbol{E}_I$ in Figure 3.28 does not change with deformation and is assumed to be aligned with beams

axial strain $\bar{\varepsilon}_{11}$ to reach configuration C_1, then a curvature $\bar{K}_3$ around basis $\boldsymbol{e}_3$ to finally reach configuration C_2 such that the total difference in cross section orientation is $\bar{K}_3 dS_0$ in $\boldsymbol{e}_3$ direction. If we are interested in evaluating the deformation gradient at a material point located at distance X_2 from centroid, the deformation gradient of configuration C_1 will be:

$$\left[{}^1_0\bar{\boldsymbol{F}}\right]_{\boldsymbol{e}\otimes\boldsymbol{E}} = \begin{bmatrix} 1+\bar{\varepsilon}_{11} & 0 \\ \bar{\gamma}_{21} & 1 \end{bmatrix} \tag{3.148}$$

The axial strain $\bar{\varepsilon}_{11}(curv)$ due to curvature results from change in the length of the longitudinal fiber located at X_2 as follows:

$$\bar{\varepsilon}_{11}(curv) = \frac{\text{Change in beam length}}{\text{original length}} \tag{3.149}$$

$$= \frac{\text{Change in beam orientation} \times \text{Point position relative to centroid}}{\text{original length}} \tag{3.150}$$

$$= \frac{(\bar{K}_3 dS_0)\ (X_2)}{dS_0} = \bar{K}_3 X_2 \tag{3.151}$$

Such that overall deformation gradient at configuration C_1 is:

$$\left[{}^2_0\bar{\boldsymbol{F}}\right]_{\boldsymbol{e}\otimes\boldsymbol{E}} = \begin{bmatrix} 1+\bar{\varepsilon}_{11}+\bar{K}_3 X_2 & 0 \\ \bar{\gamma}_{21} & 1 \end{bmatrix} \tag{3.152}$$

For a three dimensional beam, the axial strain $\bar{\boldsymbol{\varepsilon}}(curv)$ due to 3D curvature resolved in material basis $\boldsymbol{E}$ as $\left[\bar{\boldsymbol{K}}\right]_{\boldsymbol{E}} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 & \bar{K}_3 \end{bmatrix}^T$ is defined as:

$$\bar{\boldsymbol{\varepsilon}}(curv) = \tilde{\bar{\boldsymbol{K}}}\boldsymbol{X} \rightarrow [\bar{\boldsymbol{\varepsilon}}(curv)]^{\boldsymbol{E}} = \begin{bmatrix} \bar{K}_2 X_3 - \bar{K}_3 X_2 \\ -\bar{K}_1 X_3 \\ \bar{K}_1 X_2 \end{bmatrix} \tag{3.153}$$

Where $\boldsymbol{X}$ defines the position of a material point. When it is resolved in material frame, it will be $[\boldsymbol{X}]_{\boldsymbol{E}} = \begin{bmatrix} 0 & X_2 & X_3 \end{bmatrix}^T$, where X_2 and X_3 define beam position along the beam principle axes as shown in Figure 3.29, such that the resulting deformation gradient in index notation will be:

$$\left[{}^2_0\bar{\boldsymbol{F}}\right]_{\boldsymbol{e}\otimes\boldsymbol{E}} = \begin{bmatrix} 1+\bar{\varepsilon}_{11}+\bar{K}_2 X_3 - \bar{K}_3 X_2 & 0 & 0 \\ \bar{\gamma}_{21} - \bar{K}_1 X_3 & 1 & 0 \\ \bar{\gamma}_{31} + \bar{K}_1 X_2 & 0 & 1 \end{bmatrix} \tag{3.154}$$

And in tensorial form:

$${}^2_0\bar{\boldsymbol{F}} = \bar{\boldsymbol{F}}_{iI}\boldsymbol{e}_i \otimes \boldsymbol{E}_I = \boldsymbol{1} + \bar{\boldsymbol{\varepsilon}} \otimes \boldsymbol{E}_1 = \boldsymbol{e}_i \otimes \boldsymbol{E}_1 + \bar{\varepsilon}_i \boldsymbol{e}_i \otimes \boldsymbol{E}_1 \tag{3.155}$$

where $\bar{\varepsilon}_1 = \bar{\varepsilon_{11}}, \quad \bar{\varepsilon}_2 = \bar{\varepsilon}_{21} = \bar{\gamma}_{21} - \bar{K}_1 X_3, \quad \bar{\varepsilon}_3 = \bar{\varepsilon}_{31} = \bar{\gamma}_{31} + \bar{K}_1 X_2$ [5]

Applying virtual strain $[\delta\bar{\boldsymbol{\varepsilon}}]_{\boldsymbol{E}} = \begin{bmatrix} \delta\bar{\varepsilon}_1 & \delta\bar{\varepsilon}_2 & \delta\bar{\varepsilon}_3 \end{bmatrix}^T$ and curvature $\left[\delta\bar{\boldsymbol{K}}\right]_{\boldsymbol{E}} = \begin{bmatrix} \delta\bar{K}_1 & \delta\bar{K}_2 & \delta\bar{K}_3 \end{bmatrix}^T$ to the beam in the final configuration, the internal resulting virtual work δW_{int} in terms of first Piola Kirchhoff stress tensor will be:

principle axes and cross section normal at the undeformed configuration C_0 which, in this case, is identical to inertia frame $\boldsymbol{e}_i$ as the line of undeformed beam centroids is straight and directed along $\boldsymbol{e}_1$, while co-rotational or moving frame (beam triad) $\boldsymbol{t}_i$ is attached to the beam and its orientation changes with deformations (change in cross section normal $\boldsymbol{t}_1$) and principle axes orientation $\boldsymbol{t}_2, \boldsymbol{t}_3$.

[5]The strains $\bar{\varepsilon}_{22}, \bar{\varepsilon}_{33}$ and $\bar{\varepsilon}_{23}$ vanish from the rigid cross section assumption in Timoshenko beam theory

$$\begin{aligned}\delta W_{int} &= \int_{V_0} \bar{\boldsymbol{P}} : \delta\bar{\boldsymbol{F}} dV_0 \\ &= \int_{V_0} \bar{P}_{11} \left(\delta\bar{\varepsilon}_{11} + \delta\bar{K}_2 X_3 - \delta\bar{K}_3 X_2\right) dV_0 \\ &+ \int_{V_0} \bar{P}_{21} \left(\delta\bar{\gamma}_{21} - \delta\bar{K}_1 X_3\right) dV_0 \\ &+ \int_{V_0} \bar{P}_{31} \left(\delta\bar{\gamma}_{13} + \delta\bar{K}_1 X_2\right) dV_0\end{aligned} \tag{3.156}$$

Where $\bar{P}_{iI}$ forms the components of first Piola Kirchhoff stress tensor ($\bar{\boldsymbol{P}} = \bar{\boldsymbol{P}}_i \otimes \boldsymbol{E}_I = \bar{P}_{iI} \boldsymbol{e}_i \otimes \boldsymbol{E}_I$) and $\bar{\boldsymbol{P}}_i$ is stress traction vector applied on the beam cross section surface. As beam strain and curvature are only function of arc length s along the line of centroids, the integration can be simplified to:

$$\delta W_{int} = \int_{S_0} \left(\boldsymbol{N}.\delta\bar{\boldsymbol{\varepsilon}} + \boldsymbol{M}.\delta\bar{\boldsymbol{K}}\right) dS_0 = \int_{S_0} \left([\boldsymbol{N}]_{\boldsymbol{E}}.[\delta\bar{\boldsymbol{\varepsilon}}]_{\boldsymbol{E}} + [\boldsymbol{M}]_{\boldsymbol{E}}.[\delta\bar{\boldsymbol{K}}]_{\boldsymbol{E}}\right) dS_0 \tag{3.157}$$

The last equality comes from the fact that work is a scalar value, so we can resolve its terms in any frame of reference. Terms $[\boldsymbol{N}]_{\boldsymbol{E}}$ and $[\boldsymbol{M}]_{\boldsymbol{E}}$ represent the cross section resultant force and moment resolved in basis $\boldsymbol{E}$ defined as follows:

$[\boldsymbol{N}]_{\boldsymbol{E}} = [N_1 \;\; N_2 \;\; N_3]^T$, $\quad [\boldsymbol{M}]_{\boldsymbol{E}} = [M_1 \;\; M_2 \;\; M_3]^T$, $\quad \left[\bar{\boldsymbol{P}}_1\right]_{\boldsymbol{E}} = [\bar{P}_{11} \;\; \bar{P}_{21} \;\; \bar{P}_{31}]^T$
$[\bar{\boldsymbol{\varepsilon}}]_{\boldsymbol{E}} = [\bar{\varepsilon}_{11} \;\; \bar{\gamma}_{21} \;\; \bar{\gamma}_{31}]^T$, $\quad \left[\bar{\boldsymbol{K}}\right]_{\boldsymbol{E}} = [\bar{K}_1 \;\; \bar{K}_2 \;\; \bar{K}_3]^T$
Where
$N_1 = \int_{A_0} \bar{P}_{11} dA_0$, $\quad N_2 = \int_{A_0} \bar{P}_{21} dA_0$, $\quad N_3 = \int_{A_0} \bar{P}_{31} dA_0$
$M_1 = \int_{A_0} \left(\bar{P}_{31} X_2 - \bar{P}_{21} X_3\right) dA_0$, $\quad M_2 = -\int_{A_0} \bar{P}_{11} X_3 dA_0$, $\quad M_3 = \int_{A_0} \bar{P}_{11} X_2 dA_0$

If a rigid body rotation $\boldsymbol{R}$ is superimposed on the configuration C_2 as shown in Figure 3.30, the beam triads (co-rotational basis) $\boldsymbol{t}_i$, the stress traction vector applied on the beam cross section surface $\boldsymbol{P}_1$, the strains and curvature, the resultant force and moment on the cross section, and the new deformation gradient will be:

$$\boldsymbol{t}_i = \boldsymbol{R}\boldsymbol{E}_i, \quad \boldsymbol{P}_1 = \boldsymbol{R}\bar{\boldsymbol{P}}_1, \quad \boldsymbol{n} = \boldsymbol{R}\boldsymbol{N}, \quad \boldsymbol{m} = \boldsymbol{R}\boldsymbol{M}, \quad \boldsymbol{\varepsilon} = \boldsymbol{R}\bar{\boldsymbol{\varepsilon}}, \quad \boldsymbol{K} = \boldsymbol{R}\bar{\boldsymbol{K}}, \quad \boldsymbol{F} = \boldsymbol{R}\bar{\boldsymbol{F}} \tag{3.158}$$

As all the above terms except deformation gradient in the last equality are vectors, they transform like vector, while the last equity can be deduced using subsection 3.2.2 or using section 3.4. The above expressions can also be interpreted as shown in Figure 3.31, such that the components of stress vector $\boldsymbol{P}_1$ resolved in the local triad $\boldsymbol{t}_I$ is identical to the components of stress vector $\bar{\boldsymbol{P}}_1$ resolved in the material frame $\boldsymbol{I}_I$ ($\bar{P}_{I1}$) and it follows:

$$\boldsymbol{P}_1 = P_{I1} \boldsymbol{e}_I = \bar{P}_{I1} \boldsymbol{t}_I \tag{3.159}$$

With
$[\boldsymbol{n}]_{\boldsymbol{E}} = [n_1 \;\; n_2 \;\; n_3]^T$, $\quad [\boldsymbol{m}]_{\boldsymbol{E}} = [m_1 \;\; m_2 \;\; m_3]^T$, $\quad [\boldsymbol{P}_1]_{\boldsymbol{E}} = [P_{11} \;\; P_{21} \;\; P_{31}]^T$
$[\boldsymbol{\varepsilon}]_{\boldsymbol{E}} = [\varepsilon_{11} \;\; \gamma_{21} \;\; \gamma_{31}]^T$, $\quad [\boldsymbol{K}]_{\boldsymbol{E}} = [K_1 \;\; K_2 \;\; K_3]^T$
In the same manner:

$$\boldsymbol{n} = n_I \boldsymbol{e}_I = N_I \boldsymbol{t}_I \tag{3.160}$$

$$\boldsymbol{m} = m_I \boldsymbol{e}_I = M_I \boldsymbol{t}_I \tag{3.161}$$

$$\boldsymbol{\varepsilon} = \varepsilon_{11} \boldsymbol{e}_1 + \gamma_{21} \boldsymbol{e}_2 + \gamma_{31} \boldsymbol{e}_3 = \bar{\varepsilon}_{11} \boldsymbol{t}_1 + \bar{\gamma}_{21} \boldsymbol{t}_2 + \bar{\gamma}_{31} \boldsymbol{t}_3 \tag{3.162}$$

$$\boldsymbol{K} = K_I \boldsymbol{e}_I = \bar{K}_I \boldsymbol{t}_I \tag{3.163}$$

$$\boldsymbol{F} = F_{iI} \boldsymbol{e}_i \otimes \boldsymbol{E}_I = \bar{F}_{iI} \boldsymbol{t}_i \otimes \boldsymbol{E}_I \tag{3.164}$$

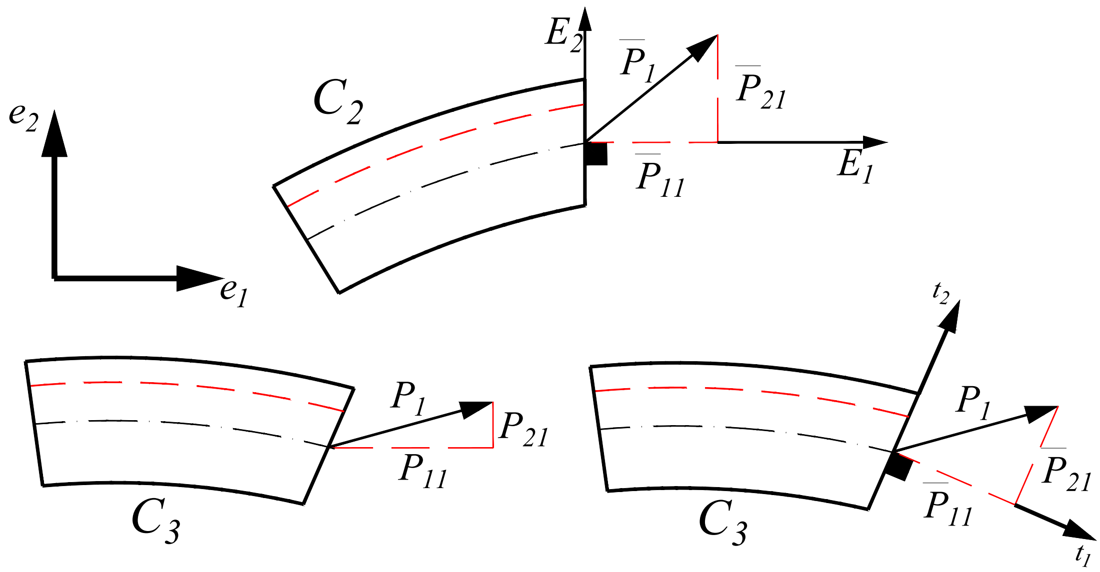


Figure 3.31: Applying a rigid body rotation on configuration C_2 with surface first Piola Kirchhoff stress $\bar{\boldsymbol{P}}_1$ resolved in material frame $\boldsymbol{E}_I$ as $(\bar{P}_{11}, \bar{P}_{21}, \bar{P}_{31})$ to get configuration C_3 with surface first Piola Kirchhoff stress $\boldsymbol{P}_1$ defined through the transformation rule $\boldsymbol{P}_1 = \boldsymbol{R}\bar{\boldsymbol{P}}_1$ and resolved in the inertia frame $\boldsymbol{e}_i$ as (P_{11}, P_{21}, P_{31}) and in the co-rotational frame $\boldsymbol{t}_i$ as $(\bar{P}_{11}, \bar{P}_{21}, , \bar{P}_{31})$ which is identical to this vector in C_2 and resolved in $\boldsymbol{E}_I$

The last equality results from using ($\boldsymbol{R} = \boldsymbol{t}_I \otimes \boldsymbol{E}_I$) and ($\boldsymbol{E}_I.\boldsymbol{e}_i = \delta_{Ii}$) as follows:

$$\boldsymbol{F} = \boldsymbol{R}\bar{\boldsymbol{F}} = (\boldsymbol{t}_I \otimes \boldsymbol{E}_I)\left(\bar{F}_{iJ}\boldsymbol{e}_i \otimes \boldsymbol{E}_J\right) = \bar{F}_{iJ}\delta_{Ii}\boldsymbol{t}_I \otimes \boldsymbol{E}_J = \bar{F}_{iJ}\boldsymbol{t}_i \otimes \boldsymbol{E}_J \tag{3.165}$$

In the same manner:

$$\boldsymbol{P} = \boldsymbol{R}\bar{\boldsymbol{P}} \tag{3.166}$$

Note that first Piola Kirchhoff stress tensor and deformation gradient are called two-point tensors and they follow the transformation rule described in the above expressions. We also need to note that the virtual work created by these spatial vectors $\boldsymbol{n}$, $\boldsymbol{m}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{K}$ described in Equation 3.158 are not effected by rigid body rotation and it should be equivalent to the virtual work generated by Equation 3.157 as follows:

$$\delta W_{int} = \int_{V_0} \boldsymbol{P} : \delta\boldsymbol{F}\, dV_0 = \int_{S_0} (\boldsymbol{n}.\delta\boldsymbol{\varepsilon} + \boldsymbol{m}.\delta\boldsymbol{K})\, dS_0 \tag{3.167}$$

Such that

$$\boldsymbol{P} : \delta\boldsymbol{F} = \bar{\boldsymbol{P}} : \delta\bar{\boldsymbol{F}} \tag{3.168}$$

Using Equation 3.158, Equation 3.166, it yields:

$$\bar{\boldsymbol{P}} : \delta\bar{\boldsymbol{F}} = \boldsymbol{R}^T\boldsymbol{P} : \delta\left(\boldsymbol{R}^T\boldsymbol{F}\right) = \boldsymbol{P} : \boldsymbol{R}\ \delta(\boldsymbol{R}^T\boldsymbol{F}) \tag{3.169}$$

Term $\boldsymbol{R}\delta(\boldsymbol{R}^T\boldsymbol{F})$ is called the co-rotational variation in deformation gradient and dented by $\delta\mathring{\boldsymbol{F}}$. It is defined as a variation of spatial property recorded by an observer attached to the moving frame to get $\delta(\boldsymbol{R}^T\boldsymbol{F})$ and pulled forward to the spatial form (It will be farther discussed in section 3.4). The relation between the co-rotational variation and ordinary variation is defined as follow:

$$\delta\mathring{\boldsymbol{F}} = \boldsymbol{R}\ \delta(\boldsymbol{R}^T\boldsymbol{F}) = \delta\boldsymbol{F} + \left(\boldsymbol{R}\delta\boldsymbol{R}^T\right)\bar{\boldsymbol{F}} = \delta\boldsymbol{F} - \delta\tilde{\boldsymbol{\omega}}\bar{\boldsymbol{F}} \tag{3.170}$$

Where $\delta\tilde{\boldsymbol{\omega}}$ is the variational spatial spin [6].

From Equation 3.168, it follows that

$$\boldsymbol{P}:\delta\boldsymbol{F}=\boldsymbol{P}:\boldsymbol{R}^T\delta\left(\boldsymbol{R}\bar{\boldsymbol{F}}\right)=\boldsymbol{P}:\delta\overset{\circ}{\boldsymbol{F}} \tag{3.171}$$

We see that the first Piola Kirchhoff tensor $\boldsymbol{P}$ is conjugate to co-rotational variation of deformation gradient $\delta\overset{\circ}{\boldsymbol{F}}$ in exerting the virtual work.

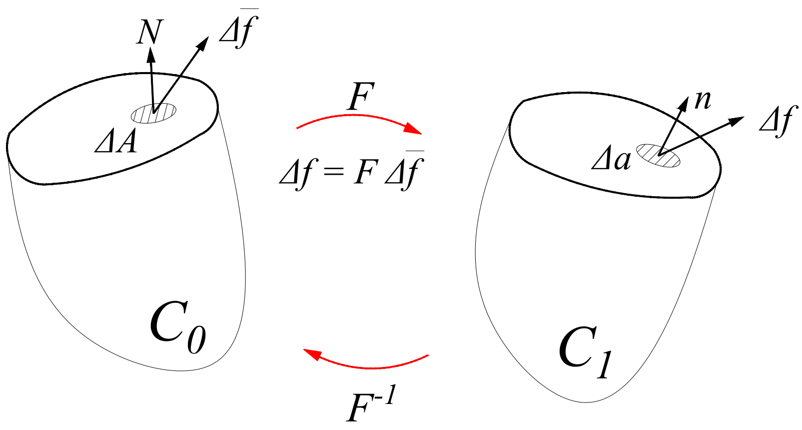


Figure 3.32

The physical meaning of the first Piola Kirchhoff stress tensor $\boldsymbol{P}$ is obvious from the previous examples, while it is hard to imagine a sensible definition for second Piola Kirchhoff stress tensor $\boldsymbol{S}$ which performs work over the variation in Green-Lagrange strain tensor $\delta\boldsymbol{E}$, see equation Equation 3.127. However, being a symmetric tensor makes it desirable in finite element formulation (see chapter 3). Also ${}^1_0\boldsymbol{S}^{\boldsymbol{N}}$ can be defined using Figure 3.32 as the current force at configuration C_1 affecting a section area with current normal $\boldsymbol{n}$ and **unit** initial area with initial normal $\boldsymbol{N}$ at configuration C_0 after being subjected to inverse mapping via deformation gradient (pulled back to the initial configuration C_0) as shown in Figure 3.32. From this definition, second Piola Kirchhoff stress tensor can be defined as follows:

$${}^1_0\boldsymbol{S}^{(\boldsymbol{N})}=\frac{d\bar{\boldsymbol{f}}}{d\boldsymbol{A}}=\frac{\boldsymbol{F}^{-1}\partial\boldsymbol{f}}{\partial\boldsymbol{a}}\frac{\partial\boldsymbol{a}}{\partial\boldsymbol{A}}=J\boldsymbol{F}^{-1}\frac{d\boldsymbol{f}}{d\boldsymbol{a}}\boldsymbol{F}^{-T}=J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^{(\boldsymbol{n})}\boldsymbol{F}^{-T} \tag{3.172}$$

Where $d\boldsymbol{f}$ is the applied force of current area $d\boldsymbol{a}$ with unit normal vector $\boldsymbol{n}$ and initial area $d\boldsymbol{A}$ with unit normal vector $\boldsymbol{N}$ as shown in Figure 3.32. Applying inverse mapping on this force results ($\boldsymbol{F}^{-1}d\boldsymbol{f}=d\bar{\boldsymbol{f}}$). We used Nanson's formula to prove the above equation ($\frac{\partial\boldsymbol{a}}{\partial\boldsymbol{A}}=J\boldsymbol{F}^{-T}$).

[6]For a spatial vector $\boldsymbol{v}=\bar{v}_i\boldsymbol{t}_i=v_i\boldsymbol{e}_i=\boldsymbol{R}\bar{\boldsymbol{v}}$, we get $\delta\boldsymbol{v}=\delta\boldsymbol{R}\bar{\boldsymbol{v}}+\boldsymbol{R}\delta\bar{\boldsymbol{v}}=\delta\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{v}+\boldsymbol{R}\delta\left(\boldsymbol{R}^T\boldsymbol{v}\right)=\delta\tilde{\boldsymbol{\omega}}\boldsymbol{v}+\delta\overset{\circ}{\boldsymbol{v}}$, while for second order tensor $\boldsymbol{T}=\boldsymbol{a}_1\otimes\boldsymbol{a}_2$. If each vector $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is induced from individual rigid body rotation ($\boldsymbol{a}_1=\boldsymbol{R}_1\bar{\boldsymbol{a}}_1$ and $\boldsymbol{a}_2=\boldsymbol{R}_2\bar{\boldsymbol{a}}_2$), the resulting tensor $\boldsymbol{T}$ will be:
$\boldsymbol{T}=(\boldsymbol{R}_1\bar{\boldsymbol{a}}_1)\otimes(\boldsymbol{R}_2\bar{\boldsymbol{a}}_2)=\boldsymbol{R}_1\bar{\boldsymbol{a}}_1\otimes\bar{\boldsymbol{a}}_2\boldsymbol{R}_2^T=\boldsymbol{R}_1\bar{\boldsymbol{T}}\boldsymbol{R}_2^T$. Where $\bar{\boldsymbol{T}}=\bar{\boldsymbol{a}}_1\otimes\bar{\boldsymbol{a}}_2$, the variation of $\boldsymbol{T}$ will be:
$\delta\boldsymbol{T}=\delta\boldsymbol{R}_1\bar{\boldsymbol{T}}\boldsymbol{R}_2^T+\boldsymbol{R}_1\delta\bar{\boldsymbol{T}}\boldsymbol{R}_2^T+\boldsymbol{R}_1\bar{\boldsymbol{T}}\delta\boldsymbol{R}_2^T=\delta\boldsymbol{R}_1\boldsymbol{R}_1^T\boldsymbol{T}+\boldsymbol{R}_1\delta(\boldsymbol{R}_1^T\bar{\boldsymbol{T}}\boldsymbol{R}_2)\boldsymbol{R}_2^T+\bar{\boldsymbol{T}}\boldsymbol{R}_2\delta\boldsymbol{R}_2^T=\delta\tilde{\boldsymbol{w}}_1\boldsymbol{T}+\boldsymbol{R}_1\delta(\boldsymbol{R}_1^T\bar{\boldsymbol{T}}\boldsymbol{R}_2)\boldsymbol{R}_2^T-\bar{\boldsymbol{T}}\delta\tilde{\boldsymbol{w}}_2$
Where $\delta\tilde{\boldsymbol{w}}_1=\delta\boldsymbol{R}_1\boldsymbol{R}_1^T$ and $\delta\tilde{\boldsymbol{w}}_2=\delta\boldsymbol{R}_2\boldsymbol{R}_2^T$

For Cauchy stress tensor $\boldsymbol{\sigma}$, the transformation rule $\boldsymbol{\sigma}=\boldsymbol{R}\,\bar{\boldsymbol{\sigma}}\,\boldsymbol{R}^T$ makes ($\boldsymbol{R}_1=\boldsymbol{R}_2=\boldsymbol{R}$) and
$\delta\boldsymbol{\sigma}=\delta\left(\boldsymbol{R}\,\bar{\boldsymbol{\sigma}}\,\boldsymbol{R}^T\right)=\boldsymbol{R}\delta\bar{\boldsymbol{\sigma}}\boldsymbol{R}^T+\delta\boldsymbol{R}\bar{\boldsymbol{\sigma}}\boldsymbol{R}^T+\delta\boldsymbol{R}\bar{\boldsymbol{\sigma}}\boldsymbol{R}^T=\boldsymbol{R}\,\delta\left(\boldsymbol{R}^T\boldsymbol{\sigma}\boldsymbol{R}\right)\boldsymbol{R}^T+\delta\tilde{\boldsymbol{w}}\boldsymbol{\sigma}-\boldsymbol{\sigma}\delta\tilde{\boldsymbol{w}}=\delta\overset{\circ}{\boldsymbol{\sigma}}+\delta\tilde{\boldsymbol{w}}\boldsymbol{\sigma}-\boldsymbol{\sigma}\delta\tilde{\boldsymbol{w}}$

For two-point tensor $\boldsymbol{A}$, $\boldsymbol{R}_1=\boldsymbol{R}$ and $\boldsymbol{R}_2=\boldsymbol{1}$, the resulting variation will be:
$\delta\boldsymbol{A}=\delta\left(\boldsymbol{R}\bar{\boldsymbol{A}}\right)=\boldsymbol{R}\delta\bar{\boldsymbol{A}}+\delta\boldsymbol{R}\bar{\boldsymbol{A}}=\boldsymbol{R}\,\delta\left(\boldsymbol{R}^T\boldsymbol{A}\right)+\delta\tilde{\boldsymbol{w}}\boldsymbol{A}=\delta\overset{\circ}{\boldsymbol{A}}+\delta\tilde{\boldsymbol{w}}\boldsymbol{A}$, where $\delta\overset{\circ}{\boldsymbol{A}}$ represents co-rotational variation of tensor $\boldsymbol{A}$ (farther explanation in section 3.4).

■ **Example 3.12** Assume four-node element shown in Figure 3.33 with axial stress $\sigma_{11} = \frac{P}{A}$ and then subjected to the rigid body rotation, such that the resulting stress will be:

$$\boldsymbol{\sigma} = \boldsymbol{R}^T \overline{\boldsymbol{\sigma}} \boldsymbol{R}, \quad \boldsymbol{R} = \begin{bmatrix} C & -S \\ S & C \end{bmatrix}, \quad \overline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } C = \cos(\theta), \quad S = \sin(\theta) \tag{3.173}$$

It follows that

$$\overline{\boldsymbol{\sigma}} = \sigma_{11} \begin{bmatrix} C^2 & SC \\ SC & S^2 \end{bmatrix} \tag{3.174}$$

As the deformation gradient is identical to rotation matrix, second Piola Kirchhoff stress tensor $\boldsymbol{S}$ will be:

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T\boldsymbol{F}^{-T} = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.175}$$

Which is identical to the co-rotational Cauchy stress tensor $\overline{\boldsymbol{\sigma}}$.
Reciting the definition of second Piola Kirchhoff stress tensor in the previous paragraph, $\boldsymbol{S}$ is the force applied in the current configuration is $(P\cos(\theta), P\sin(\theta))$ is subjected to inverse mapping through deformation gradient $\boldsymbol{F} = \boldsymbol{R}$ to be $(P, 0)$ applied on the initial area A which yields the same results in the above equation.

■

3.3.6 The material form of equilibrium equation of motion

Substituting with $\left(\frac{\partial}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{X}}\boldsymbol{F}^{-1}\right)$ into Equation 3.106 results in:

$$\boldsymbol{F}_s = \int_V \boldsymbol{\nabla}.\boldsymbol{\sigma} dV = \int_V \boldsymbol{\nabla}_0.\left(\boldsymbol{F}^{-1}\boldsymbol{\sigma}\right) dV = \int_{V_0} \boldsymbol{\nabla}_0.\boldsymbol{P}^T dV_0 \tag{3.176}$$

The above expression can be proven using index notation and first Piola Kirchhoff stress tensor definition as follows:

$$\boldsymbol{P} = P_{ij}\boldsymbol{e}_i \otimes \boldsymbol{E}_j = J\sigma_{ki}^T F_{kj}^{-T}\boldsymbol{e}_i \otimes \boldsymbol{E}_j; \quad F_{ij} = \frac{\partial x_i}{\partial X_j} \rightarrow F_{ij}^{-1} = \frac{\partial X_j}{\partial x_i} = F_{ji}^{-T} \tag{3.177}$$

$$\boldsymbol{\nabla}_0.\boldsymbol{P}^T = \frac{\partial P_{ij}}{\partial X_j}\boldsymbol{e}_i = J\frac{\partial \sigma_{ki}}{\partial X_j}\frac{\partial X_k}{\partial x_j}\boldsymbol{e}_i = J\frac{\partial \sigma_{ki}}{\partial X_k}\frac{\partial X_k}{\partial x_j}\delta_{jk}\boldsymbol{e}_i = J\frac{\partial \sigma_{ki}}{\partial x_j}\delta_{jk}\boldsymbol{e}_i = J\frac{\partial \sigma_{ji}}{\partial x_j}\boldsymbol{e}_i = J\boldsymbol{\nabla}.\boldsymbol{\sigma} \tag{3.178}$$

If $\boldsymbol{f}_{b0}$ is the body force per unit volume of the initial configuration, the total body force will be defined as follows:

$$\boldsymbol{F}_b = \int_V \boldsymbol{f}_b dV = \int_{V_0} \boldsymbol{f}_{b0} dV_0 \tag{3.179}$$

Which leads to the material form the motion equation of equilibrium in terms of the first Piola Kirchhoff stress tensor $\boldsymbol{P}$ as follows:

$$\boldsymbol{\nabla}_0.\boldsymbol{P}^T + \boldsymbol{f}_{\boldsymbol{b0}} - \rho_0\boldsymbol{a} = \boldsymbol{0} \tag{3.180}$$

Where $\boldsymbol{f}_{b0}$, ρ_0 are the body force and density referred to the initial configuration. Also from expression ($\boldsymbol{P} = \boldsymbol{F}.\boldsymbol{S}$), the material form of equilibrium equation of motion in terms of the second Piola Kirchhoff stress tensor will be defined as follows:

$$\boldsymbol{\nabla}_0.\left(\boldsymbol{S}^T\boldsymbol{F}^T\right) + \boldsymbol{f}_{b0} - \rho_0\boldsymbol{a} = \boldsymbol{0} \tag{3.181}$$

As second Piola Kirchhoff stress tensor is symmetric tensor, it yields that:

$$(\nabla_0.\boldsymbol{S}).\boldsymbol{F}^T + \boldsymbol{S}.\left(\nabla_0.\boldsymbol{F}^T\right) + \boldsymbol{f}_{b0} - \rho_0 \boldsymbol{a} = \boldsymbol{0} \tag{3.182}$$

Where $\nabla_0.\boldsymbol{F}^T$ can be written in index notation as follows:

$$\nabla_0.\boldsymbol{F}^T = \nabla_0.\left(\frac{\partial x_i}{\partial X_j}\boldsymbol{E}_j \otimes \boldsymbol{E}_i\right) = \frac{\partial^2 x_i}{\partial X_j \partial X_j}\boldsymbol{E}_i \tag{3.183}$$

3.3.7 Constitutive equation in the rate form

For a linear elastic body with Young modulus E and Poisson's ratio ν, the constitutive relation between the infinitesimal strain $\boldsymbol{\varepsilon}$ and Cauchy stress $\boldsymbol{\sigma}$ is defined as follows:

$$\boldsymbol{\varepsilon} = \frac{1}{E}\left[(1+\nu)\boldsymbol{\sigma} - \nu\, trace(\boldsymbol{\sigma})\right] \leftrightarrow \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \tag{3.184}$$

with index notation defined as follows:

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{ii}\right] \tag{3.185}$$

But its time rate form does not follow the above constitutive equation or:

$$\boldsymbol{D} \neq \frac{1}{E}\left[(1+\nu)\dot{\boldsymbol{\sigma}} - \nu\, trace(\dot{\boldsymbol{\sigma}})\right] \leftrightarrow \dot{\boldsymbol{\sigma}} \neq \boldsymbol{C} : \boldsymbol{D} \tag{3.186}$$

For example, if the body is subjected to rigid body rotation, $\boldsymbol{D} = \boldsymbol{0}$ as stated in the subsection 3.2.5,

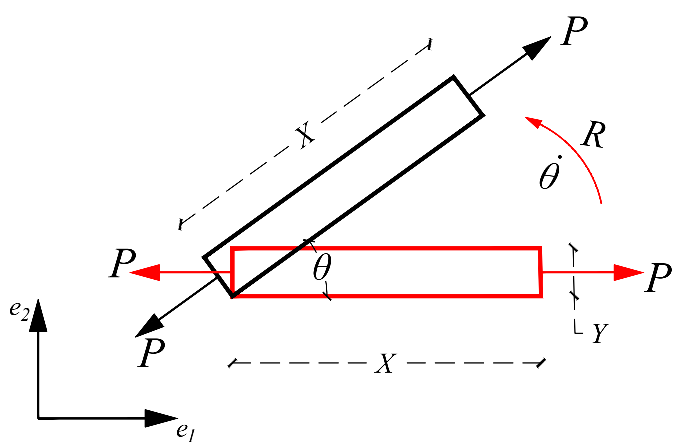


Figure 3.33

while $\dot{\boldsymbol{\sigma}}$ changes according to the transformation rule (see in the next example).

■ **Example 3.13** If we have a bar shown in Figure 3.33 with cross section area A and axial load P inclined at angle θ and under rigid body rotation with time rate $\dot{\theta}$, the current stress rate $\boldsymbol{\sigma}(\theta(t))$ is defined as:

$$\begin{aligned} \boldsymbol{\sigma}(\theta(t)) &= \boldsymbol{R}(\theta(t))\,\boldsymbol{\sigma}(\theta=0)\,\boldsymbol{R}(\theta(t))^T \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{P}{A} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \frac{P}{A}\begin{bmatrix} \cos(\theta)^2 & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin(\theta)^2 \end{bmatrix} \end{aligned} \tag{3.187}$$

As (P/A) remains the same with time and angle θ changes, the time rate of change of stress will be:

$$\dot{\boldsymbol{\sigma}} = \dot{\theta}\frac{P}{A}\begin{bmatrix} -2CS & C^2 - S^2 \\ C^2 - S^2 & 2SC \end{bmatrix} \qquad \text{Using } C = \cos(\theta),\ S = \sin(\theta) \tag{3.188}$$

While $\boldsymbol{D}$ vanishes if we used the same procedures defined in subsection 3.2.5. Consequently, Cauchy stress rate and rate of deformation tensor behave incompatibly in the presence of finite rotation. This problem forces us to search for new objective rates for stresses and strains. Using an *objective stress rate* is an essential step in nonlinear finite element analysis. In the next section, we will find out other objective stress measures which can be also used in nonlinear analysis. For example, we can relate the time rate of change of Green-Lagrange strain $\dot{\boldsymbol{E}}$ and time rate of change of second Piola Kirchhoff stress tensor $\dot{\boldsymbol{S}}$ as follows:

$$\dot{\boldsymbol{E}} = \frac{1}{E}\left[(1+\nu)\dot{\boldsymbol{S}} - \nu\, trace\left(\dot{\boldsymbol{S}}\right)\right] \leftrightarrow \dot{\boldsymbol{S}} = \boldsymbol{C} : \dot{\boldsymbol{E}} \tag{3.189}$$

The above expression can be used as stress-strain constitutive relation in the rate form, as this relation is not effected by finite rotation and consequently are considered objective quantity. ■

3.4 Change of observer and objectivity

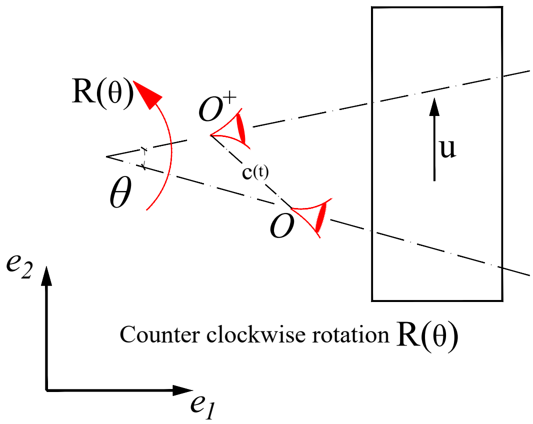


Figure 3.34: Two observers tracking a rectangular block

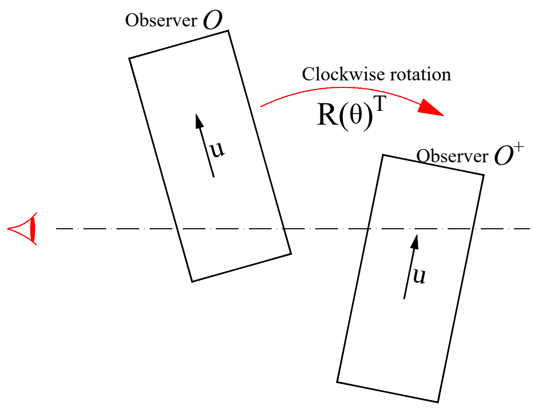


Figure 3.35: What observers O and O^+ see in Figure 3.34

Any physical phenomena should remain unchanged even if we change the observer or the point of view from which we observe it. This is called objectivity or frame-indifference which is necessary part in nonlinear continuum mechanics. We can describe a phenomena or an event by choosing an observer which has the ability to record the position and the time of the event, and track its change with time. Assume we have two observers O and O^+ monitoring the same event (two-dimension event) (e.g. a rectangular block) through their eyes as shown in Figure 3.34. If we asked both observers to take a snapshot of what they see, we find that each every observer sees a different picture (e.g. observer O finds the rectangular block inclined toward him or her, while the other sees away inclination for the block as shown in Figure 3.35). Assuming the relative position between the two observers is $\boldsymbol{c}(t)$ and the orientation of observer O^+ is formed through applying rotation $\boldsymbol{R}(t)$ on the observer O by rotating an angle θ about axis $\boldsymbol{e}_3$. These terms $\boldsymbol{c}(t), \boldsymbol{R}(t)$ may change with time t as one of the two observers may be moving relative to the other. For vector $\boldsymbol{u}$ attached to the rectangular block as shown in Figure 3.35 and observed by observer O as $\boldsymbol{u}$, it will be observed by O^+ defined as follows:

$$\boldsymbol{u}^+ = \boldsymbol{R}(t)^T \boldsymbol{u} = \boldsymbol{Q}(t)\,\boldsymbol{u} \tag{3.190}$$

For a general position $\boldsymbol{X}$ in space, if this position is monitored by the two observers as $\boldsymbol{X}$ and $\boldsymbol{X}^+$, these two observations are related through the following:

$$\boldsymbol{X}^+ = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\,\boldsymbol{X} \tag{3.191}$$

Where $\boldsymbol{Q}(t)$ represents the transformation tensor from observer O to observer O^+ which is equivalent to the transpose or inverse of rotation tensor $\boldsymbol{Q}(t) = \boldsymbol{R}(t)^T$. Any vector that transforms like the above expression is called objective. We also conclude that the change in observer preserves the scalar quantities like material properties at the point of interest, the distance between two points and the angle between two vectors.

The velocity and acceleration vector are not objective as the time rate of change of Equation 3.190 results:

$$\begin{aligned} \boldsymbol{v}^+ = \dot{\boldsymbol{c}}(t) + \boldsymbol{Q}(t)\,\boldsymbol{v} + \dot{\boldsymbol{Q}}(t)\,\boldsymbol{X} &= \dot{\boldsymbol{c}}(t) + \boldsymbol{Q}(t)\,\boldsymbol{v} + \dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T\left(\boldsymbol{X}^+ - \boldsymbol{c}(t)\right) \\ &= \dot{\boldsymbol{c}}(t) + \boldsymbol{Q}(t)\,\boldsymbol{v} - \boldsymbol{W}\left(\boldsymbol{X}^+ - \boldsymbol{c}(t)\right) \\ &\neq \dot{\boldsymbol{c}}(t) + \boldsymbol{Q}(t)\,\boldsymbol{v} \\ as \quad \dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}(t)^T &= \dot{\boldsymbol{R}}(t)^T \boldsymbol{R}(t) = \tilde{\boldsymbol{w}}^T = \boldsymbol{W}^T = -\boldsymbol{W} \qquad \text{See chapter 2} \end{aligned} \tag{3.192}$$

Where $(\dot{\boldsymbol{A}})$ signifies the time derivative of the quantity $(\boldsymbol{A})$. The non objectivity results from the effect of spin appeared in the last term in the above equation $-\boldsymbol{W}\,(\boldsymbol{X}^+ - \boldsymbol{c}(t))$.

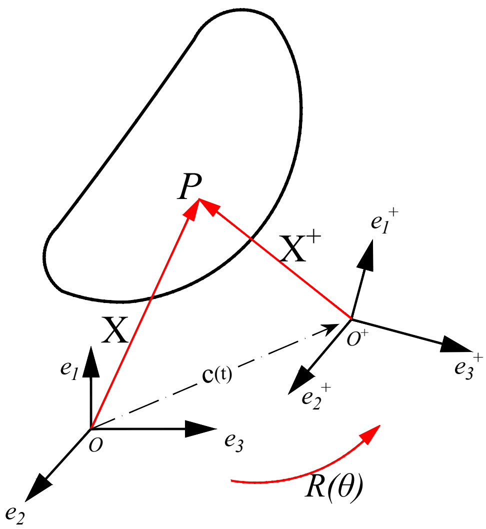


Figure 3.36: One event monitored by two observers

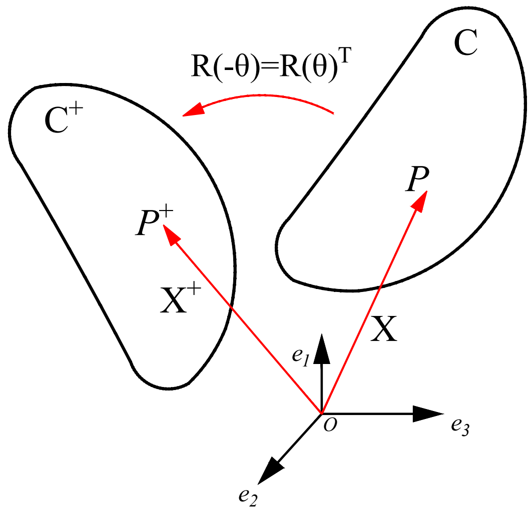


Figure 3.37: Two events monitored by a single observer. The second event C^+ is formed through superimposing a rigid body rotation $\boldsymbol{R}(\boldsymbol{\theta})^T$ on configuration C

To simplify the idea, a single motion monitored by two observers can be equivalent two different events observed by the same observer via rotating the event in reverse direction the observer is rotated as pictured in Figure 3.36, so the same results can be obtained if we assume two different events observed by single observer O as shown in Figure 3.37, such that the second event or configuration C^+ is formed via superimposing a rigid body rotation $\boldsymbol{R}^T(\theta)$ on the first

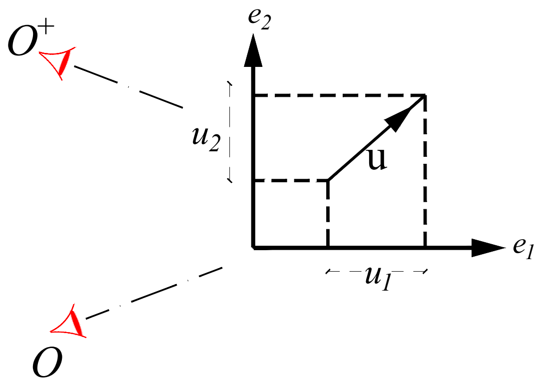


Figure 3.38: The components of vector $\boldsymbol{u}$ resolved in a particular basis, e.g. $\boldsymbol{e}_i$ do not change with changing the observer

configuration C. This rotation makes orientation of vector $\boldsymbol{u}$ attached to the body rotate to $\boldsymbol{u}^+$ in the final configuration as follows:

$$\boldsymbol{X}^+ = \boldsymbol{c}(t) + \boldsymbol{R}(t)^T \boldsymbol{X} \tag{3.193}$$

which is identical to the results of Equation 3.191. Term $\boldsymbol{c}(t)$ is the position vector linking observer O and observer O^+ as shown in Figure 3.36.

To describe any physical event in three dimensional space, we have to assign a frame of reference (rectangular coordinate system) for each observers. If we choose a single inertia frame (e.g. $\boldsymbol{e}_i$) for both observers as shown in Figure 3.38, the vector $\boldsymbol{u}$ seen by observers O and O^+ can be resolved in this frame through:

$$\boldsymbol{u} = u_i \boldsymbol{e}_i, \qquad \boldsymbol{u}^+ = u_i^+ \boldsymbol{e}_i^+ \tag{3.194}$$

Where $\boldsymbol{u}$ and $\boldsymbol{e}_i$ defines, respectively, the vectors $\boldsymbol{u}$ and basis $\boldsymbol{e}_i$ monitored by observer O, while $\boldsymbol{u}^+$ and $\boldsymbol{e}_i^+$ defines the same vectors monitored by observer O^+ with relation defined as:

$$\boldsymbol{u}^+ = \boldsymbol{Q}u, \qquad \boldsymbol{e}_i^+ = Q\boldsymbol{e}_i \tag{3.195}$$

Substituting the above expressions into Equation 3.191, we get:

$$\begin{aligned} \boldsymbol{u}^+ &= \boldsymbol{Q}u \\ u_i^+ \boldsymbol{e}_i^+ &= \boldsymbol{Q} u_i \boldsymbol{e}_i \\ u_i^+ \boldsymbol{Q}\boldsymbol{e}_i &= \boldsymbol{Q} u_i \boldsymbol{e}_i \rightarrow u_i^+ = u_i \end{aligned} \tag{3.196}$$

We conclude that the components of vector $\boldsymbol{u}$ observed by two different observers and resolved in the same frame are identical and independent of the observer as the projection of some vector on some basis is a scalar value which does not change with changing the observer.

If we have two vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that transform according to the above rule like $\boldsymbol{u}_1^+ = \boldsymbol{Q}_1 \boldsymbol{u}_1$ and $\boldsymbol{u}_2^+ = \boldsymbol{Q}_2 \boldsymbol{u}_2$ and a second order tensor $\boldsymbol{A}$ defined through dyadic product ($\boldsymbol{u}_1 \otimes \boldsymbol{u}_2$), this tensor can be seen by both observers as follows:

$$\boldsymbol{A}^+ = \boldsymbol{u}_1^+ \otimes \boldsymbol{u}_2^+ = (\boldsymbol{Q}_1 \boldsymbol{u}_1) \otimes (\boldsymbol{Q}_2 \boldsymbol{u}_2) = \boldsymbol{Q}_1 (\boldsymbol{u}_1 \otimes \boldsymbol{u}_2) \boldsymbol{Q}_2^T \tag{3.197}$$

For Cauchy stress tensor $\boldsymbol{\sigma}$, it is written in index notation as $\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ where $\boldsymbol{e}_i^+ = \boldsymbol{Q}\boldsymbol{e}_i$, it follows that

$$\boldsymbol{\sigma}^+ = \sigma_{ij} \boldsymbol{e}_i^+ \otimes \boldsymbol{e}_j^+, \qquad \text{where } \sigma_{ij} \text{ and } \sigma_{ij}^+ \text{ are identical as they are components} \tag{3.198}$$

$$\boldsymbol{\sigma}^+ = \sigma_{ij}\left(\boldsymbol{Q}\boldsymbol{e}_i\right)\otimes\left(\boldsymbol{Q}\boldsymbol{e}_j\right) = \boldsymbol{Q}\left(\sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right)\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T \tag{3.199}$$

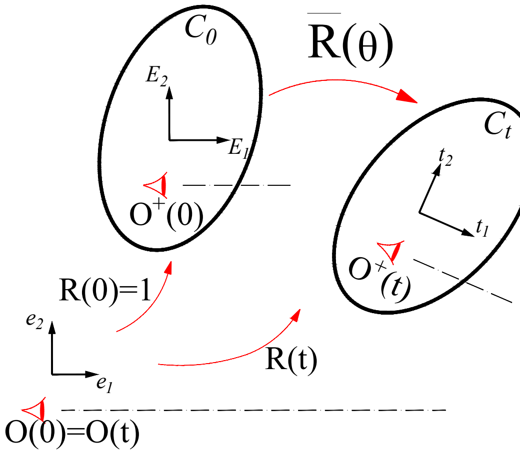


Figure 3.39

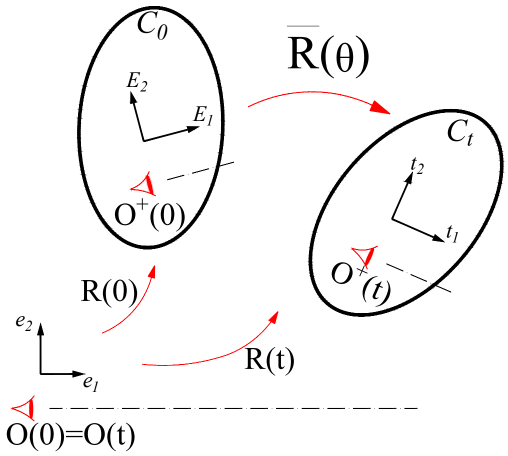


Figure 3.40

■ **Example 3.14** Assume we have two frame of references shown in Figure 3.39, one is inertia frame fixed in the space $\boldsymbol{e}_i$ and other is co-rotational or moving frame $\boldsymbol{t}_I$ attached to the body. We note that the co-rotational frame $\boldsymbol{t}_I(t)$ is changing with time t and is identical to the material frame $\boldsymbol{E}_I$ at time $(t=0)$ such that:

$$\boldsymbol{t}_I(t=0) = \boldsymbol{E}_I \tag{3.200}$$

Assume we have two observers; one fixed in the space (observer O) and the other attached to the body (observer O^+). The orientation of observer O^+ is formed through the rotation of the body with time $\bar{\boldsymbol{R}}(t)$ superimposed on observer o such that $\bar{\boldsymbol{R}}(t=0)=\boldsymbol{1}$ and they have the same orientation at time $t=0$. If the initial configuration of the body is C_0 and is rotated by rotation tensor $\bar{\boldsymbol{R}}(t)$ to configuration C_t, the moving frame will be related to the material frame through:

$$\boldsymbol{t}_I(t) = \bar{\boldsymbol{R}}(t)\boldsymbol{E}_I, \qquad \text{or} \qquad \bar{\boldsymbol{R}}(t) = \boldsymbol{t}_I(t)\otimes\boldsymbol{E}_I \tag{3.201}$$

We can observe this tensor rotation through observer O^+ as follows:

$$\bar{\boldsymbol{R}}(t)^+ = \boldsymbol{t}_I(t)^+\otimes\boldsymbol{E}_I^+ \tag{3.202}$$

As the two observers orientation are identical in the initial configuration C_0 ($\boldsymbol{R}(t=0)=\boldsymbol{1}$) we get $\boldsymbol{E}_I^+ = \boldsymbol{E}_I$. This results can be observed in Figure 3.39 (both observers O and O^+ are directed in the same directions at C_0), while in the final configuration C_t, the moving frame $\boldsymbol{t}_i$ seen by the two observers $O(t)$, $O^+(t)$ follows this relation:

$$\boldsymbol{t}_I(t)^+ = \boldsymbol{Q}(t)\boldsymbol{t}_I(t) = \boldsymbol{R}(t)^T\boldsymbol{t}_I(t) \tag{3.203}$$

From above expression, Equation 3.202 will be:

$$\bar{\boldsymbol{R}}(t)^+ = \boldsymbol{t}_I(t)^+\otimes\boldsymbol{E}_I^+ = \boldsymbol{Q}(t)\left(\boldsymbol{t}_I(t)\otimes\boldsymbol{E}_I\right) = \boldsymbol{Q}\bar{\boldsymbol{R}} = \boldsymbol{R}^T\bar{\boldsymbol{R}} = \boldsymbol{1} \tag{3.204}$$

as $\bar{\boldsymbol{R}}$ and $\boldsymbol{R}$ are identical from Figure 3.39 $\left(\boldsymbol{R}(t) = \bar{\boldsymbol{R}}(t)\boldsymbol{R}(0) = \bar{\boldsymbol{R}}(t)\right)$. Even if the orientation of observer O^+ is not identical to that of observer O in the initial configuration C_0 as shown in Figure 3.40, we get also the same above result $\bar{\boldsymbol{R}}(t)^+ = \mathbf{1}$. As if we use the same above example with both $\boldsymbol{E}_I$ and observer O^+ are formed through superimposing a rotation tensor $\boldsymbol{R}_0$ on $\boldsymbol{e}_i$, it results:

$$\boldsymbol{E}_i = \boldsymbol{R}_0\boldsymbol{e}_i, \qquad \boldsymbol{E}_I^+ = \boldsymbol{Q}_0\boldsymbol{E}_I, \qquad \text{where} \quad \boldsymbol{Q}_0 = \boldsymbol{Q}(t=0) = \boldsymbol{R}_0^T = \boldsymbol{R}(t=0)^T \tag{3.205}$$

For a rotation tensor $\bar{\boldsymbol{R}}$ imposed on the body in configuration C_0 to form configuration C_1, the observations by observers O and O^+ will be:

$$\bar{\boldsymbol{R}}^+ = \boldsymbol{t}_I^+ \otimes \boldsymbol{E}_I^+ \tag{3.206}$$

As $\boldsymbol{t}_I^+ = \boldsymbol{Q}_t\boldsymbol{t}_I$ and $\boldsymbol{E}_I^+ = \boldsymbol{Q}_0\boldsymbol{E}_I$, we get

$$\bar{\boldsymbol{R}}^+ = \boldsymbol{Q}_t\boldsymbol{t}_I \otimes \boldsymbol{E}_I^+ = \boldsymbol{Q}_t\bar{\boldsymbol{R}}\boldsymbol{Q}_0^T = \mathbf{1} \tag{3.207}$$

The last equality comes from the fact that $\boldsymbol{Q}_t^T = \boldsymbol{R}_t = \bar{\boldsymbol{R}}\boldsymbol{R}_0 = \bar{\boldsymbol{R}}\boldsymbol{Q}_0^T$.
From above, rotation vector is called two-point tensor and transforms like vector field as follows:

$$\boldsymbol{R}^+ = \boldsymbol{Q}\boldsymbol{R} \tag{3.208}$$

Also rotation tensor $\boldsymbol{R}$ is composed of three orthonormal unit vectors, e.g. $[\boldsymbol{R}]_I = [\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3]$, each vector transform like vector, so we can get the same findings of the above equation. ■

In the same manner, deformation gradient $\boldsymbol{F} = F_{iI}\boldsymbol{e}_i \otimes \boldsymbol{E}_I$ transforms like vector field ($\boldsymbol{F}^+ = \boldsymbol{Q}\boldsymbol{F}$). Using spectral decomposition for deformation:

$$\boldsymbol{F}^+ = \boldsymbol{R}^+\boldsymbol{U}^+ = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{U} = \boldsymbol{Q}\boldsymbol{F} \tag{3.209}$$

as $\boldsymbol{R}^+ = \boldsymbol{Q}\boldsymbol{R}$, while $\boldsymbol{U}$ is not effected with rotation ($\boldsymbol{U}^+ = \boldsymbol{U}$) or from

$$\boldsymbol{F} = \frac{\partial x}{\partial X} \rightarrow \boldsymbol{F}^+ = \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right)^+ = \frac{\partial \boldsymbol{x}^+}{\partial \boldsymbol{X}} = \boldsymbol{Q}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{Q}\boldsymbol{F} \tag{3.210}$$

As rigid body rotation transform vector $d\boldsymbol{x}$ through $d\boldsymbol{x}^+ = \boldsymbol{Q}d\boldsymbol{x}$ Similarly, the first Piola stress tensor $\boldsymbol{P} = P_{iI}\boldsymbol{e}_i \otimes \boldsymbol{E}_I$ transform like vector field ($\boldsymbol{P}^+ = \boldsymbol{Q}\boldsymbol{P}$) as:

$$\boldsymbol{P}^+ = J\boldsymbol{\sigma}^+.\boldsymbol{F}^{+-T} = J\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F}^{-T} = \boldsymbol{Q}\boldsymbol{P} \tag{3.211}$$

Another type of second-order tensor is called material tensors or tensors parameterized only by material coordinates only like stretch tensor $\boldsymbol{U} = U_{IJ}\boldsymbol{E}_I \otimes \boldsymbol{E}_J$, Green-Lagrangian strain tensor $\boldsymbol{E}$, and second Piola Kirchhoff $\boldsymbol{S}$ that transforms as follows:

$$\boldsymbol{E}^+ = \frac{1}{2}\left(\boldsymbol{F}^{+T}\boldsymbol{F}^+ - \mathbf{1}\right) = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \mathbf{1}\right) = \boldsymbol{E} \rightarrow \boldsymbol{F}^{+T}\boldsymbol{F}^+ = \boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{F} \tag{3.212}$$

In the same manner:

$$\boldsymbol{S}^+ = J\boldsymbol{F}^{+-1}\boldsymbol{\sigma}^+.\boldsymbol{F}^{+-T} = J\boldsymbol{F}^{+-1}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F}^{-T} = \boldsymbol{S} \tag{3.213}$$

All stresses and strains measure mentioned above are objective, while the time rate of change of Cauchy tress is not, as differentiating $\boldsymbol{\sigma}^+ = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$ with time results in:

$$\dot{\boldsymbol{\sigma}}^+ = \dot{\boldsymbol{Q}}\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\dot{\boldsymbol{Q}}^T \neq \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T \tag{3.214}$$

Also we need to check the objectivity of different types of strain rates like $\dot{\boldsymbol{F}}$, $\boldsymbol{L}$, $\boldsymbol{D}$, and $\boldsymbol{W}$ as follows:

$$\boldsymbol{F}^{+} = \boldsymbol{QF} \leftrightarrow \dot{\boldsymbol{F}}^{+} = \dot{\boldsymbol{Q}}\boldsymbol{F} + \boldsymbol{Q}\dot{\boldsymbol{F}} \neq \boldsymbol{Q}\dot{\boldsymbol{F}} \tag{3.215}$$

$$\boldsymbol{L}^{+} = \dot{\boldsymbol{F}}^{+}\boldsymbol{F}^{+-1} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} + \boldsymbol{QLQ}^{T} \neq \boldsymbol{QLQ}^{T} \tag{3.216}$$

$$\boldsymbol{D}^{+} = \frac{\boldsymbol{L}^{+} + \boldsymbol{L}}{2} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{Q}}^{T} + \boldsymbol{Q}\frac{(\boldsymbol{L}+\boldsymbol{L}^{T})}{2}\boldsymbol{Q}^{T} = \boldsymbol{Q}\frac{(\boldsymbol{L}+\boldsymbol{L}^{T})}{2}\boldsymbol{Q}^{T} = \boldsymbol{QDQ}^{T} \tag{3.217}$$

$$\boldsymbol{W}^{+} = asym(\boldsymbol{L}^{+}) = \boldsymbol{QWQ}^{T} + \dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} \neq \boldsymbol{QWQ}^{T} \tag{3.218}$$

We find that all the above time rate of change of strains mentioned above are non-objective and do not follow the transformation rules except the rate of deformation $\boldsymbol{D}$.

■ **Example 3.15** Lets assume a bar shown in Figure 3.33 with area A and subjected to axial load $\boldsymbol{P}$ and aligned horizontally in the initial orientation, and its orientation is changing with time t such that the bar only undergoes rigid body rotation. We need to write down the Cauchy stress referred to two frame of reference; spatial frame and co-rotational frame attached to the body

If the stress at the initial configuration and final configuration at time t are denoted by $\boldsymbol{\sigma}(0)$, and $\boldsymbol{\sigma}(t)$, respectively, the relation between them will be:

$$\boldsymbol{\sigma}(t) = \boldsymbol{R}(t)\,\boldsymbol{\sigma}(0)\,\boldsymbol{R}(t)^{T} \tag{3.219}$$

Where $R(t)$ infers the rotation tensor that defines the orientation of the bar. This orientation is a function of the time t. To sense the values in the problem and describe it, we have to choose a suitable coordinate system, e.g. coordinate system $\boldsymbol{E}$, such that the Cauchy stress resolved in this coordinate system at the initial and final configuration will be:

$$[\boldsymbol{\sigma}(0)]_{\boldsymbol{E}\otimes\boldsymbol{E}} = \begin{bmatrix} \frac{P}{A} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.220}$$

$$\begin{aligned}[\boldsymbol{\sigma}(t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} &= \left[\boldsymbol{R}(t)\,\boldsymbol{\sigma}(0)\,\boldsymbol{R}(t)^{T}\right]_{\boldsymbol{E}\otimes\boldsymbol{E}} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} \frac{P}{A} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \frac{P}{A}\begin{bmatrix} \cos(\theta)^{2} & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin(\theta)^{2} \end{bmatrix}\end{aligned} \tag{3.221}$$

If we observe the same stress using the co-rotational frame attached to the body, the stress will be called co-rotational stress resolved as follows:

$$[\check{\boldsymbol{\sigma}}(t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} = [\boldsymbol{\sigma}(t)]_{\boldsymbol{t}\otimes\boldsymbol{t}} \tag{3.222}$$

$$= \boldsymbol{R}(t)\,[\boldsymbol{\sigma}(t)]_{\boldsymbol{E}\otimes\boldsymbol{E}}\boldsymbol{R}(t)^{T} \tag{3.223}$$

$$= [\boldsymbol{\sigma}(0)]_{\boldsymbol{E}\otimes\boldsymbol{E}} \tag{3.224}$$

$$= \begin{bmatrix} \frac{P}{A} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.225}$$

So we conclude that co-rotational Cauchy stress is an objective quantity as it is independent of the bar orientation as follows: ■

Also its time rate of change is objective such that it can follow the **material constitutive relation** in the rate form (e.g. for linear elastic material):

$$\boldsymbol{D} = \frac{1}{E}\left[(1+\nu)\,\mathring{\dot{\boldsymbol{\sigma}}} - \nu\; trace\left(\mathring{\dot{\boldsymbol{\sigma}}}\right)\right] \leftrightarrow \mathring{\dot{\boldsymbol{\sigma}}} = \boldsymbol{C}:\boldsymbol{D} \tag{3.226}$$

We shall now introduce another type of stress measure known as Jamann stress rate $\boldsymbol{\sigma^o}$ which is considered as a push forward to the time rate of change of co-rotational Cauchy stress:

$$\boldsymbol{\sigma^o} = \boldsymbol{R}\left(\mathring{\dot{\boldsymbol{\sigma}}}\right)\boldsymbol{R}^T \tag{3.227}$$

$$= \boldsymbol{R}\left(\boldsymbol{Q\sigma Q}^T\right)_{,t}\boldsymbol{R}^T \tag{3.228}$$

$$= \boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{\sigma R}\right)_{,t}\boldsymbol{R}^T \tag{3.229}$$

$$= \dot{\boldsymbol{\sigma}} + \boldsymbol{R}\dot{\boldsymbol{R}}^T\boldsymbol{\sigma} + \boldsymbol{\sigma}\dot{\boldsymbol{R}}\boldsymbol{R}^T \tag{3.230}$$

$$= \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega\sigma} + \boldsymbol{\sigma\Omega} \tag{3.231}$$

We used ($\boldsymbol{RR}^T = \boldsymbol{1} \rightarrow \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \boldsymbol{0} \rightarrow \boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T$) in the above expression
In the same manner, for vector $\boldsymbol{v}$, the co-rotational time rate of change of this vector $\boldsymbol{v}^o$ is defined as:

$$\boldsymbol{v^o} = \boldsymbol{R}\left(\mathring{\dot{\boldsymbol{v}}}\right) = \boldsymbol{R}(\boldsymbol{Qv})_{,t} = \boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{v}\right)_{,t} = \dot{\boldsymbol{v}} + \boldsymbol{R}\dot{\boldsymbol{R}}^T\boldsymbol{v} = \dot{\boldsymbol{v}} - \boldsymbol{\Omega v} \tag{3.232}$$

Which $\boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{v}\right)_{,t}$ $\left[\boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{\sigma R}\right)_{,t}\boldsymbol{R}^T\right]$ means rate change of spatial tensor $\boldsymbol{v}\,[\boldsymbol{\sigma}]$ taken by an observer attached to the body. For a fixed observer in space, he or she needs to pull-back the object to the material form $\boldsymbol{R}^T\boldsymbol{v}$ $\left[\boldsymbol{R}^T\boldsymbol{\sigma R}\right]$ to perform the usual derivative operation and then push-forward to the spatial form $\boldsymbol{R}\left(\boldsymbol{R}^T\boldsymbol{v}\right)_{,t}$ $\left[\boldsymbol{R}\left(\boldsymbol{Q\sigma Q}^T\right)_{,t}\boldsymbol{R}^T\right]$; or equivalently removing the spin effect $\boldsymbol{Wv}$ $[\boldsymbol{\Omega\sigma} - \boldsymbol{\sigma\Omega}]$ from the usual derivative $\dot{\boldsymbol{v}}\,[\dot{\boldsymbol{\sigma}}]$ to have the same objective observation seen by an observer fixed in the moving frame .Another application to co-rotated derivative of basis $\boldsymbol{e}_i$ attached to the body is null

$$\boldsymbol{e}_i^o = \dot{\boldsymbol{e}}_i - \boldsymbol{\Omega e}_i = \boldsymbol{0} \tag{3.233}$$

Such that:

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega e}_i \tag{3.234}$$

The objectivity of Jamann stress rate can be proven as follows:
For tensor $\boldsymbol{\sigma}^o$ observed by O and O^+ as follows

$$\boldsymbol{\sigma}^o = \dot{\boldsymbol{\sigma}} - \boldsymbol{W\sigma} + \boldsymbol{\sigma W} \tag{3.235}$$

$$\boldsymbol{\sigma}^{+o} = \dot{\boldsymbol{\sigma}}^+ - \boldsymbol{W}^+\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^+\boldsymbol{W}^+ \tag{3.236}$$

Where

$$\boldsymbol{W}^+ = \boldsymbol{QWQ}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T \qquad \boldsymbol{\sigma}^+ = \boldsymbol{Q\sigma Q}^T \qquad \dot{\boldsymbol{\sigma}}^+ = \dot{\boldsymbol{Q}}\boldsymbol{\sigma Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q\sigma}\dot{\boldsymbol{Q}}^T \tag{3.237}$$

Substituting into Equation 3.236 results in:

$$\begin{aligned}\boldsymbol{\sigma}^{+o} &= \dot{\boldsymbol{\sigma}}^+ - \boldsymbol{W}^+\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^+\boldsymbol{W}^+\\ &= \dot{\boldsymbol{Q}}\boldsymbol{\sigma Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q\sigma}\dot{\boldsymbol{Q}}^T - \left(\boldsymbol{QWQ}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)\boldsymbol{Q\sigma Q}^T - \boldsymbol{Q\sigma Q}^T\left(\boldsymbol{QWQ}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\right)\\ &= \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T - \boldsymbol{QW\sigma Q}^T - \boldsymbol{Q\sigma WQ}^T = \boldsymbol{Q\sigma}^o\boldsymbol{Q}^T\end{aligned} \tag{3.238}$$

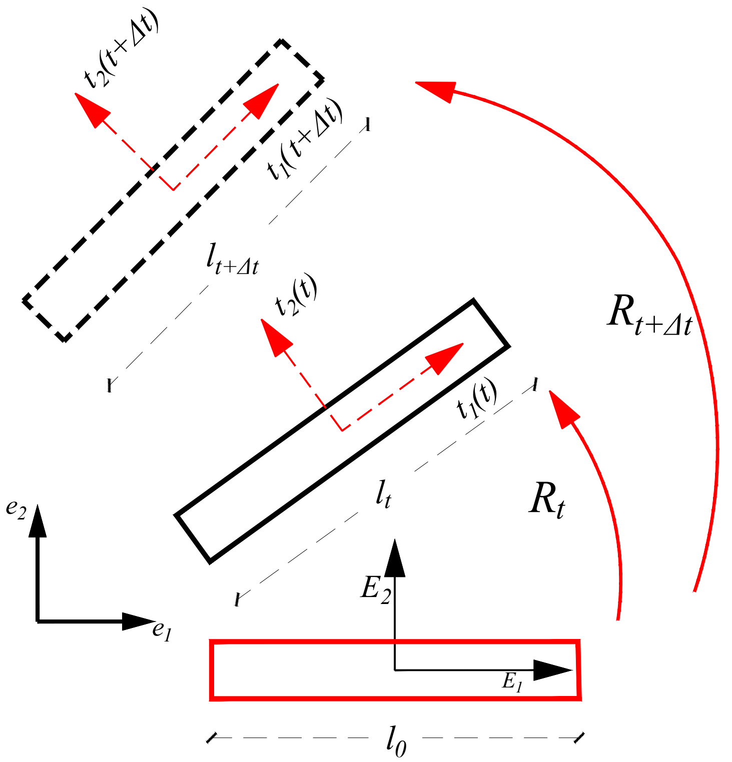


Figure 3.41

From above equation, we conclude that Jamann stress rate is an objective rate.

Assume a bar shown in Figure 3.41 with cross section area A which is aligned horizontally in the unstressed configuration C_0 with material frame $\boldsymbol{E}$ attached to it, then rotated to configuration C_t at time t with axial load P_1 with length l_t and co-rotational frame $\boldsymbol{t}$ attached to the body at the other configuration C_t. If the bar is further stretched to $l_{t+\triangle t} = l_t + \frac{\partial l}{\partial t}\triangle t$ to form final the configuration $C_{t+\triangle t}$ with final axial load P_2. The co-rotational stress at two different times $t,\ t+\triangle t$ is defined as:

$$[\check{\boldsymbol{\sigma}}(t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} = \begin{bmatrix} \frac{P_1}{A} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{as} \qquad \check{\boldsymbol{\sigma}}(t) = \boldsymbol{Q}_t\boldsymbol{\sigma}_t\boldsymbol{Q}_t^T \leftrightarrow \boldsymbol{\sigma}_t = \boldsymbol{R}_t\check{\boldsymbol{\sigma}}(t)\boldsymbol{R}_t^T \tag{3.239}$$

$$[\check{\boldsymbol{\sigma}}(t+\triangle t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} = \begin{bmatrix} \frac{P_2}{A} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.240}$$

$$\text{as} \qquad \check{\boldsymbol{\sigma}}(t+\triangle t) = \boldsymbol{Q}_{t+\triangle t}\boldsymbol{\sigma}_{t+\triangle t}\boldsymbol{Q}_{t+\triangle t}^T \leftrightarrow \boldsymbol{\sigma}_{t+\triangle t} = \boldsymbol{R}_{t+\triangle t}\check{\boldsymbol{\sigma}}(t+\triangle t)\boldsymbol{R}_{t+\triangle t}^T \tag{3.241}$$

If the body co-rotational coordinate system rotates with rate $\dot{\boldsymbol{\theta}}$, the rotation tensor at time $t+\triangle t$ will be:

$$\boldsymbol{R}_{t+\triangle t} = \boldsymbol{R}\left(\dot{\boldsymbol{\theta}}\triangle t\right)\boldsymbol{R}_t = \left(\mathbf{1}+\widetilde{\dot{\boldsymbol{\theta}}}\triangle t\right)\boldsymbol{R}_t = (\mathbf{1}+\boldsymbol{\Omega}\triangle t)\boldsymbol{R}_t \tag{3.242}$$

We assumed in the second equality in the above equation that $\dot{\boldsymbol{\theta}}\triangle t$ is infinitesimal due to the infinitesimal change in time $\triangle t$ such that ($\boldsymbol{R}\left(\dot{\boldsymbol{\theta}}\triangle t\right) = \left(\mathbf{1}+\widetilde{\dot{\boldsymbol{\theta}}}\triangle t\right)$).
Using the following

$$(\mathbf{1}+\boldsymbol{\Omega}\triangle t)^T = (\mathbf{1}-\boldsymbol{\Omega}\triangle t) \tag{3.243}$$

We can evaluate the time rate of change of Cauchy stress $\dot{\boldsymbol{\sigma}}$ as follows:

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \frac{\boldsymbol{\sigma}_{t+\triangle t} - \boldsymbol{\sigma}_t}{\triangle t} = \frac{\boldsymbol{R}_{t+\triangle t}\check{\boldsymbol{\sigma}}(t+\triangle t)\boldsymbol{R}^T_{t+\triangle t} - \boldsymbol{R}_t\check{\boldsymbol{\sigma}}(t)\boldsymbol{R}^T_t}{\triangle t} \\
&= \frac{(\mathbf{1}+\boldsymbol{\Omega}\triangle t)\left(\boldsymbol{R}_t\{\check{\boldsymbol{\sigma}}(t) + \boldsymbol{R}^t_t[\boldsymbol{C}:\boldsymbol{D}\triangle t]\boldsymbol{R}_t\}\boldsymbol{R}^T_t\right)(\mathbf{1}-\boldsymbol{\Omega}\triangle t) - \boldsymbol{R}_t\check{\boldsymbol{\sigma}}(t)\boldsymbol{R}^T_t}{\triangle t} \\
&= \frac{(1+\boldsymbol{\Omega}\triangle t)\left(\boldsymbol{R}_t\check{\boldsymbol{\sigma}}(t)\boldsymbol{R}^T_t + \boldsymbol{R}_t(\boldsymbol{R}^t_t[\boldsymbol{C}:\boldsymbol{D}]\boldsymbol{R}_t\triangle t)\boldsymbol{R}^T_t\right)(1-\boldsymbol{\Omega}\triangle t) - \boldsymbol{R}_t\check{\boldsymbol{\sigma}}(t)\boldsymbol{R}^T_t}{\triangle t} \\
&= \frac{(1+\boldsymbol{\Omega}\triangle t)(\boldsymbol{\sigma}_t + \boldsymbol{C}:\boldsymbol{D})(1-\boldsymbol{\Omega}\triangle t) - \boldsymbol{\sigma}_t}{\triangle t} \\
&= \frac{(\boldsymbol{\Omega}\boldsymbol{\sigma}_t - \boldsymbol{\sigma}_t\boldsymbol{\Omega} + (\boldsymbol{C}:\boldsymbol{D}))\triangle t + O(\triangle t^2)}{\triangle t} \\
&\simeq \boldsymbol{C}:\boldsymbol{D} + \boldsymbol{\Omega}\boldsymbol{\sigma}_t - \boldsymbol{\sigma}_t\boldsymbol{\Omega} \to \boldsymbol{C}:\boldsymbol{D} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\boldsymbol{\sigma}_t + \boldsymbol{\sigma}_t\boldsymbol{\Omega} = \boldsymbol{\sigma}^o \\
&\qquad \boxed{\boldsymbol{\sigma}^o = \boldsymbol{C}:\boldsymbol{D}}
\end{aligned} \tag{3.244}$$

Such that the constitutive relation will be:

$$\boldsymbol{D} = \frac{1}{E}\left[(1+\nu)(\boldsymbol{\sigma}^o) - \nu\, trace(\boldsymbol{\sigma}^o)\right] \tag{3.245}$$

We used $\check{\boldsymbol{\sigma}}(t+\triangle t) = \check{\boldsymbol{\sigma}}(t) + \boldsymbol{R}^t_t[\boldsymbol{C}:\boldsymbol{D}\triangle t]\boldsymbol{R}_t$ as the deformation rate $\boldsymbol{D}$ resolved in the co-rotational frame of reference $\boldsymbol{t}_t$ at configuration C_1 as follows:

$$[\boldsymbol{D}]_{\boldsymbol{t}\otimes\boldsymbol{t}} = \begin{bmatrix} \frac{\partial l}{\partial t} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.246}$$

While it is resolved in inertia frame $\boldsymbol{e}_i$ as follows:

$$[\boldsymbol{D}]_{\boldsymbol{e}\otimes\boldsymbol{e}} = \boldsymbol{R}_t\,[\boldsymbol{D}]_{\boldsymbol{t}\otimes\boldsymbol{t}}\boldsymbol{R}^T_t \tag{3.247}$$

As from Equation 3.239 and Equation 3.240

$$[\check{\boldsymbol{\sigma}}(t+\triangle t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} - [\check{\boldsymbol{\sigma}}(t)]_{\boldsymbol{E}\otimes\boldsymbol{E}} = \begin{bmatrix} \frac{P_2-P_1}{A} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.248}$$

and from

$$\left[\boldsymbol{R}^T_t[\boldsymbol{C}:\boldsymbol{D}\triangle t]\boldsymbol{R}_t\right]_{\boldsymbol{E}\otimes\boldsymbol{E}} = \boldsymbol{R}^T_t[\boldsymbol{C}:\boldsymbol{D}\triangle t]_{\boldsymbol{E}\otimes\boldsymbol{E}}\boldsymbol{R}_t = [\boldsymbol{C}:\boldsymbol{D}]_{\boldsymbol{t}\otimes\boldsymbol{t}}\triangle t = [\boldsymbol{C}]_{\boldsymbol{E}\otimes\boldsymbol{E}} : \begin{bmatrix} \frac{\partial l}{\partial t} & 0 \\ 0 & 0 \end{bmatrix}\triangle t \tag{3.249}$$

Which, using the constitutive relation, gives the same findings of Equation 3.248.
Co-rotational deformation gradient rate of change $\boldsymbol{F}^O$. Deformation gradient is a two point tensor, so co-rotational rate will be:

$$\boldsymbol{F}^o = \dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{F} \tag{3.250}$$

The conjugate pairs $\boldsymbol{P}:\dot{\boldsymbol{F}}$ can be reduced to

$$\boldsymbol{P}:\dot{\boldsymbol{F}} = \boldsymbol{P}:(\boldsymbol{\Omega}\boldsymbol{F} + \boldsymbol{F}^o) \tag{3.251}$$

$$= \boldsymbol{P}:\boldsymbol{\Omega}\boldsymbol{F} + \boldsymbol{P}:\boldsymbol{F}^o \tag{3.252}$$

$$= \boldsymbol{P}:\boldsymbol{F}^o + \boldsymbol{P}\boldsymbol{F}^T:\boldsymbol{\Omega} \tag{3.253}$$

$$= \boldsymbol{P}:\boldsymbol{F}^o \tag{3.254}$$

$$\boxed{\boldsymbol{P}:\dot{\boldsymbol{F}} = \boldsymbol{P}:\boldsymbol{F}^o} \tag{3.255}$$

Note that $\boldsymbol{PF}$ is symmetric[7] and $\boldsymbol{\Omega}$ is skew-symmetric, so we find that $\boldsymbol{PF}^T : \boldsymbol{\Omega}$ vanishes, (see Equation 1.100) and

$$\int_{V_0} \boldsymbol{P} : \dot{\boldsymbol{F}} dV_0 = \int_{V_0} \boldsymbol{P} : \boldsymbol{F}^o dV_0 \tag{3.256}$$

So $\boldsymbol{P} : \boldsymbol{F}^o$ can be considered as conjugate pairs which was proven in the geometrically exact beam theory in subsection 3.3.5.

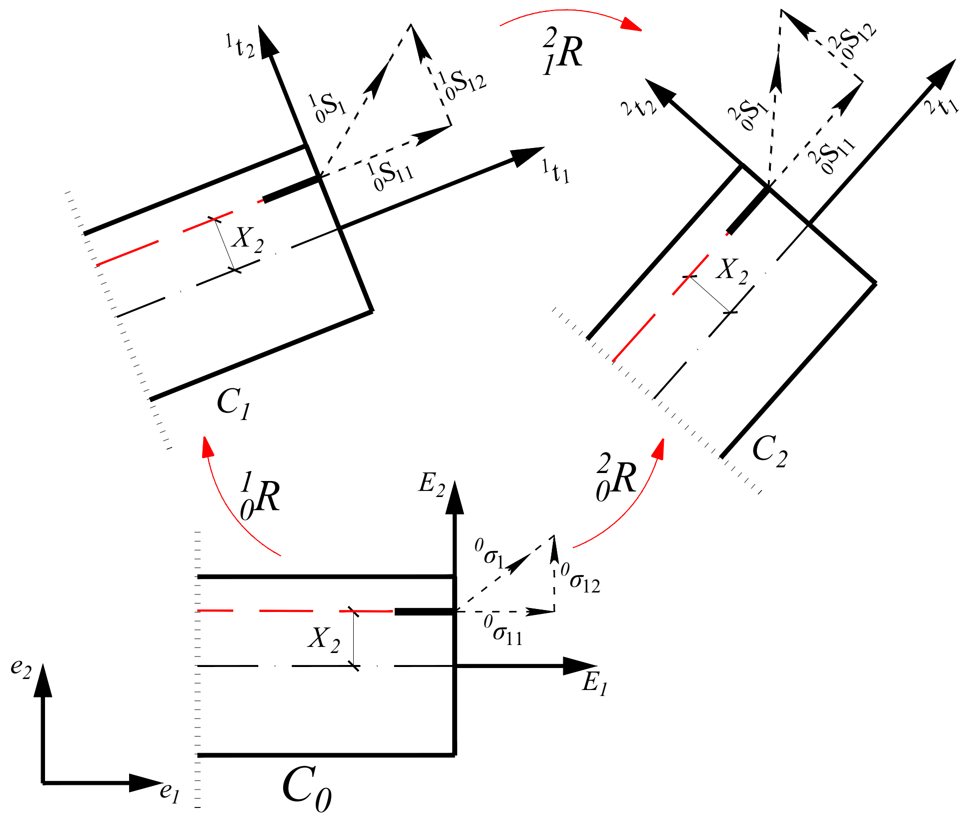


Figure 3.42

3.4.1 Second Piola Kirchhoff Stress update and force resultant in beam element

There are two methods to update second Piola Kirchhoff stresses, namely total Lagrangian and updated Lagrangian formulations. Assume a rigid cross section of a beam shown in Figure 3.42 with Cauchy stress resolved in the inertia frame and co-rotational frame as follows:

$$^1\boldsymbol{\sigma} = {}^1\boldsymbol{\sigma}_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = {}^1\overline{\sigma}_{ij}\,{}^1\boldsymbol{t}_i \otimes {}^1\boldsymbol{t}_j \tag{3.257}$$

While second Piola Kirchhoff stress tensor is defined as follows:

$${}^2_1\boldsymbol{S} = {}^2_1S_{ij}\,{}^1\boldsymbol{t}_i \otimes {}^1\boldsymbol{t}_j \tag{3.258}$$

Defining a stress vectors ${}^1_0\boldsymbol{S}_1$ and ${}^2_0\boldsymbol{S}_1$ at configurations C_1 and C_2 as shown in Figure 3.42 as follows:

$${}^1_0\boldsymbol{S}_1 = {}^1_0S_{1I}\,{}^1\boldsymbol{t}_I \tag{3.259}$$

$${}^2_0\boldsymbol{S}_1 = {}^2_0S_{1I}\,{}^2\boldsymbol{t}_I \tag{3.260}$$

[7] $\boldsymbol{PF}=J\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}\boldsymbol{F}^T = J\boldsymbol{\sigma}^T$ is symmetric quantity due to the symmetry of Cauchy stress

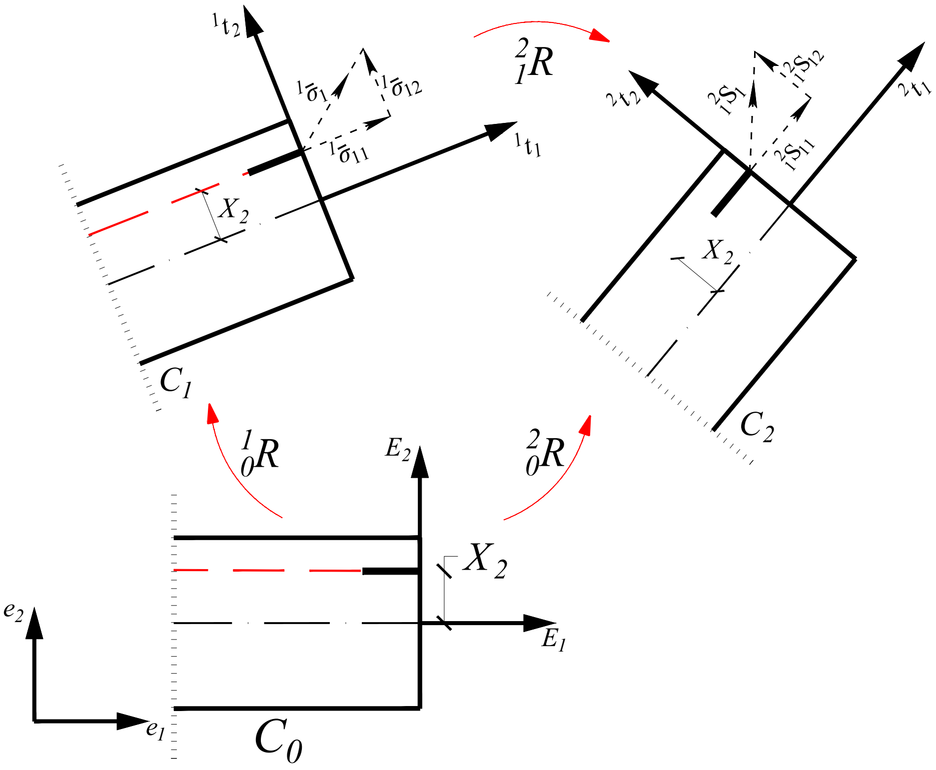


Figure 3.43

Where superscript signifies the time or configuration of measure, while subscript indicates the reference configuration the property referred to. Due to the objectivity of second Piola Kirchhoff stress, the update form of total Lagrangian formulation is defined as follows:

$$ {}^2_0S_{IJ} = {}^1_0S_{IJ} + \triangle \left({}^{12}_{0}S_{IJ}\right) \tag{3.261} $$

Where the constitutive relation is defined as follows:

$$ \triangle \left({}^{12}_{0}S_{ij}\right) = {}^{12}_{0}C_{ijrs}\triangle \left({}^{12}_{0}E_{rs}\right) \tag{3.262} $$

The resultant forces and moments applied on beam section at configurations C_1 and C_2 are defined as follows:

$$ {}^1\boldsymbol{F} = {}^1F_i\boldsymbol{e}_i = {}^1\overline{F}_i{}^1\boldsymbol{t}_i \tag{3.263} $$

$$ {}^1\boldsymbol{M} = {}^1M_i\boldsymbol{e}_i = {}^1\overline{M}_i{}^1\boldsymbol{t}_i \tag{3.264} $$

The co-rotational components ${}^1\overline{F}_i$ and ${}^1\overline{M}_i$ is defined as follows:

$$ {}^1\overline{F}_I = \int_A {}^1_0S_{1I}dA \tag{3.265} $$

$$ {}^1\overline{M}_I = \int_A \left[\widetilde{\boldsymbol{X}}\right]_I \left[{}^1_0\boldsymbol{S}_1\right]_{{}^1\boldsymbol{t}_i} dA = \int_A X_J\, {}^1_0S_{1K}\, e_{JKI}dA \tag{3.266} $$

Where $\boldsymbol{X} = X_I\boldsymbol{E}_I = X_2\boldsymbol{E}_2 + X_3\boldsymbol{E}_3$ or $\left[\widetilde{\boldsymbol{X}}\right]_I = [0 \;\; X_2 \;\; X_3]$ as shown in Figure 3.44, while $\left[{}^1_0\boldsymbol{S}_1\right]_{{}^1\boldsymbol{t}_i} = \left[{}^1_0S_{11} \;\; {}^1_0S_{12} \;\; {}^1_0S_{13}\right]$ and e_{JKI} is Permutation symbol.

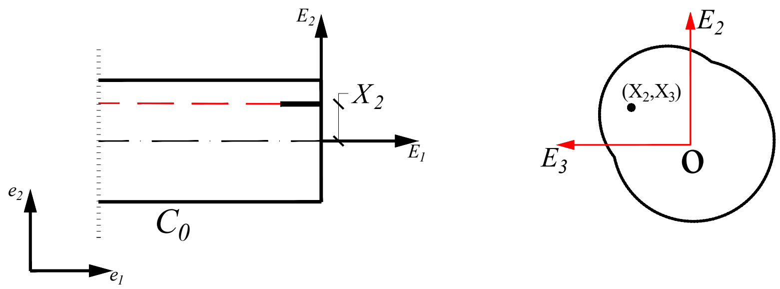


Figure 3.44

The spatial components of Equation 3.263 and Equation 3.264 can be defined using ($\boldsymbol{t}_i = {}^1_0\boldsymbol{R}\ \boldsymbol{e}_i$) as follows:

$$^1F_i = {}^1_0R_{iI}\,{}^1\overline{F}_I = {}^1_0R_{iI}\int_A {}^1_0S_{1I}dA \tag{3.267}$$

$$^1M_i = {}^1_0R_{iI}\,{}^1\overline{M}_I = {}^1_0R_{iI}\int_A X_J\ {}^1_0S_{1K}\ e_{JKI}dA \tag{3.268}$$

Where

$${}^1_0\boldsymbol{R} = {}^1_0R_{iI}\ \boldsymbol{e}_i\otimes\boldsymbol{E}_I = {}^1\boldsymbol{t}_i\otimes\boldsymbol{E}_I \tag{3.269}$$

$$^1\overline{F}_1 = \int_A {}^1_0S_{11}dA \tag{3.270}$$

$$^1\overline{F}_2 = \int_A {}^1_0S_{12}dA \tag{3.271}$$

$$^1\overline{F}_3 = \int_A {}^1_0S_{13}dA \tag{3.272}$$

$$^1\overline{M}_1 = \int_A \left(X_2\ {}^1_0S_{13} - X_3\ {}^1_0S_{12}\right) dA \tag{3.273}$$

$$^1\overline{M}_2 = \int_A X_3\ {}^1_0S_{11}dA \tag{3.274}$$

$$^1\overline{M}_3 = \int_A -X_2\ {}^1_0S_{11}dA \tag{3.275}$$

In the same manner configuration C_2, with:

$$^2\boldsymbol{F} = {}^2F_i\boldsymbol{e}_i = {}^2\overline{F}_i{}^2\boldsymbol{t}_i \tag{3.276}$$

$$^2\boldsymbol{M} = {}^2M_i\boldsymbol{e}_i = {}^2\overline{M}_i{}^2\boldsymbol{t}_i \tag{3.277}$$

We get the following:

$$^2F_i = {}^2_0R_{iI}\,{}^2\overline{F}_I = {}^2_0R_{iI}\int_A {}^2_0S_{1I}dA \tag{3.278}$$

$$^2M_i = {}^2_0R_{iI}\,{}^2\overline{M}_I = {}^2_0R_{iI}\int_A X_J\ {}^2_0S_{1K}\ e_{JKI}dA \tag{3.279}$$

Where ${}^2_0\boldsymbol{R} = {}^2_0R_{iI}\ \boldsymbol{e}_i\otimes\boldsymbol{E}_I = {}^2\boldsymbol{t}_i\otimes\boldsymbol{E}_I$.

Using Figure 3.43 to define the following force and moment resultants:

$$^2\boldsymbol{F} = {}^2F_i{}^1\boldsymbol{t}_i = {}^2\overline{F}_i{}^2\boldsymbol{t}_i \tag{3.280}$$

$$^2\boldsymbol{M} = {}^2M_i{}^1\boldsymbol{t}_i = {}^2\overline{M}_i{}^2\boldsymbol{t}_i \tag{3.281}$$

The components of forces and moment resultants can be defined as follows:

$$ {}^{2}F_i = {}^{2}_{1}R_{iI}\,{}^{2}\overline{F}_I = {}^{2}_{1}R_{iI}\int_A {}^{2}_{1}S_{1I}\,dA \tag{3.282}$$

$$ {}^{2}M_i = {}^{2}_{1}R_{iI}\,{}^{2}\overline{M}_i = {}^{2}_{1}R_{iI}\int_A X_j\,{}^{2}_{1}S_{1j}\,e_{JKI}\,dA \tag{3.283}$$

Where

$$ {}^{2}_{1}\boldsymbol{R} = {}^{2}_{1}R_{iI}\,{}^{1}\boldsymbol{t}_I\otimes{}^{1}\boldsymbol{t}_I = {}^{2}\boldsymbol{t}_i\otimes{}^{1}\boldsymbol{t}_I \tag{3.284}$$

The update form of updated Lagrangian formulation is defined as follows:

$$ {}^{2}_{1}S_{IJ} = {}^{1}_{1}S_{IJ} + \triangle\left({}^{12}_{1}S_{IJ}\right) \tag{3.285}$$

Where the constitutive relation is defined as follows:

$$ \triangle\left({}^{12}_{1}S_{ij}\right) = {}^{12}_{1}C_{ijrs}\triangle\left({}^{12}_{1}E_{rs}\right) \tag{3.286}$$

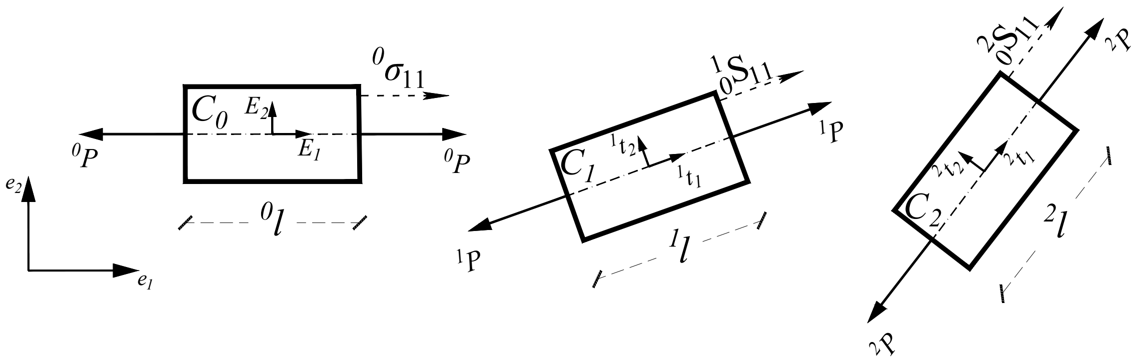


Figure 3.45

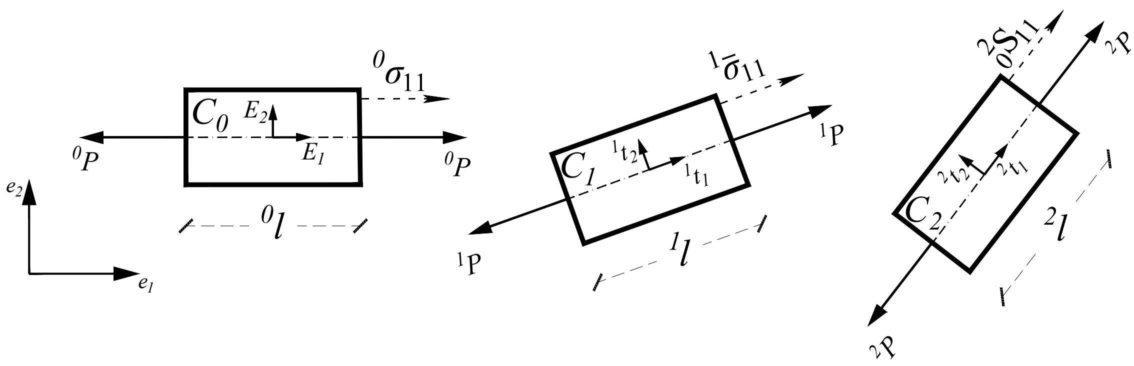


Figure 3.46

■ **Example 3.16** If we have a beam shown in Figure 3.45 subjected to only axial loads with ${}^{0}P$, ${}^{1}P$ and ${}^{2}P$ and lengths ${}^{0}l$, ${}^{1}l$ and ${}^{2}l$ at configurations C_0, C_1 and C_2, respectively, the only generated second Piola Kirchhoff stress components is ${}^{t}_{0}S_{11}$ at configuration C_t at time t with corresponding Green Lagrange strain ${}^{t}_{0}E_{11}$ defined as follows:

$$ {}^{0}_{0}E_{11} = {}^{0}e_{11} \tag{3.287}$$

$$ {}^{1}_{0}E_{11} = \frac{1}{2}\frac{{}^{1}l^2 - {}^{0}l^2}{{}^{0}l^2} \tag{3.288}$$

$$ {}^{2}_{0}E_{11} = \frac{1}{2}\frac{{}^{1}l^2 - {}^{0}l^2}{{}^{0}l^2} \tag{3.289}$$

The update form of total Lagrange formulation is defined as follows:

$$ {}^{2}_{0}S_{11} = {}^{1}_{0}S_{11} + \triangle\left({}^{12}_{0}S_{11}\right) \tag{3.290}$$

$$ {}^{1}_{0}S_{11} = {}^{0}_{0}S_{11} + \triangle\left({}^{01}_{0}S_{11}\right) = {}^{0}\sigma_{11} + \triangle\left({}^{01}_{0}S_{11}\right) \tag{3.291}$$

Where ${}^{0}e_{11}$ is the infinitesimal strain and $\triangle\left({}^{01}_{0}S_{11}\right)$ and $\triangle\left({}^{01}_{0}S_{11}\right)$ are defined as

$$ \triangle\left({}^{01}_{0}S_{11}\right) = {}^{01}_{0}C_{1111}\left({}^{1}_{0}E_{11} - {}^{0}e_{11}\right) \tag{3.292}$$

$$ \triangle\left({}^{12}_{0}S_{11}\right) = {}^{12}_{0}C_{1111}\left({}^{2}_{0}E_{11} - {}^{1}_{0}E_{11}\right) \tag{3.293}$$

For linear elastic material ${}^{12}_{0}C_{1111} = E$, where E is Young modulus.
While The update form of updated Lagrange formulation is defined using Figure 3.46 as follows:

$$ {}^{1}_{1}E_{11} = {}_{1}\bar{e}_{11} \tag{3.294}$$

$$ {}^{2}_{1}E_{11} = \frac{1}{2}\frac{{}^{1}l^2 - {}^{1}l^2}{{}^{1}l^2} \tag{3.295}$$

$$ {}^{2}_{1}S_{11} = {}^{1}_{1}S_{11} + \triangle\left({}^{01}_{1}S_{11}\right) = {}^{1}\overline{\sigma}_{11} + \triangle\left({}^{01}_{1}S_{11}\right) \tag{3.296}$$

Where $\overline{e}_{ij}$ and $\overline{\sigma}_{ij}$ are the co-rotational components or the components of the infinitesimal strain and Cauchy stress resolved in the co-rotational frame ${}^{1}\boldsymbol{t}_1$ as shown in Figure 3.46. $\triangle\left({}^{01}_{1}S_{11}\right)$ is defined as

$$ \triangle\left({}^{12}_{1}S_{11}\right) = {}^{12}_{1}C_{1111}\left({}^{2}_{1}E_{11} - {}_{1}\bar{e}_{11}\right) \tag{3.297}$$

■

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4. Energy Principles and Introduction to FEA

4.1 Introduction

4.1.1 Work

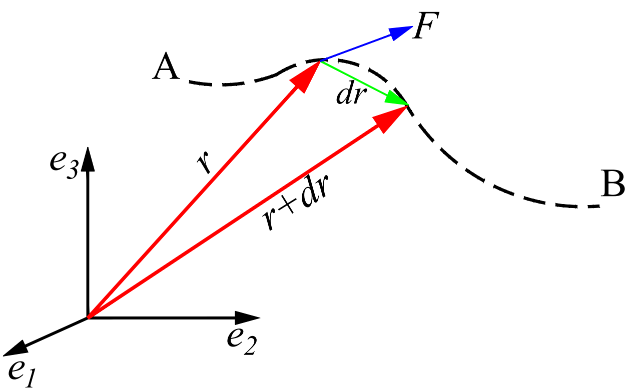


Figure 4.1

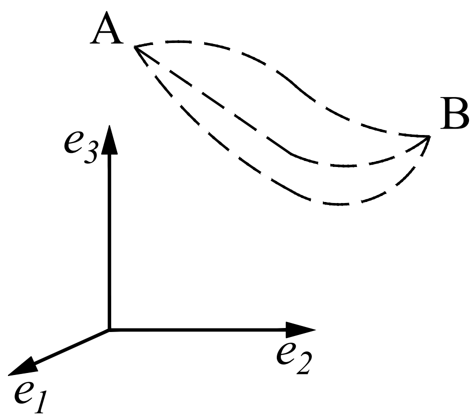


Figure 4.2

Assume a particle moving through path AB with position vector $\boldsymbol{r}$ relative to fixed frame of reference under an influence of force $\boldsymbol{F}$, such that the infinitesimal work dW on the particle through moving from position $\boldsymbol{r}$ to position $\boldsymbol{r}+d\boldsymbol{r}$ will be the dot product of the force vector at position $\boldsymbol{r}$ and the infinitesimal movement $d\boldsymbol{r}$ or the product of the displacement and force in displacement direction.

$$dW = \boldsymbol{F}.d\boldsymbol{r} = F_1 dr_1 + F_2 dr_2 + F_3 dr_3 \tag{4.1}$$

So total work done through the entire path AB will be:

$$W = \int_A^B \boldsymbol{F}.d\boldsymbol{r} \tag{4.2}$$

The work carries positive sign if projection of the force vector on displacement and displacement vector has the same direction. Bear in mind that this quantity is a scalar value which does not change with changing coordinate system, even if the components of $d\boldsymbol{r}$ and $\boldsymbol{F}$ (vectors) depend on the coordinate system chosen.

Like above, work done by moment vector $\boldsymbol{M}$ through an infinitesimal rotational spin $d\boldsymbol{\phi}$ will be:

$$dW = \boldsymbol{M}.d\boldsymbol{\phi} \tag{4.3}$$

The total work done from point A to B will be:

$$W = \int_A^B \boldsymbol{M}.d\boldsymbol{\phi} \tag{4.4}$$

See Appendix 4.5.5 for different types of moments and the corresponding work done for each type. For example, the work done by particle's weight mg elevated a distance y equal to $-mgy$. Also the work done on linear elastic spring with stiffness k stretched or compressed by displacement x is $-\frac{1}{2}kx^2$. The work is negative in both cases as the force and its displaced distance have different direction. For flexible bodies, the total work performed on the body contains two parts, work done by internal forces W_I and other by external forces W_E defined as:

$$W = W_I + W_E \tag{4.5}$$

4.1.2 Power

The time rate of change of the work done by force $\boldsymbol{F}$ to move a particle through an infinitesimal distance $d\boldsymbol{r}$ for an infinitesimal time dt leads to definition of the power P given by:

$$P = \frac{dW}{dt} = \boldsymbol{F}.\frac{d\boldsymbol{r}}{dt} = \boldsymbol{F}.\boldsymbol{v} \tag{4.6}$$

Where $\boldsymbol{v}$ is velocity of the particle. As a result, the total work done through path AB can be converted to time integral with interval $[t_A,\ t_B]$ given by:

$$W = \int_A^B \boldsymbol{F}.d\boldsymbol{r} = \int_{t_A}^{t_B} \boldsymbol{F}.\boldsymbol{v}dt = \int_{t_A}^{t_B} Pdt \tag{4.7}$$

Where t_A and t_B represent the start and end time of path AB spent by the particle. Newton's second law of motion for particle with mass m moving under an influence of force $\boldsymbol{F}$ is given by:

$$\boldsymbol{F} = m\boldsymbol{a} \tag{4.8}$$

So power exerted by force $\boldsymbol{F}$ contributes to change in kinetic energy $K.E$ as follow:

$$P = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}(K.E) \tag{4.9}$$

From above equation or using principle of work and energy, work W is converted to a change in kinetic energy as follows:

$$W = \int_{t_A}^{t_B} Pdt = \triangle K.E \tag{4.10}$$

4.1.3 Potential energy and conservative forces

A force $\boldsymbol{F}$ is considered conservative, if the work done by it is independent on the path taken, but it depends only on the initial and final positions of the force, e.g. work done by particle weight depends only on the vertical displacement. This work is stored in the weight as a potential energy, such that if the weight mg lifted a distance y which means that negative work $-mgy$ is exerted by weight (as weight force is downward and the displacement is the opposite direction), the weight acquires a positive potential energy ($\Pi = mgy$) as it has the potential or capacity of doing positive work mgy when returning back down to its initial position so the change in potential energy is defined as

$$\triangle\Pi = -\int_A^B \boldsymbol{F}.d\boldsymbol{r} = -W \tag{4.11}$$

Also, when elastic spring with stiffness k is stretched or compressed by distance x from its unstretched position, an elastic potential energy is stored in the spring equal to $\frac{1}{2}kx^2$ (linear elastic spring), as in any deformed position, the spring has the potential to do positive work when moving back to its undeformed position. From above equation, the conservative force $\boldsymbol{F}$ can be evaluated from the gradient of its potential Π in the direction of its displacement as follows:

$$\boldsymbol{F}(\boldsymbol{x}) = -\nabla\Pi(\boldsymbol{x}) \qquad where\ \nabla(A)\ is\ the\ gradient\ of\ a\ scalar\ A \tag{4.12}$$

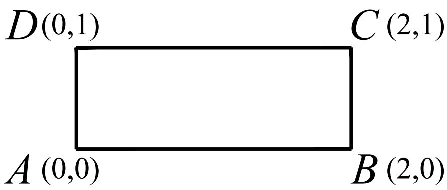


Figure 4.3

■ **Example 4.1 — Conservative force.** Consider a force field $\boldsymbol{F}(x,y) = (y+2x)i + xj$ affecting a particle moving from point A to point C shown in Figure 4.3, check whether the force is conservative or not, then calculate the work done through two paths ABC and ADC.
The components of force $\boldsymbol{F}(x,y)$ are:

$$F_x = y+2x,\ F_y = x \tag{4.13}$$

Applying Equation 4.12 to get the potential as follows:

$$F_x = -\frac{d\Pi}{dx} \rightarrow \Pi = -yx - x^2 + f_1(y) \tag{4.14}$$

$$F_y = -\frac{d\Pi}{dy} \rightarrow \Pi = -yx - f_1(x) \tag{4.15}$$

So we can conclude that

$$\Pi = -yx - x^2 + C \quad \text{where } C \text{ is constant} \tag{4.16}$$

So the force is conservative.
The work done through path AB is

$$W = \int_A^B \boldsymbol{F}.d\boldsymbol{r} = \int_1^2 (y+2x).dx\Big|_{y=0} = \left(\left[yx+\frac{x^2}{2}\right]\Big|_{x=0}^{x=2}\right)\Big|_{y=0} = (2y+2)|_{y=0} = 2 \quad (4.17)$$

Similarly, work done through path BC, AD, and DC is 1, 0, 3, respectively. so the work done through path ABC and ADC is equal to 3 which makes the force $\boldsymbol{F}$ conservative.
■

■ **Example 4.2 — Non-conservative force.** Force $\boldsymbol{F} = xyi + yx^2$ is not conservative as

$$F_x = -\frac{d\Pi}{dx} \rightarrow \Pi = -\frac{1}{2}x^2y + f_1(y) \quad (4.18)$$

$$F_y = -\frac{d\Pi}{dy} \rightarrow \Pi = -\frac{1}{2}y^2x^2 - f_1(x) \quad (4.19)$$

There is no potential function that can achieve the two equations which make the force field nonconservative. ■

Another example of non-conservative force is friction forces which depend on many parameters like path length.

4.1.4 Conservation of energy

From Equation 4.10 and Equation 4.11, we get

$$\triangle(\Pi + K.E) = 0 \quad (4.20)$$

Conservation of energy states that the total energy (sum of the system potential energy Π and kinetic energy $K.E$) for a conservative system remains stationary. Conservation of energy needs the external forces to be conservative or have a field, so we can evaluate the change of its potential from end points of the path moved. For flexible bodies, Another requirement to apply the conservation of energy is that the body should be elastic, such that a unique internal forces can be extracted for the given body deformation. In this case a unique force field will be a function of the deformation and independent of the path, such that we can extract the internal potential (potential strain energy) for any particular deformation.

■ **Example 4.3** Assume an object of mass m located at an earth gravity field and thrown upward from level x_1 with velocity v_1 to reach level x_2, what is its velocity at level x_2? The object is subjected to force field or gravity force $(F(x) = mg)$ pointing downward (constant with x), where g is the gravity acceleration of the earth.

The change in potential energy $\triangle\Pi = -\int_{x_1}^{x_2} F(x)\,dx = -\int_{x_1}^{x_2} -mgdx = mg(x_2 - x_1)$, the negative sign of mg inside the integral due to the applied force is opposite in the direction to displacement moved.

The change in kinetic energy $K.E$ will be $\triangle K.E = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$.

As the total energy is constant, we get.

$$mgx + \frac{1}{2}mv^2 = constant \quad (4.21)$$

Also differentiating the equation, so the acceleration of the object (a) is as follow:

$$mg\dot{x} + mv\dot{v} = mgv + mva = 0 \leftrightarrow a = -g \tag{4.22}$$

Also the power of the gravity force equals to the rate of change of kinetic energy which leads to the same results.

$$mgv = \boldsymbol{F}.\boldsymbol{v} = P = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mva \leftrightarrow a = -g \tag{4.23}$$

From the last above two equation, we can check that acceleration is identical to the gravity acceleration. ■

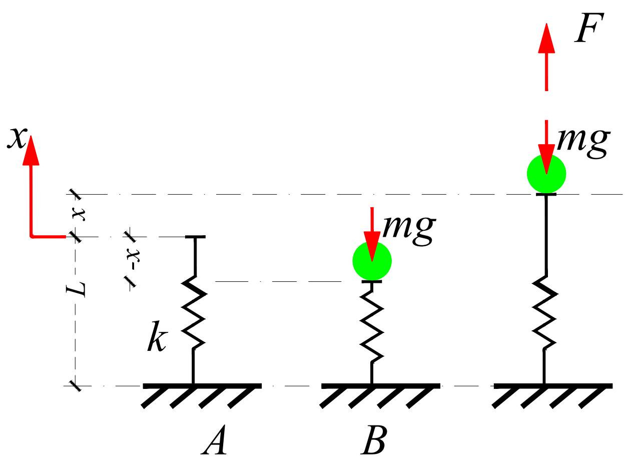


Figure 4.4

■ **Example 4.4 — Flexible body.** Assume unstressed vertical linear elastic spring shown in Figure 4.4 of length L and stiffness K then glued with gravity load mg to displace downward distance mg/K, then this mass is pulled at distance x added to L $(x+L)$ then left to vibrate freely. We need to evaluate the mass velocity when spring reaches its unstressed length L.

When the mass is vibrating, it is subjected to two force fields, gravity force field and force exerted from the spring equal to Kx where x is the distance the spring stretches, such that the force field will be:

$$F(x) = mg + Kx \tag{4.24}$$

The change in potential energy for the mass moving from point A to point B will be:

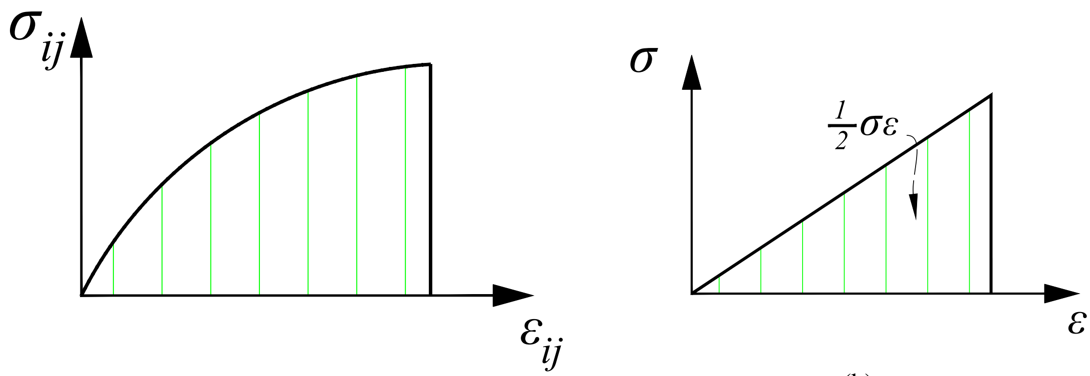
$$\triangle\Pi = -\int_{x_1}^{x_2} F(x)\,dx = -\left(mgx + \frac{1}{2}Kx^2\right) \tag{4.25}$$

The negative sign resulting from the above equation because the motion of the mass in the direction of the force field. Note that the first term of the equation called *increase in load potential energy* $\triangle V$, while the second term is called *increase in the strain energy* $\triangle U$.

The change in the kinetic energy $\triangle K.E = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2$, such that the total change of the energy of the system will be:

$$\frac{1}{2}mv_B^2 - \left(mgx + \frac{1}{2}Kx^2\right) = 0 \tag{4.26}$$

4.1.5 Strain energy for different types of loading



(a) (b)

Figure 4.5

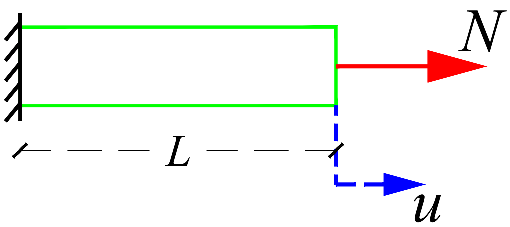


Figure 4.6

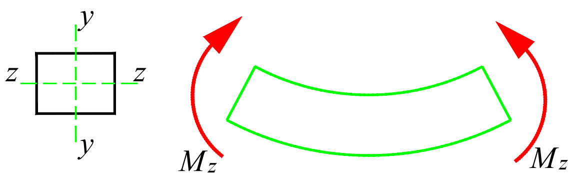


Figure 4.7

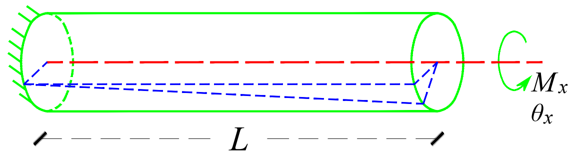


Figure 4.8

Applying loads on elastic body results in internal stresses and strains. Strain (potential) energy stored in the body per unit volume $(\overline{U})$ is defined as the area under stress strain curve shown in Figure 4.5a as follows:

$$\overline{U} = \int_0^{\varepsilon_f} \sigma_{ij} d\varepsilon_{ij} \tag{4.27}$$

For linear elastic body shown in Figure 4.5b, this energy will be:

$$\overline{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2E}\sigma_{ij}^2 \tag{4.28}$$

Stain energy for the total volume of the body will be:

$$U = \int_V \left(\overline{U}\right) dV = \int_V \left(\int_0^{\varepsilon_f} \sigma_{ij} d\varepsilon_{ij}\right) dV \tag{4.29}$$

For linear elastic body, the total stain energy is:

$$U = \int_V \frac{1}{2E} \sigma_{ij}^2 dV \tag{4.30}$$

Strain energy due to axial loading

Assume a linear elastic bar problem shown in Figure 4.6 with length L, area A and modulus of elasticity E fixed at one support and subjected to axial load N at the other free end. The stress and strain distributions along the bar is defined as follow:

$$\sigma = \frac{N}{A} \tag{4.31}$$

$$\varepsilon = \frac{\sigma}{E} = \frac{N}{EA} \tag{4.32}$$

Also the kinematic relation for the axial strain is defined as:

$$\varepsilon = \frac{du}{dx} = u' \tag{4.33}$$

$$\begin{aligned} U = \int_V \frac{1}{2E} \sigma^2 dV = \int_V \frac{1}{2E} \left(\frac{N}{A}\right)^2 dV = \int_0^L \left(\int_A \frac{1}{2E} \left(\frac{N}{A}\right)^2 dA\right) dx \\ = \int_0^L \frac{1}{2E} \left(\frac{N}{A}\right)^2 A \, dx = \int_0^L \frac{N^2}{2EA} dx \end{aligned} \tag{4.34}$$

From Equation 4.32, it follows:

$$U = \frac{1}{2} \int_0^L EA \left(u'\right)^2 dx \tag{4.35}$$

Strain energy due to bending moment

For a linear elastic beam directed along x direction subjected to moment M_z about its major axis z with inertia I_z, the stress and strain distributions is defined as:

$$\sigma = -\frac{M_z}{I_z} y \tag{4.36}$$

$$\varepsilon = -\frac{\sigma}{E} = \frac{M_z y}{EI_z} \tag{4.37}$$

Where y is the vertical distance away from the geometric natural axis]of the beam. Also the strain can be related to beam curvature v'' using this expression:

$$\varepsilon = -\frac{du}{dx} = v'' y \tag{4.38}$$

$$U = \int_V \frac{1}{2E} \sigma^2 dV = \int_V \frac{1}{2E} \left(\frac{M_z}{I_z} y\right)^2 dV = \int_0^L \int_V \frac{1}{2E} \left(\frac{M_z}{I_z} y\right)^2 dA \, dx \tag{4.39}$$

$$= \int_0^L \frac{1}{2E} \left(\frac{M_z}{I_z}\right)^2 \left(\int_A y^2 dA\right) dx = \int_0^L \frac{1}{2E} \left(\frac{M_z}{I_z}\right)^2 I_z \, dx = \int_0^L \frac{M_z^2}{2EI_z} dx = \int_0^L \frac{1}{2} EI_z \left(v''\right)^2 dx \tag{4.40}$$

Strain energy due to shear stresses

For a linear elastic beam subjected to shear force Q with area A, length L and shear modulus of elasticity G, using the concept of an *equivalent shear area* $A_s = kA$, where k is area shear factor. The shear force is equal to the shear stress τ_{NA} calculated at the neutral axis times this area as follows:

$$Q = \tau_{NA} A_s = k \tau_{NA} A \tag{4.41}$$

So the corresponding shear strain at the neutral axis will be:

$$\gamma = \frac{\tau_{NA}}{G} = \frac{Q}{A_s G} \tag{4.42}$$

$$U = \int_V \frac{1}{2E} \sigma^2 dV = \int_0^L \int_V \frac{1}{2G} \left(\frac{Q}{A_s} \right)^2 dA\, dx = \int_0^L \frac{Q^2}{2GA_s} dx = \int_0^L \frac{1}{2} GA_s \gamma^2 dx \tag{4.43}$$

Strain energy due to uniform torsion

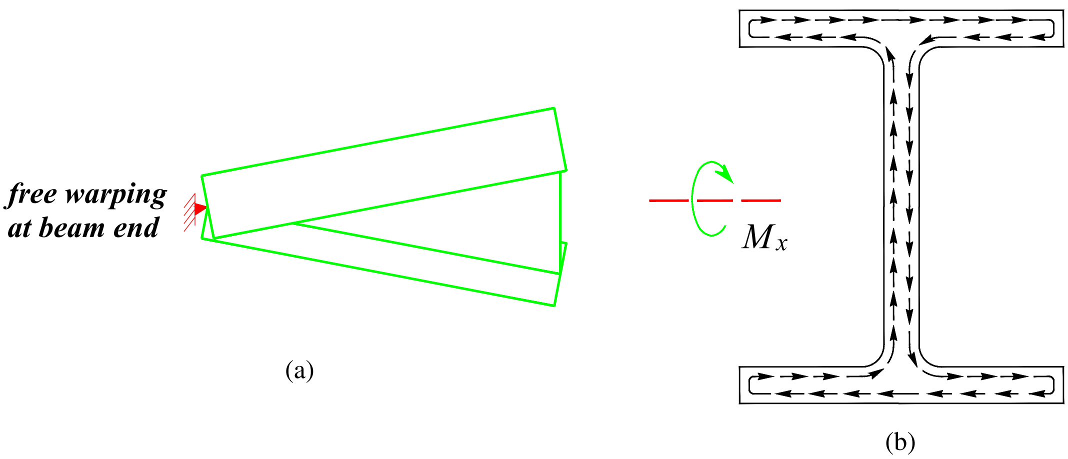


Figure 4.9

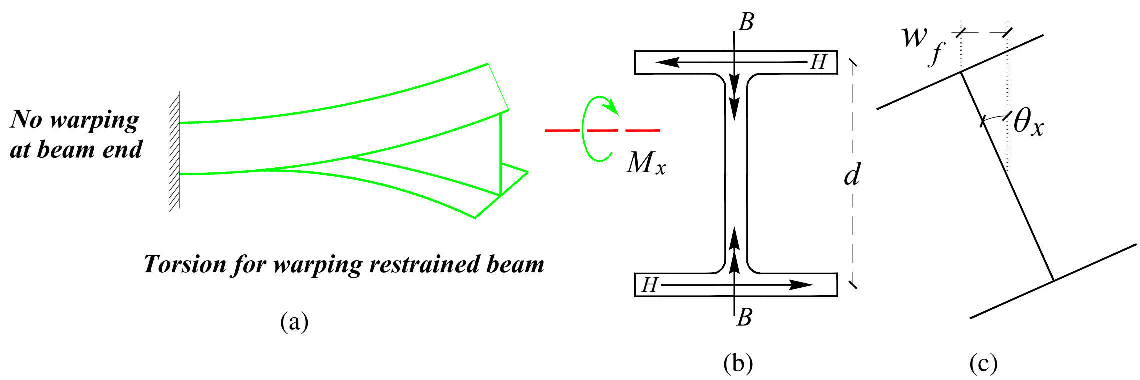


Figure 4.10

For I-section shown in Figure 4.9a with length L, torsional rigidity GJ, fixed from axial rotation at the left end and subjected to torsional moment M_x at the other end, the rate of beam twist β is defined as:

$$\beta = \theta_x' = \frac{M_x}{GJ} \tag{4.44}$$

As the torsion moment is constant along the beam length, the rate of twist from above equation is also constant with rotation θ_x at the right end defined as

$$\theta_x = \beta L = \frac{M_x L}{GJ} \tag{4.45}$$

M_x is called ST. Venant or pure torque $M_{sv} = GJ\theta_x'$ with shear stress shown in Figure 4.9b, so the strain energy is defined as:

$$U = \int_0^L \frac{{M_x}^2}{2GJ} dx \tag{4.46}$$

Strain energy due to non-uniform (warping) torsion

In some cases, torsion can be carried by axial stresses in addition to shear stresses. This occurs when the cross section is prevented from warping, which is supposed to happen in the section when subjected to torsion as in Figure 4.10a at the left end of the beam. Warping out of plane means that the axial displacement of fiber appears as shown in Figure 4.9a. Preventing section from warping results in longitudinal stresses and corresponding torsional resistance called warping torsion. Consider two beams in Figure 4.9a and Figure 4.10a subjected to moment M_x at the right end, and restrained from twisting at the other end but one beam is warping restrained and the other is not. The first warping free beam has the freedom to displace axially without any restriction and exhibits a similar warping distribution at any cross section along beam length. Also the rate of twist is constant across the beam length, and all the cross section is subjected to shear stresses. While the warping-fixed beam shows that the rate of twist is not constant starting from null at the wrapping-restrained section end reaching to its maximum at the right end which forces the two flange of the beam to display laterally in a bending form. As a result, axial stresses is formed in the bending flange and participates in resisting the applied torsion besides shear stresses. The torsion portion resisted by axial stresses is called warping torsion which is defined as

$$M_w = V.d \tag{4.47}$$

Where V is the horizontal shear force resulted due to the resistance of the flange to the bending and d is the distance between two flange as shown in Figure 4.10b.

$$V = -\frac{dM_f}{dx} \tag{4.48}$$

$$M_f = EI_{yf} \frac{d^2 w_f}{dx^2} \tag{4.49}$$

From Figure 4.10c, $w_f = \frac{\theta_x d}{2}$, the warping torsion is defined as:

$$T_w = -\frac{dM_f}{dx} d = -EI_{yf} \frac{d^3 w_f}{dx^3} d = -\frac{EI_{yf} d^2}{2} \frac{d^3 \theta_x}{dx^3} = -EC_w \theta_x''' \tag{4.50}$$

Where C_w is defined as warping constant equal to $\frac{I_{yf} d^2}{2} = \frac{I_y d^2}{4}$ for beams with I-sections. So the total torsion resistance will be:

$$T = T_{sv} + T_w = GJ\theta_x' - EC_w \theta_x''' \tag{4.51}$$

stress distribution across the flange will be:

$$\sigma = \frac{M_f}{I_{yf}} z \tag{4.52}$$

From Equation 4.49 and $w_f = \frac{\theta_x d}{2}$, the stress distribution across the top flange will be:

$$\sigma = E\frac{d^2 w_f}{dx^2} z = E\frac{d^2}{4}\frac{d\theta}{dx^2} z = Ed\theta'' z \tag{4.53}$$

So the resulting strain energy form the top flange U_T will be:

$$U_T = \int_V \frac{1}{2E}\sigma^2 dV = \int_0^L \frac{1}{8}Ed^2\left(\theta''\right)^2\left(\int_A z^2 dA\right)dx = \int_0^L \frac{1}{8}Ed^2 I_{yf}\left(\theta''\right)^2 dx = \int_0^L \left(\frac{1}{4}EC_w\theta_x''^2\right)dx \tag{4.54}$$

Similarly the bottom flange stores the same strain energy, so the total strain energy for beam subjected to torsion moment is defined as

$$U = \int_0^L \left(\frac{1}{2}GJ\theta_x'^2 + \frac{1}{2}EC_w\theta_x''^2\right)dx \tag{4.55}$$

In finite element analysis, we can consider the rate of twist θ'_x as an additional DOF with a force variable conjugate to it called bi-moment. Bi-moment B is considered an auxiliary quantity represented by two equal and self-equilibrating moments appears at the two flange as shown in Figure 4.10b and defined as:

$$B = M_f d = EI_{yf}\frac{d^2 w_f}{dx^2}d = \frac{EI_{yf}d^2}{4}\frac{d^2\theta_x}{dx^2} = EC_w\theta_x'' \tag{4.56}$$

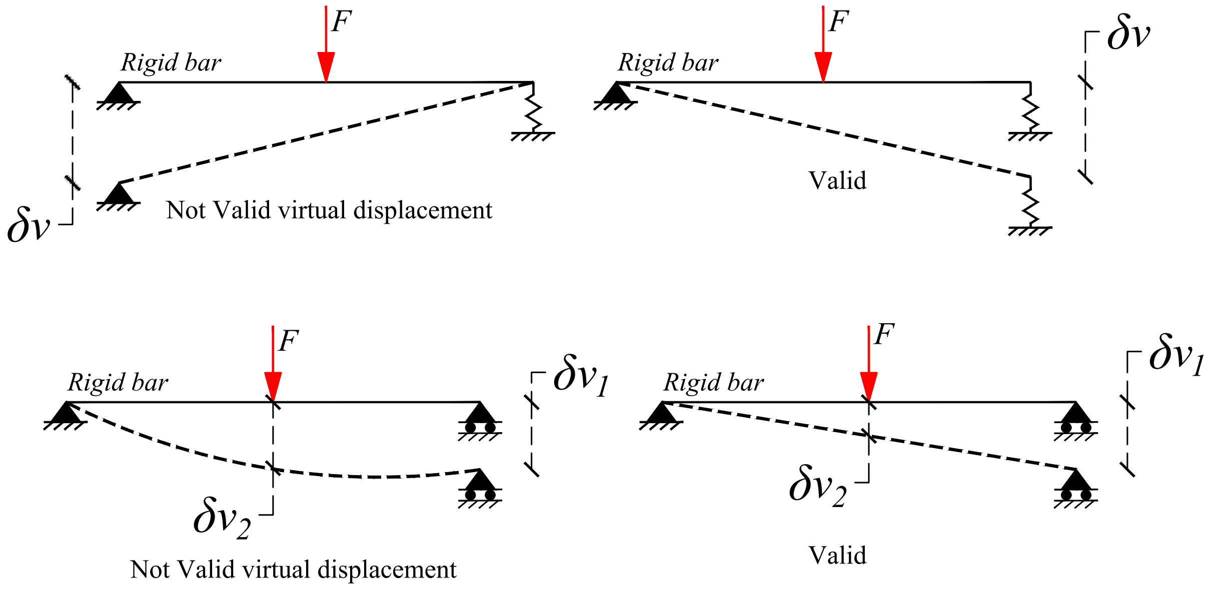
The objective of bi-moment is to formulate an expression similar to the one used in beam theory $M_z = EI_z(v'')$.

For open cross section like I-sections, out-of-plane warping resistance is large compared to its torsional rigidity and can not be neglected.

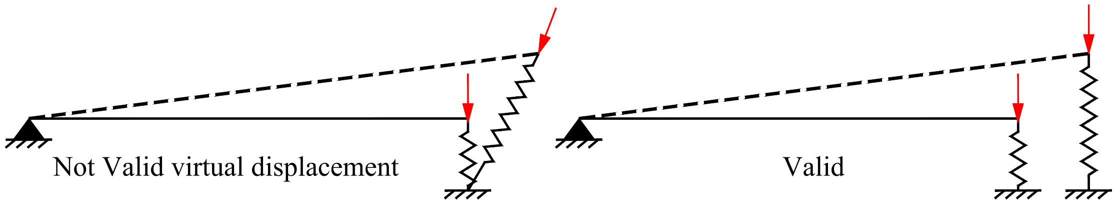
4.2 Virtual work

Any system restrained at some locations on its boundary and subjected to external forces takes many configuration. The set of configurations that satisfies the geometric boundary condition is called set of admissible configurations. For elastic bodies, there is only one equilibrium or true configuration in this set that corresponds to these applied forces. We can also assume that the admissible configuration is obtained by infinitesimal variations of the true configuration. These displacement variations are completely imaginary or virtual and does not have any relation with the true displacement. However, these variations do not violate the boundary conditions (B.C) as shown in Figure 4.11a. Also the applied loads should be the same in the magnitude and direction during these variations as shown in Figure 4.11b. Also it should be independent as shown in Figure 4.11c(As it is a rigid body, the δv_2 is related to δv_1 and both displacements can not be used together in formulating the virtual displacement of the beam). The principle of virtual work states that, for a body configuration under equilibrium of external loads and for any virtual displacement added to this equilibrium configuration, the sum of the virtual work exerted through this virtual displacement vanishes. We can verify this principle via the following examples.

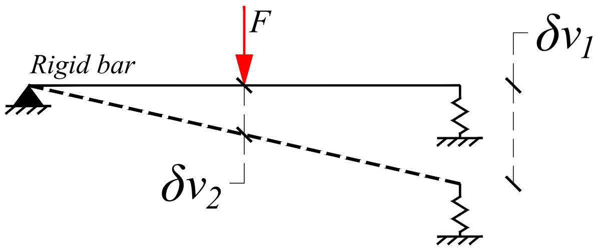
■ **Example 4.5 — Rigid body.** Let us assume a rigid rectangular plate shown in Figure 4.12 with dimensions a and b subjected to external concentrated forces F_1, F_2, F_3, and concentrated moment M, then it undergoes three independent virtual displacements δu, δv , and $\delta\theta$ [a]. For



(a)



(b) The left case exhibits a change in load direction



(c) The virtual displacement δv_1 and δv_2 are dependent if the beam is rigid

Figure 4.11

equilibrium case, the resulting virtual work should vanish as follows:

$$\delta W = F_1\left(-\delta u - \frac{1}{2}b\delta\theta\right) + F_2\left(-\delta v + \frac{1}{2}a\delta\theta\right) \tag{4.57}$$

$$+ F_3 \sin\theta\left(\delta v + \frac{1}{2}a\delta\theta\right) + F_3\cos\theta\left(\delta u - \frac{1}{2}b\delta\theta\right) = 0 \tag{4.58}$$

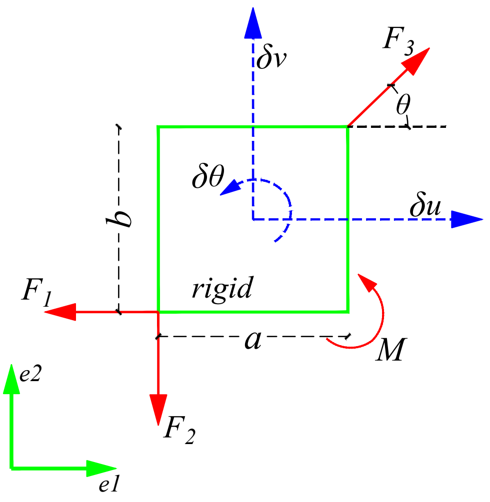


Figure 4.12

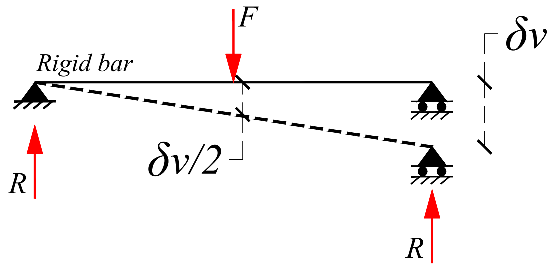


Figure 4.13

$$(-F_1 + F_3 \cos\theta)\,\delta u + (F_2 + F_3 \sin\theta)\,\delta v + \left(-\frac{1}{2}F_1\ b + \frac{1}{2}F_2\ a + \frac{1}{2}F_3\ a\sin\theta - \frac{1}{2}F_3\ b\cos\theta\right)\delta\theta = 0 \tag{4.59}$$

As the virtual displacements are independent and arbitrary, so their coefficients will vanish also as follows:

$$-F_1 + F_3 \cos\theta = 0 \tag{4.60}$$

$$F_2 + F_3 \sin\theta = 0 \tag{4.61}$$

$$-\frac{1}{2}F_1\ b + \frac{1}{2}F_2\ a + \frac{1}{2}F_3\ a\sin\theta - \frac{1}{2}F_3\ b\cos\theta = 0 \tag{4.62}$$

From above, the principle of virtual work provides the three equilibrium equations. ■

[a] We note that any rigid planar element has three independent displacements; two displacements to express displacement in x and y direction and the third one to expresses rotation. We can choose any three independent displacements to express this motion like using two displacements in x direction and one in y direction, such that we can fully describe the planar body motion

In some cases, the assumed virtual displacement could violate the boundary conditions as shown in Figure 4.13. In this case, the reaction related to the violated boundary point will be considered as an external loads and the virtual work will be defined as follows:

$$\delta W = R\delta v + F\frac{\delta V}{2} = 0 \rightarrow R = \frac{F}{2} \tag{4.63}$$

This violated virtual displacements is used to calculate the reactions of structures.

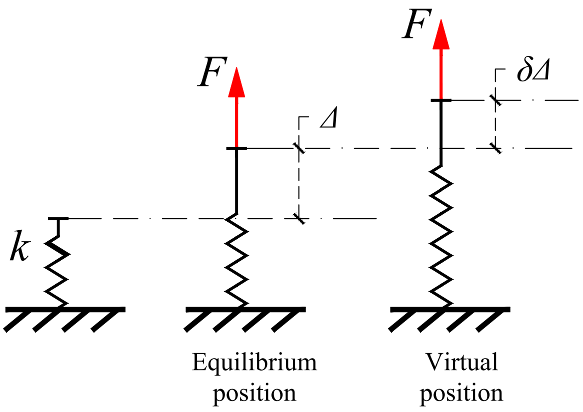


Figure 4.14

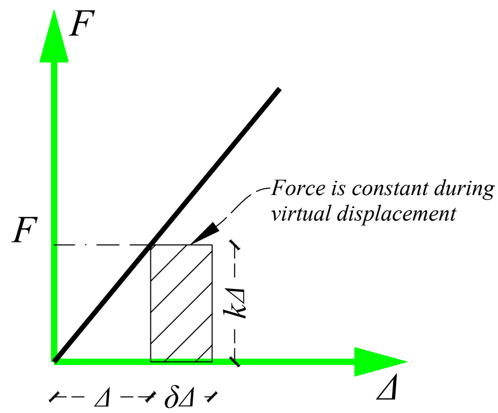
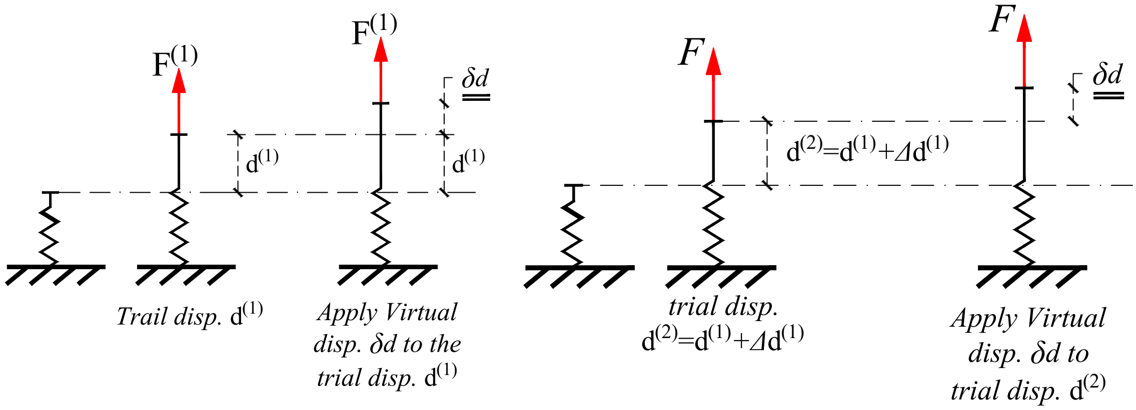


Figure 4.15



(a)

(b)

Figure 4.16

■ **Example 4.6 — Flexible bodies.** Assume a linear elastic spring with stiffness k and subjected to external force F stretching the spring a displacement Δ as shown in Figure 4.14. To evaluate this displacement, we assume a virtual displacement.

The virtual work includes two components; one results from internal stresses W_I and other comes from the external loads δW_{ext}, such that the total virtual work will be:

$$\delta W = \delta W_I + \delta W_{ext} \tag{4.64}$$

Each component is calculated from the area shown in Figure 4.15

$$\delta W = (K\Delta)\delta\Delta - (F)\delta\Delta = 0 \to \Delta = \frac{F}{K}, \quad for\ arbitrary\ \delta\Delta \tag{4.65}$$

■

Virtual work principle is used for solving nonlinear problems and non conservative systems. The above examples are very simple compared to its powerful use in solid mechanics and finite element analysis. The next two examples provide an insight into its use in nonlinear analysis.

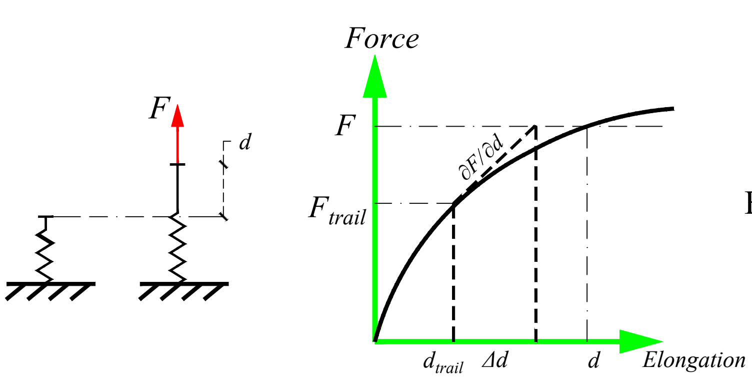


Figure 4.17

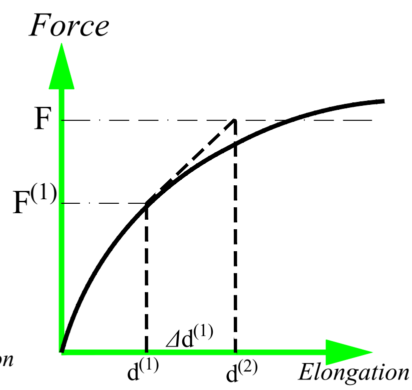


Figure 4.18

■ **Example 4.7** Assume a nonlinear elastic spring with such that the internal force is a function of the spring elongation d ($F = F(d)$) as shown in Figure 4.17. This relation is irreversible such that we cannot calculate the elongation for a particular force directly[a]. It is required to evaluate the displacement d for applied external force F. In this example we will evaluate this force using Newton Raphson method or Taylor's theorem as follows:

First we start at a assumed trial displacement d_{try} and evaluate the corresponding force F_{try}, then applying Taylor's Theorem after neglecting the higher order terms of $\triangle d$ than first as follows:

$$F = F_{try} + \left.\frac{\partial F}{\partial d}\right|_{d=d_{try}} \triangle d \tag{4.66}$$

The $\left.\frac{\partial F}{\partial d}\right|_{d=d_{try}}$ represents the slope of tangent at d_{try} which could be evaluated from function $F = F(d)$ and is called the tangent stiffness of the spring as shown in Figure 4.17. From the above equation we can evaluate an approximate solution to $\triangle d$. Repeating this process using $d_{try} = d_{try} + \triangle d$ many times leads to an accurate result for displacement d. Also it can be solved using virtual work principle as follows:

As shown in Figure 4.18, we can assume the first trial solution is $d^{(1)}$ and it is required to evaluate a better approximation for the displacement $d^{(2)}$. Applying a virtual displacement δd on both cases. As shown in Figure 4.16a and Figure 4.16b, this virtual displacement is identical in both cases and independent on $\triangle d$. The virtual work in the both cases will be:

$$\delta W|_{d=d^{(1)}} = \delta d F^{(1)}, \quad \delta W|_{d=d^{(2)}} = \delta d F \tag{4.67}$$

In the second case, the virtual work can be evaluated using Taylor's expression as follows:

$$\delta W|_{d=d^{(2)}} = \delta W|_{d=d^{(1)}} + \left.\frac{\partial \delta W}{\partial d}\right|_{d=d^{(1)}} \triangle d \tag{4.68}$$

As a result:

$$\delta d F = \delta d F^{(1)} + \delta d \left.\frac{\partial F^{(1)}}{\partial d}\right|_{d=d^{(1)}} \triangle d \tag{4.69}$$

$$\delta d\left(F - F^{(1)}\right) = \delta d\left(\left.\frac{\partial F^{(1)}}{\partial d}\right|_{d=d^{(1)}} \triangle d\right) \tag{4.70}$$

For an arbitrary displacement δd, we get an equation similar to Equation 4.66.

$$F - F^{(1)} = \left.\frac{\partial F^{(1)}}{\partial d}\right|_{d=d^{(1)}} \triangle d \tag{4.71}$$

The process above is called linearization of virtual work which is used to evaluate the tangent stiffness of the structures. ■

[a] In most structures, if the displacement of the structure is known, we can evaluate the corresponding strains and stresses which is integrated over the body volume to evaluate the external loads, but real problems have the displacements unknowns for given external loads and this irreversible function ($F = F(d)$) is an example of a real problem.

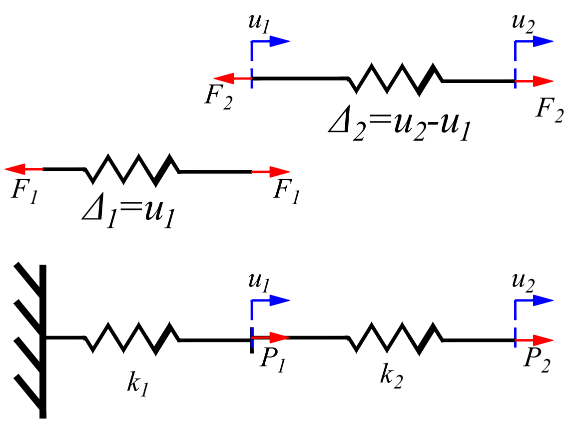


Figure 4.19

■ **Example 4.8** Assume two linear springs connected in series as shown in Figure 4.19 and subjected to two concentrated loads P_1 and P_2 with corresponding displacements u_1 and u_2. From equilibrium at each node, we get:

$$\begin{aligned} P_2 &= k_2(u_2 - u_1) \\ P_1 + P_2 = k_1 u_1 \rightarrow P_1 &= k_1 u_1 - k_2(u_2 - u_1) \end{aligned} \tag{4.72}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.73}$$

If the two springs are nonlinear, the forces generated in each spring are

$$F_1 = 0.1\triangle_1^2 + \triangle_1, \quad F_2 = 0.2\triangle_2^2 + \triangle_2 \tag{4.74}$$

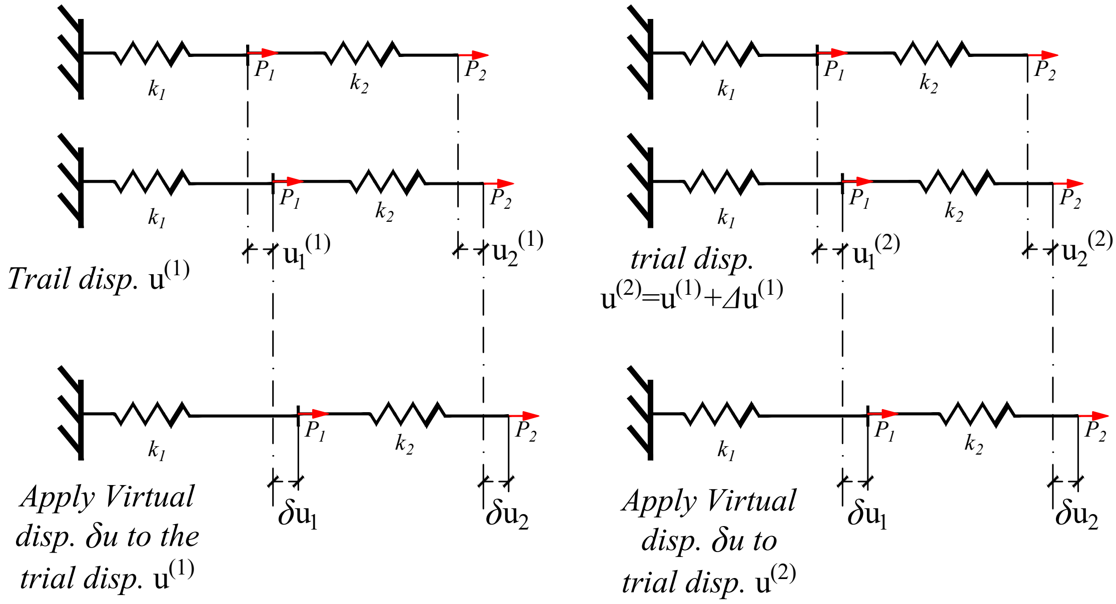


Figure 4.20

Where $\triangle_1$ and $\triangle_2$ represent the elongation undergone in each spring which are related to the nodal displacements through:

$$\triangle_1 = u_1, \quad \triangle_2 = u_2 - u_1 \tag{4.75}$$

So the forces in each spring will be:

$$F_1 = 0.1u_1^2 + u_1, \quad F_2 = 0.2(u_2 - u_1)^2 + (u_2 - u_1) \tag{4.76}$$

The nodal forces is related to the internal forces in springs as follows:

$$p_2 = F_2 = 0.2(u_2 - u_1)^2 + u_2 - u_1 \tag{4.77}$$

$$p_1 + p_2 = F_1 \tag{4.78}$$

$$p_1 = F_1 - F_2 = 0.1u_1^2 + u_1 - \left(0.2(u_2 - u_1)^2 + u_2 - u_1\right) \tag{4.79}$$

So the stiffness of each spring is defined as:

$$k_1 = \frac{\partial F_1}{\partial \triangle_1} = 0.2\triangle_1 + 1 = 0.2u_1 + 1 \tag{4.80}$$

$$k_2 = 0.4\triangle_1 + 1 = 0.4(u_2 - u_1) + 1 \tag{4.81}$$

Using virtual work principle as in the previous example as follows:
Starting with trial solution $u^{(1)^T} = \left[\begin{array}{cc} u_1^{(1)} & u_1^{(2)} \end{array}\right]$

$$\delta W = \delta W|_{u=u^{(1)}} + \left.\frac{\partial \delta W}{\partial u}\right|_{u=u^{(1)}} \triangle u \tag{4.82}$$

As shown from Figure 4.20, applying identical virtual displacements on the real displacements before and after the current trail and rewriting the upper equation in terms of theses virtual displacements as follows:

$$\delta u_1 P_1 + \delta u_2 P_2 = \delta u_1 P_1^{(1)} + \delta u_2 P_2^{(1)} + \left.\frac{\partial \delta W}{\partial u}\right|_{u=u^{(1)}} \triangle u \tag{4.83}$$

$$\begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \begin{bmatrix} P_1 - P_1^{(1)} \\ P_2 - P_2^{(1)} \end{bmatrix} = \left.\frac{\partial \delta W}{\partial u}\right|_{u=u^{(1)}} \triangle u \tag{4.84}$$

But

$$\left.\frac{\partial \delta W}{\partial u}\right|_{u=u^{(1)}} = \left.\frac{\partial \left(\delta u_1 P_1 + \delta u_2 P_2\right)}{\partial u}\right|_{u=u^{(1)}} = \begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \left.\begin{bmatrix} \frac{\partial P_1}{\partial u_1} & \frac{\partial P_1}{\partial u_2} \\ \frac{\partial P_2}{\partial u_1} & \frac{\partial P_2}{\partial u_2} \end{bmatrix}\right|_{u=u^{(1)}} \begin{bmatrix} \triangle u_1 \\ \triangle u_2 \end{bmatrix} \tag{4.85}$$

Then it follows:

$$\begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \left(\begin{bmatrix} P_1 - P_1^{(1)} \\ P_2 - P_2^{(1)} \end{bmatrix} - \left.\begin{bmatrix} \frac{\partial P_1}{\partial u_1} & \frac{\partial P_1}{\partial u_2} \\ \frac{\partial P_2}{\partial u_1} & \frac{\partial P_2}{\partial u_2} \end{bmatrix}\right|_{u=u^{(1)}} \begin{bmatrix} \triangle u_1 \\ \triangle u_2 \end{bmatrix} \right) = 0 \tag{4.86}$$

As δu_1 and δu_2 are arbitrary, it follows that:

$$\begin{bmatrix} P_1 - P_1^{(1)} \\ P_2 - P_2^{(1)} \end{bmatrix} - \left.\begin{bmatrix} \frac{\partial P_1}{\partial u_1} & \frac{\partial P_1}{\partial u_2} \\ \frac{\partial P_2}{\partial u_1} & \frac{\partial P_2}{\partial u_2} \end{bmatrix}\right|_{u=u^{(1)}} \begin{bmatrix} \triangle u_1 \\ \triangle u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.87}$$

The second term is called stiffness matrix and can be defined using Equation 4.72 as follows:

$$\left.\begin{bmatrix} \frac{\partial P_1}{\partial u_1} & \frac{\partial P_1}{\partial u_2} \\ \frac{\partial P_2}{\partial u_1} & \frac{\partial P_2}{\partial u_2} \end{bmatrix}\right|_{u=u^{(1)}} = \left.\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\right|_{u=u^{(1)}} \tag{4.88}$$

While the first term of Equation 4.87 is called the unbalanced forces at nodes which approaches zero with iterations as follow:
Assuming the nodal forces $P_1 = -0.1$ and $P_2 = 1.2$ and we need to evaluate the nodal displacement due to nodal forces . Assuming the first iteration ${u^{(1)}}^T = \begin{bmatrix} 0.5 & 1.5 \end{bmatrix}$ and using Equation 4.76, Equation 4.77 and Equation 4.80, the stiffness matrix and nodal forces will be:

$$\begin{gathered} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 0.4u_2 - 0.2u_1 + 2 & -0.4(u_2 - u_1) - 1 \\ -0.4(u_2 - u_1) - 1 & 0.4(u_2 - u_1) + 1 \end{bmatrix} \\ \begin{bmatrix} P_1 - P_1^{(1)} \\ P_2 - P_2^{(1)} \end{bmatrix} = \begin{bmatrix} -0.1 - 0.1u_1^2 - 2u_1 + \left(0.2(u_2 - u_1)^2 + u_2\right) \\ 1.2 - 0.2(u_2 - u_1)^2 - u_2 + u_1 \end{bmatrix} \end{gathered} \tag{4.89}$$

$$\left.\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\right|_{u=u^{(1)}} = \begin{bmatrix} 2.5 & -1.4 \\ -1.4 & 1.4 \end{bmatrix} \; \& \; \begin{bmatrix} P_1 - P_1^{(1)} \\ P_2 - P_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0.575 \\ 0 \end{bmatrix} \tag{4.90}$$

Applying Equation 4.87

$$\begin{bmatrix} \triangle u_1 \\ \triangle u_2 \end{bmatrix} = \begin{bmatrix} 0.5227 \\ 0.5227 \end{bmatrix} \tag{4.91}$$

The next trial start with $u^{(2)} = u^{(1)} + \triangle u = \begin{bmatrix} 1.0277 \\ 2.0277 \end{bmatrix}$, the stiffness matrix and nodal forces will be:

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\Bigg|_{u=u^{(2)}} = \begin{bmatrix} -2.6045 & -1.4 \\ -1.4 & 1.4 \end{bmatrix} \;\&\; \begin{bmatrix} P_1 - P_1^{(2)} \\ P_2 - P_2^{(2)} \end{bmatrix} = \begin{bmatrix} -0.0273 \\ 0 \end{bmatrix} \quad (4.92)$$

Applying Equation 4.87

$$\begin{bmatrix} \triangle u_1 \\ \triangle u_2 \end{bmatrix} = \begin{bmatrix} -0.0277 \\ -0.0277 \end{bmatrix} \quad (4.93)$$

The next trial start with $u^{(3)} = u^{(2)} + \triangle u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, the stiffness matrix and nodal forces will be:

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\Bigg|_{u=u^{(3)}} = \begin{bmatrix} -2.6 & -1.4 \\ -1.4 & 1.4 \end{bmatrix} \;\&\; \begin{bmatrix} P_1 - P_1^{(3)} \\ P_2 - P_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4.94)$$

The unbalance forces vanishes which mean the equilibrium configuration is reached. ■

4.2.1 Stationary potential energy

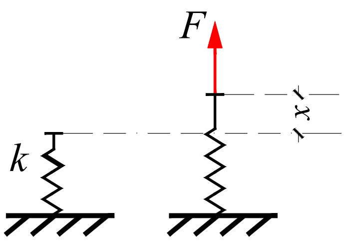


Figure 4.21

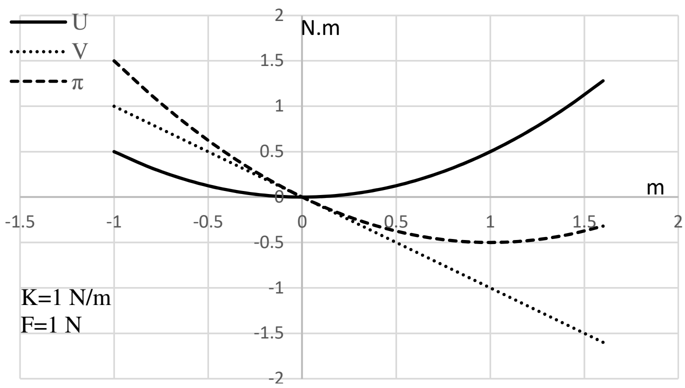


Figure 4.22

As stated in subsection 4.1.3 and Equation 4.20, for a conservative system (elastic and subjected to conservative forces), there is no change in the total potential energy for static loading as follows:

$$\delta\Pi = \delta U + \delta V = 0 \quad (4.95)$$

where U and V is the elastic strain energy stored in the system and load potential energy, respectively. In other words, the potential energy is stationary and it could be maximum or minimum. For stable structures, it undergoes minimum value with respect to displacements.

■ **Example 4.9** Assume a linear elastic spring with stiffness K subjected to axial load F as shown in Figure 4.21. Due to axial displacement x the strain energy induces is $\frac{1}{2}kx^2$, while the load potential will be $-Fx$, so the total potential will be:

$$\Pi = \frac{1}{2}kx^2 - Fx \quad (4.96)$$

Its variation will be:

$$\delta\Pi = \frac{d\Pi}{dx}\delta x = 0 \tag{4.97}$$

For arbitrary displacement δx, $\frac{d\Pi}{dx}$ will vanish as follows:

$$\frac{d\Pi}{dx} = kx - F = 0 \rightarrow x = \frac{F}{k} \tag{4.98}$$

Figure 4.22 shows each components of potential energy and the total energy for $k = 1N/m$ and $F = 1N$. The total potential reaches minimum value at $x = 1$.

4.3 Variational approach

4.3.1 Calculus of Variance

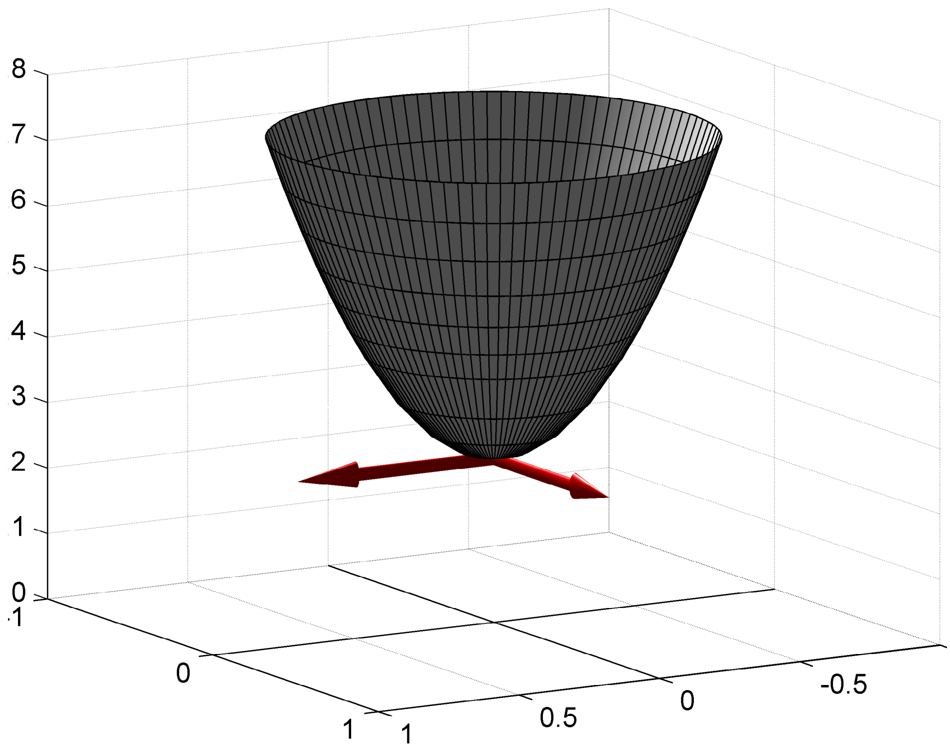


Figure 4.23

If a function $f(x)$ has an extremum (minimum or maximum) at a point x_o in the interval $x = [a,b]$, the first derivative of this function at this point vanishes as follows:

$$\left.\frac{df}{dx}\right|_{x=x_o} = 0 \tag{4.99}$$

The function is considered maximum (minimum) at this point when

$$\frac{d^2f}{dx^2} < 0 \ \left(\frac{d^2f}{dx^2} > 0\right) \tag{4.100}$$

For a differentiable function $f(x,y)$ of two variables, the necessary condition for an extremum at some point (x_0, y_0) is that the total differential of this function vanishes at this point as follows:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \ \ at\ x = x_0\ and\ y = y_0 \tag{4.101}$$

As x and y are linear independent (x and y are not related to each other), so for arbitrary values for dx and dy, it follows that:

$$\frac{\partial f}{\partial x} = 0 \ , \quad \frac{\partial f}{\partial y} = 0 \quad at\ x = x_0\ and\ y = y_0 \tag{4.102}$$

For paraboloid $z = x^2 + y^2 + 0.25$ shown in Figure 4.23, its derivatives with respect to x and y vanish at:

$$\frac{\partial f}{\partial x} = 2x = 0 \rightarrow x = 0 \tag{4.103}$$

$$\frac{\partial f}{\partial y} = 2y = 0 \rightarrow y = 0 \tag{4.104}$$

As shown in Figure 4.23, the surface tangents at point $(0,0)$ in x and y directions vanish as shown in red arrows with zero slope at point $(0,0)$.
Variational methods seek the extremum of integrals of what is called functionals or function of functions. Functional is definite integral of dependent function(s) and their derivatives that are themselves functions of other independent variables. For example:

$$F = \int_a^b I(y,z,y',z',y'',...)dx \tag{4.105}$$

$y = y(x)$ and $z = z(x)$ are dependent functions of independent variable x, and $I(y,z)$ is functionals or function of functions. The calculus of variance is used to calculate this dependent function(s) that make the functional stationary value. For example, in the real structures, the total potential energy should reach minimum value at the equilibrium configuration. For example, if functional F is given by:

$$F = \int_a^b \left(u' + 2u^2\right) dx \tag{4.106}$$

it could be written as:

$$F = \int_a^b \phi\left(u,u'\right) dx \tag{4.107}$$

Where $u = u(x)$ is dependent function of independent variable x. The purpose of calculus of variance is to evaluate the function u(x) that make functional F stationary value. First we will introduce variational operator (δ), such that δF is called the first variation of functional F. Variational operator (δ) operates like differential operator (d), but does not depend on the independent variable, such that x is fixed during variation of function δu and its derivative $\delta u'$, such that the first variation of functional δF and differential dF are defined as:

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \tag{4.108}$$

$$dF = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' + \frac{\partial F}{\partial x}dx \tag{4.109}$$

Variational calculus operates similar to differential calculus as follows:

$$\delta\left(F_1 \pm F_2\right) = \delta F_1 \pm \delta F_2 \tag{4.110}$$

$$\delta\left(F_1 F_2\right) = \delta F_1\ F_2 + F_1\ \delta F_2 \tag{4.111}$$

Also (δ) can be interchanged with differential operator or integral operator as follows:

$$\delta\left(\frac{du}{dx}\right) = \frac{d\left(\delta u\right)}{dx}, \qquad \delta\left(\int_a^b u\ dx\right) = \int_a^b \delta u\ dx \tag{4.112}$$

For functional $F = \int_a^b \phi\left(u, v, w\right) dx$ defined in terms of several dependent functions u, v, and w, its variation

$$\delta F = \delta F_u + \delta F_v + \delta F_w \tag{4.113}$$

Functional is called linear (quadratic) functional as follow

$$F\left(\alpha u\right) = \alpha F\left(u\right) \left(F\left(\alpha u\right) = \alpha^2 F\left(u\right)\right) \tag{4.114}$$

■ **Example 4.10** $F = \int \left(au + bu' + cw\right) dx$ is a linear functional, while $F = \int \left(au^2 + bu'^2 + cw''^2\right) dx$ is a quadratic functional. ■

The first variation δF also called Gateaux derivative of function in direction δu takes these forms

$$\delta F\left(\boldsymbol{u}, \delta\boldsymbol{u}\right) = D_{\delta\boldsymbol{u}} F\left(\boldsymbol{u}\right) = DF\left(\boldsymbol{u}, \delta\boldsymbol{u}\right) = DF\left(\boldsymbol{u}\right)\left[\delta\boldsymbol{u}\right] = \frac{d}{d\varepsilon} F\left(\boldsymbol{u} + \delta\boldsymbol{u}\right)\Big|_{\varepsilon=0} \tag{4.115}$$

Note 4.1 The following expressions are useful for nonlinear analysis:

$$\nabla\left(\delta\boldsymbol{u}\right) = \frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial\boldsymbol{x}} = \frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial\boldsymbol{X}}\frac{\partial\boldsymbol{X}}{\partial\boldsymbol{x}} = \nabla_0\left(\delta\boldsymbol{u}\right)\boldsymbol{F}^{-1} \tag{4.116}$$

$$\nabla\left(\delta\boldsymbol{v}\right) = \frac{\partial\left(\delta\boldsymbol{v}\right)}{\partial\boldsymbol{x}} = \frac{\partial\left(\delta\boldsymbol{v}\right)}{\partial\boldsymbol{X}}\frac{\partial\boldsymbol{X}}{\partial\boldsymbol{x}} = \nabla_0\left(\delta\boldsymbol{v}\right)\boldsymbol{F}^{-1} \tag{4.117}$$

Using the above expressions:

$$\delta\boldsymbol{F} = \delta\left(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}} + \boldsymbol{1}\right) = \frac{\partial\left(\delta\boldsymbol{u}\right)}{\partial\boldsymbol{X}} = \nabla_0\left(\delta\boldsymbol{u}\right) = \nabla\left(\delta\boldsymbol{u}\right)\boldsymbol{F} \tag{4.118}$$

$$\delta\dot{\boldsymbol{F}} = \delta\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{X}}\right) = \frac{\partial\left(\delta\boldsymbol{v}\right)}{\partial\boldsymbol{X}} = \nabla_0\left(\delta\boldsymbol{v}\right) \rightarrow \delta\dot{F} = \nabla\left(\delta\boldsymbol{v}\right)\boldsymbol{F} \tag{4.119}$$

$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}\delta\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) = \frac{1}{2}\left(\nabla\left(\delta\boldsymbol{u}\right)^T + \nabla\left(\delta\boldsymbol{u}\right)\right) \tag{4.120}$$

$$\delta \boldsymbol{D} = \frac{1}{2}\delta\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\right) = \frac{1}{2}\left(\nabla(\delta \boldsymbol{v})^T + \nabla(\delta \boldsymbol{v})\right) = \frac{1}{2}\left(\delta \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\delta \dot{\boldsymbol{F}}^T\right) \quad (4.121)$$

using Equation 4.120, we reach:

$$\delta \boldsymbol{E} = \frac{1}{2}\delta\left(\boldsymbol{F}^T\boldsymbol{F} - 1\right) = \frac{1}{2}\left(\delta \boldsymbol{F}^T \boldsymbol{F} + \boldsymbol{F}^T \delta \boldsymbol{F}\right) = \frac{1}{2}\boldsymbol{F}^T\left(\nabla(\delta \boldsymbol{u})^T + \nabla(\delta \boldsymbol{u})\right)\boldsymbol{F} = \boldsymbol{F}^T \delta \boldsymbol{\varepsilon} \boldsymbol{F} \quad (4.122)$$

■

■ **Example 4.11**

$$F = \int \left(c_1 u^2 + c_2 u (u')^2 + c_3 u'' + c_4 u v\right) dx \quad (4.123)$$

We find that the above expression can be expressed as follows:

$$F = \int \phi\left(u, u', u'', v\right) dx \quad (4.124)$$

With variation:

$$\delta F = \delta F_u + \delta F_v = \int \left(\frac{\partial \phi}{\partial u}\delta u + \frac{\partial \phi}{\partial u'}\delta u' + \frac{\partial \phi}{\partial u''}\delta u''\right) + \left(\frac{\partial \phi}{\partial v}\delta v\right) dx \quad (4.125)$$

$$= \int \left(\left(2c_1 u + c_2 (u')^2 + c_4 v\right)\delta u + \left(2c_2 u\, u'\right)\delta u' + c_3 \delta u''\right) + \left(c_4 u \delta v\right) dx \quad (4.126)$$

■

In structural problems, variational approach is used to find the displacement (dependent) function that make the potential energy stationary value (principle of minimum potential energy).

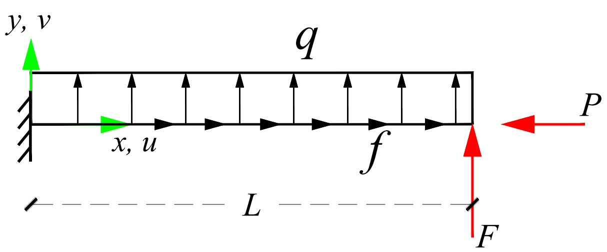


Figure 4.24

■ **Example 4.12** The total potential energy of a fixed beam shown in Figure 4.24 with length L, bending rigidity EI_z and axial rigidity EA subjected to axial load P, transverse load F at its right end $x = L$, distributed axial load f and transverse loads q is defined using Equation 4.95, Equation 4.35 and Equation 4.40 as follows:

$$\Pi = U + V = \int \left(\frac{1}{2} EAu'^2 + \frac{1}{2} EI_z v''^2 - qv - fu \right) dx - Fv(L) - Pu(L) \tag{4.127}$$

The variation in the total potential energy will be:

$$\delta\Pi = \int^{L} \left(\frac{1}{2} EAu' \delta u' + \frac{1}{2} EI_z v'' \delta v'' - q\delta v - f\delta u \right) dx - F\delta v(L) - P\delta u(L) \tag{4.128}$$

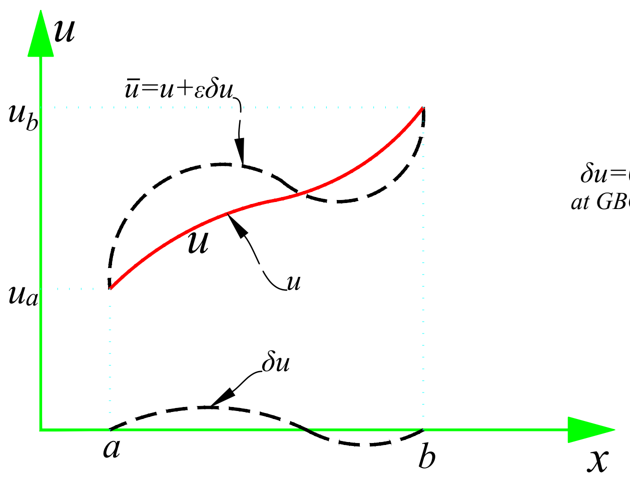


Figure 4.25

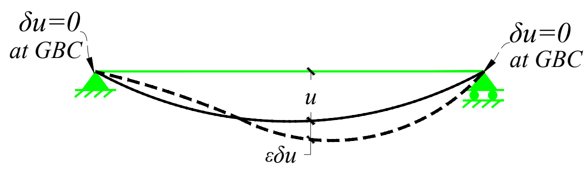


Figure 4.26

Let us assume a beam with true (equilibrium) configuration u(x) needed to be evaluated. We can get what is called an admissible configuration $\bar{u}$ by applying an infinitesimal variation ε to the true configuration in the direction δu as shown in Figure 4.25 as follows:

$$\bar{u} = u + \varepsilon \delta u \tag{4.129}$$

ε is very small variation, such that it does not disturb the equilibrium. δu is an arbitrary kinematically admissible function that satisfies the geometric boundary condition (GBC) as shown in Figure 4.26. δu is an assumed or imaginary (displacement) function field and does not have any relation with the true configuration. We are not interested in all functions $\bar{u}$, but the one that satisfies the geometric boundary condition. These geometric boundary condition can be defined as follows:

$$\delta u|_{s_u} = 0 \qquad or \qquad \bar{u}|_{S_u} = u|_{S_u} \tag{4.130}$$

Where S_u represents the location of restrained boundary. The above equation means that the assumed displacements must be equal to the assigned displacements at this restrained boundary. As shown in the Figure 4.26.
There is an infinite number of admissible configurations even for the same δu via changing ε.
Using Taylor series, the change $\triangle F$ in a functional $F = \int_a^b \phi(u, u')\, dx$ due to disturbance ε in

direction of δu will be defined as:

$$\begin{aligned}
\triangle F &= \int_a^b \phi\left(u+\varepsilon\delta u, u'+\varepsilon\delta u'\right)dx - \int_a^b \phi\left(u,u'\right)dx \\
&= \frac{\partial F}{\partial u}\varepsilon\delta u + \frac{1}{2}\frac{\partial^2 F}{\partial u^2}(\varepsilon\delta u)^2 + \cdots + \frac{\partial F}{\partial u'}\varepsilon\delta u' + \frac{1}{2}\frac{\partial^2 F}{\partial u'^2}\left(\varepsilon\delta u'\right)^2 + \ldots \\
&= \left(\underbrace{\frac{\partial F}{\partial\phi\phi}\delta u + \frac{\partial F}{\partial u'}\delta u'}_{\delta F}\right)\varepsilon + \left(\underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial u^2}(\delta u)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial u'^2}\left(\delta u'\right)^2}_{\delta^2 F}\right)\varepsilon^2 + \ldots \\
&= \delta F\varepsilon + \delta^2 F\varepsilon^2 + \ldots
\end{aligned} \tag{4.131}$$

For a functional to be stationary or extremum (minimum or maximum) at a particular configuration, the first variation of the functional δF should vanish, while the second variation $\delta^2 F$ defines if the function is minimum (maximum) at this configuration as follows:

$$\delta F = 0, \qquad \delta^2 F > 0\ (\delta^2 F < 0) \tag{4.132}$$

So we get:

$$0 = \delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' = \int_a^b \left(\frac{\partial\phi}{\partial u}\delta u + \frac{\partial\phi}{\partial u'}\delta u'\right)dx \tag{4.133}$$

Using integration by part for the second term, it follows:

$$0 = \int_a^b \left(\frac{\partial\phi}{\partial u} - \frac{d}{dx}\left(\frac{\partial\phi}{\partial u'}\right)\right)\delta u dx + \left.\frac{\partial\phi}{\partial u'}\delta u\right|_a^b \tag{4.134}$$

Generally, the last term $\frac{\partial\phi}{\partial u'}\delta u$ vanishes at boundaries as for geometric boundary conditions δu vanishes, while for essential boundary conditions $\frac{\partial\phi}{\partial u'}$ vanishes (see the next example), so the above equation will be:

$$\int_a^b \left(\frac{\partial\phi}{\partial u} - \frac{d}{dx}\left(\frac{\partial\phi}{\partial u'}\right)\right)\delta u dx = 0 \tag{4.135}$$

Using the following Lemma for any arbitrary function δu:

$$If \int_a^b G\delta u dx = 0,\ it\ follows\ that\ G = 0\ at\ any\ point\ on\ the\ domain\ of\ integral\ [a,b] \tag{4.136}$$

While, for two independent arbitrary functions δu, and δv,

$$if \int_a^b (G\delta u + H\delta v)\,dx = 0 \leftrightarrow Both\ G\ and\ H\ vanish\ at\ any\ point\ on\ the\ domain\ of\ integral\ [a,b] \tag{4.137}$$

As a result of this Lemma, it follows:

$$\boxed{\frac{\partial\phi}{\partial u} - \frac{d}{dx}\left(\frac{\partial\phi}{\partial u'}\right) = 0} \tag{4.138}$$

This equation is Euler equation of functional. Of all admissible functions, there is only one solution that satisfies the above equation which express the true function that minimize the functional F.

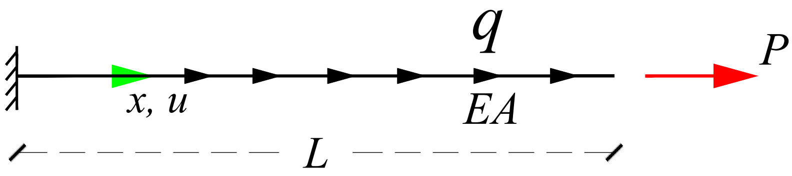


Figure 4.27

■ **Example 4.13** Let us assume a rod shown in Figure 4.27 with length L and axial rigidity EA and loaded with axial load P and with distributed axial load q. Using Figure 4.27, the total potential energy will be:

$$\Pi = \int_0^L \left(\frac{1}{2}EAu'^2 - qu\right) dx - Pu(0) \tag{4.139}$$

The equilibrium path that makes $\delta\Pi = 0$ as follows:

$$0 = \delta\Pi = \int_0^L \left(EAu'\delta u' - q\delta u\right) dx - P\delta u(0) \tag{4.140}$$

Using integration by part over the first term leads to:

$$0 = \delta\Pi = EAu'\delta u\big|_L - (EAu' + P)\delta u\big|_0 - \int_0^L \left(\frac{d}{dx}\left(EAu'\right) + q\right) \delta u dx \tag{4.141}$$

At the left end $x = 0$, using Equation 4.32 and Equation 4.33 it follows:

$$EAu'(0) + P = 0 \tag{4.142}$$

This condition is called the essential boundary condition, while $\delta u|_L$ vanishes to satisfy the geometric boundary condition leading to finally:

$$\int_0^L \left(\frac{d}{dx}\left(EAu'\right) + q\right) \delta u dx = 0 \tag{4.143}$$

Using the above lemma, it follows:

$$\frac{d}{dx}\left(EAu'\right) + q = 0 \tag{4.144}$$

The above equation corresponds to the Euler Equation 4.138. ■

■ **Example 4.14** For a hinged-hinged beam shown in Figure 4.28 with length L, bending rigidity EI and axial rigidity EA subjected to axial load P at the right end, Moments M_0 and M_L at its ends, distributed axial load q_o and transverse loads q, the total potential energy of the beam is

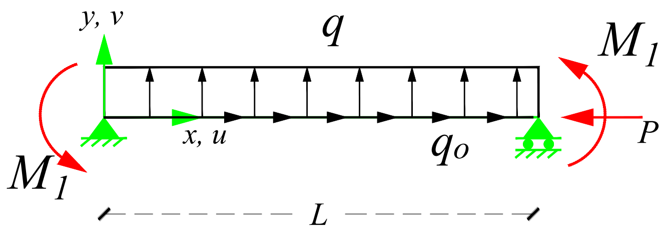


Figure 4.28

defined using Equation 4.95, Equation 4.35 and Equation 4.40 as follows:

$$\Pi = \int_0^L \left(\frac{1}{2} EAu'^2 + \frac{1}{2} EI_x v''^2 - q_o u - qv \right) dx - M_0\theta_0 - M_L\theta_L \tag{4.145}$$

Its variation will vanish (see subsection 4.2.1) as follows:

$$\delta\Pi = \int_0^L \left(EAu'\delta u' + EI_z v''\delta v' - q_o\delta u - q\delta v \right) dx - P\delta u(L) - M_0\delta\theta_0 - M_L\delta\theta_L = 0 \tag{4.146}$$

Integrating once and twice by part for the first and second term, respectively.

$$\begin{aligned}\delta\Pi = &-\int_0^L \left(EAu'' + q_o \right) \delta u dx + \int_0^L \left(EI_z v'''' - q \right) \delta v dx \\ &+ \left(EAu' \right) \delta u\big|_0^L - P\delta u(L) - \left(EI_z v''' + Pv' \right) \delta v\big|_0^L + \left(EI_z v'' - M \right) \delta v'\big|_0^L\end{aligned} \tag{4.147}$$

As δu δv and $\delta v'$ are arbitrary and independent so their coefficients vanish. This leads to the following differential equations associated with simple beam.

$$\left(EAu'' + q_o \right) = 0, \quad x = [0, L] \left(EI_z v'''' - q \right) = 0, \quad x = [0, L] \tag{4.148}$$

The boundary conditions at ends may be essential or geometric as follows:

$$\begin{aligned}\left(EAu' - P \right) = 0 \quad & or \quad \delta u = 0 \quad & at\ x = 0, L \\ \left(EI_z v'' - M \right) = 0 \quad & or \quad \delta v' = 0 \quad & at\ x = 0, L\end{aligned} \tag{4.149}$$

In this beam, the left end has two GBC and one EBC as follows:

$$\delta u = 0|_{x=0}, \quad \delta v = 0|_{x=0}, \quad \left(EI_z v'' - M \right)\big|_{x=0} = 0 \rightarrow EI_z v''(0) = M_0 \tag{4.150}$$

Similarly, the right end has one GBC and two EBC as follows:

$$\delta v = 0|_{x=L}, \quad \left(EAu' - P \right)\big|_{x=L} = 0 \rightarrow EAu'(L) = P, \quad \left(EI_z v'' - M \right)\big|_{x=L} = 0 \rightarrow EI_z v''(L) = M_L \tag{4.151}$$

For the same above beam, if we need to evaluate the buckling load P (Stability problem), term $u(L)$ should be split into two parts; part due to axial strain $\int u' dx$ and other due to beam bowing (shortening due to bending) $\frac{1}{2} \int v'^2 dx$. The last part comes from the change in length of the beam. For an infinitesimal beam ds, the change in its length will be $ds - dx = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + v'^2} - dx$. Using Taylor series and neglecting higher order effect, the change in length will be $\frac{1}{2}v'^2dx$. Integrating this term over the length results in the bowing effect as follows:

$$u(L) = \frac{1}{2}\int v'^2 dx \tag{4.152}$$

Then the potential energy will be:

$$\Pi = \int_0^L \left(\frac{1}{2}EAu'^2 + \frac{1}{2}EI_z v''^2 - \frac{1}{2}Pv'^2 - Pu' - q_o u - qv\right) dx - M_0\theta_0 - M_L\theta_L \tag{4.153}$$

$$\delta\Pi = \int_0^L \left(EAu'\delta u' + EI_z v''\delta v'' - Pv'\delta v' - P\delta u' - q_o\delta u - q\delta v\right) dx - M_0\delta\theta_0 - M_L\delta\theta_L \tag{4.154}$$

Integrating once by part the first and third terms, and twice by part the second term leads to:

$$\begin{aligned}\delta\Pi = &-\int_0^L \left(EAu'' + q_o\right)\delta u dx + \int_0^L \left(EI_z v'''' - q + Pv''\right)\delta v dx \\ &+ \left(EAu' - P\right)\delta u\Big|_0^L - \left(EI_x v''' + Pv'\right)\delta v\Big|_0^L + \left(EI_z v'' - M\right)\delta v'\Big|_0^L\end{aligned} \tag{4.155}$$

As δu, δv and $\delta v'$ are arbitrary and independent, so their coefficients vanish.

$$\left(EAu'' + q_o\right) = 0, \quad x = [0, L] \tag{4.156}$$

$$\left(EI_z v'''' - q + Pv''\right) = 0, \quad x = [0, L] \tag{4.157}$$

The second differential equation expresses the beam buckling (Eigen value problem). The boundary conditions at ends may be essential (EBC) or geometric (GBC) as follows:

$$\begin{aligned}\left(EAu' - P\right) = 0 \quad &or \quad \delta u = 0 \quad at\ x = 0, L \\ \left(EI_z v''' + Pv'\right) = 0 \quad &or \quad \delta v = 0 \quad at\ x = 0, L \\ \left(EI_z v'' - M\right) = 0 \quad &or \quad \delta v' = 0 \quad at\ x = 0, L\end{aligned} \tag{4.158}$$

In this beam, the left end has two GBC and one EBC as follows:

$$\delta u = 0, \quad \delta v = 0, \quad \left(EI_z v'' - M\right)\Big|_{x=0} = 0 \rightarrow EI_z v''(0) = M_0 \tag{4.159}$$

Similarly, the right end has one GBC and two EBC as follows:

$$\delta v = 0, \quad \left(EAu' - P\right)\Big|_{x=L} = 0 \rightarrow EAu'(L) = P, \quad \left(EI_z v'' - M\right)\Big|_{x=L} = 0 \rightarrow EI_z v''(L) = M_L \tag{4.160}$$

■

Differential equation of motion associated with continuum body can also be derived from variational principles as follow:

■ **Example 4.15** The total potential energy contains the stored strain energy and external loads potential energy. The external loads include surface loads $\boldsymbol{t}$ and body forces $\boldsymbol{f}$ as shown in

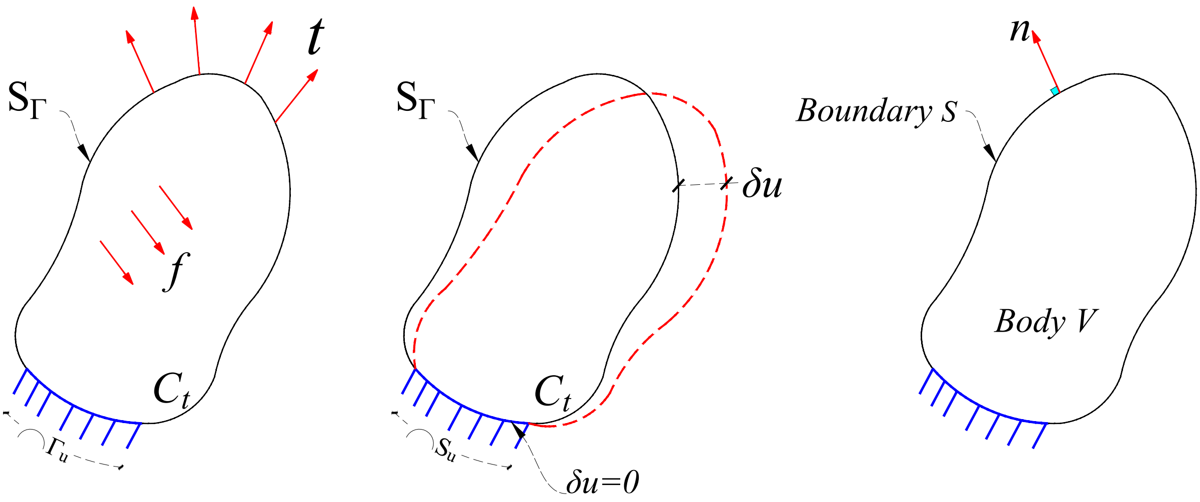


Figure 4.29

Figure 4.29, so the total potential will be:

$$\delta\Pi = \delta U + \delta V = \int_V \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\, dV - \int_V \boldsymbol{f}^*.\delta\boldsymbol{u}\, dV - \int_{S_\Gamma} \boldsymbol{t}.\delta\boldsymbol{u}\, dA \tag{4.161}$$

To include the dynamic effect, we use fictitious body force $\boldsymbol{f}^* = \boldsymbol{f} - \rho\ddot{\boldsymbol{u}}$. Boundary S_Γ represents the loaded (not constrained) boundary of the body. For symmetric tensor $\boldsymbol{\sigma}$, it follows using Equation 1.100:

$$\int_V \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon} dV = \int_V \boldsymbol{\sigma}:\delta\frac{\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right)}{2} dV = \int_V \boldsymbol{\sigma}:\delta\left(\nabla\boldsymbol{u}\right) dV = \int_V \boldsymbol{\sigma}:\nabla\left(\delta\boldsymbol{u}\right)\, dV \tag{4.162}$$

Using divergence theorem and Equation 1.198:

$$\int_V \boldsymbol{\sigma}:\nabla\left(\delta\boldsymbol{u}\right)\, dV = \int_{S_u} \delta\boldsymbol{u}.\left(\boldsymbol{\sigma}.\boldsymbol{n}\right)\, dA_0 + \int_{S_\Gamma} \delta\boldsymbol{u}.\left(\boldsymbol{\sigma}.\boldsymbol{n}\right)\, dA_0 - \int_V \delta\boldsymbol{u}.\left(\nabla.\boldsymbol{\sigma}\right)\, dV_0 \tag{4.163}$$

Then the variation in the total potential energy will be:

$$\delta\Pi = \int_{S_u} \delta\boldsymbol{u}.\left(\boldsymbol{\sigma}.\boldsymbol{n}\right)\, dA_0 + \int_{S_\Gamma} \delta\boldsymbol{u}.\left(\boldsymbol{\sigma}.\boldsymbol{n}-\boldsymbol{t}\right)\, dA_0 - \int_V \delta\boldsymbol{u}.\left(\nabla.\boldsymbol{\sigma}+\boldsymbol{f}^*\right)\, dV_0 = 0 \tag{4.164}$$

Which leads to Euler equation of motion, and natural and geometric boundary conditions as follows:

$$\begin{aligned} \nabla.\boldsymbol{\sigma} + \boldsymbol{f}^* &= \boldsymbol{0} \quad \textit{on } V \\ \boldsymbol{t} - \boldsymbol{n}.\boldsymbol{\sigma} &= \boldsymbol{0} \quad \textit{on boundary } S_\Gamma \\ \boldsymbol{u}|_{S_u} &= \boldsymbol{u} \quad \textit{on boundary } S_u \;\; \textit{as } \delta\boldsymbol{u}|_{S_u} = \boldsymbol{0} \end{aligned} \tag{4.165}$$

■

■ **Example 4.16** For Lagrangian differential equation of motions $\boldsymbol{P}:\boldsymbol{F}^O$ are considered conjugate pairs as stated in Equation 3.256 where $\boldsymbol{F}^O = \dot{\boldsymbol{F}} - \boldsymbol{W}\boldsymbol{F}$

$$\int_{V_0} \boldsymbol{P}:\dot{\boldsymbol{F}}\, dV_0 = \int_{V_0} \boldsymbol{P}:\boldsymbol{F}^O\, dV_0 \tag{4.166}$$

$$\delta\Pi = \delta U + \delta V = \int_{V_0} \boldsymbol{P}:\delta\mathring{\boldsymbol{F}}\, dV_0 - \int_{V_0} \boldsymbol{f}_0^*.\delta\boldsymbol{u}\, dV_0 - \int_{S_{\Gamma 0}} \boldsymbol{t}_0.\delta\boldsymbol{u}\, dA_0 \tag{4.167}$$

Where $\boldsymbol{f}_0^*$ and $\boldsymbol{t}_0$ are the body force per unit volume of the initial configuration and traction stress affecting the area of the same configuration, respectively, while $\delta\mathring{\boldsymbol{F}}$ is defined as

$$\delta\mathring{\boldsymbol{F}} = \delta\boldsymbol{F} - \delta\boldsymbol{\phi}\boldsymbol{F} = \frac{\partial(\delta\boldsymbol{x})}{\partial\boldsymbol{X}} - \widetilde{\delta\boldsymbol{\phi}}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}} = \frac{\partial(\delta\boldsymbol{u})}{\partial\boldsymbol{X}} - \widetilde{\delta\boldsymbol{\phi}}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}} \tag{4.168}$$

Where $\widetilde{\delta\boldsymbol{\phi}} = \delta\boldsymbol{R}\boldsymbol{R}^T$ and $S_{\Gamma 0}$ initial boundary of stress

$$\int_V \boldsymbol{P}:\delta\boldsymbol{F}^O\, dV_0 = \int_V \boldsymbol{P}:\frac{\partial(\delta\boldsymbol{u})}{\partial\boldsymbol{X}}\, dV_0 - \int_V \boldsymbol{P}:\left(\delta\boldsymbol{\phi}\times\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}}\right) dV_0 \tag{4.169}$$

Using divergence theorem and Equation 1.198:

$$\int_V \boldsymbol{P}:\frac{\partial(\delta\boldsymbol{u})}{\partial\boldsymbol{X}}\, dV_0 = \int_{S_{\Gamma 0}} \delta\boldsymbol{u}.(\boldsymbol{P}.\boldsymbol{N})\, dA_0 - \int_V \delta\boldsymbol{u}.(\nabla_0.\boldsymbol{P})\, dV_0 \tag{4.170}$$

Where N is normal to body boundary surface $S_{\Gamma 0}$ at initial configuration For $\boldsymbol{P} = [\ T_1 \quad T_2 \quad T_3\ ]$, and from Equation 1.205

$$\int_V \boldsymbol{P}:\left(\delta\boldsymbol{\phi}\times\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}}\right) dV_0 = \int_V \delta\boldsymbol{\phi}.\left(\frac{\partial\boldsymbol{x}}{\partial X_i}\times T_i\right) dV_0 \tag{4.171}$$

$$\delta\Pi = \int_{S_{\Gamma 0}} \delta\boldsymbol{u}.(\boldsymbol{P}.\boldsymbol{N})\, dA_0 - \int_V \delta\boldsymbol{u}.\nabla_0.\boldsymbol{P}\, dV_0 - \int_V \delta\boldsymbol{\phi}.\left(\frac{\partial\boldsymbol{x}}{\partial X_i}\times T_i\right) dV_0 \tag{4.172}$$

$$- \int_{V_0} \boldsymbol{f}_0^*.\delta\boldsymbol{u}\, dV_0 - \int_{S_{\Gamma 0}} \boldsymbol{t}_0.\delta\boldsymbol{u}\, dA_0 \tag{4.173}$$

$$= \int_{S_{\Gamma 0}} \delta\boldsymbol{u}.(\boldsymbol{P}.\boldsymbol{N} - \boldsymbol{t}_0)\, dA_0 - \int_V \delta\boldsymbol{u}.(\nabla_0.P + \boldsymbol{f}_0^*)\, dV_0 - \int_V \delta\boldsymbol{\phi}.\left(\frac{\partial\boldsymbol{x}}{\partial X_i}\times T_i\right) dV_0 \tag{4.174}$$

Euler Equations or balance of linear momentum in the material form (balance of angular momentum)

$$\nabla_0.\boldsymbol{P} + \boldsymbol{f}_0^* = 0 \qquad \left(\frac{\partial x}{\partial X_i}\times T_i = 0\right) \qquad on\ V_0 \tag{4.175}$$

Natural boundary condition

$$\boldsymbol{t}_0 - \boldsymbol{P}.\boldsymbol{N} = 0 \qquad on\ boundary \quad S_{\Gamma 0} \tag{4.176}$$

■

Note that the variational approach produces the differential equations and natural (essential) boundary conditions, but it does not provide the function shape that minimizes the functional (potential energy). However, for some complicated systems, it is very hard to get the controlling differential equation, and implementing variational principle will requires the help of other methods such as Rayleigh Ritz or weighted residual methods which find an approximate solution to these complicated problems (see the next sections).

4.3.2 Rayleigh Ritz method

This method uses an assumed solution for dependent function u such that it satisfies the boundary conditions. This assumed function is generally polynomial as follows:

$$\left(u = \sum_{i=0}^{n} a_i \phi_i = a_0 + a_1 x + a_2 x^2 \dots \right) \tag{4.177}$$

Which converts the variational functional Π to simple differential function of parameters a_i as follows:

$$\Pi = \Pi(a_0, a_1, a_2, \dots) \tag{4.178}$$

To make Π extremum, $\delta\Pi$ should vanish as follows:

$$0 = \delta\Pi = \frac{\partial \Pi}{\partial a_0}\delta a_0 + \frac{\partial \Pi}{\partial a_1}\delta a_1 + \frac{\partial \Pi}{\partial a_2}\delta a_2 + \dots . \tag{4.179}$$

As δa_i are independent variables, it yields that their coefficients vanish as follows:

$$\frac{\partial \Pi}{\partial a_i} = 0 \quad for\ i = 0, 1, 2, \dots \tag{4.180}$$

The assumed solution may be approximate, but its accuracy can be increased with increasing the order of polynomial function.

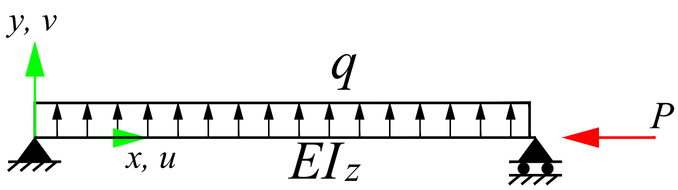


Figure 4.30

■ **Example 4.17** Assume a beam shown in Figure 4.30 with flexural rigidity EI_z and subjected to uniform distributed load q, and its required to find the deflection function using Rayleigh Ritz method.
First, assume a polynomial function for the lateral displacement as follows:

$$v = a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 \tag{4.181}$$

As the assumed solution should follows the boundary conditions $v(0) = 0$ and $v(L) = 0$, it leads to:

$$v = a_1(x^4 - xL^3) + a_2(x^3 - xL^2) + a_3(x^2 - xL) \tag{4.182}$$

As the virtual axial and lateral displacements are independent we can neglect the potential of loads in the axial direction, the total potential equation will be:

$$\Pi = \int_0^L \left(\frac{1}{2}EI_z v''^2 - qv\right) dx$$
$$= \left(\frac{72}{5}L^5a_1^2 + 6L^3a_2^2 + 2La_3^2 + 18L^4a_1a_2 + 6L^2a_2a_3 + 8L^3a_1a_3\right) EI_z \tag{4.183}$$
$$+ \frac{3}{10}qL^5a_1 + \frac{1}{4}qL^4a_2 + \frac{1}{6}qL^3a_3$$

As $\frac{\partial \Pi}{\partial a_i}$ vanishes for $i = 0, 1, 2, \ldots$, it follows:

$$\frac{\partial \Pi}{\partial a_1} \rightarrow \frac{144}{5}L^5a_1 + 18L^4a_2 + 8L^3a_3 + \frac{3}{10EI_z}L^5q = 0$$
$$\frac{\partial \Pi}{\partial a_2} \rightarrow 18L^4a_1 + 12L^3a_2 + 6L^2c + \frac{1}{4EI_z}L^4q = 0 \tag{4.184}$$
$$\frac{\partial \Pi}{\partial a_3} \rightarrow 8L^3a_1 + 6L^2a_2 + 4La_3 + \frac{1}{6EI_z}L^3q = 0$$

Or in matrix form

$$\begin{bmatrix} \frac{144}{5}L^5 & 18L^4 & 8L^3 \\ 18L^4 & 12L^3 & 6L^2 \\ 8L^3 & 6L^2 & 4L \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10EI_z}L^5q \\ -\frac{1}{4EI_z}L^4q \\ -\frac{1}{6EI_z}L^3q \end{bmatrix} \tag{4.185}$$

Solving the above equation for a_i leads to the following displacement function

$$v = \frac{qx\left(L^3 - 2*L*x^2 + x^3\right)}{24EI_z} \tag{4.186}$$

Which corresponds to the exact solution. ■

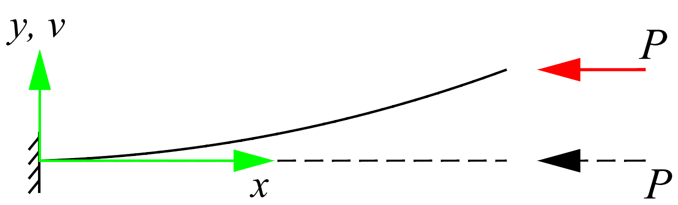


Figure 4.31

Also Rayleigh Ritz method can be used to solve stability problems and determining the buckling loads.

■ **Example 4.18** Assume beam shown in Figure 4.31 with assumed solution defined as follows:

$$v = a_1 + a_2x + a_3x^2 \tag{4.187}$$

To satisfy the GBC ($v(0) = v'(0) = 0$), the assumed solution will be $v = a_3x^2$, neglecting the potential of loads in the axial direction, the total potential energy will be defined as follows:

$$\Pi = \int_0^L \left(\frac{1}{2}EI_z v''^2 - \frac{1}{2}Pv'^2\right) dx = \int_0^L \left(\frac{1}{2}EI_z(2a_3)^2 - \frac{1}{2}P(2a_3x)^2\right) dx \tag{4.188}$$

$$= 2a_3^2EIL - \frac{2}{3}a^2L^3P = \left(2 - \frac{2\lambda}{3}\right) a_3^2EIL \tag{4.189}$$

Assume $\frac{PL^2}{EI} = \lambda$

$$\delta\Pi = 0 = \frac{\partial\Pi}{\partial a_2}\delta a_2 \leftrightarrow \frac{\partial\Pi}{\partial a_2} = 0 \leftrightarrow \lambda = 3 \tag{4.190}$$

While the exact solution is $\frac{PL^2}{EI} = \lambda = 2.47$, using higher order polynomial equation $v = a_1 + a_2x + a_3x^2 + a_4x^3$ increases the accuracy of calculated buckling loads. In this case, to satisfy the GBC, displacement function will be $v = a_3x^2 + a_4x^3$ and using the same procedures results in $\lambda = 2.49$ which is very close to the exact solutions when using polynomial equations with higher order.
■

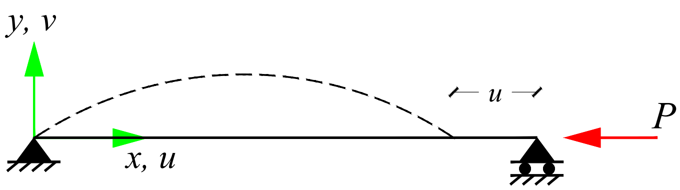


Figure 4.32

■ **Example 4.19** For beam shown in Figure 4.32, assume a polynomial function for lateral displacement of forth degree as follows:

$$v = a_0 + a_1x + a_2x^2 + a_3x^3 \tag{4.191}$$

Satisfying GBC:

$$v(0) = 0, v(L) = 0 \tag{4.192}$$

This results in

$$v = a_2\left(x^2 - xL\right) + a_3(x^3 - xL^2) \tag{4.193}$$

Substituting into Equation 4.189 results in:

$$\begin{aligned}\Pi &= \int_0^L \left(\frac{1}{2}EI_z(a_2 + 6a_3x)^2 - \frac{1}{2}P\left(a_2(2x - L) + a_3(3x^2 - l^2)\right)^2\right) dx \\ &= \left(2La_2^2 + 6L^3a_3^2 + 6L^2a_3a_2\right) EI_z + \left(\frac{1}{6}L^3a_2^2 + \frac{2}{5}L^5a_3^2 + \frac{1}{2}L^4a_3a_2\right) P\end{aligned} \tag{4.194}$$

Using Rayleigh Ritz principle results in:

$$\begin{aligned}\frac{\partial \Pi}{\partial a_2} &= 0 \rightarrow 4La_2 \quad +6L^2a_3 - \left(\frac{1}{3}L^3a_2 + \frac{1}{2}L^4a_3\right)\frac{P}{EI} = 0\\ \frac{\partial \Pi}{\partial a_3} &= 0 \rightarrow 6L^2a_2 + 12L^3a_3 - \left(\frac{1}{2}L^4a_2 + \frac{4}{5}L^5a_3\right)\frac{P}{EI} = 0\end{aligned} \tag{4.195}$$

Assuming $\frac{PL^2}{EI} = \lambda$ result in this matrix form:

$$\begin{bmatrix} \left(4-\frac{\lambda}{3}\right)L & \left(6-\frac{1\lambda}{2}\right)L^2 \\ \left(6-\frac{\lambda}{2}\right)L^2 & \left(12-\frac{4\lambda}{5}\right)L^3 \end{bmatrix}\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.196}$$

The non-trivial solution for above equation is that the determinant of the left matrix vanishes as follows:

$$\begin{vmatrix} \left(4-\frac{\lambda}{3}\right)L & \left(6-\frac{1\lambda}{2}\right)L^2 \\ \left(6-\frac{\lambda}{2}\right)L^2 & \left(12-\frac{4\lambda}{5}\right)L^3 \end{vmatrix} = 0 \tag{4.197}$$

Leads to $\lambda_1 = \frac{PL^2}{EI} = 12$; $\frac{PL^2}{EI} = \lambda_2 = 60$. While the exact solution $\lambda_1 = \pi^2 = 9.81$; $\lambda_2 = 4\pi^2 = 39.24$. Increasing the order of polynomial function leads to more accurate results. ■

Note 4.2 Rayleigh Ritz method gives upper bound value for calculated load P because assuming a solution other than the exact one provides more constraint to the displacement which in turn results in higher stiffness of the problem and higher load capacity. ■

4.3.3 Weighted residual methods

These methods are used for the system of known governing differential equation. Let us assume a system with known differential equation like beam defined as follows:

$$EI_z v'''' - q = 0 \tag{4.198}$$

Generally, the differential equation is called the strong form. If we choose an approximate polynomial function

$$\left(\bar{v} = \sum_{i=0}^{n} a_i\phi_i = a_0 + a_1x + a_2x^2 \dots\right) \tag{4.199}$$

That satisfies the GBC like Rayleigh Ritz method, for the above equation, it will produce an error e gien by:

$$e(x) = EI_z\bar{v}'''' - q\phi 0 \tag{4.200}$$

The above error does not have to vanish as we substitute with an approximate solution. Integrating error over the beam domain results in the total error E_T as follows:

$$E_T = \int e(x)^2\, dx \tag{4.201}$$

The error is squared to make sure that the internal error at any point on the beam domain; either be positive or negative; contributes to the total error. This method represents one type of weighted residual methods called least-square method. We are concerned in Minimizing the total error as follows:

$$0 = \delta E_T = \int e(x)\delta e \, dx \tag{4.202}$$

Generally, weighted residual methods are obtained through this general expression:

$$\int e(x)\delta w \, dx = 0 \tag{4.203}$$

Where w is called the weight function. One of the weighted residual methods that is generally used in the structural analysis is called Galerkin method, in which weight functions w equal to the functions used to approximate the solution $\delta\bar{v}$, but $\delta\bar{v}$ can be evaluated from variations of its parameters δa_i from Equation 4.199 as follows:

$$\delta w = \delta\bar{v} = \frac{\partial\bar{v}}{\partial a_i}\delta a_i \tag{4.204}$$

For independent parameters δa_i and using Equation 4.203 and Equation 4.204, we get the following:

$$\int e(x)\frac{\partial\bar{v}}{\partial a_i} \, dx = 0, \quad for\ i = 0, 1, 2, \ldots. \tag{4.205}$$

To include the natural boundary conditions, Galerkin variational equation can be written in this form:

$$\sum \int e(x)\delta\bar{v} \, dx + \sum j(x)\delta\bar{v} = 0 \tag{4.206}$$

Where $j(x)$ represents the natural boundary condition.

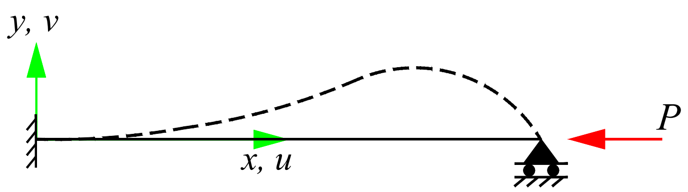


Figure 4.33

■ **Example 4.20** Solve the differential equation

$$v''(x) + v(x) = 0 \quad on\ x = [0, 1] \tag{4.207}$$

With these geometric boundary conditions $v(0) = 0$ and $v(1) = 1$. First, we assume the polynomial function for the solution $v = a_1 + a_2 x + a_3 x^2$. Satisfying geometric boundary condition leads to:

$$v = x + a_3\left(x^2 - x\right) \tag{4.208}$$

$$e(x) = v''(x) + v(x) = 2a_3 + x + a_3\left(x^2 - x\right) \tag{4.209}$$

$$\frac{\partial v}{\partial a_3} = (x^2 - x) \tag{4.210}$$

Applying Galerkin method results in:

$$0=\int_0^1 e(x)\frac{\partial \overline{v}}{\partial a_i}\,dx=\int_0^1 \left(2a_3+x+a_3\left(x^2-x\right)\right)\left(x^2-x\right)dx=0 \tag{4.211}$$

Solving the above equation leads to $a_3=-\frac{5}{18}$ Also the Galerkin can be applied to structural systems.

■ **Example 4.21** Let us us assume the beam shown in Figure 4.33. Substituting the differential equations into Equation 4.148 and natural boundary conditions in Equation 4.149 in Equation 4.206 results in:

$$\int_0^L \left(EAu''+q_o\right)\delta\overline{u}dx+\int_0^L \left(EI_z\overline{v}''''-q+Pv''\right)\delta\overline{v}dx \tag{4.212}$$

$$+\left(P-EAu'\right)\delta\overline{u}\big|_0^L+\left(EIv''-M\right)\delta\overline{v}'\big|_0^L=0 \tag{4.213}$$

As $\delta\overline{u}$ and $\delta\overline{v}$ are independent variables, we could neglect the coefficients of variational axial displacement δu as follows:

$$\int_0^L \left(EI_z\overline{v}''''-q+Pv''\right)\delta\overline{v}dx+\left(EIv''-M\right)\delta\overline{v}'|_0^L=0 \tag{4.214}$$

The chosen equation must have a derivative up to 4^{th} order to be used in evaluating ($\overline{v}''''$) as follows:

$$v=a_1+a_2x+a_3x^2+a_4x^3+a_5x^4 \tag{4.215}$$

For the beam satisfying GBC, it follows:

$$y(0)=0\rightarrow a_1=0,\ y'(0)=0\rightarrow a_2=0 \tag{4.216}$$

$$y(L)=0\rightarrow a_3=-\left(a_4L+a_5L^2\right) \tag{4.217}$$

Also we can use essential boundary condition[a] ($y''(L)=0$) as the moment vanishes at this end (see Equation 4.38) which results in:

$$y''(L)=0\rightarrow 2a_3+6a_4L+12a_5L^2=0\rightarrow v=\frac{a_5}{2}\left(3L^2x^2-5Lx^3+2X^4\right) \tag{4.218}$$

$$e(x)=EI_x\overline{v}''''-q+Pv''=EI\ \left(48a_5\right)+P\left(6L^2-30Lx+24x^2\right)*a_5 \tag{4.219}$$

$$\frac{\partial v}{\partial a_5}=\frac{1}{2}\left(3L^2x^2-5Lx^3+2X^4\right) \tag{4.220}$$

Applying Galerkin method using Equation 4.206 results in:

$$0=\int_0^L e(x)\frac{\partial \overline{v}}{\partial a_i}\,dx=a_5\int_0^L \left(EI\ (48)+P\left(6L^2-30Lx+24x^2\right)\right)\left(\frac{1}{2}\left(3L^2x^2-5Lx^3+2X^4\right)\right)dx \tag{4.221}$$

Which results in:

$$\left(\frac{36}{5}L^5EI-\frac{12}{35}PL^3\right)a_5=0\rightarrow P=\frac{21EI}{L^2} \qquad (4.222)$$

While the exact solution $P=\frac{20.2EI}{L^2}$ ■

[a]Using essential boundary conditions is not necessary, but we can implement them to simplify the problem

4.3.4 Weak form

The above example required the solution to be 4^{th} order differentiable, but we can elevate this condition using what is called the weak form corresponding to the differential equation.

■ **Example 4.22** Let us assume this differential equation defined as:

$$v''(x)+v(x)=0 \qquad (4.223)$$

This above form is called the strong form. Using Galerkin method, it follows:

$$\int\left(v''(x)+v(x)\right)\delta v\,dx=0 \qquad (4.224)$$

Using integrating by part for the first term results in:

$$v'\delta v\big|_a^b-\int\left(-v'(x)\,\delta v'+v(x)\,\delta v\right)\,dx=0 \qquad (4.225)$$

The above equation is called the weak form corresponding to the differential Equation 4.223. If we integrating the above expression again by part, it leads to:

$$\int\left(v(x)\,\delta v''+v(x)\,\delta v\right)dx+v'\delta v\big|_a^b-\;v\delta v'\big|_a^b=0 \qquad (4.226)$$

■

The first term of expressions Equation 4.223 and Equation 4.225 need the function v to be 2^{nd} differentiable, while expression Equation 4.224 requires this function to be only 1^{st} differentiable which alleviate the condition required for the assumed solution chosen. Generally, in Galerkin method, it is preferred to use the weak form in which the function $\delta v'$ and the weight functions $v'(x)$ have the same order of derivative. We also note that the weak form is identical to the first variation of the potential energy, so the weak form is also called the variational form. For the two hinged beam in Figure 4.28, using the Galerkin method (see Equation 4.206), we reach to the same result in Equation 4.147. Using integration by part leads to the following weak form:

$$\delta\Pi=\int_0^L\left(EAu'\delta u'+EI_zv''\delta v''-Pv'\delta v'-P\delta u'-q_o\delta u-q\delta v\right)dx-M_0\delta\theta_0-M_L\delta\theta_L \qquad (4.227)$$

Note that integration by part is used twice for the bending strain energy such that v'', *and* $\delta v''$ have the same degree of differentiable equation degree, so it reduce the requirement for using high order polynomial approximation solution (just polynomial of second order). Also we note that using the weak form in Galerkin method or Rayleigh Ritz method leads to identical results for the same polynomial function used for the assumed displacement, but Rayleigh Ritz method is better used for problems with known formulations for the total potential energy, while Galerkin method is used for problems with available governing differential equations. Using variational methods when solving geometrically complex structures to get an approximate solution is not a proper way, as there will be a large number of dependent variables, which is impossible to find a suitable differential equation or a formulation for the total potential energy. In this case, we are force to use finite element method through dividing the body into small parts and applying the variational principles over each part.

4.4 Using energy principles in dynamic problems

4.4.1 Introduction

We will first introduce the linear momentum $\boldsymbol{L}$, angular momentum $\boldsymbol{H}_c$ about point c and their rate of change with time defined as

$$\boldsymbol{L} = \int \boldsymbol{v}\, dm \;\rightarrow \dot{\boldsymbol{L}} = \int \boldsymbol{a}\, dm = \sum \boldsymbol{F} \tag{4.228}$$

$$\boldsymbol{H}_c = \int \boldsymbol{x}_c \times \boldsymbol{v}\, dm \;\rightarrow \dot{\boldsymbol{H}}_c = \int \boldsymbol{x}_c \times \boldsymbol{a}\, dm = \sum \boldsymbol{M}_c \tag{4.229}$$

Where $\boldsymbol{v}$ and $\boldsymbol{a}$ are velocity and acceleration of infinitesimal point with mass dm. The last equality in the two above equations represents the Newton's second law of motion in which $\sum \boldsymbol{F}$ and $\sum \boldsymbol{M}_c$ define the resultant forces and moment about point c. Angular momentum of body about an arbitrary point c can be calculated in terms of angular momentum about its center of gravity (point o) as follows:

$$\boldsymbol{H}_c = \boldsymbol{H}_o + \boldsymbol{x}_{co} \times \boldsymbol{L} \quad or \quad \dot{\boldsymbol{H}}_c = \boldsymbol{H}_o + \boldsymbol{x}_{co} \times \dot{\boldsymbol{L}} \tag{4.230}$$

where $\boldsymbol{H}_o = \int \boldsymbol{x}_o \times \boldsymbol{a}\, dm$ is the angular momentum around mass centroid and x_{co} represents a position vector from point c to point x as shown in Figure 4.34.

■ **Example 4.23** Assume a rigid body shown in Figure 4.35 rotating about point c with center of gravity o with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$. As it is a rigid body, the velocity of point o is $\boldsymbol{v}_o = \boldsymbol{\omega} \times \boldsymbol{x}_{co}$, then $\boldsymbol{L}$, $\dot{\boldsymbol{L}}$, $\boldsymbol{H}_c$ and $\dot{\boldsymbol{H}}_c$ are defined as follows:

$$\boldsymbol{L} = m\boldsymbol{v}_o \rightarrow \dot{\boldsymbol{L}} = m\dot{\boldsymbol{v}}_o = m\,(\dot{\boldsymbol{\omega}} \times \boldsymbol{x}_{co} + \boldsymbol{\omega} \times \dot{\boldsymbol{x}}_{co}) \tag{4.231}$$

$$= m\,(\dot{\boldsymbol{\omega}} \times \boldsymbol{x}_{co} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}_{co})) \tag{4.232}$$

$$= m(\dot{\boldsymbol{\omega}} \times \boldsymbol{x}_{co} - \boldsymbol{\omega}^2 \boldsymbol{x}_{co}) \tag{4.233}$$

[a] Where $\boldsymbol{v}_o$ and m are the velocity of its mass centroid and the total mass of the object. If point o is located on the c, the net force on the body ($\dot{\boldsymbol{L}}$) vanishes and it rotates to infinity.

$$\begin{aligned}
\boldsymbol{H}_c &= \int \boldsymbol{x} \times \boldsymbol{v}\, dm = \int \boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}) dm = \int \left((\boldsymbol{x}.\boldsymbol{x})\, \mathbf{1} - \boldsymbol{x}\boldsymbol{x}^{\boldsymbol{T}} \right) \boldsymbol{\omega} dm^a \\
&= \int \left((x_1^2 + x_2^2 + x_3^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_2x_3 & x_1x_3 & x_3^2 \end{bmatrix} \right) \boldsymbol{\omega} dm \\
&= \int \left(\begin{bmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_2x_3 & -x_1x_3 & x_1^2 + x_2^2 \end{bmatrix} \right) \boldsymbol{\omega} dm
\end{aligned} \tag{4.234}$$

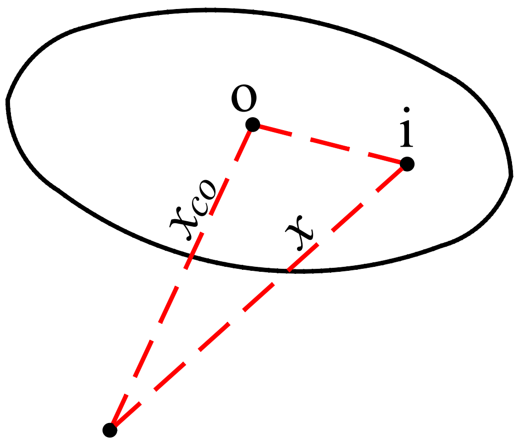


Figure 4.34

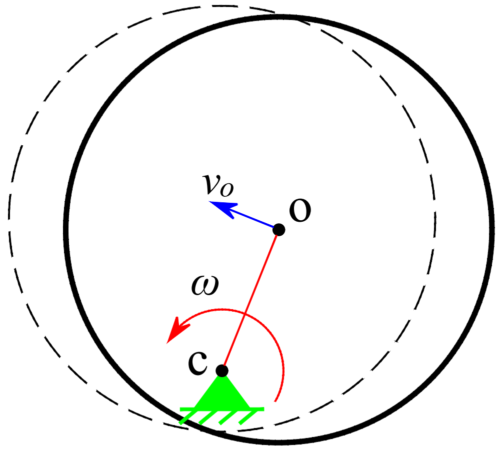


Figure 4.35

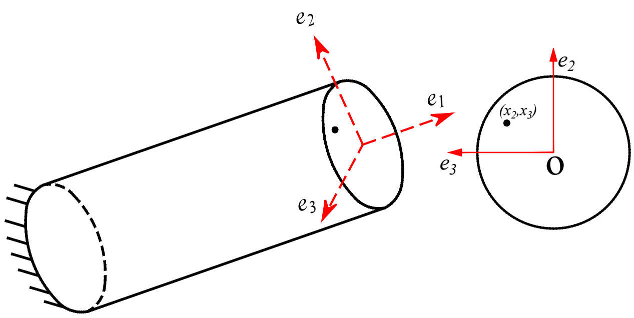


Figure 4.36

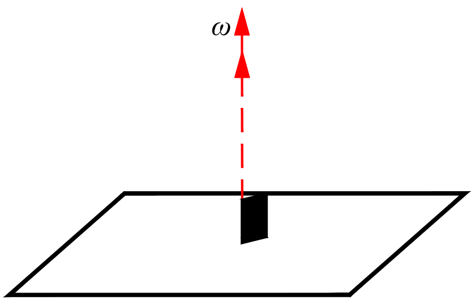


Figure 4.37

For rigid bodies, ω has the same value over the body volume which results in:

$$\boldsymbol{H}_c = \int \left(\begin{bmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2+x_3^2 & -x_2x_3 \\ -x_2x_3 & -x_1x_3 & x_1^2+x_2^2 \end{bmatrix} \right) dm\, \boldsymbol{\omega} = \boldsymbol{I}_p\, \boldsymbol{\omega} \tag{4.235}$$

■

[a] The last equality comes from this expression $(a \times b) \times c = ((a.c)\, b - (b.c)\, a)$

■ **Example 4.24** For line element like beam[a] as shown in Figure 4.36, the angular momentum around its mass centroid o (using $x = (0, x_2, x_3)$) will be:

$$\boldsymbol{H}_o = \int_0^L \boldsymbol{\omega} \left\{ \int \left(\begin{bmatrix} x_2^2 & -x_1x_2 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & x_1^2+x_2^2 \end{bmatrix} \right) \rho\, dA \right\} dx = \int_0^L \boldsymbol{I}_p^B\, \boldsymbol{\omega}\, dx$$

$$\text{Where } \boldsymbol{I}_p^B = \int \left(\begin{bmatrix} x_2^2 & -x_1x_2 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & x_1^2+x_2^2 \end{bmatrix} \right) \rho\, dA \tag{4.236}$$

Where ρ is the beam density at point (x_1, x_2) located on the cross section. ■

[a] Assuming that the plane section remains the same after deformation, so it can be considered rigid in section direction

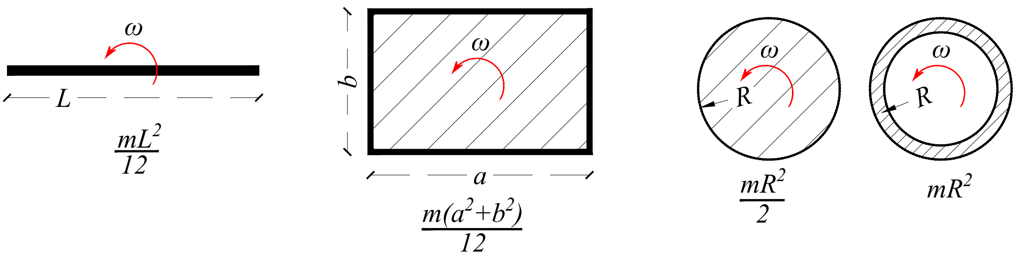


Figure 4.38

For planar elements subjected to angular velocity perpendicular to its plane as shown in Figure 4.37, the magnitude of angular momentum around its mass centroid will be:

$$H_c = \omega \int_A \left(x_1^2 + x_2^2\right) \overline{m} dA = I_p\ \omega \ \textit{where } \overline{m} \textit{ is mass per unit area.} \tag{4.237}$$

With direction perpendicular to the element plane. Figure 4.38 shows values of mass moment of inertia I_p for some planer elements around its mass centroid. To evaluate the I_p around other point than the mass centroid, we use parallel axis theorem which states:

$$I = I_o + md^2 \tag{4.238}$$

where I_o and I are the mass moment of inertia around the point of interest and centroid point, respectively. m and d are the total mass of the element and the distance between point of interest and centroid point.

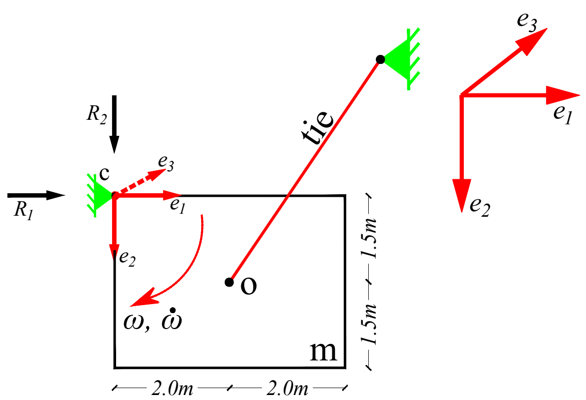


Figure 4.39

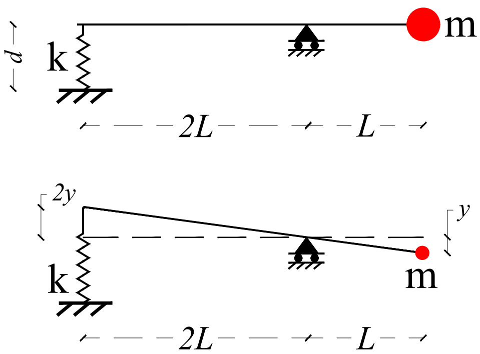


Figure 4.40

■ **Example 4.25** Assume rectangular rigid plate shown in Figure 4.39 with mass m and dimensions shown supported with hinge and cable, we need the force induced in the hinge after cutting the cable.
For angular momentum about hinge (point c); using Equation 4.237 and parallel axis theorem

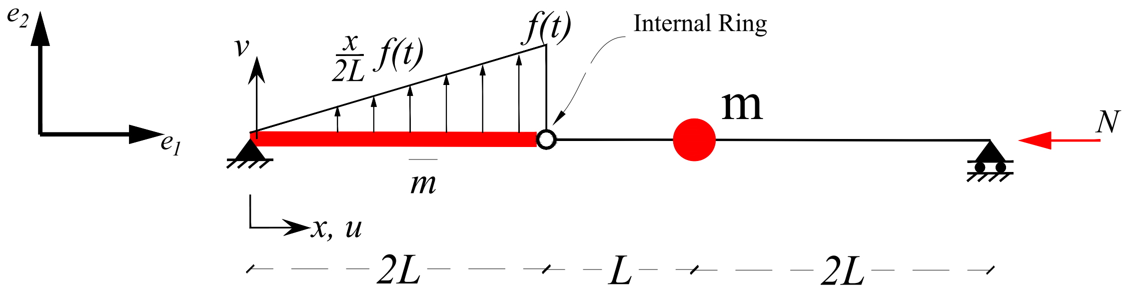


Figure 4.41

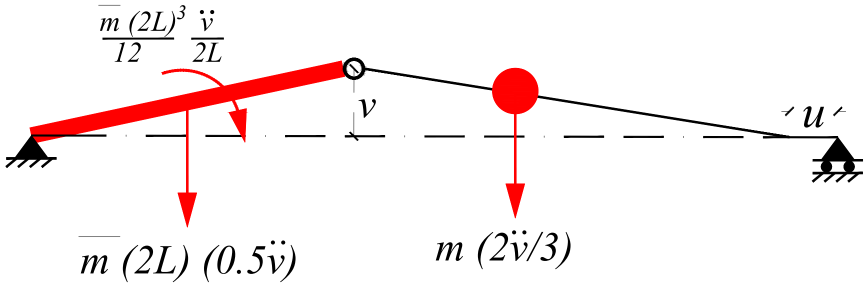


Figure 4.42

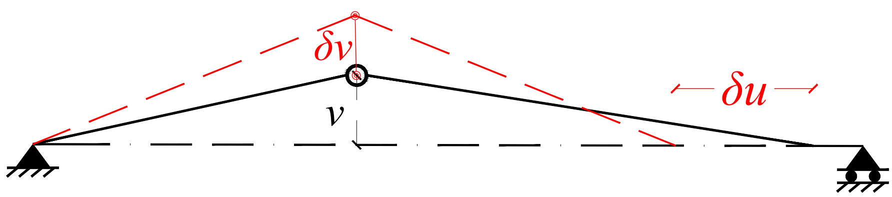


Figure 4.43

results in:

$$2*mg = \sum M = \dot{H}_c = \dot{\omega}\int x^2 dm = \dot{\omega}\left(I_o + md^2\right) = \dot{\omega}\left(m\left(\frac{a^2+b^2}{12}\right) + m(2.5)^2\right) = \frac{25}{3}\dot{\omega}m \tag{4.239}$$

From linear momentum:

$$R_x\boldsymbol{e}_1 + (R_y - mg)\boldsymbol{e}_2 = \sum F = ma \tag{4.240}$$

Velocity of rotating plate is defined as $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{x}$, where $\boldsymbol{x}$ is a position vector from hinge location to plate centroid, so the acceleration will be $\boldsymbol{a} = \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})$, but its initial angular velocity is zero at the time of releasing the plate, so for $\boldsymbol{x} = (2, 1.5, 0)$, $\boldsymbol{\omega} = (0, 0, \omega)$, the acceleration will be $\boldsymbol{a} = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} = (1.5, -2, 0)\dot{\omega}$, then the above equation will be:

$$R_1\boldsymbol{e}_1 + (R_2 + mg)\boldsymbol{e}_2 = \sum F = ma = m\dot{\omega} \times \boldsymbol{x} = m\dot{\omega}(1.5e_1 - 2e_2) \tag{4.241}$$

From the Equation 4.239, $R_1 = \frac{9}{25}mg$ and $R_2 = -\frac{37}{25}mg$.

■

■ **Example 4.26** Assume a massless rigid bar with properties shown in Figure 4.40. If the system is in static equilibrium in this condition, the spring stretch $\triangle = \frac{F_{spring}}{k} = \frac{0.5mg}{k}$. The mass m is pulled down a distance y then released to produce free vibration for the mass, as a result, the angular momentum time rate of change around point o will be:

$$\sum M_o = \dot{H}_c \to \left(mgL - k\left(2y + \frac{mg}{2k}\right)L\right)\boldsymbol{e}_3 = r \times m\ddot{y} \to \; m\ddot{y} - 4ky = 0 \; [Equation\ of\ motion] \tag{4.242}$$

■

4.4.2 Virtual work in dynamic analysis

We will use the same principles used in section 4.2. In addition, we will add the virtual work resulting from inertia forces as shown in the following example.

■ **Example 4.27** Let us us assume two rigid bars shown in Figure 4.41 connected with an internal hinge supported by hinge at A and roller at B subjected to axial load P and excited with varied lateral loads varied with time $\frac{x}{2L}f(t)$ and it is required to write the equation of motion. Applying virtual lateral displacement δv added to the the true lateral displacement v as shown in Figure 4.42 leads to virtual work defined as:

$$\begin{aligned} \delta W = -m\left(\frac{2}{3}\ddot{v}\right)\delta v - \overline{m}\,(2L)\,\frac{\ddot{v}}{2}\frac{\delta v}{2} - \frac{\overline{m}(2L)^3}{12}\left(\frac{\ddot{v}}{2L}\right)\left(\frac{\delta v}{2L}\right) - kv\delta v \\ + \left(\frac{f(t)(2L)}{2}\right)\left(\frac{2\delta v}{3}\right) + N\delta u(D) = 0 \end{aligned} \tag{4.243}$$

The first two terms represent the virtual work resulting from the inertia forces for mass m and bar mass $\overline{m}$, while the third one represents the inertia couple resulting from rotation of the bar by angle $\frac{v}{2L}$. From Equation 4.152 and Figure 4.43, the variation of axial displacement at end D is $\delta u(D) = \sum \frac{v_i \delta v_i}{L_i}$, where the sum is done over each rigid element with length L_i and difference in lateral displacement between its ends v_i, so $\delta u(D) = \frac{v\delta v}{2L} + \frac{v\delta v}{3L}$ and the resulting equation of motion will be:

$$\left(\frac{2}{3}\overline{m}a + \frac{4}{9}m\right)\ddot{v} + \left(k - \frac{5N}{6L}\right)v = \frac{2}{3}f(t)L \tag{4.244}$$

■

4.4.3 Hamilton's principle

Let us assume a particle with mass m moving along a real path shown in Figure 4.44 from point A at time t_1 to point B at time t_2, such that the particle position at any time t is $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))$

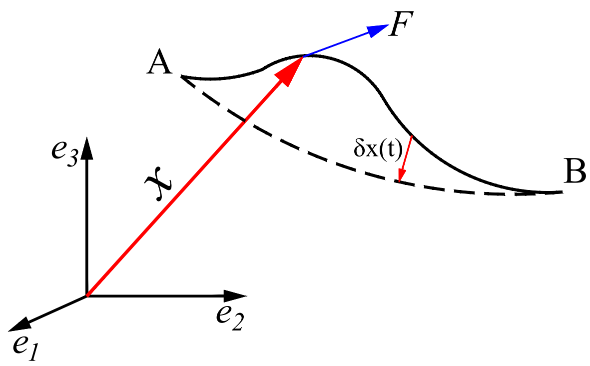


Figure 4.44

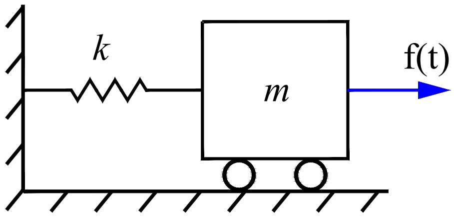


Figure 4.45

and subjected to force F varied with time $\boldsymbol{F}(t) = (F_1(t), F_2(t), F_3(t))$. During this real path, the inertia forces, structural forces and external forces are in equilibrium (d'Alembert's principle). If the particle path is subjected to virtual displacement $\boldsymbol{\delta x}(t)$, the virtual work of these forces must vanish as follows:

$$[\boldsymbol{F} - m\ddot{\boldsymbol{x}}(t)] . \boldsymbol{\delta x}(t) = 0 \tag{4.245}$$

Integrating the above equation over the path results in:

$$\int_{t_1}^{t_2} (\boldsymbol{F}.\boldsymbol{\delta x} - m\ddot{x}.\boldsymbol{\delta x})\, dt = 0 \tag{4.246}$$

Using integration by part over the second term leads to:

$$\int_{t_1}^{t_2} (\boldsymbol{F}.\delta x + m\dot{\boldsymbol{x}}.\boldsymbol{\delta\dot{x}})\, dt - m\dot{\boldsymbol{x}}\boldsymbol{\delta x}|_{t_1}^{t_2} = 0 \tag{4.247}$$

The last term vanishes as $\boldsymbol{\delta x} = 0$ at t_1 and t_2. The second term equal to

$$\int_{t_1}^{t_2} (m\dot{\boldsymbol{x}}.\boldsymbol{\delta\dot{x}})\, dt = \int_{t_1}^{t_2} \delta\left(\frac{1}{2}m\dot{\boldsymbol{x}}^2\right) dt = \int_{t_1}^{t_2} \delta T\, dt \tag{4.248}$$

Where T is the kinematic energy of the particle, while force $\boldsymbol{F}(t)$ may be conservative $\boldsymbol{F}_c(t)$ or nonconservative $\boldsymbol{F}_{nc}(t)$ or both as follows:

$$\boldsymbol{F}(t) = \boldsymbol{F}_c(t) + \boldsymbol{F}_{nc}(t) \tag{4.249}$$

We can define a potential energy Π for the conservative forces using Equation 4.12 as follows:

$$\frac{\partial \Pi}{\partial x} = -\boldsymbol{F}_c \rightarrow \boldsymbol{F}_c.\delta x = \delta \Pi \tag{4.250}$$

From above, Equation 4.247 will be:

$$\int_{t_1}^{t_2} \delta(\Pi - T) dt - \int_{t_1}^{t_2} \delta W_{nc} = 0 \tag{4.251}$$

Where the total potential energy $\Pi = U + V$ includes stored strain energy U in elastic bodies and potential energy of external conservative loads$V$, while $\delta W_{nc} = \boldsymbol{F}_{nc}.\boldsymbol{\delta x}$ represents the virtual work done by nonconservative forces like friction, damping, external forces varied with time, etc. Hamilton's principle can be used to solve linear and nonlinear, static and dynamic problems.

■ **Example 4.28** Assume a rigid block shown in Figure 4.45 with mass m vibrating in $\boldsymbol{x}$ direction under the influence of external dynamic loading $f(t)$ and tied with linear elastic spring with stiffness k. We can use Hamilton's principle to solve for the equation of motion as follows. The total potential energy results from the spring ($\frac{1}{2}kx^2$), the kinematic energy T equals to ($\frac{1}{2}m\dot{x}^2$), while the variation in work done by non conservative (external dynamic) force δW_{nc} is $f(t).\delta x$, so applying Hamilton's Equation 4.251 results in:

$$\int_{t_1}^{t_2} \delta\left(\frac{1}{2}kx^2 - \frac{1}{2}m\dot{x}^2)dt\right) - \int_{t_1}^{t_2} f(t).\delta x = 0 \tag{4.252}$$

$$\int_{t_1}^{t_2} kx.\delta x - m\dot{x}.\delta\dot{x} - f(t).\delta x dt = 0 \tag{4.253}$$

Using integration by part for the second term yields:

$$\int_{t_1}^{t_2} kx.\delta x + m\ddot{x}.\delta x - f(t).\delta x dt + m\dot{x}.\delta x|_{t_1}^{t_2} = 0 \tag{4.254}$$

The last term vanishes yielding the equation of motion as follows:

$$m\ddot{x} + kx = f(t) \tag{4.255}$$

■

It can also be applied to static analysis as shown in the next example. In this condition, the kinematic energy T vanish and Hamilton's equation reduces to:

$$\delta\Pi - \delta W_{nc} = 0 \tag{4.256}$$

which reduces to the virtual work principle for static problems.

■ **Example 4.29** Let us assume a mass m rested on the ground as shown in Figure 4.46 (state 1), then lifted a distance L (state 2) by a rigid tie (change in its length is negligible), and put on a linear elastic spring with stiffness K and unstressed length L very slowly (to neglect the developed kinetic energy) until the force in the tie vanishes and the mass weight entirely rested on the spring (state 4).
In state 2, the body acquires gravitational potential energy $\triangle\Pi$ from lifting the weight, while the tie force (external source) exerts work $\triangle W_{nc}$ defined as:

$$\triangle\Pi = \triangle V = mgL,\ \triangle W_{nc} = mgL \tag{4.257}$$

From above equation, Hamilton's principle is achieved ($\triangle\Pi - \triangle W_{nc} = 0$).

State 3 is an intermediate state between state 2 and state 4 when the spring carries a part of the weight (kx) when compressed distance x. At that position, there is a reduction in gravitational potential energy of the weight by ($\triangle V = -mgx$) from state 2, but another potential energy is stored in the spring ($\frac{1}{2}kx^2$), so the change in total potential energy from state 2 to state 3 is:

$$\triangle\Pi = \frac{1}{2}kx^2 - mgx \tag{4.258}$$

The work done by the tie force F_t is the area of force-displacement history for the tie as shown in Figure 4.47. The force in tie in state 2 and state 4 is mg and zero, respectively, while, in state 3, the force in tie becomes $(F_t = mg - kx)$ (mg minus the force carried by the spring), so the work done by the tie force is the hatched area in Figure 4.47 defined as follow:

$$\triangle W_{nc} = -area = \left(\frac{kx^2}{2} - mgx\right) \qquad (4.259)$$

The negative sign is used as the force direction and mass displacement have different directions. From Equation 4.258, the variation in total potential energy and work done by non conservative force (tie force) are identical which prove the validation of Hamilton's principle $\triangle\Pi - \triangle W_{nc} = 0$. At state 4; the spring carries the weight of the mass and compressed to $(\Delta = \frac{mg}{k})$, so the variation in total potential from state 2 to state 4 will be:

$$\triangle\Pi = \frac{1}{2}k\Delta^2 - mg\Delta = -0.5mg\Delta \qquad (4.260)$$

While the work done by tie force will be:

$$\triangle W_{nc} = -area = (-0.5mg\Delta) \quad \rightarrow \quad \triangle W_{nc} = \triangle\Pi \qquad (4.261)$$

The last equality can be derived directly using Hamilton's principle without need to evaluate the work done by non-conservative forces $\triangle W_{nc}$. For structural systems with complicated loads, it is hard to find the work done by external loads, so we can use Hamilton's principle $(\triangle W_{nc} = \triangle\Pi)$ in the static problems. ■

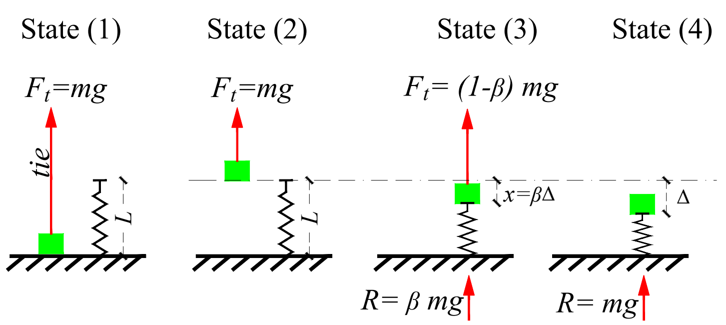


Figure 4.46: The reaction is defined as $R = \beta mg$, where $\beta = \frac{x}{\triangle}$, $\triangle = \frac{mg}{k}$ and k is spring stiffness

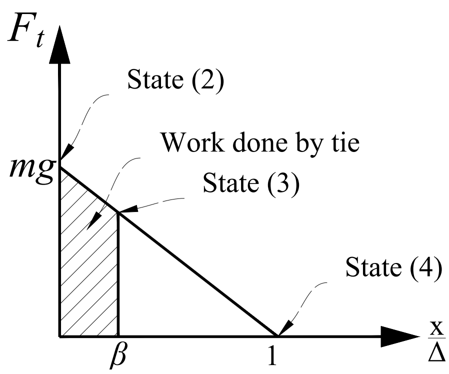


Figure 4.47: The hatched area expresses the work done by the tie force F_t from state 2 to state 3

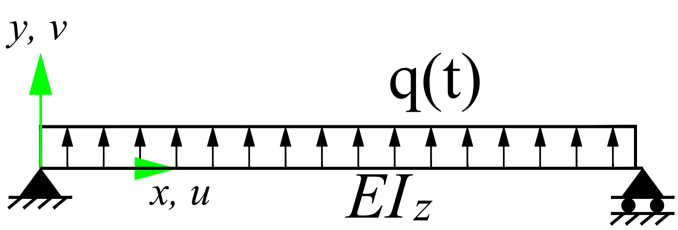


Figure 4.48

■ **Example 4.30** Assume a beam shown in Figure 4.48 with length L, mass $\overline{m}$ per unit length and bending stiffness EI_z subjected to distributed dynamic load $q(t)$, it is required to evaluate the equation of motion, its total potential, kinematic energy and variation in nonconservative work are defined as:

$$\Pi = U = \frac{1}{2}\int_0^L EI_z v''^2 dx, \quad T = \frac{1}{2}\int_0^L \overline{m}\dot{v}^2 dx, \quad \delta W_{nc} = \int_0^L q(t)\delta v dx \tag{4.262}$$

Applying Hamilton's Equation 4.251 results in

$$\int_{t_1}^{t_2}\left\{\int_0^L \left(EI_z v''\delta v'' - \overline{m}\dot{v}\delta\dot{v} - q(t)\delta v\right) dx\right\} dt = 0 \tag{4.263}$$

Using integration by part twice for the first term as follow:

$$\int_0^L \left(EI_z v''\delta v''\right) dx = EI_z v''\delta v'|_0^L - EI_z v'''\delta v|_0^L + \int_0^L EI_z v''''\delta v dx \tag{4.264}$$

The first and second terms (boundary terms) generally vanishes as the left and right moment vanishes $(0 = M = EIv'')$, also (δv) vanishes as each end is restrained from lateral displacement (GBC). Using integration by part once for the second term in Equation 4.263 results in

$$\int_{t_1}^{t_2}\left(\int_0^L (\overline{m}\dot{v}\delta\dot{v})\, dx\right) dt = \int_0^L \left(\int_{t_1}^{t_2} (\overline{m}\dot{v}\delta\dot{v})\, dt\right) dx = \int_0^L \left(\overline{m}\dot{v}\delta v|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\overline{m}\ddot{v}\delta v)\, dt\right) dx \tag{4.265}$$

The first term in the last equality vanishes. Using the above expressions, the Hamilton's Equation 4.263 reduces to:

$$\int_{t_1}^{t_2}\left\{\int_0^L \left(EI_z v'''' + \overline{m}\ddot{v} - q(t)\right)\delta v dx\right\} dt = 0 \tag{4.266}$$

which yields the beam equation of motion as follows:

$$EI_z v'''' + \overline{m}\ddot{v} - q(t) = 0 \tag{4.267}$$

■

4.4.4 Lagrange equations of motion

Assuming a displacement function that satisfies the geometric boundary conditions like the one used in subsection 4.3.2 $\left(u = \sum_{i=0}^{n} a_i\phi_i = a_0 + a_1 x + a_2 x^2 \ldots\right)$, the kinematic energy and potential energy will be converted to functions of parameters a_i and $\dot{a}_i$ as follows:

$$T = T(a_i, \dot{a}_i), \quad \Pi = \Pi(a_i) \tag{4.268}$$

And their variation will be:

$$\delta T = \frac{\partial T}{\partial a_i}\delta a_i + \frac{\partial T}{\partial \dot{a}_i}\delta\dot{a}_i, \quad \delta\Pi = \frac{\partial \Pi}{\partial a_i}\delta a_i \tag{4.269}$$

While the variation in the nonconservative work can be defined as follows:

$$W_{nc} = F_i^*\delta a_i \tag{4.270}$$

Using Hamilton's Equation 4.251, it follows:

$$\int_{t_1}^{t_2}\left[\left(\frac{\partial\Pi}{\partial a_i}-\frac{\partial T}{\partial a_i}-F_i^*\right)\delta a_i-\frac{\partial T}{\partial \dot{a}_i}\delta\dot{a}_i\right]dt=0 \tag{4.271}$$

Integrating by part the forth term results in:

$$\int_{t_1}^{t_2}\left[\left(\frac{\partial\Pi}{\partial a_i}-\frac{\partial T}{\partial a_i}+\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_i}\right)-F_i^*\right)\delta a_i\right]dt-\frac{\partial T}{\partial \dot{a}_i}\delta a_i\Big|_{t_1}^{t_2}=0 \tag{4.272}$$

The last term vanishes as $\delta\boldsymbol{u}$ at t_1 and t_2 is null as follows:

$$\frac{\partial\Pi}{\partial a_i}-\frac{\partial T}{\partial a_i}+\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_i}\right)-F_i^*=0 \tag{4.273}$$

This equation is called Lagrange equations of motion.

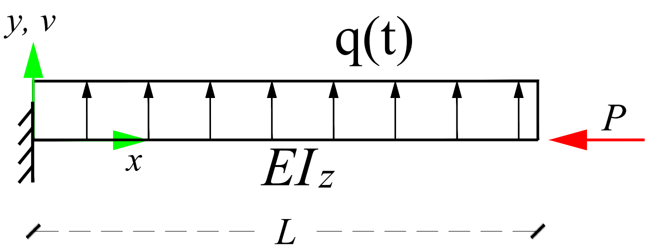


Figure 4.49

■ **Example 4.31** Let us assume a beam fixed at the left end, while the right end is subjected to axial load P and lateral dynamic uniform distributed load $q(t)$ as pictured in Figure 4.49. The beam mass per unit length is $\overline{m}$. Assuming a suitable displacement function that satisfies the GBC as follows:

$$v=a_1x^2+a_2x^3 \tag{4.274}$$

Using virtual work method or Hamilton's equation and Equation 4.152 results in the weak form of equation of motion for the beam as follows:

$$\int_0^L\left(EI_zv''\delta v''-pv'\delta v'-q(t)\delta v+\overline{m}\ddot{v}\delta v\right)dx=0 \tag{4.275}$$

From the displacement function, $v'=2a_1x+3a_2x^2$, $v''=2a_1+6a_2x$ and $\ddot{v}=\ddot{a}_1x^2+\ddot{a}_2x^3$. Substituting these functions into the above equation results in:

$$\begin{aligned}&\begin{bmatrix}\delta a_1 & \delta a_2\end{bmatrix}\overline{m}\begin{bmatrix}\frac{L^5}{5} & \frac{L^6}{6}\\ \frac{L^6}{6} & \frac{L^7}{7}\end{bmatrix}\begin{bmatrix}\ddot{a}_1\\ \ddot{a}_2\end{bmatrix}\\ &+\begin{bmatrix}\delta a_1 & \delta a_2\end{bmatrix}\left\{EI_z\begin{bmatrix}\frac{4L^3}{3} & \frac{3L^4}{2}\\ \frac{3L^4}{2} & \frac{9L^5}{5}\end{bmatrix}-P\begin{bmatrix}4L & 6L^2\\ 6L^2 & 12L^3\end{bmatrix}\right\}\begin{bmatrix}a_1\\ a_2\end{bmatrix}\\ &=\begin{bmatrix}\delta a_1 & \delta a_2\end{bmatrix}q(t)\begin{bmatrix}\frac{L^3}{3}\\ \frac{L^4}{4}\end{bmatrix}\end{aligned} \tag{4.276}$$

$$\delta a.M^*\ddot{a}+\delta a.(K^*+K_g^*)a=\delta a.f(t)^* \tag{4.277}$$

Where M^*, K^*, K_g^*, and $f(t)^*$ represent the generalized mass, generalized elastic stiffness, generalized geometric stiffness matrix, and generalized force vector, respectively, defined as:

$$M^*=\overline{m}\begin{bmatrix}\frac{L^5}{5} & \frac{L^6}{6}\\ \frac{L^6}{6} & \frac{L^7}{7}\end{bmatrix} \tag{4.278}$$

$$K^*=EI_z\begin{bmatrix}\frac{4L^3}{3} & \frac{3L^4}{2}\\ \frac{3L^4}{2} & \frac{9L^5}{5}\end{bmatrix} \tag{4.279}$$

$$K_g^*=-P\begin{bmatrix}4L & 6L^2\\ 6L^2 & 12L^3\end{bmatrix} \tag{4.280}$$

$$f(t)^*=q(t)\begin{bmatrix}\frac{L^3}{3}\\ \frac{L^4}{4}\end{bmatrix} \tag{4.281}$$

■

4.5 Introduction to finite element method

This method implements the same idea used for variational methods through using approximate functions, but these functions are used for subdomains of the body or finite elements with simple shapes that allows us to use a simple approximate polynomial function for it (not all domain). Generally, the subdomains are chosen to be similar in shape, so we can use the same calculation procedures for each subdomains making the solution systemic. Also, when using Rayleigh Ritz or Galerkin method, the undetermined parameters are a_i (coefficients of assumed polynomial function), while, in finite element method (FEM), they are in terms of a common property between the adjacent elements in the domain at prescribed points, e.g. the displacements. As these elements share the same nodes and from continuity, they have the same displacements at these nodes. In this case, undetermined parameters are known property like displacements which is considered an advantage to reduce the time of post-processing analysis. We will provide how to use FEM in 1-D elements in the following sections.

4.5.1 Finite element analysis (FEA) of simple bars

Shape function

Let us assume a bar with uniform axial distributed load (f) divided into subdomains, each one of length L. If the prescribed points for each element are three, the number of degree of freedom DOF for each element will be the number of (DOF) associated with each node times the number of nodes per element which shall be three. Similarly, like Rayleigh Ritz, we will define an approximate solution of polynomial function over the element as follows:

$$u=\sum_{i=1}^{n}u_iN_i(x) \tag{4.282}$$

In FEA approximate function, the parameters used will be in terms of the nodal displacement u_i of the element associated with the DOF (i), while n the number of DOF per element and, in this case, N_i represents what is called the shape function. For example, Let us assume a bar element of 2 nodes, so the number of DOF, local displacements, and shape function per element shall be two.

Using the following approximate linear solution for axial displacement:

$$u = \sum_{i=1}^{2} a_i \phi_i = a_1 \phi_1 + a_2 \phi_2 = a_1 + a_2 x \tag{4.283}$$

applying the boundary condition using $u(x_1) = u_1 \rightarrow u_1 = a_1$ and $u(x_2) = u_2 \rightarrow u_2 = a_1 + a_2 x$ results in the approximate solution in terms of displacement at ends as follows:

$$u = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2 = \sum_{i=1}^{n} u_i N_i(x) \tag{4.284}$$

So the shape functions associated with each DOF for two-node element will be:

$$N_1(x) = \left(1 - \frac{x}{L}\right), \quad N_2(x) = \frac{x}{L} \tag{4.285}$$

The shape function is shown in Figure 4.50, from above equation the properties of shape function

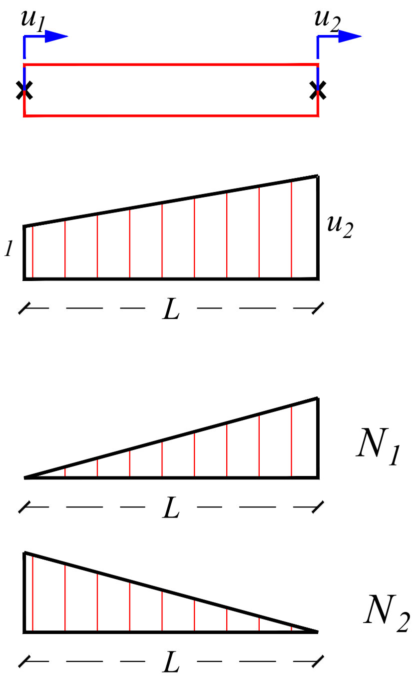


Figure 4.50

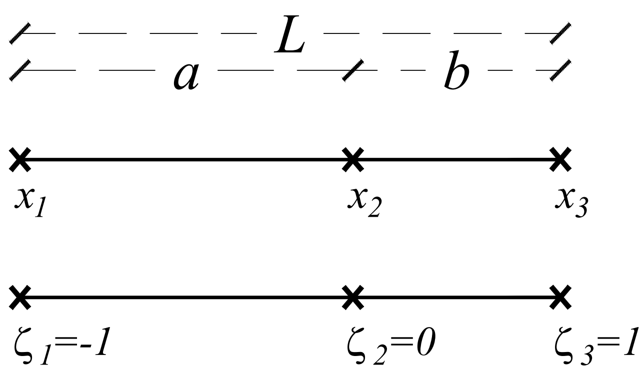


Figure 4.51

are:

$$N_i(x_j) = \delta_{ij} \rightarrow N_1(x_1) = N_2(x_2) = 1, \; N_1(x_2) = 0, \; N_2(x_1) = 0 \tag{4.286}$$

We can us what is called normalized or natural coordinate with range $\xi = [-1, 1]$ instead of using the local coordinate $x = [x_1, x_2]$, such that $\xi = \frac{2x}{L} - 1$ and the shape function in terms of natural coordinates will be:

$$N_1(\xi) = \frac{1}{2}(1 - \xi), \quad N_2(\xi) = \frac{1}{2}(1 + \xi) \tag{4.287}$$

If we need to use larger number of nodes per element, we can follow the same above procedures or use Lagrange interpolation formula defined as follows:

$$N_i(x) = \prod_{j=1, j \neq i}^{n-1} \left(\frac{x - x_j}{x_i - x_j} \right) \tag{4.288}$$

And in terms of natural coordinate, it will be:

$$N_i(\xi) = \prod_{j=1, j\neq i}^{n-1} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right) \tag{4.289}$$

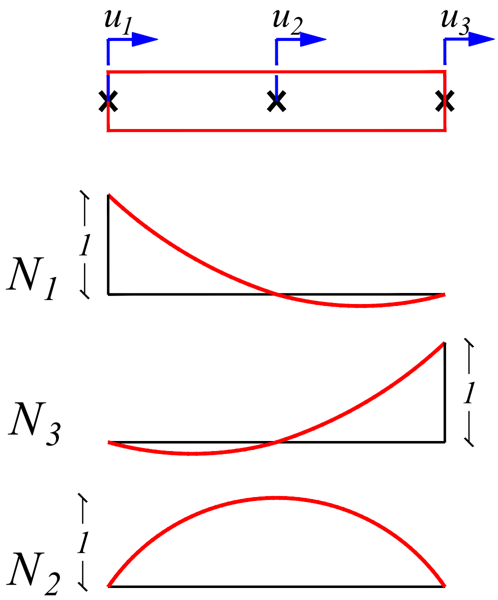


Figure 4.52

■ **Example 4.32** Assume a line element of 3 nodes not equally spaced as shown in Figure 4.51[a]. If the local coordinates of element nodes are x_1, x_2 and x_3, the corresponding natural coordinates are $\xi_1 = -1$, $\xi_2 = 0$ and $\xi_3 = 1$, respectively with shape function defined for each node in Figure 4.52 as follows:

$$\begin{aligned}
N_1(\xi) &= \prod_{j=1, j\neq 1}^{2} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right) = \left(\frac{\xi - \xi_2}{\xi_1 - \xi_2} \right) \left(\frac{\xi - \xi_3}{\xi_1 - \xi_3} \right) = \frac{1}{2}\xi(\xi - 1) \\
N_2(\xi) &= \prod_{j=1, j\neq 2}^{2} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right) = \left(\frac{\xi - \xi_1}{\xi_2 - \xi_1} \right) \left(\frac{\xi - \xi_3}{\xi_2 - \xi_3} \right) = 1 - \xi^2 \\
N_3(\xi) &= \prod_{j=1, j\neq 3}^{2} \left(\frac{\xi - \xi_j}{\xi_i - \xi_j} \right) = \left(\frac{\xi - \xi_1}{\xi_3 - \xi_1} \right) \left(\frac{\xi - \xi_1}{\xi_3 - \xi_1} \right) = \frac{1}{2}\xi(\xi + 1)
\end{aligned} \tag{4.290}$$

■

[a] The distance between any two subsequent nodes a or b should not be less than or equal to $\frac{1}{4}$ the element length to avoid singularity problems

Stiffness matrix and load vector

■ **Example 4.33** For n-node element shown in Figure 4.53 with axial stiffness (EA), length L and distributed axial load q, and from Equation 4.140, the variation of total potential energy or

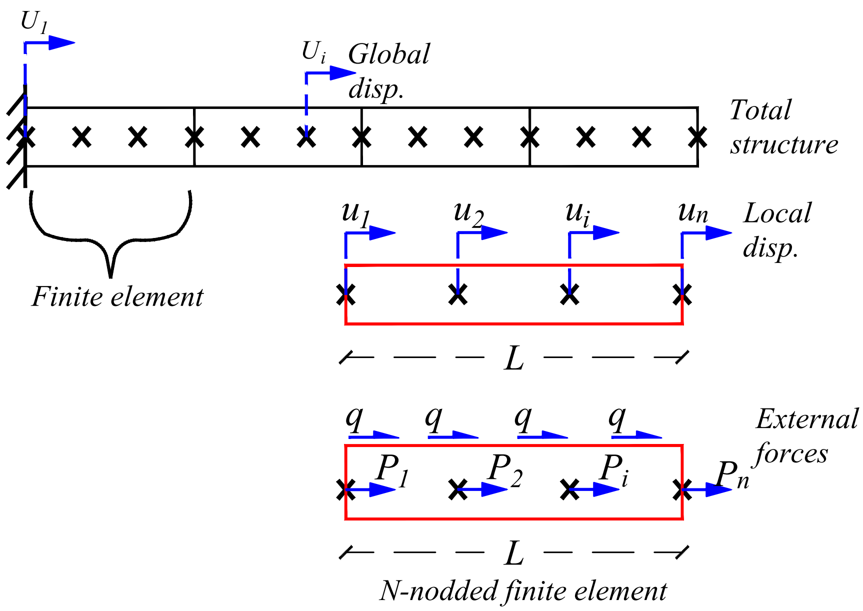


Figure 4.53

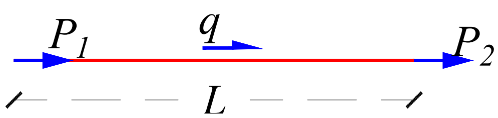


Figure 4.54

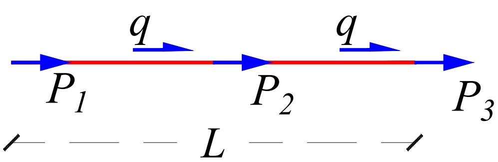


Figure 4.55

the weak form for the bar problem is defined as follows:

$$\delta\Pi = \int_0^L \left(EAu'\delta u' - q_o\delta u\right)dx - P_1\delta u_1 - P_2\delta u_2 \cdots - P_n\delta u_n = \int_0^L \left(EAu'\delta u' - q_o\delta u\right)dx - \sum_{i=1}^{n} P_i\delta u_i \tag{4.291}$$

$$u' = \sum_{j=1}^{n} u_j N_j'(x) \qquad \delta u = \sum_{i=1}^{n} \delta u_i N_i(x) \qquad \delta u' = \sum_{i=1}^{n} \delta u_i N_i'(x) \tag{4.292}$$

$$\delta\Pi = \int_0^L \left(EA\sum_{j=1}^{n} u_j N_j'(x) \sum_{i=1}^{n} \delta u_i N_i'(x) - q_o \sum_{i=1}^{n} \delta u_i N_i(x)\right)dx - \sum_{i=1}^{n} P_i\delta u_i \tag{4.293}$$

Using index notation

$$\delta\Pi = \delta u_i \underbrace{\left(\int_0^L \left(EAN_i'(x)N_j'(x)\right)dx\right)}_{k_{ij}^e} u_j - \delta u_i \underbrace{\left(\int_0^L \left(q_o N_i(x)\right)dx + P_i\right)}_{F_i^e} = \delta u_i.\left(k_{ij}^e.u_j - F_i^e\right)$$

(4.294)

Where k_{ij}^e and F_i^e are called the stiffness matrix and load vector, respectively, defined as:

$$k_{ij}^e = \int_0^L \left(EAN_i'(x)\,N_j'(x)\right)dx, \quad F_i^e = \int_0^L \left(q_o N_i(x)\right)dx + P_i \tag{4.295}$$

we not that the stiffness matrix is symmetric as $k_{ij}^e = k_{ji}^e$. For two-node element shown in Figure 4.54, the stiffness matrix and load vector are defined as:

$$[N_i(x)] = \begin{bmatrix} \left(1-\frac{x}{L}\right) & \frac{x}{L} \end{bmatrix} \tag{4.296}$$

$$[N_i'(x)] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \tag{4.297}$$

$$[k_{ij}^e] = \left[\int_0^L \left(EAN_i'(x)\,N_j'(x)\right)dx\right] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.298}$$

$$[F_i^e] = \int_0^L \left(q_o N_i(x)\right)dx = \frac{q_0 L}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{4.299}$$

While, for three-node element shown in Figure 4.55, it will be:

$$\xi = \frac{2x}{L} - 1, d\xi = \frac{2dx}{L} \tag{4.300}$$

$$[N_i(\xi)] = \begin{bmatrix} \frac{1}{2}\xi(\xi-1) & 1-\xi^2 & \frac{1}{2}\xi(\xi+1) \end{bmatrix} \tag{4.301}$$

$$[N_i'(\xi)] = \begin{bmatrix} \xi-\frac{1}{2} & -2\xi & \xi+\frac{1}{2} \end{bmatrix} * \frac{d\xi}{dx} = \begin{bmatrix} \xi-\frac{1}{2} & -2\xi & \xi+\frac{1}{2} \end{bmatrix} * \frac{2}{L} \tag{4.302}$$

$$[k_{ij}^e] = \left[\int_0^L \left(EAN_i'(x)\,N_j'(x)\right)dx\right] = \left[\int_{-1}^1 \left(EAN_i'(\xi)\,N_j'(\xi)\right)\left(\frac{L}{2}\,d\xi\right)\right] \tag{4.303}$$

$$= \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{4.304}$$

$$[F_i^e] = \int_0^L \left(q_o N_i(x)\right)dx = \frac{q_0 L}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{4.305}$$

■

Assembly of elements and applying boundary conditions

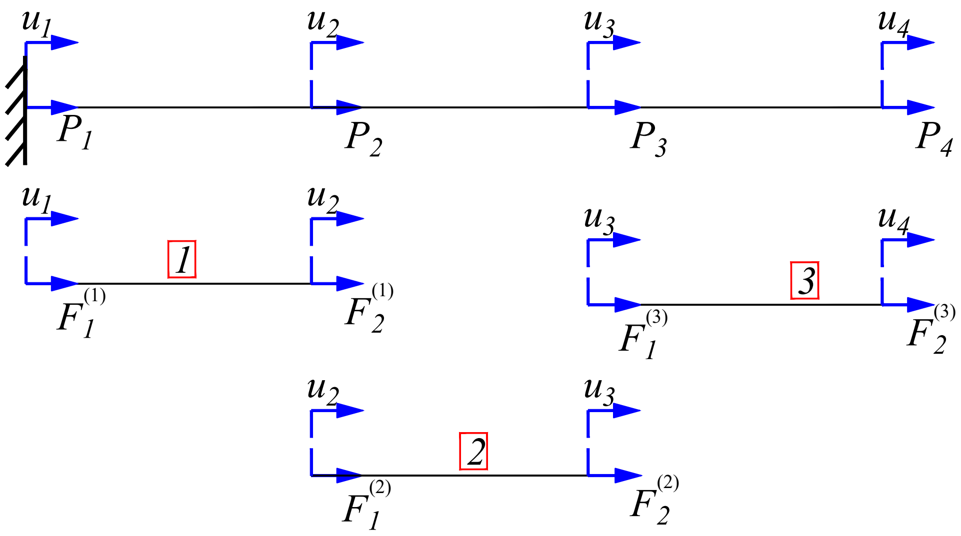


Figure 4.56

■ **Example 4.34** Let us assume three two-node bars subjected only to joint loads P_i as shown in Figure 4.56. Using Equation 4.294, the resulting variation of potential energy for element e will be:

$$\delta\Pi^e = \delta u_i.k^e_{ij}.u_j - \delta u_i.F^e_i = \delta u_1.\left(k^e_{11}u_1 + k^e_{12}u_2\right) + \delta u_2.\left(k^e_{21}u_1 + k^e_{22}u_2\right) - \delta u_1.F^e_1 - \delta u_1.F^e_2 \tag{4.306}$$

Summing this variation of three elements, such that the total variation of body potential energy should vanish as follows:

$$\begin{aligned} 0 = \sum_{e=1}^{m} \delta\Pi^e &= \delta u_1.\left(k^1_{11}u_1 + k^1_{12}u_2\right) + \delta u_2.\left(k^1_{21}u_1 + k^1_{22}u_2\right) - \delta u_1.F^1_1 - \delta u_2.F^1_2 \\ &+ \delta u_2.\left(k^2_{11}u_2 + k^2_{12}u_3\right) + \delta u_3.\left(k^2_{21}u_2 + k^2_{22}u_3\right) - \delta u_2.F^2_1 - \delta u_3.F^2_1 \\ &+ \delta u_3.\left(k^3_{11}u_3 + k^3_{12}u_4\right) + \delta u_4.\left(k^3_{21}u_3 + k^3_{22}u_4\right) - \delta u_3.F^3_1 - \delta u_4.F^3_1 \end{aligned} \tag{4.307}$$

$$= \begin{bmatrix} \delta u_1 & \delta u_2 & \delta u_3 & \delta u_4 \end{bmatrix} \left\{ \begin{bmatrix} k^1_{11} & k^1_{12} & 0 & 0 \\ k^1_{21} & k^1_{22} + k^2_{11} & k^2_{12} & 0 \\ 0 & k^2_{21} & k^2_{22} + k^3_{11} & k^3_{12} \\ 0 & 0 & k^3_{21} & k^3_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} - \begin{bmatrix} F^1_1 \\ F^1_2 + F^2_1 \\ F^2_2 + F^3_1 \\ F^3_2 \end{bmatrix} \right\} = 0 \tag{4.308}$$

$$= \begin{bmatrix} \delta u_1 & \delta u_2 & \delta u_3 & \delta u_4 \end{bmatrix} \left(Ku - P_{ext}\right) = 0 \tag{4.309}$$

Note that $F^1_2 + F^2_1$ gives the sum of the forces over node 2 coming from the both elements sharing this node with external load P_2 as shown in fig , so the total external loads will be:

$$P_{ext} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 \end{bmatrix} \tag{4.310}$$

So we reach finally

$$= \begin{bmatrix} \delta u_1 & \delta u_2 & \delta u_3 & \delta u_4 \end{bmatrix} (Ku - P_{ext}) = 0 \tag{4.311}$$

As δu_i is an arbitrary displacement for $i = 1,\ 2,\ 3,\ 4$, we get the following equation (equilibrium equation):

$$Ku = P_{extt} \tag{4.312}$$

■

The above equation can not be solved directly, as shown in Figure 4.56, the deflection vector u includes known geometric boundary condition u_1 , while the rest displacements (u_i, for $i = 2,3,4$) are unknown. Similarly, the load vector P_{ext}, P_1 is unknown, (P_i, for $i = 2,3,4$) are known. As a result, we shall divide the degree of freedom into two parts; free DOF (f) and restrained DOF (r) at which GBC is defined and reactions needs to calculated. Similarly we will divide the stiffness matrix, load and displacement vector in the same manner as follows:

$$\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix} \begin{bmatrix} \underline{u_r} \\ u_f \end{bmatrix} = \begin{bmatrix} P_r \\ \underline{P_f} \end{bmatrix} \rightarrow \begin{matrix} K_{rr}\ \underline{u_r} + K_{rf}\ u_f = P_r \\ K_{fr}\ \underline{u_r} + K_{ff}\ u_f = \underline{P_f} \end{matrix} \tag{4.313}$$

The underlined terms are known like the restrained displacement u_r and loads at free points P_f. Using the second equation in above equation, u_f will be:

$$K_{ff}u_f = P_f - K_{fr}\ u_r \rightarrow u_f = K_{ff}^{-1}\,(P_f - K_{fr}\ u_r) \tag{4.314}$$

After calculating the u_f, we can evaluate the reaction at restrained nodes P_r form the first equation in Equation 4.313.

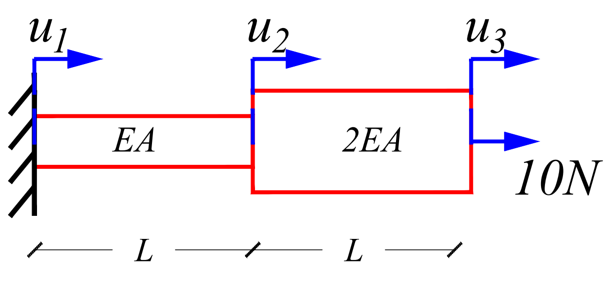


Figure 4.57

■ **Example 4.35** Let us assume a bar with properties shown in Figure 4.57 with $\frac{EI}{L} = 100N/m$. The right end is subjected to force $10N$, while the left end has initial axial displacement

$u_1 = 0.01m$.

$$[k^1_{ij}] = \frac{(EA)_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 100 \begin{array}{c} \textcircled{1} \quad \textcircled{2} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \tag{4.315}$$

$$[k^2_{ij}] = 200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200 \begin{array}{c} \textcircled{2} \quad \textcircled{3} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array} \tag{4.316}$$

$$[K] = 100 \begin{array}{c} \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array} \tag{4.317}$$

$$[k^2_{ij}] = 200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 200 \begin{array}{c} \textcircled{2} \quad \textcircled{3} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array} \tag{4.318}$$

$$[K_{ff}] = 100 \begin{array}{c} \textcircled{2} \quad \textcircled{3} \\ \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array}, \quad K_{fr} = 100 \begin{array}{c} \textcircled{1} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array} \tag{4.319}$$

$$P_f = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array}, \quad u_r = 0.01 \tag{4.320}$$

$$K_{ff}u_f = P_f - K_{fr}\, u_r \rightarrow u_f = K^{-1}_{ff}\,(P_f - K_{fr}\, u_r) \tag{4.321}$$

$$= 1/200\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 10 \end{bmatrix} - 100\begin{bmatrix} -1 \\ 0 \end{bmatrix} \times 0.01\right) \tag{4.322}$$

$$= \begin{bmatrix} 0.11 \\ 0.16 \end{bmatrix} m \tag{4.323}$$

Euler Bernoulli beam

The shape functions defined in Lagrange interpolation use one type of degree of freedom, e.g displacements at nodal points or their derivatives like rotation in beams not both. There is another type of interpolation that uses both types. For example, Let us assume two-node Euler Bernoulli beam shown in Figure 4.58 with four DOF (lateral displacement and rotation for each node), with approximate solution for lateral deformation defined as follows:

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \sum_{i=1}^{4} u_i N_i(x) \tag{4.324}$$

The displacements associated with each DOF are defined as follows:

$$u_i = \left\{ \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \right\}^T \tag{4.325}$$

Using the following boundary condition:

$$\begin{aligned} &v_1 = v(0) = a_0,\ v_2 = v(L) = a_0 + a_1 L + a_2 L^2 + a_3 L^3, \\ &\theta_1 = v'(0) = a_1,\ \theta_2 = v'(L) = a_1 + 2a_2 L + 3a_3 L^2 \end{aligned} \tag{4.326}$$

which results the following shape functions:

$$N_i(x) = \left[\begin{array}{cccc} 1 - 3r^2 + 2r^3 & x(1-r)^2 & 3r^2 - 2r^3 & x(r^2 - r) \end{array} \right] \tag{4.327}$$

Where $r = \frac{x}{L}$, and these shape functions above are called Hermite cubic interpolation functions as shown in Figure 4.59.

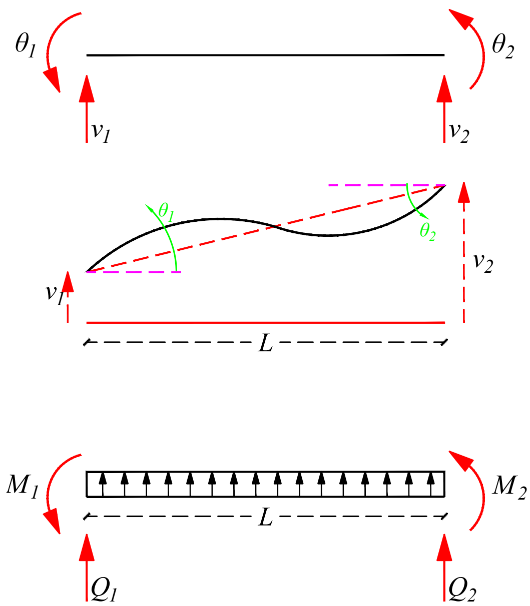


Figure 4.58

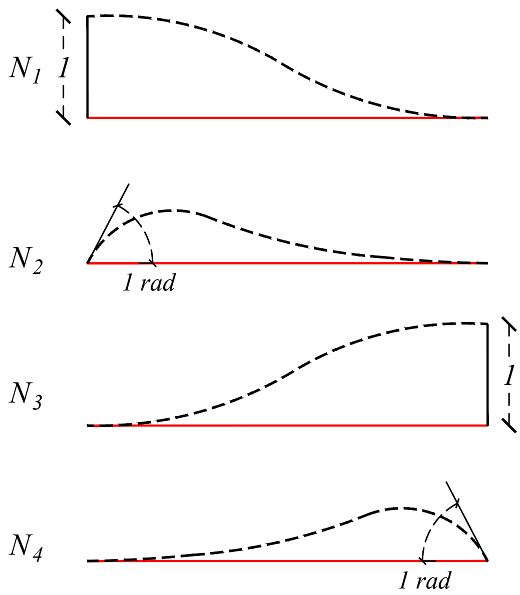


Figure 4.59

$$\delta\Pi = \int_0^L \left(EI_x v'' \delta v'' - q \delta v \right) dx - Q_1 \delta v_1 - Q_2 \delta v_2 - M_1 \delta v_1' - M_2 \delta v_2' \tag{4.328}$$

$$v'' = \sum_{j=1}^{4} u_j N_j''(x) \qquad \delta v = \sum_{i=1}^{4} \delta u_i N_i(x) \qquad \delta v'' = \sum_{i=1}^{4} \delta u_i N_i''(x) \tag{4.329}$$

$$\delta\Pi = \int_0^L \left(EI_x \sum_{j=1}^{4} u_j N_j''(x) \sum_{i=1}^{4} \delta u_i N_i''(x) \ - q_o \sum_{i=1}^{4} \delta u_i N_i(x) \right) dx - \sum_{i=1}^{4} P_i \delta u_i \tag{4.330}$$

$$N_i(x)'' = \frac{1}{L^2} \left[\begin{array}{cccc} 6 + 12r & L(4 - 6r) & 6 - 12r & L(6r - 2) \end{array} \right] \tag{4.331}$$

Using index notation

$$\delta\Pi = \delta u_i \underbrace{\left(\int_0^L \left[EI_x N_i''(x) N_j''(x)\right] dx\right)}_{k_{ij}^e} u_j - \delta u_i \underbrace{\left(\int_0^L \left[q_o N_i(x)\right] dx + P_i\right)}_{F_i^e} = \delta u_i . k_{ij}^e . u_j - \delta u_i . F_i^e \tag{4.332}$$

The stiffness matrix

$$k_{ij}^e = \int_0^L \left(EI_z N_i''(x) N_j''(x)\right) dx = \frac{EI_z}{L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & 6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \tag{4.333}$$

The load vector

$$F_i^e = \int_0^L \left(q_o N_i(x)\right) dx + P_i = \begin{bmatrix} \frac{qL}{2} \\ -\frac{qL^2}{12} \\ \frac{qL}{2} \\ \frac{qL^2}{12} \end{bmatrix} + \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} \tag{4.334}$$

Beam torsional stiffness matrix

As stated in section 4.1.5 and section 4.1.5, there are two types of torsion; pure torsion and warping torsion. For a two-node beam element with torsional rigidity GJ and length L shown in Figure 4.60, if we neglect the warping torsion and assume a linear interpolation for angle of twist as follows:

$$\left[k_{ij}^e\right] = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.335}$$

which is similar to bar stiffness subjected to axial load in Equation 4.298 If we take into account the warping rigidity EC_w, the angle of twist can be represented by a cubic polynomial similar to beam interpolation function as follows:

$$\theta_x(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \sum_{i=1}^{4} u_i N_i(x) \tag{4.336}$$

Using two degree of freedom $\theta_x,\ \theta_x'$ for each end shown in Figure 4.61 as follows:

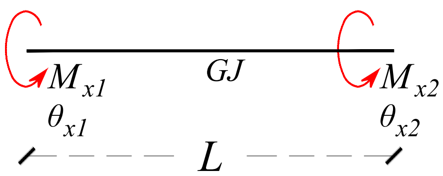


Figure 4.60

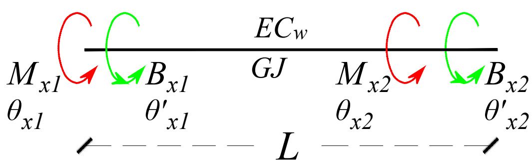


Figure 4.61

$$u_i = \left\{ \theta_{x1} \quad \theta_{x2} \quad \theta_{x1}' \quad \theta_{x2}' \right\}^T \tag{4.337}$$

Using Equation 4.327, the shape functions will be:

$$N_i(x) = \left[\ 1 - 3r^2 + 2r^3 \quad 3r^2 - 2r^3 \quad x(1-r)^2 \quad x(r^2 - r) \ \right] \tag{4.338}$$

$$N_i(x)' = \begin{bmatrix} \frac{-6r+6r^2}{L} & \frac{6r-6r^2}{L} & 1-4r+3r^2 & 3r^2-2r \end{bmatrix} \tag{4.339}$$

Using Equation 4.46 and Equation 4.55, the variation in stored potential energy will be:

$$\delta U = \int_0^L GJ\theta_x'\delta\theta_x' + EC_w\theta_x''\delta\theta_x''dx = \delta u_i.k_{ij}^e.u_j \tag{4.340}$$

So, the resulting stiffness matrix will be:

$$k_{ij}^e = \int_0^L \left(GJN_i'(x)N_j'(x) + EC_wN_i''(x)N_j''(x)\right)dx \tag{4.341}$$

$$[K] = \frac{GJ}{L}\begin{bmatrix} 1.2 & 1.2 & 0.1L & 0.1L \\ -1.2 & 1.2 & -0.1L & -0.1L \\ 0.1L & -0.1L & 2/15L & -1/30L^2 \\ 0.1L & -0.1L & -1/30L^2 & 2/15L \end{bmatrix} + \frac{EC_w}{L^3}\begin{bmatrix} 12 & -12 & 6L & 6 \\ -12 & 12 & -6L & -6L \\ 6L & -6L & 4L^2 & 2L^2 \\ 6L & -6L & 2L^2 & 4L^2 \end{bmatrix} \tag{4.342}$$

Warping resistance using the above stiffness leads to a good approximation to the exact solution. Sufficient number of elements can converge to the exact solution. Warping resistance can be used for open section with sufficient warping resistance like wide steel I-section, while we can neglect it for open section with component elements meeting at a point like angles and tee sections. Also the above stiffness matrix can apply for a number of finite element beams that form a straight line, such that beam ends can be warping fixed or free as shown in Figure 4.9a and Figure 4.10a, while taking the effect of the corner beam-column connection is beyond our scope of study.

4.5.2 Flexibility matrix D_{ij}, and Forced based FEA

The flexibility matrix D_{ij} is equivalent to the inverse of stiffness matrix K_{ij} for element formulation defined as follows:

$$K_{ij}.u_j = F_i \quad \rightarrow \quad D_{ij}.F_j = u_i \tag{4.343}$$

We can reach the flexibility matrix in another form. Suppose if we have a beam with fixed right end and free left end subjected to M_1, Q_1 shown in Figure 4.62, the shear force and moment at any section x, lying from the left end as shown in Figure 4.62 will be:

$$Q = Q_1 \tag{4.344}$$

$$M = M1 - xQ_1 = \begin{bmatrix} -x & 1 \end{bmatrix}\begin{bmatrix} Q_1 \\ M_1 \end{bmatrix} = N_iF_i \tag{4.345}$$

At the right end, the shear force and moment will be $Q_2 = -Q_1$, $M_2 = LQ_1 - M_1$ or:

$$\begin{bmatrix} Q_2 \\ M_2 \end{bmatrix} = [\phi]\begin{bmatrix} Q_1 \\ M_1 \end{bmatrix} \tag{4.346}$$

Where

$$[\phi] = \begin{bmatrix} -1 & 0 \\ L & -1 \end{bmatrix} \tag{4.347}$$

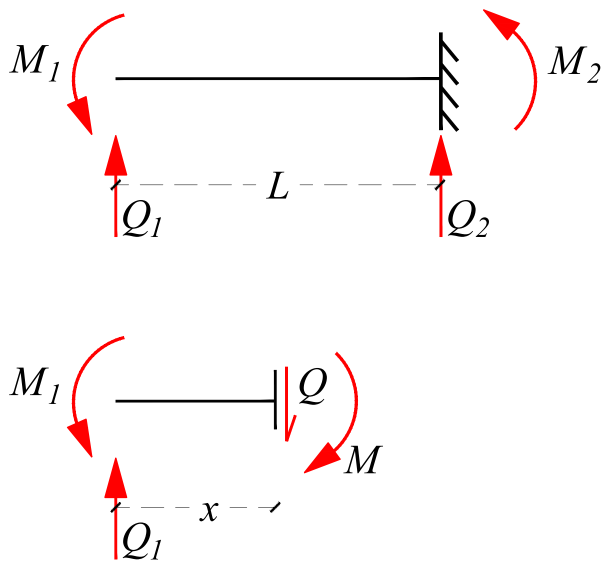


Figure 4.62

Also

$$\begin{bmatrix} Q_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix} \begin{bmatrix} Q_2 \\ M_2 \end{bmatrix} \tag{4.348}$$

Using principle of virtual work, using virtual force instead of virtual displacement results in:

$$\delta\Pi^* = \int_0^L \left(EI_z v'' \delta v''\right) dx - \delta Q_1 v_1 - \delta M_1 v_1' \tag{4.349}$$

Substituting $v'' = \frac{M}{EI_z}, \delta v'' = \frac{\delta M}{EI_z}$ into the above equation results in:

$$\delta\Pi^* = \int_0^L \left(\frac{M\delta M}{EI_z}\right) dx - \delta Q_1 v_1 - \delta M_1 v_1' \tag{4.350}$$

$$= \delta F_i . \int_0^L \left(\frac{N_i N_j \delta M}{EI_z}\right) dx.F_j - \delta F_i.u_i \tag{4.351}$$

$\delta\Pi^*$ is called complementary virtual work.

$$\delta F_i \left[D_{ij} F_j - u_i\right] = 0 \tag{4.352}$$

Where D_{ij} is the flexibility matrix corresponding to forces Q_1 and M_1 defined as:

$$[D_{ij}] = \int_0^L \left(\frac{N_i N_j \delta M}{EI_z}\right) dx = \frac{1}{EI_z} \begin{bmatrix} \frac{L^3}{3} & -\frac{L^2}{2} \\ -\frac{L^2}{2} & L \end{bmatrix} \tag{4.353}$$

So we get:

$$D_{ij} F_j = u_i \tag{4.354}$$

So the stiffness matrix is:

$$[\overline{K}] = [D]^{-1} = EI_z \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \tag{4.355}$$

This is the stiffness matrix for the left two DOF (compare it with Equation 4.333). To get the total stiffness matrix of the beam, it can be divided into two parts; part associated with the left two DOF and another associated with the right two DOF.

$$\begin{bmatrix} F_l \\ F_r \end{bmatrix} = \begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & K_{rr} \end{bmatrix} \begin{bmatrix} u_l \\ u_r \end{bmatrix} \tag{4.356}$$

or

$$f_l = k_{ll}u_l + k_{lr}u_r \tag{4.357}$$

$$f_r = k_{rl}u_l + k_{rr}u_r \tag{4.358}$$

Where

$$F_l = \begin{bmatrix} Q_1 \\ M_1 \end{bmatrix}, \ F_r = \begin{bmatrix} Q_2 \\ M_2 \end{bmatrix}, \ u_l = \begin{bmatrix} v_1 \\ \theta_1 \end{bmatrix}, \ u_r = \begin{bmatrix} v_2 \\ \theta_2 \end{bmatrix} \tag{4.359}$$

In this case, k_{ll} refers to $\overline{K}$ in Equation 4.355. From Equation 4.346, we get $f_r = [\phi] f_l$. Substituting it into Equation 4.358 results in:

$$[\phi] f_l = k_{rl}u_l + k_{rr}u_r \tag{4.360}$$

Multiplying Equation 4.357 by $[\phi]$, and subtracting it from the above equation results in:

$$0 = (\phi k_{ll} - k_{rl})\, u_l + (\phi k_{lr} - k_{rr})\, u_r \tag{4.361}$$

As u_l, u_r are independent terms, their coefficients vanish for nontrivial solution as follows:

$$k_{rl} = \phi k_{ll}, \qquad \phi k_{lr} = k_{rr} \tag{4.362}$$

From symmetry of stiffness matrix, it will be

$$K = \begin{bmatrix} K_{ll} & k_{ll}\phi^T \\ \phi k_{ll} & \phi k_{ll}\phi^T \end{bmatrix} \tag{4.363}$$

With $k_{ll} = \overline{K}$, we get the total stiffness as follows::

$$K = EI_z \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \tag{4.364}$$

The above method used in formulating the stiffness matrix is called forced-based finite element method, while the traditional method described in subsection 4.5.1 is called displacement-based finite element method. There is another method that combine using these two previous methods called Mixed finite element which is described in subsection 4.5.5.

Timoshenko beam

For thick beams shown in Figure 4.63, the angle between section normal n and the tangent to beam centerline changes after deformation. This change is defined as shear deformation ($\gamma_{xy} = v' - \theta$) and the deformation field follows this expression:

$$u(x,y) = -y\phi(x) \tag{4.365}$$

$$v(x,y) = v(x) \tag{4.366}$$

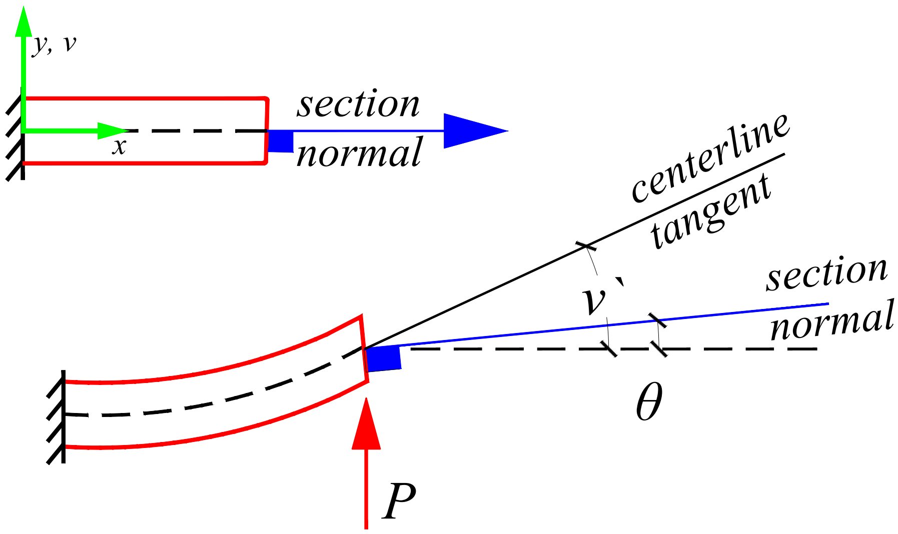


Figure 4.63

Such that the axial and shear strains and stresses and their resultants are defined as follows:

$$\varepsilon_{xx} = -y\theta' \rightarrow \sigma_{xx} = E\varepsilon_{xx} \rightarrow M_x = -\int_A \sigma_{xx} y dA = EI_z\theta' \tag{4.367}$$

$$\gamma_{xy} = v' - \theta \rightarrow \tau_{xy} = G\gamma_{xy} \rightarrow Q = \int_A \tau_{xy} dA = GA_s\gamma_{xy} = k_S GA\gamma_{xy} \tag{4.368}$$

And the corresponding variations in strain energy are defined as

$$\delta\Pi_{bending} = \int_0^L \left(EI_z\theta'\delta\theta'\right) dx \tag{4.369}$$

$$\delta\Pi_{shear} = \int_0^L \left(k_S GA\gamma_{xy}\delta\gamma_{xy}\right) dx = \int_0^L \left(k_S GA\left(v' - \theta\right)\delta\left(v' - \theta\right)\right) dx \tag{4.370}$$

Such that the total variation in potential energy will be:

$$\delta\Pi = \int_0^L \left(EI_z\theta'\delta\theta' + k_S GA\left(v' - \theta\right)\delta\left(v' - \theta\right) - q\delta v\right) dx - Q_1\delta v_1 - Q_2\delta v_2 - M_1\delta\theta_1 - M_2\delta\theta_2 \tag{4.371}$$

Where Q_1, Q_2, M_1 and M_2 are beam end forces. Using linear Lagrange interpolation function for lateral displacement v and section rotation θ in terms of the two ends DOF as follows:

$$v = \sum_{i=1}^{2} v_i N_i(x) = \begin{bmatrix} \left(1 - \frac{x}{L}\right) & \frac{x}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.372}$$

$$\theta = \sum_{i=1}^{2} \theta_i N_i(x) = \begin{bmatrix} \left(1 - \frac{x}{L}\right) & \frac{x}{L} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{4.373}$$

Or generally, the lateral displacement will be:

$$v = \sum_{i=1}^{4} u_i N_i^1(x) = \begin{bmatrix} \left(1 - \frac{x}{L}\right) & 0 & \frac{x}{L} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \tag{4.374}$$

With variation:

$$\delta v' = \sum_{i=1}^{4} \delta u_i N_i^{1'}(x) \tag{4.375}$$

Similarly, section rotation θ and its derivative with respect to beam length x (θ') will be:

$$\theta = \sum_{i=1}^{4} u_i N_i^2(x) = \begin{bmatrix} 0 & \left(1-\frac{x}{L}\right) & 0 & \frac{x}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \tag{4.376}$$

$$\theta' = \sum_{j=1}^{4} u_j N_j^{2'}(x) = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \tag{4.377}$$

And variations defined as:

$$\delta\theta = \sum_{i=1}^{4} \delta u_i N_i^2(x) \qquad \delta\theta' = \sum_{i=1}^{4} \delta u_i N_i^{2'}(x) \tag{4.378}$$

For end beam lateral displacements v_1, v_2 and section rotations θ_1, θ_2 , the linear interpolation for lateral displacement and section rotation forces the beam to displace as shown in Figure 4.64. The resulting variation in total potential energy will be:

$$\delta\Pi = \int_0^L \left(\begin{array}{c} EI_z \sum_{j=1}^{4} u_j N_j^{2'}(x) \sum_{i=1}^{4} \delta u_i N_j^{2'}(x) \\ +k_S GA \sum_{j=1}^{4} u_j \left(N_j^{1'}(x) - N_j^2(x) \right) \delta \sum_{i=1}^{4} \delta u_i \left(N_i^{1'}(x) - N_i^2(x) \right) \\ -q_o \sum_{i=1}^{n} \delta u_i N_i^1(x) \end{array} \right) dx - \sum_{i=1}^{n} P_i \delta u_i \tag{4.379}$$

Substituting with the interpolation functions in Equation 4.372 to Equation 4.378 results into the following stiffness matrix:

$$k = \frac{EI_z}{12\lambda L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2(1+3\lambda) & -6L & 2L^2(1-6\lambda) \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2(1-6\lambda) & -6L & 4L^2(1+3\lambda) \end{bmatrix} \tag{4.380}$$

Where

$$\lambda = \frac{EI_z}{K_s GAL^2} \tag{4.381}$$

But the above formulation and the assumed deformed shape in Figure 4.64 can not be used for thin beams (Bernoulli beam theory). As thin beam exhibit zero shear deformation as follows:

$$0 = \gamma_{xy} = v' - \theta \rightarrow v' = \theta \tag{4.382}$$

$$v' = \sum_{i=1}^{2} v_i N_i(x) = \frac{v_1 - v_2}{L} \tag{4.383}$$

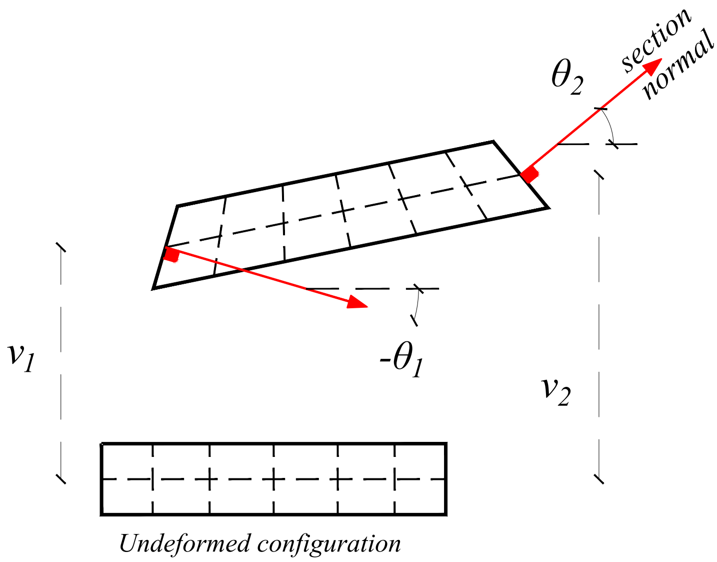


Figure 4.64

The tangent to beam centerline in the above equation is constant which contradicts the linear interpolation function assumed for the lateral displacement in Equation 4.374, so assuming a linear interpolation for lateral displacement and section rotation produces inconsistent beam element. The linear interpolation for lateral displacement forces section rotation to be constant all over the beam which leads to zero curvature (change in section rotation $\theta' = 0$). Zero curvature means no bending deformation or bending strain energy ($\int_0^L (EI_z\theta'\delta\theta')\,dx = 0$) and the beam exhibits only shear deformation, as shown in the beam deformed shape at the lower part of Figure 4.65. This deformed shape shows that, for zero lateral displacement, the rotation is varied linearly ($\theta = (1-x/L)\theta_1 + (x/L)\theta_2$), while v' is horizontal. This shape is different from the expected shape of deformation for Bernoulli beam as shown in the upper part of Figure 4.65. This problem is called shear locking. We remark that for any bending element like beam or shell element, using Lagrange interpolation function of the same order for deflection and rotation produces shear lock, especially for thin elements. To solve this problem, we need to choose a consistent interpolation

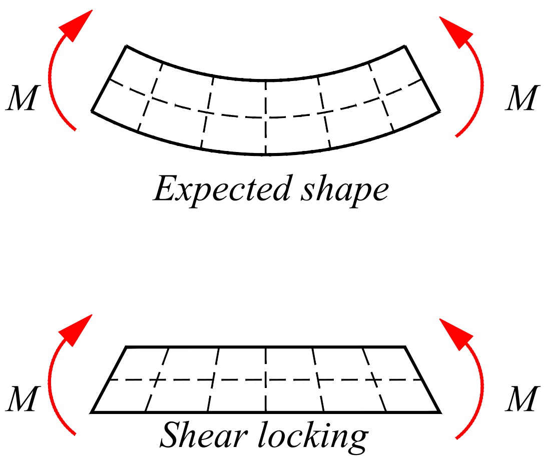


Figure 4.65: Shear locking is expected when using linear interpolation functions for both lateral displacement and section rotation in Bernoulli beam theory

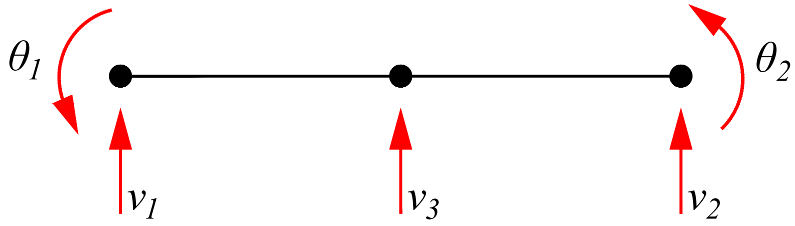


Figure 4.66: Consistent interpolation element

for both v and θ, such that lateral displacement derivative v' and section rotation θ should have the same interpolation function. For example, if we choose a linear interpolation function for section rotation, we need to assume a quadratic interpolation function for lateral displacement. We need to an additional node (e.g. at beam element mid-span) with lateral displacement as an undetermined parameter and the beam element will have five degree of freedom as shown in Figure 4.66. This element is called consistent interpolation element with interpolation functions defined using Equation 4.290 as follows:

$$v = \sum_{i=1}^{2} v_i N_i(x) = \begin{bmatrix} \frac{1}{2}\xi(\xi-1) & 1-\xi^2 & \frac{1}{2}\xi(\xi+1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_2 \end{bmatrix}, \quad (4.384)$$

$$\theta = \sum_{i=1}^{2} \theta_i N_i(x) = \begin{bmatrix} (1-\xi) & \xi \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad (4.385)$$

Where $\xi = \frac{x}{L}$, and the stiffness matrix can be evaluated like the same above procedures using Equation 4.379, but the element will have five DOF (two rotational at ends and three lateral displacements)

Another way to solve shear locking is to make both v' and θ to be constant instead of being linearly varied as stated in the previous five-DOF element. Using an average section rotation $\theta^* = \frac{(\theta_1+\theta_2)}{2}$ as a constant value in evaluating shear stiffness, the variation in total potential energy will be:

$$\delta\Pi = \int_0^L \left(EI_z\theta'\delta\theta' + k_S GA\left(v'-\theta^*\right)\delta\left(v'-\theta^*\right) - q\delta v\right)dx - Q_1\delta v_1 - Q_2\delta v_2 - M_1\delta\theta - M_2\delta\theta \quad (4.386)$$

And the interpolation function is defined as:

$$v = \sum_{i=1}^{4} u_i N_i^1(x) = \begin{bmatrix} \left(1-\frac{x}{L}\right) & 0 & \frac{x}{L} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (4.387)$$

$$\delta v' = \sum_{i=1}^{4} \delta u_i N_i^{1'}(x) \quad (4.388)$$

$$\theta^* = \sum_{i=1}^{4} u_i N_i^3(x) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (4.389)$$

$$\theta' = \sum_{j=1}^{4} u_j N_j^{2'}(x) = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (4.390)$$

$$\theta = \sum_{i=1}^{4} u_i N_i^{2}(x) = \begin{bmatrix} 0 & \left(1-\frac{x}{L}\right) & 0 & \frac{x}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (4.391)$$

θ^* can not be used in Equation 4.386, as for constant value for θ^*, it results that $(\theta'^* = 0)$ will vanish resulting no bending stiffness.
The resulting variation in total potential energy in Equation 4.386 will be:

$$\delta\theta^* = \sum_{i=1}^{4} \delta u_i N_i^3(x) \qquad \delta\theta' = \sum_{i=1}^{4} \delta u_i N_i^{2'}(x) \quad (4.392)$$

$$\delta\Pi = \int_0^L \left(\begin{array}{c} EI_z \sum_{j=1}^{4} u_j N_j^{2'}(x) \sum_{i=1}^{4} \delta u_i N_j^{2'}(x) \\ +k_S GA \sum_{j=1}^{4} u_j \left(N_j^{1'}(x) - N_j^3(x)\right) \delta \sum_{i=1}^{4} \delta u_i \left(N_i^{1'}(x) - N_i^3(x)\right) \\ -q_o \sum_{i=1}^{4} \delta u_i N_i^1(x) \end{array} \right) dx - \sum_{i=1}^{4} P_i \delta u_i \quad (4.393)$$

Substituting with the interpolation functions in Equation 4.387 to Equation 4.390 results into the following stiffness matrix:

$$K = \frac{EI_z}{12\lambda L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 3L^2(1+4\lambda) & -6L & 3L^2(1-4\lambda) \\ -12 & -6L & 12 & -6L \\ 6L & 3L^2(1-4\lambda) & -6L & 3L^2(1+4\lambda) \end{bmatrix} \quad (4.394)$$

These findings can be achieved through evaluating the integral corresponding to shear deformation in Equation 4.393 $\left[\int_0^L k_S GA \sum_{j=1}^{4} u_j \left(N_j^{1'}(x) - N_j^3(x)\right) \delta \sum_{i=1}^{4} \delta u_i \left(N_i^{1'}(x) - N_i^3(x)\right) dx\right]$ using one Gauss integration point (at mid-point) and the interpolation functions defined in Equation 4.372 to Equation 4.378 without the need to define a separate interpolation function for section rotation θ^*, as the rotation at beam mid-point from Equation 4.391 is $\theta\left(\frac{L}{2}\right) = \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2$ is equivalent to using an average value for rotation ($\theta^* = \frac{(\theta_1+\theta_2)}{2}$). This type of integration used in evaluating the finite element stiffness is called reduced integration.

For a beam free of body forces, this element does not lock but does not also yield the exact displacements as the section rotation θ is assumed to be varied linearly, while the curvature and moment have to be linearly varied for Bernoulli beam (see Hermite cubic interpolation functions in Equation 4.324 and Equation 4.327). This lower polynomial interpolation function used effect solution accuracy. Using reduced integration with finer mesh (by increasing the number of finite elements for each beam), solution will converge to more accurate results.

Another way to evaluate the stiffness matrix free of shear locking is to use forced-based finite element procedures (see stiffness matrix derived for Bernoulli beam from Equation 4.343 to

Equation 4.364). The complementary virtual work of shear force is defined as:

$$\Pi^* = \Pi = \int_0^L \frac{Q^2}{2GA_s} dx \tag{4.395}$$

With variation:

$$\delta\Pi^* = \int_0^L \frac{Q\delta Q}{GA_s} dx = \frac{L}{GA_s} \delta Q_1 Q_1 \tag{4.396}$$

As seen in Figure 4.62, $Q = Q_1$, $\delta Q = \delta Q_1$. Adding the resulting shear flexibility to bending flexibility in Equation 4.355 yields:

$$D = \frac{1}{EI_z} \begin{bmatrix} \frac{L^3}{3} & -\frac{L^2}{2} \\ -\frac{L^2}{2} & L \end{bmatrix} + \frac{1}{GA_s} \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{EI_z} \begin{bmatrix} \left(\frac{1}{3} + \lambda\right) L^3 & -\frac{L^2}{2} \\ -\frac{L^2}{2} & L \end{bmatrix} \tag{4.397}$$

Where $\lambda = \frac{EI_z}{GA_s L^2}$. The stiffness matrix corresponding to the first DOF at the left end of the beam.

$$\overline{K} = D^{-1} = \frac{EI_z}{L^3 (12\,\lambda + 1)} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 (3\,\lambda + 1) \end{bmatrix} \tag{4.398}$$

Using the same procedures from Equation 4.357 to Equation 4.363, we get the total stiffness as follows:

$$K = \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 (1 + 3\lambda) & -6L & 2L^2 (1 - 6\lambda) \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 (1 - 6\lambda) & -6L & 4L^2 (3\lambda + 1) \end{bmatrix} \tag{4.399}$$

For a beam free of body forces, this formulation for stiffness matrix gives the exact solution for displacements and section rotations even if we use one finite element for each beam of the structure, unlike using reduced integration in which it requires a finer mesh for structure to force the solution to converge to the exact solution.

For very thin beam ($\lambda \to 0$), the stiffness matrix in the above equation will be identical to the one used for the Bernoulli beam element in Equation 4.333.

4.5.3 Formulation of continuum mechanics incremental equations of motion

Total and updated Lagrangian formulation

As stated in chapter 3, we use Lagrangian description in solid bodies especially when they are subjected to large displacements and rotations. Consider a body shown in Figure 4.67 with initial configuration C_0 and then subjected to some external forces yielding configuration C_1. Assume that the deformation is known until this configuration C_1, while the deformation in configuration C_2 is unknown, such that a material point P attached to this body has coordinates $P_0 = ({}^0X_1, {}^0X_2, {}^0X_3)$, $P_1 = ({}^1X_1, {}^1X_2, {}^1X_3)$ and $P_2 = ({}^2X_1, {}^2X_2, {}^2X_3)$ in configuration C_0, C_1 and C_2, respectively. If the coordinate system used remains constant during body motion, the coordinates of point P in different configurations are related through following:

$$\begin{aligned} {}^1X_i &= {}^0X_i + {}^1u_i \\ {}^2X_i &= {}^0X_i + {}^2u_i \\ u_i &= {}^2u_i - {}^1u_i \end{aligned} \tag{4.400}$$

Where ${}^1u_i$,${}^2u_i$ and u_i represent the incremental displacements from configuration C_0 to C_2, C_0 to C_2 and C_1 to C_2, respectively. The superscript is used generally to define the configuration at which the

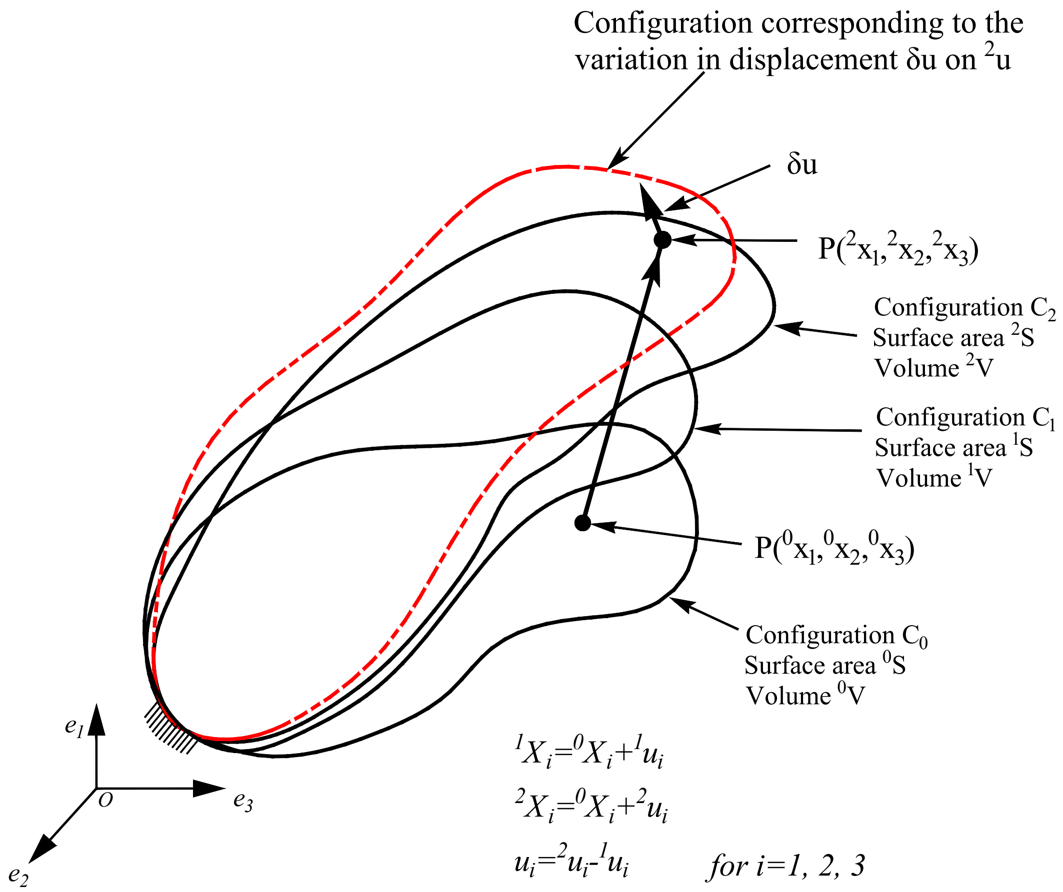


Figure 4.67: Body motion

property is measured. Applying virtual (variational) displacement δu on the unknown configuration C_2 that satisfies the boundary conditions to get an admissible configuration shown in Figure 4.35. These virtual displacements undergo virtual strain denoted by $\delta^2\boldsymbol{\varepsilon}$ and virtual work defined using Equation 4.161 integrated over the unknown configuration C_2 as follows:

$$^2\delta\Pi = {}^2\delta\Pi_{int} - {}^2\delta\Pi_{ext} = \int_{^2V} {}^2\sigma_{ij}\delta^2\varepsilon_{ij}\, d^2V - \int_{^2V} {}^2f_i^*\delta^2u_i\, d^2V - \int_{^2S_\Gamma} {}^2t_i\delta^2u_i\, d^2A = 0 \quad (4.401)$$

Where $^2\boldsymbol{\sigma}$ is Cauchy stress at configuration t_2 and $\delta^2\boldsymbol{\varepsilon}$ defines the infinitesimal virtual strain referred to configuration C_2 defined as follows:

$$\delta^2\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\delta u_i}{\partial\, ^2X_j} + \frac{\partial\delta u_j}{\partial\, ^2X_i}\right) \quad (4.402)$$

In Equation 4.401, we face two problems. First, we can not evaluate the integration over unknown volume ^2V and second, Cauchy stress can not be used in an incremental analysis as its rate is not an objective (see section 3.4), such that there is no direct expression for the increment in stress $\triangle\boldsymbol{\sigma}$ from configuration C_1 to C_2 that satisfies the following equation:

$$^2\boldsymbol{\sigma} = {}^1\boldsymbol{\sigma} + \triangle\boldsymbol{\sigma} \quad (4.403)$$

Therefore, we should use an alternative expression for the internal virtual work. We can write the internal virtual work in the material form as follows:

$$^2\delta\Pi_{int} = \int_{^0V} {}^2_0S_{ij}\delta^2_0E_{ij}\, d^0V \quad (4.404)$$

Where ${}^2_0S_{ij}$ represents second Piola-Kirchhoff stress tensor and and ${}^2_0E_{ij}$ is Green-Lagrange strain tensor. The superscript 2 indicates that they are measured at configuration C_2, while subscript 0 signifies that they are referred to configuration C_0. Green-Lagrange strain tensor is defined as:

$$ {}^2_0E_{ij} = \frac{1}{2}\left({}^2_0U_{i,j} + {}^2_0U_{j,i} + {}^2_0U_{k,i}\, {}^2_0U_{k,j}\right), \quad where \;\; {}^m_nU_{i,j} = \frac{\partial\, {}^m u_i}{\partial\, {}^n X_j} \tag{4.405} $$

While its variation will be:

$$ \delta {}^2_0E_{ij} = \frac{1}{2}\left(\delta {}^2_0U_{i,j} + \delta {}^2_0U_{j,i} + \delta {}^2_0U_{k,i}\, {}^2_0U_{k,j} + {}^2_0U_{k,i}\, \delta {}^2_0U_{k,j}\right) \tag{4.406} $$

From above equations, the alternative virtual work is expressed in terms of a known configuration. Also the displacement U in Equation 4.405 is differentiated with respect to known configuration. In addition, we can decompose second Piola-Kirchhoff stress and Green-Lagrange strain tensors because of their objective rate as follows:

$$ {}^2_0S_{ij} = {}^1_0S_{ij} + {}_0S_{ij}, \quad {}^2_0E_{ij} = {}^1_0E_{ij} + {}_0E_{ij} \tag{4.407} $$

Where

$$ {}^1_0E_{ij} = \frac{1}{2}\left({}^1_0U_{i,j} + {}^1_0U_{j,i} + {}^1_0U_{k,i}\, {}^1_0U_{k,j}\right) \tag{4.408} $$

Substituting Equation 4.405 and the above equation into Equation 4.407, we get the increment in Green-Lagrange strain as follows:

$$ {}_0E_{ij} = \frac{1}{2}\left({}_0U_{i,j} + {}_0U_{j,i} + {}^1_0U_{k,i}\, {}_0U_{k,j} + {}_0U_{k,i}\, {}^1_0U_{k,j}\right) + \frac{1}{2}\left({}_0U_{k,i}\, {}_0U_{k,j}\right) \tag{4.409} $$

The above increment can be decomposed to two parts as follows:

$$ {}_0E_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \tag{4.410} $$

$$ {}_0e_{ij} = \frac{1}{2}\left({}_0U_{i,j} + {}_0U_{j,i} + \underbrace{{}^1_0U_{k,i}\, {}_0U_{k,j} + {}_0U_{k,i}\, {}^1_0U_{k,j}}_{\text{initial displacement effect}} \right) \tag{4.411} $$

$$ {}_0\eta_{ij} = \frac{1}{2}\, {}_0U_{k,i}\, {}_0U_{k,j} \tag{4.412} $$

With the variations $\delta_0 e_{ij}$ and $\delta_0\eta_{ij}$ defined as:

$$ \delta_0 e_{ij} = \frac{1}{2}\left(\delta_0U_{i,j} + \delta_0U_{j,i} + {}^1_0U_{k,i}\, \delta_0U_{k,j} + \delta_0U_{k,i}\, {}^1_0U_{k,j}\right), \;\; \delta_0\eta_{ij} = \frac{1}{2}\left(\delta_0U_{k,i}\, {}_0U_{k,j} + {}_0U_{k,i}\delta_0U_{k,j}\right) \tag{4.413} $$

Also from Equation 4.407, the variation δ_0E_{ij} is defined as:

$$ \delta {}^2_0E_{ij} = \delta_0E_{ij} + \delta {}^1_0E_{ij} = \delta_0E_{ij} \tag{4.414} $$

The first term in Equation 4.410 (${}_0e_{ij}$) defines the linear incremental strain in ${}_0U_{i,j}$ (see Equation 4.411) as ${}^1_0U_{i,j} = \frac{\partial^1 u_i}{\partial^0 X_j}$ is known and considered constant through applying u_i or $U_{i,j}$, while the second term (${}_0\eta_{ij}$) is nonlinear incremental strain denoted as seen in Equation 4.412.

As shown in Figure 4.67, the displacement field ${}^1u_i$ can be interpolated in terms of nodal point variables (degree of freedom) which may be displacements or rotations or both. For body undergoing large rotation, ${}^1u_i$ will be a linear function in nodal point displacement and a nonlinear one in nodal point rotation (see Equation 2.179), which in turn makes a part of ${}_0e_{ij}$ associated with nodal point rotation to be nonlinear and ${}_0\eta_{ij}$ is not the full story of all nonlinear strain increment. Also we need to note that the external forces are assumed constant during displacement increment. Some loads like pressures are deformation dependent and it will add additional stiffness to the total stiffness (see Appendix 4.5.5). The resulting principle of virtual work is:

$$\int_{{}^0V} {}_0S_{ij}\delta_0E_{ij}\, d^0V + \int_{{}^0V} {}_0^1S_{ij}\delta_0\eta_{ij}\, d^0V = {}^2\delta\Pi_{ext} - \int_{{}^0V} {}_0^1S_{ij}\delta_0e_{ij}\, d^0V \tag{4.415}$$

For given variation δu_i, the right hand side in the above is known, while the left hand side contains unknown displacement increments which is responsible for the stiffness matrix. Deriving the stiffness matrix requires that neglecting all higher-order terms in U_i, such that all linear terms in U_i remain. This process is called linearization which leads to:

$$\left({}_0^tK\ {}_0U\right)\delta_0U_i = \left({}^2R - {}^1F\right)\delta_0U_i \rightarrow {}_0^tK\ {}_0U = {}^2R - {}^1F \tag{4.416}$$

Where ${}_0^tK$, ${}_0U$, ${}^2R$ and ${}^1F$ are the stiffness matrix, incremental displacement, external applied force at configuration C_2 and internal forces at configuration C_1, respectively. The following term ${}_0^1S_{ij}\delta_0\eta_{ij}$ is linear in ${}_0U_i$ as ${}_0^1S_{ij}$ is known from the configuration C_1, while the term η_{ij} is linear in ${}_0U_i$ and δ_0U_i as seen in Equation 4.413. The term ${}_0S_{ij}\delta_0E_{ij}$ is non-linear in ${}_0U_i$, as the first part ${}_0S_{ij}$ is generally nonlinear function in δ_0E_{ij} according to the constitutive relation, so neglecting higher order will make δ_0E_{ij} a linear function in ${}_0U_i$ as follows:

$${}_0S_{ij} = \left.\frac{\partial_0^tS_{ij}}{\partial_0^tE_{rs}}\right|_{t1} {}_0E_{rs} + higher\ order\ terms \tag{4.417}$$

The above equation can be expanded using Taylor series. The term ${}_0E_{rs} = {}_0e_{rs} + {}_0\eta_{rs}$ is quadratic function in ${}_0U_i$ because of the nonlinearity of ${}_0\eta_{rs}$ as stated in Equation 4.412, which requires neglecting ${}_0\eta_{rs}$. By equating $\left.\frac{\partial_0^tS_{ij}}{\partial_0^tE_{rs}}\right|_{t1}$ with ${}_0C_{ijrs}$, the resulting linear term ${}_0S_{ij}$ will be:

$${}_0S_{ij} = {}_0C_{ijrs0}e_{rs} \tag{4.418}$$

while the second part δ_0E_{ij} contains linear and non-linear terms as follow:

$$\delta_0E_{ij} = \underbrace{\delta_0e_{ij}}_{\text{constant}} + \underbrace{\delta_0\eta_{ij}}_{\text{linear}} \tag{4.419}$$

As the first part of is linear, second part is needed to be constant by neglecting the second term in the above equation, such that term ${}_0S_{ij}\delta_0E_{ij}$ can be linear only through the following approximation:

$$\underbrace{{}_0C_{ijrs0}e_{rs}}_{\text{linear}}\ \underbrace{\delta_0e_{ij}}_{\text{constant}} = \underbrace{{}_0C_{ijrs0}e_{rs}\delta_0e_{ij}}_{\text{linear}} \tag{4.420}$$

So the final linearized equation of Equation 4.415 can be written as follows:

$$\underbrace{\int_{{}^0V} {}_0C_{ijrs}\ {}_0e_{rs}\delta_0e_{ij}\, d^0V + \int_{{}^0V} {}_0^1S_{ij}\delta_0\eta_{ij}\, d^0V}_{\text{linear}} = \underbrace{{}^2\delta\Pi_{ext} - \int_{{}^0V} {}_0^1S_{ij}\delta_0e_{ij}\, d^0V}_{\text{Constant}} \tag{4.421}$$

The left side of the above equation is responsible for the material and geometric stiffness matrices, while the right side represents out of balance virtual work term. This term, the difference between the external virtual work and internal virtual work, can be reduced by performing some iterations in which the solution step is repeated until this difference can be neglected within a certain convergence measure as follows:

$$\int_{^0V} {}_0C_{ijrs}^{(k-1)} {}_0e_{rs}^{(k)} \delta_0 e_{ij} \, d^0V + \int_{^0V} {}_0^2S_{ij}^{(k-1)} \delta_0 \triangle \eta_{ij}^{(k)} \, d^0V = {}^2\delta\Pi_{ext} - \int_{^0V} {}_0^2S_{ij}^{(k-1)} \delta_0^2 \varepsilon_{ij}^{(k-1)} \, d^0V \quad (4.422)$$

The superscript k indicates the iteration at which the term is calculated.
The last term ($\int_{^0V} {}_0^2S_{ij}^{(k-1)} \delta_0^2 \varepsilon_{ij}^{(k-1)} \, d^0V$) corresponds to the current internal stresses in th element at configuration C_1. Although, we are forced to use linearization (approximation through neglecting higher order terms) to get the stiffness matrix in the predictor phase of the finite element analysis, we can achieve the exact solution as long as the unbalance force is evaluated accurately in the corrector phase. These exact results can be guaranteed through calculating accurately the last term of the unbalance virtual work equation ($\int_{^0V} {}_0^2S_{ij}^{(k-1)} \delta_0^2 \varepsilon_{ij}^{(k-1)} \, d^0V$) . This term is an essential quantity that controls the final results of the finite element analysis that we have to calculate accurately, as our ultimate goal is equilibrating this term with ${}^2\delta\Pi_{ext}$. If we mistake in calculating this term, the analysis will converge to a wrong solution. However, the approximation used in evaluating the stiffness matrix has no effect on the solution results and just increases the number of iterations to for solution to converge or reach the equilibrium state in the loading step.

Equation 4.421 can be simplified using the symmetry property of second Piola Kirchhoff stress tensor (using Equation 1.100) and Equation 4.413 as follows:

$$\begin{aligned} {}_0^1S_{ij} \delta_0 \eta_{ij} &= {}_0^1S_{ij} \left(\delta_0 U_{k,i} \, {}_0U_{k,j} \right) \\ {}_0^1S_{ij} \delta_0 e_{ij} &= {}_0^1S_{ij} \left(\delta_0 U_{i,j} + {}_0^1U_{k,i} \, \delta_0 U_{k,j} \right) \end{aligned} \quad (4.423)$$

The above formulation is called *Total Lagrangian (TL) Formulation* in which the initial configuration C_0 is used as a reference configuration. We can use instead the last converged configuration C_1 as a reference configuration which leads to so-called *Updated Lagrangian (UL) Formulation*. In this formulation, the internal virtual work will be defined as:

$${}^2\delta\Pi_{int} = \int_{^0V} {}_1^2S_{ij} \delta_1^2 E_{ij} \, d^0V \quad (4.424)$$

Which ${}_1^2S_{ij}$ and ${}_1^2E_{ij}$ are conjugate pairs defined as follows:

$$\begin{aligned} {}_1^2\boldsymbol{S} &= det({}_1^2\boldsymbol{F}) {}_1^2\boldsymbol{F}^{-1} \left({}^2\boldsymbol{\sigma} \right)^T {}_1^2\boldsymbol{F} \\ {}_1^2\boldsymbol{E} &= \frac{1}{2} \left({}_1^2\boldsymbol{F}^T {}_1^2\boldsymbol{F} - \boldsymbol{1} \right) \end{aligned} \quad (4.425)$$

With

$${}_1^2F_{ij} = \frac{\partial^2 X_i}{\partial^1 X_j} = \frac{\partial u_i}{\partial^1 X_j} + \delta_{ij} = {}_1U_{i,j} + \delta_{ij} \quad (4.426)$$

We get

$${}_1^2E_{ij} = \frac{1}{2} \left({}_1U_{i,j} + {}_1U_{j,i} + {}_1U_{k,i} \, {}_1U_{k,j} \right) \quad (4.427)$$

${}_1^2E_{ij}$ can be split into two terms as stated before:

$${}_1E_{ij} = {}_1e_{ij} + {}_1\eta_{ij} = \frac{1}{2} \left({}_1U_{i,j} + {}_1U_{j,i} \right) + \frac{1}{2} \left({}_1U_{k,i} \, {}_1U_{k,j} \right) \quad (4.428)$$

$${}_1e_{ij} = \frac{1}{2} \left({}_1U_{i,j} + {}_1U_{j,i} \right) \quad (4.429)$$

$${}_1\eta_{ij} = \frac{1}{2} \left({}_1U_{k,i} \, {}_1U_{k,j} \right) \quad (4.430)$$

With variation define as:

$$\delta_1 E_{ij} = \frac{1}{2}\left(\delta_1 U_{i,j} + \delta_1 U_{j,i} + \delta_1 U_{k,i}\ {}_1U_{k,j} + {}_1U_{k,i}\ \delta_1 U_{k,j}\right) \tag{4.431}$$

$$\delta_1 e_{ij} = \frac{1}{2}\left(\delta_1 U_{i,j} + \delta_1 U_{j,i}\right) \tag{4.432}$$

$$\delta_1 \eta_{ij} = \frac{1}{2}\left(\delta_1 U_{k,i}\ {}_1U_{k,j} + {}_1U_{k,i}\delta_1 U_{k,j}\right) \tag{4.433}$$

Second Piola Kirchhoff stress tensor at the current configuration can be resolved into two components:

$${}^2_1S_{ij} = {}^1_1S_{ij} + {}_1S_{ij} = {}^1\sigma_{ij} + {}_1S_{ij} \tag{4.434}$$

As from Equation 4.425, ${}^1_1S_{ij} = {}^1\sigma_{ij}$. Virtual work equation will be:

$$\int_{^1V} {}_1S_{ij}\delta_1 E_{ij}\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta_1\eta_{ij}\, d^1V = {}^2\delta\Pi_{ext} - \int_{^1V} {}^1\sigma_{ij}\delta_1 e_{ij}\, d^1V \tag{4.435}$$

Linearization of the above equation results in:

$$\int_{^1V} {}_1C_{ijrs}\ {}_1e_{rs}\delta_1 e_{ij}\, d^1V + \int_{^1V} {}^1\sigma_{ij}\delta_1\eta_{ij}\, d^1V = {}^2\delta\Pi_{ext} - \int_{^1V} {}^1\sigma_{ij}\delta_1 e_{ij}\, d^1V \tag{4.436}$$

With incremental form defined as:

$$\int_{^2V^{(k-1)}} {}_2C^{(k-1)}_{ijrs}\ {}_2e^{(k)}_{rs}\delta_2 e_{ij}\, d^2V + \int_{^2V^{(k-1)}} {}^2\sigma^{(k-1)}_{ij}\delta_2\eta^{(k)}_{ij}\, d^2V = {}^2\delta\Pi_{ext} - \int_{^2V^{(k-1)}} {}^2\sigma^{(k-1)}_{ij}\delta_2 e^{(k-1)}_{ij}\, d^2V \tag{4.437}$$

The difference between updated Lagrangian (UL) and total Lagrangian (TL) formulations is that TL formulation includes initial displacement effect as stated in Equation 4.411 which makes the stress-displacement matrix more complicated than UL formulation, but they gave the same results.

Lagrangian formulation of displacement-based finite elements

For a general body described in the previous section, the linearized virtual work Equation 4.421 can be written in terms of the nodal point variables (displacement rotation) as follows:

$$\begin{aligned}
\int_{^0V} {}_0C_{ijrs}\ {}_0e_{rs}\delta_0 e_{ij}\, d^0V &= \delta\hat{\boldsymbol{u}}\left(\int_{^0V} {}^1_0\boldsymbol{B_L}^T\ {}_0\boldsymbol{C}\ {}^1_0\boldsymbol{B_L}\, d^0V\right)\hat{\boldsymbol{u}} \\
\int_{^0V} {}^1_0S_{ij}\delta_0\eta_{ij}\, d^0V &= \delta\hat{\boldsymbol{u}}\left(\int_{^0V} {}^1_0\boldsymbol{B_{NL}}^T\ {}^1_0\boldsymbol{S}^1_0\boldsymbol{B_{NL}}\, d^0V\right)\hat{\boldsymbol{u}} \\
\int_{^0V} {}^1_0S_{ij}\delta_0 e_{ij}\, d^0V &= \delta\hat{\boldsymbol{u}}\left(\int_{^0V} \boldsymbol{B_L}^T\ {}^1_0\hat{\boldsymbol{S}}\, d^0V\right)
\end{aligned} \tag{4.438}$$

Where ${}^1_0\boldsymbol{B_L}$ and ${}^1_0\boldsymbol{B_{NL}}$ is the linear and nonlinear strain-displacement transformation matrices. Term ${}_0\boldsymbol{C}$ defines stress-strain constitutive relation. ${}^1_0\boldsymbol{S}$ and $\hat{\boldsymbol{S}}$ are matrix and vector of second Piola Kirchhoff stress. While vectors $\hat{\boldsymbol{u}}$ and $\delta\hat{\boldsymbol{u}}$ signify the nodal point variables nodes and their variations, respectively, defined as follows:

$$\hat{\boldsymbol{u}} = \left[u^1_1\ \ u^1_2 \ldots u^1_i \ldots u^1_m \mid u^1_2\ \ u^2_2 \ldots \mid u^n_1 \ldots u^n_m\right]^T \tag{4.439}$$

$$\delta\hat{\boldsymbol{u}} = \left[\delta u^1_1\ \ \delta u^1_2 \ldots \delta u^1_i \ldots \delta u^1_m \mid \delta u^1_2\ \ \delta u^2_2 \ldots \mid \delta u^n_1 \ldots \delta u^n_m\right]^T \tag{4.440}$$

Where m is the number of DOF associated with each node of the finite element, while n is the number of nodes in the finite element, such that u^j_i defines the displacement at node j associated with DOF i at this node. Generally, $m = 3$ for continuum finite element (three nodal point displacement at each node) and $m = 6$ for structural element (three nodal point displacement and three nodal point rotation at each node).

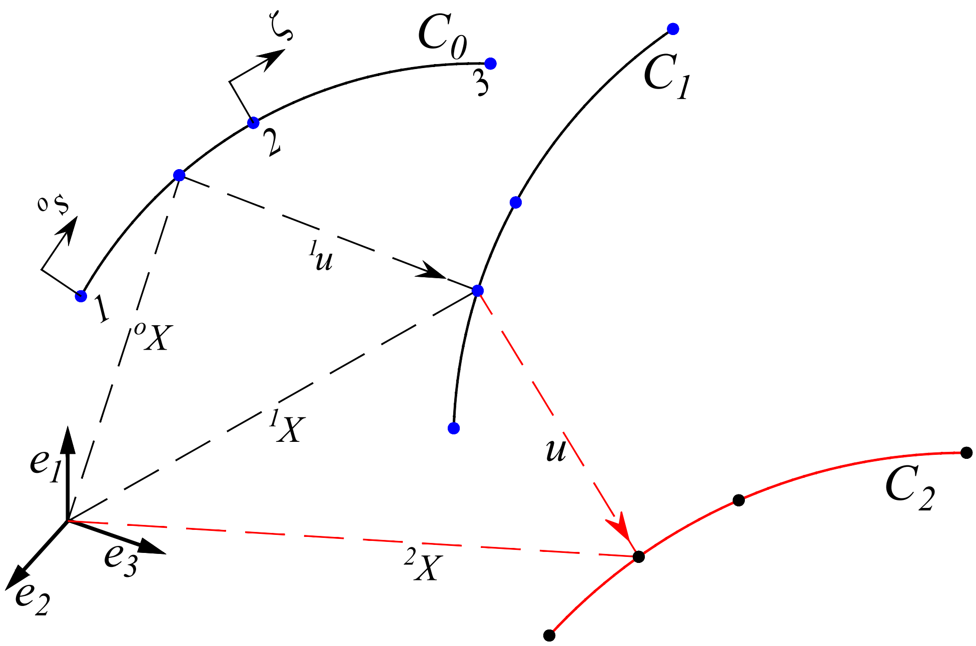


Figure 4.68

■ **Example 4.36** Assume a three-node curved truss element shown in Figure 4.68. As the only stress considered in truss element is the normal stress on its cross section, we are interested in the corresponding longitudinal Green Lagrangian strain E_{11}. Assume an infinitesimal vector $d^0\boldsymbol{s}$ of the truss element at the initial configuration C_0 along its centroid and is deformed to $d^1\boldsymbol{s}$ in the deformed configuration C_1, such that E_{11} is defined using its expression in chapter 3 as follows:

$$d^1s^2 - d^0s^2 = 2{}_0^1E_{11}d^0s^2 \tag{4.441}$$

Where d^0s and d^1s represent the arc length of undeformed and deformed infinitesimal vectors $d^0\boldsymbol{s}$, $d^1\boldsymbol{s}$, respectively. If the truss has initial position ${}^0\boldsymbol{X}$ and is subjected to displacement vector ${}^1\boldsymbol{u}$ reaching to position ${}^1\boldsymbol{X}$ in the deformed configuration C_1, the length square of d^0s and d^1s can be defined as follows:

$$\begin{aligned}
(d^0s)^2 &= d^0\boldsymbol{s}.d^0\boldsymbol{s} = d^0\boldsymbol{X}_i.d^0\boldsymbol{X}_i\\
(d^1s)^2 &= d^1\boldsymbol{s}.d^1\boldsymbol{s} = d^1\boldsymbol{X}_i.d^1\boldsymbol{X}_i\\
d^0X_i &= \frac{d^0X_i}{d^0S}d^0S\\
d^1X_i &= \frac{d^1X_i}{d^0S}d^0S = \left(\frac{d^0X_i}{d^0S} + \frac{d^1u_i}{d^0S}\right)d^0S\\
(d^1s)^2 - (d^0s)^2 &= \left(2\frac{d^0X_i}{d^0S}\frac{d^1u_i}{d^0S} + \frac{d^1u_i}{d^0S}\frac{d^1u_i}{d^0S}\right)(d^0s)^2 = 2E_{11}(d^0s)^2
\end{aligned} \tag{4.442}$$

Then, we get the axial strain as follows:

$${}_0^1E_{11} = \frac{d^0X_i}{d^0S}\frac{d^1u_i}{d^0S} + \frac{1}{2}\frac{d^1u_i}{d^0S}\frac{d^1u_i}{d^0S} \tag{4.443}$$

In the same manner, the element is deformed to the final configuration C_2 through additional displacement u_i, such that the Green-Lagrange strain will be:

$$ {}_0^2E_{11} = \frac{d{}^0X_i}{d{}^0S}\frac{d({}^1u_i+u_i)}{d{}^0S} + \frac{1}{2}\frac{d({}^1u_i+u_i)}{d{}^0S}\frac{d({}^1u_i+u_i)}{d{}^0S} \tag{4.444} $$

The incremental in this strain ${}_0E_{11} = {}_0^2E_{11} - {}_0^1E_{11}$ will be:

$$ \frac{d{}^0X_i}{d{}^0S}\frac{du_i}{d{}^0S} + \frac{d{}^1u_i}{d{}^0S}\frac{du_i}{d{}^0S} + \frac{1}{2}\frac{du_i}{d{}^0S}\frac{du_i}{d{}^0S} \tag{4.445} $$

The increment in strain can be decomposed into linear ${}_0e_{11}$ and nonlinear part ${}_0\eta_{11}$ as follows:

$$ {}_0e_{11} = \frac{d{}^0X_i}{d{}^0S}\frac{du_i}{d{}^0S} + \frac{d{}^1u_i}{d{}^0S}\frac{du_i}{d{}^0S} = \frac{d{}^1X_i}{d{}^0S}\frac{du_i}{d{}^0S} \tag{4.446} $$

$$ {}_0\eta_{11} = \frac{1}{2}\frac{du_i}{d{}^0S}\frac{du_i}{d{}^0S} \tag{4.447} $$

Where the arc length at the initial configuration ${}^0S(\xi)$ can be defined in terms of natural coordinate $\xi = [-1, 1]$ using Lagrange interpolation functions as follows:

$$ {}^0S(\xi) = \sum_{j=1}^{n} N_j(\xi){}^0S^j \tag{4.448} $$

$N_j(\xi)$ is the interpolation function defined for three-node element (see Equation 4.290). In the same manner, the following vectors can be interpolated as follows:

$$ \begin{aligned} {}^0X_i(\xi) &= \sum_{j=1}^{n} N_j(\xi){}^0X_i^j = \boldsymbol{N}.{}^0\hat{\boldsymbol{X}} \\ {}^1X_i(\xi) &= \sum_{j=1}^{n} N_j(\xi){}^1X_i^j = \boldsymbol{N}.{}^1\hat{\boldsymbol{X}} \\ u_i(\xi) &= \sum_{j=1}^{n} N_j(\xi)u_i^j = \boldsymbol{N}.\hat{\boldsymbol{u}} \end{aligned} \tag{4.449} $$

With $N_j(\xi)$ defined in Equation 4.290 and $\boldsymbol{N}$ defined as:

$$ \begin{aligned} \boldsymbol{N} &= [N_1(\xi)I_3 \mid N_2(\xi)I_3 \mid \ldots \mid N_n(\xi)I_3], \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ {}^0\hat{\boldsymbol{X}} &= \left[{}^0X_1^1 \ {}^0X_2^1 \ {}^0X_3^1 \ \ldots \ {}^0X_1^3 \ {}^0X_2^3 \ {}^0X_3^3\right]^T \\ {}^1\hat{\boldsymbol{X}} &= \left[{}^1X_1^1 \ {}^1X_2^1 \ {}^1X_3^1 \ \ldots \ {}^1X_1^3 \ {}^1X_2^3 \ {}^1X_3^3\right]^T \\ {}^1\hat{\boldsymbol{u}} &= \left[u_1^1 \ u_2^1 \ u_3^1 \ \ldots \ u_1^3 \ u_2^3 \ u_3^3\right]^T \end{aligned} \tag{4.450} $$

Where ${}^0\hat{\boldsymbol{X}}$, ${}^1\hat{\boldsymbol{X}}$ and ${}^1\hat{\boldsymbol{u}}$ represents the initial and final position, and displacement at point j in i

direction, such that

$$\frac{d^1X_i}{d^0S} = \frac{d\xi}{d^0S}\frac{d^1X_i}{d\xi} \tag{4.451}$$

$$= \frac{d\xi}{d^0S}\sum_{j=1}^{n} N_{j,\xi}(\xi)^1X_i^j \tag{4.452}$$

$$= \frac{d\xi}{d^0S}\boldsymbol{N}_{,\xi}{}^1\hat{\boldsymbol{X}} \tag{4.453}$$

Similarly:

$$\frac{du_i}{d^0S} = \frac{d\xi}{d^0S}\sum_{j=1}^{n}\boldsymbol{N}_{,\xi}\,\hat{\boldsymbol{u}} \tag{4.454}$$

Note that subscript $(,\xi)$ in $N_{j,\xi}$ signifies the derivative of N_j with respect to the natural coordinate ξ. Equation 4.446 and Equation 4.447 can also be interpolated as follows:

$${}_0e_{11} = \frac{d^1X_i}{d^0S}\frac{du_i}{d^0S} \tag{4.455}$$

$$= \left(\frac{d\xi}{d^0S}\right)^2 \left(\boldsymbol{N}_{,\xi}\,{}^1\hat{\boldsymbol{X}}\right).\left(\boldsymbol{N}_{,\xi}\,\hat{\boldsymbol{u}}\right) \tag{4.456}$$

$$= \left(\frac{d\xi}{d^0S}\right)^2 \left({}^1\hat{\boldsymbol{X}}^T\boldsymbol{N}_{,\xi}^T\boldsymbol{N}_{,\xi}\,\hat{\boldsymbol{u}}\right) \tag{4.457}$$

$${}_0\eta_{11} = \frac{1}{2}\left(\frac{d\xi}{d^0S}\right)^2 \left(\hat{\boldsymbol{u}}^T\boldsymbol{N}_{,\xi}^T\boldsymbol{N}_{,\xi}\,\hat{\boldsymbol{u}}\right) \tag{4.458}$$

With variations defined using Equation 4.438 as follows:

$$\begin{aligned}\delta_0e_{11} &= \left(\frac{d\xi}{d^0S}\right)^2 \left({}^1\hat{\boldsymbol{X}}^T\boldsymbol{N}_{,\xi}^T\boldsymbol{N}_{,\xi}\,\delta\hat{\boldsymbol{u}}\right) = \boldsymbol{B_L}\delta\hat{\boldsymbol{u}}\\ \delta_0\eta_{11} &= \left(\frac{d\xi}{d^0S}\right)^2 \left(\hat{\boldsymbol{u}}^T\boldsymbol{N}_{,\xi}^T\boldsymbol{N}_{,\xi}\,\delta\hat{\boldsymbol{u}}\right) = \frac{1}{2}\hat{\boldsymbol{u}}^T\boldsymbol{B_{NL}}^T\boldsymbol{B_{NL}}\delta\hat{\boldsymbol{u}}\end{aligned} \tag{4.459}$$

Where $\boldsymbol{B_L}$ and $\boldsymbol{B_{NL}}$ are defined as:

$$\boldsymbol{B_L} = \left(\frac{d\xi}{d^0S}\right)^2 \left({}^1\hat{\boldsymbol{X}}^T\boldsymbol{N}_{,\xi}^T\boldsymbol{N}_{,\xi}\right) \tag{4.460}$$

$$\boldsymbol{B_{NL}} = \left(\frac{d\xi}{d^0S}\right)\left(\boldsymbol{N}_{,\xi}\right) \tag{4.461}$$

While second Piola Kirchhoff stress vector is defined as:

$${}_0^1\hat{\boldsymbol{S}} = \left[{}_0^1S_{11}^1\;\; {}_0^1S_{11}^2\;\; {}_0^1S_{11}^3\right]^T \tag{4.462}$$

Where ${}^{1}_{0}S^{i}_{11}$ is the second Piola Kirchhoff longitudinal stress tensor at node i. The corresponding nonzero stress ${}^{1}_{0}S_{11}$ can be a function of strain ${}^{1}_{0}E_{11}$, such that the tangent stress-strain relation is defined as follows:

$$ {}_{0}C_{1111} = \frac{\partial\, {}^{1}_{0}S_{11}}{\partial\, {}^{1}_{0}E_{11}}, \qquad or \qquad \triangle{}^{1}_{0}S_{11} = {}_{0}C_{1111}\triangle{}^{1}_{0}E_{11} \tag{4.463} $$

For linear elastic material, ${}_{0}C_{1111}$ will be identical to Young's modulus. As the axial stress is constant over the cross section A, volume integration in Equation 4.464 can be simplified to a line integral as follows:

$$ \begin{aligned} \int_{{}^{0}V} {}_{0}C_{ijrs}\ {}_{0}e_{rs}\delta_{0}e_{ij}\ d^{0}V &= \int_{{}^{0}V} {}_{0}C_{1111}\ {}_{0}e_{11}\delta_{0}e_{11}\ d^{0}V = \delta\hat{\boldsymbol{u}}\left(\int_{{}^{0}S} {}^{1}_{0}\boldsymbol{B_L}^{T}\ {}_{0}\boldsymbol{C}A\ {}^{1}_{0}\boldsymbol{B_L}\ d^{0}S\right)\hat{\boldsymbol{u}} \\ \int_{{}^{0}V} {}^{1}_{0}S_{11}\delta_{0}\eta_{11}\ d^{0}V &= \int_{{}^{0}V} {}^{1}_{0}S_{ij}\delta_{0}\eta_{ij}\ d^{0}V = \delta\hat{\boldsymbol{u}}\left(\int_{{}^{0}S} {}^{1}_{0}\boldsymbol{B_{NL}}^{T}\ {}^{1}_{0}\boldsymbol{S}A\ {}^{1}_{0}\boldsymbol{B_{NL}}\ d^{0}S\right)\hat{\boldsymbol{u}} \\ \int_{{}^{0}V} {}^{1}_{0}S_{ij}\delta_{0}e_{ij}\ d^{0}V &= \int_{{}^{0}V} {}^{1}_{0}S_{11}\delta_{0}e_{11}\ d^{0}V = \delta\hat{\boldsymbol{u}}\left(\int_{{}^{0}S} \boldsymbol{B_L}^{T}\ {}^{1}_{0}\hat{\boldsymbol{S}}A\ d^{0}S\right) \end{aligned} \tag{4.464} $$

The integration of above expressions is generally performed using Gauss integration. ■

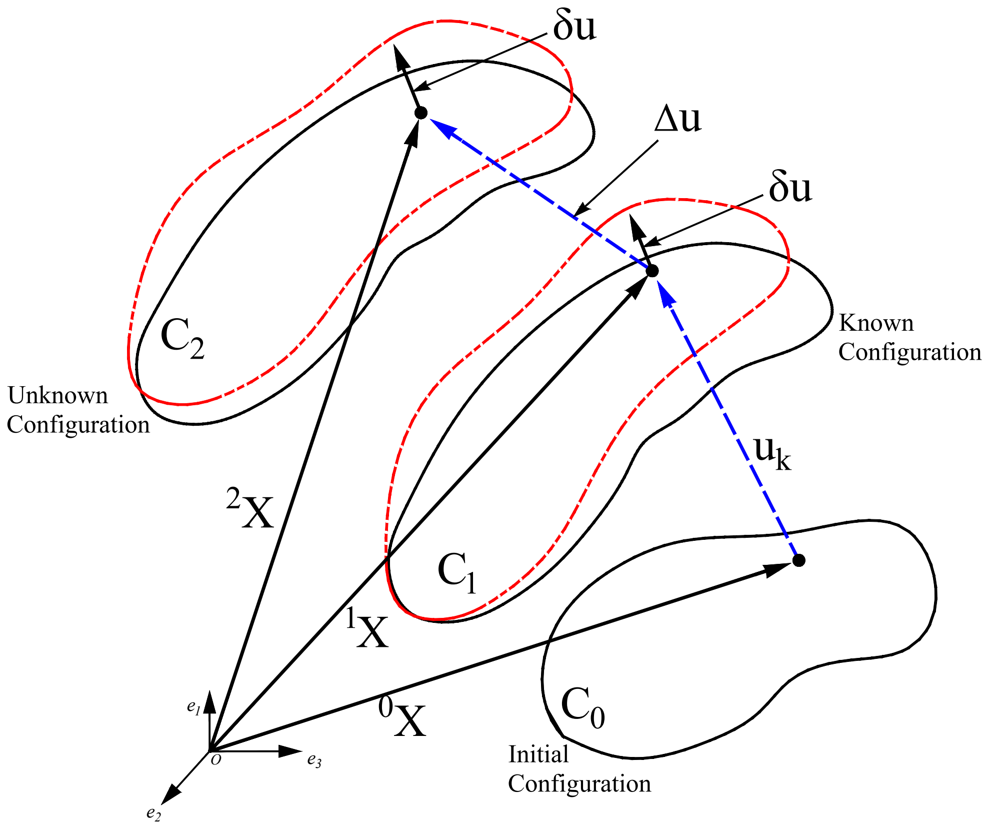


Figure 4.69

Newton Raphson linearization

Assume a body shown in Figure 4.69 with initial configuration C_0, and it is required to find its equilibrium configuration under the applied external forces. The principle of virtual work in terms of second Piola Kirchhoff stress tensor states that:

$$\delta W(\boldsymbol{u},\delta\boldsymbol{u}) = \int_V \boldsymbol{S}{:}\delta\boldsymbol{E}\ dV - \int_V \boldsymbol{f}^{*}.\delta\boldsymbol{u}\ dV - \int_{S_\Gamma} \boldsymbol{t}.\delta\boldsymbol{u}\ dA = 0 \tag{4.465}$$

Assuming a trial solution $\boldsymbol{u}_k$ and using Taylor series to evaluate incremental solution $\triangle u$ that makes variation in total potential vanish as follows:

$$\delta W(\boldsymbol{u}_k+\triangle\boldsymbol{u},\delta\boldsymbol{u}) = \delta W(\boldsymbol{u}_k,\delta\boldsymbol{u}) + D\delta W(\boldsymbol{u}_k,\delta\boldsymbol{u})[\triangle\boldsymbol{u}] + \text{higher order terms} = 0 \tag{4.466}$$

Where $D\delta W(\boldsymbol{u}_k,\delta\boldsymbol{u})[\triangle\boldsymbol{u}]$ represents the directional derivative of virtual work in direction $\triangle\boldsymbol{u}$. We need to note that, in the first term of the above equation ($\delta W(\boldsymbol{u}_k+\triangle\boldsymbol{u},\delta\boldsymbol{u})$), $\boldsymbol{u}$ is changed to $\boldsymbol{u}_k+\triangle\boldsymbol{u}$, while solution variation $\delta\boldsymbol{u}$ is not as shown in Figure 4.69. See 4.7 and 4.8 for further explanation. Linearization means neglecting higher order terms and the above expression reduces to:

$$\delta W(\boldsymbol{u},\delta\boldsymbol{u}) + D\delta W(\boldsymbol{u},\delta\boldsymbol{u})[\triangle\boldsymbol{u}] = 0 \tag{4.467}$$

Assuming the external forces is deformation independent during incremental displacement $\triangle\boldsymbol{u}$, the directional derivative of virtual work in direction $\triangle\boldsymbol{u}$ will correspond only to the internal virtual work defined as:

$$\delta W_{int}(\boldsymbol{u},\delta\boldsymbol{u}) = \int_V \boldsymbol{S}{:}\delta\boldsymbol{E}\ dV \tag{4.468}$$

With directional derivative defined as:

$$D\delta W_{int}(\boldsymbol{u},\delta\boldsymbol{u})[\triangle\boldsymbol{u}] = \int_V D\boldsymbol{S}[\triangle\boldsymbol{u}]{:}\delta\boldsymbol{E}\ dV + \int_V \boldsymbol{S}{:}D\delta\boldsymbol{E}[\triangle\boldsymbol{u}]\ dV \tag{4.469}$$

Using Equation 4.118 and Equation 4.122, we can get the following

$$\begin{aligned}
\delta\boldsymbol{E} &= \frac{1}{2}\left(\boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{\nabla}_0(\delta\boldsymbol{u})\right)\\
D(\delta\boldsymbol{E})[\Delta\boldsymbol{u}] &= \frac{1}{2}\left(\boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\Delta\boldsymbol{F} + \Delta\boldsymbol{F}^T\boldsymbol{\nabla}_0(\delta\boldsymbol{u})\right)\\
&= \frac{1}{2}\left(\boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\Delta\boldsymbol{u}) + \boldsymbol{\nabla}_0(\Delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\delta\boldsymbol{u})\right)
\end{aligned} \tag{4.470}$$

In the above expression, we used $\triangle(\delta\boldsymbol{u}) = 0$ as the variation $\delta\boldsymbol{u}$ remains the same after incremental displacement $\triangle\boldsymbol{u}$ as state before in Figure 4.69. From symmetry of second Piola Kirchhoff stress tensor and using Equation 1.100, it results in:

$$\boldsymbol{S}{:}\left(\frac{1}{2}\left(\boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\Delta\boldsymbol{u}) + \boldsymbol{\nabla}_0(\Delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\delta\boldsymbol{u})\right)\right) = \boldsymbol{S} : \boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\Delta\boldsymbol{u}) \tag{4.471}$$

Using constitutive stress-strain relation $D\boldsymbol{S}[\triangle\boldsymbol{u}] = \boldsymbol{C} : \triangle\boldsymbol{E}$ and Equation 4.469 results in:

$$D\delta W_{int}(\boldsymbol{u},\delta\boldsymbol{u})[\triangle\boldsymbol{u}] = \int_V \triangle\boldsymbol{E}[\triangle\boldsymbol{u}] : \boldsymbol{C} : \delta\boldsymbol{E}\ dV + \int_V \boldsymbol{S} : \boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\Delta\boldsymbol{u})\ dV \tag{4.472}$$

The first term $\int_V \triangle\boldsymbol{E}[\triangle\boldsymbol{u}] : \boldsymbol{C} : \delta\boldsymbol{E}\ dV$ signifies the source of material stiffness, while second one $\int_V \boldsymbol{S} : \boldsymbol{\nabla}_0(\delta\boldsymbol{u})^T\boldsymbol{\nabla}_0(\Delta\boldsymbol{u})\ dV$ represents the geometric stiffness of the body. The above expression

gives identical findings to the one used in Lagrangian formulation Equation 4.421 in the previous section. If the external force is deformation dependent (changes with body deformation), it will contribute to the directional derivative and produce what is called load stiffness matrix (see Appendix 4.5.5).
Also, virtual work principle can be rewritten in terms of first Piola Kirchhoff stress tensor as follows:

$$\delta W(\boldsymbol{u}, \delta \boldsymbol{u}) = \int_V \boldsymbol{P}:\delta \boldsymbol{F}\, dV - \int_V \boldsymbol{f}^*.\delta \boldsymbol{u}\, dV - \int_{S_\Gamma} \boldsymbol{t}.\delta \boldsymbol{u}\, dA = 0 \tag{4.473}$$

In this case, the direction derivative of internal virtual work in direction of $\triangle \boldsymbol{u}$ will be:

$$D\delta W_{int}(\boldsymbol{u}, \delta \boldsymbol{u})[\triangle \boldsymbol{u}] = \int_V D\boldsymbol{P}[\triangle \boldsymbol{u}]:\delta \boldsymbol{F}\, dV + \int_V \boldsymbol{P}:D\delta \boldsymbol{F}[\triangle \boldsymbol{u}]\, dV \tag{4.474}$$

Using constitutive stress-strain relation $D\boldsymbol{P}[\triangle \boldsymbol{u}] = \underline{\boldsymbol{C}} : \triangle \boldsymbol{F}$ and Equation 4.118, the above equation will be:

$$D\delta W_{int}(\boldsymbol{u}, \delta \boldsymbol{u})[\triangle \boldsymbol{u}] = \int_V \triangle \boldsymbol{F}:\underline{\boldsymbol{C}} : \delta \boldsymbol{F}\, dV + \int_V \boldsymbol{P} : D\left(\boldsymbol{\nabla}_0 \left(\delta \boldsymbol{u}\right)\right)[\triangle \boldsymbol{u}]\, dV \tag{4.475}$$

As stated before, the virtual displacement $\delta \boldsymbol{u}$ does not change during the incremental displacement $\triangle \boldsymbol{u}$, it yields $D(\delta \boldsymbol{u})[\triangle \boldsymbol{u}] = 0$ which forces the second term to vanish and the above equation reduces to:

$$D\delta W_{int}(\boldsymbol{u}, \delta \boldsymbol{u})[\triangle \boldsymbol{u}] = \int_V \triangle \boldsymbol{F}:\underline{\boldsymbol{C}} : \delta \boldsymbol{F}\, dV \tag{4.476}$$

4.5.4 Co-rotational approach

The main purpose of co-rotational formulation is to decompose the body displacement into a rigid body and pure deformation parts. The pure deformation part is responsible for the internal forces. It is measured with respect to element triad as stated in subsection 2.3.1. The merit of using co-rotational approach is to separate material and geometric nonlinearities in deriving formulations for internal forces and tangent stiffness.

For a two-node beam element Figure 2.44 as stated in subsection 2.3.1 variation in the natural deformation measured with respect to the moving (element) triad $\boldsymbol{E}$ is defined as

$$[\delta \boldsymbol{d}_l] = \boldsymbol{B}[\delta \boldsymbol{d}_g] \tag{4.477}$$

The natural deformation is responsible for the internal forces in the local coordinate system $\boldsymbol{f}_l$ and local tangent stiffness $\boldsymbol{K}_l$, while the internal forces calculated in the global coordinate system $\boldsymbol{f}_g$ can be calculated through equating the variational work performed by two forces through its corresponding displacement as follows:

$$\delta W = \delta \boldsymbol{d}_l \boldsymbol{f}_l = \delta \boldsymbol{d}_g \boldsymbol{f}_g \tag{4.478}$$

We note that rotations in $\delta \boldsymbol{d}_l$ and $\delta \boldsymbol{d}_g$ are incremental spin (non-additive rotation $\delta \boldsymbol{\phi}$), as the moment is work conjugate to the incremental spin not the change in rotation vector (additive rotation vector) $\delta \boldsymbol{\theta}$. From Equation 4.477, the local and global internal forces are related through:

$$\boldsymbol{f}_g = \boldsymbol{B}^T \boldsymbol{f}_l \tag{4.479}$$

The global tangent stiffness $\boldsymbol{K}_g$ will be defined from the variation of the global internal forces with respect to the global displacement as follows:

$$\delta \boldsymbol{f}_g = \boldsymbol{B}^T \delta \boldsymbol{f}_l = \boldsymbol{K}_g \delta \boldsymbol{d}_g + \delta\left(\boldsymbol{B}^T \boldsymbol{f}_l\right)|_{constant\ \boldsymbol{f}_l} \tag{4.480}$$

Where

$$\delta \boldsymbol{f}_l = \boldsymbol{K}_l \delta \boldsymbol{d}_l = \boldsymbol{K}_l \boldsymbol{B} \delta \boldsymbol{d}_g \tag{4.481}$$

$$\delta \left(\boldsymbol{B}^T \boldsymbol{f}_l\right) |_{constant\ \boldsymbol{f}_l} = \frac{\partial \left(\boldsymbol{B}^T \boldsymbol{f}_l\right)}{\partial \boldsymbol{d}_g} \delta \boldsymbol{d}_g \tag{4.482}$$

So the resulting general stiffness matrix will be:

$$\boldsymbol{K}_g = \boldsymbol{B}^T \boldsymbol{K}_l \boldsymbol{B} + \left. \frac{\partial \left(\boldsymbol{B}^T \boldsymbol{f}_l\right)}{\partial \boldsymbol{d}_g} \right|_{constant\ \boldsymbol{f}_l} \tag{4.483}$$

Two dimensional beam element

If we have two dimensional beam element as mentioned in subsection 2.3.1, using Equation 2.319 and Equation 2.319 and for , the second term of tangent stiffness will be:

$$\delta \left(\boldsymbol{B}^T \boldsymbol{f}_l\right) |_{constant\ \boldsymbol{f}_l} = \delta \boldsymbol{b}_1^T \overline{n} - \delta \boldsymbol{b}_2^T \left(\overline{m_1} + \overline{m_2}/l_n\right) \tag{4.484}$$

Where the local internal forces $\boldsymbol{f}_l$ include the beam axial force $\overline{n}$ and end moments $\overline{m_1}$ and $\overline{m_2}$ and defined as follows:

$$[\boldsymbol{f}_l] = \begin{bmatrix} \overline{n} & \overline{m_1} & \overline{m_2} \end{bmatrix}^T \tag{4.485}$$

The variation in $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ can be evaluated as follows:

$$\delta \boldsymbol{b}_1^T = \boldsymbol{b}_2^T \delta \beta = \frac{1}{l_n} \boldsymbol{b}_2^T \boldsymbol{b}_2 \ \delta \boldsymbol{d}_g \tag{4.486}$$

$$\delta \boldsymbol{b}_2^T = -\boldsymbol{b}_1^T \delta \beta = -\frac{1}{l_n} \boldsymbol{b}_1^T \boldsymbol{b}_2 \ \delta \boldsymbol{d}_g + \frac{\partial \boldsymbol{b}_2^T}{\partial l_n} \delta l_n = -\frac{1}{l_n} \boldsymbol{b}_1^T \boldsymbol{b}_2 \ \delta \boldsymbol{d}_g - \boldsymbol{b}_2^T \boldsymbol{b}_1 \delta \boldsymbol{d}_g \tag{4.487}$$

So, we get:

$$\delta \left(\boldsymbol{B}^T \boldsymbol{f}_l\right) |_{constant\ \boldsymbol{f}_l} = \frac{1}{l_n} \left(\boldsymbol{b}_2^T \boldsymbol{b}_2 \overline{n} + \left(\boldsymbol{b}_1^T \boldsymbol{b}_2 + \boldsymbol{b}_2^T \boldsymbol{b}_1\right) \frac{(\overline{m_1} + \overline{m_2})}{l_n} \right) \delta \boldsymbol{d}_g \tag{4.488}$$

And the resulting stiffness matrix will be:

$$\boldsymbol{K}_g = \boldsymbol{B}^T \boldsymbol{K}_l \boldsymbol{B} + \frac{1}{l_n} \left(\boldsymbol{b}_2^T \boldsymbol{b}_2 \overline{n} + \left(\boldsymbol{b}_1^T \boldsymbol{b}_2 + \boldsymbol{b}_2^T \boldsymbol{b}_1\right) \frac{(\overline{m_1} + \overline{m_2})}{l_n} \right) \tag{4.489}$$

Three dimensional beam element

As stated in Equation 2.355, the variation in natural deformation is related to the variation of the global displacement through the following:

$$[\delta \boldsymbol{d}_l] = \boldsymbol{P} \boldsymbol{E}_4^T [\delta \boldsymbol{d}_g]^{\boldsymbol{I}} \tag{4.490}$$

So the following term will be:

$$\begin{aligned} \delta \left(\boldsymbol{B}^T \boldsymbol{f}_l\right) |_{constant\ \boldsymbol{f}_l} &= \delta \left(\boldsymbol{E}_4 [\boldsymbol{r}^T \ \ \boldsymbol{P}_1^T \ \ \boldsymbol{P}_2^T] \begin{bmatrix} \overline{n} \\ \overline{\boldsymbol{m}_1} \\ \overline{\boldsymbol{m}_2} \end{bmatrix} \right) \\ &= \delta \left(\boldsymbol{E}_4\right) \left([\boldsymbol{r}^T \ \ \boldsymbol{P}_1^T \ \ \boldsymbol{P}_2^T] \begin{bmatrix} \overline{n} \\ \overline{\boldsymbol{m}_1} \\ \overline{\boldsymbol{m}_2} \end{bmatrix} \right) + \boldsymbol{E}_4 \left(\delta \left(\boldsymbol{r}^T \overline{n}\right) + \delta \left(\boldsymbol{P}_1^T \overline{\boldsymbol{m}_1}\right) + \delta \left(\boldsymbol{P}_2^T \overline{\boldsymbol{m}_2}\right) \right) \end{aligned} \tag{4.491}$$

Where

$$\delta\left(\boldsymbol{r}^T\overline{n}\right) = 0 \tag{4.492}$$

$$\delta\boldsymbol{P}_1^T = \delta\boldsymbol{P}_2^T = -\delta\boldsymbol{A}^T \tag{4.493}$$

As $\boldsymbol{A}$ is function only of the beam length, we get:

$$\delta\boldsymbol{A}^T = \frac{\partial\boldsymbol{A}^T}{\partial l_n}\delta l_n = \frac{\partial\boldsymbol{A}^T}{\partial l_n}\boldsymbol{r}E_4^T\left[\delta\boldsymbol{d}_g\right]^{\boldsymbol{I}} \tag{4.494}$$

Where

$$\frac{\partial\boldsymbol{A}}{\partial l_n} = \frac{1}{{l_n}^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.495}$$

So we get the following:

$$\delta\left(\boldsymbol{P}_1^T\overline{\boldsymbol{m}_1}\right) + \delta\left(\boldsymbol{P}_2^T\overline{\boldsymbol{m}_2}\right) = \frac{\partial\boldsymbol{A}^T}{\partial l_n}(\overline{\boldsymbol{m}_1}+\overline{\boldsymbol{m}_2})\boldsymbol{r}E_4^T\left[\delta\boldsymbol{d}_g\right] \tag{4.496}$$

As $\delta l_n = \boldsymbol{r}E_4^T\left[\delta\boldsymbol{d}_g\right]$ is a scalar term it can be flipped with any vector or tensor terms. assuming that:

$$\begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{M}_1 \\ \boldsymbol{N}_2 \\ \boldsymbol{M}_2 \end{bmatrix} = \left([\boldsymbol{r}^T \;\; \boldsymbol{P}_1^T \;\; \boldsymbol{P}_2^T] \begin{bmatrix} \overline{n} \\ \overline{\boldsymbol{m}_1} \\ \overline{\boldsymbol{m}_2} \end{bmatrix} \right) \tag{4.497}$$

And from

$$[\delta\boldsymbol{E}_4]^{\boldsymbol{I}} = [\delta\widetilde{\boldsymbol{\phi}_{e4}}]^{\boldsymbol{I}}\boldsymbol{E}_4 = \boldsymbol{E}_4[\delta\widetilde{\boldsymbol{\phi}_{e4}}]^{\boldsymbol{E}}\boldsymbol{E}_4^T\boldsymbol{E}_4 = \boldsymbol{E}_4[\delta\widetilde{\boldsymbol{\phi}_{e4}}]^{\boldsymbol{E}} \tag{4.498}$$

Where

$$\delta\widetilde{\boldsymbol{\phi}_{e4}} = \begin{bmatrix} \delta\widetilde{\boldsymbol{\phi}_e} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \delta\widetilde{\boldsymbol{\phi}_e} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \delta\widetilde{\boldsymbol{\phi}_e} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \delta\widetilde{\boldsymbol{\phi}_e} \end{bmatrix} \tag{4.499}$$

and from Equation 2.351, We get

$$[\delta\boldsymbol{E}_4]^{\boldsymbol{I}}\begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{M}_1 \\ \boldsymbol{N}_2 \\ \boldsymbol{M}_2 \end{bmatrix} = \boldsymbol{E}_4[\delta\widetilde{\boldsymbol{\phi}_{e4}}]^{\boldsymbol{E}}\begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{M}_1 \\ \boldsymbol{N}_2 \\ \boldsymbol{M}_2 \end{bmatrix} \tag{4.500}$$

$$= -\boldsymbol{E}_4\begin{bmatrix} \tilde{\boldsymbol{N}}_1 \\ \tilde{\boldsymbol{M}}_1 \\ \tilde{\boldsymbol{N}}_2 \\ \tilde{\boldsymbol{M}}_2 \end{bmatrix}[\delta\boldsymbol{\phi}_e]^{\boldsymbol{E}} \tag{4.501}$$

$$= -\boldsymbol{E}_4\begin{bmatrix} \tilde{\boldsymbol{N}}_1 \\ \tilde{\boldsymbol{M}}_1 \\ \tilde{\boldsymbol{N}}_2 \\ \tilde{\boldsymbol{M}}_2 \end{bmatrix}\boldsymbol{A}\boldsymbol{E}_4^T[\delta\boldsymbol{d}_g]^{\boldsymbol{I}} \tag{4.502}$$

So the stiffness matrix will be:

$$\boldsymbol{K}_g = \boldsymbol{E}_4 \left(\boldsymbol{P}^T \boldsymbol{K}_l \boldsymbol{P} + \frac{\partial \boldsymbol{A}^T}{\partial l_n} (\overline{\boldsymbol{m}_1} + \overline{\boldsymbol{m}_2}) \boldsymbol{r} - \begin{bmatrix} \tilde{\boldsymbol{N}}_1 \\ \tilde{\boldsymbol{M}}_1 \\ \tilde{\boldsymbol{N}}_2 \\ \tilde{\boldsymbol{M}}_2 \end{bmatrix} \boldsymbol{A} \right) \boldsymbol{E}_4^T = \boldsymbol{E}_4 (\boldsymbol{K}_L) \boldsymbol{E}_4^T \tag{4.503}$$

Where $\boldsymbol{P}$, and $\boldsymbol{A}$ are defined in Equation 2.358 and Equation 2.343, respectively, and $\boldsymbol{K}_L$ defines the total local stiffness as follows:

$$\boldsymbol{K}_L = \boldsymbol{P}^T \boldsymbol{K}_l \boldsymbol{P} + \frac{\partial \boldsymbol{A}^T}{\partial l_n} (\overline{\boldsymbol{m}_1} + \overline{\boldsymbol{m}_2}) \boldsymbol{r} - \begin{bmatrix} \tilde{\boldsymbol{N}}_1 \\ \tilde{\boldsymbol{M}}_1 \\ \tilde{\boldsymbol{N}}_2 \\ \tilde{\boldsymbol{M}}_2 \end{bmatrix} \boldsymbol{A} \tag{4.504}$$

4.5.5 Mixed finite element

For a linear elastic body subjected to body force $\boldsymbol{f}^{\boldsymbol{B}}$ with constrained boundary at S_U. The remaining free boundary S_Γ is subjected to traction forces $\boldsymbol{F}^{S_\Gamma}$, such that the total potential energy is defined as:

$$\Pi = \frac{1}{2} \int_V \boldsymbol{\varepsilon}^T \boldsymbol{C} \boldsymbol{\varepsilon} dV - \int_V \boldsymbol{u}^T \boldsymbol{f}^{\boldsymbol{B}} dV - \int_{S_\Gamma} \boldsymbol{u}^{S_\Gamma T} \boldsymbol{f}^{S_\Gamma} dA \tag{4.505}$$

with strain-displacement relation and displacement boundary conditions:

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial}_{\boldsymbol{\varepsilon}} \boldsymbol{u}, \quad \boldsymbol{u}|_{S_U} = \underline{\boldsymbol{u}} \tag{4.506}$$

where ∂_ε ia a differential operator on displacement $\boldsymbol{u}$ to get the strain components ε, $\underline{\boldsymbol{u}}$ represents the vector of prescribed displacements at S_U. In displacement-based finite element solution, the stationary of potential energy (with respect to the displacements) makes its variation on $\boldsymbol{u}$ that achieves the prescribed displacements to vanish. Also it should be noted that the solution variables are only displacements, while other variables like strains and stresses are evaluated in the post-processing stage. There are other extended variational principles that use not only displacements but also other variables such as stresses and/or strain as a primary variables so-called mixed finite element method. In this method, the variational principle is rewritten using Equation 4.505 and Equation 4.506 as follows:

$$\underline{\Pi} = \Pi - \int_V \boldsymbol{\lambda}_{\boldsymbol{\varepsilon}}{}^T (\boldsymbol{\varepsilon} - \boldsymbol{\partial}_{\boldsymbol{\varepsilon}} \boldsymbol{u}) \, dV - \int_{S_U} \boldsymbol{\lambda}_{\boldsymbol{U}}{}^T \left(\boldsymbol{u}^{S_U} - \underline{\boldsymbol{u}} \right) dv \tag{4.507}$$

Where $\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\lambda}_{\boldsymbol{U}}$ are considered as Lagrange multipliers which are implemented to insure the conditions Equation 4.506. To make sure that each term of the above equation has the same units, Lagrange multipliers $\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}}$ and $\boldsymbol{\lambda}_{\boldsymbol{U}}$ can be considered, respectively, as the stresses $\boldsymbol{\sigma}$ and traction stress vector over boundary S_U, $\boldsymbol{f}^{S_U}$, so the extended potential energy will be:

$$\underline{\Pi}_{HW} = \Pi - \int_V \boldsymbol{\sigma}^T (\boldsymbol{\varepsilon} - \boldsymbol{\partial}_{\boldsymbol{\varepsilon}} \boldsymbol{u}) \, dV - \int_{S_U} \boldsymbol{f}^{S_U T} \left(\boldsymbol{u}^{S_U} - \underline{\boldsymbol{u}} \right) dv \tag{4.508}$$

This potential functional is called Hu-Washizu functional. Stationary of this functional requires $\delta\underline{\Pi}_{HW} = 0$ as follows:

$$0 = \delta\underline{\Pi}_{HW} = \underbrace{\int_V \delta\boldsymbol{\varepsilon}^T (\boldsymbol{C}\boldsymbol{\varepsilon} - \boldsymbol{\tau})\, dV - \int_V \delta\boldsymbol{\sigma}^T (\boldsymbol{\varepsilon} - \boldsymbol{\partial}_{\varepsilon}\boldsymbol{u})\, dV + \int_V (\boldsymbol{\partial}_{\varepsilon}\delta\boldsymbol{u})^T \boldsymbol{\sigma}}_{\text{Stiffness terms}}$$
$$\underbrace{-\int_V \delta\boldsymbol{u}^T \boldsymbol{f}^{\boldsymbol{B}} dV}_{\text{Body force terms}} \underbrace{- \int_V \delta\boldsymbol{u}^T \boldsymbol{f}^{\boldsymbol{B}} dV - \int_{S_\Gamma} \delta\boldsymbol{u}^{S_\Gamma T} \boldsymbol{f}^{S_\Gamma} dA - \int_{S_U} \left(\delta\boldsymbol{u}^{S_U T} \boldsymbol{f}^{S_U} + \delta\boldsymbol{f}^{S_U T} \left(\boldsymbol{u}^{S_U} - \underline{\boldsymbol{u}}\right)\right) dA}_{\text{Boundary terms}} \quad (4.509)$$

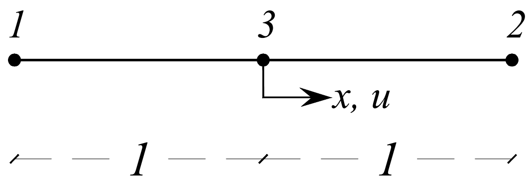


Figure 4.70

■ **Example 4.37** Assume a three-node truss element shown in Figure 4.70. Consider a parabolic approximation for displacement and linear approximation of stress and strain as follows

$$\boldsymbol{u}(x) = \boldsymbol{N}(x)\hat{\boldsymbol{u}} \quad (4.510)$$
$$\boldsymbol{\sigma}(x) = \underline{\boldsymbol{N}}(x)\hat{\boldsymbol{\sigma}} \quad (4.511)$$
$$\boldsymbol{\varepsilon}(x) = \underline{\boldsymbol{N}}(x)\hat{\boldsymbol{\varepsilon}} \quad (4.512)$$

Where

$$\boldsymbol{N} = \left[\frac{1}{2}(1+x)x \quad \frac{1}{2}(x-1)x \quad 1-x^2\right] \quad (4.513)$$
$$\underline{\boldsymbol{N}} = \left[\frac{1}{2}(1+x) \quad \frac{1}{2}(1-x)\right] \quad (4.514)$$
$$\hat{\boldsymbol{u}} = [u_1 \quad u_2 \quad u_3]^T \quad (4.515)$$
$$\hat{\boldsymbol{\sigma}} = [\sigma_1 \quad \sigma_2]^T \quad (4.516)$$
$$\hat{\boldsymbol{\varepsilon}} = [\varepsilon_1 \quad \varepsilon_2]^T \quad (4.517)$$

The stiffness part of Equation 4.509 will be: $\delta\hat{\boldsymbol{\varepsilon}}^T \left[\int_V \underline{\boldsymbol{N}}^T \boldsymbol{C}\underline{\boldsymbol{N}} dV\right] \hat{\boldsymbol{\varepsilon}} - \delta\hat{\boldsymbol{\varepsilon}}^T \left[\int_V \underline{\boldsymbol{N}}^T \underline{\boldsymbol{N}} dV\right] \hat{\boldsymbol{\sigma}}$
While the second term will be: $-\delta\hat{\boldsymbol{\sigma}}^T \left[\int_V \underline{\boldsymbol{N}}^T \underline{\boldsymbol{N}}\ dV\right] \hat{\boldsymbol{\varepsilon}} + \delta\hat{\boldsymbol{\sigma}}^T \left[\int_V \underline{\boldsymbol{N}}^T \boldsymbol{B} dV\right] \hat{\boldsymbol{u}}$
The third term is: $\delta\hat{\boldsymbol{u}}^T \left[\int_V \boldsymbol{B}^T \underline{\boldsymbol{N}}\ dV\right] \hat{\boldsymbol{\sigma}}$
(4.518)

Where

$$\boldsymbol{B} = \left[\left(\frac{1}{2} + x\right) \quad \left(x - \frac{1}{2}\right) \quad -2x\right] \quad (4.519)$$

The stiffness part of Equation 4.509 will be:

$$\begin{bmatrix} \delta\hat{\boldsymbol{u}} & \delta\hat{\boldsymbol{\varepsilon}} & \delta\hat{\boldsymbol{\sigma}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{K_{u\sigma}} \\ \mathbf{0} & \boldsymbol{K_{\varepsilon\varepsilon}} & \boldsymbol{K_{\varepsilon\sigma}} \\ \boldsymbol{K_{u\sigma}}^T & \boldsymbol{K_{\varepsilon\sigma}}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{\varepsilon}} \\ \hat{\boldsymbol{\sigma}} \end{bmatrix} \tag{4.520}$$

Where

$$\boldsymbol{K_{\varepsilon\varepsilon}} = \int_V \underline{\boldsymbol{N}}^T \boldsymbol{C} \underline{\boldsymbol{N}} dV \tag{4.521}$$

$$\boldsymbol{K_{u\sigma}} = \int_V \boldsymbol{B}^T \underline{\boldsymbol{N}} dV \tag{4.522}$$

$$\boldsymbol{K_{\varepsilon\sigma}} = \int_V \underline{\boldsymbol{N}}^T \underline{\boldsymbol{N}} dV \tag{4.523}$$

substituting for $\boldsymbol{B}$ and $\underline{\boldsymbol{N}}$ in Equation 4.523 results in

$$\boldsymbol{K_{\varepsilon\varepsilon}} = \tfrac{EA}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{K_{u\sigma}} = \tfrac{A}{6}\begin{bmatrix} 5 & 1 \\ -1 & -5 \\ -4 & 4 \end{bmatrix}, \quad \boldsymbol{K_{\varepsilon\sigma}} = \tfrac{A}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

By eliminating the stress and strain degree of freedom ($\hat{\boldsymbol{\varepsilon}}$, $\hat{\boldsymbol{\sigma}}$), Equation 4.520 becomes:

$$[\delta u_1 \quad \delta u_2 \quad \delta u_3]^T \left(\frac{EA}{6}\right) \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \to K = \frac{EA}{6} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \tag{4.524}$$

The stiffness matrix obtained above is identical to the one obtained from displacement-based truss element Equation 4.304 as it assumes a parabolic interpolation for displacement Equation 4.301 and consequently a linear strain distribution in Equation 4.302.
The degree of interpolation for each degree of freedom should be 'wisely' chosen. ■

■ **Example 4.38** If we assume a parabolic displacement, linear strain, and constant stress assumptions, the interpolation functions will be:

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{N}\hat{\boldsymbol{u}}; \qquad \boldsymbol{\varepsilon}(x) = \underline{\boldsymbol{N}}(x)\hat{\boldsymbol{\varepsilon}} \qquad \text{where } \boldsymbol{N}, \underline{\boldsymbol{N}} \text{ are as stated before} \\ &\boldsymbol{\varepsilon}(x) = \boldsymbol{T}(x)\hat{\boldsymbol{\varepsilon}}; \quad \hat{\boldsymbol{\sigma}} = \sigma_3; \qquad \boldsymbol{T} = [1] \end{aligned} \tag{4.525}$$

with

$$\boldsymbol{K_{\varepsilon\varepsilon}} = \textstyle\int_V \underline{\boldsymbol{N}}^T \boldsymbol{C} \underline{\boldsymbol{N}} dV, \qquad \boldsymbol{K_{u\sigma}} = \int_V \boldsymbol{B}^T \boldsymbol{T} dV, \qquad \boldsymbol{K_{\varepsilon\sigma}} = \int_V \underline{\boldsymbol{N}}^T \boldsymbol{T} dV$$

$$\boldsymbol{K_{\varepsilon\varepsilon}} = \tfrac{EA}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{K_{u\sigma}} = \begin{bmatrix} A \\ -A \\ 0 \end{bmatrix}, \quad \boldsymbol{K_{\varepsilon\sigma}} = \begin{bmatrix} -A \\ -A \end{bmatrix}$$

And the resulting stiffness will be:

$$K = \frac{EA}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.526}$$

The resulting element reduces to a two-node truss displacement-based element Equation 4.298 which is not sufficient for a three-node truss. ■

The extended variational principle can include only displacements and strains as primary variables unlike Hu-Washizu functional which includes stress as unknown variable in addition to displacements and strains. This functional is called Hellinger-Reissner proved from Equation 4.508 and using $\boldsymbol{\varepsilon} = \boldsymbol{C}^{-1}\boldsymbol{\sigma}$ as follows:

$$\underline{\Pi}_{HR} = \int_V \left(-\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\sigma}^T\boldsymbol{\partial_\varepsilon u} - \boldsymbol{u}^T\boldsymbol{f_b}\right) dV - \underbrace{\int_{S_u} \boldsymbol{f}^{S_f\,T}\boldsymbol{u}^{S_f}\,dA - \int_{S_u} \boldsymbol{f}^{S_u\,T}\left(\boldsymbol{u}^{S_u} - \underline{\boldsymbol{u}}\right) dA}_{\text{Boundary terms}} \quad (4.527)$$

Applying divergence theorem on the second term results in:

$$\int_V \boldsymbol{\sigma}^T\boldsymbol{\partial_\varepsilon u}\,dV = \int_S \boldsymbol{\varepsilon u}^T\left(\boldsymbol{\sigma n}\right) dV - \int_V \boldsymbol{\partial_\varepsilon \sigma}^T\boldsymbol{u}\,dV \quad (4.528)$$

Where $S = S_\Gamma + S_U$. Including the stationary of potential functional leads to:

$$\begin{aligned} 0 = & \delta\underline{\Pi}_{HR} = \int_V \left[\delta\boldsymbol{\sigma}^T\left(-\boldsymbol{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\partial_\varepsilon u}\right) - \delta\boldsymbol{u}^T\left(\boldsymbol{\partial_\varepsilon \sigma} + \boldsymbol{f^b}\right)\right] dV \\ & - \int_{S_\Gamma} \delta\boldsymbol{u}^{S_\Gamma\,T}\left(\boldsymbol{f}^{S_\Gamma} - \boldsymbol{\sigma n}\right) dA - \int_{S_U}\left[\delta\boldsymbol{u}^{S_U\,T}\left(\boldsymbol{f}^{S_U} - \boldsymbol{\sigma n}\right) + \delta\boldsymbol{f}^{S_U\,T}\left(\boldsymbol{u}^{S_U} - \underline{\boldsymbol{u}}\right)\right] dA \end{aligned} \quad (4.529)$$

We get:

$$\begin{array}{lll} \text{Stress-strain relation} & \boldsymbol{\partial_\varepsilon u} = \boldsymbol{C}^{-1}\boldsymbol{\sigma} & \text{on } V \\ \text{Equilibruim equation} & \boldsymbol{\partial_\varepsilon \sigma} + \boldsymbol{f^b} = 0 & \text{on } V \\ \text{prescribed tractions} & \boldsymbol{f}^{S_\Gamma} = \boldsymbol{\sigma n} & \text{on } S_\Gamma \\ \text{Boundary equilibruim} & \boldsymbol{f}^{S_U} = \boldsymbol{\sigma n} & \text{on } S_U \\ \text{prescribed displacements} & \boldsymbol{u}^{S_U} = \underline{\boldsymbol{u}} & \text{on } S_U \end{array} \quad (4.530)$$

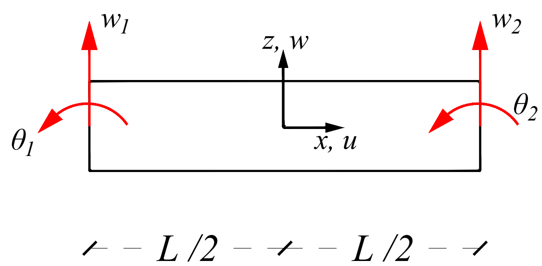


Figure 4.71

■ **Example 4.39** Assume a two-node Timoshenko beam shown in Figure 4.71. Consider the transverse displacement and beam rotation are distributed linearly, while shear strain is constant over the beam $\hat{\gamma}_{xz}$

$$w = \boldsymbol{N}\hat{\boldsymbol{w}} \quad \theta = \boldsymbol{N}\hat{\boldsymbol{\theta}}; \quad \boldsymbol{N} = \left[\ \left(\tfrac{1}{2} - \tfrac{x}{L}\right) \quad \left(\tfrac{1}{2} + \tfrac{x}{L}\right)\ \right] \quad (4.531)$$

$$u = -z\theta = -z\boldsymbol{N}\hat{\boldsymbol{\theta}} \quad (4.532)$$

The assumed strains are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} = -z\boldsymbol{B}\hat{\boldsymbol{\theta}} \tag{4.533}$$

$$\gamma_{xz} = \hat{\gamma} \tag{4.534}$$

While applying operator on displacement $\boldsymbol{u}$ results in:

$$\overline{\varepsilon}_{xx} = \frac{\partial u}{\partial x} = -z\boldsymbol{B} = \varepsilon_{xx}; \quad \overline{\gamma}_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \boldsymbol{N}\hat{\boldsymbol{\theta}} - \boldsymbol{B}\hat{\boldsymbol{w}} \tag{4.535}$$

Neglecting the boundary terms in Equation 4.527 to be:

$$\int_V \left(-\frac{1}{2} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xz} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{bmatrix} + \begin{bmatrix} \varepsilon_{xx} & \gamma_{xz} \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \overline{\gamma}_{xz} \end{bmatrix} \right) dV = \int_V \boldsymbol{u}^T \boldsymbol{f^b} dV \tag{4.536}$$

Taking the variation of above equation results in

$$\int_V \delta\varepsilon_{xx} E \varepsilon_{xx} + \delta\overline{\gamma}_{xz} G \gamma_{xz} + \delta\gamma_{xz} G \left(\overline{\gamma}_{xz} - \gamma_{xz}\right) dV = \int_V \delta\boldsymbol{u}^T \boldsymbol{f^b} dV \tag{4.537}$$

substituting Equation 4.535 into the above equation results in:

$$\begin{aligned} &\int_V \left(\delta\hat{\boldsymbol{\theta}}^T \boldsymbol{B}^T E \boldsymbol{B}\hat{\boldsymbol{\theta}} + \delta\hat{\boldsymbol{\theta}}^T \boldsymbol{N}^T G\hat{\boldsymbol{\gamma}} - \delta\hat{\boldsymbol{w}}^T \boldsymbol{B^T} \boldsymbol{G}\hat{\boldsymbol{\gamma}} + \hat{\boldsymbol{\gamma}}^T G\boldsymbol{N}\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\gamma}}^T G\boldsymbol{B}\hat{\boldsymbol{w}} - \hat{\boldsymbol{\gamma}}^T G\hat{\boldsymbol{\gamma}} \right) dV \\ &= \int_V \delta\boldsymbol{u}^T \boldsymbol{f^b} dV \\ &\begin{bmatrix} \delta\hat{\boldsymbol{w}} & \delta\hat{\boldsymbol{\theta}} & \delta\hat{\boldsymbol{\gamma}} \end{bmatrix} \left(\int_V \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}^T G \\ \boldsymbol{0} & \boldsymbol{B}^T E\boldsymbol{B} & \boldsymbol{N}^T G \\ -G\boldsymbol{B} & G\boldsymbol{N} & -G \end{bmatrix} dV \right) \begin{bmatrix} \hat{\boldsymbol{w}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} \\ &= \begin{bmatrix} \delta\hat{\boldsymbol{w}} & \delta\hat{\boldsymbol{\theta}} & \delta\hat{\boldsymbol{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{Q}} \\ \hat{\boldsymbol{M}} \\ 0 \end{bmatrix} \end{aligned} \tag{4.538}$$

The resulting stiffness will be:

$$K = \int_V \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{B}^T G \\ \boldsymbol{0} & \boldsymbol{B}^T E\boldsymbol{B} & \boldsymbol{N}^T G \\ -G\boldsymbol{B} & G\boldsymbol{N} & -G \end{bmatrix} dV = \begin{bmatrix} 0 & 0 & 0 & 0 & -GA \\ 0 & 0 & 0 & 0 & GA \\ 0 & 0 & EI/L & -EI/L & GAL/2 \\ 0 & 0 & -EI/L & EI/L & GAL/2 \\ -GA & GA & GAL/2 & GAL/2 & -GAL \end{bmatrix} \tag{4.539}$$

Applying static condensation on $\hat{\boldsymbol{\gamma}}$ in Equation 4.538, the resulting stiffness matrix will be:

$$K = \begin{bmatrix} \frac{GA}{L} & -\frac{GA}{L} & -\frac{GA}{2} & \frac{GA}{2} \\ -\frac{GA}{L} & \frac{GA}{L} & -\frac{GA}{2} & \frac{GA}{2} \\ -\frac{GA}{2} & -\frac{GA}{2} & \frac{GAL}{4} + \frac{EI}{L} & \frac{GAL}{4} - \frac{EI}{L} \\ \frac{GA}{2} & \frac{GA}{2} & \frac{GAL}{4} - \frac{EI}{L} & \frac{GAL}{4} + \frac{EI}{L} \end{bmatrix} \tag{4.540}$$

If we assumed a linear variation in transverse shear strain γ_{xz} instead of the constant one assumed in Equation 4.534 and repeated the above equations with new assumed γ_{xz}, it results in:

$$K = \begin{bmatrix} \frac{GA}{L} & -\frac{GA}{L} & -\frac{GA}{2} & \frac{GA}{2} \\ -\frac{GA}{L} & \frac{GA}{L} & -\frac{GA}{2} & \frac{GA}{2} \\ -\frac{GA}{2} & -\frac{GA}{2} & \frac{GAL}{3}+\frac{EI}{L} & \frac{GAL}{6}-\frac{EI}{L} \\ \frac{GA}{2} & \frac{GA}{2} & \frac{GAL}{6}-\frac{EI}{L} & \frac{GAL}{3}+\frac{EI}{L} \end{bmatrix} \tag{4.541}$$

Which exhibits a stiffer behavior. This behavior exaggerates for thin elements (beam depth $<<$ its length), so this previous assumption results in shear locking as stated in subsection 4.5.2. Also the same stiffness matrix in Equation 4.541 will be obtained if we use the displacement-based finite element formulation.

Using last row of Equation 4.539 in conjunction with Equation 4.538 result that

$$GAL\left[\frac{w_2-w_1}{L}+\frac{\theta_1+\theta_2}{2}-\hat{\gamma}_{xz}\right]=0 \rightarrow -\hat{\gamma}_{xz}=\frac{w_2-w_1}{L}+\frac{\theta_1+\theta_2}{2} \tag{4.542}$$

■

The resulting assumed constant shear strain is equal to the shear strain at the beam midpoint if evaluated from Equation 4.535 at $x=0$.

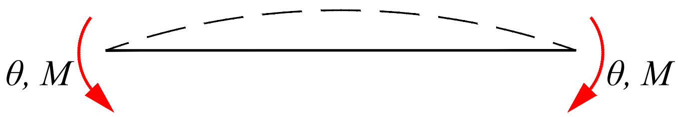


Figure 4.72

■ **Example 4.40** Assume the same above beam with only end moments M_1, M_2 and its corresponding rotations θ_1, θ_2 as shown in Figure 4.72. If the applied moments and rotations at beam ends are equal, the shear stresses and consequently shear strains vanish. Through theses displacement and based on the mixed finite element formulation in Equation 4.540, the resulting nodal forces $\boldsymbol{F}=\boldsymbol{K}\hat{\boldsymbol{u}}=[0\ \ \frac{E}{L}\theta\ \ 0-\frac{E}{L}\theta]^T$, while using pure displacement-based element results in $\boldsymbol{F}=[0\ \ \left(\frac{GAL}{6}+\frac{E}{L}\right)\theta\ \ 0\ \ -\left(\frac{GAL}{6}+\frac{E}{L}\right)\theta]^T$ which results in erroneous shear contribution or shear locking Figure 4.65 ■

Also mixed formulation is much more powerful than the traditional displacement-based finite element in constructing plate or shell finite element formulation and the analysis of incompressible media.

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Appendix A: Derivation of T

As the axis of rotation is not effected by rotation

$$\mathbf{R}\boldsymbol{\theta} = \boldsymbol{\theta} \tag{4.543}$$

$$\Delta\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta} \tag{4.544}$$

As $\Delta\mathbf{R} = \widetilde{\Delta\boldsymbol{\phi}}\mathbf{R}$

$$\widetilde{\Delta\boldsymbol{\phi}}\mathbf{R}\boldsymbol{\theta} + \mathbf{R}\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta} \tag{4.545}$$

$$\widetilde{\Delta\boldsymbol{\phi}}\boldsymbol{\theta} + \mathbf{R}\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta} \tag{4.546}$$

$$\Delta\boldsymbol{\phi} \times \boldsymbol{\theta} = (\mathbf{1} - \mathbf{R})\Delta\boldsymbol{\theta} \tag{4.547}$$

But, if $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$, it follows that $\boldsymbol{a} = \frac{\boldsymbol{b}\times\boldsymbol{c}}{|\boldsymbol{b}|^2} + \boldsymbol{\lambda}\boldsymbol{b}$, similarly:

$$\Delta\boldsymbol{\phi} = \frac{\tilde{\boldsymbol{\theta}}(\mathbf{1} - \mathbf{R})\Delta\theta}{|\boldsymbol{\theta}|^2} + \lambda\boldsymbol{\theta} \tag{4.548}$$

From Equation 2.54 formula, the trace of rotation tensor is:

$$\mathbf{R} : \mathbf{1} = 1 + 2\cos\theta \tag{4.549}$$

$$\Delta\mathbf{R} : \mathbf{1} = -2\sin\theta\Delta\theta = -2\sin\theta\frac{\boldsymbol{\theta}.\Delta\boldsymbol{\theta}}{\theta} \tag{4.550}$$

The last expression results from

$$\theta^2 = \boldsymbol{\theta}.\boldsymbol{\theta} \rightarrow 2\theta\Delta\theta = \boldsymbol{\theta}.\Delta\boldsymbol{\theta} + \Delta\boldsymbol{\theta}.\boldsymbol{\theta} = 2\boldsymbol{\theta}.\Delta\boldsymbol{\theta} \rightarrow \Delta\theta = \frac{\boldsymbol{\theta}.\Delta\boldsymbol{\theta}}{\theta}$$

From expressionEquation 2.89 and Equation 2.54, $\Delta\mathbf{R} = \widetilde{\Delta\boldsymbol{\phi}}\mathbf{R}$ and $skew\left(\mathbf{R}\right) = \frac{\sin\theta}{\theta}\widetilde{\boldsymbol{\theta}}$

$$\begin{aligned}
\Delta\mathbf{R} : \mathbf{1} &= \widetilde{\Delta\boldsymbol{\phi}}\mathbf{R} : \mathbf{1} \\
&= \widetilde{\Delta\boldsymbol{\phi}} : \mathbf{R}^T \\
&= \widetilde{\Delta\boldsymbol{\phi}} : sym\left(\mathbf{R}^T\right) + \widetilde{\Delta\boldsymbol{\phi}} : skew\left(\mathbf{R}^T\right) \\
&= \widetilde{\Delta\boldsymbol{\phi}} : skew\left(\mathbf{R}^T\right) \\
&= \widetilde{\Delta\boldsymbol{\phi}}\ skew\left(\mathbf{R}\right) : \mathbf{1} \\
&= \sin\theta\left(\widetilde{\Delta\boldsymbol{\phi}}\widetilde{\mathbf{n}} : \mathbf{1}\right)
\end{aligned} \tag{4.551}$$

$\Delta\widetilde{\boldsymbol{\phi}} : sym\left(\mathbf{R}^T\right)$ vanishes as double product of skew symmetric and symmetric tensor is null.

$$\widetilde{\Delta\boldsymbol{\phi}}\widetilde{\mathbf{n}} = \mathbf{n}\otimes\Delta\boldsymbol{\phi} - \left(\Delta\boldsymbol{\phi}.\mathbf{n}\right)\mathbf{1} \tag{4.552}$$

$$\widetilde{\Delta\boldsymbol{\phi}}\widetilde{\mathbf{n}} : \mathbf{1} = \left(\mathbf{n}\otimes\Delta\boldsymbol{\phi}\right) : \mathbf{1} - \left(\Delta\boldsymbol{\phi}.\mathbf{n}\right)\mathbf{1} : \mathbf{1} = \Delta\boldsymbol{\phi}.\mathbf{n} - 3\Delta\boldsymbol{\phi}.\mathbf{n} = -2\Delta\boldsymbol{\phi}.\mathbf{n} = -2\frac{\Delta\boldsymbol{\phi}.\theta}{\theta} \tag{4.553}$$

$$\boldsymbol{\theta}.\Delta\boldsymbol{\theta} = \boldsymbol{\theta}.\Delta\boldsymbol{\phi} \tag{4.554}$$

$$\boldsymbol{\theta}.\Delta\boldsymbol{\theta} = \boldsymbol{\theta}.\left(\frac{\widetilde{\boldsymbol{\theta}}(\mathbf{1}-\mathbf{R})\Delta\boldsymbol{\theta}}{|\theta|^2} + \lambda\boldsymbol{\theta}\right) = \lambda\boldsymbol{\theta}^2 \leftrightarrow \lambda = \frac{\boldsymbol{\theta}.\Delta\boldsymbol{\theta}}{\boldsymbol{\theta}^2} \tag{4.555}$$

$$\Delta\boldsymbol{\phi} = \frac{\widetilde{\boldsymbol{\theta}}(\mathbf{1}-\mathbf{R})\Delta\boldsymbol{\theta}}{\boldsymbol{\theta}^2} + \frac{\boldsymbol{\theta}.\Delta\boldsymbol{\theta}}{\boldsymbol{\theta}^2}\boldsymbol{\theta} = T\left(\boldsymbol{\theta}\right)\Delta\boldsymbol{\theta} \tag{4.556}$$

Appendix B: load stiffness matrix

Rotation-dependent moments

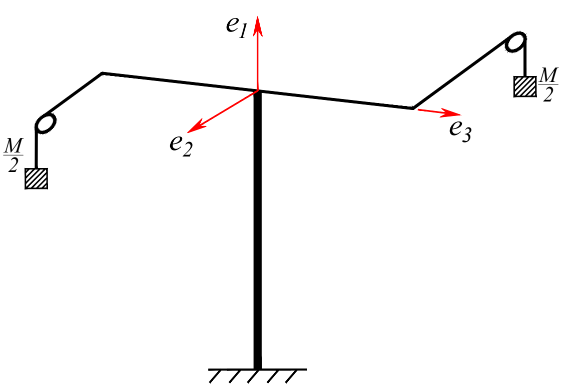

Figure 4.73: Pseudo tangential moment

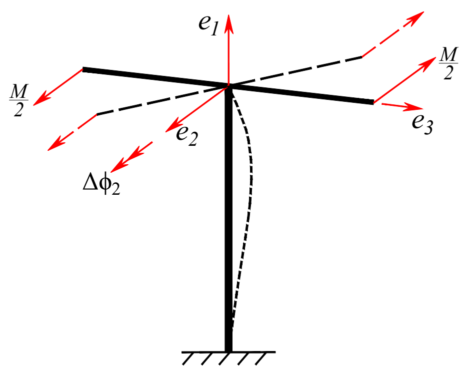

Figure 4.74: Induced moment due to rotation $\triangle\phi_2$ around axis $\boldsymbol{e}_2$

Ziegler presented three types of conservative moments named pseudo tangential moment, quasi tangential moment and semi tangential moment. They are elaborated by Argyris through using mechanical devices including conservative forces like gravity forces. Assume we have two equal gravity loads $M/2$ applied through two parallel strings tied at the end point of rigid levers, each of unit length, attached to a vertical shaft along axis $\boldsymbol{e}_1$ and hanging from a fixed pulley as shown in Figure 4.73. The distance between the pulleys and the corresponding lever ends is infinitely long, such that the strings direction remain the same after shaft rotation. For small rotations $\triangle\phi_1$ around axis $\boldsymbol{e}_1$, the induced (change in the) moment is negligible and vanishes for rotation $\triangle\phi_3$ around axis $\boldsymbol{e}_3$, while the induced moment due to rotation $\triangle\phi_2$ around axis $\boldsymbol{e}_2$ as shown in Figure 4.74 is defined as

$$\triangle\boldsymbol{M} = -M\triangle\phi_2\boldsymbol{e}_3 \tag{4.557}$$

This moment is named pseudo tangential moment, while quasi tangential moment is generated

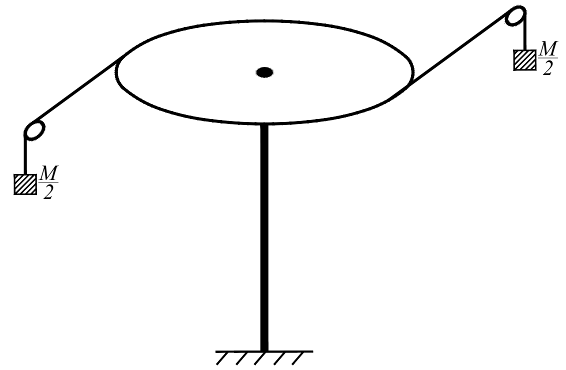

Figure 4.75: Quasi tangential moment

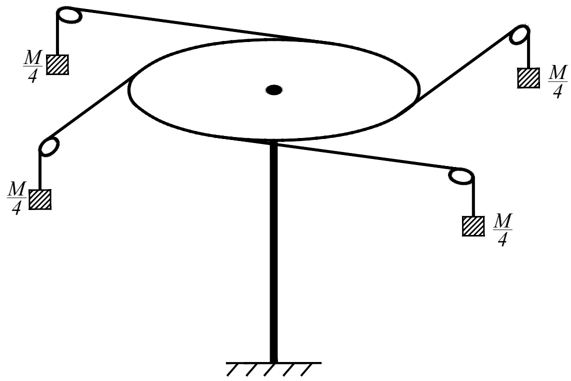

Figure 4.76: Semi tangential moment

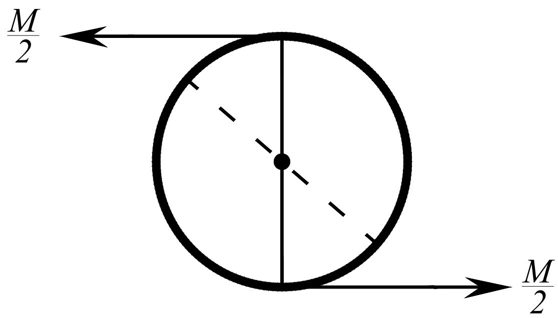

Figure 4.77: Pseudo tangential moment

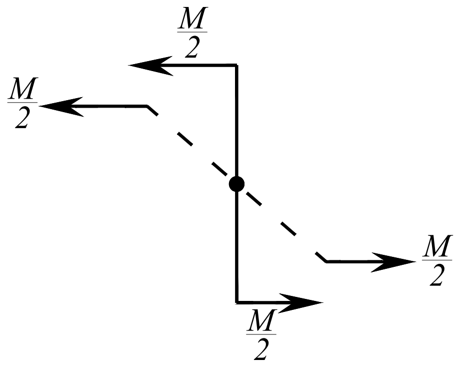

Figure 4.78: Quasi tangential moment

through the same strings stated above but wrapped around a disk of unit radius attached to the vertical shaft as shown in Figure 4.75. The induced moment is approximately same as the one induced in pseudo tangential moment for small rotation, but the difference appears for finite rotation, e.g. a rotation around the shaft axis shown in Figure 4.77 and Figure 4.78 shorten the couple arm of pseudo tangential moment resulting a reduction in the moment around axis $\boldsymbol{e}_1$, while it remains the same for quasi tangential mechanism. The third conservative moment introduced by Ziegler or semi tangential moment is generated by four equal forces $(M/4)$ distributed at each quarter of the disk of unit radius as shown in Figure 4.76. Due to small rotation or incremental spin around axis $\boldsymbol{e}_1$, the induced moment is negligible, while, for incremental spin $\triangle\phi_2$ $[\triangle\phi_3]$ around axis $\boldsymbol{e}_2$ $[\boldsymbol{e}_3]$, the induced moment will be $-\frac{1}{2}M\triangle\phi_2\boldsymbol{e}_3$ $[\frac{1}{2}M\triangle\phi_3\boldsymbol{e}_2]$, so the resulting moment due to spin $(\triangle\boldsymbol{\phi} = \triangle\phi_1\boldsymbol{e}_1 + \triangle\phi_2\boldsymbol{e}_2 + \triangle\phi_3\boldsymbol{e}_3)$ will be:

$$\triangle\boldsymbol{M} = \frac{1}{2}M\triangle\phi_3\boldsymbol{e}_2 - \frac{1}{2}M\triangle\phi_2\boldsymbol{e}_3 \tag{4.558}$$

For moment around a general axis $\boldsymbol{M}$, the induced moment due to spin $\triangle\boldsymbol{\phi}$ will be:

$$\triangle\boldsymbol{M} = \frac{1}{2}\widetilde{\triangle\boldsymbol{\phi}}\boldsymbol{M} \tag{4.559}$$

We will introduce another mechanism with conservative moment generated by four equal forces $M/4$ attached to a rigid arm of L shape with unit length and width a as shown in Figure 4.79. We will called it forth kind conservative moment. In the plane view in Figure 4.81, the induced moment due to incremental spin $\triangle\phi_1$ around axis $\boldsymbol{e}_1$ will be:

$$\triangle\boldsymbol{M} = -\triangle\phi_1 a\,\boldsymbol{e}_1 \tag{4.560}$$

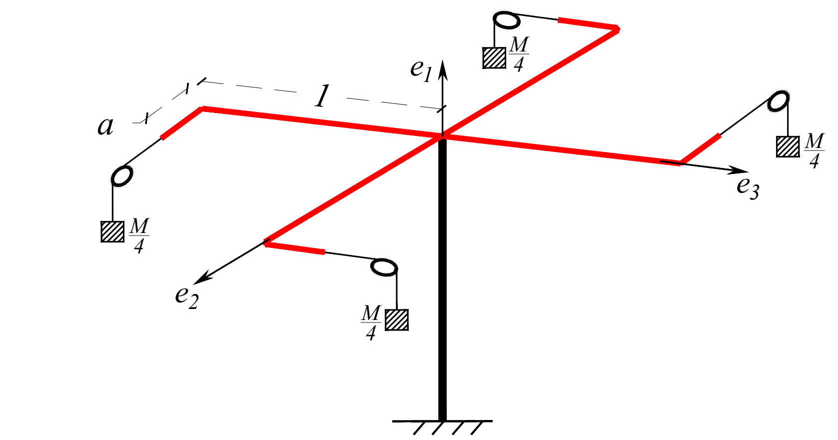

Figure 4.79: Forth kind moment

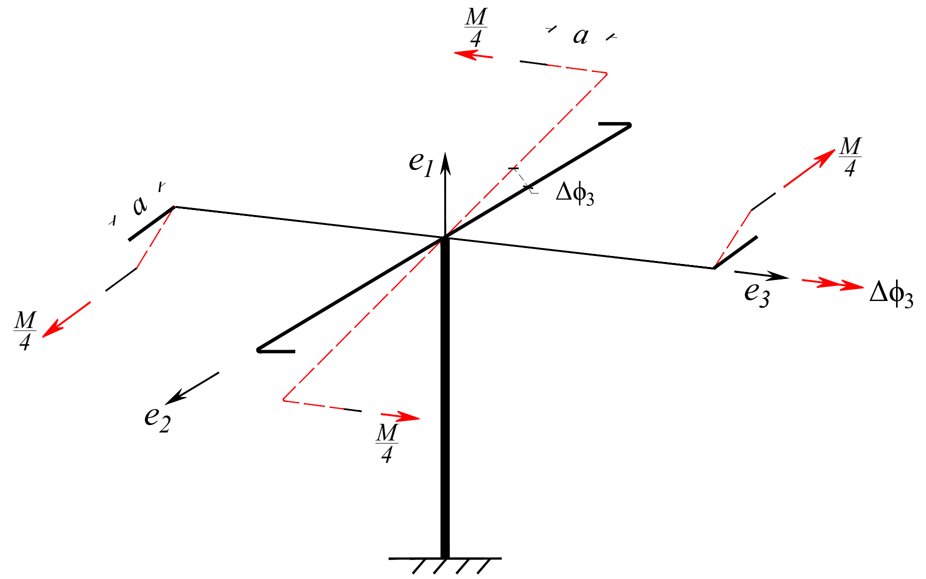

Figure 4.80: Induced moment due to rotation $\triangle\phi_2$ around axis $\boldsymbol{e}_2$

while, for incremental spin ($\triangle\phi_2$ [$\triangle\phi_3$]) around axis $\boldsymbol{e}_2$ [$\boldsymbol{e}_3$] as shown in Figure 4.80, the induced moment will be

$$\triangle\boldsymbol{M} = -\frac{1}{2}M\triangle\phi_2\boldsymbol{e}_3 - \frac{1}{2}M\triangle\phi_2\ a\boldsymbol{e}_2 \qquad \text{due to spin } \triangle\phi_2 \tag{4.561}$$

$$\triangle\boldsymbol{M} = +\frac{1}{2}M\triangle\phi_3\boldsymbol{e}_2 - \frac{1}{2}M\triangle\phi_3\ a\boldsymbol{e}_3 \qquad \text{due to spin } \triangle\phi_3 \tag{4.562}$$

So the total resulting moment due to spin will be: ($\triangle\boldsymbol{\phi} = \triangle\phi_1\boldsymbol{e}_1 + \triangle\phi_2\boldsymbol{e}_2 + \triangle\phi_3\boldsymbol{e}_3$) will be:

$$\triangle\boldsymbol{M} = -\triangle\phi_1 a\ \boldsymbol{e}_1 - \frac{1}{2}M\triangle\phi_2\ a\ \boldsymbol{e}_2 - \frac{1}{2}M\triangle\phi_3\ a\ \boldsymbol{e}_3 - \frac{1}{2}M\triangle\phi_3\boldsymbol{e}_2 - \frac{1}{2}M\triangle\phi_2\boldsymbol{e}_3 \tag{4.563}$$

For a general moment initially defined as $\boldsymbol{M}_0$, the induced moment due to spin ($\triangle\boldsymbol{\phi} = \triangle\phi_1\boldsymbol{e}_1 + \triangle\phi_2\boldsymbol{e}_2 + \triangle\phi_3\boldsymbol{e}_3$) will be:

$$\triangle\boldsymbol{M} = -\triangle\boldsymbol{A}\boldsymbol{M}_0 + \frac{1}{2}\widetilde{\triangle\boldsymbol{\phi}}\boldsymbol{M}_0 \tag{4.564}$$

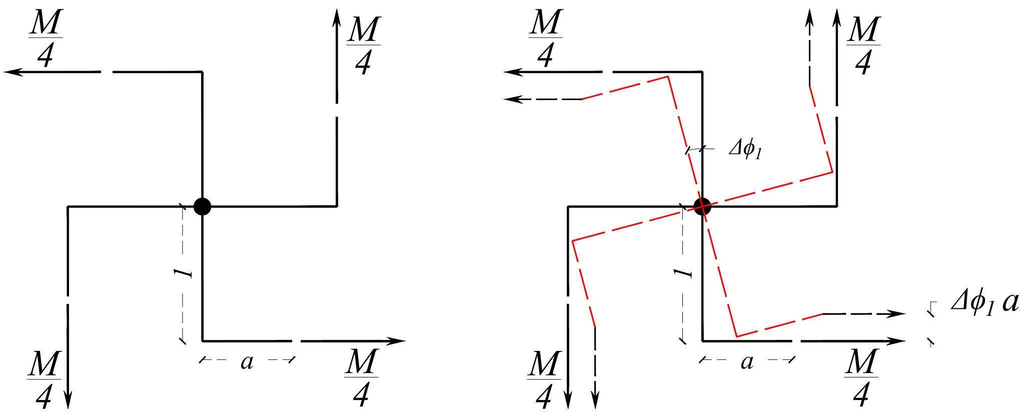

Figure 4.81

Where matrix $\triangle\boldsymbol{A}$ is defined as follows:

$$\triangle\boldsymbol{A} = a\begin{bmatrix} \triangle\phi_1 & \frac{1}{2}\triangle\phi_1 & \frac{1}{2}\triangle\phi_1 \\ \frac{1}{2}\triangle\phi_2 & \triangle\phi_2 & \frac{1}{2}\triangle\phi_2 \\ \frac{1}{2}\triangle\phi_3 & \frac{1}{2}\triangle\phi_3 & \triangle\phi_3 \end{bmatrix} \tag{4.565}$$

Also it can be defined as follows:

$$\triangle\boldsymbol{M} = \left(-\boldsymbol{B} - \frac{1}{2}\widetilde{\boldsymbol{M}}_0\right)\triangle\boldsymbol{\phi} = \hat{\boldsymbol{B}}\triangle\boldsymbol{\phi} \tag{4.566}$$

Where $\boldsymbol{B}$ is symmetric matrix defined as:

$$\boldsymbol{B} = a\begin{bmatrix} M_1+M_2/2+M_3/3 & 0 & 0 \\ 0 & M_2+M_1/2+M_3/3 & 0 \\ 0 & 0 & M_3+M_1/2+M_2/3 \end{bmatrix} \tag{4.567}$$

Generally, the change in a conservative moment due to small rotation follows Equation 4.566

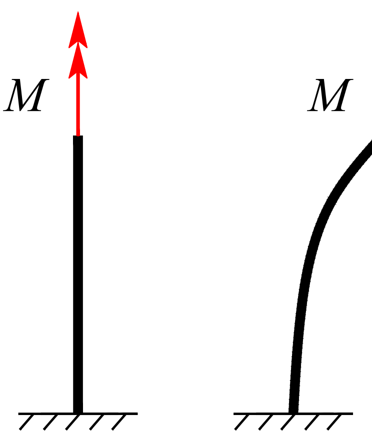

Figure 4.82: Axial moment

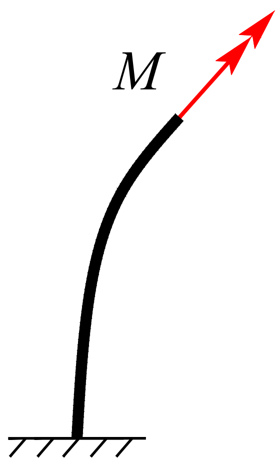

Figure 4.83: Follower moment

with symmetric $\boldsymbol{B}$ under small rotations. In other words, the skew symmetric part of matrix $\hat{\boldsymbol{B}}$ is $-\frac{1}{2}\widetilde{\boldsymbol{M}}$.The symmetry condition of matrix $\boldsymbol{B}$ will be proven in the next sections. There are other types of moment that are considered non conservative such as the axial moment as shown in Figure 4.82 which remains the same after rotation (no induced moment) and follower moment that follows completely the rotation applied at its point of application as shown in Figure 4.83, such that the induced moment due to joint rotation $\triangle\boldsymbol{\phi}$ is defined as:

$$\triangle\boldsymbol{M} = \boldsymbol{R}(\triangle\boldsymbol{\phi})\boldsymbol{M} - \boldsymbol{M} \tag{4.568}$$

Which can be approximated for small rotation using Equation 2.1.7 as follows:

$$\triangle\boldsymbol{M} \cong \widetilde{\triangle\boldsymbol{\phi}}\boldsymbol{M} \tag{4.569}$$

From above the induced moment in axial and follower moment, applying equation Equation 4.566 results un-symmetric $\boldsymbol{B}$ matrix.

Work performed by moment

Assume a moment M_0 applying on a point subjected to incremental spin $\triangle\boldsymbol{\phi}$, such that the change in point spatial rotation is defined through:

$$\triangle\boldsymbol{R} = \widetilde{\triangle\boldsymbol{\phi}_1}\boldsymbol{R}(\boldsymbol{\theta}) \tag{4.570}$$

Where $\triangle\boldsymbol{\phi}$ is defined using Equation 2.90 as follows:

$$\triangle\boldsymbol{\phi} = \boldsymbol{T}(\boldsymbol{\theta})\triangle\boldsymbol{\theta} \tag{4.571}$$

Work performed by a moment M_0 through a spin $\triangle\phi$ is:

$$\triangle W = \boldsymbol{M}.\triangle\boldsymbol{\phi} \tag{4.572}$$

Spin $\triangle\boldsymbol{\phi}$ is not a total differential as there is no $\boldsymbol{\phi}$ to derive. Also, $\triangle W$ does not has to be a total differential either. From previous section, we can assume the moment change from through the following:

$$\boldsymbol{M} = \triangle\boldsymbol{M} + \boldsymbol{M}_0 = \boldsymbol{Q}(\triangle\boldsymbol{\phi})\boldsymbol{M}_0 \tag{4.573}$$

For example, for semi tangential moment mentioned in the previous section, $\boldsymbol{Q} = \mathbf{1} + \frac{1}{2}\widetilde{\triangle\boldsymbol{\phi}}$. From above equations, the resulting work can be rewritten in this form:

$$\triangle W = \boldsymbol{M}_0.\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta})\triangle\boldsymbol{\theta} = \boldsymbol{M}_0.\triangle\boldsymbol{a} \tag{4.574}$$

From above, initial moment $\boldsymbol{M}_0$ is work conjugate to $(\triangle\boldsymbol{a} = \boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta})\triangle\boldsymbol{\theta})$.

Required condition for conservativeness

Assume that $\triangle\boldsymbol{a}(\theta)$ is a total differential and initial moment $\boldsymbol{M}_0$ is constant, such that there is a moment potential V as follows:

$$\triangle V(\boldsymbol{a}) = -\triangle W \qquad \text{with} \qquad \triangle V(a) = -\boldsymbol{M}_0.\triangle\boldsymbol{a} \tag{4.575}$$

If a two successive incremental rotations $\delta\boldsymbol{\theta}$ and $\triangle\boldsymbol{\theta}$ are applied on moment, initially $\boldsymbol{M}_0$, we find that the second variation (directional derivative) of $V(\theta)$ is defined as:

$$\delta(\triangle V(\boldsymbol{\theta})) = \delta\boldsymbol{\theta}.\frac{\partial^2 V}{\partial\delta(\boldsymbol{\theta})\partial\triangle(\boldsymbol{\theta})}\triangle\boldsymbol{\theta} = -\boldsymbol{M}_0.\delta(\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta}))\triangle\boldsymbol{\theta} = \delta\boldsymbol{\theta}.\boldsymbol{K}\triangle\boldsymbol{\theta} \tag{4.576}$$

Due to existence of moment potential V, the tangent load stiffness matrix defined as the second partial derivative of V is symmetric and the order of differentiation is not important.

$$\frac{\partial^2 V}{\partial\delta(\boldsymbol{\theta})\partial\triangle(\boldsymbol{\theta})} = \frac{\partial^2 V}{\partial\triangle(\boldsymbol{\theta})\partial\delta(\boldsymbol{\theta})} \qquad \text{or} \qquad \boldsymbol{K} = \boldsymbol{K}^T \tag{4.577}$$

So symmetry of stiffness matrix ensure the conservativeness of the applied moment. Assume a semi tangential moment $\boldsymbol{M}$ ($\boldsymbol{Q} = \boldsymbol{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}$). Using Equation 2.95 and neglecting second order terms, we get that:

$$\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta}) \cong (\boldsymbol{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}})^T(\boldsymbol{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}) = (\boldsymbol{1} - \frac{1}{2}\widetilde{\boldsymbol{\theta}})(\boldsymbol{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}) \cong \boldsymbol{1} \tag{4.578}$$

So the resulting load stiffness matrix will be:

$$\delta(\triangle V(\boldsymbol{\theta})) = -\boldsymbol{M}_0.\delta(\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta}))\triangle\boldsymbol{\theta} = 0 \qquad \text{or} \qquad \boldsymbol{K} = 0 \tag{4.579}$$

For the forth kind moment $\boldsymbol{M}_0$, the $\boldsymbol{Q}$ matrix is defined through Equation 4.564 as follows:

$$\boldsymbol{Q} = \boldsymbol{1} - \boldsymbol{A} + \frac{1}{2}\widetilde{\boldsymbol{\theta}} \tag{4.580}$$

Where $\boldsymbol{A}$ defined through Equation 4.565 as follows:

$$\boldsymbol{A} = a\begin{bmatrix} \theta_1 & \frac{1}{2}\theta_1 & \frac{1}{2}\theta_1 \\ \frac{1}{2}\theta_2 & \theta_2 & \frac{1}{2}\theta_2 \\ \frac{1}{2}\theta_3 & \frac{1}{2}\theta_3 & \theta_3 \end{bmatrix} \tag{4.581}$$

and $[\boldsymbol{\theta}] = [\theta_1 \;\; \theta_2 \;\; \theta_3]$ is the angle rotated, So

$$\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta}) \cong (\boldsymbol{1} - \boldsymbol{A} + \frac{1}{2}\widetilde{\boldsymbol{\theta}})^T(\boldsymbol{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}) \cong \boldsymbol{1} - \boldsymbol{A}^T \tag{4.582}$$

So using Equation 4.564 and Equation 4.566 results in:

$$\delta(\triangle V(\boldsymbol{\theta})) = -\boldsymbol{M}_0.\delta(\boldsymbol{Q}^T\boldsymbol{T}(\boldsymbol{\theta}))\triangle\boldsymbol{\theta} = \boldsymbol{M}_0.\delta\boldsymbol{A}^T\triangle\boldsymbol{\theta} = \delta\boldsymbol{\theta}.\boldsymbol{B}\triangle\boldsymbol{\theta} \qquad \text{or} \qquad \boldsymbol{K} = 0 \tag{4.583}$$

Matrix $\boldsymbol{B}$ has to be symmtric for a conservative moment as stated before.

Work performed by off-axis force

Assume a force $\boldsymbol{F}$ linked through a rigid bar (${}^1\boldsymbol{X}$) to point O at configuration C_1 as shown in Figure 4.84, such that it produces a moment around point O defined as follows:

$$\boldsymbol{M}_1 = {}^1\boldsymbol{X} \times \boldsymbol{F} = \widetilde{{}^1\boldsymbol{X}}\boldsymbol{F} = -\widetilde{\boldsymbol{F}}\,{}^1\boldsymbol{X} \tag{4.584}$$

If a small rigid body rotation with spin $\triangle\boldsymbol{\phi}$ is induced on the arm ${}^1\boldsymbol{X}$ to produce configuration C_2 with new arm ${}^2\boldsymbol{X}$ defined as follows:

$${}^2\boldsymbol{X} = \boldsymbol{R}(\boldsymbol{\phi})^1\boldsymbol{X} = (\boldsymbol{1} + \widetilde{\triangle\boldsymbol{\phi}})^1\boldsymbol{X} \tag{4.585}$$

The last equality assumes small rotation for $\triangle\boldsymbol{\phi}$. If the force is constant in magnitude and direction during the rigid body rotation, the resulting moment in configuration C_2 will be:

$$\boldsymbol{M}_2 = -\widetilde{\boldsymbol{F}}{}^2\boldsymbol{X} = -\widetilde{\boldsymbol{F}}\left(\boldsymbol{1} + \widetilde{\triangle\boldsymbol{\phi}}\right){}^1\boldsymbol{X} \tag{4.586}$$

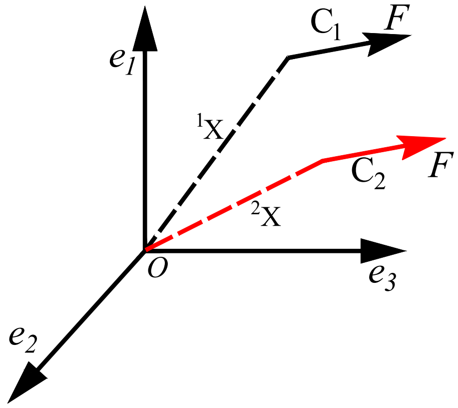

Figure 4.84

With incremental moment $\triangle \boldsymbol{M}$ defined as:

$$\triangle \boldsymbol{M} = \boldsymbol{M_2} - \boldsymbol{M_1} = -\widetilde{\boldsymbol{F}}{}^2\boldsymbol{X} = -\widetilde{\boldsymbol{F}}\widetilde{\triangle\boldsymbol{\phi}}{}^1\boldsymbol{X} = \widetilde{\boldsymbol{F}}{}^1\widetilde{\boldsymbol{X}}\triangle\boldsymbol{\phi} \tag{4.587}$$

The above load type is changing with rotation called deformation dependent load. This load type produces load stiffness matrix.
In Figure 4.85, if the known configuration C_1 is formed through rotation of th initial configuration by angle $\boldsymbol{\theta}$, then subjected to virtual rotation $\boldsymbol{\delta\theta}$ with corresponding spin $\boldsymbol{\delta\phi_1}$, such that the final rotation is defined as:

$$\boldsymbol{R}(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) = \boldsymbol{R}(\delta\boldsymbol{\phi_1})\boldsymbol{R}(\boldsymbol{\theta}) \tag{4.588}$$

For small rotations $\boldsymbol{\theta}$ and infinitesimal spin $\boldsymbol{\delta\phi_1}$, $\boldsymbol{\delta\phi_1}$ can be approximated using Equation 2.1.7 and Equation 2.66, such that the above equation will be resolved to the following:

$$\mathbf{1} + \left(\widetilde{\boldsymbol{\theta} + \delta\boldsymbol{\theta}}\right) + \frac{1}{2}\left(\widetilde{\boldsymbol{\theta} + \delta\boldsymbol{\theta}}\right)\left(\widetilde{\boldsymbol{\theta} + \delta\boldsymbol{\theta}}\right) = \left(\mathbf{1} + \widetilde{\delta\boldsymbol{\phi_1}}\right)\left(\mathbf{1} + \widetilde{\boldsymbol{\theta}} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\widetilde{\boldsymbol{\theta}}\right) \tag{4.589}$$

Which results in:

$$\widetilde{\delta\boldsymbol{\phi_1}}\left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right) = \left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)\widetilde{\delta\boldsymbol{\theta}} \tag{4.590}$$

Which can approximated for small rotations as follows:

$$\widetilde{\delta\boldsymbol{\phi_1}} = \left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)\widetilde{\delta\boldsymbol{\theta}}\left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)^{-1} \tag{4.591}$$

$$\cong \left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)\widetilde{\delta\boldsymbol{\theta}}\left(\mathbf{1} - \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right) \tag{4.592}$$

$$\cong \left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)\widetilde{\delta\boldsymbol{\theta}}\left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)^{T} \tag{4.593}$$

Which leads to

$$\delta\boldsymbol{\phi_1} = \left(\mathbf{1} + \frac{1}{2}\widetilde{\boldsymbol{\theta}}\right)\delta\boldsymbol{\theta} \tag{4.594}$$

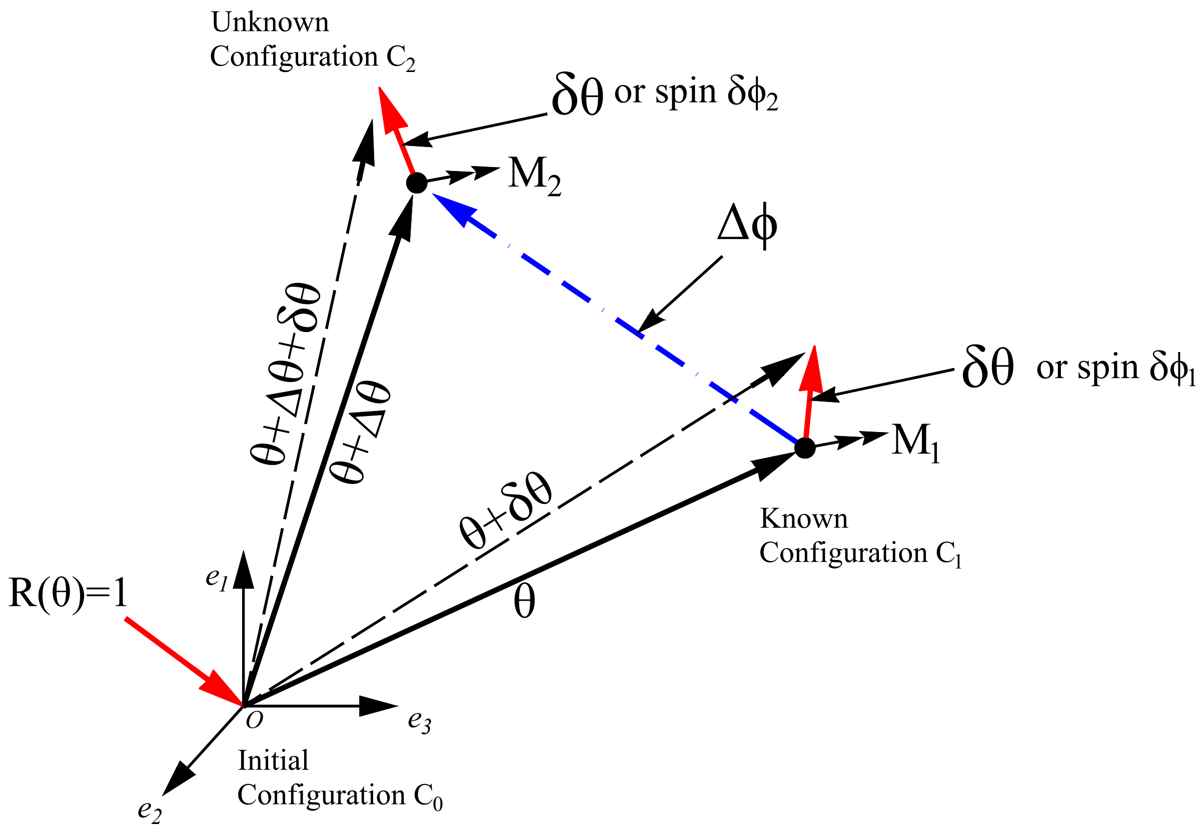

Figure 4.85

Comparing the above equation with Equation 2.95, we get the same results.
The corresponding virtual work in configuration C_1 will be:

$$\delta W_1 = \boldsymbol{M}_1 . \delta \boldsymbol{\phi}_1 \tag{4.595}$$

If the configuration C_1 is subjected to an incremental spin $\triangle\boldsymbol{\phi}$ to form configuration C_2 with rotation defined as:

$$\boldsymbol{R}\left(\boldsymbol{\theta} + \triangle\boldsymbol{\theta}\right) = \boldsymbol{R}\left(\triangle\boldsymbol{\phi_1}\right)\boldsymbol{R}\left(\boldsymbol{\theta}\right) \tag{4.596}$$

Where $\triangle\boldsymbol{\theta}$ is an additive incremental rotation vector corresponding to incremental spin $\triangle\boldsymbol{\phi}$. If this formed configuration is subjected to virtual rotation $\delta\boldsymbol{\theta}$ with corresponding spin $\delta\boldsymbol{\phi_2}$, such that the final rotation is defined as:

$$\boldsymbol{R}\left(\boldsymbol{\theta} + \triangle\boldsymbol{\theta} + \delta\boldsymbol{\theta}\right) = \boldsymbol{R}\left(\delta\boldsymbol{\phi_2}\right)\boldsymbol{R}\left(\boldsymbol{\theta} + \delta\boldsymbol{\theta}\right) \tag{4.597}$$

Using Equation 2.90 and from Figure 2.20a, we will define spin $\delta\boldsymbol{\phi_2}$ as follows:

$$\delta\boldsymbol{\phi_2} = \boldsymbol{T}\left(\boldsymbol{\theta} + \triangle\boldsymbol{\theta}\right)\delta\boldsymbol{\theta} \tag{4.598}$$

$$= \boldsymbol{T}\left(\triangle\boldsymbol{\phi}\right)\delta\boldsymbol{\phi_1} \tag{4.599}$$

$$\cong \left(\boldsymbol{1} + \frac{1}{2}\widetilde{\triangle\boldsymbol{\phi}}\right)\delta\boldsymbol{\phi_1} \tag{4.600}$$

And the corresponding virtual work to configuration C_2 is defined as:

$$\delta W_2 = \boldsymbol{M}_2 . \delta\boldsymbol{\phi}_2 = \left(\boldsymbol{M}_1 + \triangle\boldsymbol{M}\right) . \left(\boldsymbol{1} + \widetilde{\triangle\boldsymbol{\phi}}\right)\delta\boldsymbol{\phi_1} \tag{4.601}$$

The increment virtual work will be:

$$\delta(\triangle W) = (\boldsymbol{M}_1 + \triangle\boldsymbol{M}) . \left(\mathbf{1} + \widetilde{\triangle\boldsymbol{\phi}}\right) \delta\boldsymbol{\phi}_{\mathbf{1}} - \boldsymbol{M}_1 . \delta\boldsymbol{\phi}_1 \tag{4.602}$$

Neglecting second order terms results in:

$$\delta(\triangle W) = \triangle\boldsymbol{M} . \delta\boldsymbol{\phi}_1 + \frac{1}{2}\boldsymbol{M}_1 . \left(\widetilde{\triangle\boldsymbol{\phi}}\delta\boldsymbol{\phi}_1\right) \tag{4.603}$$

The above equation can be concluded through linearization of Equation 4.595 ($\triangle(\delta W_1) = \triangle\boldsymbol{M}_1 . \delta\boldsymbol{\phi}_1 + \boldsymbol{M}_1 . \triangle(\delta\boldsymbol{\phi}_1)$). Using Equation 4.584 and Equation 4.587 results in:

$$\delta(\triangle W) = \delta\boldsymbol{\phi}_1 . \left(\widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}}\triangle\boldsymbol{\phi}\right) + \frac{1}{2}\delta\phi_1 . \left(\widetilde{\boldsymbol{M}}_1\triangle\boldsymbol{\phi}\right) \tag{4.604}$$

$$= \delta\boldsymbol{\phi}_1 . \left(\widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}}\triangle\boldsymbol{\phi}\right) + \frac{1}{2}\delta\phi_1 . \left(-\widetilde{\widetilde{\boldsymbol{F}}\,{}^1\boldsymbol{X}}\triangle\boldsymbol{\phi}\right) \tag{4.605}$$

As $\widetilde{\tilde{a}b} = \tilde{a}\tilde{b} - \tilde{b}\tilde{a}$, the above equation reduces to:

$$\delta(\triangle W) = \delta\boldsymbol{\phi}_1 . \left(\frac{1}{2}\left[\widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}} + \widetilde{{}^1\boldsymbol{X}}\widetilde{\boldsymbol{F}}\right]\triangle\boldsymbol{\phi}\right) \tag{4.606}$$

The symmetry of term $\frac{1}{2}\left[\widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}} + \widetilde{{}^1\boldsymbol{X}}\widetilde{\boldsymbol{F}}\right]$ is due to the mechanism used to create moment is applied through conservative force So the resulting load stiffness matrix is:

$$\delta(\triangle V) = -\delta(\triangle W) \qquad \text{or} \qquad \boldsymbol{K} = -\frac{1}{2}\left(\widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}} + \widetilde{\boldsymbol{F}}\widetilde{{}^1\boldsymbol{X}}\right) \tag{4.607}$$

For force $\boldsymbol{F}$ and arm $\boldsymbol{X}$ resolved in the same frame of reference $\boldsymbol{e}_i$ as follows:

$$\boldsymbol{F} = F_i\boldsymbol{e}_i, \qquad {}^1\boldsymbol{X} = {}^1X_i\boldsymbol{e}_i \tag{4.608}$$

The above load stiffness will be:

$$[\boldsymbol{K}] = -\frac{1}{2}\begin{bmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ -F_2 & F_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -{}^1X_3 & {}^1X_2 \\ {}^1X_3 & 0 & -{}^1X_1 \\ -{}^1X_2 & {}^1X_1 & 0 \end{bmatrix} \tag{4.609}$$

$$-\frac{1}{2}\begin{bmatrix} 0 & -{}^1X_3 & {}^1X_2 \\ {}^1X_3 & 0 & -{}^1X_1 \\ -{}^1X_2 & {}^1X_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ -F_2 & F_1 & 0 \end{bmatrix} \tag{4.610}$$

$$= \begin{bmatrix} F_3{}^1X_3 + F_2{}^1X_2 & -\frac{1}{2}(F_1{}^1X_2 + F_2{}^1X_1) & -\frac{1}{2}(F_1{}^1X_3 + F_3{}^1X_1) \\ & F_3{}^1X_3 + F_1{}^1X_1 & -\frac{1}{2}(F_3{}^1X_2 + F_2{}^1X_3) \\ Symmtric & & F_2{}^1X_2 + F_1{}^1X_1 \end{bmatrix} \tag{4.611}$$

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