Averaging wins again

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The averaging method used to establish the classical inequality between the means works just as well to show domination by the Root-Mean-Square.

To begin, Cauchy's inequality [2]

$$\sum_1^N x_j y_j < \left(\sum_1^N x_j^2 \sum_1^N y_j^2\right)^{1/2}$$

using the sequence of all ones $\{y_j\}=\{1_j\}$ yields immediately domination by the Root-Mean-Square (RMS) over the Arithmetic-Mean (AM):

$$\sum_1^N x_j/N < \left(\sum_1^N x_j^2/N\right)^{1/2}$$

However, for only two numbers this inequality is obvious:

$(x_1 - x_2)^2 > 0$ implies that $2x_1^2 + 2x_2^2 > (x_1 + x_2)^2$, so $½(x_1^2 + x_2^2) > (½(x_1 + x_2))^2$

and the result follows by taking square roots.

Now, let's emulate the averaging method in [1] and apply it carefully to the case for three numbers. We get

$x_1 + x_2 + x_3 = ½(x_1 + x_2) + ½(x_2 + x_3) + ½(x_1 + x_3) < (½(x_1^2 + x_2^2))^{½} + (½(x_2^2 + x_3^2))^{½} + (½(x_1^2 + x_3^2))^{½} = ½((½(x_1^2 + x_2^2))^{½} + (½(x_2^2 + x_3^2))^{½}) + ½((½(x_2^2 + x_3^2))^{½} + (½(x_3^2 + x_1^2))^{½}) + ½((½(x_3^2 + x_1^2))^{½} + (½(x_1^2 + x_2^2))^{½}) < (1/4(x_1^2 + 2x_2^2 + x_3^2))^{½} + (1/4(x_1^2 + x_2^2 + 2x_3^2))^{½} + (1/4(2x_1^2 + x_2^2 + x_3^2))^{½} < (1/8(2x_1^2 + 3x_2^2 + 2x_3^2))^{½} + (1/8(3x_1^2 + 2x_2^2 + 3x_3^2))^{½} + (1/8(3x_1^2 + 3x_2^2 + 2x_3^2))^{½} < (1/16(5x_1^2 + 5x_2^2 + 6x_3^2))^{½} + (1/16(6x_1^2 + 5x_2^2 + 5x_3^2))^{½} + (1/16(5x_1^2 + 6x_2^2 + 5x_3^2))^{½}$

and so on. Thus the components of the 3x3 matrix

$V = ½(I + S),\ I = (\delta_{ij}),\ S = (\delta_{i+1,j})$

and its powers appear now as multipliers of quadratic sums in each iteration. Let

$$f_n = f(V^n, x^2) = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} V^n_{ij} x_j^2\right)^{1/2}$$

Since V is doubly stochastic, an appeal to the fundamental theorem for transition matrices [3] (or directly, as in [1]), shows that

$$\lim_{n\to\infty} V^n = J/3$$

where $J = (1)_{ij}$ is the matrix of all 1's. Thus

$$\sum_{i=1}^{3} x_i < f_1$$

while

$$\lim_{n\to\infty} f_n = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}(1/3)1_{ij}x_j^2\right)^{1/2} = 3\sum_{i=1}^{3}\left((1/3)x_i^2\right)^{1/2}$$

so the RMS-AM inequality will follow after showing that f_n is increasing. First, note that

$$f(V^n,X^2) = f(SV^n,X^2)$$

since the shift matrix S just re-orders the sequence f_n - albeit in a useful way. The 'doublet' averaging procedure amounts to expressing f_n as

$$f_n = 1/2(f(V^n,X^2) + f(SV^n,X^2))$$

so an application of RMS-AM inequality for two numbers yields finally that

$$f_n < f(1/2(I + S)V^n,X^2) = f(V^{n+1},X^2) = f_{n+1}$$

Now that the inequality for three numbers has been established, let's try 'triplet' averaging on four numbers. We get

$x_1 + x_2 + x_3 + x_4 = 1/3(x_1 + x_2 + x_3) + 1/3(x_2 + x_3 + x_4) + 1/3(x_3 + x_4 + x_1) + 1/3(x_4 + x_1 + x_2) < (1/3(x_1^2 + x_2^2 + x_3^2))^{½} + (1/3(x_2^2 + x_3^2 + x_4^2))^{½} + (1/3(x_3^2 + x_4^2 + x_1^2))^{½} + (1/3(x_4^2 + x_1^2 + x_2^2))^{½} < (1/9(2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_4^2))^{½} + (1/9(2x_1^2 + 2x_2^2 + 2x_3^2 + 3x_4^2))^{½} + (1/9(3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2))^{½} + (1/9(2x_1^2 + 3x_2^2 + 2x_3^2 + 2x_4^2))^{½}$

and so on. This time the 4x4 doubly-stochastic matrix V is

$$V = 1/3(I + S + S^2)$$

so that again

$$\lim_{n\to\infty} V^n = J/4$$

and letting

$$f_n = f(V^n,X^2) = \sum_{i=1}^{4}\left(\sum_{j=1}^{4} V^n_{ij}x_j^2\right)^{1/2}$$

we get as before

$$\sum_{i=1}^{4} x_i < f_1$$

and

$$\lim_{n\to\infty} f_n = 4\sum_{i=1}^{4}\left((1/4)x_i^2\right)^{1/2}$$

Finally, monotonicity for this f_n sequence follows similarly: since

$$f(V^n,X^2) = f(SV^n,X^2) = f(S^2V^n,X^2)$$

'triplet' averaging means that

$$f_n = 1/3(f(V^n,X^2) + f(SV^n,X^2) + f(S^2V^n,X^2))$$

to which an application of the RMS-AM inequality for three numbers yields the desired result:

$$f_n < f(1/3(V^n + SV^n + S^2V^n),X^2) = f_{n+1}$$

The proof for N numbers merely requires increasing the size of the matrix V accordingly, using 'doublet' or 'triplet' averaging. In fact, the path is clear to an induction proof as well: at step N+1 use averaging for step N. Here are the details.

Since the RMS-AM inequality is obvious when N=2, assume validity at step N and let V be the (N+1) x (N+1) doubly stochastic matrix

$$V = (1/N) \sum_{k=0}^{N-1} S^k$$

where $S^0 = I$ as usual. Next, define

$$f_n = f(V^n, x^2) = \sum_{i=1}^{N+1} \Big(\sum_{j=1}^{N+1} V^n_{ij} x_j^2 \Big)^{1/2}$$

The convergence

$$\lim_{n \to \infty} V^n = J/(N+1)$$

implies that

$$\lim_{n \to \infty} f_n = (N+1) \sum_{i=1}^{N+1} \big(x_i^2/(N+1) \big)^{1/2}$$

and since

$$\sum_{i=1}^{N+1} x_i < f_1$$

it only remains to show sequence monotonicity. But for any k =1, 2, ..., N

$$f(V^n, x^2) = f(S^k V^n, x^2)$$

so that N-averaging becomes

$$f_n = (1/N) \sum_{k=0}^{N-1} f(S^k V^n, x^2)$$

hence a final application of the RMS-AM inequality for N numbers yields

$$f_n < f\Big(\sum_{k=0}^{N-1} ((1/N) S^k) V^n, x^2 \Big) = f(VV^n, x^2) = f_{n+1}$$

and induction carries through.

REFERENCES

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[3] J. G. Kemeny, J. L. Snell, Finite Markov Chains, Van Nostrand Reinhold, New York, 1960.