

Imaginary doings, Real results in Special Relativity

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Rotation with imaginary angle in imaginary time produces the relativistic transformation in hyperbolic form, but neither by itself does it. A purely real approach is also provided. The transformation in Einstein form is further derived without invoking the Relativity Principle.

The relativistic transformation in hyperbolic form was given by Minkowski [1] as

$$M(\theta) = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix}$$

obtained ingeniously by shifting variables into the complex domain, yet in the end magically producing a purely real result; but first things first.

It was Poincaré [2] who first remarked - almost in passing near the very end - that Cartesian space-time rotation and imaginary time substitution would lead to a relativistic transformation; he apparently did not pursue his insight in detail, so let us do it.

Rotation of Cartesian space-time (x, t) - the y and z components are unaffected by collinear motion - is provided by the orthogonal matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

In component form, the action of Cartesian rotation on frame transition becomes

$$\begin{aligned} x' &= x\cos(\theta) + t\sin(\theta) \\ t' &= -x\sin(\theta) + t\cos(\theta) \end{aligned}$$

so that making time imaginary ($t \rightarrow it$) yields

$$\begin{aligned} x' &= x\cos(\theta) + it\sin(\theta) \\ it' &= -x\sin(\theta) + it\cos(\theta) \end{aligned}$$

Multiplication of the time component by $-i$ produces

$$\begin{aligned} x' &= x\cos(\theta) + it\sin(\theta) \\ t' &= ix\sin(\theta) + t\cos(\theta) \end{aligned}$$

so the resulting frame transformation

$$P(\theta) = \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}$$

is unfortunately complex. More is needed to bring off the real result; but what?

Although aware of Poincaré's prior contribution, Minkowski [1] traveled the imaginary route without attribution, perhaps knowing that imaginary time usage alone made for a complex result. The missing idea was to make the rotational angle imaginary as well, and this does it. The elementary identities

$$\begin{aligned} \cos(i\theta) &= \cosh(\theta) \\ i\sin(i\theta) &= -\sinh(\theta) \end{aligned}$$

show that $P(i\theta)$ is indeed real, producing the Minkowski matrix $M(\theta)$.

Actually Minkowski followed the imaginary path the other way around: angle first, then time. Let's see what happens. Clearly

$$R(i\theta) = \begin{pmatrix} \cosh(\theta) & i\sinh(\theta) \\ -i\sinh(\theta) & \cosh(\theta) \end{pmatrix}$$

which is still complex; note that $\mathbf{P}(\theta) \neq \mathbf{R}(i\theta)$.

In component form, the frame transition becomes

$$\begin{aligned} x' &= x\cosh(\theta) + it\sinh(\theta) \\ t' &= -ix\sinh(\theta) + t\cosh(\theta) \end{aligned}$$

Next, imaginary time substitution produces

$$\begin{aligned} x' &= x\cosh(\theta) - t\sinh(\theta) \\ it' &= -ix\sinh(\theta) + it\cosh(\theta) \end{aligned}$$

so multiplying the time component by $-i$ does indeed yields the relativistic transformation in hyperbolic form.

Now, recast the Minkowski matrix $\mathbf{M}(\theta)$ as

$$M(v) = \gamma \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}$$

by first factoring out $\cosh(\theta)$, then using the hyperbolic identity

$$\cosh^2(\theta) = (1 - \tanh^2(\theta))^{-1/2}$$

and finally the critical assignment $\tanh(\theta) = v/c$, first proposed by Varićak [3]. Here

$$\gamma(v) = \gamma = (1 - v^2/c^2)^{-1/2}$$

as usual. In component form the frame transitions are

$$\begin{aligned} x' &= \gamma(x - vt/c) \\ t' &= \gamma(-vx/c + t) \end{aligned}$$

Now, the variable change $t \rightarrow ct$ makes time and position have similar units. Thus

$$\begin{aligned} x' &= \gamma(x - vt) \\ ct' &= \gamma(-vx/c + ct) \end{aligned}$$

Lastly, division of the time component by C produces

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(-vx/c^2 + t) \end{aligned}$$

the relativistic transformation first derived (very differently) by Einstein himself [4]; the matrix form is of course

$$E(v) = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix}$$

The Principle of Relativity (Law invariance and Light-speed constancy) was not invoked at all in deriving the Einstein form; merely the acceptance that there is a velocity upper bound (since $|\tanh(\theta)| < 1$), a mild though sensible departure from classical relativity.

References

- [1] H. Minkowski: 'The fundamental equations for electromagnetic processes in moving bodies', *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1908), pp. 53-111; English translation in *Space and Time* – Minkowski Institute Press (2012), pp. 51-110, esp. Sec §3.
- [2] H. Poincaré: 'On the dynamics of the electron', *Rendiconti del Circolo Matematico di Palermo*, **21** (1906), pp. 129-176, esp. Sec §9.; English translation by Wikisource.
- [3] V. Varičák: 'Application of Lobachevskian geometry to the theory of relativity', *Physikalische Zeitschrift*, **11** (1910), pp. 93-96.; English translation by Wikisource.
- [4] A. Einstein: 'On the electrodynamics of moving bodies', *Annalen der Physik*, **17** (1905), pp. 891-921; English translation in *The Principle of Relativity* – Dover Publications (1952), pp. 37-65, esp. p. 48.