AN INTEGRATED MEDIA, INTEGRATED PROCESSES WATERSHED MODEL – WASH123D: PART 4 – A CHARACTERISTICS-BASED FINITE ELEMENT METHOD FOR 2-D OVERLAND FLOW

GUOBIAO HUANG[1] GOUR-TSYH YEH[2]

[1]*Sutron Corporation, 6903 Vista Parkway Suite 5, West Palm Beach, FL 33411 USA*

[2]*Dept of Civil and Environmental Engineering, University of Central Florida, Orlando, FL 32816 USA*

ABSTRACT

The Method of Characteristics (MOC) in the context of finite element method was applied to the complete 2-D shallow water equations for 2-D overland flow. For two-dimensional overland flow, finite element or finite volume methods are more flexible in dealing with complex boundary. Recently, finite volume methods have been very popular in numerical solution of the shallow water equations. Some have pointed out that finite volume methods for 2-D flow are fundamentally one-dimensional (normal to the cell interface). The results may rely on the grid orientation. The search for genuinely multidimensional numerical schemes for 2-D flow is an active topic. We consider the Method of Characteristics (MOC) in the context of finite element method as a good alternative. Many researchers have pointed out the advantage of MOC in solving 2-D shallow water equations that are of the hyperbolic type that has wave-like solutions and at same time, considered MOC for 2-D overland flow being non-tractable on complex topography. The intrinsic difficulty in implementing MOC for 2-D overland flow is that there are infinite numbers of wave characteristics in the 2-D context, although only three independent wave directions are needed for a well-posed solution to the characteristic equations. We have implemented a numerical scheme that attempts to diagonalize the characteristic equations based on pressure and velocity gradient relationship. This new scheme was evaluated by comparison with other choice of wave characteristic directions in the literature. Example problems of mixed sub-critical flow/super-critical flow in a channel with approximate analytical solution was used to verify the numerical algorithm. Then experiments of overland flow on a cascade of three planes (Iwagaki 1955) were solved by the new method. The circular dam break problem was solved with different selections of wave characteristic directions and the performance of each selection was evaluated based on accuracy and numerical stability. Finally, 2-D overland flow over complex topography in a wetland setting with very mild slope was solved by the new numerical method to demonstrate its applicability.

1. INTRODUCTION

The simplified form of the two-dimensional shallow water equations, e.g., the diffusion wave or kinematic wave approximation, has been frequently used in modeling the two-dimensional shallow overland flow originating from rainfall-runoff process, irrigation and flows in flood plains and wetlands. On the other hand, the full two-dimensional shallow water equations have been extensively studied for fast transient flow processes (dam break type flood propagation and hydraulic jumps) or deep surface water flows in estuary and ocean.

Chow and Ben-Zvi (1973) reported the first two-dimensional hydrodynamic model for overland flow using the Lax-Wendroff scheme. Since then, many numerical schemes based on finite difference or finite volume methods have been studied (Zhang and Cundy, 1989; Fielder and Ramirez, 2000; Zhao et al., 1994; among others) and Katopodes and Strelkoff (1978; 1979) developed a numerical scheme based on the method of characteristics in the framework of finite difference method for two-dimensional dam break simulations.

In the finite element framework, it is well known that Galerkin finite element methods perform very poorly for advection-dominant shallow water flows. The streamline upwind finite element methods (SUPG) apply selective dissipation to dampen numerical oscillation. Lately, the discontinuous Galerkin finite element method has also been applied for trans-critical shallow water flows (Schwanenberg and Harms, 2004).

All above-mentioned methods are Eulerian methods with some stabilization schemes. Since the shallow water equations are PDEs of hyperbolic type, characteristics-based or Eulerian-Largragian methods are more appropriate. The characteristic Galerkin method (Zienkiewicz et al., 1999) was developed for the scalar advection equation but is not directly applicable to the shallow water equations. Since more than one characteristic speeds are involved, the characteristic-based split (CBS) scheme was proposed to resolve this difficulty. Paillere et al. (1998), Brufau and Garcia-Navarro (2003) and Garcia-Navarro et al. (1999) studied genuinely multidimensional upwinding schemes for the 2D shallow water equations based on a residual distribution scheme with wave models.

2. GOVERNING EQUATIONS

The governing equations for two-dimensional overland flow are the shallow water equations based on the conservation law of mass and momentum. Comparing to the conservative form, the primitive form is more revealing for the intrinsic physical property of the shallow water equations and amendable to advective schemes. The governing equations written in the primitive form:

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}+v\frac{\partial h}{\partial y}+h\frac{\partial v}{\partial y}=R \tag{1}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial h}{\partial x}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g(S_{0x}-S_{fz})-\frac{uR}{h} \tag{2}$$

$$\frac{\partial v}{\partial t}+g\frac{\partial h}{\partial y}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=g(S_{0x}-S_{fz})-\frac{vR}{h} \tag{3}$$

where h is water depth; u is the velocity component in the x-direction; v is the velocity component in the y-velocity, respectively. R is the source/Sink term as a result of rainfall, evapotranspiration and infiltration, etc. Without losing generality, the eddy turbulent term, momentum exchange flux, surface shear stress (wind effect), etc. have been omitted.

The bed slopes and frictional slopes are given as :

$$S_{0x} = -\frac{\partial Z_0}{\partial x}, \quad S_{0y} = -\frac{\partial Z_0}{\partial y} \text{ and } S_{fx} = \frac{n^2 u\sqrt{u^2+v^2}}{h^{4/3}}, \quad S_{fy} = \frac{n^2 v\sqrt{u^2+v^2}}{h^{4/3}} \tag{4}$$

where g is gravitational acceleration, Z_0 is the bed elevation above a datum, n is the Manning's roughness coefficient.

Equations (1) through (3) can be written in matrix form as

$$\frac{\partial \mathbf{E}}{\partial \mathrm{t}} + \mathbf{A_x}\frac{\partial E}{\partial x} + \mathbf{A_y}\frac{\partial E}{\partial y} = \mathbf{R}$$

$$\mathbf{E} = \{\mathrm{h\ u\ v}\}^T; \quad \mathbf{A_x} = \begin{bmatrix} u\ h\ 0 \\ g\ u\ 0 \\ 0\ 0\ u \end{bmatrix}; \quad \mathbf{A_y} = \begin{bmatrix} v\ 0\ h \\ 0\ v\ 0 \\ g\ 0\ v \end{bmatrix} \text{ and } \mathbf{R} = \begin{Bmatrix} R \\ g(S_{0x} - S_{fz}) - \dfrac{uR}{h} \\ g(S_{0y} - S_{fy}) - \dfrac{vR}{h} \end{Bmatrix} \tag{5}$$

For an arbitrary wave propagation direction **k** $=(k_x, k_y) = (\cos\theta, \sin\theta)$; θ is the angle of the wave direction from x-direction, let the matrix **B** be the linear combination of the matrices $\mathbf{A_x}$ and $\mathbf{A_y}$ as follows

$$\mathbf{B} = \mathbf{A}\cdot\mathbf{k} = \mathbf{A_x}\cos\theta + \mathbf{A_y}\sin\theta = \begin{bmatrix} u\cos\theta + v\sin\theta & h\cos\theta & h\sin\theta \\ g\sin\theta & u\cos\theta + v\sin\theta & 0 \\ g\sin\theta & 0 & u\cos\theta + v\sin\theta \end{bmatrix} \tag{6}$$

The three eigenvalues of matrix **B** are

$$\lambda_1 = u\cos\theta + v\sin\theta, \quad \lambda_2 = u\cos\theta + v\sin\theta + c,, \text{ and } \lambda_3 = u\cos\theta + v\sin\theta - c \tag{7}$$

The wave celerity is defined $c = \sqrt{gh}$. The primitive form can be recast in the characteristic form by using the eigenvectors associated with matrix **B**

$$\frac{\partial \mathbf{W}}{\partial \mathrm{t}} + \begin{bmatrix} u & 0 & 0 \\ 0 & u + c\cos\theta & 0 \\ 0 & 0 & u - c\cos\theta \end{bmatrix}\frac{\partial \mathbf{W}}{\partial \mathrm{x}} + \begin{bmatrix} v & 0 & 0 \\ 0 & v + c\sin\theta & 0 \\ 0 & 0 & v - c\sin\theta \end{bmatrix}\frac{\partial \mathbf{W}}{\partial \mathrm{y}} = -\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} + \mathbf{L}^{-1}\mathbf{R} \tag{8}$$

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} = \begin{Bmatrix} g\left(\dfrac{\partial h}{\partial x}\sin\theta - \dfrac{\partial h}{\partial y}\cos\theta\right) \\ \dfrac{c}{g}\left[\dfrac{\partial u}{\partial x}\sin^2\theta + \dfrac{\partial v}{\partial y}\cos^2\theta - \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)\cos\theta\sin\theta\right] \\ \dfrac{-c}{g}\left[\dfrac{\partial u}{\partial x}\sin^2\theta + \dfrac{\partial v}{\partial y}\cos^2\theta - \left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)\cos\theta\sin\theta\right] \end{Bmatrix} \tag{9}$$

The characteristic variable vector **W** is defined as

$$\mathbf{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix}^T = L^{-1}E = \begin{bmatrix} 0 & \sin\theta & -\cos\theta \\ \dfrac{1}{c} & \dfrac{\cos\theta}{g} & \dfrac{\sin\theta}{g} \\ -\dfrac{1}{c} & \dfrac{\cos\theta}{g} & \dfrac{\sin\theta}{g} \end{bmatrix} \begin{Bmatrix} h \\ u \\ v \end{Bmatrix} = \begin{Bmatrix} u\sin\theta - v\cos\theta \\ \dfrac{u\cos\theta + v\sin\theta + c}{g} \\ \dfrac{u\cos\theta + v\sin\theta - c}{g} \end{Bmatrix} \tag{10}$$

where W_1 is a characteristic variable associated with a shear wave, which has no equivalent in one-dimensional flow (θ=0). W_2 and W_3 are characteristic variables associated with the positive and negative gravity waves, respectively.

This is the characteristic form of two-dimensional shallow water equations with an arbitrary wave direction **K**=(cosθ, sinθ). The left hand side terms represent water wave propagation in the characteristic wave directions and can be written with the total derivative along the characteristics:

$$\begin{Bmatrix} \dfrac{D_{\vec{V}} W_1}{Dt} \\ \dfrac{D_{\vec{V}+c\vec{k}} W_2}{Dt} \\ \dfrac{D_{\vec{V}-c\vec{k}} W_3}{Dt} \end{Bmatrix} = -\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} + \begin{bmatrix} 0 & \sin\theta & -\cos\theta \\ \dfrac{1}{c} & \dfrac{\cos\theta}{g} & \dfrac{\sin\theta}{g} \\ -\dfrac{1}{c} & \dfrac{\cos\theta}{g} & \dfrac{\sin\theta}{g} \end{bmatrix} R \tag{11}$$

The coupling terms (S_1, S_2, S_3) cannot be simultaneously eliminated and as in the case of the Euler equations (Hirsh et al., 1987), this results in the non-unique selection of upwind directions for two-dimensional flows. It is noteworthy that the above characteristic equations in Lagrangian form (10) are identical to both the original conservative and primitive forms of the shallow water equations. No numerical approximations have been introduced.

The governing equations must be supplemented with initial condition and appropriate boundary conditions for a well-posed two-dimensional overland flow problem. Wave characteristic directions at the boundary determine the required boundary conditions.

3. NUMERICAL METHODS

Equation (11) is the basis of the characteristics-based finite element scheme. At the interior nodes, backward tracking along the three characteristics is performed by a subelement tracking scheme (Cheng et al., 1997). The solution values at the foot of the characteristic curve are interpolated by linear finite elements. At the boundary nodes, characteristic directions and flow directions are used to determine the needed boundary conditions. Details on implementation can be found in (Yeh et al., 2006) and only the choice of characteristic wave directions will be discussed.

3.1 Characteristic wave directions

After the selection of two specific characteristic directions with the propagation angles, θ_1 and θ_2, the new characteristic equations are defined as:

$$\frac{\partial \mathbf{W}}{\partial t} + \begin{bmatrix} u & 0 & 0 \\ 0 & u + c\cos\theta_2 & 0 \\ 0 & 0 & u - c\cos\theta_2 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial x} + \begin{bmatrix} v & 0 & 0 \\ 0 & v + c\sin\theta_2 & 0 \\ 0 & 0 & v - c\sin\theta_2 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial y} = -\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} + \mathbf{L}^{*-1}\mathbf{R} \qquad (12)$$

$$\mathbf{L}^* = \begin{bmatrix} 0 & \dfrac{c}{2} & -\dfrac{c}{2} \\ \sin\theta_2/\omega & \dfrac{g\cos\theta_2}{2\omega} & \dfrac{g\cos\theta_2}{2\omega} \\ -\cos\theta_2 & \dfrac{g\sin\theta_2}{2\omega} & \dfrac{g\sin\theta_2}{2\omega} \end{bmatrix} \quad W = L^{*-1} E \qquad (13)$$

where $\omega = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$. Since ω should not be zero, the two wave directions cannot be orthogonal.

Choosing the two wave characteristic directions (θ_1 and θ_2) is the most critical part of the characteristics-based finite element method. It can be seen that the first characteristic speed (u, v) is along the streamlines and only the two characteristic directions associated with the gravity waves need to be chosen.

The first approach is the wave directions based on maximum diagonalization. The first choice is based on the minimization of the coupling term (Equation (12). This follows the diagonalization approach for the Euler equations suggested by Hirsh et al. (1987). By setting the coupling terms to zero, we have the following relationships.

$$\begin{aligned} & g\left(\frac{\partial h}{\partial x}\sin\theta_1 - \frac{\partial h}{\partial y}\cos\theta_1\right) = 0 \\ & \frac{c}{g}\left[\frac{\partial u}{\partial x}\sin^2\theta_2 + \frac{\partial v}{\partial y}\cos^2\theta_2 - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\theta_2 \sin\theta_2\right] = 0 \\ & \frac{-c}{g}\left[\frac{\partial u}{\partial x}\sin^2\theta_2 + \frac{\partial v}{\partial y}\cos^2\theta_2 - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\theta_2 \sin\theta_2\right] = 0 \end{aligned} \qquad (14)$$

From the above algebraic equations, we observe that the first characteristic direction is related to the shear wave, and the two gravity waves share the same second characteristic direction.

The first characteristic direction is determined by: $\tan\theta_1 = \dfrac{\partial h/\partial y}{\partial h/\partial x}$ that is in the pressure gradient direction. The second characteristic direction, if exists, is based on the solution of the following equation:

$$\frac{\partial u}{\partial x}\sin^2\theta_2 + \frac{\partial v}{\partial y}\cos^2\theta_2 - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\theta_2 \sin\theta_2 = 0 \tag{14}$$

It can have two, one or zero solutions. If no solutions can be found, the following particular characteristic direction as suggested in Roe (1986) for the treatment of the Euler equations will be used: $\tan\theta_2 = \dfrac{\partial u/\partial y + \partial v/\partial x}{\partial u/\partial x - \partial v/\partial y}$

It is noteworthy that by choosing wave decomposition following the flow gradients (depth and velocity), it may be possible to minimize the coupling terms and obtain diagonalization of the shallow water equations. However, numerical experiments show that this approach often suffer from convergence problem (e.g., Paillere et al., 1998). The characteristic directions are dependent on the numerical solution and sensitive to the accurate evaluation of the gradients of water depth and velocity components. Numerical stability and convergence are the major concern.

The second approach is the characteristic decomposition proposed by (Paillere et al., 1998). When the flow is supercritical, the angle θ_2 is taken to be along the Froude line. The Froude number is defined as ($Fr^2 = (u^2+v^2)/c^2$) and the Froude angle is ($\sin\theta_2 = 1/Fr$). If the flow is sub-critical, the propagation angle is taken as $\tan\theta_2 = \dfrac{1}{\sqrt{1-Fr^2}}$.

The first characteristic direction θ_1 is chosen to be equal to the angle θ_2 in order to maximize the determinant of the transformation ($\omega = 1.0$). The coupling terms will not be zero with this selection of characteristic directions. On the other hand, numerical experiments shows that this approach is more stable.

Another approach is the ad-hoc wave directions based on some geometric parameters. For example, the characteristic directions can be specified to be along the x-direction or y-direction or along the steepest elevation gradients. This approach is less accurate and grid orientation of the numerical solutions may occur.

4. NUMERICAL EXAMPLES

The performance of the new numerical scheme was verified and tested with several typical overland flow examples. Only the circular dam break problem will be presented due to the page limit.

Two-dimensional Circular dam break problem is an academic test problem. A circular dam with a radius of 11 m is located in the center of a 50 m x 50 m computational domain. The bed is horizontal and frictionless. The initial water depth in the dam is 10 m and 1 m outside the dam (Figure 1). The dam is instantaneously removed at time = 0.

It has been widely used in hydraulics literature to test performance of different numerical methods (e.g., Schwanenberg and Harms, 2004; Tseng and Chu, 2000 and Alcrudo and Garcia-Navarro, 1993, among others). The dominant wave propagation direction is known *a* *priori*. It is along the radial directions. So it is a good example to test impact of chosen wave directions on numerical solutions.

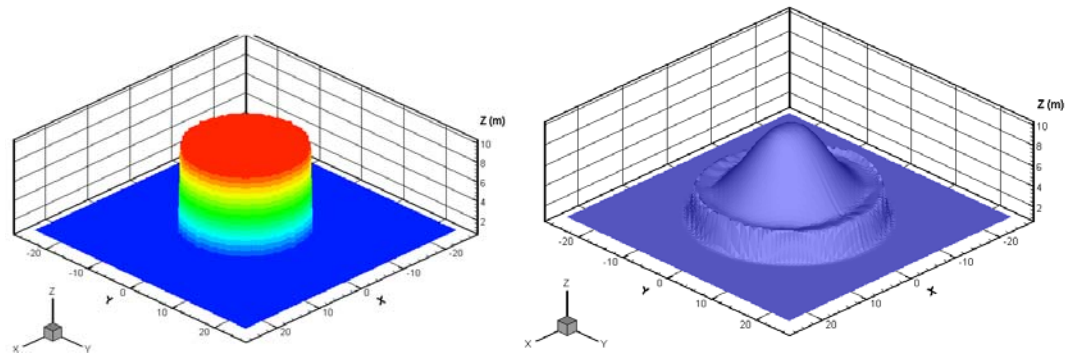

FIGURE 1. Water Depth of Circular Dam-break Problem (t=0 and 0.69 s)

As pointed out by Schwanenberg and Harms (2004), the solution between the shock wave and the rarefaction wave is not flat as in the corresponding one-dimensional dam break. This difference arises from the two-dimensional nature of the flow and is a good test for the correctness of a numerical solution. As shown in Figure 1, the solution solved by the characteristics based finite element method was able to capture this aspect of the solution quite well. The radial symmetry of water depth is also preserved well considering the use of triangular elements and a perfect symmetry could not be set at the beginning of the simulation.

The grid orientation effect of selected arbitrary wave directions was demonstrated in a solution that used the x-direction as the second characteristic direction. As can be seen in Figure 2, a pre-specified characteristic direction cannot capture the two-dimensional nature as well as by a dynamically computed wave direction (the second approach).

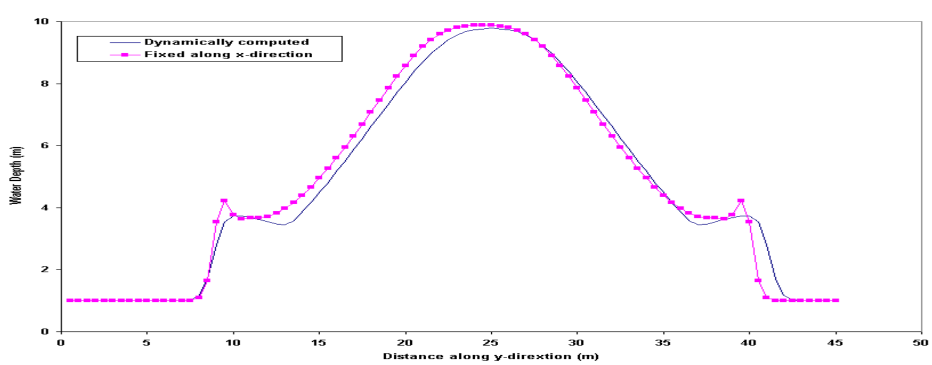

FIGURE 2. Impact of Selected Characteristic Direction on Computed Water Depth

5. CONCLUSION

We have demonstrated that it is feasible to apply the method of characteristics in the framework of finite element method for the shallow water equations for two-dimensional overland flow. The advantages in such a numerical scheme include the straightforward and physics-based treatment of boundary conditions; the source terms are easily handled and numerical instabilities and oscillation in Galerkin or simple upwind finite element methods are avoided. The judicious choice of wave characteristic directions is the critical aspect.

ACKNOWLEDGEMENTS

This research is supported by U.S. EPA-Science To Achieve Results (STAR) Program under Grant # R-82795602 with University of Central Florida.

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