A SYMMETRIC MIXED ADAPTIVE VARIATIONAL MULTISCALE METHOD

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Abstract

We present a mixed adaptive variational multiscale method for solving elliptic second order problems. This work is an extension of the adaptive variational multiscale method (AVMS), introduced by Larson and Målqvist [9, 10, 11], to a mixed formulation. The method is based on a particular splitting into coarse and fine scales together with a systematic technique for approximation of the fine scale part based on solution of decoupled localized subgrid problems. We present the mixed AVMS method and derive a posteriori error estimates for both linear functionals and the energy norm. Based on the estimates we propose an adaptive algorithm for automatic tuning of critical discretization parameters.

1. INTRODUCTION

We consider the Poisson equation on mixed form with positive diffusion coefficient $a \in L^\infty(\Omega)$:

$$\begin{cases} \frac{1}{a}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ n \cdot \boldsymbol{\sigma} = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

We assume the integral over the polygonal domain Ω of the right hand side to be zero, $\int_\Omega f\,dx = 0$, in order to get a well posed problem with a solution $u \in H^1(\Omega)/\mathbf{R}$ and $\boldsymbol{\sigma} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : n \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$, see [1] for definitions of these spaces. The boundary of Ω is denoted Γ. Our main focus is to develop a method for solving this problem in the case when a has fine scale features that can not be resolved by a single mesh.

The variational multiscale method was introduced in 1995 by Hughes [7, 8]. Here the spaces are divided into a coarse and a fine part and then analytical methods are used to approximate the effect of the fine scales on the coarse scale. See also the work by Arbogast [2] for mixed formulations.

In [9, 10, 11] we develop a new multiscale method for the standard formulation of Poisson's equation with diffusion coefficient a. We also derive a posteriori error estimates both in the energy norm and for linear functionals and based on these estimates we propose and implement adaptive algorithms.

In this paper we present the mixed adaptive variational method. We also derive a posteriori error estimates for both linear functionals and the energy norm of the error and based on the estimate we formulate an adaptive algorithm that automatically tunes

the parameters in the method according to the error estimate. This is a very important feature of the method.

The remainder of the paper is organized as follows: in Section 2 we present the method and discuss implementation issues; in Section 3 we derive a posteriori error estimates and in Section 4 we present an adaptive algorithm.

2. THE MIXED VARIATIONAL MULTISCALE METHOD

We start by deriving the variational formulation of equation (1.1) by multiplying the first equation with a test function $\boldsymbol{v} \in \boldsymbol{V} = \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : n \cdot \boldsymbol{v} = 0 \text{ on } \Gamma\}$, integrate over the domain Ω, and integrate by parts. We also multiply the second equation by a test function $w \in W = L^2(\Omega)/\mathbf{R}$ and integrate over Ω. The weak form reads: find $\boldsymbol{\sigma} \in \boldsymbol{V}$ and $u \in W$ such that,

$$\left\{ \begin{array}{rl} (\frac{1}{a}\boldsymbol{\sigma}, \boldsymbol{v}) + (u, \nabla \cdot \boldsymbol{v}) = 0, \\ -(\nabla \cdot \boldsymbol{\sigma}, w) = (f, w), \end{array} \right. \tag{2.1}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $w \in W$, where $(v, w) = (v, w)_\Omega = \int_\Omega v\, w\, dx$. Since we focus on problems that features fine scale behavior it is natural to assume that we will not be able to get enough accuracy by just solving the problem on a single mesh using a single processor. With this in mind we split the spaces $\boldsymbol{V} = \boldsymbol{V}_c \oplus \boldsymbol{V}_f$ and $W = W_c \oplus W_f$ into a discrete coarse part, $\boldsymbol{V}_c$ and W_c, that we can resolve using a standard finite element method on a single mesh and a fine part, $\boldsymbol{V}_f$ and W_f, that needs to be taken care of in a non-standard way.

By this argument we end up with the variational multiscale formulation, see [7] for an overview, of equation (1.1): find $\boldsymbol{\sigma}_c \in \boldsymbol{V}_c$, $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$, $u_c \in W_c$, and $u_f \in W_f$ such that,

$$\left\{ \begin{array}{rl} (\frac{1}{a}(\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), \boldsymbol{v}_c + \boldsymbol{v}_f) + (u_c + u_f, \nabla \cdot (\boldsymbol{v}_c + \boldsymbol{v}_f)) = 0, \\ -(\nabla \cdot (\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_f), w_c + w_f) = (f, w_c + w_f), \end{array} \right. \tag{2.2}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

2.1. Splitting into Coarse and Fine Scales.

There are many possible ways of splitting the spaces $\boldsymbol{V}$ and W into coarse and fine parts. In this paper we will use the lowest order Raviart-Thomas elements on rectangles together with piecewise constants for the coarse spaces. For the fine scale we use the natural hierarchical basis.

We note that some products present in equations (2.2) will actually vanish using this split,

$$(w_c, \nabla \cdot \boldsymbol{v}_f) = \sum_{K \in \mathcal{K}} (w_c, \nabla \cdot \boldsymbol{v}_f)_K = \sum_{K \in \mathcal{K}} w_c^K \int_{\partial K} \boldsymbol{n} \cdot \boldsymbol{v}_f\, dx = 0, \tag{2.3}$$

where w_c^K is the constant value w_c has at coarse element K and $\mathcal{K} = \cup K$ is the set of coarse elements on Ω. We also have,

$$(w_f, \nabla \cdot \boldsymbol{v}_c) = \sum_{K} (w_f, \nabla \cdot \boldsymbol{v}_c)_K = \sum_{K} \nabla \cdot \boldsymbol{v}_c^K \int_K w_f\, dx = 0, \tag{2.4}$$

where $\nabla \cdot \boldsymbol{v}_c^K$ is the constant vector $\nabla \cdot \boldsymbol{v}_c$ at coarse element K. Equation (2.3) and (2.4) holds for all $\boldsymbol{v}_c \in \boldsymbol{V}_c$, $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_c \in W_c$, and $w_f \in W_f$.

2.2. **The fine scale equations.** If we in particular study the set of equations with fine scale test functions in equation (2.2) i.e. letting $\boldsymbol{v}_c = 0$ and $w_c = 0$ and use the identities in equation (2.3) and (2.4) we get the following system: find $\boldsymbol{\sigma}_f \in \boldsymbol{V}_f$ and $u_f \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_f, \boldsymbol{v}_f) + (u_f, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\sigma}_c, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\sigma}_f, w_f) = (f, w_f), \end{cases} \tag{2.5}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$ and $w_f \in W_f$. We will have to somehow split (2.5) in to independent localized problems that can be solved in parallel.

We let $\boldsymbol{\phi}_i$ be the lowest order Raviart-Thomas basis functions on the coarse mesh. This means that we can write $\boldsymbol{\sigma}_c = \sum_{i \in \mathcal{N}} \sigma_c^i \boldsymbol{\phi}_i$, where $\sigma_c^i \in \mathbf{R}$ and $\mathcal{N}$ is the set of coarse edges. Further we let $\psi_i = I_{\text{supp}\boldsymbol{\phi}_i}/2d$, where I is the indicator function and d is he dimension of Ω. We note that ψ is piecewise constant on the coarse mesh and that $\sum_{i \in \mathcal{N}} \psi_i = 1$ since we use rectangular meshes. These sets of functions will help us localize the fine scale equations.

In order to do this we need to introduce two help problems. The first one reads: find $\boldsymbol{\xi}_i \in \boldsymbol{V}_f$ and $\zeta_i \in W_f$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\xi}_i, \boldsymbol{v}_f) + (\zeta_i, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\xi}_i, w_f) = 0, \end{cases} \tag{2.6}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$, and $w_f \in W_f$ and $i \in \mathcal{N}$. Here f is replaced by 0 and $\boldsymbol{\sigma}_c$ is replaced by $\boldsymbol{\phi}_i$ compared to equation (2.5). We are going to introduce T as the solution operator to equation (2.6),

$$\boldsymbol{\xi}_i = T\boldsymbol{\phi}_i. \tag{2.7}$$

The second problem reads: find $\boldsymbol{\beta}_i \in \boldsymbol{V}_f$ and $\rho_i \in W_f$ such that

$$\begin{cases} (\frac{1}{a}\boldsymbol{\beta}_i, \boldsymbol{v}_f) + (\rho_i, \nabla \cdot \boldsymbol{v}_f) = 0, \\ -(\nabla \cdot \boldsymbol{\beta}_i, w_f) = (f\psi_i, w_f), \end{cases} \tag{2.8}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$, and $w_f \in W_f$ and $i \in \mathcal{N}$. Here instead $\boldsymbol{\sigma}_c$ is replaced by 0 and f by $f\psi_i$. We introduce the following notations,

$$\boldsymbol{\sigma}_{f,i} = \sigma_c^i \boldsymbol{\xi}_i + \boldsymbol{\beta}_i, \tag{2.9}$$

and,

$$u_{f,i} = \sigma_c^i \zeta_i + \rho_i, \tag{2.10}$$

and state the localized fine scale equations: find $\boldsymbol{\sigma}_{f,i} \in \boldsymbol{V}_f$ and $u_{f,i} \in W_f$ such that,

$$\begin{cases} (\frac{1}{a}\boldsymbol{\sigma}_{f,i}, \boldsymbol{v}_f) + (u_{f,i}, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\sigma_c^i\boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\sigma}_{f,i}, w_f) = (f\psi_i, w_f), \end{cases} \tag{2.11}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_f$, $w_f \in W_f$, and $i \in \mathcal{N}$. We let $\boldsymbol{\beta} = \sum_i \boldsymbol{\beta}_i$, $\rho = \sum_i \rho_i$. Since all equations are linear we have,

$$\boldsymbol{\sigma}_f = \sum_{i \in \mathcal{N}} \boldsymbol{\sigma}_{f,i} = \sum_{i \in \mathcal{N}} \sigma_c^i \boldsymbol{\xi}_i + \boldsymbol{\beta}, \tag{2.12}$$

and for the scalar variable,

$$u_f = \sum_{i \in \mathcal{N}} u_{f,i} = \sum_{i \in \mathcal{N}} \sigma_c^i \zeta_i + \rho. \tag{2.13}$$

2.3. **Approximation by solving local Neumann problems on patches.** At this point we have been able to split the fine scale equations into lots of subproblems, equation (2.11), and in order to solve these we need to compute solutions to equations (2.6) and (2.8). Each of these still needs to be solved in an infinite dimensional space. In this section we discuss how to compute approximate solutions to these equations.

We let $\boldsymbol{V}_c = \boldsymbol{V}_H$ be the lowest order Raviart-Thomas finite element space and $W_c = W_H$ be the space of piecewise constants on a rectangular mesh $\cup K = \mathcal{K}$ with mesh function H defined as a piecewise constant function equal to the diameter of the current element. Since $\boldsymbol{V}_f$ and W_f are infinite dimensional they also need to be discretized. We introduce a new mesh function $h < H$ individually for the decoupled problems by refining the coarse mesh once or several times.

Our aim is to find optimal coarse scale approximations $\boldsymbol{\Sigma}_c \in \boldsymbol{V}_H$ and $U_c \in W_H$ in the spaces $\boldsymbol{V}_H$ and W_H which are discrete. For the approximation of the fine scale solutions $\boldsymbol{\Sigma}_{f,i}$ and $U_{f,i}$ we introduce corresponding solution spaces, $\boldsymbol{V}_h(\omega_i)$ and $W_h(\omega_i)$ where $\{\omega_i\}$ are domains (patches) such that,

$$\text{supp}(\boldsymbol{\phi}_i) = \text{supp}(\psi_i) \subset \omega_i \subset \Omega. \tag{2.14}$$

The fine scale spaces $\boldsymbol{V}_h(\omega_i)$ and $W_h(\omega_i)$ are constructed on the patch ω_i as a hierarchical extension of the spaces $\boldsymbol{V}_H$ and W_H restricted to the patch, with homogenous Neumann conditions $\boldsymbol{n} \cdot \boldsymbol{v}_f = 0$ on the boundary of the patch, $\partial\omega_i$, for all $v_f \in \boldsymbol{V}_h(\omega_i)$.

We will frequently refer to layers in the rest of this paper. Layers are a measure of how large the patches are. One layer will be the two coarse element on which the coarse Raviart-Thomas base function has its support, two layers are these two coarse element together with all coarse element that neighbors the first two and so on.

The computable versions of equations (2.6) and (2.8) reads: find $\boldsymbol{\xi}_i^h \in \boldsymbol{V}_h(\omega_i)$ and $\zeta_i^h \in W_h(\omega_i)$ such that

$$\left\{ \begin{array}{rl} (\frac{1}{a}\boldsymbol{\xi}_i^h, \boldsymbol{v}_f) + (\zeta_i^h, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\xi}_i^h, w_f) = 0, \end{array} \right. \tag{2.15}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, and $w_f \in W_h(\omega_i)$ and $i \in \mathcal{N}$ and: find $\boldsymbol{\beta}_i^h \in \boldsymbol{V}_h(\omega_i)$ and $\rho_i^h \in W_h(\omega_i)$ such that

$$\left\{ \begin{array}{rl} (\frac{1}{a}\boldsymbol{\beta}_i^h, \boldsymbol{v}_f) + (\rho_i^h, \nabla \cdot \boldsymbol{v}_f) = 0, \\ -(\nabla \cdot \boldsymbol{\beta}_i^h, w_f) = (f\psi_i, w_f), \end{array} \right. \tag{2.16}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, and $w_f \in W_h(\omega_i)$ and $i \in \mathcal{N}$.

If we write the method in terms of the approximate fine scale solutions $\boldsymbol{\Sigma}_f$ and U_f we get: find $\boldsymbol{\Sigma}_{f,i} \in \boldsymbol{V}_h(\omega_i)$ and $U_{f,i} \in W_h(\omega_i)$ such that

$$\left\{ \begin{array}{rl} (\frac{1}{a}\boldsymbol{\Sigma}_{f,i}, \boldsymbol{v}_f) + (U_{f,i}, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\Sigma_c^i\boldsymbol{\phi}_i, v_f), \\ -(\nabla \cdot \boldsymbol{\Sigma}_{f,i}, w_f) = (f\psi_i, w_f), \end{array} \right. \tag{2.17}$$

for all $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, and $w_f \in W_h(\omega_i)$ and $i \in \mathcal{N}$. The real numbers Σ_c^i builds up the discrete coarse scale solution $\boldsymbol{\Sigma}_c = \sum_{i\in\mathcal{N}} \Sigma_c^i \boldsymbol{\phi}_i$. This gives us a discrete version of the fine scale equations. The next step is to present the full numerical method of approximating the coarse part of the solution $\boldsymbol{\Sigma}_c$ and U_c.

We use equation (2.2) as a starting point and replace the fine scale test functions v_f by Tv_c, where T is given from equation (2.7). Together with equation (2.3), equation (2.4),

the fact that $(w_f, \nabla \cdot T v_c) = 0$ for all $w_f \in W_h(\omega_i)$, and $-(\nabla \cdot \boldsymbol{\Sigma}_f, w_f) = (f, w_f)$ from equation (2.17) we state the following approximate method: find $\boldsymbol{\Sigma}_c \in \boldsymbol{V}_H, \boldsymbol{\Sigma}_{f,i} \in \boldsymbol{V}_h(\omega_i)$, $U_c \in W_H$, and $U_{f,i} \in W_h(\omega_i)$ such that

$$\begin{cases} (\frac{1}{a}(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_f), \boldsymbol{v}_c + T\boldsymbol{v}_c) + (U_c, \nabla \cdot \boldsymbol{v}_c) = 0, \\ -(\nabla \cdot \boldsymbol{\Sigma}_c, w_c) = (f, w_c), \\ (\frac{1}{a}\boldsymbol{\Sigma}_{f,i}, \boldsymbol{v}_f) + (U_{f,i}, \nabla \cdot \boldsymbol{v}_f) = -(\frac{1}{a}\Sigma_c^i \boldsymbol{\phi}_i, \boldsymbol{v}_f), \\ -(\nabla \cdot \boldsymbol{\Sigma}_{f,i}, w_f) = (\psi_i f, w_f), \end{cases} \tag{2.18}$$

for all $\boldsymbol{v}_c \in \boldsymbol{V}_H$, $\boldsymbol{v}_f \in \boldsymbol{V}_h(\omega_i)$, $w_c \in W_H$, $w_f \in W_h(\omega_i)$, and $i \in \mathcal{N}$.

3. A POSTERIORI ERROR ESTIMATE

3.1. **Postprocessing of $U_{f,i}$.** In this section we introduce functions $U_{f,i}^*$ that are post processed (improved) versions of $U_{f,i}$. We use the method described in [13] and [12].

To state the estimates we also need a technical definition to state the error representation formula. We let F_i be the coarse face on which $|\boldsymbol{\phi}_i| = 1$ i.e. $F_i = \{x : |\boldsymbol{\phi}_i| = 1\}$. We next define the auxiliary function $U_{c,i}$.

Definition 3.1. *For each fine element $K \in \mathcal{K}(\omega_i)$ we let U_c^K be the constant interior value of U_c on K and define $U_{c,i}$ on ∂K as,*

$$\begin{cases} U_{c,i} = U_c^K \quad \textit{when } x \in F_i, \\ U_{c,i} = 0 \quad \textit{otherwise.} \end{cases} \tag{3.1}$$

Thus $U_{c,i}$ is defined on all faces of every fine scale element K. It takes the value zero on all faces except F_i. On F_i it takes either value of the discontinuous function U_c depending on which side of F_i element K is on.

3.2. **Error Representation for Linear Functionals.** We introduce the following dual problem,

$$\begin{cases} \frac{1}{a}\boldsymbol{\chi} - \nabla \eta = \boldsymbol{\omega}, \\ -\nabla \cdot \boldsymbol{\chi} = 0. \end{cases} \tag{3.2}$$

Using the dual problem we can derive an error representation formula for a linear functional.

Proposition 3.1. *For arbitrary $U_{f,i}^* \in \bigoplus_{K \in \mathcal{K}} P_1(K)$ it holds,*

$$\begin{aligned} (\boldsymbol{\sigma} - \boldsymbol{\Sigma}, \boldsymbol{\omega}) = \sum_{i \in \mathcal{N}} (-\frac{1}{a}(\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}) + \nabla U_{f,i}^*, \boldsymbol{\chi}) \\ - \sum_{i \in \mathcal{N}} \sum_{K \in \mathcal{K}(\omega_i)} (U_{f,i}^* + U_{c,i}, \boldsymbol{n} \cdot \boldsymbol{\chi})_{\partial K \setminus \partial \omega_i} \\ - \sum_{i \in \mathcal{N}} (U_{f,i}^*, \boldsymbol{n} \cdot \boldsymbol{\chi})_{\partial \omega_i \setminus \Gamma} \\ - \sum_{i \in \mathcal{N}} (f\psi_i + \nabla \cdot (\Sigma_c^i \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}), \eta - \pi_c \eta - \pi_{f,i} \eta). \end{aligned} \tag{3.3}$$

The error representation formula (3.3) can be used to create an adaptive algorithm as it stands. The dual problem needs to be solved numerically. The more effort we put into the calculation of the solution to the dual problem the better control of the error we get. We may also proceed on the calculation using estimates to get abound of the error.

3.3. **Error estimates.** In this paper we are primarily interested in deriving a posteriori error estimates which are easy to implement and scale properly with the error so that they can serve as a basis for adaptive algorithms. We are thus not interested in tracking constants C independent of the mesh size.

We present an error estimate in the energy norm,

$$\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|_a = (\frac{1}{a}\boldsymbol{\sigma}-\boldsymbol{\Sigma},\boldsymbol{\sigma}-\boldsymbol{\Sigma})^{1/2}. \tag{3.4}$$

Theorem 3.1. *For arbitrary $U^*_{f,i} \in \bigoplus_{K\in\mathcal{K}} P_1(K)$ it holds,*

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\Sigma}\|_a \leq & \left(\sum_{i\in\mathcal{N}} C\|\sqrt{a}\|^2_{L^\infty(\omega_i)}\|\nabla U^*_{f,i}-\frac{1}{a}(\Sigma^i_c\boldsymbol{\phi}_i+\boldsymbol{\Sigma}_{f,i})\|^2_{\omega_i}\right)^{1/2} \\ & + \left(\sum_{i\in\mathcal{N}} C\|\sqrt{a}\|^2_{L^\infty(\omega_i)}\sum_{K\in\mathcal{K}(\omega_i)} h^{-1}\|[U^*_{f,i}+U_{c,i}]\|^2_{\partial K\setminus\partial\omega_i}\right)^{1/2} \\ & + \left(\sum_{i\in\mathcal{N}} C\|\sqrt{a}\|^2_{L^\infty(\omega_i)}\|h^{-1/2}U^*_{f,i}\|^2_{\partial\omega_i\setminus\Gamma}\right)^{1/2} \\ & + \left(\sum_{i\in\mathcal{N}} C\|\sqrt{a}\|^2_{\tilde{L}^\infty(\omega_i)}\|\frac{h}{a}(f\psi_i+\nabla\cdot(\Sigma^i_c\boldsymbol{\phi}_i+\boldsymbol{\Sigma}_{f,i}))\|^2_{\omega_i}\right)^{1/2},\end{aligned} \tag{3.5}$$

where we use the notation,

$$\|\sqrt{a}\|_{\tilde{L}^\infty(\omega_i)} = \max_{K\in\mathcal{K}(\omega_i)} \|a\|_{L^\infty(K)}\|\frac{1}{\sqrt{a}}\|_{L^\infty(K)}. \tag{3.6}$$

4. The Adaptive Algorithm

We will base our adaptive strategy on the energy norm estimate presented in Theorem 3.1. The same ideas may be used in the duality based case. The parameters we will tune to get an improved solution are the individual mesh size on each patch and the individual size of each patch. We do not discuss how the coarse mesh is chosen. The idea is that we have reached a point where we can not afford a richer coarse space and need to consider parallel methods. The coarse mesh will in practise probably be constructed using a standard adaptive algorithm for a single mesh, see [5, 3] for such methods.

We will use the error indicators from Theorem 3.1 to refine and extend the patches. The refinements does not need to be uniform since we can calculate indicators for individual fine element on the patch. However in this section we will present a simple algorithm that uses different but uniform meshes on each patch.

We have the following four error indicators if we ignore the constants,

$$\begin{cases} \|\sqrt{a}\|^2_{L^\infty(\omega_i)} \|\nabla U^*_{f,i} - \frac{1}{a}(\Sigma^i_c \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i})\|^2_{\omega_i}, \\ \|\sqrt{a}\|^2_{L^\infty(\omega_i)} \sum_{K\in\mathcal{K}(\omega_i)} h^{-1} \|[U^*_{f,i} + U_{c,i}]\|^2_{\partial K\setminus\partial\omega_i}, \\ \|\sqrt{a}\|^2_{L^\infty(\omega_i)} \|h^{-1/2} U^*_{f,i}\|^2_{\partial\omega_i\setminus\Gamma}, \\ \|\sqrt{a}\|^2_{\tilde{L}^\infty(\omega_i)} \|\frac{h}{a}(f\psi_i + \nabla\cdot(\Sigma^i_c\boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|^2_{\omega_i}. \end{cases} \tag{4.1}$$

We are mainly interested in creating an adaptive algorithm that automatically improves the solution in an iterative fashion based on an error estimate that scales correctly in the parameters of interest. The main goal is therefore not to calculate a good approximation of the error. This means that we are more interested in how the four indicators compare to each other than of their absolute value. But when we look at the four term we immediately see that the fourth term has a different a dependent term in front of it. However this is not a big issue since if the a coefficient is constant on the fine mesh, and we can assume that it is close to constant since we have resolved the coefficient with the fine mesh, we have the following identity,

$$\|\sqrt{a}\|^2_{L^\infty(\omega_i)} = \max_{K\in\mathcal{K}(\omega_i)} \|a\|^2_{L^\infty(K)} \|\frac{1}{\sqrt{a}}\|^2_{L^\infty(K)} := \|\sqrt{a}\|^2_{\tilde{L}^\infty(\omega_i)}. \tag{4.2}$$

With this result in mind we are ready to present four indicators that will be used in the adaptive algorithm,

$$\begin{cases} X_i = \|\nabla U^*_{f,i} - \frac{1}{a}(\Sigma^i_c \boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i})\|^2_{\omega_i}, \\ Y_i = \sum_{K\in\mathcal{K}(\omega_i)} h^{-1} \|[U^*_{f,i} + U_{c,i}]\|^2_{\partial K\setminus\partial\omega_i}, \\ Z_i = \|h^{-1/2} U^*_{f,i}\|^2_{\partial\omega_i\setminus\Gamma}, \\ W_i = \|\frac{h}{a}(f\psi_i + \nabla\cdot(\Sigma^i_c\boldsymbol{\phi}_i + \boldsymbol{\Sigma}_{f,i}))\|^2_{\omega_i}. \end{cases} \tag{4.3}$$

We do not take the a dependent coefficient into account since it multiplies all expressions with the same factor. The four error indicators in equation (4.3) are easy and cheep to calculate.

Since we are not primarily interested in the absolute size of these estimators, just how they compare to each other, we do not use a tolerance as a stopping criteria. Below we will sketch a simple adaptive algorithm based on the four estimators in equation (4.3).

(1) Calculate the solution to equation (2.18) $\boldsymbol{\Sigma}$ using small patches with low resolution.
(2) Calculate the four error indicators on each patch, equation (4.3).
(3) For large values in X_i, Y_i, and, W_i refine the mesh once for patch i.
(4) For large values in Z_i increase the patch size by one layer for patch i.
(5) Stop if the solution is sufficiently good or go back to 1.

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