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C O N T E N T S

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Chapter V

SPHERICAL COORDINATES

Chapter VI

PROJECTION OF THE SPHERICAL SURFACE ON THE PLANE

Sketch of the history of geodetic survey

According to the childish concept which is expressed in the songs by Homer (800-900 B.C.), the earth was a disk surrounded by the Okeanos; and this concept has long been retained without letting interfere the direct view which, for instance, the curvature of the surface of the sea offers when a ship vanishes.

Pythagoras (born 582 B.C.) declared the earth to be a globe.

Aristotle (384-322 B.C.) develops the reasons for and against the global shape in the work *περὶ οὐρανοῦ* [*Concerning the Heavens*], B, 13-14, and arrives at the conclusion that the shape of the earth is necessarily that of a globe (Aristotle, Greek and German by Prantl, p. 178): "*ἀναγκαῖον εἶναι τὸ σχῆμα σφαιροειδές*" ["the necessity of the figure (of the earth) being spherical"].

Then as far as the question of the *size* of the earth is concerned, i.e. after the spherical shape had been recognized, the question of the circumference or the radius of the terrestrial sphere, we are to mention the Alexandrine scientist *Eratosthenes* (276-195 B.C.) as one of the first to whom we owe a historically warranted measurement or, as the case may be, estimate.

Eratosthenes used for the determination of the circumference of the earth the accidental favorable circumstance that in Upper Egypt in Assuan (today Syene, cf. Fig. 1), at the time of the summer solstice, the rays of the sun shone vertically into a well while, at the same time in Alexandria, the rays of the sun made a considerable angle with the direction of the plumb, which was measured at $\frac{1}{50}$ of 360°. The distance between the two points, Alexandria and Syene, was estimated from the number of one-day journeys at 5000 stadia.

On these data, Eratosthenes based a determination of the circumference of the earth according to the fundamental theorem which, from then on through thousands of years, has been used for carrying out "degree measurements."

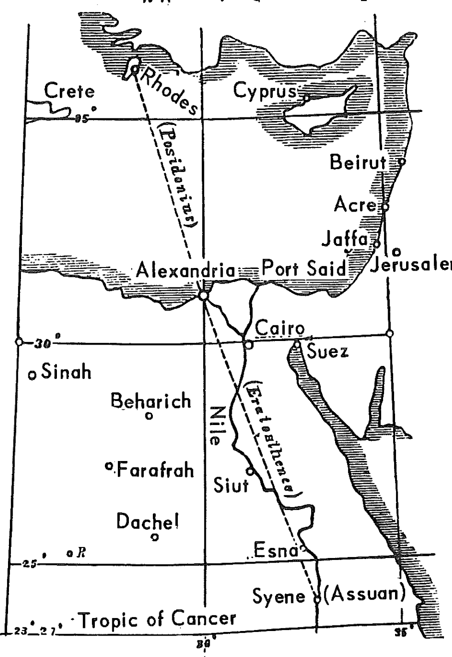

Fig. 1.
Measuring of a degree by Eratosthenes and Posidonius.
Scale 1:18,000,000.

We shall explain this most thoroughly by using Fig. 2:

A and A' here denote two points of the spherical surface of the earth (say, A = Alexandria, A' = Assuan) which are hit by the sun rays S and S'. Because of the great distance of the sun, the sun rays emanating from S and S' are to be considered *parallel*.

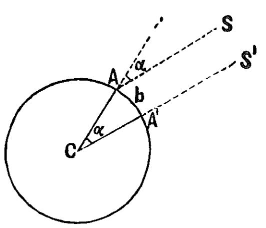

Fig. 2.
Measurement of degrees of latitude.

Let at A' the direction of the sun rays coincide accidentally with the vertical or the direction of the radius of the earth $A'C$ (well at Assuan); and let at A the sun rays make the angle α with the radius of the earth, which can be determined by shadow observation. This angle α at A then equals also the angle at the center of the earth ACA'; and if we further determine, in any way, the arc of meridian $AA' = b$, then we can determine from this the circumference of the earth:

$$U = \frac{360^\circ}{\alpha} b .$$

In the case of the measurement of degrees by Eratosthenes, we had $\alpha = \frac{1}{50}\,360^\circ$ and $b = 5000$ stadia, hence the

$$\text{circumference of the earth} = 50 \times 5000 = 250{,}000 \text{ stadia} .$$

As is seen from Fig. 1, Alexandria and Syene do not lie in the same meridian; also Syene does not lie precisely under the tropic, as was assumed by Eratosthenes; these secondary circumstances, however, are not important in the *first* approximate answer to the question of the circumference of the earth.

If we assume 1 stadium = 185 meters approximately (cf. Karsten, Allgemeine Encyclopädie der Physik, Vol. I, *Einleitung in die Physik* [*Introduction to physics*], Leipzig, 1869, p. 433, together with bibliography, p. 441), then we obtain: Circumference of the earth = 46,250,000 meters, or:

$$\text{Quadrant of the earth according to Eratosthenes} = 11{,}562{,}500 \text{ meters ;}$$

hence 16% too great since in reality the quadrant of the earth equals nearly 10,000,000 meters.

Like Eratosthenes, *Posidonius* (135-51 B.C.) made a determination of the circumference of the earth by means of the arc Alexandria-Rhodes (cf. Fig. 1) in the following way:

The star Canopus could barely be seen on the horizon at Rhodes, whereas it rose over the horizon by $\frac{1}{48}$ of a celestial great circle (= 7°30') at Alexandria; hence it can be concluded that the circumference of the earth is 48 times the arc of the terrestrial meridian Alexandria-Rhodes, and since this arc was estimated to be just as large as the former Alexandria-Assuan, namely = 5000 stadia, the circumference of the earth was found = 48 × 5000 stadia = 240,000 stadia; hence, calculating again the stadium as approximately = 185 meters, the circumference of the earth = 44,400,000 meters, or:

$$\text{Quadrant of the earth according to Posidonius} = 11{,}100{,}000 \text{ meters .}$$

In order to obtain a general view of the accuracy of these old measurements or, as the case may be, estimates, we bring them together with the numbers ΔB and b as known now:

	Latitudes B	ΔB	Arc of Meridian b	$\Delta B'$	Arc of Meridian b'
Rhodes	36°26'				
		5°14'	580 km	7°30'	5000 stadia = 925 km (Posidon.)
Alexandria	31°12'				
		7° 7'	789 km	7°12'	5000 stadia = 925 km (Eratosth.)
Syene	24° 5'				

The errors 925 km against 580 km and 925 km against 789 km are thus quite considerable.

In this connection we mention further a work on field and land survey of ancient times, namely *Heron* (approximately 200 B.C.) about the diopter (περὶ διόπτρας), cf. *Zeitschr. f. Verm.*, 1876, p. 120; 1887, pp. 553, 674; 1888, pp. 282, 325, 365; 1903, pp. 556, 591.

After this, we have to report about a geodetic survey carried out in the Middle Ages which we owe to the Arabs. Approximately around 827 A.D. they made a measurement of degrees of latitude about which the Dutchman Snell reports in his work, *Eratosthenes Batavus*, pp. 107-112, based on an Arabian writer, Abelfedeas (1322), somewhat as follows: The measurement was carried out by order of the Calif *Almanun* in the plane of Zinjar (northwest of Bagdad) under the latitude 36°20'.

The result was the following:

$$1 \text{ meridional degree} = 56\frac{2}{3} \text{ miles} = \frac{170}{3} \text{ miles}$$

$$1 \text{ mile} = 4000 \text{ ells ,}$$

hence the quadrant of the meridian of the earth:

$$Q = 90 \text{ degrees} = 90\,\frac{170}{3}\,4000 = 20{,}400{,}000 \text{ ells .}$$

Further let: 1 ell = 24 inches [zoll] and 1 inch = 6 breadths of a barleycorn [Gerstenkornbreiten]; but now what is 1 breadth of a barleycorn? Snell assumed: 1 breadth of a barleycorn = $\frac{1}{89}$ Rhineland foot, hence $\frac{0.313853}{89}$ = 0.00352644 meters, and this yields the following: 1 meridional degree = 115,103 meters, and

the quadrant of the meridian = 10,359 kilometers .

Instead of this barleycorn computation, we have obtained, in recent times, a determination of the Arabian ell according to the Nile meter of Cairo, which leads approximately to the correct way of determining the circumference of the earth.

For the Arabian ell with its 24 inches still exists in Egypt at the Nile meter (Mikyas) on the Nile island Rodah near Cairo. The Mikyas was constructed in 97 of the [era of the] Hejirah (716 A.D.) by order of the Omayyadian Calif Suleman (715-717) and repaired by the Abbassid Calif Manun in 199 of the Hejirah (814 A.D.). (This *Manun* seems to be *Al-Manun* of The Measurement of the Degree.)

In 1874 Jordan saw the column of the Nile meter near Cairo and made a cursory drawing and measurement of it, which is discussed in *Zeitschr. f. Verm.*, 1889, pp. 106-107. The Arabian ell has been found here to be approximately ≐ 0.52 meter; hence the quadrant of the meridian of the earth = 0.52 × 20,400,000 = 10,608,000 meters.

Meanwhile the Nilometer was examined and measured more accurately (at Jordan's suggestion) by Dr. *Reiss* in Cairo, and a detailed report was given by Dr. Reiss about it in *Zeitschrift f. Verm.*, 1889, pp. 439-445, which has also been partly printed in the fourth edition of this volume, 1896, p. 4.

According to this, 1 ell = 0.54 meter; and hence we have according to the Arabian measurement of the degree:

Quadrant of the meridian = 0.54 × 20,400,000 = 11,016,000 meters .

This is 10% too great, which is not a splendid result. In order to obtain the quadrant of the earth = 10,000,000 meters, we would have to set the ell = 0.49 meter, which is not practicable according to the condition of the Nilometer.

Since that time, nothing more has been done for 700 years.

We owe the first geodetic survey after this long interval of time to the French physician, *Fernel*, who, in 1525, measured the latitudes of Paris and Amiens by means of a quadrant and the distance between the two places by means of the rotations of his carriage wheel and, by so doing, obtained a result which incidentally is nearly correct, namely according to Lalande's recomputation:

1 meridional degree = 57,070 toises = 111,232 meters
quadrant of the meridian = 10,011 kilometers ;

hence the error = +0.1% only.

A new epoch of geodetic survey begins with the Dutchman *Willebrord Snell* (1580-1626).

As follows from his work, *Eratosthenes Batavus*, de terrae ambitus vera quantitate, a Willebrordo Snellio, Lugduni-Batavorum, 1617, Snell was not only from the mathematical viewpoint a very understanding man, but he was also generally very well educated and sagacious. Geodesy owes to him, if not the "invention," yet the first triangulation handed down to posterity, measured to approximately 1' at the angles and correctly computed trigonometrically; this has already been discussed in our first volume, *Handb. d. Verm.*, 8th edition, 1935, p. 507.

The whole triangulation by Snell includes 33 triangles which were used, essentially in the way still customary today, for a measurement of degrees of latitude between Alkmaar and Bergen op Zoom, as the following numbers show:

Point	Latitude	Difference in Latitude	Arc of Meridian (Triang.)
Alkmaar	52°40' 30"		
		1°11' 30"	33,930 Rhineland rods
Bergen op Zoom	51°29' 0"		

Hence 1 degree = $\frac{60'}{71.5'}$ 33,930 = 28,473 rods.

After this result, in connection with a second similar measurement, Snell assumed the degree of the meridian = 28,500 Rhineland rods. This is = 107,338 meters, and hence we also calculate

the quadrant of the meridian = 9660 kilometers .

According to this result, the determination of the earth by Snell has an error of 3.4%. Snell himself made verifications, but only his successor, *Musschenbroek*, brought Snell's work to a conclusion; he found in 1719 "secundum mensuram ultimam Snellii et nostram," 1 degree of the meridian = 29,514 rods or = 111,157 meters, hence

the quadrant of the meridian = 10,004 kilometers .

In the meantime, two measurements of the degree, remarkable by the nature of their methods, had followed Snell's. In 1633, *Norwood* measured the arc between London and York directly with the chain.

In Italy, in 1645, *Grimaldi* and *Riccioli* determined one degree of the meridian by reciprocal *terrestrial* zenith distances.

This measurement of terrestrial zenith distances which Snell also already mentions in his closing chapter would be the simplest and best means for the measurement of the earth if the refraction of rays did not exist or, at least, were better amenable to calculation than is the case thus far.

As we now approach the precise measurements of the earth, we also have to mention briefly *which* surface of the earth is to be determined: We are to consider, as surface of the earth in the sense of these measurements, the surface of the sea imagined at rest, together with its continuation below the continents which is assumed to be steady.

The most important geodetic surveys of the 17th and 18th centuries are the French. They were suggested by the Paris Academy, founded in 1666, and were directed by *Picard*. The purpose of these measurements was a double one, first the preparation of a good map of France, and secondly the determination of the size of the earth.

From the continuation of Picard's measurements which was directed by *Lahire, Dominique Cassini* and *Jacques Cassini* (1683-1716, to the south as far as Collioure, to the north as far as Dunkirk), it seemed to follow that the earth is pointed at the poles, whereas Newton's theory and Richer's pendulum experiments asserted the contrary. The question was decided by the geodetic survey expeditions sent by the French to Peru and Lapland in 1735 through which it was established that the terrestrial meridian is more strongly curved at the equator than it is in the neighborhood of the pole, which agrees with the Newtonian theory.

The measuring of degrees in Peru by *Godin, La Condamine* and *Bouguer*, 1735-1741, is described in the work, *Mesure des trois premiers degrés du méridien dans l'hémisphère australe, tirée des observations de Mrs. de l'académie royale des sciences, envoyés par le roi sous l'équateur*, par M. de la Condamine, Paris, 1751 [*Measurement of the first three degrees of the meridian in the southern hemisphere, drawn from observations by men of the Royal Academy of Sciences who were sent to the equator by the king*, by Monsieur de la Condamine, Paris, 1751].

The measuring of the degree in Lapland, carried out 1736-1737 by *Maupertuis* and *Clairaut*, is described by Maupertuis, *La figure de la terre*, Paris, 1738 [*The figure of the earth*, Paris, 1738].

This measuring of degrees made a verification and addition necessary which was carried out, by order of the Swedish Academy, by *Svanberg*, and which is described by him in *Exposition des opérations faites en Lapponie en 1801-1803 pour la détermination d'un arc du méridien*, Stockholm, 1805 [*Exposition of the operations made in Lapland in 1801-1803 for the determination of an arc of the meridian*, Stockholm, 1805]. In 1928, the astronomical determinations were repeated by the Finnish Geodetic Institute and a report by Leinberg has been published about it in the *Verh. d. 4. Tagung der Balt. Geod. Komm. in Berlin 1928*, Helsinki, 1929 [*Proceedings of the Fourth Session of the Baltic Geodetic Commission in Berlin 1928*, Helsinki, 1929], pp. 148-160.

In 1740 there followed a verification of the French arc of the meridian by Cassini de Thury (III) and Lacaille.

The following are the most important numerical values referring to it:

Picard's measurement of the arc between Paris and Amiens which began in 1669 yielded a degree of latitude = 57,060 toises and hence:

Quadrant of the meridian = 10,009,081 meters .

The northern and southern continuation yielded the following:

	Mean Latitude	1 Degree of the Meridian
North between Paris and Dunkirk	49°56'	56,960 toises
Between Paris and Amiens	49°22'	57,060 toises
South between Paris and Bourges	47°57'	57,098 toises .

Hence there seemed to follow a shape of the earth which is pointed toward the poles.

We tried to calculate from this a greatly elongated ellipsoid which thus obtains a *negative* flattening. It was found:

Quadrant of the meridian = 10,042,650 meters, flattening = 1:−66 .

The Peruvian and Lapp measurements of degrees (the latter with the correction and addition by Svanberg, 1801-1803) yield the following:

	Mean Latitude	1 Degree of the Meridian
Lapland	+ 66°20' 10"	57,196 toises
Peru	− 1°31' 30"	56,734 toises .

Hence we calculate:

Quadrant of the meridian = 10,000,157 meters, flattening = 1:310.3 .

Of the non-European geodetic surveys which now follow, we mention here especially that by Mason and Dixon in North America, 1764-1768.

It is distinguished by the fact that a straight line, 434,011.64 feet (= 132,286 meters) long, was measured directly with measuring rods (hence without triangulation) approximately in the direction of the meridian. In all, the arc of meridian between the latitudes 39°56' 19" and 38°27' 34" was determined.

(For a more recent communication about it, see *Zeitschr. f. Verm.*, 1888, pp. 33-39.)

The measuring of degrees by Lacaille at the Cape of Good Hope, Beccaria's measurement in Turin, Liesganig's measurement in Hungary, then the beginnings of the English surveys in England proper and in India, also occurred during the second half of the eighteenth century.

The most important geodetic survey at the end of the eighteenth and beginning of the nineteenth centuries was again French, however, namely by *Delambre* and *Méchain*, 1792-1808, for the final determination of the *meter*. The work published on this subject is *Base du système métrique décimal, ou mesure du méridien compris entre les parallèles de Dunkerque et Barcelone, exécutée en 1792 et années suivantes par M. M. Méchain et Delambre*, rédigée par M. Delambre. Tome premier Paris janvier 1806, tome second Paris juillet 1807, tome troisième Paris novembre 1810 [*Base of the decimal metric system, or measurement of the meridian comprised between the parallels of Dunkirk and Barcelona, carried out in 1792 and the following years by Messrs. Méchain and Delambre*, edited by Monsieur Delambre. First volume, Paris, January 1806; second volume, Paris, July 1807; third volume, Paris, November 1810].

The meter was supposed to be, as precisely as possible, the ten-millionth part of the quadrant of the earth, for the determination of which the measurement of the degree was undertaken by Delambre and Méchain in 1792. It yielded the arc between Dunkirk and Montjouy = 275,792.36 modules (1 module = 2 toises); and this arc of nearly 10° between the latitudes 51°2' 8.85" and 41°21' 44.96" was used as the foundation for the metric system. In order to obtain the flattening, this arc was combined with the Peruvian measurement of degrees, of which the flattening 1:334 was obtained.

Now the quadrant of the meridian was computed = 2,565,370 modules = 5,130,740 toises, and since one toise has 864 Paris lines, hence we have:

$$1 \text{ meter} = \frac{5,130,740 \times 864}{10,000,000} = 443.295936 \text{ Paris lines },$$

which was rounded to 443.296.

Since, vice versa, we assume the meter = 443.296 Paris lines as the unit for the Delambre measurement, we have

the quadrant of the meridian = 10,000,000 meters; flattening = 1:334 .

(The above numerical data are found in the work, *Base du système métrique*, Volume III, pp. 433, 619-622.)

We add to these older French enterprises two further geodetic surveys at the beginning of the nineteenth century, the Hannover degree-measurement by *Gauss*, 1821-1823, and the East Prussian degree-measurement by *Bessel*, 1831-1838, the results of which have likewise but a historical significance only, which for the rest, however, count among classical geodetic work. (Further details about them are given in the first volume, eighth edition, 1935, pp. 522-529.)

Determination of the ellipse of the meridian

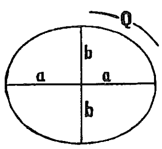

Fig. 3.
Ellipse of the terrestrial meridian.

After the flattening of the earth had been decided upon, it was no longer a question of *one* unknown only, as before, namely the radius of the terrestrial sphere, but of *two* unknowns, say, the two semiaxes a and b of the ellipse of the meridian (Fig. 3), or else of one semiaxis a and, in addition, the eccentricity $e = \sqrt{\frac{a^2 - b^2}{a^2}}$ or the quadrant of the meridian Q and the flattening $\alpha = \frac{a-b}{a}$.

As we shall develop later, we calculate the quadrant of the meridian Q from the major semiaxis a and the flattening α according to the formula

$$Q = \frac{a\pi}{2}\left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{16}\right).$$

If two measurements of degrees exist in the sense applied thus far, namely an arc of meridian and the two geographic latitudes of the end points for each measurement of degrees, then, considered from the mathematical viewpoint, it is an easy task to determine, with sufficient approximation, an ellipse which satisfies the *two* measurements of degrees, and later we shall be able to develop, in a few equations, for instance, the calculation of the measurements of the degree of Lapland and Peru mentioned above.

In the past century, however, when we started to take into account more than two measurements of the degree at the same time, we met with serious discrepancies which could hardly be explained by the unavoidable error of measurement and soon gave rise to the thought that the earth is not precisely an ellipsoid of rotation.

Nevertheless a period of time of a good hundred years (say, from 1740-1840) was now dedicated to the task of attaining an *adjustment* of the numerous measurements of degrees, such that the remaining discrepancies in the measurements of geographic latitudes turn out as small as possible as a whole.

In this respect, the history of the computations of measurements of degrees coincides with the history of the method of least squares which we have discussed in our first volume, *Handb. d. Verm.*, 8th edition, 1935 (Introduction).

The first published article about the method of least squares, namely *Legendre's* treatise, *Sur la méthode des moindres carrés*, which appeared in 1806 as an appendix to *Nouvelles méthodes pour la détermination des orbites des comètes*, also contains, at the same time, the first adjustment of measurements of degrees according to this method. Legendre takes 5 geographic latitudes between Dunkirk (51°2' 10.50") and

Montjouy (41°21' 44.80") with the 4 intermediate French arcs of meridian and thus makes a theoretically correct adjustment which yielded however the very large value of flattening of 1:148, and also too small a quadrant of the meridian = 9,997,780 meters; yet the result of the computation of the earth concerns us here less than does the theoretically correct method applied here.

Walbeck made the next adjustment of this kind in 1819 in a small treatise, "De forma et magnitudine telluris, ex dimensis arcubus meridiani, definiendis" ["On the form and size of the earth defined from measured arcs of the meridian"] (Abo 1819), which has long been known only from a citation by Gauss in the *Bestimmung des Breitenunterschieds zwischen Göttingen und Altona* [*Determination of the difference in latitude between Göttingen and Altona*], 1823, pp. 72 and 82, was later reprinted in Helsingfors in 1891 and from there also reprinted in the *Zeitschr. f. Verm.*, 1893, pp. 426-434.

Walbeck submitted to computation the Peruvian, the two East Indian, the French, English and the newer Lapp measurements of degrees; in the case of each individual measurement of degrees, however, he took into account only the whole arc or the latitudes observed at the *end* points, without considering the intermediate points. The result was flattening = 1:302.78, and mean degree of the meridian = 57,009.76 toises; this yields:

Quadrant of the meridian = 10,000,268 meters; flattening = 1:302.78 .

Nine years later, in 1828, we have an adjustment, further improved and extended, which *Schmidt* made in Göttingen, at Gauss' suggestion, as discussed by Gauss in *Bestimmung des Breitenunterschieds zwischen Göttingen und Altona*, pp. 82-84. Schmidt took into account also the polar heights observed at the intermediate points, included also the measuring of degrees of Hannover, so that he adjusted the Peruvian, first and second East Indian, French, English, Hanoverian and Swedish measurements of degrees together with 25 latitudes according to the fundamental theorem that the sum of squares of the remaining errors of latitudes become a minimum. The result was as follows: Flattening 1:(298.39 ± 12.5) and mean degree of the meridian = 57,010.35 ± 5 toises, or:

Quadrant of the meridian = 10,000,372 meters; flattening = 1:298.39
±88 ±12.50 .

The mean error of latitude is ±3.18" .

Schmidt carried such calculations still further and discussed them in *Lehrbuch der mathematischen und physischen Geographie* by Dr. J. C. Eduard Schmidt, Privatdozent auf der Universität Göttingen. First Part, 1829, Foreword, pp. IV-V, and *Astronomische Nachrichten*, Volume 7 (1829), No. 161, pp. 329-332.

Schmidt's last determination dates from the year 1830 and yields [according to Listing (3)]:

Quadrant of the meridian = 10,000,061 meters; flattening = 1:297.648 .

In 1830, the Englishman *Airy* made a determination from 14 measurements of degrees of latitude, including a few degrees of longitude measured. The results are the following, according to *Ordnance trigonometrical survey of Great Britain and Ireland*, London, 1858, Introduction, p. XVI: a = 20,923,713 feet and b = 20,853,810 feet, from which we calculate:

Quadrant of the meridian = 10,001,012 meters; flattening = 1:299.325 .

Now we arrive from 1837-1841 at Bessel's adjustment of degree-measurements, which is based on a very profound examination and study of the material of geodetic surveys collected until then. Bessel used the following measurements:

Degree-Measurement	Mean Latitude	Amplitude	Number of Latitudes
1. Peruvian	−1°31'	3° 7'	2
2. First East Indian	12°32'	1°35'	2
3. Second East Indian	16° 8'	15°58'	7
4. French	44°51'	12°22'	7
5. English	52° 2'	2°50'	5
6. Hanoverian	52°32'	2° 1'	2
7. Danish	54° 8'	1°32'	2
8. Prussian	54°58'	1°30'	3
9. Russian	56° 4'	8° 2'	6
10. Swedish	66°20'	1°37'	2
	Total	50°34'	38

The details of the measurements and adjustment (sum of squares of the 38 corrections of latitudes) are given by Bessel in a treatise in the *Astr. Nachr.*, Volume 14, 1837, No. 333, pp. 333-346. A new computation, however, became necessary because an error was found in the French measurement of degrees, which Bessel discusses in Volume 19 of the *Astr. Nachr.*, 1842, No. 438, pp. 97-116. Bessel's final results are (2 December 1841):

$$\text{Quadrant of the meridian} = 10{,}000{,}855.76 \text{ meters} \pm 498.23; \quad \text{flattening} = 1:299.1528 \pm 4.667 .$$

These terrestrial dimensions by Bessel soon found the most general recognition and the widest distribution; they are especially important because auxiliary tables in great numbers and extent for practical surveys and computations are based on them.

The *English* triangulations to which we shall refer once again on p. 9 gave rise to a whole series of new computations of the terrestrial ellipsoid by A. R. Clarke.

In the great work about the English land survey (cf. p. 9) on p. 771, there were given at first as the result of an adjustment of eight measurements of degrees with 66 latitudes

$$a = 20{,}926{,}348 \text{ feet} \qquad b = 20{,}855{,}233 \text{ feet}$$

or

$$a = 6{,}378{,}294 \text{ meters} \qquad b = 6{,}356{,}618 \text{ meters} ,$$

from which we calculate:

$$\text{Quadrant of the meridian} = 10{,}001{,}983 \text{ meters}; \text{ flattening} = 1:294.261 .$$

A later computation dating from the year 1866 yielded the following values:

$$a = 6{,}378{,}206.4 \text{ meters} \qquad b = 6{,}356{,}583.8 \text{ meters}$$
$$\text{flattening} = 1:295.0 .$$

The triangulations of the Coast and Geodetic Survey in the United States were based on these dimensions of the earth.

J. B. Listing gives an account on these and further computations by Clarke, which are also based in part on the hypothesis of a triaxial ellipsoid, in *Nachr. v. d. Kgl. Ges. d. W. u. d. Univ. zu Göttingen 1873* [Read: Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Universität zu Göttingen, 1873 – *Information bulletin by the Royal Society of Sciences and the University at Göttingen 1873*], pp. 33-98.

We only mention further the last computation which is published in Clarke, *Geodesy*, 1880, p. 313. According to it, we have

$$a = 20{,}926{,}202 \text{ feet} \qquad b = 20{,}854{,}895 \text{ feet}$$

or

$$a = 6{,}378{,}249 \text{ meters} \qquad b = 6{,}356{,}515 \text{ meters} .$$

Next to Bessel's dimensions of the earth, these last values by Clarke obtained the greatest significance.

Newer geodetic surveys

In the information given thus far about the *calculations* we went in part ahead of the history of the *measurements*. In this connection, we shall now list all those measurements whose results form the material for the newer calculations of the measurement of the earth.

The *Russo-Scandinavian measurement of degrees of latitude* was begun by *F. G. W. Struve*, Director of the Dorpat Observatory, in 1821-1831, and extended over the 44th meridian from Jakobstadt to the island of Mäkipäälys in the Finnish Gulf of the Baltic Sea. This arc of the meridian was extended by General *Tenner*, until 1852, to the north as far as the Arctic Ocean and to the south as far as the mouth of the Danube so that he obtains an amplitude of 25°. (Cf. Vol. I, 8th edition, 1935, p. 516.)

The *West European measurement of degrees of latitude* consists of the English, French and Spanish chains of triangulation.

The English surveys began in 1783 under General *Roy*. We have about them the large work, "Ordnance trigonometrical survey of Great Britain and Ireland. Account of the observations and calculations of the principal triangulation and of the figure, dimensions and mean specific gravity of the earth as derived from, etc., by Captain *Alexander Ross Clarke* under the direction of Colonel *H. James*, Superintendent of the Ordnance survey. London, 1858." (Cf. Vol. I, 8th edition, 1935, pp. 514-515.)

In 1860 the English Government suggested in France the attachment of the triangulations of the two countries over the Channel, and this gave rise in France to a new measurement of the old meridian chain by Delambre and Méchain, which was carried out in 1870-1894 with the astronomical measurements necessary.

The continuation of the triangulation chain to the south coast of Spain is part of the Spanish triangulation net which was measured in 1859-1877.

The connection of Spain and Algeria prepared under difficult conditions in 1879, by means of a few triangles of extraordinary size, by means of which the further extension of the measurement of degrees of latitude on the African continent was made possible, is of great significance. The chain reaches its provisional end point at Laghouat in Algeria and comprises an arc of the meridian of 27° of extent with 38 astronomical latitude stations.

The measurement of degrees of longitude at 52° northern latitude between Valentia on the west coast of Ireland and Orsk south of the Ural Mountains with an extent of 69 degrees of longitude was suggested by *W. Struve* in 1857. The triangulation chains are, in substance, part of the land surveys of England, Belgium, Prussia and Russia, in addition to which a few more connecting nets were measured. For the part of the measurement of degrees of longitude lying between Greenwich and Warsaw, we have a publication of the Geodetic Institute in Potsdam, *Die europäische Längengradmessung in 52 Grad Breite von Greenwich bis Warschau*, Heft I, 1893, Heft II, 1896. In regard to the Russian chains of triangulation, cf. also Vol. I, 8th edition, 1935, p. 516.

The measurement of degrees of longitude at 47½° northern latitude runs likewise through the whole of Europe, from Brest to Astrachan. The trigonometric operations for it have been nearly completed.

The measurement of degrees in India was begun already in the first years of the 19th century during which an arc of the meridian of 1°35' as well as an arc perpendicular to the meridian were measured by Major *Lambton* west of Madras; the first one forms the first Indian measurement of degrees which has already been mentioned on pp. 7 and 8. The measurement of a meridian arc of an extent of 22½° which begins at the southern point of India at Cape Comorin was attached to this, as well as the measurement of an arc of the parallel which runs through the whole of India at 24° northern latitude. The southern part of the latter measurement of degrees of latitude with an extent of approximately 16° was used in the adjustments previously mentioned (pp. 7 and 8) as second Indian degree-measurement.

The North American degree-measurements, in their earliest beginnings, have already been mentioned on p. 5. In 1807 there was founded a land surveying bureau, the *United States Coast Survey*, which set to work in 1816, and by which, at first, chains of triangles along the coasts, later also extended triangulations in the interior were measured. With this enlarged field of work, the bureau obtained the name of *Coast and Geodetic Survey*.

The first geodetic survey work resulted from the nets of triangles for the measurement of the Great Lakes from 1841-1878; this work, however, remained without a permanent value because of the deficient determination of the unit of length. (*Zeitschr. f. Verm.*, 1888, pp. 385-395.)

Two further measurements of degrees of latitude of an extent of 3½° and 4½°, respectively, of the years 1844-1860 and 1844-1876, which, starting from the coast, extended over the meridians 70°20' and 76°10' likewise did not obtain further significance. (Cf. *Rep. of the Superintendent of the U. S. Coast Survey 1868*. Washington, 1871, pp. 147-153, and 1877; Washington, 1880, pp. 84-95.)

During the last years of the nineteenth century, two extensive works have been completed:

1. The measurement of degrees of longitude at 39° northern latitude from the Atlantic to the Pacific Ocean with an extent of nearly 49° was carried out in the years from 1871-1898. It includes 109 latitude stations, 73 azimuth stations and 37 longitude stations, as well as 10 base line measurements. The details are presented in the large work, *The Transcontinental Triangulation and the American Arc of the Parallel*, Washington, 1900. (Cf. the reference in *Vierteljahrschr. d. Astr. Ges.* [*Quarterly publication of the Astronomical Society*], Vol. 36, pp. 242-250.) The longest triangle side measures 294 kilometers.

2. The oblique degree-measurement along the east coast of the United States from 30° to 45° northern latitude has an amplitude of 23½° and 71 latitude stations, 17 longitude stations and 55 azimuth stations. The official publication forms the work, *The Eastern Oblique Arc of the United States and Osculating Spheroid*, Washington, 1902.

3. The measurement of degrees of latitude at the meridian 98° from the south coast of Mexico to the Arctic Ocean has been carried out only to a minor extent.

These extensive measurements offered the foundation for a new calculation of the ellipsoidal constants carried out in the United States under the direction of Hayford and published in 1909 and 1910. It yielded

$$\text{Major semiaxis:} \quad a = 6{,}378{,}388 \text{ meters}$$

$$\text{Flattening:} \quad \alpha = 1{:}297 \ .$$

The ellipsoid thus found is called the *International Ellipsoid* because it adapts itself best to the actual surface of the earth at present.

The African measurement of degrees of latitude at the 30th meridian east of Greenwich was suggested by Sir *David Gill* and is to extend from Cairo to the Cape of Good Hope, thus comprising an arc of the meridian of 65°. Of this, the southern part as far as Lake Tanganyika has been carried out by the English surveys in the Cape Colony, in Transvaal and Rhodesia, in the years from 1883-1906, so that a stretch of the meridian of an extent of 23° is available complete at present. Additional short stretches in northern Africa, especially Egypt, are likewise already worked up; reports on them are included in *Verhandlungen der 16. allgem. Konf. d. Internat. Erdmessung in London und Cambridge 1909* [*Proceedings of the 16th general conference of the Internationale Erdmessung at London and Cambridge 1909*]. First Part, Berlin, 1910, pp. 219-225, and in *Bulletin géodésique*, 1933, pp. 84-91.

The measurement of degrees of latitude on Spitzbergen is an undertaking carried out in common by Russia and Sweden in 1898-1902 which, in view of extreme climatic conditions, had to fight against greatest difficulties. The measurement of degrees extends over a meridian arc of approximately 4° with the mean latitude of 79°. In spite of all pains, the value of ±1.8" was reached for the mean error of an angle of a triangle in the angular measurement. (Cf. *Verhandl. d. 16. allg. Konf. d. Intern. Erdm. 1909*, First Part, Berlin, 1910, pp. 287-296 and 301-308.)

The French *measurement of the degree in South America*, which took place at the instigation of the Internationale Erdmessung in the years from 1899-1906 at an extent of 5°52', represents a new measurement of the arc of the meridian in Peru. The measurements were carried out by officers of the Service géographique de l'armée under the supervision of the Paris Academy of Sciences. This undertaking also met with great difficulties, partly by climatic conditions, partly by the hostile attitude of the natives. The results of this undertaking are published in the work, *Mission du Service géographique de l'armée pour la mesure d'un arc de méridien équatorial en Amérique du Sud sous le contrôle scientifique de l'Académie des Sciences 1899-1906,* of which several volumes already appeared.

As a supplement to the above information, we give, in Fig. 4, below, a general outline map of the most important measurements of degrees which we took from the *Verhandl. d. 17. allg. Konf. d. Intern. Erdm.,* Vol. I, 1913.

At the same time we refer to the historical sketches in Vol. I, Eighth Edition, 1935, pp. 507-529.

Measurements of gravity

As the most important supplement to measurements of degrees, which always are tied to the continents, we have the measurements of gravity from which a contribution to the knowledge of the figure of the earth likewise results. In 1743, the French physicist, *Clairaut,* proved that the gravity on the equator and the gravitational decrease from the pole to the equator are in simple ratio to the flattening of the earth. This theorem by Clairaut thus offers the possibility of determining the flattening of the earth from gravitational measurements. Especially in recent times, since a convenient way for the determination of gravity had been found by pendulum measurements, this method has reached a rather great importance. In 1900, measurements of gravity at 1400 places of the surface of the earth were already available, from which *F. R. Helmert* derived a formula for the gravity at sea level as a function of geographic latitude. Hence there resulted the value 1:298.3 for the flattening of the earth which deviates only a little from Bessel's value.

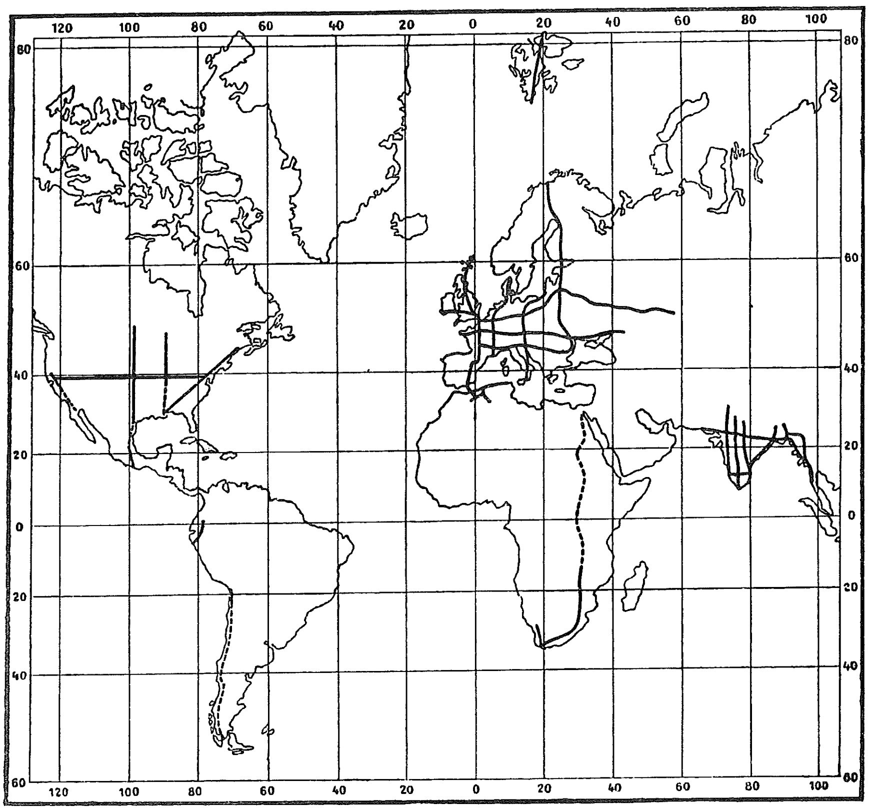

Fig. 4.
General map of measurements of the degree.

Meanwhile, measurements of gravity increased constantly. Whereas the measurements had thus far been limited to the continent, successful experiments started at the beginning of the twentieth century to determine the force of gravity at sea. A systematic measurement of gravity on the oceans which is of decisive importance for the determination of the figure of the earth has thus been brought within our grasp.

The first computations of the dimensions of the earth from more than two measurements of degrees already showed differences which could not be explained by the inaccuracies of measurement alone and finally lead to the certainty that the surface of the earth is by no means accurately represented by an ellipsoid of rotation. In 1873, *J. B. Listing* introduced the denotation, *geoid,* for the actual mathematical surface of the earth, i.e. the surface of the sea at rest and its continuation beneath the continents. Now since the surface of the geoid has approximately the form of an ellipsoid of rotation, it is convenient, for the study of this surface, to determine the small deviations from an ellipsoid of reference appropriately chosen, e.g. the ellipsoid by Bessel. The plumb directions of the geoid which are given by the direction of the gravity make here small angles with the normals to the ellipsoid, the deflections of the plumb, and now the task of modern degree-measurement is to determine, in addition to the dimensions of the ellipsoid of rotation best suited to the geoid, also the deflections of the plumb for as many points of the terrestrial surface as possible.

Whereas formerly astronomical position determinations at a few points were sufficient, for this enlarged task it is necessary that we have a net of astronomical points as dense as possible at which geographic longitude and latitude are to be determined. If we assume now that for one of these points the direction of the plumb coincides with the normal to the ellipsoid, then we can calculate for all remaining points, with the help of triangle measurements, the ellipsoidal longitudes and latitudes which, compared with the astronomical determinations, yield the deflections of the plumb. If in these calculations we introduce further corrections of the dimensions of the ellipsoid as unknowns, these corrections can be determined in such a way that the deflections of the plumb become as small as possible. This last requirement can be satisfied better still if the deflections of the plumb of the starting point also are not set equal to zero, but taken into the calculation as unknowns.

According to this fundamental principle, *Helmert* developed the method of calculation for such an astronomic-geodetic net, and for some time the preparation of a large deflection-of-the-vertical system, comprising the whole of central Europe, has been under way.

In a similar way, the astronomic-geodetic measurements in the United States which extend over an extremely large territory have been used for the calculation of new elements of the ellipsoid (cf. p. 10).

Formerly it was believed that the cause of the deflections of the vertical had to be assumed in the visible irregularities of the distribution of the masses on the surface of the earth. When in the middle of the past century, however, *Henry Pratt* calculated the influence of the masses of the Himalaya Mountains for several stations of the Indian degree-measurement and compared them with the astronomic-geodetic measurements, he arrived at the result that these masses of mountains would have to be compensated by a lesser density of the subterranean masses.

This hypothesis set up by Pratt was also found to be confirmed in the case of later investigations, and we can assume that the continents as well as the oceans are compensated beneath the surface, and that only the secondary irregularities in the distribution of masses cause the deflections of the vertical.

In the case of the calculations of the deflections of the vertical in the United States, just mentioned, the question was raised by *Hayford* as to what depth the compensating masses reach, and it followed that we can assume a surface, approximately 120 kilometers under the surface of the earth, such that the masses lying on it exert the same pressure in every place.

The attraction of the masses influences not only the direction but also the size of the gravitational force, and consequently the measurements of gravity also offer an excellent means for the study of the distribution of the masses on the surface of the earth. These measurements of gravity, including those thus far carried out at sea, are likewise a confirmation of the hypothesis by Pratt.

With the constant expansion of the measurements of gravity, we shall also arrive at a direct determination of the elevation of the geoid with respect to the reference ellipsoid. In 1849, the English physicist, *Stokes*, developed a formula from which we can calculate this elevation for every point of the surface of the earth if the deviations of the force of gravity from its normal value are known for the whole surface of the earth. Although at present we are still far from such a complete knowledge of the force of gravity, we are now in a position to make use, at least approximately, of Stoke's formula. Such calculations, which were undertaken by Helmert in 1910, by Schumann in 1911, showed that the geoid hardly deviates from the reference ellipsoid in any place by 100 meters.

With the measurements of gravity, and especially with the hypothesis of isostasy, the study of the distribution of the masses within the terrestrial body has entered the range of geodetic work. In close connection with this subject are the small motions of the two poles on the terrestrial surface, which were continuously determined by the uniform co-operation of a series of astronomical stations during the last decades. These measurements led to the result that the two poles move around a central position in more or less regular circles, with a period of approximately 434 days. In the case of a completely rigid, terrestrial body, these motions would have to take place in the so-called Euler's period of 303 days, and from the extension of this period it has been proved, especially by the work of W. Schweydar, that we must assume a central rigidity for the earth which is three to four times as great as the rigidity of steel.

The measurements by means of the horizontal pendulum with which we can determine the deformations of the earth under the attracting effect of the moon and the sun led to nearly the same result.

Internationale Erdmessung

The great progress achieved during the last centuries in the study of the figure of the earth is due, in the first place, to the common work of all civilized countries which joined in a large scientific union, the "Internationale Erdmessung."

Lieutenant General *Baeyer*, Bessel's co-worker in the degree-measurement in East Prussia, submitted in 1861, to the Prussian Government the plan of uniting the chains of triangulation existing in Central Europe, in connection with the necessary astronomical measurements, as a Central European degree-measurement. The Prussian Government agreed to this plan and in April 1862, at its invitation, the delegates of Prussia, Saxony and Austria met at a conference in Berlin, on the basis of which Baeyer published, still in the same year, the first "General Report on the State of the Central European Degree-Measurement," with 16 states participating.

In the fall of 1864, the "Erste allgemeine Konferenz der Bevollmächtigten zur mitteleuropäischen Gradmessung" ["First general conference of the delegates for the Centraleuropean measurement of degrees"] took place in Berlin, in which Baeyer was elected president. The scientific leadership of the union was assigned to a "permanente Kommission"; in addition, the foundation of a Central Office, which was to assist the Permanent Commission as the executive organ, was decided upon.

In 1869, by the order of Baeyer, the foundation of the Geodetic Institute took place, and the Central Office of the union, expanded in the meantime to become the "European Degree-Measurement," was attached to it.

After Baeyer's death, *F. R. Helmert* stepped to the head of the Geodetic Institute and the Central Office, on the 1st of January, 1886. At the general conference which was held at Berlin in the fall of the same year, a new organization of the union was decided upon, through which the Permanent Commission was provided with its own financial means for the promotion of geodetic survey work. At the same time, the union expanded by the joining of non-European countries, to become the "Internationale Erdmessung" ["International Measurement of the Earth"]. A reorganization again took place at the eleventh general conference in Berlin, in 1895. (*Zeitschr. f. Verm.*, 1895, pp. 569-586 and 625-630.)

Before the beginning of the World War, the following countries belonged to the Internationale Erdmessung: Argentina, Australia, Belgium, Chile, Denmark, Germany, France, Greece, Great Britain, Italy, Japan, Mexico, the Netherlands, Norway, Austria, Portugal, Romania, Russia, Sweden, Switzerland, Spain, Hungary, and the United States of America.

The general conference met every three years; on the occasion of the last conference in the fall of 1912, a celebration took place, at the same time, in honor of the 50th anniversary of the existence of the organization.

On the 15th of June, 1917, Friedrich Robert Helmert, the most successful promoter of geodetic science and the Internationale Erdmessung, died.

The agreements on which the Internationale Erdmessung was based ceased to exist during the World War in 1916 and were not renewed again. The continuation of the most important activities, especially the international latitude service, was first taken over by some countries not participating in the World War, and in July 1919, within the International Geodetic and Geophysical Union, a new geodetic organization was founded in Brussels, which received the inheritance of the International Geodetic Survey. In 1937, Germany also announced her entrance in the International Union, to which 36 countries belong at present.

* * * *

Chapter II

MATHEMATICAL AUXILIARY MEANS OF GEODETIC DEVELOPMENTS

We insert this small chapter here in order to have on hand the most customary formulae and numerical values for geodetic developments and to be able to use them as may be required.

Section 33. Spherical Trigonometry

1. Rectangular spherical triangle

We assume the following denotations with reference to Fig. 1:

Hypotenuse	$= c$	Opposite angle $= 90°$
Leg	$= a$	Opposite angle $= \alpha$
Leg	$= b$	Opposite angle $= \beta$.

With these we have the following equations:

$$\cos c = \cos a \cos b$$
$$\cos c = \cot \alpha \cot \beta$$
$$\sin \alpha = \frac{\sin a}{\sin c} \quad \text{and} \quad \sin \beta = \frac{\sin b}{\sin c}$$
$$\cos \alpha = \frac{\tan b}{\tan c} \quad \text{and} \quad \cos \beta = \frac{\tan a}{\tan c}$$
$$\tan \alpha = \frac{\tan a}{\sin b} \quad \text{and} \quad \tan \beta = \frac{\tan b}{\sin a}$$
$$\cos \alpha = \sin \beta \cos a \text{ and } \cos \beta = \sin \alpha \cos b \, .$$

These equations are easily impressed on the memory in this form if we bear in mind the analogies with the formulae of *plane* trigonometry.

For instance, according to Fig. 1 we would write for a *plane* right triangle:

$$\sin \alpha = \frac{a}{c} \qquad \cos \alpha = \frac{b}{c} \qquad \tan \alpha = \frac{a}{b}$$

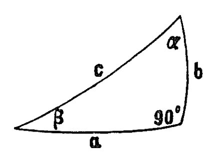

Fig. 1.
Right spherical triangle.

and now we only need to remember that sine has sine in the numerator and sine in the denominator, just as, accordingly, tangent has tangent and sine, finally, that *cos* has *tan* in the numerator and in the denominator both times, in order to be able to write these formulae always from memory.

Let us remember further that the first formula for *cos c* corresponds to the Pythagorean theorem of the plane, i.e. $c^2 = a^2 + b^2$, and that the second and sixth formulae correspond to the relationship in the plane $\alpha + \beta = 90°$.

Furthermore, we have Napier's rule: If we consider the parts a, c, β, $90° - a$, $90° - b$, a . . . in cyclic succession, then the cosine of any part is equal to the product of the sines of the opposite parts and equal to the product of the cotangents of the adjacent parts.

II. General spherical triangle

According to Fig. 2, we denote

side a with the opposite angle α
side b with the opposite angle β
side c with the opposite angle γ .

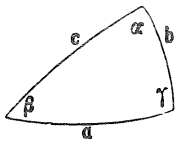

Fig. 2. Spherical triangle.

We have at first the following four groups of equations which contain *four* parts each and which are sufficient for the determination of a triangle from three given parts:

Cosine Law

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos \beta$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

Sine Law

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

Cotangent Law

$$\cot a \sin b = \cos b \cos \gamma + \sin \gamma \cot \alpha$$
$$\cot b \sin c = \cos c \cos \alpha + \sin \alpha \cot \beta$$
$$\cot c \sin a = \cos a \cos \beta + \sin \beta \cot \gamma$$

$$\cot a \sin c = \cos c \cos \beta + \sin \beta \cot \alpha$$
$$\cot b \sin a = \cos a \cos \gamma + \sin \gamma \cot \beta$$
$$\cot c \sin b = \cos b \cos \alpha + \sin \alpha \cot \gamma$$

Polar Cosine Law

$$\cos \alpha = - \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$$
$$\cos \beta = - \cos \gamma \cos \alpha + \sin \gamma \sin \alpha \cos b$$
$$\cos \gamma = - \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c .$$

A relationship between *five* parts of the triangle, which yields the following group in six applications, is not a direct one for the solution of a triangle but often usable for other operations:

$$\sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha$$
$$\sin b \cos \gamma = \cos c \sin a - \sin c \cos a \cos \beta$$
$$\sin c \cos \alpha = \cos a \sin b - \sin a \cos b \cos \gamma$$

$$\sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha$$
$$\sin b \cos \alpha = \cos a \sin c - \sin a \cos c \cos \beta$$
$$\sin c \cos \beta = \cos b \sin a - \sin b \cos a \cos \gamma .$$

In addition, the following two systems of equations belong to it:

$$\sin \alpha \cos b = \cos \beta \sin \gamma + \sin \beta \cos \gamma \cos a$$
$$\sin \beta \cos c = \cos \gamma \sin \alpha + \sin \gamma \cos \alpha \cos b$$
$$\sin \gamma \cos a = \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos c$$

$$\sin\alpha\cos\beta = \cos b\sin\gamma - \sin\beta\cos\alpha\cos c$$
$$\sin\beta\cos\gamma = \cos c\sin\alpha - \sin\gamma\cos\beta\cos a$$
$$\sin\gamma\cos\alpha = \cos a\sin\beta - \sin\alpha\cos\gamma\cos b\,.$$

We have further the important equations by Gauss:

$$\sin\frac{a}{2}\cos\frac{\beta-\gamma}{2} = \sin\frac{b+c}{2}\sin\frac{\alpha}{2}$$
$$\sin\frac{a}{2}\sin\frac{\beta-\gamma}{2} = \sin\frac{b-c}{2}\cos\frac{\alpha}{2}$$
$$\cos\frac{a}{2}\cos\frac{\beta+\gamma}{2} = \cos\frac{b+c}{2}\sin\frac{\alpha}{2}$$
$$\cos\frac{a}{2}\sin\frac{\beta+\gamma}{2} = \cos\frac{b-c}{2}\cos\frac{\alpha}{2}$$

From these result Napier's analogies:

$$\tan\frac{b+c}{2} = \tan\frac{a}{2}\,\frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}} \qquad \tan\frac{b-c}{2} = \tan\frac{a}{2}\,\frac{\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}$$

$$\tan\frac{\beta+\gamma}{2} = \cot\frac{\alpha}{2}\,\frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}} \qquad \tan\frac{\beta-\gamma}{2} = \cot\frac{\alpha}{2}\,\frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}$$

and hence follows the law of the tangent:

$$\frac{\tan\frac{b+c}{2}}{\tan\frac{b-c}{2}} = \frac{\tan\frac{\beta+\gamma}{2}}{\tan\frac{\beta-\gamma}{2}}$$

The *spherical excess* of a spherical triangle is the surplus of the sum of angles over 180°, i.e.

$$\varepsilon = \alpha + \beta + \gamma - 180^\circ\,.$$

If r is the radius of the sphere and F the curved surface of the spherical triangle, then we find hence the excess ε according to the formula:

$$\varepsilon = \frac{F}{r^2}\,\rho\,,$$

where $\rho = \frac{180^\circ}{\pi}$ or for seconds $\rho = \frac{180\cdot 60\cdot 60}{\pi} = 206{,}265''$. (Cf. p. 22).

We obtain the spherical excess from the sides a, b, c by the formula

$$\tan\frac{\varepsilon}{4} = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}},$$

where $a + b + c = 2s$.

Section 34. Developments in Series

In higher geodesy, convergent series play an important role.

Let us study first the Taylor series with the variable x and its variation h as the basis of convergent power series:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$f'(x)$ means here the first derivative of $f(x)$ with respect to x, $f''(x)$ the second derivative, and so on; we have further

$1! = 1$	$5! = 120$	$9! = 362{,}880$
$2! = 1 \cdot 2 = 2$	$6! = 720$	$10! = 3{,}628{,}800$
$3! = 1 \cdot 2 \cdot 3 = 6$	$7! = 5\,040$	$11! = 39{,}916{,}800$
$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$	$8! = 40\,320$	$12! = 479{,}001{,}600$.

These expressions are in other form as follows:

$0! = 1$

$1! = 1$	$5! = 2^3 \cdot 3 \cdot 5$	$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$
$2! = 2$	$6! = 2^4 \cdot 3^2 \cdot 5$	$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
$3! = 2 \cdot 3$	$7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7$	$11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
$4! = 2^3 \cdot 3$	$8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$	$12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
$2^n =$	2	4	8	16	32	64	128	256	512	1024	2048	4096.

The trigonometric applications of the Taylor series, which are very often used, are:

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2}\sin x - \frac{h^3}{6}\cos x + \frac{h^4}{24}\sin x + \dots$$

$$\sin(x-h) = \sin x - h\cos x - \frac{h^2}{2}\sin x + \frac{h^3}{6}\cos x + \frac{h^4}{24}\sin x - \dots$$

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{2}\cos x + \frac{h^3}{6}\sin x + \frac{h^4}{24}\cos x - \dots$$

$$\cos(x-h) = \cos x + h\sin x - \frac{h^2}{2}\cos x - \frac{h^3}{6}\sin x + \frac{h^4}{24}\cos x + \dots$$

$$\tan(x+h) = \tan x + h\frac{1}{\cos^2 x} + h^2\frac{\sin x}{\cos^3 x} + \frac{h^3}{3}\frac{\cos^2 x + 3\sin^2 x}{\cos^4 x} + \dots$$

$$\cot(x+h) = \cot x - h\frac{1}{\sin^2 x} + h^2\frac{\cos x}{\sin^3 x} - \frac{h^3}{3}\frac{\sin^2 x + 3\cos^2 x}{\sin^4 x} + \dots$$

Sometimes it is convenient to express everything in $\tan x = t$, as the following examples show:

$$\cos(x+h) = \cos x\left(1 - ht - \frac{h^2}{2} + \frac{h^3}{6}t + \frac{h^4}{24} \dots\right)$$

$$\frac{1}{\cos(x+h)} = \frac{1}{\cos x}\left(1 + ht + \frac{h^2}{2}(1+2t^2) + \frac{h^3}{6}t(5+6t^2) + \frac{h^4}{24}(5+28t^2+24t^4)\dots\right)$$

$$\tan(x+h) = \tan x + h(1+t^2) + h^2 t(1+t^2) + \frac{h^3}{3}(1+4t^2+3t^4).$$

Following are the derivatives of *tan x* carried further:

$$\tan x = t$$

$$\frac{dt}{dx} = 1 + t^2$$

$$\frac{d^2 t}{dx^2} = 2t(1+t^2)$$

$$\frac{d^3 t}{dx^3} = 2(1+t^2)(1+3t^2)$$

$$\frac{d^4 t}{dx^4} = 8t(1+t^2)(2+3t^2)$$

$$\frac{d^5 t}{dx^5} = 8(1+t^2)(2+15t^2+15t^4)$$

$$\frac{d^6 t}{dx^6} = 16t(1+t^2)(17+60t^2+45t^4)$$

$$\frac{d^7 t}{dx^7} = 16(1+t^2)(17+231t^2+525t^4+315t^6)$$

$$\frac{d^8 t}{dx^8} = 128t(1+t^2)(62+378t^2+630t^4+315t^6)$$

$$\frac{d^9 t}{dx^9} = 128(1+t^2)(62+1320t^2+5040t^4+6615t^6+2835t^8)$$

$$\frac{d^{10} t}{dx^{10}} = 256t(1+t^2)(1382+12720t^2+34965t^4+37800t^6+14175t^8)$$

$$\frac{d^{11} t}{dx^{11}} = 256(1+t^2)(1382+42306t^2+238425t^4+509355t^6+467775t^8+155925t^{10}).$$

Likewise the derivatives of *cot x*:

$$\cot x = c$$

$$\frac{dc}{dx} = -(1+c^2)$$

$$\frac{d^2 c}{dx^2} = +2c(1+c^2)$$

$$\frac{d^3 c}{dx^3} = -2(1+c^2)(1+3c^2)$$

$$\frac{d^4 c}{dx^4} = +8c(1+c^2)(2+3c^2)\text{, and so forth.}$$

The coefficients are the same here as in the case of the derivatives of t; in the case of the odd derivatives, however, a change of sign takes place.

With the above we have at the same time the derivatives of *log cos x* and *log sin x*, for:

$$\frac{d\,l\cos x}{dx} = -\tan x \qquad \frac{d\,l\sin x}{dx} = \cot x$$

$$\frac{d^2\,l\cos x}{dx^2} = -\frac{dt}{dx} = -(1+t^2) \qquad \frac{d^2\,l\sin x}{dx^2} = \frac{dc}{dx} = -(1+c^2).$$

The following Maclaurin series is also based on the first mentioned Taylor series:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0),$$

where $f(0)$, $f'(0)$, $f''(0)$, and so forth, are the values which result if in $f(x)$, $f'(x)$, $f''(x)$, and so forth, the variable x is set equal to zero.

The binomial series is of special importance:

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots$$
$$(1-x)^n = 1 - \binom{n}{1} x + \binom{n}{2} x^2 - \binom{n}{3} x^3 + \dots$$

The coefficients of this series are called binomial coefficients and have the following meanings:

$$\binom{n}{1} = \frac{n}{1}$$
$$\binom{n}{2} = \frac{n}{1} \frac{n-1}{2}$$
$$\binom{n}{3} = \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3}, \text{ etc.}$$

For instance $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$
$$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x + 6x^5 + x^6 .$$

7th power yields	1	7	21	35	35	21	7	1			
8th power yields	1	8	28	56	70	56	28	8	1		
9th power yields	1	9	36	84	126	126	84	36	9	1	
10th power yields	1	10	45	120	210	252	210	120	45	10	1 ,

the recurrence is expressed by $\binom{n}{r} = \binom{n}{n-r}$.

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 6abc + 3ab^2 + 3ac^2 + 6bc^2 .$$

The binomial series holds in general for integral or fractional positive or negative exponents n and always converges if $x < 1$. Some applications of this series which are frequently used are the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$
$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$
$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \frac{33}{2048}x^7 - \dots$$
$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \dots$$
$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \frac{231}{1024}x^6 - \frac{429}{2048}x^7 + \dots$$
$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \dots$$
$$\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \dots$$
$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \frac{63}{256}x^{10} + \dots .$$

$$\left.\begin{aligned} l(1+x) &= +x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots \\ l(1-x) &= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\ldots\right). \end{aligned}\right\}$$

l is here the symbol for natural logarithms with the base $e = 2.71828\ldots$, or $l\,x$ is the natural logarithm of x. *Log* x, however, means the decadic or Briggs' logarithm with the base 10, and the following relation exists:

$$log\ x = \mu\,(l\,x)\,.$$

The factor μ is called the modulus of Briggs' system of logarithms.
We have in addition the following, accurate to 20 places:

$$\mu = 0.43429\ 44819\ 03251\ 82765 \qquad \frac{1}{\mu} = 2.30258\ 50929\ 94045\ 68402$$

$$log\ \mu = 9.63778\ 43113\ 00536\ 78912 \qquad log\ \frac{1}{\mu} = 0.36221\ 56886\ 99463\ 21088\,.$$

If we have to carry out rather large multiplications or divisions by the factor $\mu = 0.43429\ldots$, we can use a table of multiples of μ or $1{:}\mu$, as found in some older tables of logarithms, e.g. *Thesaurus*, p. 641, and Steinhauser, *20stell. Logar.*, Wien, 1880, p. XII up to $100\,\mu$ and p. XI reverse.

Exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

$$a^x = 1+\frac{x\,l\,a}{1!}+\frac{(x\,l\,a)^2}{2!}+\frac{(x\,l\,a)^3}{3!}+\frac{(x\,l\,a)^4}{4!}+\ldots$$

$$10^x = 1+\frac{x}{\mu}+\frac{1}{2!}\left(\frac{x}{\mu}\right)^2+\frac{1}{3!}\left(\frac{x}{\mu}\right)^3+\frac{1}{4!}\left(\frac{x}{\mu}\right)^4+\ldots$$

$$e = 2.71828\ 18284\ 59045 \quad log\,e = \mu = 0.43429\ldots \quad l\,10 = \frac{1}{\mu} = 2.3026\ldots \text{ (see above).}$$

Goniometric series

In the power series for *sin x, cos x,* etc., x is to be used in radian measure, for the explanation of which reference is made to the calculation of an arc of a circle corresponding to a central angle. Suppose in a circle with the radius r a central angle α is given in degrees (e.g. $\alpha = 30°$, $\alpha = 40°$, etc.), and we are to compute the arc b pertaining to it; the following proportion is used:

$$b : 2r\pi = \alpha^\circ : 360^\circ\,;$$

hence

$$b = \frac{2r\pi\alpha^\circ}{360^\circ} = \frac{r\alpha^\circ}{\rho^\circ}, \quad \text{if } \rho^\circ = \frac{180^\circ}{\pi}.$$

If we set

$$\frac{\alpha^\circ}{\rho^\circ} = \alpha\,, \quad \text{we have } b = r\alpha\,.$$

This leads in general to the following explanation: If x°, x', x'' is an angle value in geometric measure, i.e. in degrees, minutes or seconds, and x is the corresponding value in radian measure, then the following equations exist:

$$x = \frac{x^\circ}{\varrho^\circ}, \text{ where } \varrho^\circ = \frac{180}{\pi} = \frac{180}{3.14159\ldots}$$

$$x = \frac{x'}{\varrho'} \quad \text{"} \quad \varrho' = \frac{180 \,.\, 60}{\pi} = \frac{10800}{3.14159\ldots}$$

$$x = \frac{x''}{\varrho''} \quad \text{"} \quad \varrho'' = \frac{180 \,.\, 60 \,.\, 60}{\pi} = \frac{648000}{3.14159\ldots}$$

or for grades (centesimal division):

$$x = \frac{x^g}{\varrho^g}, \text{ where } \varrho^g = \frac{200}{\pi} = \frac{200}{3.14159\ldots} .$$

The more accurate numerical values for these are:

$\pi =$	3.14159 26535 89793	$\log \pi$	= 0.49714 98726 94134
$\frac{1}{\pi} =$	0.31830 98861 83791	$\log \frac{1}{\pi}$	= 9.50285 01273 05866
$\varrho^\circ =$	57.29577 95130 82321	$\log \varrho^\circ$	= 1.75812 26324 09172
$\frac{1}{\varrho^\circ} =$	0.01745 32925 19943	$\log \frac{1}{\varrho^\circ}$	= 8.24187 73675 90828
$\varrho' =$	3437.74677 07849 39253	$\log \varrho'$	= 3.53627 38827 92816
$\frac{1}{\varrho'} =$	0.00029 08882 08666	$\log \frac{1}{\varrho'}$	= 6.46372 61172 07184
$\varrho'' =$	206264.80624 70963 55156	$\log \varrho''$	= 5.31442 51331 76459
$\frac{1}{\varrho''} =$	0.00000 48481 36811	$\log \frac{1}{\varrho''}$	= 4.68557 48668 23541
$\varrho^g =$	63.66197 72367 58134	$\log \varrho^g$	= 1.80388 01229 69847
$\frac{1}{\varrho^g} =$	0.01570 79632 67949	$\log \frac{1}{\varrho^g}$	= 8.19611 98770 30153

For series developments carried far we also need $\pi^2, \pi^3 \ldots \left(\text{for the powers of } \frac{1}{\rho}\right)$.

$\pi =$	3.14159 26535 89793 23846	$\log \pi$	= 0.49714 98726 94133 85435
$\pi^2 =$	9.86960 44010 89358 61883	$\log \pi^2$	= 0.99429 97453 88267 70870
$\pi^3 =$	31.00627 66802 99820 1754	$\log \pi^3$	= 1.49144 96180 82401 56305
$\pi^4 =$	97.40909 10340 02437 24	$\log \pi^4$	= 1.98859 94907 76535 41741
$\pi^5 =$	306.01968 47852 81453	$\log \pi^5$	= 2.48574 93634 70669
$\pi^6 =$	961.38919 35753 044	$\log \pi^6$	= 2.98289 92361 64803
$\pi^7 =$	3020.29322 77768	$\log \pi^7$	= 3.48004 91089
$\pi^8 =$	9488.53101 60757	$\log \pi^8$	= 3.97719 89816
		$\log \pi^9$	= 4.47434 88542
		$\log \pi^{10}$	= 4.97149 87269 .

These values are obtained directly by multiplications of π.

We need most frequently ρ in seconds, and therefore we list in the following the logarithms of the powers of this ρ, in addition to the factor μ often used with it:

n	$\log \varrho^n$	$\log \frac{1}{\varrho^n}$	$\log \frac{\mu}{\varrho^n}$	n
			9.637 7843·113	0
1	5.314 4251·332	4.685 5748·668	4.323 3591·781	1
2	0.628 8502·664	9.371 1497·336	9.008 9340·449	2
3	5.943 2753·995	4.056 7246·005	3.694 5089·118	3
4	1.257 7005·327	8.742 2994·673	8.380 0837·786	4
5	6.572 1256·659	3.427 8743·341	3.065 6586·454	5
6	1.886 5507·991	8.113 4492·009	7.751 2335·122	6
7	7.200 9759·322	2.799 0240·678	2.436 8083·791	7
8	2.515 4010·654	7.484 5989·346	7.122 3832·459	8
9	7.829 8261·986	2.170 1738·014	1.807 9581·127	9
10	3.144 2513·318	6.855 7486·682	6.493 5329·795	10
12	3.773 1016	6.226 8984	5.864 6827	12.

In the characteristic of logarithms we only write here the values between 0 and 9, e.g., for $\log \rho^2 = 10.628 \ldots$ there is written only $0.628 \ldots$, etc., and likewise -10, etc., is simply omitted at the end; the point above, for instance 1·332, is after the 7th place. These remarks also hold for all further computations of logarithms in this book.

The above table holds for sexagesimal units and ρ in seconds. For centesimal units the corresponding power logarithms are in the author's *log.-trig. Tafeln für neue Teilung*, 4th edition, Stuttgart, 1931, p. 417, and the most important constants involving μ and ρ are:

New division

$$\frac{\mu}{\varrho} = \frac{\mu \pi}{200} = 0.00682\ 18817\ 69209\ 20674 \qquad \log = 7.83390\ 41883\ 30689$$

$$\frac{\mu}{6\varrho^2} = 178596.44708\ 17149 \qquad \log = 5.25187\ 28149\ 77198$$

$$\frac{\mu}{2\varrho^2} = 535789.34124\ 51446 \qquad \log = 5.72899\ 40696\ 96861$$

$$\frac{\mu}{180\varrho^4} = 1.46889\ 69001 \qquad \log = 0.16699\ 13143$$

$$\frac{\mu}{12\varrho^4} = 22.03345\ 35017 \qquad \log = 1.34308\ 25734\,.$$

The following series holds for x in radian measure:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925}, \quad \tan\frac{x}{2} = \frac{x}{2} + \frac{x^3}{24} + \frac{x^5}{240} + \ldots$$

$$\cot x = \frac{1}{x}\left(1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \ldots\right)$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \ldots$$

$$\operatorname{cosec} x = \frac{1}{x}\left(1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \ldots\right)$$

$$x = \arcsin y = y + \frac{y^3}{6} + \frac{3y^5}{40} + \frac{5y^7}{112} + \frac{35y^9}{1152} + \frac{63y^{11}}{2816} + \frac{231y^{13}}{13312} + \ldots (y = \sin x)$$

$$x = \arctan y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \frac{y^{13}}{13} - \ldots (y = \tan x)$$

$$l \sin x = l x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \frac{x^8}{37800} - \frac{x^{10}}{467775} - \frac{691x^{12}}{3831077250}$$

$$l \cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \frac{31x^{10}}{14175} - \frac{691x^{12}}{935550}$$

$$l \tan x = l x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \frac{127x^8}{18900} + \frac{146x^{10}}{66825} + \frac{2828954x^{12}}{3831077250}.$$

The coefficients of these last three series satisfy the conditions:

$$\frac{1}{6}+\frac{1}{3}=\frac{1}{2}, \quad \frac{1}{180}+\frac{7}{90}=\frac{1}{12}, \qquad \text{generally} \quad S_n + T_n = C_n$$

and

$$\frac{1}{6}+\frac{1}{2}=2^2\,\frac{1}{6}, \quad \frac{1}{180}+\frac{1}{12}=2^4\,\frac{1}{180}, \qquad \text{generally} \quad S_n + C_n = 2^n S_n$$

(because $\sin 2x = 2 \sin x \cos x$).

These series hold for x in radian measure, as already mentioned above; if x is counted in degrees, minutes or seconds, then we must divide by the ρ in question, and for common logarithms we are to multiply by the modulus μ as a whole, a procedure by which we have for degree-measure and common logarithms:

$$\log \sin x = \log \frac{x}{\varrho} - \left\{ \frac{\mu}{6\varrho^2} x^2 + \frac{\mu x^4}{180\varrho^4} + \frac{\mu x^6}{2835\varrho^6} + \frac{\mu x^8}{37800\varrho^8} + \frac{\mu x^{10}}{467775\varrho^{10}} \right\}$$

$$\log \cos x = - \left\{ \frac{\mu}{2\varrho^2} x^2 + \frac{\mu x^4}{12\varrho^4} + \frac{\mu x^6}{45\varrho^6} + \frac{\mu\, 17 x^8}{2520\varrho^8} + \frac{\mu\, 31 x^{10}}{14175\varrho^{10}} \right\}$$

$$\log \tan x = \log \frac{x}{\varrho} + \left\{ \frac{\mu}{2\varrho^2} x^2 + \frac{\mu\, 7 x^4}{90\varrho^4} + \frac{\mu\, 62 x^6}{2835\varrho^6} + \frac{\mu\, 127 x^8}{18900\varrho^8} + \frac{\mu\, 146 x^{10}}{66825\varrho^{10}} \right\}.$$

We have here for seconds $\log \frac{1}{\rho} = 4.685\,5748{\cdot}668$, and the remaining logarithms of coefficients are the following:

	x^2	x^4	x^6	x^8	x^{10}	x^{12}
log sin	8.230 7827·945	6.124 8112·7	4.298 6804	2.544 891	0.82350	9.1208
log cos	8.707 9040·493	7.300 9025·3	6.098 0210	4.951 432	3.83337	2.7331
log tan	8.531 8127·902	7.270 9393·1	6.091 0721	4.949 725	3.83295	2.7330 .

The corresponding values for new division are communicated in the author's *logar.-trig. Tafeln für neue Teilung*, 4th edition, 1931, p. 431.

With these series we can compute *log sin*, etc., as far as 15 places, say for angles of 0° to 10°, and if we need a rigorous value $\log \sin x$ or $\log \tan x$ for a *small* angle x, we obtain it better directly with a few terms of the series than by interpolation from the 10-place table.

We are to mention mainly the *Thesaurus logarithmorum completus* by Georg Vega, Leipzig, 1794, in general with 10-place logarithms, as the source for the fundamental numbers used above. This work gives on pp. 308 and 309 the numbers π to 140 places, $l\,10 = \frac{1}{\mu}$, μ and e to 48 places, and on p. 633 the actual series computation of π to 140 places. Some additional material is offered also in the first (oldest) editions of Vega's *7-place logar.-trig. tables*, and Vega-Hülsse, Leipzig, 1840; further Steinhauser, *Hilfstafeln zur präzisen Berechnung 20stelliger Logarithmen*, Wien, 1880 (with corrections by Nell, *Zeitschrift für Vermessungswesen 1893*, p. 603), and *Astronomische Nachrichten*, 166, 1904, pp. 285-288; Peters and Stein, *52stellige Logarithmen*, publication of the Astr. Recheninstitut at Berlin, 1919, as well as further tabular works mentioned in section 36.

Hyperbolic functions

If the following relation exists between two magnitudes x and u,

$$\mu x = \log \tan \left(45° + \frac{u}{2}\right),$$

then we have

$$\frac{1}{i} \sin i x = \tan u \qquad \frac{1}{i} \tan i x = \sin u$$

$$\cos i x = \frac{1}{\cos u} \qquad \frac{1}{i} \tan i \frac{x}{2} = \tan \frac{u}{2}.$$

We call

$$\mathfrak{Sin}\, x = \frac{1}{i} \sin i x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$
$$\mathfrak{Cos}\, x = \cos i x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$
$$\mathfrak{Tan}\, x = \frac{1}{i} \tan i x$$

the hyperbolic functions of x.

For small values of x we have the following series

$$\mathfrak{Sin}\, x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$\mathfrak{Cos}\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots ,$$

and in logarithmic form

$$\log \mathfrak{Sin}\, x = \log x + \mu\left(\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \frac{x^8}{37800} + \dots\right)$$
$$\log \mathfrak{Cos}\, x = \mu\left(\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \frac{17x^8}{2520} + \dots\right)$$
$$\log \mathfrak{Tan}\, x = \log x - \mu\left(\frac{x^2}{3} - \frac{7x^4}{90} + \frac{62x^6}{2835} - \frac{127x^8}{18900} + \dots\right).$$

In analogy to the formulae for trigonometric functions we have the following formulae for hyperbolic functions:

$$\mathfrak{Tan}\, x = \frac{\mathfrak{Sin}\, x}{\mathfrak{Cos}\, x} \qquad \mathfrak{Tan}\, x\, \mathfrak{Cot}\, x = 1 \qquad \mathfrak{Cos}^2 x - \mathfrak{Sin}^2 x = 1$$
$$\mathfrak{Sin}(-x) = -\mathfrak{Sin}\, x \qquad \mathfrak{Cos}(-x) = \mathfrak{Cos}\, x$$
$$\mathfrak{Sin}(x_1 \pm x_2) = \mathfrak{Sin}\, x_1\, \mathfrak{Cos}\, x_2 \pm \mathfrak{Cos}\, x_1\, \mathfrak{Sin}\, x_2$$
$$\mathfrak{Cos}(x_1 \pm x_2) = \mathfrak{Cos}\, x_1\, \mathfrak{Cos}\, x_2 \pm \mathfrak{Sin}\, x_1\, \mathfrak{Sin}\, x_2$$
$$\mathfrak{Tan}(x_1 \pm x_2) = \frac{\mathfrak{Tan}\, x_1 \pm \mathfrak{Tan}\, x_2}{1 \pm \mathfrak{Tan}\, x_1\, \mathfrak{Tan}\, x_2}$$
$$\mathfrak{Cot}(x_1 \pm x_2) = \frac{1 \pm \mathfrak{Cot}\, x_1\, \mathfrak{Cot}\, x_2}{\mathfrak{Cot}\, x_1 \pm \mathfrak{Cot}\, x_2}$$
$$\mathfrak{Sin}\, x_1 \pm \mathfrak{Sin}\, x_2 = 2\, \mathfrak{Sin}\frac{x_1 \pm x_2}{2}\, \mathfrak{Cos}\frac{x_1 \pm x_2}{2}$$
$$\mathfrak{Cos}\, x_1 + \mathfrak{Cos}\, x_2 = 2\, \mathfrak{Cos}\frac{x_1 + x_2}{2}\, \mathfrak{Cos}\frac{x_1 - x_2}{2}$$
$$\mathfrak{Cos}\, x_1 - \mathfrak{Cos}\, x_2 = 2\, \mathfrak{Sin}\frac{x_1 + x_2}{2}\, \mathfrak{Sin}\frac{x_1 - x_2}{2}$$
$$\mathfrak{Tan}\, x_1 \pm \mathfrak{Tan}\, x_2 = \frac{\mathfrak{Sin}(x_1 \pm x_2)}{\mathfrak{Cos}\, x_1\, \mathfrak{Cos}\, x_2}.$$

Section 35. Further Series

In the case of geodetic developments we often need to express the powers $\sin^n x$ and $\cos^n x$ by $\sin nx$ and $\cos nx$, etc., for instance, for integrating those powers and the like; the inverse conversions are needed also.

We can derive all this step by step from the simplest goniometric formulae:

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$\begin{aligned} \sin 3x &= \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \\ \sin 3x &= 3 \sin x \cos^2 x - \sin^3 x . \end{aligned}$$

In this and a similar way we could develop all formulae of pp. 28 and 29; however, we reach our goal better with the help of the imaginary expressions for $\sin x$ and $\cos x$, to which we pass now.

The series for $\sin x$ and for $\cos x$ are related to the exponential series e^x as follows:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots .$$

If we replace x by ix here (where $i = \sqrt{-1}$), then x^2 changes to $-x^2$, x^3 to $-ix^3$, x^4 to $+x^4$, x^5 to ix^5, etc., and thus we obtain from the above three series:

$$e^{+ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

or

$$\cos x = \frac{e^{+ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{+ix} - e^{-ix}}{2i} ,$$

and if we replace, in addition, x by nx, then we obtain De Moivre's theorem

$$\begin{aligned} \cos nx \pm i \sin nx &= e^{\pm inx} \\ &= (e^{\pm ix})^n = (\cos x \pm i \sin x)^n . \end{aligned}$$

If we develop here according to the binomial theorem, we obtain

$$\cos nx - i \sin nx = \cos^n x + \binom{n}{1} \cos^{n-1} x \, i \sin x - \binom{n}{2} \cos^{n-2} x \sin^2 x - \ldots$$

$$\cos nx + i \sin nx = \cos^n x - \binom{n}{1} \cos^{n-1} x \, i \sin x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \ldots .$$

By subtraction and addition we find from these

$$\left.\begin{aligned} \sin nx &= \binom{n}{1} \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x + \binom{n}{5} \cos^{n-5} x \sin^5 x \,.\,. \\ \cos nx &= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \ldots . \end{aligned}\right\} \quad (1)$$

Vice versa, if we aim to have $sin^n x$ and $cos^n x$ as a function of $sin\, nx$ and $cos\, nx$, then we set

$$e^{+ix} = p = cos\, x + i\, sin\, x\,, \text{hence } p^m = cos\, m\, x + i\, sin\, m\, x$$
$$e^{-ix} = q = cos\, x - i\, sin\, x \qquad \text{"} \quad q^m = cos\, m\, x - i\, sin\, m\, x\,,$$

then

$$p\, q = 1 \qquad p + q = 2\, cos\, x \qquad p^m + q^m = 2\, cos\, m\, x$$
$$p - q = 2\, i\, sin\, x \qquad p^m - q^m = 2\, i\, sin\, m\, x\,;$$

from these we make the following applications by powers of $(p + q)$ and of $(p - q)$:

$$(2\, cos\, x)^2 = (p + q)^2 \qquad (2\, i\, sin\, x)^2 = (p - q)^2$$
$$= (p^2 + q^2) + 2\, p\, q \qquad = (p^2 + q^2) - 2\, p\, q$$
$$4\, cos^2 x = 2\, cos\, 2\, x + 2 \qquad -4\, sin^2 x = 2\, cos\, 2\, x - 2$$
$$cos^2 x = \frac{1}{2} + \frac{1}{2}\, cos\, 2\, x \qquad sin^2 x = \frac{1}{2} - \frac{1}{2}\, cos\, 2\, x\,.$$

These are the familiar goniometric formulae. We go on:

$$(2\, cos\, x)^4 = p^4 + 4\, p^3\, q + 6 p^2\, q^2 + 4\, p\, q^3 + q^4$$
$$= (p^4 + q^4) + 4\, (p^2 + q^2)\, p\, q + 6\, p^2\, q^2$$
$$16\, cos^4 x = 2\, cos\, 4\, x + 8\, cos\, 2\, x + 6 \qquad \text{or} \qquad cos^4 x = \frac{1}{8}\, cos\, 4\, x + \frac{1}{2}\, cos\, 2\, x + \frac{3}{8}\,.$$

In the same way we also obtain $sin^4 x$, and by pursuing this way further we can find any formula of this kind, for instance,

$$(2\, i\, sin\, x)^5 = (p - q)^5$$
$$32\, i^5\, sin^5 x = p^5 - 5\, p^4\, q + 10\, p^3\, q^2 - 19\, p^2\, q^3 + 5\, p\, q^4 - q^5$$
$$= (p^5 - q^5) - 5\, (p^3 - q^3)\, p\, q + 10\, (p - q)\, p^2\, q^2$$
$$32\, i\, sin^5 x = 2\, i\, sin\, 5\, x - 10\, i\, sin\, 3\, x + 20\, i\, sin\, x$$
$$sin^5 x = \frac{1}{16}\, sin\, 5\, x - \frac{5}{16}\, sin\, 3\, x + \frac{5}{8}\, sin\, x$$
$$(2\, cos\, x)^5 = (p + q)^5 = p^5 + 5\, p^4\, q + 10\, p^3\, q^2 + 10\, p^2\, q^3 + 5\, p\, q^4 + q^5$$
$$32\, cos^5 x = (p^5 + q^5) + 5\, (p^3 + q^3)\, p\, q + 10\, (p + q)\, p^2\, q^2$$
$$= 2\, cos\, 5\, x + 10\, cos\, 3\, x + 20\, cos\, x$$
$$cos^5 x = \frac{1}{16}\, cos\, 5\, x + \frac{5}{16}\, cos\, 3\, x + \frac{5}{8}\, cos\, x\,.$$

The coefficients for $sin^n x$ and $cos^n x$ are the same; in the case of $sin^n x$ only change of sign takes place.

In such a way we can quickly develop individually the formulae for use, say up to $sin^{10} x$ and $cos^{10} x$, which we shall list later. It is also possible to set up general formulae for $sin^n x$ and $cos^n x$, but the development is somewhat circumstantial because we must distinguish even and odd n. The general formulae are:

1. for even exponents:

$$\left.\begin{aligned} sin^{2n}x &= \frac{1}{2^{2n-1}}\left(\frac{1}{2}\binom{2n}{n} - \binom{2n}{n-1}\cos 2x + \binom{2n}{n-2}\cos 4x - \ldots\right) \\ cos^{2n}x &= \frac{1}{2^{2n-1}}\left(\frac{1}{2}\binom{2n}{n} + \binom{2n}{n-1}\cos 2x + \binom{2n}{n-2}\cos 4x + \ldots\right); \end{aligned}\right.$$

2. for odd exponents:

$$\left.\begin{aligned} sin^{2n+1}x &= \frac{1}{2^{2n}}\left(\binom{2n+1}{n}\sin x - \binom{2n+1}{n-1}\sin 3x + \ldots\right) \\ cos^{2n+1}x &= \frac{1}{2^{2n}}\left(\binom{2n+1}{n}\cos x + \binom{2n+1}{n-1}\cos 3x + \ldots\right). \end{aligned}\right\} \quad (2)$$

We are to go on in these series until a coefficient becomes = 0.

In accordance with the formula groups (2) and (1), the following individual formulae for use are written down up to the tenth order:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\sin^5 x = \frac{5}{8}\sin x - \frac{5}{16}\sin 3x + \frac{1}{16}\sin 5x$$

$$\sin^6 x = \frac{5}{16} - \frac{15}{32}\cos 2x + \frac{3}{16}\cos 4x - \frac{1}{32}\cos 6x$$

$$\sin^7 x = \frac{35}{64}\sin x - \frac{21}{64}\sin 3x + \frac{7}{64}\sin 5x - \frac{1}{64}\sin 7x$$

$$\sin^8 x = \frac{35}{128} - \frac{7}{16}\cos 2x + \frac{7}{32}\cos 4x - \frac{1}{16}\cos 6x + \frac{1}{128}\cos 8x$$

$$\sin^9 x = \frac{63}{128}\sin x - \frac{21}{64}\sin 3x + \frac{9}{64}\sin 5x - \frac{9}{256}\sin 7x + \frac{1}{256}\sin 9x$$

$$\sin^{10} x = \frac{63}{256} - \frac{105}{256}\cos 2x + \frac{15}{64}\cos 4x - \frac{45}{512}\cos 6x + \frac{5}{256}\cos 8x - \frac{1}{512}\cos 10x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\cos^5 x = \frac{5}{8}\cos x + \frac{5}{16}\cos 3x + \frac{1}{16}\cos 5x$$

$$\cos^6 x = \frac{5}{16} + \frac{15}{32}\cos 2x + \frac{3}{16}\cos 4x + \frac{1}{32}\cos 6x$$

$$\cos^7 x = \frac{35}{64}\cos x + \frac{21}{64}\cos 3x + \frac{7}{64}\cos 5x + \frac{1}{64}\cos 7x$$

$$\cos^8 x = \frac{35}{128} + \frac{7}{16}\cos 2x + \frac{7}{32}\cos 4x + \frac{1}{16}\cos 6x + \frac{1}{128}\cos 8x$$

$$\cos^9 x = \frac{63}{128}\cos x + \frac{21}{64}\cos 3x + \frac{9}{64}\cos 5x + \frac{9}{256}\cos 7x + \frac{1}{256}\cos 9x$$

$$\cos^{10} x = \frac{63}{256} + \frac{105}{256}\cos 2x + \frac{15}{64}\cos 4x + \frac{45}{512}\cos 6x + \frac{5}{256}\cos 8x + \frac{1}{512}\cos 10x$$

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x\\
\sin 3x &= 3\sin x\cos^2 x - \sin^3 x\\
\sin 4x &= 4\sin x\cos^3 x - 4\sin^3 x\cos x\\
\sin 5x &= 5\sin x\cos^4 x - 10\sin^3 x\cos^2 x + \sin^5 x\\
\sin 6x &= 6\sin x\cos^5 x - 20\sin^3 x\cos^3 x + 6\sin^5 x\cos x\\
\sin 7x &= 7\sin x\cos^6 x - 35\sin^3 x\cos^4 x + 21\sin^5 x\cos^2 x - \sin^7 x\\
\sin 8x &= 8\sin x\cos^7 x - 56\sin^3 x\cos^5 x + 56\sin^5 x\cos^3 x - 8\sin^7 x\cos x\\
\sin 9x &= 9\sin x\cos^8 x - 84\sin^3 x\cos^6 x + 126\sin^5 x\cos^4 x - 36\sin^7 x\cos^2 x\\
&\quad + \sin^9 x\\
\sin 10x &= 10\sin x\cos^9 x - 120\sin^3 x\cos^7 x + 252\sin^5 x\cos^5 x - 120\sin^7 x\cos^3 x\\
&\quad + 10\sin^9 x\cos x.
\end{aligned}$$

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x\\
\cos 3x &= \cos^3 x - 3\cos x\sin^2 x\\
\cos 4x &= \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x\\
\cos 5x &= \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\\
\cos 6x &= \cos^6 x - 15\cos^4 x\sin^2 x + 15\cos^2 x\sin^4 x - \sin^6 x\\
\cos 7x &= \cos^7 x - 21\cos^5 x\sin^2 x + 35\cos^3 x\sin^4 x - 7\cos x\sin^6 x\\
\cos 8x &= \cos^8 x - 28\cos^6 x\sin^2 x + 70\cos^4 x\sin^4 x - 28\cos^2 x\sin^6 x + \sin^8 x\\
\cos 9x &= \cos^9 x - 36\cos^7 x\sin^2 x + 126\cos^5 x\sin^4 x - 84\cos^3 x\sin^6 x + 9\cos x\sin^8 x\\
\cos 10x &= \cos^{10} x - 45\cos^8 x\sin^2 x + 210\cos^6 x\sin^4 x - 210\cos^4 x\sin^6 x\\
&\quad + 45\cos^2 x\sin^8 x - \sin^{10} x.
\end{aligned}$$

Abbreviated power series with mean argument

In a development in power series according to Taylor's theorem, we can always save half of the terms by the introduction of a *mean* argument, as can be shown thus:

At first we set $$x + h = \left(x + \frac{h}{2}\right) + \frac{h}{2}$$

and then $$x = \left(x + \frac{h}{2}\right) - \frac{h}{2}.$$

Then we have according to Taylor's theorem

$$f(x+h) = f\left(x+\frac{h}{2}\right) + \frac{h}{2} f'\left(x+\frac{h}{2}\right) + \frac{h^2}{8} f''\left(x+\frac{h}{2}\right) + \dots$$

$$f(x) = f\left(x+\frac{h}{2}\right) - \frac{h}{2} f'\left(x+\frac{h}{2}\right) + \frac{h^2}{8} f''\left(x+\frac{h}{2}\right) - \dots$$

Hence we find by subtraction and addition

$$f(x+h) - f(x) = h f'\left(x + \frac{h}{2}\right) + h^3 \dots \quad *) \qquad (3)$$

$$\frac{f(x+h) + f(x)}{2} = f\left(x + \frac{h}{2}\right) + h^2 \dots . \qquad (4)$$

[*) Note that the symbol . . . follows after h^3, without an intervening + or − sign, indicating that a coefficient for h^3 is omitted. The same notation is used in several subsequent equations.]

In (3) there does not occur any term with h^2, and in (4) there is no term with h; these terms were eliminated by introducing $x + \frac{h}{2}$ as argument of f and of f'.

As a simple application of equation (3) we take, for instance,

$$sin\ u - sin\ v = (u - v)\ cos\ \frac{u + v}{2} + (u - v)^3\ .\ .\ .\ .$$

If we aim to calculate here only to an accuracy of $(u - v)^2$ inclusive, then we can write u or v, optionally, in the term with $(u - v)$, for instance:

$$sin\ u = sin\ v + (u - v)\ cos\ u + (u - v)^3\ .\ .\ . \qquad (5)$$

or

$$sin\ u = sin\ v + (u - v)\ cos\ v + (u - v)^3\ .\ .\ .\ . \qquad (6)$$

These last two formulae are *equally* accurate insofar as terms of the same order are neglected in both.

Another application of this fundamental theorem is the following:

If $f(x, x')$ is a function of x and x', which can be developed in powers of $(x' - x)$, then

$$f(x, x') = f(x, x) + (x' - x)\ f'(x) + (x' - x)^2 + .\ .\ .$$

or

$$f(x, x') = f(x', x') + (x - x')\ f'(x') + (x' - x)^2 + .\ .\ .\ .$$

From these two equations together follows

$$f(x, x') = \frac{f(x, x) + f(x', x')}{2} + (x' - x)\frac{f'(x) - f'(x')}{2}\ .$$

But $f'(x)$ and $f'(x')$ themselves only differ by terms of the order $(x' - x)$; hence

$$f(x, x') = \frac{f(x, x) + f(x', x')}{2} + (x' - x)^2\ .\ .\ .\ .$$

$f(x, x)$ and $f(x', x')$ are the two values of $f(x, x')$ here, which result if we set $x' = x$ and $x = x'$, respectively.

Following are two simple examples for this:

$$\sqrt{x x'} = \frac{x + x'}{2} + (x' - x)^2\ .\ .\ .$$

$$\sqrt{\frac{x^2 + x'^2}{2}} = \frac{x + x'}{2} + (x' - x)^2\ .\ .\ .$$

or in words: the geometric mean, the R.M.S. mean (in the case of the calculation of the mean error) and many other means of two numbers x and x' are absolutely equal to the arithmetic mean up to terms of the order $(x' - x)$, inclusive.

At the close of these considerations, reference is made to the fact that approximation formulae which are to be accurate to *one* term only are written in the simplest way in the form of differential equations. For example, if we aim to have $sin\ u - sin\ v$ accurate to terms of the order $u - v$ only, then we set

$$sin\ u - sin\ v = d\ sin\ v \quad \text{or} \quad = -\ d\ sin\ u$$

and

$$u - v = dv \quad \text{or} \quad = -\ du\ ,$$

and we have thus:

$$d \sin u = \cos u \, du \quad \text{or} \quad d \sin v = \cos v \, dv \; .$$

Hence it follows:

$$\sin u - \sin v = (u - v) \cos u \quad \text{or} \quad = (u - v) \cos v$$

in agreement with the above (5) and (6).

Inversion of series

If a convergent power series exists in this form:

$$y = Ax + Bx^2 + Cx^3 + Dx^4 + \ldots , \tag{7}$$

then we can set up the problem in the inverse sense of presenting x by a convergent series in y.

In the first approximation, series (7) solved for x yields in any case:

$$x = \frac{y}{A} + y^2 \ldots ,$$

hence
$$y = Ax + B\left(\frac{y}{A} + y^2 \ldots\right)^2 + \ldots ,$$

and this solved for x yields:

$$x = \frac{y}{A} - \frac{B}{A^3} y^2 + \ldots .$$

In this way we can go on and add, step by step, higher terms $y^3 \ldots , y^4 \ldots ,$ etc., a procedure which is often useful in special cases. We also can present the method more generally, according to which the solution of series (7) is to assume this form:

$$x = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 + \ldots .$$

The coefficients $\alpha, \beta, \gamma, \delta \ldots$ have the following meanings here:

$$\left.\begin{aligned} \alpha &= \frac{1}{A} \quad , \quad & \beta &= -\frac{B}{A^3} \\ \gamma &= \frac{2B^2}{A^5} - \frac{C}{A^4} \quad , \quad & \delta &= -\frac{5B^3}{A^7} + \frac{5BC}{A^6} - \frac{D}{A^5} . \end{aligned}\right\} \tag{8}$$

Although such developments are best carried out for an individual case, we will nevertheless give here, for example, such an inversion of series with four elements (from *Zeitschr. f. Verm.*, 1894, p. 38 and p. 149), which perhaps can be used again, or else, vice versa, proves what is said below at the end.

$$\left.\begin{aligned} \Delta &= Ax - By^2 - Cx^2 - Dxy^2 + Ex^3 - Fx^2y^2 + Gy^4 \\ \lambda &= ay + byx + cyx^2 - dy^3 + eyx^3 - fy^3x . \end{aligned}\right\} \tag{9}$$

The solution of these two equations of x and y, carried out step by step, yielded:

$$x = \frac{1}{A}\Delta + \frac{B}{A\,a^2}\lambda^2 + \frac{C}{A^2}\Delta^2$$
$$+ \left(\frac{2\,B\,C}{A^3 a^2} + \frac{D}{A^2 a^2} - \frac{2\,B\,b}{A^2 a^3}\right)\Delta\lambda^2 + \left(\frac{2\,C^2}{A^5} - \frac{E}{A^4}\right)\Delta^3$$
$$+ \left(\frac{3\,B\,b^2}{A^3 a^4} - \frac{6\,B\,b\,C}{A^4 a^3} - \frac{2\,B\,c}{A^3 a^3} + \frac{3\,C\,D}{A^4 a^2} + \frac{6\,B\,C^2}{A^5 a^2} - \frac{2\,b\,D}{A^3 a^3} - \frac{3\,B\,E}{A^4 a^2} + \frac{F}{A^3 a^2}\right)\Delta^2\lambda^2$$
$$+ \left(\frac{5\,C^3}{A^7} - \frac{5\,C\,E}{A^6}\right)\Delta^4$$
$$+ \left(\frac{2\,B\,d}{A\,a^5} - \frac{2\,B^2\,b}{A^2 a^5} + \frac{B^2\,C}{A^3 a^4} + \frac{B\,D}{A^2 a^4} - \frac{G}{A a^4}\right)\lambda^4 \qquad (10)$$

$$y = \frac{1}{a}\lambda - \frac{b}{A\,a^2}\Delta\lambda + \left(\frac{b^2}{A^2 a^3} - \frac{b\,C}{A^3 a^2} - \frac{e}{A^2 a^2}\right)\Delta^2\lambda + \left(\frac{d}{a^4} - \frac{B\,b}{A\,a^4}\right)\lambda^3$$
$$+ \left(\frac{4\,B\,b^2}{A^2 a^5} - \frac{b\,D}{A^2 a^4} - \frac{2\,B\,b\,C}{A^3 a^4} - \frac{2\,B\,c}{A^2 a^4} - \frac{4\,b\,d}{A\,a^5} + \frac{f}{A\,a^4}\right)\Delta\lambda^3$$
$$+ \left(\frac{2\,b^2\,C}{A^4 a^3} - \frac{b^3}{A^3 a^4} + \frac{2\,b\,c}{A^3 a^3} - \frac{2\,b\,C^2}{A^5 a^2} + \frac{b\,E}{A^4 a^2} - \frac{2\,C\,c}{A^4 a^2} - \frac{e}{A^3 a^2}\right)\Delta^3\lambda\,. \qquad (11)$$

The former case (7) is partially included here also; we only need to set in (9)

$$A = A\,, \quad -C = B\,, \quad E = C$$

and, in addition,

$$B = 0\,, \quad D = 0\,, \quad F = 0\,, \quad G = 0\,,$$

then (10) changes to

$$x = \frac{1}{A}\Delta - \frac{B}{A^3}\Delta^2 + \left(\frac{2\,B^2}{A^5} - \frac{C}{A^4}\right)\Delta^3 + \left(-\frac{5\,B^3}{A^7} + \frac{5\,B\,C}{A^6}\right)\Delta^4,$$

in agreement with (8) within the comparable parts.

Or set in (9) $a = A$, $b = 0$, $c = 0$, $d = -C$, $e = 0$, $f = 0$, then we will have

$$y = \frac{1}{A}\lambda - \frac{C}{A^4}\lambda^3\,,$$

which also agrees with (8) as far as the comparable.

In a similar way, we also can compare *two* series with one another; let there be given:

$$A\,x + B\,x^2 + C\,x^3 + \ldots = A'y + B'y^2 + C'y^3 + \ldots\,.$$

Then we can present x thus:

$$x = \alpha y + \beta y^2 + \gamma y^3 + \ldots\,,$$

where the coefficients $\alpha, \beta, \gamma \ldots$ have the following meanings:

$$\alpha = \frac{A'}{A}\,, \quad \beta = \frac{B'}{A} - \frac{B\,A'^2}{A^3}\,, \quad \gamma = \frac{C'}{A} - \frac{2\,B\,A'\,B'}{A^3} - \frac{C\,A'^3}{A^4} + \frac{2\,B^2\,A'^3}{A^5}\,. \qquad (12)$$

Such inversions of series occur frequently; but it is seldom useful to apply generally prepared formulae with coefficients $A, B, C \ldots$ because in practical cases the coefficients mostly have simple relationships among themselves also (for example, goniometric ones), which can be used then immediately in the case of the stepwise solution.

We consider different values of a function y, which correspond to certain values of the argument x being in arithmetical progression, and use the denotations according to the following arrangement:

Argument	Function	Differences		
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		$\Delta^3 y_1$
x_3	y_3		$\Delta^2 y_2$	
		Δy_3		
x_4	y_4			

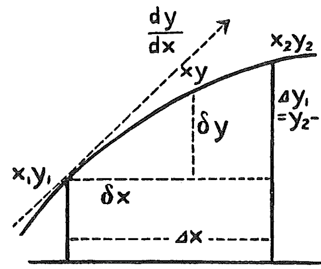

Fig. 1.

The point in question is an intermediate value of x, which, for example, lies between x_1 and x_2 and we let it be $= x_1 + \delta x$. We assume δx to be smaller here than the general interval Δx; therefore we set:

$$\frac{\delta x}{\Delta x} = \varkappa \quad \text{hence} \quad \varkappa < 1. \tag{1}$$

The functional value y pertinent to this $x_1 + \delta x$ is calculated according to the interpolation formula:

$$y = y_1 + \varkappa\,\Delta y_1 - \frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,\Delta^2 y_1 + \frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,\frac{2-\varkappa}{3}\,\Delta^3 y_1 - \frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,\frac{2-\varkappa}{3}\,\frac{3-\varkappa}{4}\,\Delta^4 y_1 \tag{2}$$

or

$$y = y_1 + z_1\,\Delta y_1 + z_2\,\Delta^2 y_1 + z_3\,\Delta^3 y_1 + z_4\,\Delta^4 y_1 + \dots$$

where we set

$$z_1 = \varkappa\,,\quad z_2 = -\frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,,\quad z_3 = +\frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,\frac{2-\varkappa}{3}\,,\quad z_4 = -\frac{\varkappa}{1}\,\frac{1-\varkappa}{2}\,\frac{2-\varkappa}{3}\,\frac{3-\varkappa}{4}\,,\quad \text{etc.} \tag{3}$$

Tables have also been computed for these coefficients, which are useful in the case of frequent interpolation work with higher than second differences. We only give here a small table for ten-part interval up to the fifth order z_5.

$\varkappa$	z_2	z_3	z_4	z_5
	—	+	—	+
0.1	0.045	0.0285	0.0207	0.016
0.2	0.080	0.0480	0.0336	0.026
0.3	0.105	0.0595	0.0402	0.030
0.4	0.120	0.0640	0.0416	0.030
0.5	0.125	0.0625	0.0391	0.027
0.6	0.120	0.0560	0.0336	0.023
0.7	0.105	0.0455	0.0262	0.017
0.8	0.080	0.0320	0.0176	0.011
0.9	0.045	0.0165	0.0087	0.005

A simple numerical example may illustrate the application here:

x	$y = \log x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
26	1.4149733	+	—	+	—	+
		163905				
27	4313638		5963			
		157942		421		
28	4471580		5542		46	
		152400		375		9
29	4623980		5167		37	
		147233		338		7
30	4771213		4829		30	
		142404		308		
31	4913617		4521			
		137883				
32	5051500					

$y = \log 26{,}4$

	1.4149733
0.4 · 163905 =	65562·0
0.120 . 5963 =	715·56
0,064 . 421 =	26·94
0.042 . 46 =	1·93
0.03 . 8 =	0·24
$\log 26.4$ =	1.4216039·67

The deviation from the correct value 4216039 is caused by rounding off.

Since the differences are positive and negative in turn, in the same sense as the z's, all product terms have become positive here.

If we exclude extended tabular computations (for which the interpolation is frequently carried out in another form), we seldom deal in geodesy with higher than second differences; with the latter, however, we deal very frequently, for instance, in the case of fundamental trigonometric calculations in ten-place logarithms of the *Thesaurus logarithmorum*, as we shall show by an example:

Let there be given $x = 15°30'34.67492''$, and we are to determine the pertinent value $y = \log \sin x$ from the ten-place table.

Assuming that the reader has on hand the ten-place table in question, we set up the computation accordingly in full detail:

x	$\log \sin x$	+	—
$x = 15° 30' 30''$	$\log \sin x = 9 . 4271265{\cdot}273$		
		758·727	
30 40	4272024·000		145
		758·582	
30 50	4272782·582		141
		758·441	
30 60	4273541·023 .		

For $15° 30' 34.67492''$ $\delta x = 4.67492''$ and $z = 0.467\,492$, hence:

$$
\begin{array}{rr}
 & 9.4271265{\cdot}273 \\
0.467492 \,.\, 758{\cdot}727 = & 354{\cdot}6988 \\
0.467 \cdot \dfrac{0{,}533}{2} \cdot 0{\cdot}143 = & 0{\cdot}0178 \\
\hline
 & 9.4271619{\cdot}9896 \\
\log \sin 15°\, 30'\, 34.67492'' = & 9.4271619{\cdot}990\ .
\end{array}
$$

The *Thesaurus logarithmorum* gives on page 2 an auxiliary table for this method of computation for $z \dfrac{1-z}{2} \delta$ with centesimal z, but $\delta = 4, 6, 8 \ldots 44$ only, which is not nearly enough in the trigonometric part. Such an auxiliary table is not necessary if we can take care of the second differences with the slide rule.

It is often useful to bring the interpolation formula to a form *other* than the original form (2), namely to this form:

$$
y = y_1 + z\left(\Delta y_1 - \frac{1-z}{2}\,\Delta^2 y\right). \tag{4}
$$

A correction $-\frac{1-z}{2}\Delta^2 y$ to the first difference Δy_1 is computed here and then we compute further with the corrected first difference as in the case of simple proportional interpolation:

On this basis the following has been computed for clear and quick illustration:

$$\left.\begin{array}{lccccccccccc} z = & 0.0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0 \\ \frac{(1-z)}{2} = & 0.50 & 0.45 & 0.40 & 0.35 & 0.30 & 0.25 & 0.20 & 0.15 & 0.10 & 0.05 & 0.00\,. \end{array}\right\} \quad (5)$$

The actual solution is made best with the slide rule in the case of this form also.

We will take once more the previous example with the computation of the proportional part with six-place logarithms:

$$\begin{array}{rll}
\text{Required } \log\sin 15°\,30'\,34.67492'' , & z = 0.467492\,, & 1 - z = 0.533 \\
 & +\ 758{\cdot}868 & \\
\log\sin 15°\,30'\,30'' = 9.427\ 1265{\cdot}273 & & -0{\cdot}141 \\
 & +\ 758{\cdot}727 & \\
 & +\ \ \ 0{\cdot}038 & (\text{correct } \frac{0.533}{2}\,0{\cdot}141) \\
 & \overline{\ \ 758{\cdot}765} & \\
 & 2.880107 = \log 758{\cdot}765 & \\
 & 9.669774 = \log 0.467492 & \\
354{\cdot}717 & \overline{2.549881} & \\
\log\sin 15°\,30'\,34.67492'' = 9.427\ \ 1619{\cdot}990 & \text{as above.} &
\end{array}$$

This method is also applicable for the inverse problem, to find the angle x pertaining to the given $\log\sin x$; then we calculate first an approximate value of z; with the slide rule in the above case $z = 0.46$ (which we write, say with pencil, in its place, i.e. 15°30' 34.6" . . .), and with this immediately further, in part mentally, in part with the slide rule, $\frac{0.54}{2} \cdot 0{\cdot}141 = 0{\cdot}038$, etc.

In order to apply the correction $\frac{1-z}{2}\Delta^2 y$ always with the correct sign, the mechanical rule should be remembered: An approach to the *previous* difference must be achieved by this correction.

For such purposes we can use a mimeographed form (according to the above example), which contains a small auxiliary table according to (5) and everything that is constant printed thereon in advance. For major computations with ten-place logarithms, the individual logarithms can be treated in the form, which then forms an enclosure to the main computation.

As usual, however, it is a matter of practice; he who has to do with such computations for an extended length of time soon becomes used to taking into account the second differences with the slide rule besides, and then it has hardly caused more additional work than the computing of the actual proportional parts.

Interpolation with differential quotients

In the explanatory figure on p. 33, there is written, in addition, the differential quotient $\frac{dx}{dy}$ next to the tangent of the point $x_1 y_1$, besides Δx, Δy and δx, δy, and we will consider how the interpolation can be carried out, if there are indicated, not the finite differences Δy, $\Delta^2 y$, but the theoretical differential quotients for the individual z's and y's themselves, hence for Δx as unit, for instance:

$$
\left.
\begin{array}{llll}
x_0 & y_0 & \Delta x \frac{dy}{dx}\Big)_0 = d_0 & \\
 & & & d_1 - d_0 = \delta_0 \\
x_1 & y_1 & \Delta x \frac{dy}{dx}\Big)_1 = d_1 & \\
 & & & d_2 - d_1 = \delta_1 \\
x_2 & y_2 & \Delta x \frac{dy}{dx}\Big)_2 = d_2 & \\
\ldots & & &
\end{array}
\right\} \quad (6)
$$

For example, we would have in a common-logarithmic table:

$$y = \log x \qquad \frac{dy}{dx} = \frac{\mu}{x}.$$

We can compute these differential quotients more rigorously than are computed the Δy's which include the rounding-off errors of two neighboring y's. We denote in the above scheme by δ_0 and δ_1 the differences of two successive d's; we will assume, however, that those δ_0's and δ_1's are equal, i.e. that it is only a question of second-order interpolation. Then we have from (4), p. 34:

$$y = y_1 + z\left(\Delta y_1 - \frac{\Delta^2 y}{2} + \frac{z}{2}\Delta^2 y\right)_1, \tag{7}$$

where $$\Delta y_1 - \frac{\Delta^2 y}{2} = \Delta y_1 - \frac{\Delta y_1 - \Delta y_0}{2} = \frac{\Delta y_1 + \Delta y_0}{2},$$

from the geometric viewpoint, however, this is $= \Delta x \frac{dx}{dy}\Big)_1$, namely because the chord from $x_0 y_0$ to $x_2 y_2$ is to be considered parallel to the tangent at $x_1 y_1$. In a similar way, we also have

$$\Delta y_1 + \Delta y_0 = y_2 - y_0 = 2\,\Delta x \frac{dy}{dx}\Big)_1 \quad \text{and} \quad \Delta y_2 + \Delta y_1 = y_3 - y_1 = 2\,\Delta x \frac{dy}{dx}\Big)_2$$
$$\Delta y_2 - \Delta y_0 = 2\,\Delta^2 y = 2\,\delta;$$

δ is assumed here according to scheme (6) with $\delta_1 = \delta_0 = \delta$, and if we also use the symbol d which is introduced there, we have now from (7):

$$y = y_1 + z\left(d_1 + \frac{z}{2}\delta\right) = y_1 + z\,d. \tag{8}$$

Hence, according to this, we would first interpolate proportionally $d = d_1 + \frac{z}{2}\delta$ in the series of the d's, but with $\frac{z}{2}$ instead of with z, and then go on computing $y = y_1 + z\,d$ proportionally.

This method of interpolation would have some advantages compared with the common one if the differential quotient $\frac{dy}{dx}$ or d, as the case may be, could be indicated rigorously; but the tables are in general not set up for this.

Logarithms of many places

We have summarized the logarithmic-trigonometric tables required for calculations in the field and land survey in Vol. II, first half-volume, 9th edition, 1931, p. 153.

Seven-place logarithms are involved primarily for the problems of land survey, for which we have the older tables by Vega-Bremiker-Tietgen, by Vega-Hülsse, by Bruhns and by Schön. The last-mentioned table also offers the possibility of approximately taking into account the eighth decimal place.

Of newer seven-place tables we have:

Siebenstellige Logarithmentafel der trigonometrischen Funktionen für jede Bogensekunde des Quadranten, prepared by Prof. Dr. J. Peters, Leipzig, 1911.

Siebenstellige Logarithmen der trigonometrischen Funktionen von 0° bis 90° für jedes Tausendstel des Grades, published by the Trigonometric Department of the Preuss. Landesaufnahme under the scientific direction of Prof. Dr. Peters, Berlin, 1921.

For the cases in which seven-place computation is not sufficient, eight- and ten-place tables have been published in more recent times. We have the very valuable work for sexagesimal units:

Logarithmische-trigonometrische Tafeln mit acht Dezimalstellen, newly computed and published by Dr. J. Bauschinger and Dr. J. Peters. First Volume: Table of the eight-place logarithms of all numbers from 1 to 200,000. Second edition, Leipzig, 1936. Second Volume: Table of the eight-place logarithms of trigonometric functions for every sexagesimal second of the quadrant, Leipzig, 1911.

Furthermore, for grades, centesimal division, likewise of eight places: Service géographique de l'armée. *Tables des logarithmes à huit décimales des nombres entiers de 1 à 120 000 et des sinus et tangentes de dix secondes en dix secondes d'arc dans le systeme de la division centésimale du quadrant*, publiées par ordre du ministre de la guerre, Paris. Imprimerie nationale, 1891.

The oldest work of the ten-place logarithmic tables is:

Thesaurus logarithmorum completus, by Georg Vega, Leipzig, 1794, with ten-place logarithms of the numbers and trigonometric functions every 10". This work can only be bought secondhand at a very high price, and therefore the Italian General Staff, under the direction of General Ferrero, made an exact new edition by the method of zincography, eliminating the printing errors which had become known (Florence, 1889, second edition, 1896).

A new ten-place tabular work is:

Zehnstellige Logarithmentafel. First Volume: Zehnstellige Logarithmen der Zahlen von 1 bis 100,000 nebst einem Anhang mathematischer Tafeln, Berlin, 1922. Second Volume: Zehnstellige Logarithmen der trigonometrischen Funktionen von 0° bis 90° für jedes Tausendstel des Grades, Berlin, 1919. Published by the Preussische Landesaufnahme under the scientific direction of Prof. Dr. J. Peters.

Compared with Vega's *Thesaurus*, this work offers the advantage that the interval of 0.001° = 3.6" (against 10" in *Thesaurus*) permits a simpler interpolation. For the further simplification of interpolation as well as for the conversion of fractions of the degree into minutes and seconds there is used *Hilfstafeln zur zehnstelligen Logarithmentafel*, published by the Preussische Landesaufnahme under the scientific direction of Prof. Dr. J. Peters, stereotype print, Berlin, 1919. Printed and published by Preussische Landesaufnahme.

In this connection let us mention the ten-place table of the numbers in U. S. Coast and Geodetic Survey, 1895-96, Washington, 1897. Append. 12, pp. 395-722. Logarithms, their nature, computation and uses, with Logarithmic Tables of numbers and circular functions to ten places of decimals. Part I.

Finally, one more word is to be said about the quite rare cases in which even ten-place logarithms are not sufficient. For these we have first the work:

Nouvelles Tables trigonométriques fondamentales contenant les logarithmes des lignes trigonométriques de centième en centième du quadrant avec dix-sept décimales, de neuf en neuf minutes avec quinze décimales, et de dix en dix secondes avec quatorze décimales, by H. Andoyer, Paris, 1911.

In addition to this work which is designed for direct logarithmic-trigonometric computations there are a number of auxiliary tables with which logarithms of many decimal places can be computed. In this connection we have first on pp. 642-684 in *Thesaurus* the natural 48-decimal logarithms of all four-decimal numbers without simple factors.

We have further *Hilfstafeln zur präzisen Berechnung 20stelliger Logarithmen*, by Anton Steinhauser, with subsidies of the k.k. Academy of Sciences. Wien, Gerolds Sohn, 1880 (with corrections see *Zeitschrift für Vermessungswesen*, 1893, p. 603; also see *Astr. Nachr.*, 166 [1904], pp. 285-288). This work gives the twenty-decimal logarithms of all numbers of four digits from 1000 to 9999 (Table A) and, in addition, the logarithms of the numbers 1.000001 to 1.000999 (Table B) and of a following similar group so that each number can be broken down into factors whose logarithms can be found.

We reach about 15 places with the first part A by Steinhauser simply with the help of the logarithmic series:

$$\log(a+b) = \log a + \log\left(1+\frac{b}{a}\right) = \log a + \Delta \log a\,, \quad \frac{b}{a} = x$$

$$\Delta \log a = \mu x - \frac{\mu x^2}{2} + \frac{\mu x^3}{3} - \frac{\mu x^4}{4}$$

$$\log \mu = 9.6377843{\cdot}113$$

$$\log\frac{\mu}{2} = 9.3367543\,, \quad \log\frac{\mu}{3} = 9.160663\,, \quad \log\frac{\mu}{4} = 9.03572\,.$$

For example, let us compute $\log \pi$ on this basis:

$$\pi = 3.14159\ 26535\ 89793\ 24;$$

the first 4-place approximation which presents itself would be 3.141 or 3.142; however, we arrive immediately two places further with the approximation 3.14160, which $= 4 \cdot 0.7854$ and is presented in the following way:

$$\begin{array}{r@{\;=\;}l} \log 4 & 0.60205\ 99913\ 27962\ 39 \\ \log 0.7854 & 9.89509\ 08969\ 34399\ 43 \\ \hline \log 3{,}1416 & 0.49715\ 08882\ 62361\ 82\,. \end{array}$$

With this approximation we have

$$\begin{array}{rl} \pi = & 3{,}14159\ 26535\ 89793\ 24 \\ \text{approximation}\quad a = & 3{,}14160 \\ \hline b = & -\,0{,}00000\ 73464\ 10206\ 76 \end{array}$$

		x^2		x^3
$\log b =$	4.866 0751·747			
$\log a =$	0.497 1508·883			
$\log x =$	4.368 9242·864	8.7378486		3.10677
$\log \mu =$	9.637 7843·113	9.3367543		9.16066
	4.006 7085·977	8.0746029		2.26743
	10155,670 405	0.011874	+	0,000 000 = 10155,682279 .

Hence, the approximation

$$\begin{array}{rl} & 0.49715\ 08882\ 62361\ 82 \\ & -\quad 10155\ 68227\ 9 \\ \hline \log \pi = & 0.49714\ 98726\ 94133\ 9 \\ \text{actually} & 0.49714\ 98726\ 94133\ 854 \end{array}$$

(see above p. 22).

Therefore, we have obtained $\log \pi$ to 16 places in this way.

For the inverse problem, namely to find the number corresponding to a logarithm of many decimals, we again assume an approximation a, and let $\log(a+b)$ be the given logarithm, consequently b the unknown, and therefore it is possible to compute

$$\frac{\log(a+b) - \log a}{\mu} = y\,,$$

where

$$\log\frac{a+b}{a} = \log\left(1+\frac{b}{a}\right) = \mu y\,;$$

therefore, according to the exponential series (p. 21):

$$1+\frac{b}{a}=10^{\mu y}=1+y+\frac{y^2}{2}+\frac{y^3}{6}+\frac{y^4}{24}$$

$$b=a\,y+\frac{a\,y^2}{2}+\frac{a\,y^3}{6}+\frac{a\,y^4}{24}.$$

According to this, if we treat the previous example backward, we have given $log\ \pi = log\ (a + b)$ $= 0.49714\ 98726\ 9413385$ with the approximation $a = 3.14160$ and $log\ a$ as above; then $log\ y$ $= 4.3689247{\cdot}942\ n$ and according to the formula for b we have

$$b = -\,73464.18797 + 0.08590 = -\,73464.10207,$$

hence

$$\pi = 3.14160 - 0.00000\ 73464\ 10207 = 3.14159\ 26535\ 89793,$$

or π is obtained from $log\ \pi$ to an accuracy of 15 places.

Since a good approximation for b is easily available, we can also calculate briefly according to the principle of the mean argument in the following way:

$$b = \left(a + \frac{b}{2}\right) y = a_0\, y.$$

Let us show this by a second example:

Suppose we have

$$\begin{aligned} log\ e^2 &= 7.824\ 4104{\cdot}237 \\ log\ \mu &= 9.637\ 7843{\cdot}113\ 00537 \\ log\ (a+b) = log\ \mu\, e^2 &= 7.462\ 1947{\cdot}350\ 00537 \end{aligned}$$

in addition, from *Thesaurus*

$$\mu\, e^2 = 0.00289\ 86430\ 302.$$

But we would like to improve the accuracy of the last places and find, in this connection, from the factor table by Vega-Hülsse, p. 378, that $28987 = 101 \times 287$ and we obtain hence, by combination with the help of Steinhauser's 20-decimal logarithms:

$$log\ a = \quad 7.462\ 2032\ 705\ 16635,$$

hence

$$log\ (a+b) - log\ a = d = -\,0.000\ 0085\ 355\ 16098.$$

The mean of the above eleven-decimal $\mu\, e^2$ and of the approximation a is $a_0 = 0.00289\ 86715\ 151$, with which $b = a_0 \frac{d}{\mu} = -\ 569.697\ 7144$; hence the required

$$\mu\, e^2 = a + b = 0.00289\ 86430\ 30229\ .$$

A more recent table for the computation of logarithms of many decimal places is *Veröffentlichungen des Astronomischen Rechen-Instituts zu Berlin*, No. 43, "Zweiundfünfzigstellige Logarithmen," berechnet von Prof. Dr. J. Peters und Dr. J. Stein, Berlin, 1919. This table serves for the computation of 50-place logarithms.

In the case of such computations there are required, above all, good approximation values, as large as possible, which can be obtained with study and some practical device by breaking down into products; a factor and prime number table, for example, in Vega-Hülsse, Leipzig, 1840, pp. 360-454, is useful here.

For the computation of values of trigonometric functions to many decimal places we have a table by J. Peters, *Einundzwanzigstellige Werte der Funktionen Sinus und Kosinus zur genauen Berechnung von zwanzigstelligen Werten sämtlicher trigonometrischen Funktionen eines beliebigen Arguments sowie ihrer Logarithmen*, Berlin, 1911. This work contains first the twenty-one-place numerical values for *sin* and *cos* for every 10' (sexagesimal units) through the whole quadrant and, in addition, the same values for every second from 0° to $0^\circ 10'$. With these tables we calculate *sin* and *cos* for an arbitrary angle according to the formulae

$$\sin(\alpha \pm \Delta\alpha) = \sin\alpha\cos\Delta\alpha \pm \cos\alpha\sin\Delta\alpha$$
$$\cos(\alpha \pm \Delta\alpha) = \cos\alpha\cos\Delta\alpha \mp \sin\alpha\sin\Delta\alpha,$$

by dividing the angle into two such parts α and $\Delta\alpha$ as are contained in the two tables. In the second table the differences of first to third order are indicated for the interpolation of fractions of a second into $\Delta\alpha$.

The change to logarithms is done here also by means of Steinhauser's table or by means of the newer table by Peters-Stein.

In the table by Peters-Stein we also find a summary of older logarithmic tables with more than 20 decimals as well as a series of auxiliary tables for the computation of such logarithms.

Trigonometric natural tables

In view of the constantly growing significance of the calculating machine for trigonometric computations, let us mention a few additional tables of natural values of trigonometric functions for the completion of our statements in Vol. II, first half-volume, 9th edition, 1931, p. 154:

Zehnstellige Tafeln der Sinus, Kosinus und Tangenten für die dezimale Teilung des Nonagesimalgrades, Generaldirektion des Grundsteuerkatasters (Österr. Triangulierungs- und Kalkülbureau), 1920.

Section de Géodésie de l'Union géodésique et géophysique internationale. Publ. spéc. No. 1. *Tables à huit décimales des valeurs naturelles des sinus, cosinus et tangentes dans le système décimal, de centigrade en centigrade de 0 à 100 grades. – Tables à 20 décimales des valeurs naturelles des six lignes trigonométriques dans le système décimal, de grade en grade de 0 à 100 grades*, Paris, 1925.

Chapter III

THE TERRESTRIAL ELLIPSOID

Section 37. Explanations and Fundamental Dimensions

The ideal surface of the earth upon which our calculations are based is an ellipsoid of rotation, i.e. that surface which is generated by the rotation of an ellipse around its minor axis.

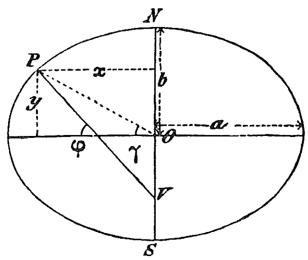

Fig. 1.

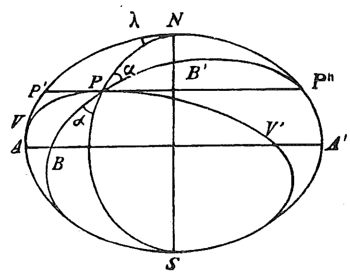

Fig. 2.

Ellipsoid of rotation.

At first the following quantities and equations which are related to the above Figs. 1 and 2 are involved:

$$\text{major semiaxis } a\,, \quad \text{minor semiaxis } b\,, \tag{1}$$

$$\text{flattening} \quad \alpha = \frac{a-b}{a}\,, \tag{2}$$

$$\text{eccentricity} \quad e = \sqrt{\frac{a^2-b^2}{a^2}}\,, \tag{3}$$

$$\text{second eccentricity} \quad e' = \sqrt{\frac{a^2-b^2}{b^2}}\,. \tag{4}$$

The eccentricity e in this sense is an absolute number and appears as the ratio of one-half of the linear eccentricity, $\sqrt{a^2-b^2}$, to the major semiaxis a.

When we also relate one-half of the linear eccentricity, $\sqrt{a^2-b^2}$, to the minor semiaxis b, we come to the value e' according to (4), which is generally more advantageous than e according to (3) for our computations.

Between e and e' there exist the following relations which are easily provable:

$$e'^2 = \frac{e^2}{1-e^2} \qquad e^2 = \frac{e'^2}{1+e'^2} \tag{5}$$

and
$$(1-e^2)(1+e'^2) = 1\,. \tag{6}$$

Between the flattening α and the related e we have

$$\alpha = 1 - \sqrt{1 - e^2} \quad \text{or} \quad e^2 = 2\alpha - \alpha^2. \tag{7}$$

In α and e the major semiaxis a is preferred and in e' the minor semiaxis b is preferred; both, a and b, occur homogenously in the values

$$n = \frac{a-b}{a+b} \qquad m^2 = \frac{a^2 - b^2}{a^2 + b^2}. \tag{8}$$

In addition to the two semiaxes a and b we introduce a further, third, quantity c, according to the equation

$$c = \frac{a^2}{b} \quad \text{or} \quad c = \frac{a}{\sqrt{1 - e^2}}. \tag{9}$$

This quantity c has the meaning of the radius of curvature at the north pole or south pole of the meridian ellipse and therefore osculates closely on the ellipsoid of rotation in the neighborhood of the poles; it will prove useful later in the case of a few developments, as is probable from the outset because the axis of rotation b is the most important one in the case of an ellipsoid of rotation.

We will conduct our geodetic developments mostly with c and e'^2 as constants.

In Fig. 1, p. 41, we have to consider further the normal PV, which indicates the direction of gravity on the ellipsoid and, in addition, the angle φ, which the normal makes with the major axis, i.e.:

the geographic latitude φ.

The angle γ, entered in Fig. 1, which is called the *geocentric* latitude and which is hardly ever used in the measurement of the earth, is different from it (γ, however, occurs in the case of astronomical parallax computations).

Furthermore, the following conceptions are to be established according to Fig. 2, p. 41:

Arc of parallel $P'P''$,
Meridians NAS and NPS,
Geographic difference of longitude λ,
Normal sections, for instance BPB',
Azimuth α of a normal section.

The Bessel dimensions of the earth

As already indicated in the introduction on p. 8, the dimensions of the earth calculated by Bessel in 1841 by adjustment from 10 latitude degree-measurements are very often applied and we will mostly use them in the following.

Bessel gave the following final values from his adjustments in the 19th volume, 1842, of the *Astronomische Nachrichten*, No. 438, Altona, 1841, 2 December, p. 116:

$$\left.\begin{array}{l}
n = \dfrac{a-b}{a+b} = 0{,}00167\ 41848 \\
\dfrac{a}{b} = \dfrac{299{,}1528}{298{,}1528} \\
a = 327{,}2077.14 \text{ toises} \qquad \log a = 6.514\,8235{\cdot}337 \text{ in toises} \\
b = 326{,}1139.33 \text{ toises} \qquad \log b = 6.513\,3693{\cdot}539 \text{ in toises} \\
\log e = 8.912\,2052 \qquad \log \sqrt{1-e^2} = 9.998\,5458{\cdot}202 \\
\text{Length of the meridian quadrant} \\
\quad = 513{,}1179.81 \text{ toises} = 10{,}000{,}855.76 \text{ meter.}
\end{array}\right\} \tag{10}$$

These are exactly Bessel's data and we could presume now that "Bessel's dimensions of the earth" which have been used for a century are thus unchangeably established; but this is not the case in the last places. These Bessel numbers do not correspond completely rigorously among themselves in the last places, as is easy to understand, and as we start now from one or the other and compute more rigorously further, we obtain deviations.

Gauss indicates Bessel's dimensions of the earth in part I of the *Untersuchungen über Gegenstände der höheren Geodäsie*, erste Abhandlung, 1843, pp. 9-10:

$$\log a = 6.514\ 8235{\cdot}337 \text{ in toises}, \quad \log \cos \varphi = \log \sqrt{1-e^2} = 9.998\ 5458{\cdot}202\ ;$$

then it reads: "Hence there follows with the help of the ten-place logarithms":

$$\varphi = 4°41'\,9.98262'' \quad \text{and} \quad \log \sin \varphi = \log e = 8.9122052{\cdot}079\ ;$$

and for the conversion of toises to the metric measure, Gauss has here the logarithm 0.2898199·300.

Gauss' table for the conformal projection of the ellipsoid on the sphere is based on these numbers.

Encke started from Bessel's $\log a$ and $\log b$ in the case of the calculation of his "Tafeln für die Gestalt der Erde" in the *Berliner Astronomisches Jahrbuch für 1852*, pp. 318-381; he says there, pp. 322-323: In the case of the following tables, the basis is the following data, according to Bessel:

$$\log a = 6.514\,8235{\cdot}337\ , \qquad \log b = 6.513\,3693{\cdot}539\ ,$$

hence there was derived:

$$a = 3{,}272{,}077.1399 \text{ toises} \qquad \log e = 8.912\,2052{\cdot}075$$
$$b = 3{,}261{,}139.3284 \text{ toises} \qquad \log \sqrt{1-e^2} = 9.998\,5458{\cdot}202$$
$$\frac{a-b}{a} = \frac{1}{299{,}152\,818} \qquad \log n = 7.223\,8033{\cdot}861$$
$$\log(1+n^2) = 0.000\,0012{\cdot}173$$
$$n = 0.001\,674184767$$

If we compare the numbers by Bessel and Encke we see that by the roundabout way via the ten-place logarithms a and b have changed by 0.0001 and 0.0016 toise, respectively.

All the numbers which have been established in well-known geodetic literature for Bessel's dimensions of the earth differ from one another more or less in the last places, as shown by us more clearly in the *Zeitschr. f. Verm.*, 1885, pp. 22-26, by listing those numbers.

The conversion of Bessel's a and b from toises to meters also caused variations of the last place. The legal ratio of the toise to the meter, which we already mentioned in the introduction, pp. 5 and 6, is $\frac{864}{443.296}$, and by calculating with the common 10-place logarithms, we obtain:

$$1 \text{ m} = \frac{443.296}{864} \text{ toises} \qquad \begin{aligned} \log 443.296 &= 2.646\,6938{\cdot}125 \\ \log 864 &= 2.936\,5137{\cdot}425 \\ \log(M,T) &= 9.710\,1800{\cdot}700 - 10 \\ \log(T,M) &= 0.289\,8199{\cdot}300 \end{aligned} \tag{11}$$

and the numbers corresponding to these logarithms

$$1 \text{ m} = 0.513\,074\,074 \text{ toise}, \quad 1 \text{ toise} = 1.949\,036\,310 \text{ m}\ .$$

If we calculate more rigorously, however, we will have by ordinary division:

$$(T,M) = \frac{443.296}{864} = 1.94903\,63098\,24587 \tag{12}$$

and the 11-place logarithm corresponding to this is:

$$\log(T,M) = \log\frac{443.296}{864} = 0.289\,8199\,2994\ ,$$

that is, rounded off to ten places, 0·001 smaller than the above 0·300 commonly used.

A. Böhm Edler von Böhmersheim carried out a new computation of the constants to 20 decimal places dependent on Bessel's dimensions of the earth in the *Abhandlungen der k. k. Geographischen Gesellschaft in Wien*, Band IX, Nr. 2, 1911, in the case of which he starts from the 10-place $\log a$ and $\log b$ in toises (10). Apart from the fact that there is rarely a need for more than ten-place logarithms of the constants of the earth, we cannot make use of these computations because in the following we have used everywhere the numerical values (11) assumed by the Preussische Landesaufnahme and by the Geodätisches Institut for the conversion from toises to meters.

The numerical sharpness of all these data goes far beyond the material precision, for after what we have seen about Bessel's adjustment itself in the introduction, p. 8, the quadrant of the meridian 10,000,856 m has a mean error of approximately 500 m and the number for the flattening 299 has a mean error of 5 whereas we calculate with $\log a$ and $\log b$ to 10 places. – Yet there exist good reasons for the adherence to certain 10-place numbers assumed unchangeably for the dimensions of the terrestrial ellipsoid upon which all computations are based as an ideal comparison and projection surface.

It is especially disturbing in the case of the computation of geodetic numerical tables where, because of the frequency of rounding off, we often write down 3 to 4 places more than we finally want, when the last places do not correspond in the case of one or the other computer.

We base all our computations in this volume as far as they refer to Bessel's terrestrial ellipsoid on the values which are indicated by Helmert in *Veröffentl. d. Kgl. Preuss. Geod. Instituts*, "Lotabweichungen Heft I, Formeln und Tafeln sowie einige Ergebnisse für Norddeutschland," Berlin, 1886, p. 4.

$$\left.\begin{aligned} \log a &= 6.804\,6434{\cdot}637 \\ \log b &= 6.803\,1892{\cdot}839\,. \end{aligned}\right\} \qquad (13)$$

All further fundamental values which are listed in the following result from these. As the most important ones we put on top the value of e^2 and its logarithm:

$$\left.\begin{aligned} e^2 &= 0.00667\,43722\,30614 \\ \log e^2 &= 7.82441\,04236\,54369 - 10\,. \end{aligned}\right\} \qquad (14)$$

The fundamental values used by the Preussische Landesaufnahme are published in *Rechenvorschriften für die trig. Abt. der Landesaufnahme*, "Formeln und Tafeln zur Berechnung der geographischen Koordinaten aus den Richtungen und Längen der Dreiecksseiten. Erste Ordnung," Berlin, 1878 [p. 4, $\log a$ and $\log e^2$ to 10 places in agreement with (13) and (14)].

We find further in *Die konforme Doppelprojektion der trig. Abt. der Kgl. Preuss. Landesaufnahme, Formeln und Tafeln*, by Dr. O. Schreiber, Berlin, 1897, p. 5, $\log a$ to 10 places and $\log e^2$ to 15 places, as indicated above in (13) and (14). We must thus assume that the above values (13) are also to be regarded as the starting values for all computations carried out by the Preussische Landesaufnahme.

Bessel's dimensions of the earth and mathematical constants

$$a = 6{,}377{,}397.15500\ m \qquad \log a = 6.804\,6434{\cdot}637$$

$$\log \frac{\varrho''}{c} = 8.508\,3274{\cdot}897$$

$$b = 6{,}356{,}078.96325\ \text{„} \qquad \log b = 6.803\,1892{\cdot}839$$

$$\log \frac{c}{\varrho''} = 1.491\,6725{\cdot}103$$

$$\frac{a^2}{b} = c = 6{,}398{,}786.84939\ \text{„} \qquad \log c = 6.806\,0976{\cdot}435$$

$$c = a\sqrt{1 + e'^2} = b\,(1 + e'^2) \qquad \log c^2 = 13.612\,1952{\cdot}870$$

$$\alpha = \frac{a-b}{a} = \frac{1}{299.15281285} \qquad log \frac{1}{\alpha} = 2.475\,8930{\cdot}907$$

$$\alpha = 0.00334\,27731\,81579 \qquad log\, \alpha = 7.524\,1069{\cdot}093$$

$$\frac{a-b}{a+b} = n = 0.00167\,41848\,00816 \qquad log\, n = 7.223\,8033{\cdot}949$$

$$\frac{a^2-b^2}{a^2} = e^2 = 0.00667\,43722\,30614 \qquad log\, e^2 = 7.824\,4104{\cdot}237$$

$$\frac{a^2-b^2}{b^2} = e'^2 = 0.00671\,92187\,97971 \qquad log\, e'^2 = 7.827\,3187{\cdot}833$$

$$1-e^2 = \frac{1}{1+e'^2} = 0.99332\,56277\,68685 \qquad log\,(1-e^2) = 9.997\,0916{\cdot}404$$

$$\frac{1}{1-e^2} = 1+e'^2 = 1.00671\,92187\,98677 \qquad log\,(1+e'^2) = 0.002\,9083{\cdot}596$$

$$\sqrt{1-e^2} = \frac{1}{\sqrt{1+e'^2}} = 0.99665\,72269 \qquad log\, \sqrt{1-e^2} = 9.998\,5458{\cdot}202$$

$$\frac{1}{\sqrt{1-e^2}} = \sqrt{1+e'^2} = 1.00335\,39848 \qquad log\, \sqrt{1+e'^2} = 0.001\,4541{\cdot}798$$

n	$log\, e^n$	$log\, \mu e^n$	$log\, e'^n$	$log\, \mu e'^n$	n
0		9.637 7843·113		9.637 7843·113	0
2	7.824 4104·237	7.462 1947·350	7.827 3187·833	7.465 1030·946	2
4	5.648 8208·474	5.286 6051·587	5.654 6375·666	5.292 4218·779	4
6	3.473 2312·711	3 111 0155·824	3.481 9563·499	3.119 7406·612	6
8	1.297 6416·948	0.935 4260·061	1.309 2751·332	0.947 0594·445	8
10	9.122 0521·185	8 759 8364·298	9.136 5939·165	8.874 3782·278	10
12	6.946 4625·422	6.584 2468·535	6.963 9126·998	6.601 6970·111	12

For the 7th log. place: $\mu e^2 = 28986.43030\,229$ $\qquad \mu e'^2 = 29181.19646\,966$

1 m = 0.513 074 074 toise $\qquad log\,(M, T) = 9.710\,1800{\cdot}700$ $\qquad$ (Cf. in this con-
1 toise = 1.949 036 310 m $\qquad log\,(T, M) = 0.289\,8199{\cdot}300$ $\qquad$ nection p. 43).

The numbers π, ϱ and μ (cf. also pp. 21 and 22).

$$\pi = 3.14159\,26536 \qquad log\, \pi = 0.497\,1498{\cdot}727 \qquad log \frac{1}{\pi} = 9.502\,8501{\cdot}273$$

$$\varrho^\circ = 57.29577\,95131 \qquad log\, \varrho^\circ = 1.758\,1226{\cdot}324 \qquad log \frac{1}{\varrho^\circ} = 8.241\,8773{\cdot}676$$

$$\varrho' = 3{,}437.74677\,07849 \qquad log\, \varrho' = 3.536\,2738{\cdot}828 \qquad log \frac{1}{\varrho'} = 6.463\,7261{\cdot}172$$

$$\varrho'' = 206{,}264.80624\,71 \qquad log\, \varrho'' = 5.314\,4251{\cdot}332 \qquad log \frac{1}{\varrho''} = 4.685\,5748{\cdot}668$$

For centesimal units:

$$\varrho_g = 63.66197\,72368 \qquad log\, \varrho_g = 1.803\,8801{\cdot}230 \qquad log \frac{1}{\varrho_g} = 8.196\,1198{\cdot}770$$

$$\mu = 0.43429\,44819 \qquad log\, \mu = 9.637\,7843{\cdot}113 \qquad log \frac{1}{\mu} = 0.362\,2156{\cdot}887$$

For units of the 7th place: $log\, \mu = 6.637\,7843{\cdot}113 \qquad log \frac{1}{\mu} = 3.362\,2156{\cdot}887$

$log\, 2 = 0.301\,0299{\cdot}957 \qquad log\, 5 = 0.698\,9700{\cdot}043 \qquad log\, 8 = 0.903\,0899{\cdot}870$
$log\, 3 = 0.477\,1212{\cdot}547 \qquad log\, 6 = 0.778\,1512{\cdot}504 \qquad log\, 9 = 0.954\,2425{\cdot}094$
$log\, 4 = 0.602\,0599{\cdot}913 \qquad log\, 7 = 0.845\,0980{\cdot}400 \qquad log\, 12 = 1.079\,1812{\cdot}460$

The boldface numbers (13), p. 44, and the ones calculated from them and the geodetic constants listed on p. 44 and above, form the basis for all trigonometric computations in Prussia and hence also the basis of all Prussian map works. The computations in this book also refer almost exclusively to the above dimensions of the earth by Bessel.

The international ellipsoid

Of the newer dimensions of the earth, those which were computed in the first years of this century under the direction of F. Hayford, of the Coást and Geodetic Survey in the United States, are to be considered the most accurate at present. The final results of these computations are published in the paper, *Supplementary*

Investigation in 1909 of the Figure of the Earth and Isostasy, Washington, 1910, p. 77.

These dimensions of the earth by Hayford were accepted by the geodetic section of the International Association for Geodesy and Geophysics in 1924 as the fundamental values of an "International ellipsoid" and recommended for all purposes of land survey. From these, extensive auxiliary tables, which are published in the following work, were computed by order of the geodetic section:

Association de Géodésie de l'Union Géodésique et Géophysique Internationale. Publication spéciale No. 2, "Tables de l'Ellipsoïde de Référence International adopté par l'assemblée générale de Madrid le 7 Octobre 1924 dans le système de la division sexagésimale de la circonférence, calculées sous la direction du Général G. Perrier" par E. Hasse. 2^{me} édition corrigée. Paris. Au sécrétariat de l'association, 19, rue Auber (9^{e}), 1935.

The starting values are

$$\text{major semiaxis } a_i = 6378388 \text{ m}$$
$$\text{flattening } \quad \mathfrak{a}_i = 1:297,$$

On the basis of the work mentioned, in the following we have summarized the principal constants for the international ellipsoid in the same form and in the same extent as indicated on p. 44 and p. 45, for Bessel's ellipsoid. In order to avoid confusion, the following values for the international ellipsoid are provided with the index i here.

International dimensions of the earth

$$a_i = 6{,}378{,}388 \text{ m} \qquad \log a_i = 6.804\,7109{\cdot}340$$

$$\log \frac{\varrho''}{c_i} = 8.508\,2494{\cdot}609$$

$$b_i = 6{,}356{,}911.94613 \text{ m} \qquad \log b_i = 6.803\,2461{\cdot}958$$

$$\log \frac{c_i}{\varrho''} = 1.491\,7505{\cdot}391$$

$$\frac{a_i^2}{b_i} = c_i = 6{,}399{,}936.60811 \text{ m} \qquad \log c_i = 6.806\,1756{\cdot}723$$

$$c_i = a_i\sqrt{1+e_i'^2} = b_i\,(1+e_i'^2) \qquad \log c_i^2 = 13.612\,3513{\cdot}446$$

$$\mathfrak{a}_i = \frac{a_i - b_i}{a_i} = \frac{1}{297} \qquad \log \frac{1}{\mathfrak{a}_i} = 2.472\,7564{\cdot}493$$

$$\mathfrak{a}_i = 0.00336\,70033\,67003 \qquad \log \mathfrak{a}_i = 7.527\,2435{\cdot}507$$

$$\frac{a_i - b_i}{a_i + b_i} = n_i = 0.00168\,63406\,41 \qquad \log n_i = 7.226\,9453{\cdot}067$$

$$\frac{a_i^2 - b_i^2}{a_i^2} = e_i^2 = 0.00672\,26700\,22333 \qquad \log e_i^2 = 7.827\,5417{\cdot}947$$

$$\frac{a_i^2 - b_i^2}{b_i^2} = e_i'^2 = 0.00676\,81701\,97224 \qquad \log e_i'^2 = 7.830\,4712{\cdot}712$$

$$1 - e_i^2 = \frac{1}{1+e_i'^2} = 0.99327\,73299\,77667 \qquad \log 1 - e_i^2 = 9.997\,0705{\cdot}235$$

$$\frac{1}{1-e_i^2} = 1 + e_i'^2 = 1.00676\,81701\,97224 \qquad \log 1 + e_i'^2 = 0.002\,9294{\cdot}765$$

$$\sqrt{1-e_i^2} = \frac{1}{\sqrt{1+e_i'^2}} = 0.99663\,29966\,32997 \qquad \log \sqrt{1-e_i^2} = 9.998\,5352{\cdot}617$$

$$\frac{1}{\sqrt{1-e_i^2}} = \sqrt{1+e_i'^2} = 1.00337\,83783\,78378 \qquad \log \sqrt{1+e'^2_i} = 0.001\,4647{\cdot}383$$

n	$\log e_i^n$	$\log \mu\, e_i^n$	$\log e_i'^n$	$\log \mu\, e_i'^n$	n
0		9.637 7843·113		9.637 7843·113	0
2	7.827 5417·947	7.465 3261·060	7.830 4712·712	7.468 2555·825	2
4	5.655 0835·895	5.292 8679·008	5.660 9425·424	5.298 7268·537	4
6	3.482 6253·842	3.120 4096·955	3.491 4138·136	3.129 1981·249	6
8	1.310 1671·789	0.947 9514·902	1.321 8850·848	0.959 6693·961	8
10	9.137 7089·736	8.775 4932·849	9.152 3563·560	8.790 1406·673	10
12	6.965 2507·684	6.603 0350·797	6.982 8276·272	6.620 6119·385	12

An ellipse with the two semiaxes a and b in rectangular coordinates x and y is drawn in Fig. 1. The equation of this ellipse is the following, as we know:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{or} \quad b^2x^2+a^2y^2=a^2b^2 . \tag{1}$$

The differentiation of this equation yields:

$$\frac{2\,x\,dx}{a^2}+\frac{2\,y\,dy}{b^2}=0 \quad \text{or} \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y} . \tag{2}$$

On the other hand, the differential quotient $\frac{dy}{dx}$ has a relation to the angle of the normal φ, namely (according to Fig. 1):

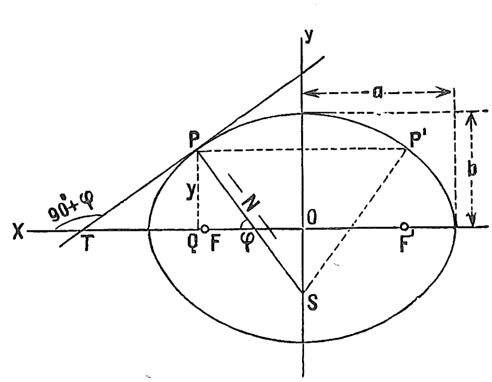

Fig. 1.
Terrestrial ellipsoid of rotation.

$$\frac{dy}{dx}=-\cot\,\varphi \quad \text{or} \quad \frac{dy}{dx}=-\frac{\cos\varphi}{\sin\varphi} . \tag{3}$$

Equations (2) and (3) together yield:

$$\frac{b^2x}{a^2y}=\frac{\cos\varphi}{\sin\varphi} \quad \text{or} \quad \frac{b^2x^2}{a^2y^2}=\frac{a^2\cos^2\varphi}{b^2\sin^2\varphi} . \tag{4}$$

Now we have in (1) and (4) two equations which can be solved for x^2 and y^2 and we write this in a very detailed form as follows:

(1) yields $\quad b^2x^2 + a^2y^2 = a^2b^2$
(4) yields $\quad b^4x^2\sin^2\varphi - a^4y^2\cos^2\varphi = 0 .$

If we multiply the first of these two equations by $a^2\cos^2\varphi$ and then add the two equations, we obtain:

$$x^2=\frac{a^4\cos^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi} . \tag{5}$$

On the other hand, if we multiply the first of the two previous equations by $b^2\sin^2\varphi$ and then subtract the two equations, we obtain:

$$y^2=\frac{b^4\sin^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi} . \tag{6}$$

Radius of curvature in the meridian M.

After this, we pass to the determination of the radius of curvature of the meridian ellipse, which we shall denote by M. In this connection, analytical geometry offers the following formula, as we know:

$$M = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}. \tag{7}$$

We already have used in (3) the first differential quotient which is needed here, namely:

$$\frac{dy}{dx} = -\cot\varphi \quad , \quad 1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{\sin^2\varphi}. \tag{8}$$

The second derivative of this yields at first:

$$\frac{d^2 y}{dx^2} = \frac{1}{\sin^2\varphi}\frac{d\varphi}{dx}, \tag{9}$$

and in order to obtain $\frac{d\varphi}{dx}$, we must derive the reciprocal value $\frac{dx}{d\varphi}$ from (5):

$$x = \frac{a^2\cos\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}.$$

According to the rules for the differentiation of a fraction, we have:

$$\frac{dx}{d\varphi} = \frac{a^2}{(\sqrt{\ldots})^2}\left(-\sin\varphi\sqrt{\ldots} - \cos\varphi\frac{-a^2\cos\varphi\sin\varphi + b^2\sin\varphi\cos\varphi}{\sqrt{\ldots}}\right)$$

$$\frac{dx}{d\varphi} = \frac{-a^2}{(\sqrt{\ldots})^3}\left(\sin\varphi\,(a^2\cos^2\varphi + b^2\sin^2\varphi) + \cos\varphi\,(-a^2+b^2)\sin\varphi\cos\varphi\right)$$

$$= \frac{-a^2}{(\sqrt{\ldots})^3}b^2\sin\varphi\,(\sin^2\varphi + \cos^2\varphi)$$

$$\frac{dx}{d\varphi} = \frac{-a^2 b^2\sin\varphi}{(a^2\cos^2\varphi + b^2\sin^2\varphi)^{\frac{3}{2}}}. \tag{10}$$

Now we can put together formula (7) from (8), (9), (10), and by so doing we obtain, if we omit the sign which is meaningless for us:

$$M = \frac{a^2 b^2}{(a^2\cos^2\varphi + b^2\sin^2\varphi)^{\frac{3}{2}}}. \tag{11}$$

Introduction of the eccentricity

The formulae developed here are those which present themselves at first; these formulae, however, are not suited for the later applications because we cannot express well the important circumstance in them that the two semiaxes a and b are nearly *equal* in the case of the terrestrial ellipsoid, or in other words: The quickly converging series developments which are indispensable later cannot be well attached to the previous formulae in a and b. Therefore, we introduce an eccentricity and a linear axis quantity. For this purpose, we have two forms which we shall treat both at first:

I. Older form with $\frac{a^2 - b^2}{a^2} = e^2$ and axis a. (12)

II. Newer form with $\frac{a^2 - b^2}{b^2} = e'^2$ and the axis quantity $\frac{a^2}{b} = c$. (13)

If we start with the older form I, we have at first, in order to express everything in a and e^2

$$\frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} = e^2 \qquad b^2 = a^2 (1 - e^2) \tag{14}$$

$$a^2 \cos^2 \varphi + b^2 \sin^2 \varphi = a^2 \cos^2 \varphi + a^2 (1 - e^2) \sin^2 \varphi = a^2 (1 - e^2 \sin^2 \varphi) .$$

We set once and for all:

$$1 - e^2 \sin^2 \varphi = W^2 \qquad W = \sqrt{1 - e^2 \sin^2 \varphi} , \tag{15}$$

with these, x and y as well as M are transformed from their first forms in (5), (6) and (11) into

$$x = \frac{a \cos \varphi}{W} \qquad y = \frac{a (1 - e^2) \sin \varphi}{W} \tag{16}$$

$$M = \frac{a (1 - e^2)}{W^3} . \tag{17}$$

We find these formulae (16) and (17) very generally in geodetic literature, but they are not the best. If we calculate according to the newer form II under (13), then we obtain:

$$\begin{gathered} \frac{a^2 - b^2}{b^2} = \frac{a^2}{b^2} - 1 = e'^2 \qquad \frac{a}{b} = \sqrt{1 + e'^2} , \; \frac{a^2}{b} = c \\ a = \frac{c}{\sqrt{1 + e'^2}} \qquad b = \frac{c}{1 + e'^2} \\ a^2 \cos^2 \varphi + b^2 \sin^2 \varphi = \frac{c^2 (1 + e'^2 \cos^2 \varphi)}{(1 + e'^2)^2} . \end{gathered} \tag{18}$$

We set once and for all:

$$1 + e'^2 \cos^2 \varphi = V^2 \qquad V = \sqrt{1 + e'^2 \cos^2 \varphi} , \tag{19}$$

with these, x, y and M from (5), (6), (11) are transformed into:

$$x = \frac{c \cos \varphi}{V} \qquad y = \frac{c \sin \varphi}{V (1 + e'^2)} \tag{20}$$

$$M = \frac{c}{V^3} . \tag{21}$$

The two auxiliary quantities W and V, which we have introduced through equations (15) and (19) can be explained by a simple method according to a communication by Schmehl. If we denote in Fig. 2 the perpendicular distance of the tangent to the meridian at the point P from the center point O of the terrestrial ellipsoid by p, the angle between p and the x-axis is equal to the geographic latitude φ, and we have the relation

$$p = x \cos \varphi + y \sin \varphi . \tag{22}$$

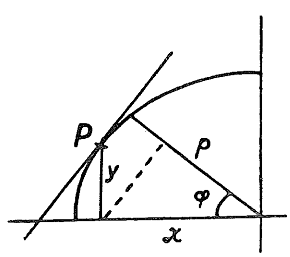

Fig. 2.

If we substitute for x and y the values (16), p. 49, there follows

$$p = \frac{a \cos^2 \varphi}{W} + \frac{a(1 - e^2) \sin^2 \varphi}{W} = \frac{a}{W}(1 - e^2 \sin^2 \varphi)$$

or

$$p = aW . \tag{23}$$

On the other hand, if we substitute for x and y the values (20), then we obtain

$$p = \frac{c \cos^2 \varphi}{V} + \frac{c \sin^2 \varphi}{V(1 + e'^2)} = \frac{c(1 + e'^2) \cos^2 \varphi + c \sin^2 \varphi}{V(1 + e'^2)}$$

$$= \frac{c(1 + e'^2 \cos^2 \varphi)}{V(1 + e'^2)} = \frac{cV}{1 + e'^2}$$

or according to (18)

$$p = bV . \tag{24}$$

Thus there exist the following two simple relations for W and V

$$W = \frac{p}{a} \qquad V = \frac{p}{b} . \tag{25}$$

Cf. in this connection: *Veröff. d. Pr. Geod. Inst.*, N. F. Nr. 98, Schmehl, "Untersuchungen über ein allgemeines Erdellipsoid," Potsdam, 1927, p. 22.

The radius of curvature in the prime vertical N

We have defined by the above the first principal radius of curvature of the ellipsoid of rotation for the meridian; the second principal radius of curvature, which refers to the curvature across to the meridian, that is from west to east, can be found without further computation by a very simple geometric consideration.

At first we consider in Fig. 1, p. 47, the arc of parallel PP' for the latitude φ, and see that all surface normals drawn in this arc of parallel intersect in *one* point S of the axis.

The prime vertical which is vertical to the meridian at P obviously must touch that arc of parallel PP' at P, and therefore two straight lines PS lying infinitely near to one another are at the same time normals to the prime vertical at P. But since the intersection point of two normals to a curve infinitely near to one another represents the center of curvature of the curve, PS is the radius of curvature of the prime vertical, or briefly, $PS = N$ is the radius of curvature in the prime vertical of the ellipsoid of rotation at the point P.

By denoting the length of this radius of curvature in the prime vertical once and for all by N, we have

$$N = \frac{x}{\cos \varphi} ,$$

and this yields either according to the old or the new form (16) or (20), as the case may be

$$N = \frac{a}{W} \qquad \text{or} \qquad N = \frac{c}{V} . \tag{26}$$

Ratio of curvature N: M

[Note: Probably a printing error in the original; should read: "*Mean radius of curvature r.*"]

In geodesy we understand by the mean radius of curvature the geometric mean of the two principal radii of curvature M and N, that is:

$$r = \sqrt{MN} \tag{27}$$

or by substituting the meanings of M and N

$$r = \frac{a\sqrt{1-e^2}}{W^2} \quad \text{or} \quad r = \frac{c}{V^2}\,. \tag{28}$$

Ratio of curvature $N : M$

After the two principal radii of curvature M and N are defined, we shall also consider their quotients, i.e. in two forms, from (17) and (21) with (26):

$$\frac{N}{M} = \frac{W^2}{1-e^2} \quad \text{or} \quad \frac{N}{M} = V^2\,. \tag{29}$$

This quotient is very important in geodesy, for the nearer this quotient is equal to 1 the more it is justified to consider the earth as a sphere at the latitude in question. In order to obtain a clearer picture, let us compute a few values for it:

$\varphi = 0°$	$\frac{N}{M} = 1.0067 = V^2$
. . 30	. . . . 1.0050
. . 45	. . . . 1.0034
. . 60	. . . . 1.0017
. . 90	. . . . 1.0000.

The values V^2 are considerably different from 1, at 45° about 1/3%; and only at the poles ($\varphi = 90°$) will we have $V^2 = 1$.

Nevertheless there are cases where only small corrections of *second* order are involved, in which the quotient $N : M$ can be set with sufficient accuracy = 1, that is, the earth can be treated as a sphere. In such cases we use then the mean radius of curvature according to (27) or (28), above, as the radius of such a sphere.

Since the ratio $N : M$ is always larger than 1, N is always larger than M, that is, the radius of curvature in the prime vertical is always larger than the radius of curvature in the meridian, or vice versa, the curvature $1 : M$ is always larger in the meridian than is the curvature $1 : N$ in the prime vertical. Only at the pole do both become equal, namely $N = M = \frac{a^2}{b} = c$ [as already mentioned in (9), section 37, p. 42], and the best working field for a geodesist would be at the pole of the earth because there all spheroidal corrections disappear.

Geocentric radius and geocentric latitude

In connection with the above, we shall determine two further values which are rarely used in geodesy but which are used in astronomy for parallax computations, namely the distance of a point on the earth from the center point of the earth, called the geocentric radius = C, and the angle of this radius with the equator, geocentric latitude = γ.

According to Fig. 1, section 37, p. 41, we have immediately the following formulae for these two magnitudes:

$$C = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\gamma = \frac{y}{x}\,. \tag{a}$$

We shall continue computing with a and e^2, i.e. according to (16), p. 49,

$$x = \frac{a \cos \varphi}{W} \quad \text{and} \quad y = \frac{a(1-e^2) \sin \varphi}{W}, \tag{b}$$

and hence

$$C^2 = \frac{a^2}{W^2} (\cos^2 \varphi + (1-e^2)^2 \sin^2 \varphi) \quad , \quad \tan \gamma = (1-e^2) \tan \varphi , \tag{c}$$

$$C^2 = \frac{a^2}{W^2} (\cos^2 \varphi + \frac{\tan^2 \gamma}{\tan^2 \varphi} \sin^2 \varphi) = \frac{a^2}{W^2} \frac{\cos^2 \varphi}{\cos^2 \gamma} . \tag{d}$$

From (c) we have further:

$$1 - e^2 = \frac{\tan \gamma}{\tan \varphi} \quad , \quad e^2 = \frac{\sin(\varphi - \gamma)}{\sin \varphi \cos \gamma}$$

$$W^2 = 1 - e^2 \sin^2 \varphi = 1 - \sin(\varphi - \gamma) \frac{\sin \varphi}{\cos \gamma} = \frac{\cos \varphi}{\cos \gamma} \cos(\varphi - \gamma) ,$$

and hence from (d):

$$C^2 = \frac{a^2 \cos \varphi}{\cos \gamma \cos(\varphi - \gamma)} . \tag{e}$$

With these formulae (c) to (e) we have sufficient means for the rigorous computation of C and γ; however, we need more frequently formulae of approximation which can be found by development in series. From (c) we have:

$$C^2 = \frac{a^2}{W^2} \left(1 + e^2 (e^2 - 2) \sin^2 \varphi\right),$$

and to terms with e^4 according to p. 20:

$$C = \frac{a}{W} \left(1 + \frac{1}{2} e^2 (e^2 - 2) \sin^2 \varphi - \frac{1}{2} e^4 \sin^4 \varphi\right).$$

With the same accuracy we also develop

$$\frac{1}{W} = \frac{1}{\sqrt{1 - e^2 \sin^2 \varphi}}$$

according to p. 20, namely:

$$\frac{1}{W} = 1 + \frac{1}{2} e^2 \sin^2 \varphi + \frac{3}{8} e^4 \sin^4 \varphi ,$$

and if this is set into the expression for C, we obtain easily:

$$C = a \left(1 - \frac{1}{2} e^2 \sin^2 \varphi + \frac{1}{2} e^4 \sin^2 \varphi - \frac{5}{8} e^4 \sin^4 \varphi\right). \tag{f}$$

By neglecting e^4 we can find another convenient formula for C. From (c) we have

$$\tan \varphi - \tan \gamma = e^2 \tan \varphi$$

$$\frac{\varphi - \gamma}{\cos^2 \varphi} = e^2 \tan \varphi \quad , \quad \varphi - \gamma = e^2 \sin \varphi \cos \varphi ,$$

or by including ρ:

$$\varphi - \gamma = \frac{1}{2} e^2 \varrho \sin 2\varphi = [2.8378056] \sin 2\varphi . \tag{g}$$

Tables for $\log \frac{C}{a}$ and for $\varphi - \gamma$ are found in astronomical textbooks; in geodesy, these values are hardly ever used.

Reduced latitude. In geodesy, another angle plays an important role, which is called the "reduced latitude" and which is determined by the equation:

$$\tan \psi = \sqrt{1 - e^2} \tan \varphi .$$

But it is not until later that we shall have to do with this.

Section 39. Radius of Curvature for Any Arbitrary Azimuth

After the radius of curvature in the meridian M and the radius of curvature in the prime vertical N are determined, we can also easily indicate the radius of curvature R for any arbitrary azimuth α if we assume that Euler's theorem is known to us, namely:

$$\frac{1}{R} = \frac{\cos^2 \alpha}{M} + \frac{\sin^2 \alpha}{N} . \tag{1}$$

This theorem is proved in analytical geometry, and we shall give, from the geometric point of view, an illustrative foundation of the theorem here, which can even be developed as a rigorous proof by a few side considerations.

In Fig. 1 let P be a point of the ellipsoid with a tangent plane AA' and a secant plane BB' parallel to AA'.

The plane BB' yields an ellipse of intersection which is presented in the lower part of Fig. 1 with its main axes PM, PN and a third direction s in the azimuth α. Now if the distance between the two planes AA' and BB' is very small, $= z$, the ordinate z can be expressed in three ways by the radii of curvature $M, R, N,$ which represent the three directions under consideration, by an approximation known to us (which is also used, for instance, for the curvature of the earth in the case of trigonometric elevation measurement), namely:

$$z = \frac{m^2}{2M} = \frac{s^2}{2R} = \frac{n^2}{2N} . \tag{2}$$

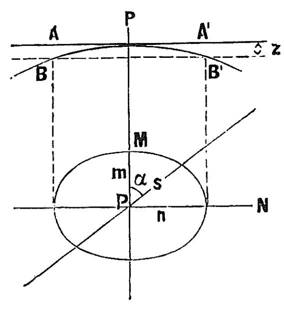

Fig. 1.

The following equation exists here for the ellipse of intersection with the semiaxes m and n:

$$\frac{(s \cos \alpha)^2}{m^2} + \frac{(s \sin \alpha)^2}{n^2} = 1 . \tag{3}$$

By combining (2) and (3) we obtain Euler's theorem already written above under (1).

In order to apply this theorem by Euler of equation (1), which is valid for any arbitrary surface, to our ellipsoid, we must put in the expressions for M and N from (21) and (26), section 38, pp. 49 and 50, namely:

$$M = \frac{c}{V^3} \quad \text{and} \quad N = \frac{c}{V} , \text{ where } V^2 = 1 + e'^2 \cos^2 \varphi .$$

With these, (1) yields:

$$\frac{1}{R} = \frac{\cos^2 \alpha}{c} V + \frac{\sin^2 \alpha}{c} V = \frac{V}{c}\left(\cos^2 \alpha\,(1 + e'^2 \cos^2 \varphi) + \sin^2 \alpha\right)$$

$$R = \frac{c}{V}\,\frac{1}{1 + e'^2 \cos^2 \varphi \cos^2 \alpha} \quad \text{or} \quad = \frac{N}{1 + e'^2 \cos^2 \varphi \cos^2 \alpha}\,. \tag{4}$$

We shall consider the special case with $\alpha = 45°$, that is, set $\sin^2 \alpha = \cos^2 \alpha = \frac{1}{2}$, whereby equation (1) yields:

$$\frac{1}{R_{45}} = \frac{1}{2}\left(\frac{1}{M} + \frac{1}{N}\right) = \frac{M+N}{2\,M\,N}\,. \tag{5}$$

$M N$ has the meaning $= r^2$ here from (27), section 38, p. 51, with r as the geometric mean of M and N, and $\frac{M+N}{2} = d$ is the arithmetic mean of M and N, that is,

$$R_{45} = \frac{r^2}{d}, \tag{6}$$

hence we see that the radius of curvature for a 45° azimuth is equal to the mean radius of curvature r or the average value d in the first approximation.

The second form for R used in the case of (4) leads to a convenient logarithmic formula of approximation; in the first approximation we have:

$$\log\,(1 + e'^2 \cos^2 \varphi \cos^2 \alpha) = u\, e'^2 \cos^2 \varphi \cos^2 \alpha - \dots$$

therefore,

$$\log R - \log N = -\mu\, e'^2 \cos^2 \varphi \cos^2 \alpha + \dots \tag{7}$$

If we set here $\alpha = 0$, then R changes to the radius of curvature in the meridian M, therefore:

$$\log M - \log N = -\mu\, e'^2 \cos^2 \varphi + \dots \tag{8}$$

or accurately,

$$\log M - \log N = -\log V^2 \quad \text{[From (29), section 38, p. 51.]}$$

With these, (7) and (8) yield:

$$\log R - \log N = -(\log V^2) \cos^2 \alpha\,, \tag{9}$$

and in the same way we also find

$$\log R - \log M = +(\log V^2) \sin^2 \alpha\,. \tag{10}$$

The approximation formula (9) or (10) yields the value $\log R$ to an accuracy of 7 places. In order that we may judge this better, let us develop formula (4) to e'^4 and find

$$\left.\begin{aligned} \log R &= \log N - \mu\, e'^2 \cos^2 \varphi \cos^2 \alpha + \mu \frac{e'^4}{2} \cos^4 \varphi \cos^4 \alpha \\ &= \log N - [4{\cdot}465\ 1031] \cos^2 \varphi \cos^2 \alpha + [1{\cdot}99139] \cos^4 \varphi \cos^4 \alpha\,, \end{aligned}\right\} \tag{11}$$

where the numbers in brackets mean logarithms of coefficients and refer to units of the seventh place.

We also can bring this into the following form:

$$log\,R = log\,N - cos^2\alpha\,(log\,V^2) - \frac{\mu\,e'^4}{2}\,cos^4\varphi\,sin^2\alpha\,cos^2\alpha \qquad (12)$$

or

$$log\,R = log\,M + sin^2\alpha\,(log\,V^2) - \frac{\mu\,e'^4}{2}\,cos^4\varphi\,sin^2\alpha\,cos^2\alpha\,. \qquad (13)$$

According to these formulae (11) to (13), there has been computed the following table which yields $log\ R$ for various latitudes φ and various azimuths α.

Latitude	Azimuth α						
φ	0°	15°	30°	45°	60°	75°	90°
	log R	log R	log R	log R	log R	log R	log R
0°	6.80 1735	6.80 1929	6.80 2460	6.80 3187	6.80 3915	6.80 4448	6.80 4643
10	1866	2055	2570	3274	3980	4498	4687
20	2244	2416	2885	3527	4169	4641	4813
30	2823	2969	3368	3913	4459	4860	5006
35	6.80 3167	6.80 3298	6.80 3655	6.80 4143	6.80 4632	6.80 4990	6.80 5121
40	3534	3648	3961	4388	4815	5128	5243
45	3913	4010	4276	4641	5005	5272	5369
50	4292	4372	4592	4893	5194	5415	5496
55	4659	4723	4899	5138	5378	5554	5618
60	5004	5053	5186	5369	5551	5684	5733
65	5316	5353	5446	5577	5707	5803	5837
70	6.80 5586	6.80 5609	6.80 5671	6.80 5756	6.80 5842	6.80 5904	6.80 5927
80	5966	5972	5988	6010	6032	6048	6054
90	6098	6098	6098	6098	6098	6098	6098

A more detailed table of this kind is not needed, for the azimuthal radii of curvature R do not play a part in the geodesy of triangulations and the like in this form.

For the rigorous computation of trigonometric elevations we need these radii of curvature. We shall assume that for the purpose of a trigonometric elevation measurement between the Polytechnikum at Karlsruhe and the trigonometric point Hornisgrinde in the Black Forest we are to calculate the radius of curvature of the earth along the sight in question. The mean latitude of the two points is $\varphi = 48°48'\,26.6''$ and the mean azimuth is $\alpha = 18°55'\,3.0''$. With these values we find according to the rigorous formula (4):

$$log\ R = 6.804\ 3345\ ,$$

and according to the formula of approximation (9) or (10):

$$log\ R = 6.804\ 3347\ .$$

This value should be the basis for a further computation according to section 37 of our volume II, second half-volume, 9th edition, 1933, corresponding to $log\ R$, p. 165, of that volume.

Change of the curvature of the earth according to latitude and azimuth

If we consider the above small summary table in regard to the changes which the radius of curvature R suffers in the latitude and in the azimuth, we note that for similar changes $\Delta\varphi$ or $\Delta\alpha$ the changes $\Delta\ log\ R$ are of nearly the same order of magnitude, and the differentiation of $log\ R$ or of R in respect to φ and α also shows this; both differentiations of R in respect to φ and α yield magnitudes of the order e'^2.

If we reflect, however, what changes of R generally occur on a survey area of the earth which is limited in space, the comparison of the influences of φ and α becomes quite different, for on a limited area of the earth the latitude φ is nearly constant, the azimuth α, however, is variable within its extreme limits 0° and 90° despite this fact.

Therefore, on a limited survey area the changes in the azimuth are much more influential than the changes of the latitude, and we can say in such a case that the changes of the curvature of the earth which originate in change of latitude $\Delta\varphi$ are only magnitudes of second order compared with the changes of curvature which depend on the azimuth α.

Intermediate remark

With the developments of sections 37 to 39 we have come so far that we can pass now to section 46 and following, spherical triangulation, and of all previous material, only the mean radius of curvature will be needed there at first. We advise for the first study to pass from section 39 to section 46.

If we proceed here differently and insert also sections 40 to 45, this was done to the effect that much material needed for further-reaching purposes is supposed to be taken care of once and for all here; as the need arises, we can turn back to individual points, for instance, arcs of meridian, section 41, which are needed later for coordinates, and similarly with the arcs of parallel and surfaces.

In the case of geodetic measurements and calculations in a more limited sense, we have no need of knowing the surfaces of individual zones or graticules of the ellipsoid; for cartography and geography, however, there is in general such a need, and therefore, the surface computation of the trapezoids of the graticule in section 43 was attached also. Later as the need arises, we also shall refer back to the very important auxiliary tables of the Appendix whose computation will be taught in the following sections.

Section 40. Functions W and V

The following two functions have appeared in the case of the development of the radii of curvature in section 38:

$$W = \sqrt{1 - e^2 \sin^2 \varphi} \qquad\qquad V = \sqrt{1 + e'^2 \cos^2 \varphi}, \tag{1}$$

which are related to one another as follows:

$$\frac{W^2}{1 - e^2} = V^2 \qquad \text{or} \qquad W^2 = \frac{V^2}{1 + e'^2}, \tag{2}$$

where

$$- \log (1 - e^2) = \log (1 + e'^2) = 0.002\ 9083{\cdot}596004$$
$$- \log \sqrt{1 - e^2} = \log \sqrt{1 + e'^2} = 0.001\ 4541{\cdot}798002.$$

These functions are used in geodesy so often that we must study them more closely and, especially, develop them in series.

On the development in series, we have from (1):

$$W^2 = 1 - e^2 \sin^2 \varphi,$$

therefore, according to the logarithmic series, p. 21:

$$\log \frac{1}{W^2} = \mu e^2 \sin^2 \varphi + \frac{\mu e^4}{2} \sin^4 \varphi + \frac{\mu e^6}{3} \sin^6 \varphi + \frac{\mu e^8}{4} \sin^8 \varphi + \frac{\mu e^{10}}{5} \sin^{10} \varphi + \dots \tag{3}$$

The computation of the coefficients with Bessel's $\log e^2 = 7.824\ 4104{\cdot}237$ yields for units of the seventh place of logarithm:

$$\log \frac{1}{W^2} = \begin{matrix} 28986{\cdot}430302\, \sin^2\varphi + 96{\cdot}733112\, \sin^4\varphi + 0{\cdot}430422\, \sin^6\varphi \\ + 0{\cdot}002155\, \sin^8\varphi + 0{\cdot}000012\, \sin^{10}\varphi \end{matrix} \tag{4}$$

or with half-coefficients

$$\log \frac{1}{W} = \begin{matrix} 14493{\cdot}215151\, \sin^2\varphi + 48{\cdot}366556\, \sin^4\varphi + 0{\cdot}215211\, \sin^6\varphi \\ + 0{\cdot}001077\, \sin^8\varphi + 0.000006\, \sin^{10}\varphi\,. \end{matrix} \tag{5}$$

The same series with the logarithms of the coefficients are:

$$\log \frac{1}{W^2} = \begin{matrix} [4.462\,1947{\cdot}350]\sin^2\varphi + [1.985\,5751{\cdot}590]\sin^4\varphi + [9.633\,8943{\cdot}3]\sin^6\varphi \\ + [7.333\,3660]\sin^8\varphi + [5.06087]\sin^{10}\varphi\,. \end{matrix} \tag{6}$$

By halving we also have:

$$\log \frac{1}{W} = \begin{matrix} [4.161\,1647{\cdot}393]\sin^2\varphi + [1.684\,5451{\cdot}633]\sin^4\varphi + [9.332\,8643{\cdot}3]\sin^6\varphi \\ + [7.032\,3360]\sin^8\varphi + [4.75984]\sin^{10}\varphi\,. \end{matrix} \tag{7}$$

In the same way, we also have the other function:

$$V^2 = 1 + e'^2 \cos^2\varphi$$

$$\log V^2 = \mu\, e'^2 \cos^2\varphi - \frac{\mu\, e'^4}{2}\cos^4\varphi + \frac{\mu\, e'^6}{3}\cos^6\varphi - \frac{\mu\, e'^8}{4}\cos^8\varphi + \frac{\mu\, e'^{10}}{5}\cos^{10}\varphi - \ldots \tag{8}$$

If we put in here $\log e'^2 = 7.827\ 3187{\cdot}833$, we obtain:

$$\log V^2 = \begin{matrix} 29\,181{\cdot}196\,469 \cos^2\varphi - 98{\cdot}0374220 \cos^4\varphi + 0.4391567 \cos^6\varphi \\ - 0{\cdot}0022131 \cos^8\varphi + 0{\cdot}0000119 \cos^{10}\varphi\,, \end{matrix} \tag{9}$$

$$\log V = \begin{matrix} 14\,590{\cdot}598235 \cos^2\varphi - 49{\cdot}0187110 \cos^4\varphi + 0{\cdot}2195783 \cos^6\varphi \\ - 0{\cdot}0011065 \cos^8\varphi + 0{\cdot}0000059 \cos^{10}\varphi\,. \end{matrix} \tag{10}$$

Further with the logarithms of the coefficients again:

$$\log V^2 = \begin{matrix} [4.465\,1030{\cdot}946] \cos^2\varphi - [1.991\,3918{\cdot}822] \cos^4\varphi + [9.642\,6194{\cdot}1] \cos^6\varphi \\ - [7.344\,9995] \cos^8\varphi + [5.07541] \cos^{10}\varphi\,, \end{matrix} \tag{11}$$

$$\log V = \begin{matrix} [4.164\,0730{\cdot}989] \cos^2\varphi - [1.690\,3618{\cdot}865] \cos^4\varphi + [9.341\,5894{\cdot}2] \cos^6\varphi \\ - [7.043\,9695] \cos^8\varphi + [4.77438] \cos^{10}\varphi\,. \end{matrix} \tag{12}$$

If we set the limiting value $\varphi = 90°$ in the case of $\log W^2$ and the limiting value $\varphi = 0$ in the case of $\log V^2$, then we obtain $\log(1 - e^2)$ and $\log(1 + e'^2)$, which are already indicated in another connection in section 37, p. 45.

Series (5) and (10) yield for $\varphi = 45°$:

$$\begin{aligned} -\log W &= 7246{\cdot}607576 + 12{\cdot}091639 + 0{\cdot}026901 + 0{\cdot}000067 = 7258{\cdot}726183 \\ \log V &= 7295{\cdot}299117 - 12{\cdot}254678 + 0{\cdot}027447 - 0{\cdot}000069 = \underline{7283{\cdot}071817} \\ &\qquad \log V : W = 14541{\cdot}798000\,. \end{aligned} \tag{13}$$

According to a more rigorous computation, this ought to be $= 14541{\cdot}798002\,.$

The check is correct to 0·000002, i.e. to 2 units of the 13th place of the logarithm, which is satisfactory here.

Since the V's and W's are simply related to one another, $V^2 = W^2 (1 + e'^2)$, we have the choice to compute V or W and to derive the other from it. This choice presents itself in the case of the above series in such a way that for small values φ we compute more conveniently $\log W$, for which we need only very few terms at the beginning of the quadrant whereas in the neighborhood of $\varphi = 90°$ the calculation with $\log V$ is the more convenient one; in the case of $\varphi = 45°$ both calculations are equally good.

We can bring the series for $\log W$ and $\log V$ also into another form, by expressing $\sin^2 \varphi$, $\cos^2 \varphi$, etc., in terms of $\cos 2\varphi$, $\cos 4\varphi$, etc., for which the formulae of section 35, pp. 28 and 29, are used, namely:

$$\log W^2 = -\mu e^2 \sin^2\varphi - \frac{\mu e^4}{2}\sin^4\varphi - \frac{\mu e^6}{3}\sin^6\varphi - \frac{\mu e^8}{4}\sin^8\varphi - \frac{\mu e^{10}}{5}\sin^{10}\varphi$$

$$\sin^2\varphi = \frac{1}{2} - \frac{1}{2}\cos 2\varphi$$

$$\sin^4\varphi = \frac{3}{8} - \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi$$

$$\sin^6\varphi = \frac{5}{16} - \frac{15}{32}\cos 2\varphi + \frac{3}{16}\cos 4\varphi - \frac{1}{32}\cos 6\varphi$$

$$\sin^8\varphi = \frac{35}{128} - \frac{7}{16}\cos 2\varphi + \frac{7}{32}\cos 4\varphi - \frac{1}{16}\cos 6\varphi + \frac{1}{128}\cos 8\varphi$$

$$\sin^{10}\varphi = \frac{63}{256} - \frac{105}{256}\cos 2\varphi + \frac{15}{64}\cos 4\varphi - \frac{45}{512}\cos 6\varphi + \frac{5}{256}\cos 8\varphi - \frac{1}{512}\cos 10\varphi\,.$$

Introduced into the series $\log W^2$, these $\sin^2\varphi$, $\sin^4\varphi$, etc., yield:

$$\begin{aligned}\log W^2 = &-\left(\frac{1}{2}\mu e^2 + \frac{3}{16}\mu e^4 + \frac{5}{48}\mu e^6 + \frac{35}{512}\mu e^8 + \frac{63}{1280}\mu e^{10}\right)\\ &+\left(\frac{1}{2}\mu e^2 + \frac{1}{4}\mu e^4 + \frac{5}{32}\mu e^6 + \frac{7}{64}\mu e^8 + \frac{21}{256}\mu e^{10}\right)\cos 2\varphi\\ &-\left(\frac{1}{16}\mu e^4 + \frac{1}{16}\mu e^6 + \frac{7}{128}\mu e^8 + \frac{3}{64}\mu e^{10}\right)\cos 4\varphi\\ &+\left(\frac{1}{96}\mu e^6 + \frac{1}{64}\mu e^8 + \frac{9}{512}\mu e^{10}\right)\cos 6\varphi\\ &-\left(\frac{1}{512}\mu e^8 + \frac{1}{256}\mu e^{10}\right)\cos 8\varphi\\ &+\left(\frac{1}{2560}\mu e^{10}\right)\cos 10\varphi\,.\end{aligned} \tag{14}$$

If we calculate this with Bessel's $\log e^2 = 7.824\,4104{\cdot}237$ (use of $\log \mu e^n$ on p. 45), then we obtain in units of the 7th place of the logarithm:

$$\left.\begin{aligned}\log W^2 = &-14529{\cdot}625\,1671 + 14541{\cdot}784\,4150\cos 2\varphi - 12{\cdot}1728172\cos 4\varphi\\ &+ 0{\cdot}013\,5864\cos 6\varphi - 0{\cdot}000\,0170\cos 8\varphi + \ldots\end{aligned}\right\} \tag{15}$$

The same development made for $\log V^2$ yields from (8), p. 57, and section 35, pp. 28 and 29, the following:

$$\log V^2 = \mu\left(e'^2 \cos^2\varphi - \frac{e'^4}{2}\cos^4\varphi + \frac{e'^6}{3}\cos^6\varphi - \frac{e'^8}{4}\cos^8\varphi + \frac{e'^{10}}{5}\cos^{10}\varphi - \ldots\right)$$

$$\cos^2\varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$$

$$\cos^4\varphi = \frac{3}{8} + \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi$$

$$\cos^6\varphi = \frac{5}{16} + \frac{15}{32}\cos 2\varphi + \frac{3}{16}\cos 4\varphi + \frac{1}{32}\cos 6\varphi$$

$$\cos^8\varphi = \frac{35}{128} + \frac{7}{16}\cos 2\varphi + \frac{7}{32}\cos 4\varphi + \frac{1}{16}\cos 6\varphi + \frac{1}{128}\cos 8\varphi$$

$$\cos^{10}\varphi = \frac{63}{256} + \frac{105}{256}\cos 2\varphi + \frac{15}{64}\cos 4\varphi + \frac{45}{512}\cos 6\varphi + \frac{5}{256}\cos 8\varphi + \frac{1}{512}\cos 10\varphi$$

$$\begin{aligned}\log V^2 = &\left(\frac{1}{2}\mu e'^2 - \frac{3}{16}\mu e'^4 + \frac{5}{48}\mu e'^6 - \frac{35}{512}\mu e'^8 + \frac{63}{1280}\mu e'^{10}\right) \\ &+\left(\frac{1}{2}\mu e'^2 - \frac{1}{4}\mu e'^4 + \frac{5}{32}\mu e'^6 - \frac{7}{64}\mu e'^8 + \frac{21}{256}\mu e'^{10}\right)\cos 2\varphi \\ &-\left(\frac{1}{16}\mu e'^4 - \frac{1}{16}\mu e'^6 + \frac{7}{128}\mu e'^8 - \frac{3}{64}\mu e'^{10}\right)\cos 4\varphi \\ &+\left(\frac{1}{96}\mu e'^6 - \frac{1}{64}\mu e'^8 + \frac{9}{512}\mu e'^{10}\right)\cos 6\varphi \\ &-\left(\frac{1}{512}\mu e'^8 - \frac{1}{256}\mu e'^{10}\right)\cos 8\varphi \\ &+\left(\frac{1}{2560}\mu e'^{10}\right)\cos 10\varphi\,.\end{aligned} \tag{16}$$

If we calculate this with Bessel's $\log e'^2 = 7.8273187{\cdot}833$ (use of $\log \mu\, e^n$ on p. 45), then we obtain:

$$\begin{aligned}\log V^2 = 14553{\cdot}9708333 + 14541{\cdot}7844155 \cos 2\varphi - 12{\cdot}1728170 \cos 4\varphi \\ + 0{\cdot}0135863 \cos 6\varphi - 0{\cdot}0000171 \cos 8\varphi + \ldots\end{aligned} \tag{17}$$

Series (17) and (15) agree with sufficient accuracy in the coefficients, and the absolute terms yield $\log V^2 - \log W^2 = 29083{\cdot}5960004$, which agrees with $\log(1+e'^2)$ of p. 45 as it should.

By halving we also have $\log W$ and $\log V$; and at the same time with slight adjustments between the final figures in (15) and (17):

$$\begin{aligned}\log V = 7276{\cdot}9854166 + 7270{\cdot}8922076 \cos 2\varphi - 6{\cdot}0864086 \cos 4\varphi \\ + 0{\cdot}0067932 \cos 6\varphi - 0{\cdot}0000085 \cos 8\varphi\,,\end{aligned} \tag{18}$$

and since we calculate with common logarithms, we shall also indicate the logarithms of coefficients:

$$\begin{aligned}\log V^2 = 14553{\cdot}9708333 + [4{\cdot}1626177{\cdot}018] \cos 2\varphi - [1{\cdot}0853911{\cdot}0] \cos 4\varphi \\ + [8{\cdot}1331028] \cos 6\varphi - [5{\cdot}23172] \cos 8\varphi\,,\end{aligned} \tag{19}$$

$$\begin{aligned}\log V = 7276{\cdot}9854166 + [3{\cdot}8615877{\cdot}062] \cos 2\varphi - [0{\cdot}7843611{\cdot}0] \cos 4\varphi \\ + [7{\cdot}8320728] \cos 6\varphi - [4{\cdot}93069] \cos 8\varphi\,.\end{aligned} \tag{20}$$

In order to count from the center, we shall also set $\varphi = 45° + (\varphi - 45°)$; therefore:

$$\begin{aligned}\log V = 7276{\cdot}9854166 - [3{\cdot}8615877{\cdot}062] \sin 2(\varphi - 45°) + [0{\cdot}7843611{\cdot}0] \\ \cos 4(\varphi - 45°) + [7{\cdot}8320728] \sin 6(\varphi - 45°) - [4{\cdot}93069] \cos 8(\varphi - 45°)\,.\end{aligned} \tag{21}$$

This form offers the advantage of always obtaining with it in *one* computation the constituent parts for two values φ which lie symmetrically around 45°; for instance, $\varphi - 45° = +15°$ and $-15°$ yield $log\ V$ for $\varphi = 30°$ and for $\varphi = 60°$ as follows:

$$\text{for } \varphi = 30° \quad log\ V = 7276{\cdot}9854166 + 3635{\cdot}4461038 + 3{\cdot}0432043 - 0{\cdot}0067932 + 0{\cdot}0000042,$$

$$\text{for } \varphi = 60° \quad log\ V = 7276{\cdot}9854166 - 3635{\cdot}4461038 + 3{\cdot}0432043 + 0{\cdot}0067932 + 0{\cdot}0000042,$$

summarizing:

$$\text{for } \varphi = 30° \quad log\ V = 0{\cdot}0010915{\cdot}4679357,$$

$$\text{for } \varphi = 60° \quad log\ V = 0{\cdot}0003644{\cdot}5893145.$$

These values agree with those calculated from (7) or (12), p. 57, within 0·000001.

Interpolation for log V

If we aim to set up a table of the $log\ V$'s, we calculate directly according to the above formulae (7), (12) or (20) certain principal values, say for φ for every 1°, and as to the rest, insert further values. Now if we already know approximate values of the $log\ V$'s to be inserted, which often is the case, we can set up a good formula for the more rigorous insertion according to the principle of the mean argument in the following way:

We assume that a value V refers to the latitude φ, furthermore V'' to the latitude $\varphi + \frac{\Delta\varphi}{2}$; and V' to the latitude $\varphi - \frac{\Delta\varphi}{2}$; the following two equations exist then according to Maclaurin's series:

$$log\ V'' = log\ V + \frac{\Delta\varphi}{2}\frac{d\ log\ V}{d\varphi} + \frac{1}{2}\left(\frac{\Delta\varphi}{2}\right)^2\frac{d^2 log\ V}{d\varphi^2} + \frac{1}{6}\left(\frac{\Delta\varphi}{2}\right)^3\frac{d^3 log\ V}{d\varphi^3}$$

$$log\ V' = log\ V - \frac{\Delta\varphi}{2}\frac{d\ log\ V}{d\varphi} + \frac{1}{2}\left(\frac{\Delta\varphi}{2}\right)^2\frac{d^2 log\ V}{d\varphi^2} - \frac{1}{6}\left(\frac{\Delta\varphi}{2}\right)^3\frac{d^3 log\ V}{d\varphi^3}.$$

The difference yields: $log\ V'' - log\ V' = \Delta\ log\ V$:

$$\Delta\ log\ V = \Delta\varphi\frac{d\ log\ V}{d\varphi} + \frac{\Delta\varphi^3}{24}\frac{d^3 log\ V}{d\varphi^3}. \tag{22}$$

For the application we have to derive $log\ V$ three times, where we shall always write in the future, in order to abbreviate, $tan\ \varphi = t$ and

$$V^2 = 1 + e'^2 cos^2\varphi = 1 + \eta^2 \quad \text{, hence } \eta^2 = e'^2 cos^2\varphi \quad , \quad \frac{d\eta^2}{d\varphi} = -2\eta^2 t \tag{23}$$

$$V = \sqrt{1+\eta^2} \quad , \quad \frac{dV}{d\varphi} = \frac{1}{2V}(-2\eta^2 t) = -\frac{\eta^2 t}{V}$$

$$\frac{d\ log\ V}{d\varphi} = \frac{1}{V}\frac{dV}{d\varphi} = -\frac{\eta^2 t}{V^2}. \tag{24}$$

In this treatment, the two following differentiations become

$$\frac{d^2 \log V}{d\varphi^2} = \frac{\eta^2}{V^4}(-1 + t^2 - \eta^2 - \eta^2 t^2)$$

$$\frac{d^3 \log V}{d\varphi^3} = \frac{2\eta^2 t}{V^6}(2 + \eta^2 + 3\eta^2 t^2 - \eta^4 - \eta^4 t^2). \qquad (25)$$

The differentiations (24) and (25) are to be introduced into (22), and at the same time let us take $\Delta\varphi = 10'$, for which purpose we are to divide by $\rho = 3437.7 \ldots$, and since we are also to multiply by $\mu = 0.43429 \ldots$, we have

$$\Delta \log V = -\frac{\mu}{\varrho} 10 \frac{\eta^2 t}{V^2} + \frac{\mu \, 1000}{12 \varrho^3} \frac{\eta^2 t}{V^6}(2 + \eta^2 + 3\eta^2 t^2 - \eta^4 - \eta^4 t^2).$$

Since $\eta^2 = e'^2 \cos^2 \varphi$, therefore $\eta^2 t = e'^2 \sin\varphi \cos\varphi$, we have, in order to collect the constants, neglecting at the same time η^4:

$$\Delta \log V = -5 \frac{\mu}{\varrho} e'^2 \frac{\sin 2\varphi}{V^2} + \frac{1000 \mu e'^2}{24 \varrho^3} \frac{\sin 2\varphi}{V^6}(2 + \eta^2 + 3\eta^2 t^2). \qquad (26)$$

The calculation yields for units of the 7th logarithmic place:

$$\Delta \log V = -[1.6277992{\cdot}161] \frac{\sin 2\varphi}{V^2} + [5.4760703] \frac{\sin 2\varphi}{V^6}(2 + \eta^2 + 3\eta^3 t^2). \qquad (27)$$

The second term amounts to very little, namely:

$\varphi = 10°$ 2nd term =	0·0000 2015	$\varphi = 50°$ 2nd term =	0·0000 5888
20	3795	60	5201
30	5133	70	3874
40	5862	80	2063.

This term thus does not enter until the 12th logarithmic place and turns out to be a small difference between the sums of every 6 intermediate values of $\Delta \log V$ [on the one hand] and [on the other] the interval between two fixed $\log V$'s, which they are to fill; [a difference] which is nearly constant within wide limits. The V^2's which we need in the first term of (27) must correspond to the mean argument φ; in our case we proceeded here in such a way that those $\log V^2$'s which were already available, computed to eight places for φ for every 10' in the former 3rd edition of the volume, furnished, when interpolated for every 5', the necessary approximate values for equation (27), in order to give the interpolation to an accuracy of 12 to 13 places; the following is an example:

		log V	φ
$\varphi = 48°$		6522·92572	48° 0'
	42·07059 — 6 = 42·07053		
		6480·85519	48 10
	42·04425 — 6 = 42·04419		
		6438·81100	48 20
	42·01646 — 6 = 42·01640		
		6396·79460	48 30
	41·98728 — 6 = 41·98722		
		6354·80738	48 40
	41·95667 — 6 = 41·95661		
		6312·85077	48 50
	41·92464 — 6 = 41·92458		
$\varphi = 49°$		6270·92619	49 0
	251·99989 — 36 = 251·99953	251·99953.	

The differences 42·07059 – 6, etc., are computed according to formula (27); 6 is the second term, rounded off in the above example (actually, we calculated with one more place).

In this way, our table of the values $\log V$, which is given in the Appendix* on pages [2] to [7], was computed to 12 to 13 places and afterwards rounded off to 10 places.

On pages [8] to [11] of the Appendix we have inserted a further table with which we can obtain the values $\log V_i$ for the international ellipsoid. With these, all other fundamental values for the international ellipsoid, for which no special tables are given in this handbook, can also be computed.

The differentiations of η^2 and V in respect to φ

The differentiations of the functions η^2 and V also will often be used later, and therefore we shall mention them here in order to have them on hand, and shall introduce fixed signs at the same time:

$$\tan\varphi = t \qquad \frac{dt}{d\varphi} = 1 + t^2 , \tag{a}$$

$$V^2 = 1 + e'^2 \cos^2\varphi = 1 + \eta^2 , \quad \text{hence} \quad \eta^2 = e'^2 \cos^2\varphi . \tag{b}$$

$$\frac{d\eta^2}{d\varphi} = -2\, e'^2 \cos\varphi \sin\varphi = -2\,\eta^2 t , \tag{c}$$

and more generally,

$$\frac{d\eta^n}{d\varphi} = -n\,\eta^n t . \tag{d}$$

Furthermore, there follows easily:

$$\frac{d^2\eta^2}{d\varphi^2} = -2\,\eta^2(1-t^2), \tag{e}$$

$$\frac{d^3\eta^2}{d\varphi^3} = +8\,\eta^2 t, \tag{f}$$

$$\frac{d^4\eta^2}{d\varphi^4} = +8\,\eta^2(1-t^2). \tag{g}$$

The differentiations of V in respect to φ are

$$\frac{dV}{d\varphi} = \frac{1}{2V}\frac{d\eta^2}{d\varphi} = -\frac{\eta^2 t}{V} \tag{h}$$

$$\frac{d^2V}{d\varphi^2} = -\left(\frac{d\eta^2}{d\varphi}\frac{t}{V} + \frac{\eta^2}{V}\frac{dt}{d\varphi} - \frac{\eta^2 t}{V^2}\frac{dV}{d\varphi}\right)$$

$$\frac{d^2V}{d\varphi^2} = +\frac{2\eta^2 t^2}{V} - \frac{\eta^2}{V}(1+t^2) - \frac{\eta^4 t^2}{V^3}.$$

If we bring here everything to the denominator V^3 and consider that $V^2 = 1 + \eta^2$, we obtain:

$$\frac{d^2V}{d\varphi^2} = \frac{\eta^2}{V^3}(1 - t^2 + \eta^2) . \tag{i}$$

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* The Appendix, which consists of tables [2] through [69], is not included in the selected portions of vol. 2 translated.

If we differentiate further in these formulae, we obtain:

$$\frac{d^3 V}{d\varphi^3} = \frac{\eta^2 t}{V^5}(4 + 5\eta^2 + 3\eta^2 t^2 + \eta^4), \tag{k}$$

$$\frac{d^4 V}{d\varphi^4} = \frac{\eta^2}{V^7}(4 - 4t^2 + 9\eta^2 + 10\eta^2 t^2 - 3\eta^2 t^4 + 6\eta^4 + 14\eta^4 t^2 + 12\eta^4 t^4 + \eta^6). \tag{l}$$

With the help of these differentiations, we can also reduce η^2 and V from a latitude φ to a neighboring latitude, for we have, according to Taylor's theorem, for a latitude φ' a value η'^2 according to the following series:

$$\eta'^2 = \eta^2 + \frac{d\eta^2}{d\varphi}(\varphi' - \varphi) + \frac{1}{2}\frac{d^2\eta^2}{d\varphi^2}(\varphi' - \varphi)^2 + \frac{1}{6}\frac{d^3\eta^2}{d\varphi^3}(\varphi' - \varphi)^3 + \frac{1}{24}\frac{d^4\eta^2}{d\varphi^4}(\varphi' - \varphi)^4 + \ldots$$

or

$$\eta'^2 = \eta^2 - 2\eta^2 t(\varphi' - \varphi) - (1 - t^2)\eta^2(\varphi' - \varphi)^2 + \frac{4}{3}\eta^2 t(\varphi' - \varphi)^3 + \frac{1}{3}(1 - t^2)\eta^2(\varphi' - \varphi)^4 + \ldots \tag{m}$$

We have likewise:

$$V' = V - \frac{\eta^2 t}{V}(\varphi' - \varphi) - \frac{\eta^2}{2V^3}(1 - t^2 + \eta^2)(\varphi' - \varphi)^2 + \frac{2\eta^2 t}{3V^5}(\varphi' - \varphi)^3 + \frac{\eta^2}{6V^7}(1 - t^2)(\varphi' - \varphi)^4 + \ldots \tag{n}$$

where terms with $\eta^4(\varphi' - \varphi)^3$ are already omitted, however.

With this, we also have the reduction of the radii of curvature from a latitude φ to a latitude φ', for instance:

$$N = \frac{c}{V} \qquad N' = \frac{c}{V'}$$

$$\frac{N}{N'} = \frac{V'}{V} = 1 - \frac{\eta^2 t}{V^2}(\varphi' - \varphi) - \frac{\eta^2}{2V^4}(1 - t^2 + \eta^2)(\varphi' - \varphi)^2 + \frac{2\eta^2 t}{3V^6}(\varphi' - \varphi)^3 + \frac{\eta^2}{6V^8}(1 - t^2)(\varphi' - \varphi)^4 + \ldots \tag{o}$$

Later we shall make use of these several times.

Section 41. Lengths of Meridional Arcs

The radius of curvature in the meridian at the latitude φ is from (17) and (21), section 38, p. 49:

$$M = \frac{a(1-e^2)}{W^3} \quad \text{or} \quad M = \frac{c}{V^3} . \tag{1}$$

An infinitely small arc of meridian for the difference of latitude $d\varphi$ is therefore $= M\,d\varphi$, and the whole arc of meridian from the equator with $\varphi = 0$ to the latitude φ is

$$B = \int_0^{\varphi} M\,d\varphi = a(1-e^2)\int_0^{\varphi}\frac{d\varphi}{\sqrt{(1-e^2\sin^2\varphi)^3}} \quad \text{or} \quad = c\int_0^{\varphi}\frac{d\varphi}{\sqrt{(1+e'^2\cos^2\varphi)^3}} . \tag{2}$$

This is an elliptic integral of the second kind; we do not speak of this, however, in the case of the evaluation for geodetic purposes, since developments in series which are broken off, according to circumstances, after a few or several terms are used here.

Integration only to e^2, inclusive, with a and e^2

If we aim to develop only to e^2, inclusive, then we write function (2) to be integrated briefly this way:

$$\frac{1}{(1-e^2\sin^2\varphi)^{\frac{3}{2}}} = (1-e^2\sin^2\varphi)^{-\frac{3}{2}} = 1 + \frac{3}{2}e^2\sin^2\varphi + e^4\ldots \tag{3}$$

We have here according to the well-known goniometric formula:

$$\sin^2\varphi = \frac{1}{2} - \frac{1}{2}\cos 2\varphi \tag{4}$$

and the integral:

$$\int \cos 2\varphi\, d\varphi = \frac{1}{2}\sin 2\varphi . \tag{5}$$

[Jordan has here falsely $\frac{1}{2}\cos 2\varphi$.]

With this, the general integral in (2) becomes:

$$\int\frac{d\varphi}{\sqrt{(1-e^2\sin^2\varphi)^3}} = \left(1+\frac{3}{4}e^2\right)\varphi - \frac{3}{8}e^2\sin 2\varphi + \ldots \tag{6}$$

The definite integral between the limits φ_1 and φ_2 is therefore:

$$\int_{\varphi_1}^{\varphi_2}\ldots d\varphi = \left(1+\frac{3}{4}e^2\right)(\varphi_2-\varphi_1) - \frac{3}{8}e^2(\sin 2\varphi_2 - \sin 2\varphi_1),$$

or if the second term is converted goniometrically:

$$\int_{\varphi_1}^{\varphi_2}\ldots d\varphi = \left(1+\frac{3}{4}e^2\right)(\varphi_2-\varphi_1) - \frac{3}{4}e^2\sin(\varphi_2-\varphi_1)\cos(\varphi_2+\varphi_1), \tag{7}$$

consequently the arc of meridian m itself between the limits φ_1 and φ_2 according to (2) and (7):

$$m = a(1-e^2)\left(\left(1+\frac{3}{4}e^2\right)(\varphi_2-\varphi_1) - \frac{3}{4}e^2 \sin(\varphi_2-\varphi_1)\cos(\varphi_2+\varphi_1)\right). \tag{8}$$

If we apply the sine series here, namely its first two terms only:

$$\sin(\varphi_2-\varphi_1) = (\varphi_2-\varphi_1) - \frac{(\varphi_2-\varphi_1)^3}{6} + \ldots,$$

then we obtain by introducing these terms into the above (8):

$$m = a(\varphi_2-\varphi_1)(1-e^2)\left(1+\frac{3}{4}e^2 - \frac{3}{4}e^2\cos(\varphi_2+\varphi_1) + \frac{1}{8}e^2(\varphi_2-\varphi_1)^2\cos(\varphi_2+\varphi_1)\right). \tag{9}$$

Although all terms of the order e^4 and above are neglected in this expression, we can make use of it for many purposes; with a small additional sacrifice in accuracy, we can even derive a very practical formula which refers to the radius of curvature in the meridian of the *mean* latitude $\frac{\varphi_1+\varphi_2}{2}$. This radius of curvature in the meridian is according to (1):

$$M' = \frac{a(1-e^2)}{\left(1-e^2\sin^2\frac{\varphi_1+\varphi_2}{2}\right)^{\frac{3}{2}}},$$

this yields, developed also to e^2 inclusive:

$$M' = a(1-e^2)\left(1+\frac{3}{2}e^2\sin^2\frac{\varphi_1+\varphi_2}{2}\right)$$

and with the use of the goniometric formula (4) for $\sin^2\varphi$:

$$M' = a(1-e^2)\left(1+\frac{3}{4}e^2 - \frac{3}{4}e^2\cos(\varphi_1+\varphi_2)\right).$$

Now if we take this radius of curvature M' as radius of the *arc* pertaining to a central angle $\varphi_2-\varphi_1$, then we obtain a corresponding arc of meridian:

$$m' = M'(\varphi_2-\varphi_1) = a(\varphi_2-\varphi_1)(1-e^2)\left(\left(1+\frac{3}{4}e^2\right) - \frac{3}{4}e^2\cos(\varphi_1+\varphi_2)\right). \tag{10}$$

If we compare this expression with the former (9), we find complete agreement in the first terms; therefore, we also have immediately the difference:

$$m - m' = a(\varphi_2-\varphi_1)(1-e^2)\frac{e^2}{8}(\varphi_2-\varphi_1)^2\cos(\varphi_1+\varphi_2)$$

or by adding the necessary ρ:

$$m - m' = a(1-e^2)\frac{e^2}{8}\left(\frac{\varphi_2-\varphi_1}{\varrho}\right)^3 \cos(\varphi_1+\varphi_2).$$

Therefore, we also can write:

$$m = M'\left(\frac{\varphi_2-\varphi_1}{\varrho}\right) + a(1-e^2)\frac{e^2}{8}\left(\frac{\varphi_2-\varphi_1}{\varrho}\right)^3 \cos(\varphi_1+\varphi_2),$$

and since we can set with sufficient accuracy in the second term

$$a(1-e^2) = M'$$

we have

$$m = M'\left(\frac{\varphi_2-\varphi_1}{\varrho}\right)\left(1+\frac{e^2}{8}\left(\frac{\varphi_2-\varphi_1}{\varrho}\right)^2 \cos(\varphi_1+\varphi_2)\right), \quad (11)$$

where M' represents now the mean latitude $\frac{\varphi_1+\varphi_2}{2}$.

The second term in (11) indicates the error which is committed if, according to (10), p. 65, we treat an elliptic arc of the meridian as an arc of a circle whose radius is the radius of curvature in the meridian for the mean latitude $\frac{\varphi_1+\varphi_2}{2}$.

At first we see that the error disappears if $\varphi_1+\varphi_2 = 90°$, i.e. if the mean latitude is $= 45°$ (with the exception of the neglected terms of the order e^4, etc.).

As for the rest, we calculate according to (11) with the understanding that for $\varphi_2-\varphi_1 = 1°$ and for $\frac{\varphi_1+\varphi_2}{2} = 30°$ or $60°$ the error becomes $= -0.014$ m or $= +0.014$ m, and in the extreme case with $\varphi = 0$ or $\varphi = 90°$, the error for $\varphi_2-\varphi_1 = 1°$ yields only -0.028 m or $+0.028$ m. Therefore, we can say briefly that at the latitudes of Central Europe an arc of the meridian of 1° extent is sufficiently accurately computed according to the method of approximation with M' for the mean latitude within 1 cm.

Integration to e^{10}

We can continue the method used above as far as we like; in general, it consists in developing the function $(1-e^2\sin^2\varphi)^{-\frac{3}{2}}$ to be integrated with respect to powers of $e^2\sin^2\varphi$ and then expressing the powers $\sin^2\varphi$, $\sin^4\varphi$, $\sin^6\varphi$, etc., in $\cos 2\varphi$, $\cos 4\varphi$, $\cos 6\varphi$, etc., and thus making them integrable.

Accordingly, we have

$$\frac{1}{W^3} = (1-e^2\sin^2\varphi)^{-\frac{3}{2}} = 1 + \frac{3}{2}e^2\sin^2\varphi + \frac{3}{2}\frac{5}{4}e^4\sin^4\varphi + \frac{3}{2}\frac{5}{4}\frac{7}{6}e^6\sin^6\varphi$$
$$+ \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{9}{8}e^8\sin^8\varphi + \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{9}{8}\frac{11}{10}e^{10}\sin^{10}\varphi + \dots$$

$$\frac{1}{W^3} = 1 + \frac{3}{2}e^2\sin^2\varphi + \frac{15}{8}e^4\sin^4\varphi + \frac{35}{16}e^6\sin^6\varphi + \frac{315}{128}e^8\sin^8\varphi + \frac{693}{256}e^{10}\sin^{10}\varphi. \quad (12)$$

According to section 35, p. 28, we are to set here:

$$sin^2\ \varphi = \frac{1}{2} - \frac{1}{2} cos\ 2\ \varphi$$

$$sin^4\ \varphi = \frac{3}{8} - \frac{1}{2} cos\ 2\ \varphi + \frac{1}{8} cos\ 4\ \varphi$$

$$sin^6\ \varphi = \frac{5}{16} - \frac{15}{32} cos\ 2\ \varphi + \frac{3}{16} cos\ 4\ \varphi - \frac{1}{32} cos\ 6\ \varphi$$

$$sin^8\ \varphi = \frac{35}{128} - \frac{7}{16} cos\ 2\ \varphi + \frac{7}{32} cos\ 4\ \varphi - \frac{1}{16} cos\ 6\ \varphi + \frac{1}{128} cos\ 8\ \varphi$$

$$sin^{10}\ \varphi = \frac{63}{256} - \frac{105}{256} cos\ 2\ \varphi + \frac{15}{64} cos\ 4\ \varphi - \frac{45}{512} cos\ 6\ \varphi + \frac{5}{256} cos\ 8\ \varphi - \frac{1}{512} cos\ 10\ \varphi\ .$$

If we set these expressions into (12) and arrange according to $cos\ 2\ \varphi$, $cos\ 4\ \varphi$, etc., then we obtain:

$$\frac{1}{W^3} = A - B\ cos\ 2\ \varphi + C\ cos\ 4\ \varphi - D\ cos\ 6\ \varphi + E\ cos\ 8\ \varphi - F\ cos\ 10\ \varphi\ , \tag{13}$$

where the coefficients A, B, etc., have the following meaning:

$$A = 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11025}{16384} e^8 + \frac{43659}{65536} e^{10} + \ldots$$

$$B = \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2205}{2048} e^8 + \frac{72765}{65536} e^{10} + \ldots$$

$$C = \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2205}{4096} e^8 + \frac{10395}{16384} e^{10} + \ldots$$

$$D = \frac{35}{512} e^6 + \frac{315}{2048} e^8 + \frac{31185}{131072} e^{10} + \ldots$$

$$E = \frac{315}{16384} e^8 + \frac{3465}{65536} e^{10} + \ldots$$

$$F = \frac{693}{131072} e^{10} + \ldots$$

If we calculate these values with Bessel's eccentricity according to section 37, p. 44 ($log\ e^2$ = 7.8244104·237), then we find:

$$\left.\begin{array}{ll} A = 1.00503\ 73060.48555 & log\ A = 0.002\ 1821{\cdot}827 \\ B = 0.00504\ 78492.40300 & log\ B = 7.703\ 1063{\cdot}757 \\ C = 0.00001\ 05637.86831 & log\ C = 5.023\ 8196{\cdot}289 \\ D = 206.33322 & log\ D = 2.314\ 5691{\cdot}6 \\ E = 0.38853 & log\ E = 9.589\ 4246 \\ F = 0.00070 & log\ F = 6.845\ 10\ . \end{array}\right\} \tag{14}$$

The last coefficient F, which only depends on e^{10}, becomes infinitesimally small, but in the remaining coefficients the terms with e^{10} still yield small amounts which still partly influence the final rounding off.

By integrating function (13), we have:

$$\int cos\ 2\ \varphi\ d\ \varphi = \frac{1}{2} sin\ 2\ \varphi \qquad \int cos\ 4\ \varphi\ d\ \varphi = \frac{1}{4} sin\ 4\ \varphi, \text{ etc.;}$$

therefore, with the addition of factor $a(1-e^2)$ of (1) and (2), the arc of meridian from the equator to the latitude φ is expressed by the series:

$$B = a(1-e^2)\left(\frac{A\varphi}{\varrho} - \frac{B}{2}\sin 2\varphi + \frac{C}{4}\sin 4\varphi - \frac{D}{6}\sin 6\varphi + \frac{E}{8}\sin 8\varphi - \frac{F}{10}\sin 10\varphi\right). \tag{15}$$

We have to calculate with the coefficients (14):

$$\left.\begin{aligned}
\frac{A\,a(1-e^2)}{\varrho^\circ} &= 111120.61962 \qquad && \log = 5.045\,7946{\cdot}544\\
\frac{B}{2}a(1-e^2) &= 15988.63853 && \log = 4.203\,8114{\cdot}841\\
\frac{C}{4}a(1-e^2) &= 16.72995\,380 && \log = 1.223\,4947{\cdot}417\\
\frac{D}{6}a(1-e^2) &= 0.02178\,4772 && \log = 8.338\,1530{\cdot}1\\
\frac{E}{8}a(1-e^2) &= 0.00003\,07659 && \log = 5.488\,0696\\
\frac{F}{10}a(1-e^2) &= 0.00000\,0044344 && \log = 2.646\,84\,.
\end{aligned}\right\} \tag{16}$$

Let us refer also here to the development in series in *Theorie der Projektionsmethode der hannoverschen Landesvermessung* by Oskar Schreiber, Hannover, 1866, p. 13: Arc of meridian $B = a(A\varphi - A_1 \sin 2\varphi + A_2 \sin 4\varphi - \ldots)$.

Integration to e'^{10}

We shall at first retain the coefficients (16) and start the integration once again all over in the second form with e and e'^2; there will again come out the same as by the first method, which is desirable as a check.

According to the second form of (1) or (2) we have to treat:

$$\frac{1}{V^3} = (1 + e'^2\cos^2\varphi)^{-\frac{3}{2}} = 1 - \frac{3}{2}e'^2\cos^2\varphi + \frac{3}{2}\frac{5}{4}e'^4\cos^4\varphi - \frac{3}{2}\frac{5}{4}\frac{7}{6}e'^6\cos^6\varphi$$

$$+ \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{9}{8}e'^8\cos^8\varphi - \frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{9}{8}\frac{11}{10}e'^{10}\cos^{10}\varphi$$

$$\frac{1}{V^3} = 1 - \frac{3}{2}e'^2\cos^2\varphi + \frac{15}{8}e'^4\cos^4\varphi - \frac{35}{16}e'^6\cos^6\varphi + \frac{315}{128}e'^8\cos^8\varphi - \frac{693}{256}e'^{10}\cos^{10}\varphi\,. \tag{17}$$

From section 35, pp. 28 and 29, we have:

$$\cos^2\varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$$

$$\cos^4\varphi = \frac{3}{8} + \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi$$

$$\cos^6\varphi = \frac{5}{16} + \frac{15}{32}\cos 2\varphi + \frac{3}{16}\cos 4\varphi + \frac{1}{32}\cos 6\varphi$$

$$\cos^8\varphi = \frac{35}{128} + \frac{7}{16}\cos 2\varphi + \frac{7}{32}\cos 4\varphi + \frac{1}{16}\cos 6\varphi + \frac{1}{128}\cos 8\varphi$$

$$\cos^{10}\varphi = \frac{63}{256} + \frac{105}{256}\cos 2\varphi + \frac{15}{64}\cos 4\varphi + \frac{45}{512}\cos 6\varphi + \frac{5}{256}\cos 8\varphi + \frac{1}{512}\cos 10\varphi\,.$$

If we introduce these expressions into (17) and arrange according to $\cos 2\varphi$, $\cos 4\varphi$, etc., then we obtain:

$$\frac{1}{V^3} = A' - B' \cos 2\varphi + C' \cos 4\varphi - D' \cos 6\varphi + E' \cos 8\varphi - F' \cos 10\varphi\,, \tag{18}$$

where the coefficients A', B', etc., have the following meanings:

$$\begin{aligned}
A' &= 1 - \frac{3}{4}e'^2 + \frac{45}{64}e'^4 - \frac{175}{256}e'^6 + \frac{11025}{16384}e'^8 - \frac{43659}{65536}e'^{10} + \ldots\\
B' &= \quad + \frac{3}{4}e'^2 - \frac{15}{16}e'^4 + \frac{525}{512}e'^6 - \frac{2205}{2048}e'^8 + \frac{72765}{65536}e'^{10} - \ldots\\
C' &= \quad + \frac{15}{64}e'^4 - \frac{105}{256}e'^6 + \frac{2205}{4096}e'^8 - \frac{10395}{16384}e'^{10} + \ldots\\
D' &= \quad + \frac{35}{512}e'^6 - \frac{315}{2048}e'^8 + \frac{31185}{131072}e'^{10} - \ldots\\
E' &= \quad + \frac{315}{16384}e'^8 - \frac{3465}{65536}e'^{10} + \ldots\\
F' &= \quad + \frac{693}{131072}e'^{10} - \ldots
\end{aligned}$$

In all these developments of (17) to F' there occur the same numerical coefficients as previously in (12) to F, only with different signs. We also note that in the group of the coefficients $A' B' C' \ldots$ each column yields the sum = zero so that altogether we have:

$$A' + B' + C' + D' + E' + F' = 1\,.$$

This has also a deeper sense, for with $\varphi = 90°$ we will have

$$\cos 2\varphi = -1\,, \quad \cos 4\varphi = +1\,, \quad \cos 6\varphi = -1\,, \quad \text{etc.,}$$

and with these we will have from (18):

$$\frac{1}{V^3} = A' + B' + C' + D' + E' + F' = 1\,.$$

This is a satisfactory check for the coefficients A', $B' \ldots$, with which also the previous coefficients A, $B \ldots$ are checked at the same time.

The calculation of the numerical values of $A' B'$, etc., yields with Bessel's $\log e'^2 = 7.827\ 3187{\cdot}833$:

$$\left.\begin{aligned}
A' &= 0.99499\ 21245.07507 & \log A' &= 9.997\ 8196{\cdot}433\\
B' &= 0.00499\ 73968.23275 & \log B' &= 7.698\ 7438{\cdot}364\\
C' &= 0.00001\ 04582.03528 & \log C' &= 5.019\ 4570{\cdot}894\\
D' &= 204.27152 & \log D' &= 2.310\ 2078{\cdot}2\\
E' &= 0{,}38465 & \log E' &= 9.585\ 0657\\
F' &= 0.00072 & \log F' &= 6{\cdot}857\ 33\,.
\end{aligned}\right\} \tag{19}$$

The further computation according to the integration then yields, just as previously in the case of the computation with W:

$$\left.\begin{aligned}
\frac{A'}{\varrho^\circ}\,c &= 111120.61962 & \log\frac{A'}{\varrho^\circ}\,c &= 5.045\ 7946{\cdot}544\\
\frac{B'}{2}\,c &= 15988.63853 & \log\frac{B'}{2}\,c &= 4.203\ 8114{\cdot}842\\
\frac{C'}{4}\,c &= 16.72995\ 380 & \log\frac{C'}{4}\,c &= 1.223\ 4947{\cdot}417\\
\frac{D'}{6}\,c &= 0.02178\ 4832 & \log\frac{D'}{6}\,c &= 8.338\ 1542{\cdot}1\\
\frac{E'}{8}\,c &= 0.00003\ 07662 & \log\frac{E'}{8}\,c &= 5.488\ 0733\\
\frac{F'}{10}\,c &= 0.00000\ 00460.71 & \log\frac{F'}{10}\,c &= 2.663\ 43\,.
\end{aligned}\right\}\qquad (20)$$

The two calculations (16) and (20) agree as nearly as can be expected in the case of the unavoidable roundings off. The whole computation is sufficiently checked by it, and hence we develop in the mean: the formula for the arc of meridian B from the equator to the latitude φ:

$$\left.\begin{aligned}
B &= \alpha\varphi + \beta \sin 2\varphi + \gamma \sin 4\varphi + \delta \sin 6\varphi + \varepsilon \sin 8\varphi + \zeta \sin 10\varphi\\
B &= 11\,1120.61962\,\varphi - 15988.63853 \sin 2\varphi + 16.72995\ 38 \sin 4\varphi\\
&\quad - 0.02178\ 480 \sin 6\varphi + 0.00003\ 0766 \sin 8\varphi\\
&\quad - 0.00000\ 00452 \sin 10\varphi\,.
\end{aligned}\right\}\qquad (21)$$

The last term with $\sin 10\ \varphi$ is no longer of importance for all calculations planned in the following; therefore, we shall omit it entirely in the following formula which gives, instead of the coefficients themselves, their logarithms:

$$\left.\begin{aligned}
B = [5.045\ 7946{\cdot}544]\,\varphi - [4.203\ 8114{\cdot}842] \sin 2\varphi + [1.223\ 4947{\cdot}4] \sin 4\varphi\\
- [8.338\ 1536] \sin 6\varphi + [5.48807] \sin 8\varphi\,.
\end{aligned}\right\}\qquad (22)$$

In the first term of (21) or (22), φ is to be taken in degrees; if we aim to compute in minutes or seconds, then the first term becomes:

$$\left.\begin{aligned}
&\text{for } 1' && 1852.01032\ 72\ \text{m} && \log = 3.267\ 6434{\cdot}040\\
&\text{for } 1'' && 30.86683\ 879 && \log = 1.489\ 4921{\cdot}536\,.
\end{aligned}\right\}\qquad (23)$$

Formula (21) yields at once an important application; with $\varphi = 90°$, $\sin 2\ \varphi$, $\sin 4\ \varphi$, etc., all become = zero, and we obtain the quadrant of the meridian:

$$Q = 10\ 000\ 855.7658\ \text{m}\,.$$

We express this frequently also in such a way that we say $\dfrac{Q}{90} = 111\ 120.61962$ m is the mean value of one degree of the meridian.

If we introduce once more the meaning of the first coefficient α in (21) from (16) and (13), we obtain:

$$Q = \frac{A\,a\,(1-e^2)}{\varrho^\circ}\,90^\circ = a\,(1-e^2)\,\frac{\pi}{2}\,A$$

$$Q = a\,(1-e^2)\,\frac{\pi}{2}\left(1 + \frac{3}{4}\,e^2 + \frac{45}{64}\,e^4 + \ldots\right)$$

$$Q = a\,\frac{\pi}{2}\left(1 - \frac{1}{4}\,e^2 - \frac{3}{64}\,e^4 - \ldots\right). \qquad (24)$$

We shall stop at this approximation and introduce the flattening α from (7), section 37, p. 42, namely with

$$e^2 = 2\,\alpha - \alpha^2 .$$

If we set this into (24) and arrange according to powers of α, we obtain:

$$Q = a\frac{\pi}{2}\left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{16}\right). \qquad (25)$$

We can also develop the logarithm of the factor in parentheses whereby we find:

$$log\left(1 - \frac{\alpha}{2} + \frac{\alpha^2}{16}\right) = \mu\left(-\frac{\alpha}{2} + \frac{\alpha^2}{16}\right) - \frac{\mu}{2}\left(-\frac{\alpha}{2}\right)^2 = -\mu\left(\frac{\alpha}{2} + \frac{\alpha^2}{16}\right).$$

Later we shall use formula (25), and in order to be able to compute easily the quadrant for such cases from the major axis and the flattening α or vice versa, we have prepared the following small auxiliary table for it:

$\frac{a-b}{a} = \alpha$	$log\frac{\pi}{2}\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{16}\right)$		$\frac{a-b}{a} = \alpha$	$log\frac{\pi}{2}\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{16}\right)$	
1 : 280	0.195 3440	+	1 : 300	0.195 3957	+
1 : 285	0.195 3577	137	1 : 305	0.195 4076	119
1 : 290	0.195 3708	131	1 : 310	0.195 4191	115
1 : 295	0.195 3835	127	1 : 315	0.195 4303	112
1 : 300	0.195 3957	122	1 : 320	0.195 4410	107

We obtain a second, obvious, application of (21) or (22) with $\varphi = 45°$, thereby we will have $2\,\varphi = 90°$, $4\,\varphi = 180°$, $6\,\varphi = 270°$, i.e.:

$$B_0^{45} = 5{,}000{,}427.882900 - 15988.638530 + 0.021785 = 4{,}984{,}439.266155 \text{ m} . \qquad (26)$$

This is the arc of meridian from the equator to the mean latitude $= 45°$; the other part from 45° to the pole is considerably larger, namely:

$$B_{45}^{90} = 5{,}016{,}416.4996 \text{ m} . \qquad (27)$$

According to formula (21) or (22) the 30 values of $\varphi = 30°$ to $\varphi = 60°$ are computed:

Arc of Meridian B from the Equator to the Latitude φ (28)

φ	B	φ	B	φ	B
	m		m		m
30°	3,319,786.510	40°	4,429,084.790	50°	5,540,279,543
31	3,430,636.950	41	4,540,116.998	51	5,651,505,565
32	3,541,502.523	42	4,651,168.472	52	5,762,750.675
33	3,652,386.539	43	4,762,239.302	53	5,874,014.723
34	3,763,288.290	44	4,873,329.553	54	5,985,297.540
35	3,874,208.046	45	4,984,439.266	55	6,096,598.931
36	3,985,146.054	46	5.095,568.459	56	6,207,918.679
37	4,096,102.540	47	5,206,717.124	57	6,319,256.544
38	4,207,077.708	48	5,317,885.233	58	6,430,612.266
39	4,318,071.739	49	5,429,072.732	59	6,541,985.560
40	4,429,084.790	50	5,540,279.543	60	6,653,376.122

In the following table we have put together the supplementary quantities ΔB with which we obtain $B_i = B + \Delta B$ for the computation of the arcs of meridian B_i for the international ellipsoid. For the interpolation we also have included the first and second differences so that the values B_i can be computed with an accuracy of 1 mm for all latitudes between 30° and 60°.

Conversion to the International Ellipsoid

φ	ΔB	1st Diff.	2nd Diff.	φ	ΔB	1st Diff.	2nd Diff.	φ	ΔB	1st Diff.	2nd Diff.
	m				m				m		
30°	375.190	13.905		40°	520.169	15.253		50°	679.170	16.690	
31	389.095	14.030	0.125	41	535.422	15.398	0.145	51	695.860	16.831	0,141
32	403.125	14.160	0.130	42	550.820	15.540	0.142	52	712.691	16.972	0.141
33	417.285	14.291	0.131	43	566.360	15.683	0.143	53	729.663	17.111	0.139
34	431.576	14.423	0.132	44	582.043	15.828	0.145	54	746.774	17.249	0.138
35	445.999	14.558	0,135	45	597.871	15.971	0.143	55	764.023	17.385	0.136
36	460.557	14.695	0.137	46	613.842	16.117	0.146	56	781.408	17.520	0.135
37	475.252	14,832	0.137	47	629.959	16.260	0.143	57	798.928	17.652	0,132
38	490.084	14,972	0.140	48	646.219	16.404	0.144	58	816.580	17.784	0.132
39	505.056	15.113	0.141	49	662.623	16.547	0.143	59	834.364	17.910	0.126
40	520.169		0.140	50	679.170		0.143	60	852.274		

All numerical values are positive.

The small auxiliary table on p. [40] of our Appendix which yields the arcs of meridian from the equator to the latitude φ denoted by B in the above formulae is computed only in its first part, from 40° to 44°, by the author, and in fact with somewhat different constants than in the above formula (21) or (22). The remaining part from 44° to 56° is an extract from F. G. Gauss, *Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst*, 4th edition, 1922, part II, pp. 4-27.

At the end of this section we shall come to a newly computed, detailed table of arcs of meridian from $\varphi = 45°$ to 57°.

Arc of meridian between the latitudes φ_1 and φ_2

If we want to have the length of a limited arc m, between φ_1 and φ_2, we can find this at once from (21), namely:

$$\left. \begin{aligned} m = \alpha(\varphi_2 - \varphi_1) + \beta(\sin 2\varphi_2 - \sin 2\varphi_1) + \gamma(\sin 4\varphi_2 - \sin 4\varphi_1) \\ + \delta(\sin 6\varphi_2 - \sin 6\varphi_1) + \dots \end{aligned} \right\} \quad (29)$$

We shall transform this to the goniometric form and set here:

$$\varphi_2 - \varphi_1 = \Delta\varphi \qquad \frac{\varphi_2 + \varphi_1}{2} = \varphi_0 .$$

With these, (29) becomes:

$$m = \alpha\Delta\varphi + 2\beta \sin\Delta\varphi \cos 2\varphi_0 + 2\gamma \sin 2\Delta\varphi \cos 4\varphi_0 + 2\delta \sin 3\Delta\varphi \cos 6\varphi_0 + \dots \quad (30)$$

Since the computed coefficients α, β, γ, etc., are given in (21) and (22), we can accordingly compute at once the arc m for every mean latitude φ_0. We shall set $\Delta\varphi = 1°$, and thus obtain the arc of meridian m_1 of 1° width with the mean latitude φ with computed coefficients:

$$\left. \begin{aligned} m_1 = 111120.61962 - 558.080436 \cos 2\varphi + 1.167734 \cos 4\varphi \\ - 0.002280 \cos 6\varphi + 0.0000043 \cos 8\varphi \end{aligned} \right\} \quad (31)$$

or with logarithms:

$$m_1 = 111{,}120.61962 - [2.746\ 6967{\cdot}983] \cos 2\varphi + [0.067\ 3439{\cdot}0] \cos 4\varphi \\ - [7.357\ 984] \cos 6\varphi + [4.63268] \cos 8\varphi\,. \tag{32}$$

According to these formulae (30) to (32) we can compute every portion of the arc of meridian; but if we deal with arcs of only 1° or a few degrees, and we already have a table of the radii of curvature in the meridian, we can find a much better series in the following way:

We consider an arc of meridian m which lies between the latitudes φ and $\varphi + \Delta\varphi$; then it will be possible to develop according to Maclaurin's theorem:

$$m = \frac{dm}{d\varphi}\Delta\varphi + \frac{\Delta\varphi^2}{2}\frac{d^2 m}{d\varphi^2} + \frac{\Delta\varphi^3}{6}\frac{d^3 m}{d\varphi^3} + \frac{\Delta\varphi^4}{24}\frac{d^4 m}{d\varphi^4}\,. \tag{33}$$

Now we know from (1) and (2), p. 64, with (h), p. 62:

$$\frac{dm}{d\varphi} = M = \frac{c}{V^3} \tag{34}$$

$$V = \sqrt{1 + e'^2 \cos^2\varphi} = \sqrt{1+\eta^2}\ , \quad \frac{dV}{d\varphi} = -\frac{\eta^2 t}{V}$$

$$\frac{d^2 m}{d\varphi^2} = -\frac{3c}{V^4}\frac{dV}{d\varphi} = +\frac{3c\,\eta^2 t}{V^5} = +\frac{3M}{V^2}\eta^2 t\,. \tag{35}$$

If we differentiate further in these formulae, we obtain:

$$\frac{d^3 m}{d\varphi^3} = \frac{3c}{V^7}\eta^2(1 - t^2 + \eta^2 + 4\eta^2 t^2) = \frac{3M}{V^4}\eta^2(1 - t^2 + \eta^2 + 4\eta^2 t^2) \tag{36}$$

$$\frac{d^4 m}{d\varphi^4} = \frac{3c}{V^9}\eta^2 t(-4 + 7\eta^2 - 15\eta^2 t^2 + 11\eta^4 + 20\eta^4 t^2)$$

$$\frac{d^4 m}{d\varphi^4} = \frac{3M}{V^6}\eta^2 t(-4 + 7\eta^2 - 15\eta^2 t^2 + 11\eta^4 + 20\eta^4 t^2)\,. \tag{37}$$

According to these, (34) to (37), we can put together formula (33):

$$m = M\Delta\varphi + \frac{3}{2}\frac{M}{V^2}\eta^2 t\,\Delta\varphi^2 + \frac{M}{2V^4}\eta^2(1 - t^2 + \eta^2 + 4\eta^2 t^2)\,\Delta\varphi^3 \\ + \frac{M}{8V^6}\eta^2 t(-4 + 7\eta^2 - 15\eta^2 t^2 + 11\eta^4 + 20\eta^4 t^2)\,\Delta\varphi^4\,. \tag{38}$$

In order to make sure of the signs, we shall write this also with $\Delta\varphi = \varphi' - \varphi$ and $m = B' - B$, where B and B' are the arcs of meridian from the equator to φ and φ', i.e. (38) in the second form:

$$m = B' - B = M(\varphi' - \varphi) + \frac{3}{2}\frac{M}{V^2}\eta^2 t\,(\varphi' - \varphi)^2 + (\varphi' - \varphi)^3 \ldots; \tag{39}$$

M, η^2, t refer to φ here.

In this connection, we can now set up a still better formula according to the principle of the mean latitude (cf. section 35, p. 29):

We consider an arc of meridian m which lies between the latitudes $\varphi - \frac{\Delta\varphi}{2}$ and $\varphi + \frac{\Delta\varphi}{2}$ where φ is therefore the mean latitude and $\Delta\varphi$ the width. The arc m is thereby also divided into two parts, m_1 and m_2, of which the northern is represented by the following series according to Maclaurin's theorem:

$$m_1 = \left(\frac{dm}{d\varphi}\right)\frac{\Delta\varphi}{2} + \frac{1}{2}\left(\frac{d^2 m}{d\varphi^2}\right)\left(\frac{\Delta\varphi}{2}\right)^2 + \frac{1}{6}\left(\frac{d^3 m}{d\varphi^3}\right)\left(\frac{\Delta\varphi}{2}\right)^3 + \ldots;$$

a corresponding series holds for the southern part m_2, namely:

$$-m_2 = -\left(\frac{dm}{d\varphi}\right)\frac{\Delta\varphi}{2} + \frac{1}{2}\left(\frac{d^2 m}{d\varphi^2}\right)\left(\frac{\Delta\varphi}{2}\right)^2 - \frac{1}{6}\left(\frac{d^3 m}{d\varphi^3}\right)\left(\frac{\Delta\varphi}{2}\right)^3 + \ldots;$$

by subtraction we find thence:

$$m_1 + m_2 = m = \left(\frac{dm}{d\varphi}\right)\Delta\varphi + \left(\frac{d^3 m}{d\varphi^3}\right)\frac{\Delta\varphi^3}{24}.$$

The derivatives which are necessary here are developed in the above (34) and (36); therefore, we can immediately put together the above formula, at the same time by including the necessary ρ's:

$$m = M\frac{\Delta\varphi}{\varrho} + \frac{M}{8V^4}\eta^2(1 - t^2 + \eta^2 + 4\eta^2 t^2)\frac{\Delta\varphi^3}{\varrho^3}, \qquad (40)$$

and by introducing an abbreviation g and γ we have:

$$m = M\frac{\Delta\varphi}{\varrho} + g\,\Delta\varphi^3, \qquad (41)$$

where

$$\left.\begin{aligned} g &= \frac{M\eta^2}{8V^4\varrho^3}(1 - t^2 + \eta^2 + 4\eta^2 t^2) \\ \gamma = \frac{g}{M}\varrho &= \frac{\eta^2}{8V^4\varrho^2}(1 - t^2 + \eta^2 + 4\eta^2 t^2) \end{aligned}\right\} \qquad (42)$$

$$\Delta\varphi = \frac{m}{M}\varrho - \gamma\left(\frac{m}{M}\varrho\right)^3. \qquad (43)$$

The values g and γ, calculated accordingly, are communicated in the following table. It contains the correction terms g and γ for the rectification of the arc of meridian with the radius of curvature of the mean latitude φ for $\Delta\varphi = 1°$.

φ	g	γ	φ	g	γ	φ	g	γ
	+	−					−	+
0°	0.0281 m	0.00091″	45°	+ 0.00024 m	− 0.000008″	55°	0.0095 m	0.00031″
5	0.0277	0.00090	46	− 0.00075	+ 0.000024	60	0.0140	0.00045
10	0.0264	0.00086	47	− 0.00174	+ 0.000056	65	0.0182	0.00059
			48	− 0.00273	+ 0.000088			
15	0.0244	0.00079	49	− 0.00372	+ 0.000120	70	0.0217	0.00070
20	0.0217	0.00070				75	0.0247	0.00080
25	0.0183	0.00059	50	− 0.00470	+ 0.000152	80	0.0268	0.00086
			51	− 0.00567	+ 0.000184			
30	0.0143	0.00046	52	− 0.00664	+ 0.000215	85	0.0281	0.00091
35	0.0099	0.00032	53	− 0.00761	+ 0.000246	90	0.0286	0.00092
40	0.0051	0.00017	54	− 0.00856	+ 0.000277			
45	0.0002	0.00001	55	− 0.00951	+ 0.000308			

As a numerical example, we shall calculate the arc of meridian between the latitudes 47° and 53°, i.e. 6° width with the mean latitude $\varphi = 50°$; we have at first from the table of the Appendix, pp. [24] and [25], $\log M = 6.804\ 2916{\cdot}0$ or at once $\log \frac{\rho}{M} = \log [1] = 8.510\ 1335{\cdot}3$. Or if we aim to compute more rigorously still, then we take from p. [5] of the Appendix for $\varphi = 50°$ $\log V = 0.000\ 6020{\cdot}131$, therefore $\log V^3 = 0.001\ 8060{\cdot}393$, in addition, from section 37, p. 44, $\log \rho - \log c = 8.508\ 3274{\cdot}897$, so that we have together $\log [1] = 8.510\ 1335{\cdot}290$, which yields, with $6° = 21{,}600''$, the principal term of formula (41):

$$m' = \frac{21600}{[1]} = 667{,}298.613 \text{ m} ;$$

and in addition we have, according to the above small correction table, the following amount for $\varphi = 50°$ and $\Delta\varphi = 6°$:

$$-\,0.00470 \times 6^3 = -\,1.015 \text{ m} .$$

This added to the above, results in:

$$m = 667{,}298.613 \text{ m} - 1.015 \text{ m} = 667{,}297.598 \text{ m} .$$

For comparison, we have from the table (28), p. 71,

for $\varphi = 47°$	$B =$	5,206,717.124 m
for $\varphi = 53$	$B =$	5,874,014.723
Difference	$m =$	667,297.599 m .

This agrees with the above to 1 mm, which is sufficient.

The accuracy of the computation according to the formula (41) is very great, for the next neglected term is only of the order $\frac{M}{160}\,\Delta\varphi^5\, e'^2 \cos 2\varphi$, which amounts to only 7 mm for a difference of latitude of 10° between 45° and 55°, which becomes more considerable, however, because of the factor $\cos 2\varphi$, if the mean latitude φ lies far from 45°.

For example, if we aim to interpolate further the arcs of meridian B of table (28), p. 71, which are computed for every 1°, we calculate best the differences according to formula (41), p. 74, where the term with g amounts to hardly anything, e.g., for $\varphi = 50°$ and $\Delta\varphi = 10'$, $g\,\Delta\varphi^3$ becomes only $\frac{0.00470 \text{ m}}{216} = 0.00002$ m.

Therefore, in order to compute the arc of meridian from 50°0' to 50°10', we simply take, from page [36] of the Appendix, for $\varphi = 50°5'$ the value $\log [1] = 8.510\ 1272{\cdot}8$ and compute with it

$$\Delta B = 600:[1] = 18536.339 \text{ m} .$$

A second example is to show the use of the table on page [40] and of the coefficients [1]:

Let there be given the latitude of the point Celle (which is one of the former 40 Prussian cadastral zero points of coordinates), namely:

$$\varphi_0 = 52°37'32.6705'' ,$$

and, corresponding to it, we are to find the arc of meridian B from the equator to the point Celle from the table on page [40] of the Appendix. At first, we have

$$\text{for } \varphi = 52°30': B_1 = 5{,}818{,}380.341 \text{ m and } \Delta\varphi = 7'32.6709'' = 452.6709'' .$$

The mean latitude for the excess is $52°33'46.3''$, and with this, we take from page [37] the value $\log [1] = 8.509\ 9429{\cdot}9$, with which we compute further logarithmically $\Delta B = \Delta\varphi:[1] = 13{,}990.705$. This added to the above B_1 results in $B_0 = 5{,}832{,}371.046$ m, and this is the required value of the arc of meridian pertaining to φ_0, which we must also find by the use of the second differences on page [40] (in the third edition of this volume, 1890, section 35, calculated with the second differences = 5,832,371.045 m).

Table of meridional arcs for every 1' from $\varphi = 45°$ to $\varphi = 57°$

Although the above treatment of the lengths of meridional arcs could well be regarded as sufficient, we have calculated a still more detailed table of the meridional arcs B from the equator $\varphi = 0$ to the latitude φ, between $\varphi = 45°$ to $\varphi = 57°$, for every minute, besides the lengths of minutes of arc m. This table is printed in the Appendix, pp. [41] to [44].

The arcs of minutes are computed according to (33) or (40), pp. 73 and 74, as the case may be, for $\Delta\varphi = 1'$, as was shown in detail in the fourth edition of this volume, section 91. The first term yields at first:

$$m = \frac{c}{\rho'}\frac{1}{V^3} = [3.269\ 8237{\cdot}607]\,\frac{1}{V^3} , \qquad (44)$$

where V is then to be taken from the table of the Appendix (pp. [4] to [5]) for the mean values of ½'.

The second term from (35), p. 73, yields:

$$\frac{d^2 m}{d\varphi^2} = \frac{3c}{V^5} e'^2 \cos^2\varphi \tan\varphi = \frac{3}{2}\frac{c\,e'^2}{V^5} \sin 2\varphi$$

and for the interval of 1':

$$\delta = \frac{3}{2}\frac{c\,e'^2}{\varrho'^2}\frac{\sin 2\varphi}{V^5} = [7.736\ 9599]\,\frac{\sin 2\varphi}{V^5} . \qquad (45)$$

This new table, pp. [41] to [44], gives now the values B and m corresponding to the constants of the Prussian Land Survey, section 37, p. 44, however not yet with full guarantee of the last millimeter place because the calculation of the differences referred to in the case of (45) should have been made somewhat more rigorous for it; this has been omitted in absence of a need for more rigorous data to millimeters. The inaccuracy, however, will hardly exceed 1 mm.

In order to show also an application of this new table, pp. [41] to [44], we shall take once more the example Celle, above:

$$\text{Celle } \varphi_0 = 52°37'32.6709'' .$$

We are to find B for it. We take from page [43]:

$$\text{for } \varphi = 52° \, 37' \qquad B_1 = 5{,}831{,}361.276 \text{ m}$$
$$\delta\varphi = 32.6709'' \qquad \text{with } m = 1854.441 \text{ m} .$$

According to these, we can compute:

$log\ \delta\varphi$	1.514 1611
$log\ 60$	1.778 1513
$log\ \delta\varphi : 60$	9.736 0098
$log\ m$	3.268 2130
$log\ \delta B$	3.004 2228

$$\delta B = 1009.771 \text{ m}$$
$$B = 5{,}832{,}371.047 \text{ m} .$$

We compute more conveniently and besides somewhat more rigorously by including the coefficients [1] from the table of the Appendix, pp. [34] to [39]. In our case, the mean latitude $\varphi = 52°37' \, 16.33545''$ is valid, i.e. from p. [37] $log\,[1] = 8.509\,9387$, with which we compute further:

$log\,[1]$	8.509 9387
$log\ \delta\varphi$	1.514 1611
$log\,(\delta\varphi : [1])$	3.004 2224

$$B_1 = 5{,}831{,}361.276 \text{ m}$$
$$\delta B = 1{,}009.770$$
$$B = 5{,}832{,}371.046 \text{ m} .$$

This agrees exactly with B_0 of p. 76, because the tables, p. [40] and p. [43], agree in this place; as already noted, this is otherwise not the case to an accuracy of 1 mm.

With the tables of the Appendix, p. [40] and pp. [41] to [44], the arc of meridian can now always be determined as the function of a latitude φ, just as vice versa, φ as the function of the pertinent B, which is required in our later formulae of coordinates.

Section 42. Arcs of Parallel

After we have treated in detail the arcs of meridian, we also have to take care of the arcs of parallel which are related to them. No further developments are needed for them, for the radius of parallel for the latitude φ is according to Fig. 1, p. 41, and Fig. 1, p. 47:

$$x = N \cos \varphi , \tag{1}$$

where we assume that $N = \frac{c}{V}$ has already been computed. With this, we also have the arc of parallel for the longitude λ:

$$L = x \frac{\lambda}{\rho} = N \cos \varphi \frac{\lambda}{\rho} = \frac{\lambda}{[2]} \cos \varphi . \tag{2}$$

We take the second or the third of these forms if we aim to use N or $[2] = \frac{\rho}{N}$ from our tables of the Appendix, pp. [12] to [39]. In order to compute more accurately, for example, to 10 places, we have to take $log\ V$ from the special table for it on pp. [2] to [7] of the Appendix, and then we have:

$$L = \frac{c}{\rho} \frac{\cos \varphi \, \lambda}{V} , \tag{3}$$

where the following is valid for λ in degrees, minutes or seconds:

	for degrees	for minutes	for seconds
$\log \frac{c}{\varrho} =$	5.047 9750·111	3.269 8237·607	1.491 6725·103 .

According to these, the following values are computed, for possible further applications with more places than usually needed.

Arcs of Parallel

φ	$\lambda = 1°$	$\lambda = 1'$	$\lambda = 1''$
	m	m	m
45°	78837.29341	1313.954890	21.89924817
46	77453.91115	1290.898519	21.51497532
47	76046.76765	1267.446128	21.12410212
48	74616.28344	1243.604724	20.72674540
49	73162.88715	1219.381452	20.32302421
50	71687.01462	1194.783577	19.91305962
51	70189.10917	1169.818486	19.49697477
52	68669.62128	1144.493688	19.07489480
53	67129.00870	1118.816812	18.64694685
54	65567.73593	1092.795599	18.21325998
55	63986.27472	1066.437912	17.77396520

Various tabular values of computed arcs of parallel are given in our Appendix on pp. [45] to [46], [48] and [49].

In addition to degrees, minutes and seconds, the arcs of parallel are also reduced to measure of time, hours, minutes and seconds; this corresponds to astronomical purposes. Therefore, a table is given on p. [63] for the conversion of arcs in time and vice versa, and on p. [48] there are given the arcs of parallel for 1' and 1" of arc, in addition however, also for 1 minute and 1 second of time, as approximate values, which are useful, for instance, for astronomical place determinations on travels.

Section 43. Area of the Terrestrial Ellipsoid

For the determination and representation of the area, we imagine the ellipsoid to be divided by meridians and parallels into trapezoids whose differential formula can easily be indicated.

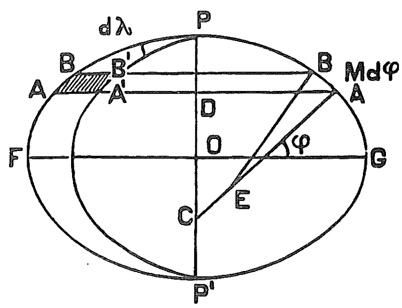

Fig. 1.

$AC = N \quad , \quad AE = M$

$DA = N \cos\varphi \qquad AB = M\,d\varphi$

$AA' = DA\,d\lambda$

$\quad = N \cos\varphi\, d\lambda\,.$

Considered as a differential, the trapezoid $ABB'A'$ has the area $dT = AB \times AA'$, i.e.:

$$dT = MN \cos\varphi\, d\lambda\, d\varphi\,, \tag{1}$$

and the whole zone $AABB$ with $\lambda = 2\pi$ between the latitudes φ and $d\varphi$ becomes:

$$dZ = 2MN\pi \cos\varphi\, d\varphi\,. \tag{2}$$

Since $MN = r^2$, the formula (2) can also be written thus:

$$dZ = 2 r^2 \pi \cos \varphi \, d\varphi \,. \tag{3}$$

If we set for r^2 its value from (28), section 38, p. 51, and at the same time $a^2(1 - e^2) = b^2$, then we will have:

$$dZ = 2 b^2 \pi \frac{\cos \varphi}{(1 - e^2 \sin^2 \varphi)^2} d\varphi \,. \tag{4}$$

This is the area differential of a zone of the ellipsoid between the latitude φ and $\varphi + d\varphi$; therefore the zone area itself, in general:

$$Z = 2 b^2 \pi \int \frac{\cos \varphi}{(1 - e^2 \sin^2 \varphi)^2} d\varphi \,. \tag{5}$$

We can develop here:

$$\frac{1}{(1 - e^2 \sin^2 \varphi)^2} = (1 - e^2 \sin^2 \varphi) - 2 = 1 - \left(\frac{-2}{1}\right) e^2 \sin^2 \varphi + \left(\frac{(-2)(-3)}{2}\right)^{*)} e^4 \sin^4 \varphi - \ldots$$
$$= 1 + 2 e^2 \sin^2 \varphi + 3 e^4 \sin^4 \varphi + 4 e^6 \sin^6 \varphi + 5 e^8 \sin^8 \varphi + \ldots$$

[*) Jordan has here falsely $\left(\frac{-2-3}{1-2}\right)$]

The function to be integrated is therefore, from (5):

$$\frac{\cos \varphi}{(1 - e^2 \sin^2 \varphi)^2} = \cos \varphi + 2 e^2 \cos \varphi \sin^2 \varphi + 3 e^4 \cos \varphi \sin^4 \varphi + 4 e^6 \cos \varphi \sin^6 \varphi + \ldots$$

These terms can be directly integrated individually, for we have in general:

$$\int \cos \varphi \sin^n \varphi \, d\varphi = \frac{1}{n+1} \sin^{n+1} \varphi \,,$$

and hence, with repeated application of this integral:

$$\int \frac{\cos \varphi}{(1 - e^2 \sin^2 \varphi)^2} d\varphi = \sin \varphi + \frac{2}{3} e^2 \sin^3 \varphi + \frac{3}{5} e^4 \sin^5 \varphi + \frac{4}{7} e^6 \sin^7 \varphi + \ldots$$

Therefore, if we turn back to (4) and introduce the limits 0 and φ, then we obtain the area of the zone from the equator to the latitude φ:

$$Z \Big]_0^{\varphi} = 2 b^2 \pi \left(\sin \varphi + \frac{2}{3} e^2 \sin^3 \varphi + \frac{3}{5} e^4 \sin^5 \varphi + \frac{4}{7} e^6 \sin^7 \varphi + \frac{5}{9} e^8 \sin^9 \varphi + \ldots \right). \tag{6}$$

If we aim to have this value itself for various φ's, then we can compute straight on according to this; for surfaces of zones between any two latitudes φ_1 and φ_2, however, it is better to transform $\sin^3 \varphi$ into $\sin 3\varphi$, etc., for according to section 35, p. 28:

$$\sin^3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi$$

$$\sin^5\varphi = \frac{5}{8}\sin\varphi - \frac{5}{16}\sin 3\varphi + \frac{1}{16}\sin 5\varphi$$

$$\sin^7\varphi = \frac{35}{64}\sin\varphi - \frac{21}{64}\sin 3\varphi + \frac{7}{64}\sin 5\varphi - \frac{1}{64}\sin 7\varphi$$

$$\sin^9\varphi = \frac{63}{128}\sin\varphi - \frac{21}{64}\sin 3\varphi + \frac{9}{64}\sin 5\varphi - \frac{9}{256}\sin 7\varphi + \frac{1}{256}\sin 9\varphi$$

$$\sin^{11}\varphi = \frac{231}{512}\sin\varphi - \frac{165}{512}\sin 3\varphi + \frac{165}{1024}\sin 5\varphi - \frac{55}{1024}\sin 7\varphi + \frac{11}{1024}\sin 9\varphi - \frac{\sin 11\varphi}{1024}$$

With these, (5) becomes:

$$Z\Big]_0^{\varphi} = 2b^2\pi(A\sin\varphi - B\sin 3\varphi + C\sin 5\varphi - D\sin 7\varphi + E\sin 9\varphi - F\sin 11\varphi), \qquad (7)$$

where the coefficients A, B, etc., are the following:

$$\left.\begin{aligned}
A &= 1 + \frac{1}{2}e^2 + \frac{3}{8}e^4 + \frac{5}{16}e^6 + \frac{35}{128}e^8 + \frac{63}{256}e^{10} = 1.00335\,39847{,}9231\\
B &= \frac{1}{6}e^2 + \frac{3}{16}e^4 + \frac{3}{16}e^6 + \frac{35}{192}e^8 + \frac{45}{256}e^{10} = 0.00112\,08040{,}9276\\
C &= \frac{3}{80}e^4 + \frac{1}{16}e^6 + \frac{5}{64}e^8 + \frac{45}{512}e^{10} = 16892{,}6070\\
D &= \frac{1}{112}e^6 + \frac{5}{256}e^8 + \frac{15}{512}e^{10} = 26{,}9384\\
E &= \frac{5}{2304}e^8 + \frac{3}{512}e^{10} = {,}0438\\
F &= \frac{3}{5632}e^{10} = {,}0001.
\end{aligned}\right\} \qquad (8)$$

The calculation was carried out with Bessel's value $\log e^2 = 7.824\;4104{\cdot}237$.

Now if we aim to have the zone between two latitudes φ_1 and φ_2, we have in (7) the differences:

$$A(\sin\varphi_2 - \sin\varphi_1) = 2A\sin\frac{\varphi_2 - \varphi_1}{2}\cos\frac{\varphi_1 + \varphi_2}{2}, \text{ etc.}$$

In order to abbreviate, we shall write thereby:

$$\varphi_2 - \varphi_1 = \Delta\varphi, \quad \frac{\varphi_1 + \varphi_2}{2} = \varphi.$$

With these, the area of the zone of the width $\Delta\varphi$ and with the mean latitude φ becomes according to (7):

$$\left.\begin{aligned}
Z = 4b^2\pi\Big\{&A\cos\varphi\sin\frac{\Delta\varphi}{2} - B\cos 3\varphi\sin 3\frac{\Delta\varphi}{2}\\
&+ C\cos 5\varphi\sin 5\frac{\Delta\varphi}{2} - D\cos 7\varphi\sin 7\frac{\Delta\varphi}{2}\\
&+ E\cos 9\varphi\sin 9\frac{\Delta\varphi}{2} - \ldots\ldots \quad \ldots\Big\}
\end{aligned}\right\} \qquad (9)$$

Formula (9) with the coefficients (8) yields with $\Delta\varphi = 1°$ the area of a ring of 1° breadth which goes around the whole earth, i.e. has 360° length. More frequently than the area of this whole ring, we need the 360th part of it, i.e. a "Gradabteilung" [one-degree division] or a trapezoid which is limited by two meridians and by two parallels, both at a 1° interval.

The curved area of such a one-degree quadrangle with the mean latitude φ is therefore

$$G = \frac{b^2\pi}{90}\Big\{A\, sin\, 30'\, cos\, \varphi - B\, sin\, 1°\, 30'\, cos\, 3\,\varphi + C\, sin\, 2°\, 30'\, cos\, 5\,\varphi - D\, sin\, 3°\, 30'\, cos\, 7\,\varphi + E\, sin\, 4°\, 30'\, cos\, 9\,\varphi - \ldots\Big\} \tag{10}$$

If we compute all constants here, then we find for a square kilometer

$$\left.\begin{aligned} G = {} & 12347.58347\; cos\, \varphi && (log\, \text{coeff.} = 4.091\,5819{\cdot}705)\\ - {} & 41.37468\; cos\, 3\,\varphi && (log\, \text{coeff.} = 1.616\,7346{\cdot}5)\\ + {} & 0.103911\; cos\, 5\,\varphi && (log\, \text{coeff.} = 9.016\,662 - 10)\\ - {} & 0.000232\; cos\, 7\,\varphi && (log\, \text{coeff.} = 6.365\,28 - 10)\\ + {} & 0.\ldots\; cos\, 9\,\varphi && (log\, \text{coeff.} = 3.678 - 10)\,. \end{aligned}\right\} \tag{11}$$

The plane-table sheets of the Prussian topography at the scale 1:25,000 have in latitude $\Delta\varphi = 6'$ and in longitude 10', and for this we have:

$$G' = \frac{b^2\pi}{540}\Big\{A\, sin\, 3'\, cos\, \varphi - B\, sin\, 9'\, cos\, 3\,\varphi + C\, sin\, 15'\, cos\, 5\,\varphi - D\, sin\, 21'\, cos\, 7\,\varphi\Big\} \tag{12}$$

or with computed coefficients, for a square kilometer:

$$G' = 205.79564\; cos\, \varphi - 0.689656\; cos\, 3\,\varphi + 0.001732\; cos\, 5\,\varphi - 0.0000039\; cos\, 7\,\varphi\,. \tag{13}$$

The logarithms of these coefficients are:

2.313 4361·8 9.838 6325 7.238 647 4.5874 .

The values calculated from these are given in our table of the Appendix, page [49].

In this connection, let us mention: E. Roedel, *Schlömilchs Zeitschr. f. Math. u. Physik*, 38th year, 1893, pp. 56-60.

Integration in a closed form

In the above development we have treated the integration at once in a series because by so doing we have been led in the shortest way to the formulae (6) and (8) which are the most convenient ones for practical computing.

The integration of (4), however, can also be carried out rigorously in a closed form whereby we obtain a mathematically more elegant formula which is less convenient for numerical application, however, than are the developed series. The integration (whose details were carried out in the former 3rd edition, 1890, pp. 227-228) yields:

$$Z\Big]_0^{\varphi} = b^2\pi\left\{\frac{sin\,\varphi}{W^2} + \frac{1}{e}\,l\left(\frac{1 + e\, sin\,\varphi}{W}\right)\right\}. \tag{14}$$

If we set $\varphi = 90°$ here, we will have $W^2 = 1 - e^2$; therefore:

$$Z\Big]_0^{90} = b^2\pi\left\{\frac{1}{1-e^2} + \frac{1}{e}\, l\, \frac{1+e}{\sqrt{1-e^2}}\right\} \quad , \quad \frac{1+e}{\sqrt{1-e^2}} = \sqrt{\frac{1+e}{1-e}}$$

$$2\,Z\Big]_0^{90} = E = 2\,a^2\pi\left\{1 + \frac{1-e^2}{2\,e}\,\frac{1}{\mu}\,log\,\frac{1+e}{1-e}\right\}.$$

This must agree with the formula (6), if we set in it $\varphi = 90°$, whereby we obtain:

$$E = 2\,b^2\pi\left(1 + \frac{2}{3}\,e^2 + \frac{3}{5}\,e^4 + \frac{4}{7}\,e^6 + \frac{5}{9}\,e^8 + \ldots\right). \tag{15}$$

The computation yields in agreement with both formulae

$$E = 509{,}950{,}714.2 \text{ km}^2\ . \tag{16}$$

Now if we imagine a sphere of the radius f, which is to have the same area E, then f is determined in the following way:

$$f = \sqrt{\frac{E}{4\,\pi}} = 6{,}370{,}289.511 \text{ m}\ . \tag{17}$$

Section 44. Mean Radius of the Earth as Sphere

The last consideration leads us further to the question of which radius we are to assign a sphere which can be used instead of the ellipsoid for some approximate computations, etc.

The next thought is to use the arithmetic mean of the three semiaxes of the ellipsoid for this purpose, i.e. to set:

$$\frac{a + a + b}{3} = r\ , \tag{1}$$

$$\left.\begin{aligned} a &= 6{,}377{,}397.155 \text{ m} \\ a &= 6{,}377{,}397.155 \\ b &= 6{,}356{,}078.963 \end{aligned}\right\} \quad \frac{a + a + b}{3} = 6{,}370{,}291.091 \text{ m}\ . \tag{2}$$

We also can develop this value r from (1) as a series, namely:

$$r = \frac{2\,a + a\sqrt{1-e^2}}{3} = \frac{a}{3}\left(2 + 1 - \frac{1}{2}\,e^2 - \frac{1}{8}\,e^4 - \frac{1}{16}\,e^6 - \ldots\right)$$

$$r = a\left(1 - \frac{1}{6}\,e^2 - \frac{1}{24}\,e^4 - \frac{1}{48}e^6 - \ldots\right). \tag{3}$$

In the second place, we can study the sphere introduced at the end of the previous section 43, which has the same area E as the terrestrial ellipsoid. From the series for E in (15), section 43, it follows that the radius f of the sphere in question must be:

$$f = a\sqrt{1-e^2}\sqrt{1+\frac{2}{3}e^2+\frac{3}{5}e^4+\frac{4}{7}e^6+\dots}$$

$$f = a\left(1-\frac{e^2}{2}-\frac{e^4}{8}-\frac{e^6}{16}\right)\left(1+\frac{1}{3}e^2+\frac{11}{45}e^4+\frac{193}{945}e^6\right)$$

$$f = a\left(1-\frac{1}{6}e^2-\frac{17}{360}e^4-\frac{67}{3024}e^6\right). \tag{4}$$

The comparison with (3) yields:

$$f = r\left(1-\frac{1}{180}e^4-\frac{17}{7560}e^6\right). \tag{5}$$

The calculation according to it yields:

$$f = 6{,}370{,}291.091\text{ m} - 1.577\text{ m} - 0.004\text{ m} = 6{,}370{,}289.510\text{ m}. \tag{6}$$

This is in sufficient agreement with the value (17), section 43, p. 82, computed previously in two other ways:

The radius k of the sphere which has the same cubic volume as the terrestrial ellipsoid presents itself as third mean value.

The volume of the ellipsoid of rotation is found, as is known, by imagining a sphere with the equatorial radius a, therefore with the volume $\frac{4}{3}\pi a^3$ compressed in the direction of the axis of rotation at the ratio $b{:}a$, i.e. we have:

$$\text{Cubic volume of the terrestrial ellipsoid} = \frac{b}{a}\left(\frac{4}{3}\pi a^3\right) = \frac{4}{3}\pi a^2 b.$$

If a sphere of the radius k is to have the same volume, then we must have:

$$k^3 = a^2 b \qquad \text{or} \qquad k = \sqrt[3]{a^2 b} = a\sqrt{1-e^2}. \tag{7}$$

We can develop this:

$$k = a\left(1-\frac{1}{6}e^2-\frac{5}{72}e^4-\frac{55}{1296}e^6\right). \tag{8}$$

Taking again the arithmetic mean r of the three semiaxes from (3) for comparison and developing it, we obtain:

$$k = r\left(1-\frac{1}{36}e^4-\frac{17}{648}e^6\right). \tag{9}$$

The computation yields:

$$k = 6{,}370{,}291.091\text{ m} - 7.8828\text{ m} - 0.0497\text{ m} = 6{,}370{,}283.158\text{ m}. \tag{10}$$

This is also in agreement with a direct computation according to (7).

For an over-all view, we put down together once again the three values found:

1. Arithmetic mean $\frac{a+a+b}{3} = r = 6{,}370{,}291.091$ m ,

2. Radius for equal area $f = 6{,}370{,}289.510$ m ,

3. Radius for equal volume $\sqrt[3]{a^2 b} = k = 6{,}370{,}283.158$ m .

These values are nearly equal, as we see, and also equally suitable for many purposes.

In his paper, *Abbildung krummer Oberflächen*, Braunschweig, 1858, p. 41, Dienger found the theorem for all radii of curvature of the whole earth that the arithmetic mean of all radii of curvature is equal to the major semiaxis a.

The whole earth is hereby taken into consideration; however, if we aim to replace by a sphere only a limited part, say, only the neighborhood of a point at the latitude φ, then it is a question of a mean value of the radii of curvature at all azimuths from a point, and for this we already have introduced, without a special theory, the "mean" radius of curvature $r = \sqrt{MN}$ in (27), section 38, p. 51.

Although it is not possible without a theory of the geodetic line to base this choice of r better, yet let us also mention here a theorem by Grunert (cf. the references to literature at the end) that the mean radius of curvature $r = \sqrt{MN}$ is at the same time the arithmetic mean of all normal section radii of curvature R at a point. This may be proved thus:

We have the sum of all values R from (1), section 39, p. 53:

$$[R] = \int_0^{2\pi} \frac{MN}{M \sin^2 \alpha + N \cos^2 \alpha} \, d\alpha ,$$

and their number is, corresponding to $n = 2\pi$, hence, the mean value:

$$\frac{[R]}{n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{MN}{M \sin^2 \alpha + N \cos^2 \alpha} \, d\alpha .$$

For the integration we introduce a new variable:

$$\sqrt{\frac{M}{N}} \tan \alpha = v \text{, and hence } \sqrt{\frac{M}{N}} \frac{d\alpha}{\cos^2 \alpha} = dv ,$$

whereby the integration is reduced to:

$$\int \frac{dv}{1+v^2} = \operatorname{arc\,tan} v .$$

And if, in addition, we introduce the limits, then we find:

$$\frac{[R]}{n} = \sqrt{MN} = r . \qquad (11)$$

A second theorem by Grunert reads:

The arithmetic mean of the reciprocal radii of curvature of all normal sections at any one point of any ellipsoid is the arithmetic mean between the reciprocal smallest and largest radius of curvature at this point.

If we bring in the radius of curvature for the mean azimuth $\alpha = 45°$ of section 39, p. 54, then we have accordingly, with $n = 2\pi$ for the summation of integrals:

$$\frac{1}{n}\left[\frac{1}{R}\right] = \frac{1}{2}\left(\frac{1}{M} + \frac{1}{N}\right) = \frac{1}{R_{45}}.$$

These Grunert theorems are developed in *Grunerts Archiv der Mathematik und Physik*, Part 40, 1863, pp. 259-354, especially p. 312, and Part 41, 1864, pp. 241-296, especially p. 292.

Reference is further made here to Helmert, *Die mathem. u. physikal. Theorien der höheren Geodäsie*, I, Leipzig, 1880, pp. 63-68; Czuber, in regard to "Mittelwerte, die Krümmung ebener Kurven und Krümmungsflächen"; Grunert-Hoppes, *Archiv der Math. u. Ph.*, Second Series, Part 6, 1888, pp. 294-304.

Section 45. Auxiliary Tables for Geodetic Computations with Bessel's Terrestrial Dimensions

Numerous tabular computations have already been founded on Bessel's data for the terrestrial dimensions, as the following summary shows:

Encke, "Über die Dimensionen des Erdkörpers nebst Tafeln nach Bessels Bestimmung," *Berl. Astr. Jahrb. f. 1852*, pp. 318-381, and the separate copy, *Enckes Astr. Abhandlungen*, Vol. II, Berlin, 1866. These Encke tables furnish first the geocentric latitude and the geocentric radius, then *log* $(N: a)$, meridian degrees and parallel degrees and degrees perpendicularly on the meridian in toises. Furthermore, Table II, meridian arcs from the equator to the latitude φ in toises to 0.001 toise.

Steinhauser, "Neue Berechnung der Dimensionen des Erdsphäroids," *Petermanns Geogr. Mitteilungen*, 1858, pp. 465-468.

Bremiker, *Logarithmisch-trigonometrische Tafeln mit sechs Dezimalstellen*, 1881, pp. 520-524. Graticules.

Bremiker, *Studien über höhere Geodäsie*, Berlin, 1869, pp. 70-81. Radii of curvature for different latitudes and azimuths.

Projection tables for the use of the United States Navy, Bureau of Navigation. Washington. Government Printing Office, 1869. Polyconic projection.

Wagner, "Die Dimensionen des Erdsphäroids nach Bessels Elementen," *Geographisches Jahrbuch*, hrsg. von Behm. Vol. III. Gotha, 1870, pp. I-LXI. Graticules, etc.

F. G. Gauss, *Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst*, Berlin, 1876, and fourth edition, Stuttgart, 1922, Part II, pp. 4-27, from $\varphi = 44°$ to $\varphi = 56°$ meridian arc, *log* M, *log* N, etc.

Schreiber, *Rechnungsvorschriften für die trigonometrische Abteilung der Landesaufnahme*. Formeln und Tafeln zur Berechnung der geographischen Koordinaten aus den Richtungen und Längen der Dreieckseiten. Erste Ordnung. Berlin, 1878. Published by the author; to be obtained through E. S. Mittler & Sohn, Booksellers to the Court, Kochstrasse 69, 70.

These tables yield from $\varphi = 47°$ to $\varphi = 57°$ with an interval of 1' 8-place *log* (1) . . . *log* (8), where (1), (2) . . . , converted into our notations,

$$\eta^2 = e'^2 \cos^2 \varphi = \frac{e^2}{1 - e^2} \cos^2 \varphi, \text{ etc.,}$$

have the following meanings:

$$(1) = \frac{\varrho}{M}, \quad (2) = \frac{\varrho}{N}, \quad (3) = \frac{V^2}{2\varrho}, \quad (4) = \frac{3}{2}\frac{\mu}{N}\eta^2 t, \quad (5) = \frac{\mu}{3r^2}$$

$$(6) = \frac{\mu\,\eta^2}{2\,e^2}(t^2 - 1), \quad (7) = \frac{\mu\,\eta^2}{6\,\varrho^2}(3 + 2\,t^2), \quad (8) = \frac{\mu\,\eta^2}{12\,\varrho^2}(31 + 3\,t^2).$$

The same values *log* (1) to *log* (4) to 7 places are published as *Rechnungsvorschriften für die trigonometrische Aufnahme der Reichsschutzgebiete*, Berlin, 1891, for the latitudes $\varphi = 0°$ to $\varphi = 13°$.

Corresponding tables for $\varphi = 47°$ to $57°$ for second order to 7 places and for third order to 6 places.

Albrecht, *Formeln und Hilfstafeln für geographische Ortsbestimmungen nebst kurzer Anleitung zur Ausführung derselben*, von Prof. Dr. Th. Albrecht, Abteilungsvorsteher am Königl. Preuss. Geodätischen Institut. 4th Edition, Leipzig, 1908. Tables on the shape of the earth, pp. 265-295. (Cf. *Zeitschrift für Vermessungswesen*, 1910, pp. 30-31.)

Helmert, *Die mathematischen und physikalischen Theorien der höheren Geodäsie*, Part I, Leipzig, 1880. Appendix, pp. 621-631, yields $\log W$ from $\varphi = 47°0'$ to $57°0'$ with an interval of $5'$ to 0.0001, furthermore $\log W$ to 8 places to an accuracy of 0.1, through the whole quadrant with $\Delta\varphi = 10'$.

Biek-Tillo, Russian translation of Jordan, *Handbuch der Vermessungskunde*, second edition, translated by A. Biek, first-class teacher of geodesy at the Messinstitut des Grossfürsten Konstantin, Moscow, 1881. N. J. Mamontowa, Bookseller. Instead of the table, pp. 424-427 of the original, this translation in its turn furnishes on pp. 651-665 a table of the coefficients for the mean latitude formulas by Gauss computed by A. A. Tillo, Colonel of the Russian General Staff; especially $\log [1]$ and $\log [2]$ for $\varphi = 34°0'$ to $\varphi = 70°0'$ with interval $10'$ to an accuracy of 0.1. Another unit of length than that by Bessel, however, is based upon here, for the Russian $\log [1]$ and $\log [2]$ have a constant difference of 371.6 against our $\log [1]$ and $\log [2]$ computed with Bessel's dimensions of the earth.

Rehm, *Mitteilungen des k. k. Milit.-geograph. Instituts*, issued by the order of the k. k. Reichskriegsministerium, Volume III, 1883, Wien, 1883. Author and publisher: K. k. Milit.-geograph. Institut, pp. 137-177. Tables of the radii of curvature of the spheroid of the earth by Bessel for the latitudes of $\varphi = 40°0'$ to $51°30'$ with an interval of $1'$ to 0.0001 (cf. Hartl below).

Schols, *Geodetische Formules en Tafels, ten gebruike bij de Triangulatie van het eiland Sumatra.* Utrecht, J. van Boeckhoven, 1884. These tables furnish the radii of curvature to 0.1 from $\varphi = 0°0'$ to $6°0'$, besides further numerical values.

Hermann Wagner's *Tafeln der Dimensionen des Erdsphäroids, auf Minutendekaden erweitert*, by A. Steinhauser, k. k. Regierungsrat, Wien, 1885. Eduard Hölzel.

Helmert, *Veröffentlichung des Kgl. Preuss. Geodätischen Instituts.* Lotabweichungen. Heft 1. Formeln und Tafeln, etc., Berlin; printed and published by P. Stankiewicz, printer, 1886. Tables in the Appendix, pp. 6-26; of these, pp. 18-24 furnish 8-place values $\log [1]$ and $\log [2]$, which are the decadal supplements of our $\log [2]$ and $\log [1]$, for $\varphi = 30°$ to $71°$.

Hartl, Tables containing the dimensions of the meridian and parallel arcs, then the logarithms of the radii of curvature of Bessel's ellipsoid of the earth, computed under the direction of Lieutenant Colonel H. Hartl in the geodetic division of the k. k. Milit.-geograph. Instituts. Reprint from the *Mitteilungen des k. k. Milit.-geograph. Instituts*, Vol. XIV, Wien, 1895. (Cf. *Zeitschr. f. Verm.*, 1896, pp. 28-30.)

Heuvelink, *Rijksdriehoeksmeting.* Formules en Tafels voor de berekening van de geografische breedten en lengten der hoekpunten en van de azimuts der zijden van het driehoeksnet, Delft, 1903.

Tafeln für Berechnungen in konformen Gauss'schen Meridianstreifen mit Benutzung der Rechenmaschine. Issued by the Bundesvermessungs-Amt, Wien, 1920.

Thilo, *Anweisungen und Tafeln zur Berechnung Gauss-Krügerscher Koordinaten.* Computational instructions for the trigonometric division of the Reichsamt für Landesaufnahme, Berlin, 1924.

Preussisches Finanzministerium, Katasterverwaltung, (XI.) *Anweisung vom 11. März 1932 für die Umformung geographischer, sphäroidischer und konformer Koordinaten.*

Lips, *Formeln und Tafeln zur Berechnung der ellipsoidischen, der konformen und der geographischen Koordinaten mit der Rechenmaschine*, Stuttgart, 1932.

All these tabular works are referred to Bessel's ellipsoid. For computations on the international ellipsoid we mention the following tables:

Y. Väisälä, "Tafeln für geodätische Berechnungen nach den Erddimensionen von Hayford," *Veröff. des Finnischen Geodätischen Instituts*, Nr. 1., Helsinki, 1923. The tables contain, among other things, the values $\log [1]$, $\log [2]$, $\log V$, $\log M$, $\log N$ and the arc of the meridian B for the latitudes $50°$ to $70°$ (according to our denotations).

Association de Géodésie de l'Union Géodesique et Géophysique Internationale. Publication spéciale Nr. 2. Tables de l'Ellipsoïde de Référence International adopté par l'assemblée générale de Madrid le 7 Octobre 1924 dans le système de la division sexagésimale de la circonférence, calculées sous la direction du Général G. Perrier par E. Hasse. 2[me] édition corrigée, Paris. Au sécrétariat de l'association, 19 rue Auber (9[e]), 1935. This fundamental work contains the logarithms of the radii of curvature of the auxiliary magnitudes V and W and the meridian arcs for all latitudes from $0°$ to $90°$ for every minute.

Coast and Geodetic, Spec. Publ. No. 200. Formulas and Tables for the computation of Geodetic Positions on the International Ellipsoid. By Walter D. Lambert and Clarence H. Swick, Washington, 1935. This very handy, small work furnishes, among other things, the logarithms of the main radii of curvature as well as the meridian arcs from $0°$ to $90°$ latitude for every minute.

In the Appendix of this volume, page [2] and following, numerous auxiliary tables are communicated which have for this purpose been computed newly and independently or, at least, the portion borrowed has been revised.

The geodetic basic function V or $\log V$, as the case may be, on pp. [2] to [7] has been computed newly and independently with the constants of the land survey (section 37, p. 44) according to the formulas indicated at the end page [7], as is shown in detail in section 40; the computation is conducted to 12-13 places and then rounded off to ten places.

The table, pp. [12] to [33], also is computed newly and independently for every $1°$; in the case of the interpolation, however, the tables by Schols and Helmert have been used at the places $0°$ to $6°$ and $47°$ to $57°$.

The special table for $\log [1]$ and $\log [2]$ on pp. [34] to [39] has only been newly computed from $45°$ to $46°$ and is a revised reprint from Schreiber's *Rechenvorschriften der Landesaufnahme* from $47°$ to $56°$.

The table, pp. [45] to [46], for the degrees of longitude and latitude and for the areas of the graticule has first been set up according to Bremiker and Wagner, but then recomputed in detail; the few errors found in this process are communicated in *Geographisches Jahrbuch* by Behm, Volume VI, 1876, p. 703.

The table of the arcs of meridian, page [40], is an extract from the larger table by F. G. Gauss from $44°$ to $56°$. The part $40°$ to $44°$ is computed in addition.

A new table of arcs of meridian from $45°$ to $57°$, computed for every minute, as indicated in section 41, p. 76 and p. 77, is given on pp. [41] to [44].

Information about further tables is given at the pertinent places of the text.

Summary of the principal symbols in the auxiliary tables of the Appendix

φ = geographic latitude

$$W = \sqrt{1 - e^2 \sin^2 \varphi}$$

$$V = \sqrt{1 + e'^2 \cos^2 \varphi} = \sqrt{1 + \eta^2} \quad , \quad \eta^2 = e'^2 \cos^2 \varphi = \frac{e^2}{1 - e^2} \cos^2 \varphi$$

$$M = \frac{a\,(1 - e^2)}{W^3} \quad \text{or} \quad = \frac{c}{V^3}$$ radius of curvature in the meridian

$$N = \frac{a}{W} \quad \text{or} \quad = \frac{c}{V}$$ radius of curvature in the prime vertical, $\quad \frac{N}{M} = V^2$

$$r = \sqrt{M N} = \frac{c}{V^2}$$ mean radius of curvature

$$[1] = \frac{\varrho''}{M}$$ coefficient of curvature in the meridian

$$[2] = \frac{\varrho''}{N}$$ coefficient of curvature in the prime vertical.

Chapter IV

COMPUTATION OF THE SPHERICAL TRIANGLE

Section 46. The Spherical Excess

In the case of the computation of the spherical triangle, the radius of the sphere is assumed according to the explanation indicated in (27), section 38, p. 51, and once again at the end of section 44, p. 84, to be:

$$r = \sqrt{MN} = \frac{c}{V^2} . \tag{1}$$

With this, the spherical excesses of the triangles are first computed.

The sum of the three angles of a spherical triangle is always larger than 180°; the surplus of the sum of angles above 180° is called the spherical excess.

If we denote the angles by α, β, γ and the spherical excess by ε, then we have hence the equation:

$$\varepsilon = \alpha + \beta + \gamma - 180° . \tag{2}$$

If the three angles α, β, γ are measured, then we find, according to these, also the excess ε, which is affected by the errors of measurement of the α's, β's, γ's, however; it is therefore desirable to have an independent rigorous determination of ε, which is independent of the small errors of measurement of the angles and, on the contrary, should serve the purpose of testing the sum of these angles α, β, γ, as, e.g., has happened in the case of the adjustment by condition equations in our Volume I, 8th Edition, 1935, sections 68 and 70, for the determination of the errors of triangle closure.

We obtain such an independent determination of the excess ε by the fundamental theorem that the excess is proportional to the area of the spherical triangle F, namely:

$$\varepsilon = \frac{F}{r^2} \rho . \tag{2a}$$

We can prove this fundamental theorem with the help of spherical lunes, and because of its importance we mention the well-known elementary proof of the theorem here:

By a lune we understand the area between two great circles, e.g. in Fig. 1 the area:

$$\text{lune } A\,C\,A'\,B\,A = (\alpha\,,\alpha) ;$$

now since the total area of the sphere is $= 4\pi r^2$, then the area

$$\text{lune } (\alpha\,,\alpha) = \frac{\alpha}{360}(4\,\pi\,r^2) .$$

If we apply this also to the other two lunes coming together in the triangle $A\,B\,C$, then we have:

$$\text{lune } (\beta\,,\beta) = \frac{\beta}{360}(4\,\pi\,r^2) ,$$

$$\text{lune } (\gamma\,,\gamma) = \frac{\gamma}{360}(4\,\pi\,r^2) ;$$

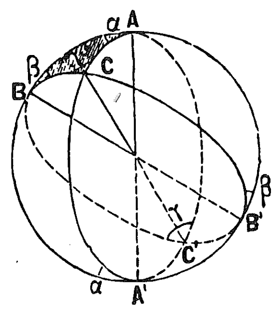

Fig. 1.

consequently the sum:

$$(\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma) = \frac{\alpha + \beta + \gamma}{360} 4 \pi r^2 . \qquad \text{(a)}$$

Now by introducing the area F of the spherical triangle $A B C$, we have according to the view of Fig. 1 (more illustrative still by designing on a spherical model):

$$\begin{array}{l} (\alpha, \alpha) = F + A' B C \\ (\beta, \beta) = F + B' A C \\ (\gamma, \gamma) = F + C' A B \\ \hline (\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma) = 3 F + A' B C + B' A C + C' A B . \end{array}$$

But now the triangle $C' A B$ lying on the other side of the spherical surface of Fig. 1 is area-equivalent to its vertical triangle $C A' B$ lying on this side; thus we have, by writing at the same time $3 F = 2 F + F$:

$$(\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma) = 2 F + F + A' B C + B' A C + C A' B' .$$

The last four terms of this equation yield together half the spherical surface $= 2 \pi r^2$, hence:

$$(\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma) = 2 F + 2 \pi r^2 . \qquad \text{(b)}$$

Now the equations (a) and (b) yield together:

$$F = (\alpha + \beta + \gamma - 180°) \frac{\pi}{180°} r^2 . \qquad \text{(c)}$$

Hence, if we apply the denotation ε according to (2), and if we write at the same time $\frac{180°}{\pi} = \varrho$, then we find from (c) the same equation as (2a), namely: (d), which was to be proved:

$$\alpha + \beta + \gamma - 180° = \varepsilon = \frac{F}{r^2} \varrho . \qquad \text{(d)}$$

Strictly speaking, by F we understand the curved (spherical) area of the triangle; instead of this, we also can use, with sufficient approximation, the area $\triangle$ of a plane triangle which is calculated from the sides of the spherical triangle, i.e. we have:

$$\text{Approximation } \varepsilon = \frac{\triangle}{r^2} \varrho . \qquad (3)$$

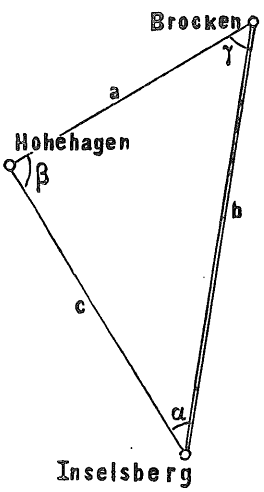

Fig. 2.
(Scale 1:2,000,000).

(We find the same also by the approximate use of the spherical-trigonometric formula for $tan \frac{\varepsilon}{4}$, which was mentioned in section 33, p. 17).

For a numerical example we shall use the triangle of Hannover, which is used several times as an example in the classical treatises by Gauss, namely the triangle represented in Fig. 2:

Inselsberg-Hohehagen-Brocken.

Let there be given:

the side Inselsberg-Brocken $b = 105{,}972.85$ m , (4)

further the angles of the triangle, approximate, and the geographic latitudes φ of the corner points likewise approximate:

Point	Angle of the Triangle		Geogr. Latitude
Inselsberg	$\alpha = 40° 39' 30''$	$(25'')$	$50° 51' 9''$
Hohehagen	$\beta = 86 \;\; 13 \;\; 59$	$(54'')$	$51 \;\; 28 \;\; 31$
Brocken	$\gamma = 53 \;\; 6 \;\; 46$	$(41'')$	$51 \;\; 48 \;\; 2$
	Sum $180° \;\; 0' 15''$	$(0'')$	$\varphi = 51° 22' 34''$ Mean.

(5)

The angles of the triangle indicated to 1" yield the sum 180°0' 15", i.e. an excess of 15" above 180°. Without knowing whether this is the spherical excess or whether this results from measuring errors, we distribute these 15" to the three angles, in order to have at least for the time being a consistent plane triangle computation, and obtain for the three angles, by so doing, the second values (25"), (54"), (41") added in parentheses in the case of (5) above.

With these angles and the base side b already indicated in the case of (4) we make an approximate preliminary triangle computation according to the law of sines of plane trigonometry:

$$a = \frac{b}{\sin\beta}\sin\alpha \quad , \quad c = \frac{b}{\sin\beta}\sin\gamma .$$

We compute here only with about 5- or 6-place logarithms:

$\log b$	5.025 195	$\log b$	5.025 195
co $\log\sin\beta$	0.000 940	co $\log\sin\beta$	0.000 940
$\log\sin\alpha$	9.813 933	$\log\sin\gamma$	9.902 983
$\log a$	4.840 068	$\log c$	4.929 118

Hence, we have now together:

$$\left.\begin{array}{ll} \alpha = 40° \, 39' \, 25'' & \log a = 4.840\,068 \\ \beta = 86 \;\; 13 \;\; 54 & \log b = 5.025\,195 \\ \gamma = 53 \;\; 6 \;\; 41 & \log c = 4.929\,118 . \end{array}\right\} \quad (6)$$

With these, we can compute the area of the triangle threefold, for we have, as we know:

$$\text{Area of the triangle } \triangle = \frac{1}{2} a\, b \sin\gamma = \frac{1}{2} a\, c \sin\beta = \frac{1}{2} b\, c \sin\alpha$$

		or		
$\log a$	4.840 068		$\log a$	4.840 068
$\log b$	5.025 195		$\log c$	4.929 118
$\log\sin\gamma$	9.902 983 — 10		$\log\sin\beta$	9.999 060 — 10
$\log 0.5$	9.698 970 — 10		$\log 0.5$	9.698 970 — 10
$\log\triangle$	9.467 216		$\log\triangle$	9.467 216 .

(7)

Furthermore, we need the mean radius of curvature for the mean latitude of the triangle. This mean latitude has already been indicated under (5): $\varphi = 51°22' 34''$, and with this, we take from the table on page [24] of the Appendix, by interpolation, the value $\log r$ or else at once:

$\log \frac{1}{r^2}$	6.390 076 — 20	
to this, adding $\log \varrho$	5.314 425	
and from (7) $\log \triangle$	9.467 216	
$\log \varepsilon$	1.171 717	$\varepsilon = 14.850''$. (8)

With these, the angles given under (5) above are confirmed in their sum, so far as they are put down only to an accuracy of 1". We shall learn the more accurate angles and the more accurate computation of the triangle sides in sections 47 and 48.

A few additional remarks can be made about the simple computation of the excess, which is given in the previous example in all thoroughness. For a triangle with the sides a, b and the included angle γ we have the excess:

$$\varepsilon = \frac{\rho}{2r^2} ab \sin \gamma \,, \tag{9}$$

and therefore, we write out, for more frequent use, the logarithms of $\frac{\rho}{2r^2}$ in the form of tables; for an over-all illustration, we put together:

$$\left.\begin{aligned} \varphi = 45° \quad & \log \frac{\varrho}{2r^2} = 1.40411 - 10 \\ 50 \quad & \log \frac{\varrho}{2r^2} = 1.40361 - 10 \\ 55 \quad & \log \frac{\varrho}{2r^2} = 1.40312 - 10 \,. \end{aligned}\right\} \tag{10}$$

For a further over-all view of the ratios we can also compute:

Area of the triangle	Spherical excess
1 sq km	$\varepsilon = 0.00507''$
1 sq mile, equilateral triangle with sides of 11-1/4 km	$\varepsilon = 0.279''$
200 sq km equilateral triangle with sides of 21-1/2 km	$\varepsilon = 1''$
Equilateral triangle with sides of 1° = 15 geographic miles = 111 km	$\varepsilon = 27''$

For small triangles up to an area of about 1 sq mile, with sides up to about 10 km, the excess thus remains far within the measuring accuracy; in such cases it is therefore often already disregarded.

The last assumption of a triangle of a 111-km side is truly the utmost for land surveys; even Gauss' triangle Inselsberg-Hohehagen-Brocken, which we used for an example in the case of (8), above, with $\varepsilon = 15''$, approximately, is one of the largest German triangles, which we have therefore already mentioned in the summary, p. 22, Chapter I (not included in this translation).

The largest triangles in Europe, namely the connecting triangles, represented on p. 22, Chapter I (not included in this translation), between Spain and Algiers across the Mediterranean Sea, have geodetic excesses of approximately 54", 1' 11", 44", 1' 0", in the several cases.

But in the case of such large triangles one must no longer compute only spherically; we shall return to this in the second half of this volume.

We consider a geodetic triangle with the sides a, b, c and the angles $\alpha, \beta, \gamma,$ as represented in Fig. 1.

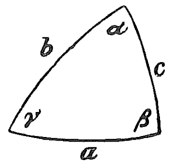

Fig. 1.
Spherical triangle.

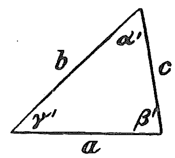

Fig. 2.
Plane triangle.

If the spherical triangle lies on a sphere of the radius r, then certain central angles of the earth correspond to the sides $a\,b\,c$, as is seen from the following summary:

$$\left.\begin{array}{lccc} \text{Side lengths in metric measure} \ldots\ldots & a, & b, & c \\ \text{Central angles of the earth in radian measure} \ldots & \frac{a}{r}, & \frac{b}{r}, & \frac{c}{r} \\ \text{Central angles of the earth in degree measure} \ldots\ldots & \frac{a}{r}\varrho, & \frac{b}{r}\varrho, & \frac{c}{r}\varrho. \end{array}\right\} \quad (1)$$

Now we could solve the spherical triangle with these central angles of the earth and the angles α, β, γ, which the arcs a, b, c form with one another on the spherical surface, according to the well-known rigorous formulae of spherical trigonometry; but for geodetic purposes one does not do that, because the central angles of the earth are very small, and therefore can be treated much more conveniently in developments in series and approximation formulae.

The oldest and most popular of these procedures is the theorem found by Legendre in Paris in 1787 and named after him:

> A small spherical triangle can be computed by approximation like a plane triangle with the same sides if one takes as the angles of the plane triangle the angles of the spherical triangle decreased by one-third of the spherical excess each.

To this theorem there corresponds the plane triangle drawn in Fig. 2, above, which has the same sides a, b, c as the spherical triangle, Fig. 1, above, and whose angles α', β', γ' are still left undetermined at first.

In order to prove the theorem mentioned, we write for the spherical triangle the equation of cosines:

$$\cos\frac{a}{r} = \cos\frac{b}{r}\cos\frac{c}{r} + \sin\frac{b}{r}\sin\frac{c}{r}\cos\alpha$$

$$\text{or} \qquad \cos\alpha = \frac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}}.$$

Now all small angles are developed according to powers (cf. the series formulae for $\sin x$ and $\cos x$, section 34, p. 23), namely to the 4th power including:

$$\cos\alpha = \frac{\left(1-\frac{a^2}{2r^2}+\frac{a^4}{24r^4}\right)-\left(1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}\right)\left(1-\frac{c^2}{2r^2}+\frac{c^4}{24r^4}\right)}{\left(\frac{b}{r}-\frac{b^3}{6r^3}\right)\left(\frac{c}{r}-\frac{c^3}{6r^3}\right)}.$$

If we multiply the parentheses occurring here, always neglecting thereby the terms of higher-than-the-fourth order, then we have:

$$\left(1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}\right)\left(1-\frac{c^2}{2r^2}+\frac{c^4}{24r^4}\right)=1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}-\frac{c^2}{2r^2}+\frac{b^2c^2}{4r^2}+\frac{c^4}{24r^4}$$

$$=1-\frac{b^2+c^2}{2r^2}+\frac{b^4+c^4}{24r^4}+\frac{b^2c^2}{4r^4}$$

$$\cos\alpha=\frac{\dfrac{b^2+c^2-a^2}{2r^2}+\dfrac{a^4-b^4-c^4}{24r^4}-\dfrac{b^2c^2}{4r^4}}{\dfrac{bc}{r^2}\left(1-\dfrac{b^2+c^2}{6r^2}\right)}.$$

The denominator $\left(1-\frac{b^2+c^2}{6r^2}\right)$ is taken into account with sufficient approximation by the fact that we add, instead, a factor $\left(1+\frac{b^2+c^2}{6r^2}\right)$ in the numerator, and if we omit, at the same time, everywhere a factor r^2 as common, we have:

$$\cos\alpha=\left(\frac{b^2+c^2-a^2}{2bc}+\frac{a^4-b^4-c^4-6b^2c^2}{24r^2bc}\right)\left(1+\frac{b^2+c^2}{6r^2}\right),$$

the two parentheses multiplied, with omission of all which goes above r^2, lead to the equation:

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}+\frac{a^4-b^4-c^4-6b^2c^2}{24r^2bc}+\frac{b^2+c^2-a^2}{2bc}\,\frac{b^2+c^2}{6r^2}. \tag{2}$$

Now the plane triangle, Fig. 2, p. 93, yields according to the law of cosines:

$$a^2=b^2+c^2-2bc\cos\alpha' \quad \text{or} \quad \cos\alpha'=\frac{b^2+c^2-a^2}{2bc}. \tag{3}$$

This together with (2) yields:

$$\cos\alpha=\cos\alpha'+\frac{a^4-b^4-c^4-6b^2c^2}{24r^2bc}+\frac{b^4+c^4+2b^2c^2-a^2b^2-a^2c^2}{12r^2bc}.$$

The two parts collected yield:

$$\cos\alpha=\cos\alpha'+\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2}{24r^2bc}. \tag{4}$$

We let this equation stand at first and consider the numerator of the fraction; this numerator is closely related to the area $\triangle$ of the plane triangle. We have, as is known:

$$\triangle=\sqrt{s(s-a)(s-b)(s-c)},$$

where $s=\dfrac{a+b+c}{2}$, $s-a=\dfrac{-a+b+c}{2}$ etc.,

consequently $$\triangle^2=\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right).$$

We have here: $(a+b+c)(-a+b+c) = -a^2+b^2+c^2+2bc$

and $(a-b+c)(+a+b-c) = a^2-b^2-c^2+2bc$,

consequently $16\triangle^2 = (-a^2+b^2+c^2+2bc)(a^2-b^2-c^2+2bc)$,

and by multiplying and arranging also these two parentheses (where all terms with odd powers cancel out), we find:

$$16\triangle^2 = -a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2. \tag{5}$$

Hence we have from (4) and (5):

$$\cos\alpha - \cos\alpha' = -\frac{16\triangle^2}{24^* r^2 b c}. \tag{6}$$

[* Jordan has here falsely 22].

But now we have in the first approximation:

$$\cos\alpha - \cos\alpha' = -(\alpha-\alpha')\sin\alpha' + \ldots, \tag{7}$$

which we can regard either directly as a differential formula according to section 35, p. 30, or else found goniometrically thus:

$$\cos\alpha - \cos\alpha' = -2\sin\frac{\alpha-\alpha'}{2}\sin\frac{\alpha+\alpha'}{2},$$

i.e. if α and α' are very nearly equal:

$$\cos\alpha - \cos\alpha' = -(\alpha-\alpha')\sin\alpha'. \tag{7a}$$

If we set this (7) or (7a), as the case may be, into (6), then we obtain:

$$\alpha-\alpha' = \frac{2}{3}\frac{\triangle^2}{r^2 b c \sin\alpha'}. \tag{8}$$

But on the other hand, we also have in the plane triangle:

$$b c \sin\alpha' = 2\triangle \tag{9}$$

and with this, (8) becomes:

$$\alpha-\alpha' = \frac{1}{3}\frac{\triangle}{r^2} \text{ or, as the case may be, } \alpha-\alpha' = \frac{1}{3}\frac{\triangle}{r^2}\rho. \tag{10}$$

The first form written here in (10) holds for radian measure, the second one for degree measure. Or if we introduce the spherical excess ε according to (3), section 46, p. 90, then we have:

$$\alpha-\alpha' = \frac{1}{3}\varepsilon \tag{11a}$$

and accordingly

$$\beta-\beta' = \frac{1}{3}\varepsilon \tag{11b}$$

$$\gamma-\gamma' = \frac{1}{3}\varepsilon \tag{11c}$$

$$\text{Sum: } \alpha+\beta+\gamma-180^\circ = \varepsilon. \tag{12}$$

The theorem expressed in words on p. 93 has thus been proved. The derivation can also be based on the law of sines of spherical trigonometry $\frac{\sin\alpha}{\sin\beta} = \frac{\sin\frac{a}{r}}{\sin\frac{b}{r}}$ with development in series for $\sin\frac{a}{r}$ and $\sin\frac{b}{r}$, as follows at once from the inversion of the development under (15) to (18) in the next section 48, p. 100.

For a numerical example, we take again the classical triangle, which we have already used in the previous section 46, p. 90, namely now with rigorous angle values:

	Spherical	Plane (Legendre's theorem)	
Inselsberg	$\alpha = 40°\ 39'\ 30.380''$	$\alpha' = 40°\ 39'\ 25.430''$	
Hohehagen	$\beta = 86\ \ 13\ \ 58.840$	$\beta' = 86\ \ 13\ \ 53.890$	(13)
Brocken	$\gamma = 53\ \ \ 6\ \ 45.630$	$\gamma' = 53\ \ \ 6\ \ 40.680$	
Sum:	$180°\ \ 0'\ 14.850''$	$180°\ \ 0'\ \ 0.000''$	
	$\varepsilon = 14.850''$		
	$\frac{\varepsilon}{3} = \ \ 4.950''$.		

Let the one given side be $b = 105{,}972.850$ m . (14)

With this, we make a computation as if the triangle were plane, according to the law of sines of plane trigonometry rigorously with 7-8 place logarithms, and for this reason we shall write here the whole computation:

$\log b$	5.025 1946·1	or	$\log b$	5.025 1946·1
$\log\sin\beta'$	9.999 0600·0		co $\log\sin\beta'$	0.000 9400·0
$\log(b:\sin\beta')$	5.026 1346·1		$\log(b:\sin\beta')$	5.026 1346·1
$\log\sin\alpha'$	9.813 9344·8		$\log\sin\gamma'$	9.902 9830·6
$\log a$	4.840 0690·9		$\log c$	4.929 1176·7
$a = 69{,}194.105$ m			$c = 84{,}941.060$ m .	(15)

Legendre's theorem was published by Legendre for the first time without proof in *Histoire de l'académie royale de sciences*, Paris, 1787. Legendre gave the proof of the theorem in the introduction to the work, *Méthodes analytiques pour la détermination d'un arc du méridien*, by D. B. J. Delambre, Paris, an VII.

A thorough exposition of the history of Legendre's theorem is given by Hauer in *Zeitschr. f. Verm.*, 1938.

Note on the rigorousness of the calculation

The above example is calculated to three decimals of the second, i.e. to an accuracy of 0.001". This happens frequently even though the measurements themselves are much less reliable. The last decimal is not to have an independent meaning here, but is only to protect the next-to-the-last decimal from rounding errors. There is no question that often with such computations to 0.001" an abundance of numbers is written and printed, but in the case of long computations of adjustment, one may be required to calculate from the outset to an accuracy of 0.001" and perhaps still more rigorously, if one aims to have at the end 0.01" sure; in the case of shorter trigonometric computations, 0.01" is sufficient as last computational place.

Corresponding accuracy is to be used in the case of logarithmic calculations. Our numerical examples are carried mostly to 8 places, i.e. with 0.0000000·1 as last logarithmic place; the last place 0·1 is obtained partly with the help of the 10-place *Thesaurus logarithnorum*, partly also only by the use of the rounding characteristics in Schrön's 7-place logarithmic table, and is used then (just as 0.001" in the case of angles) only as safety for the preceding 7th place. For the newer 8-place computations, the table by Bauschinger-Peters mentioned on p. 37 was always used.

A second method of approximation for the computation of spherical triangles whose sides are small in comparison to the spherical radius was first introduced in Bavaria at the beginning of the past century, and then generally applied also by the remaining South German land surveys. The method was denoted by "Additamentenmethode" [method of additament], because small magnitudes of correction are frequently added to (or, vice versa also, subtracted from) the logarithms.

Whereas in the case of Legendre's theorem a plane auxiliary triangle was used whose sides are equal to those of the spherical triangle and whose angles had to be assumed different from the angles of the spherical triangle, now we aim, conversely, at finding a plane auxiliary triangle which has two angles in common with the spherical triangle, but has other sides. With reference to Figs. 1 and 2, we imagine that there is given a spherical triangle with the sides a and b and the opposite angles α and β, and to this, we construct a plane auxiliary triangle which has the same angles α and β as the spherical triangle, but with these must necessarily have other sides a', b'.

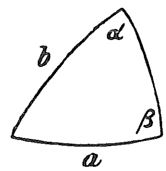

Fig. 1.
Spherical.

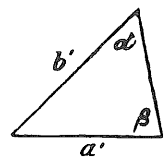

Fig. 2.
Plane.

According to the law of sines for the spherical triangle and according to the law of sines for the plane triangle we have the two equations:

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\frac{a}{r}}{\sin\frac{b}{r}} \quad \text{and} \quad \frac{\sin\alpha}{\sin\beta} = \frac{a'}{b'}\,, \tag{1}$$

whence there follows:

$$\frac{a'}{b'} = \frac{\sin\frac{a}{r}}{\sin\frac{b}{r}} = \frac{\frac{a}{r} - \frac{a^3}{6r^3} + \dots}{\frac{b}{r} - \frac{b^3}{6r^3} + \dots} \tag{2}$$

$$\frac{a'}{b'} = \frac{a - \frac{a^3}{6r^2} + \dots}{b - \frac{b^3}{6r^2} + \dots}\,. \tag{3}$$

This equation is satisfied if we set

$$a' = a - \frac{a^3}{6r^2} \quad \text{and} \quad b' = b - \frac{b^3}{6r^2}\,,$$

or, generally, for any arbitrary triangle side s we have:

$$s' = s - \frac{s^3}{6r^2}\,. \tag{4}$$

The thus defined value $\frac{s^3}{6r^2}$ is the linear additament for the side s, and if we know the radius r, we can compute a table of the values $\frac{s^3}{6r^2}$, e.g., for the mean latitude $\varphi = 50°$ we have:

$$\log r = 6.804\,894 \qquad \log\frac{1}{6r^2} = 5.612\,062\,.$$

With these, we have calculated the following for an illustration:

$$\left.\begin{array}{lllllllll} s = & 10,000 & 20,000 & 30,000 & 40,000 & 50,000 & 60,000 & 80,000 & 100,000 \text{ m} \\ \frac{s^3}{6r^2} = & 0.004 & 0.033 & 0.111 & 0.262 & 0.512 & 0.884 & 2.096 & 4.093 \text{ m}. \end{array}\right\} \quad (5)$$

At first we see that the addition can immediately be neglected if, in the case of distances up to 10 km, only data within 1 cm for the side are to be involved [cf. also the remark on the excess in the case of (10) p. 92].

If, e.g., there exists however a triangle side $s = 40,000$ m, then we will have the pertinent $s' = 40,000 - 0.262 = 39,999.738$ m, and on this we could base a triangle computation with spherical angles which has now entirely the form of a plane computation.

However, we do not do this usually in this form directly, but since we compute logarithmically, we also bring the additaments into logarithmic form, and in this connection, we go back once again to (1) and (3) and find as general relation between a triangle side s and the reduced side s' the following:

$$s' = r \sin\frac{s}{r}, \quad \text{or} \quad \frac{s'}{r} = \sin\frac{s}{r}; \qquad (6)$$

hence, logarithmically:

$$\log\frac{s'}{r} = \log\sin\frac{s}{r} = \log\left(\frac{s}{r} - \frac{s^3}{6r^3} + \frac{s^5}{120r^5} - \ldots\right). \qquad (7)$$

In this, we have still retained the fifth order in the series in order to be able to judge afterwards whether the term of fifth order is still of influence.

If we develop the last expression according to the logarithmic series, then we obtain:

$$\log\sin\frac{s}{r} = \log\frac{s}{r} + \log\left(1 - \frac{s^2}{6r^2} + \frac{s^4}{120r^4}\right)$$

$$\log\sin\frac{s}{r} = \log\frac{s}{r} + \mu\left(-\frac{s^2}{6r^2} + \frac{s^4}{120r^4}\right) - \frac{\mu}{2}\left(-\frac{s^2}{6r^2} + \ldots\right)^2$$

$$\log\sin\frac{s}{r} = \log\frac{s}{r} - \frac{\mu s^2}{6r^2} - \frac{\mu}{180}\frac{s^4}{r^4}$$ (also directly from p. 24, section 34).

Or if we use again s' according to (6):

$$\log s - \log s' = \frac{\mu s^2}{6r^2} + \frac{\mu}{180}\frac{s^4}{r^4}. \qquad (8)$$

For the mean latitude $\varphi = 50°$ we have for this, according to page [24] of the Appendix:

$$\log\frac{\mu}{6r^2} = 5.24985 - 20 \qquad \log\frac{\mu}{180r^4} = 0.16294 - 40$$

or for units of the 7th decimal of logarithm:

$$\log\frac{\mu}{6r^2} = 2{\cdot}24985 - 10 \qquad \log\frac{\mu}{180r^4} = 7{\cdot}16294 - 30.$$

For $s = 100{,}000$ m or $log\ s = 5.00000$ taken as an example, this yields:

$$\frac{\mu}{6\,r^2}\,s^2 = 0.000\ 0177{\cdot}8 \qquad \frac{\mu}{180\,r^4}\,s^4 = 0.000\ 0000{\cdot}001\ .$$

Hence it follows that for ordinary triangle sides the second term of the formula (8) is imperceptible, and that we can therefore stop at the first term of (8).

Denoting the logarithmic additament by A, we write with the omission of the second term, collectively:

$$A = log\ s - log\ s'$$

or

$$A = log\frac{s}{r} - log\ sin\frac{s}{r} = \frac{\mu}{6\,r^2}\,s^2\ , \qquad \text{where} \qquad log\frac{\mu}{6\,r^2} = 2{\cdot}249\ 846 - 10 \ \text{ for } \varphi = 50°\ . \tag{9}$$

With this, the auxiliary table on page [56] of the Appendix is computed, and, in fact, in twofold form, I as a function of $log\ s$, with the assumption $log\ r = 6.804\ 894$ for $\varphi = 50°$, and II as a function of $log\frac{s}{r}$.

The first table I, i.e. the upper part of page [56], is the more convenient one for the computation of triangles, because we have to enter the table directly with $log\ s$ (for s in meters) whereas in the case II, i.e. in the lower part of page [56], we have to form first $log\frac{s}{r}$, which otherwise is not needed. But on the other hand, table II is usable more generally because it is not limited to a definite assumption for the radius r as I, and because it is applicable also for other measures than meters (e.g. feet, toises, rods, etc., in the case of older triangulations).

For a numerical example we take again the classical triangle of the previous section, p. 96.

Inselsberg	$\alpha =$	40°39' 30.380"
Hohehagen	$\beta =$	86 13 58.840
Brocken	$\gamma =$	53 6 45.630
		180° 0' 14.850"

(10)

Base $b = 105{,}972.850$ m $\qquad log\ b = 5.025\ 1946{\cdot}1$. (11)

To this, we need the logarithmic additament which can be taken from the auxiliary table I of page [56] of the Appendix for $log\ s = 5.0252$ by interpolation $= 199.7$. Since, however, that auxiliary table I, page [56], is valid for the mean latitude $\varphi = 50°$, whereas our triangle has the mean latitude $\varphi = 51°22.6'$ with $log\ r = 6.804962$, and since our triangle sides are very large, we calculate the additament A this time specially:

$log\ b^2$	10.05039
$log\ (\mu : 6\,r^2)$	2.24971
$log\ A_b$	2.30010

$A_b = 199{\cdot}57$. (12)

This is not much different from $199{\cdot}7$ taken from the table.

Now we have a logarithmic computation substantially as on the plane, namely according to (10), (11), (12):

$log\ b$	5.025 1946·1			
Logar. Additament	— 199·6			
$log\ b'$	5.025 1746·5		$log\ b'$	5.025 1746·5
$log\ sin\ \beta$	9.999 0606·9	or	co $log\ sin\ \beta$	0.000 9393·1
$log\ (b' : sin\ \beta)$	5.026 1139·6			5.026 1139·6
$log\ sin\ \alpha$	9.813 9466·1		$log\ sin\ \gamma$	9.902 9908·8
$log\ a'$	4.840 0605·7		$log\ c'$	4.929 1048·4
Logar. Additament	+ 85·1		Logar. Add.	+ 128·2
$log\ a$	4.840 0690·8		$log\ c$	4.929 1176·6

(13)

$a = 69{,}194.105$ m $\qquad c = 84{,}941.060$ m . (14)

This agrees with (15), section 47, p. 96.

The additaments, just used in the case of (13), are taken again from the auxiliary table I of page [56] of the Appendix, or if we aim to have the last place perfectly rigorously, we compute it just as previously in the case of (12).

If we compare the computation according to this method of additament with the computation according to Legendre's theorem in regard to convenience, clearness of arrangement and the like, we are able to say:

Legendre's theorem is recommended in the case of an individual triangle or, in a rare application, due to its independence of all special aids, for it is necessary that we know the excess ε also in the case of the other method in order to check the angles, and Legendre's theorem itself, i.e. the distribution $\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$ is always in the memory.

The method of additament, however, is more advantageous in the case of whole triangle nets with many connected triangles to be computed. We reduce at first only the logarithm of the base of the net ($\log b' = \log b - A_b$), then we compute the whole triangle net with the spherical angles, and obtain, by so doing, at first only reduced values $\log a'$, $\log c'$, etc., which we can reduce, however, afterwards all at once with the table of additaments to $\log a$, $\log c$, etc.

An advantage of the method of additament consists also in the fact that we must carry only one table of the angles $\alpha, \beta, \gamma \ldots$, whereas for Legendre's theorem a second table of α', β', γ' is necessary, which not only increases the records but also can be the cause of mistakes when afterwards, for the computation of coordinates and the like, the spherical angles themselves are needed again.

Connection between Legendre's theorem and the method of additament

These two methods of computation rest on developments in series of spherical formulae to terms of the order $\frac{1}{r^2}$, and it must therefore be possible to derive the two computations in their formulae reciprocally from one another, and this we shall show now.

With the acceptance of the denotations used thus far, we have according to Legendre's theorem:

$$\frac{a}{b} = \frac{\sin\left(\alpha - \frac{\varepsilon}{3}\right)}{\sin\left(\beta - \frac{\varepsilon}{3}\right)} = \frac{\sin\alpha - \frac{\varepsilon}{3}\cos}{\sin\beta - \frac{\varepsilon}{3}\cos\beta} = \frac{\sin\alpha\left(1 - \frac{\varepsilon}{3}\cot\alpha\right)}{\sin\beta\left(1 - \frac{\varepsilon}{3}\cot\beta\right)}. \tag{15}$$

Now we have $\varepsilon = \frac{\triangle}{r^2}$, and if we interchange plane and spherical angles in the correction terms, then we have:

$$\left.\begin{aligned} &2\,b\,c\cos\alpha = b^2 + c^2 - a^2 \qquad b\,c\sin\alpha = 2\,\triangle \\ &\text{hence} \qquad \cot\alpha = \frac{b^2+c^2-a^2}{4\,\triangle}, \quad \frac{\varepsilon}{3} = \frac{\triangle}{3\,r^2} \\ &\frac{\varepsilon}{3}\cot\alpha = \frac{b^2+c^2-a^2}{12\,r^2} \text{ and } \frac{\varepsilon}{3}\cot\beta = \frac{a^2+c^2-b^2}{12\,r^2}. \end{aligned}\right\} \tag{16}$$

With these, (15) yields:

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b}\,\frac{\left(1+\frac{b^2+c^2-a^2}{12\,r^2}\right)}{\left(1+\frac{a^2+c^2-b^2}{12\,r^2}\right)} = \frac{a}{b}\left(1+\frac{1}{12\,r^2}(2\,b^2 - 2\,a^2)\right). \tag{17}$$

This can also be written in the following way:

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b}\left(1-\frac{a^2}{6\,r^2}\right)\left(1+\left(\frac{b}{6\,r^2}\right)\right) = \frac{a}{b}\,\frac{1-\frac{a^2}{6\,r^2}}{1-\frac{b^2}{6\,r^2}}. \tag{18}$$

This agrees according to (1), (3), section 48, p. 97, with the method of additament; hence, the connection mentioned above has been proved.

The "Additamentenmethode" was introduced in Bavaria by Soldner. We find further details on it in the official work, *Die Bayerische Landesvermessung in ihrer wissenschaftlichen Grundlage*, München, 1873, pp. 263 and following (on p. 262 and following, reprint of a treatise by Soldner, of 5 May 1810). This treatise by Soldner has been newly edited by J. Frischauf in *Ostwalds Klassiker der exakten Wissenschaften*, Nr. 184: Theorie der Landesvermessung, by Johann Soldner, Leipzig, 1911. (Cf. also: Bohnenberger, *De computandis dimensionibus trigonometricis in superficie terrae sphaeroidica institutis*, Tubingae, 1826, section 11.)

More detailed additament tables than our table, page [56], are found in some geodetic works, e.g. in Bremiker, *Studien über höhere Geodäsie*, Berlin, 1869. Appendix, table III, Reduction from the arc to the chord, i.e. $0.2A$, if A is the value of our table II, page [56]. — Bremiker, *Tafel zur Verwandlung von Log. Bogen in Log. Tangente.* Wissenschaftliche Begründung der Rechnungsmethoden des Zentralbureaus der europäischen Gradmessung. Supplement to the General Report of the European Degree Measurement for 1870 (yields $T = 2A$).

Also the numbers S, which are indicated in Bremiker's and Schrön's 7-place logarithmic tables and also in other tables at the foot of each page of the logarithms of the natural numbers, are in simple relation to our additaments. For we have for this S:

$$S = \log \frac{1}{\varrho''} - A = \log \sin 1'' - A\,,$$

e.g. in Schrön, p. 29, we find for $0^\circ 36' 0''$ $S = 4.685\,5669{\cdot}3$. Here is $\log(1:\varrho) = 4.685\,5748{\cdot}7$, and our table II on page [56] gives for the central angle $0^\circ 36' 0''$ the value $A = 79{\cdot}4$, which is the difference of the two numbers just written. For the numbers T of the logarithmic tables there holds the corresponding equation:

$$T = \log \frac{1}{\varrho''} + 2A = \log \sin 1'' + 2A$$

Section 49. Several Spherical Problems

After having learned, in a twofold way, the reduction of a spherical triangle to a plane auxiliary triangle, we can also solve other problems than the computation of a triangle from a side and all angles, which we have treated first. We shall discuss here, in addition, the determination of a spherical triangle from two sides and the included angle and then the spherical problem of the resection.

I. Determination of a spherical triangle b, c, α according to Legendre's theorem

If two sides b, c and the included angle α are given, we can compute hence immediately the excess ε:

$$\varepsilon = b\,c\,\sin\alpha\,\frac{\rho}{2r^2}\,. \tag{1}$$

With this, we also have the sum of the other two angles β and γ:

$$\beta + \gamma = 180^\circ + \varepsilon - \alpha\,. \tag{2}$$

Now if we consider Legendre's plane auxiliary triangle and take into account that:

$$\left(\beta - \frac{\varepsilon}{3}\right) - \left(\gamma - \frac{\varepsilon}{3}\right) = \beta - \gamma\,,$$

then we find according to Gauss' equations of plane trigonometry:

$$\tan\frac{\beta-\gamma}{2}=\frac{(b-c)\cos\frac{\alpha-{}^1\!/_3\,\varepsilon}{2}}{(b+c)\sin\frac{\alpha-{}^1\!/_3\,\varepsilon}{2}}=\frac{Z}{N} \qquad (3)$$

$$a=\frac{Z}{\sin\frac{\beta-\gamma}{2}}=\frac{N}{\cos\frac{\beta-\gamma}{2}}. \qquad (4)$$

From (2) and (4) we thus have $\beta+\gamma$ and $\beta-\gamma$, consequently also β and γ, and a, with a check, from (4), with which the problem is solved.

II. Determination of a spherical triangle b, c, α according to the additament method

If b and c are given in logarithms, then the following computation is more convenient than the previous one:

For a plane triangle with the sides b' and c' and the angles β and γ we have the following, with the introduction of an auxiliary angle λ:

$$\frac{\sin\beta}{\sin\gamma}=\frac{b'}{c'}=\frac{1}{\tan\lambda} \qquad \frac{\sin\beta-\sin\gamma}{\sin\beta+\sin\gamma}=\frac{1-\tan\lambda}{1+\tan\lambda}$$

$$\tan\frac{\beta-\gamma}{2}=\tan\frac{\beta+\gamma}{2}\cot(\lambda+45°).$$

For the spherical triangle we set accordingly:

$$\log b'=\log b-A_b, \qquad \log c'=\log c-A_c,$$

where A_b and A_c are the additaments of b and c. Now we compute the auxiliary angle λ according to the formula

$$\cot\lambda=\frac{b'}{c'}, \qquad (5)$$

then we determine the spherical excess ε by a preliminary triangle computation, and then we have

$$\left.\begin{aligned}\beta+\gamma&=180°+\varepsilon-\alpha\\ \tan\frac{\beta-\gamma}{2}&=\tan\frac{\beta+\gamma}{2}\cot(\lambda+45°).\end{aligned}\right\} \qquad (6)$$

From $\frac{\beta+\gamma}{2}$ and $\frac{\beta-\gamma}{2}$ we obtain β and γ.

The third side a can then be determined according to Legendre's theorem as well as according to the additament method.

III. Resection

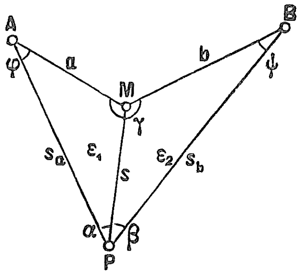

Fig. 1.

In Fig. 1 we take the same computations as previously for plane resection in Vol. II, first half-volume, 9th Edition, 1931, sections 94 and 95, and we do not have to change much from the previous computation.

Three points A, M, B are reciprocally fixed by the sides $AM=a$, $MB=b$ and the angle $BMA=\gamma$; a point P is to be determined by measurement of the angles α and β opposite A, M, B.

We first solve this problem provisionally by approximation by treating the figure as plane (i.e. we first compute according to Vol. II, first half-volume, 9th edition, 1931, sections 94-95).* Then we have as much of a basis as is needed in order to compute the spherical excesses ε_1 and ε_2 of the two triangles PAM and PMB, and with these there is also determined the sum $\varphi + \psi$, namely:

$$\varphi + \psi = 360° + \varepsilon_1 + \varepsilon_2 - (\alpha + \beta + \gamma) \; . \tag{7}$$

With the logarithmic additaments A_a and A_b we reduce:

$$log\ a - A_a = log\ a' \quad \text{and} \quad log\ b - A_b = log\ b' \; ;$$

the computation can then be carried further as for a plane quadrilateral; we set:

$$\frac{a'}{sin\ \alpha} : \frac{b'}{sin\ \beta} = tan\ \lambda$$

and find:

$$tan\ \frac{\varphi - \psi}{2} = tan\ \frac{\varphi + \psi}{2}\ cot\ (\lambda + 45°) \; . \tag{8}$$

After the two angles φ and ψ have thus been determined by (7) and (8), all triangle sides can be computed further according to Legendre's theorem or according to the additament method.

These three problems may suffice in order to show that we can compute, with Legendre's theorem and with the additament method, nearly all that we use to compute in plane trigonometry. The spherical computations of this kind, however, do not play an important role.

Section 50. Spherical-Trigonometric Developments in Series As Far As the Order $\frac{1}{r^4}$, Inclusive

Legendre's theorem and the method of additament rest on spherical-trigonometric developments in series which we have treated, in the case of Legendre's theorem, only to an accuracy of terms of the order $1{:}r^2$, inclusive, in the final result. In the case of the method of additament, in (8), section 48, p. 98, we have taken in one more term of the order $1{:}r^4$ because it thus resulted incidentally without special effort; and it has been shown that this higher term is imperceptible in the case of practical computations with triangle sides up to 100,000 m and above.

Although, by this, the probability is suggested that also in the remaining related developments the terms of the order $1{:}r^2$ are sufficient, we must learn about the higher terms in order to have a reliable judgment.

However, there is to be considered here that a *spherical* computation, carried very far and rigorously, has at first little value for geodesy as long as the spherical method of computing is not based more rigorously than it has been done in our section 44, pp. 83 and 84, for, in addition to the higher terms of the order $1{:}r^4$ we also should examine the influence of the inequality of the curvatures to the various directions and the changes of the curvatures dependent on the geographic latitude.

We cannot do this until later, and if we examine now the higher terms of the order $1{:}r^4$, this has, first, the meaning that we convince ourselves if the developments are sufficient to $1{:}r^2$, inclusive, in order to replace the *closed* spherical formulae, which we could apply also, and second, our later developments with the geodetic line are to be prepared appropriately by the following developments.

In the case of the first study of higher geodesy in the sense of the understanding of our present-day land surveys we may leave out at first entirely the section 50 following here and not take it up until later as needed.

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* Not translated.

I. *The spherical excess*

In Fig. 1 we consider the right spherical triangle ABC with the hypotenuse s, with the legs p and q and with the angles $90°$, β and α. Since one of the angles $= 90°$, the spherical excess is:

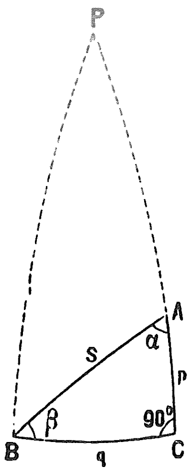

Fig. 1.

$$\varepsilon = \alpha + \beta + 90° - 180°$$

or

$$\varepsilon = \alpha + \beta - 90° . \tag{1}$$

This right triangle, Fig. 1, yields:

$$\cot\ \alpha \cot\ \beta = \cos\frac{s}{r} \quad \text{or} \quad = 1 - 2\sin^2\frac{s}{2r},$$

hence $2\sin^2\dfrac{s}{2r} = 1 - \cot\ \alpha \cot\ \beta = \dfrac{\sin\alpha \sin\beta - \cos\alpha \cos\beta}{\sin\alpha \sin\beta}$

$$2\sin^2\frac{s}{2r} = -\frac{\cos(\alpha+\beta)}{\sin\alpha\sin\beta} = \frac{\sin\varepsilon}{\sin\alpha\sin\beta} = \frac{\sin\varepsilon \sin^2\frac{s}{r}}{\sin\frac{p}{r}\sin\frac{q}{r}}$$

$$\sin\varepsilon = \frac{\sin\frac{p}{r}\sin\frac{q}{r}}{2\cos^2\frac{s}{2r}} . \tag{2}$$

This developed as far as $\dfrac{1}{r^4}$ yields

$$\varepsilon = \frac{\left(\frac{p}{r} - \frac{p^3}{6r^3}\right)\left(\frac{q}{r} - \frac{q^3}{6r^3}\right)}{2\left(1 - \frac{s^2}{8r^2}\right)^2} = \left(\frac{pq}{2r^2} - \frac{pq(p^2+q^2)}{12r^4}\right)\left(1 + \frac{s^2}{4r^2}\right).$$

But since we can set in the higher terms $s^2 = p^2 + q^2$, this yields immediately:

$$\varepsilon = \frac{pq}{2r^2} + \frac{pq}{24r^4}(p^2+q^2) \,.. \tag{3}$$

II. *The formulae for the legs [Katheten]*

The right spherical triangle, Fig. 2, yields according to section 33, p. 15:

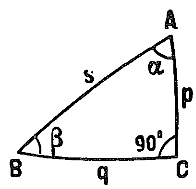

Fig. 2.

$$\sin\frac{q}{r} = \sin\frac{s}{r}\sin\alpha \ , \quad \tan\frac{p}{r} = \tan\frac{s}{r}\cos\alpha \tag{4}$$

or developed out:

$$q - \frac{q^3}{6r^2} = \left(s - \frac{s^3}{6r^2}\right)\sin\alpha \ , \quad p + \frac{p^3}{3r^2} = \left(s + \frac{s^3}{3r^2}\right)\cos\alpha . \tag{5}$$

In order to solve these equations for q or for p, as the case may be, we use at first the first approximations:

$$q = s\,sin\,\alpha + \frac{1}{r^2}\ldots \qquad p = s\,cos\,\alpha + \frac{1}{r^2}\ldots$$

$$\frac{q^3}{6r^2} = \frac{s^3\,sin^3\,\alpha}{6r^2} + \frac{1}{r^4}\ldots \qquad \frac{p^2}{3r^2} = \frac{s^3\,cos^3\,\alpha}{3r^2} + \frac{1}{r^4}\ldots$$

$$q = s\,sin\,\alpha - \frac{s^3}{6r^2}\,sin\,\alpha + \frac{s^3}{6r^2}\,sin^3\,\alpha \qquad p = s\,cos\,\alpha + \frac{s^3}{3r^2}\,cos\,\alpha - \frac{s^3}{3r^2}\,cos^3\,\alpha$$

$$q = s\,sin\,\alpha - \frac{s^3}{6r^2}\,sin\,\alpha\,cos^2\,\alpha \qquad p = s\,cos\,\alpha + \frac{s^3}{3r^2}\,sin^2\,\alpha\,cos\,\alpha\,. \tag{6}$$

We shall carry these developments further by one more term, but we shall show this in detail only in the formula for q. Instead of (5) we have then:

$$q - \frac{q^3}{6r^2} + \frac{q^5}{120\,r^4} = \left(s - \frac{s^3}{6r^2} + \frac{s^5}{120\,r^4}\right) sin\,\alpha\,; \tag{7}$$

to this, we have according to (6):

$$q^3 = s^3\;sin^3\;\alpha - \frac{3}{6}\frac{s^5}{r^2}\;sin^3\;\alpha\;cos^2\;\alpha + \ldots \qquad q^5 = s^5\;sin^5\;\alpha + \ldots.$$

If we introduce these two expressions for q^3 and q^5 in (7), then we obtain

$$\begin{aligned} q = s\,sin\,\alpha &- \frac{s^3}{6r^2}\,sin\,\alpha + \frac{s^5}{120\,r^4}\,sin\,\alpha \\ &+ \frac{s^3}{6r^2}\,sin^3\alpha - \frac{s^5}{12\,r^4}\,sin^3\,\alpha\,cos^2\,\alpha \\ &- \frac{s^5}{120\,r^4}\,sin^5\,\alpha\,. \end{aligned}$$

If we arrange this and take into account that

$$s^4 = s^4\;(sin^2\;\alpha + cos^2\;\alpha)^2\;,$$

then we find:

$$q = s\,sin\,\alpha - \frac{s^3}{6r^2}\,sin\,\alpha\,cos^2\,\alpha - \frac{s^5}{120\,r^4}\,sin\,\alpha\,cos^2\,\alpha\,(8\,sin^2\,\alpha - cos^2\,\alpha)\,. \tag{8}$$

This is the further development of the first formula of the group (6); the further development of the second formula of the group (6) is done likewise and yields:

$$p = s\,cos\,\alpha + \frac{s^3}{3r^2}\,sin^2\,\alpha\,cos\,\alpha + \frac{s^5}{15\,r^4}\,sin^2\,\alpha\,cos\,\alpha\,(2\,sin^2\,\alpha - cos^2\,\alpha)\,. \tag{9}$$

Inversion of series (8) and (9)

We can also invert the series (8) and (9), i.e. we can express $s\;sin\;\alpha$ and $s\;cos\;\alpha$ in powers of q and p. (We could apply, in this connection, the general method to which we have referred in section 35, pp. 31 and 32, but we prefer here to proceed without any auxiliary means of preparation.)

In any case we have as the first approximation from (8) and (9):

$$s\;sin\;\alpha = q + \ldots \qquad s\;cos\;\alpha = p + \ldots\;;$$

consequently, at once as a second approximation from (8) and (9)

$$s \sin \alpha = q + \frac{q p^2}{6 r^2} + \dots \qquad s \cos \alpha = p - \frac{q^2 p}{3 r^2} + \dots,$$

consequently for the introduction in the higher terms of (8) and (9)

$$s^2 \sin^2 \alpha = q^2 + \frac{q^2 p^2}{3 r^2} + \dots \qquad s^2 \cos^2 \alpha = p^2 - 2 \frac{q^2 p^2}{3 r^2} + \dots$$

$$s^3 \sin^2 \alpha \cos \alpha = \left(q^2 + \frac{q^2 p^2}{3 r^2}\right)\left(p - \frac{q^2 p}{3 r^2}\right) = q^2 p + \frac{q^2 p^3}{3 r^2} - \frac{q^4 p}{3 r^2} + \dots$$

$$s^3 \sin \alpha \cos^2 \alpha = \left(q + \frac{q p^2}{6 r^2}\right)\left(p^2 - 2 \frac{q^2 p^2}{3 r^2}\right) = q p^2 + \frac{q p^4}{6 r^2} - 2 \frac{q^3 p^2}{3 r^2} + \dots$$

If we set this into (8), then we obtain:

$$s \sin \alpha = q + \frac{q p^2}{6 r^2} + \frac{q p^4}{36 r^4} - \frac{q^3 p^2}{9 r^4} - \frac{1}{120 r^4}(q p^4 - 8 q^3 p^2) + \dots$$

This arranged, yields:

$$s \sin \alpha = q + \frac{q p^2}{6 r^2} - \frac{q p^2}{360 r^4}(16 q^2 - 7 p^2) \tag{10}$$

and in the same way we obtain from (9):

$$s \cos \alpha = p - \frac{q^2 p}{3 r^2} - \frac{q^2 p}{45 r^4}(q^2 + 2 p^2)\,. \tag{11}$$

III. The formula for the hypotenuse

From the formulae (10) and (11) just obtained we can also produce a formula for s^2 by squaring and adding $s \sin \alpha$ and $s \cos \alpha$. If we neglect here the higher terms as thus far, then we obtain:

$$s^2 \sin^2 \alpha = q^2 + \frac{q^2 p^4}{36 r^4} + \frac{q^2 p^2}{3 r^2} + \frac{q^2 p^2}{180 r^4}(-16 q^2 + 7 p^2)$$

$$s^2 \cos^2 \alpha = p^2 + \frac{q^4 p^2}{9 r^4} - 2 \frac{q^2 p^2}{3 r^2} - 2 \frac{q^2 p^2}{45 r^4}(q^2 + 2 p^2)\,.$$

If we collect and arrange this, then we find:

$$s^2 = q^2 + p^2 - \frac{q^2 p^2}{3 r^2} - \frac{q^2 p^2 (q^2 + p^2)}{45 r^4}\,. \tag{12}$$

We can find this formula also directly by the development of

$$\cos \frac{s}{r} = \cos \frac{q}{r} \cos \frac{p}{r}\,.$$

According to the indication of Fig. 3 we connect two right spherical triangles into a general spherical triangle ABC in which the two legs [Katheten] q and q' now form the side $CB = a$ to which the common leg [Kathete] p belongs as the altitude.

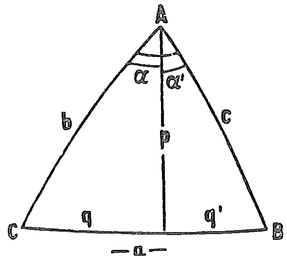

Fig. 3.

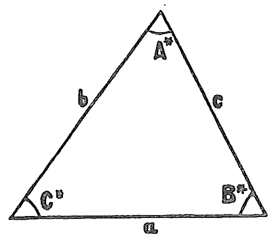

Fig. 4.

The angles A^*, B^*, C^* of the plane triangle, Fig. 4, whose sum is $A^* + B^* + C^* = 180°$, correspond to the angles α', β', γ' in the previous Fig. 2, section 47, p. 93.

Now the angle A of the triangle is composed of the two angles α and α' of the two right triangles thus:

$$A = \alpha + \alpha' , \qquad \text{hence} \qquad \cos A = \cos\alpha \cos\alpha' - \sin\alpha \sin\alpha' . \tag{13}$$

Now we have according to (11):

$$b \cos\alpha = p - \frac{p\,q^2}{3\,r^2} - \frac{p\,q^2}{360\,r^4}(16\,p^2 + 8\,q^2) \tag{14}$$

$$c \cos\alpha' = p - \frac{p\,q'^2}{3\,r^2} - \frac{p\,q'^2}{360\,r^4}(16\,p^2 + 8\,q'^2) \tag{15}$$

$$b \sin\alpha = q + \frac{p^2\,q}{6\,r^2} + \frac{p^2\,q}{360\,r^4}(7\,p^2 - 16\,q^2) \tag{16}$$

$$c \sin\alpha' = q' + \frac{p^2\,q'}{6\,r^2} + \frac{p^2\,q'}{360\,r^4}(7\,p^2 - 16\,q'^2) . \tag{17}$$

If we multiply (14) and (15), thereby neglecting the higher terms, and then arranging according to equal powers, then we obtain:

$$\begin{aligned} b\,c \cos\alpha \cos\alpha' = p^2 &- \frac{p^2\,q^2}{3\,r^2} - \frac{p^2\,q^2}{360\,r^4}(16\,p^2 + 8\,q^2) \\ &- \frac{p^2\,q'^2}{3\,r^2} + \frac{p^2\,q^2\,q'^3}{9\,r^4} \\ &- \frac{p^2\,q'^2}{360\,r^4}(16\,p^2 + 8\,q'^2) . \end{aligned}$$

In the same manner we also find from (16) and (17):

$$\begin{aligned} b\,c \sin\alpha \sin\alpha' = q\,q' &+ \frac{p^2\,q\,q'}{6\,r^2} + \frac{p^2\,q\,q'}{360\,r^4}(7\,p^2 - 16\,q^2) \\ &+ \frac{p^2\,q\,q'}{6\,r^2} + \frac{p^4\,q\,q'}{36\,r^4} \\ &+ \frac{p^2\,q\,q'}{360\,r^4}(7\,p^2 - 16\,q'^2) . \end{aligned}$$

These two expressions together yield:

$$b\,c\,(\cos\alpha\cos\alpha' - \sin\alpha\sin\alpha') = b\,c\cos(\alpha+\alpha') = b\,c\cos A = p^2 - q\,q' - \frac{p^2}{3\,r^2}(q^2+q'^2+q\,q')$$
$$-\frac{p^2}{360\,r^4}(16p^2q^2+16p^2q'^2+24p^2qq'+8q^4+8q'^4-40q^2q'^2-16q^3q'-16q'^3q)\,. \tag{18}$$

Now we have according to the formula for the hypotenuse (12):

$$\left.\begin{aligned} b^2 &= p^2+q^2-\frac{p^2\,q^2}{3\,r^2}-\frac{p^2\,q^2}{45\,r^4}(p^2+q^2) \\ c^2 &= p^2+q'^2-\frac{p^2\,q'^2}{3\,r^2}-\frac{p^2\,q'^2}{45\,r^4}(p^2+q'^2) \end{aligned}\right\} \tag{19}$$

and directly

$$a^2 = (q+q')^2 = q^2+2\,q\,q'+q'^2\,. \tag{20}$$

At the same time there is introduced in Fig. 4, p. 107, a *plane* triangle whose sides a, b, c are equal to the sides of the spherical triangle, and whose angles A^*, B^*, C^* are to be determined. For the plane triangle we have, as is known, the equation:

$$2\,b\,c\,\cos A^* = b^2+c^2-a^2\,,$$

and if we introduce here the values (19) and (20), then we obtain:

$$b\,c\cos A^* = p^2 - q\,q' - \frac{p^2}{6\,r^2}(q^2+q'^2) - \frac{p^2}{90\,r^4}(p^2\,q^2+p^2\,q'^2+q^4+q'^4)\,. \tag{21}$$

This (21) is compared with the previous (18), whereby we will find after some algebraic converting:

$$b\,c\,(\cos A^* - \cos A) = \frac{p^2}{6\,r^2}(q+q')^2 + \frac{p^2\,(q+q')^2}{90\,r^4}(3\,p^2+(q+q')^2-8\,q\,q')\,. \tag{22}$$

The product, occurring here, $p\,(q+q')$ is closely related to $b\,c\sin A$, for we have from (16) and (15) with the omission of the last terms:

$$b\,c\sin\alpha\cos\alpha' = p\,q + \frac{p^3\,q}{6\,r^2} - \frac{p\,q\,q'^2}{3\,r^2}\,.$$

Accordingly, (14) and (17) also yield:

$$b\,c\cos\alpha\sin\alpha' = p\,q' + \frac{p^3\,q'}{6\,r^2} - \frac{p\,q^2\,q'}{3\,r^2}\,.$$

The addition of these two equations yields:

$$b\,c\,(\sin\alpha\cos\alpha' + \cos\alpha\sin\alpha') = b\,c\sin A = p\,(q+q') + \frac{p\,(q+q')}{6\,r^2}(p^2-2\,q\,q')\,,$$

hence:

$$p\,(q+q') = b\,c\sin A\left(1-\frac{1}{6\,r^2}(p^2-2\,q\,q')\right)\,. \tag{23}$$

The goal of this development is the small angular difference between A and A^*, and therefore we shall set:

$$A - A^* = x \; ; \tag{24}$$

consequently, as a first approximation:

$$\cos A = \cos (A^* + x) = \cos A^* - x \sin A^* + \ldots \tag{25}$$

If we set, accordingly, $\cos A^* - \cos A = x \sin A^*$ in (22) and take into account (23) approximated, with $A = A^*$, then we obtain:

$$x = \frac{p}{6\,r^2}(q + q') \, . \tag{26}$$

With this, we develop a second approximation:

$$A = A^* + \frac{p}{6\,r^2}(q + q') \quad , \quad \sin A = \sin A^* + \frac{p}{6\,r^2}(q + q') \cos A^* + \ldots$$

$$\sin A = \sin A^* \left(1 + \frac{p}{6\,r^2}(q + q') \cot A^*\right) \tag{26a}$$

or with the substitution of $\cot A^*$ from (21) and (23) approximated:

$$\cot A^* = \frac{p^2 - q\,q'}{p\,(q + q')} \, .$$

This set into (26a) yields:

$$\sin A = \sin A^* \left(1 + \frac{1}{6\,r^2}(p^2 - q\,q')\right) . \tag{27}$$

With this, we can replace in (23) the function $\sin A$ by $\sin A^*$, and by so doing we obtain:

$$p\,(q + q') = b\,c \sin A^* \left(1 + \frac{1}{6\,r^2}(p^2 - q\,q') - \frac{1}{6\,r^2}(p^2 - 2\,q\,q')\right)$$

$$p\,(q + q') = b\,c \sin A^* \left(1 + \frac{1}{6\,r^2}\,q\,q'\right) = 2\,\triangle \left(1 + \frac{1}{6\,r^2}\,q\,q'\right) . \tag{28}$$

We have introduced here the area $\triangle$ of the plane triangle, namely:

$$\frac{b\,c \sin A^*}{2} = \triangle \, . \tag{29}$$

Now we return for the second time to (22), and form by the introduction of (28):

$$b\,c\,(\cos A^* - \cos A) = \frac{2}{3}\,\frac{\triangle^2}{r^2}\left(1 + \frac{q\,q'}{3\,r^2}\right) + \frac{4}{90}\,\frac{\triangle^2}{r^2}\left(\frac{3\,p^2 + (q + q')^2 - 8\,q\,q'}{r^2}\right) .$$

We are to take into account here according to Fig. 4, p. 107:

$$p^2 + q^2 = b^2 \; , \quad p^2 + q'^2 = c^2 \quad \text{and} \quad (q + q')^2 = a^2 \; ,$$

whereby we will find:

$$b\,c\,(\cos A^* - \cos A) = \frac{2}{3}\frac{\triangle^2}{r^2} + \frac{2}{90}\frac{\triangle^2}{r^2}\left(\frac{3\,b^2 + 3\,c^2 - a^2}{r^2}\right). \tag{30}$$

Now the previous development (25) is carried further by one more term, namely with $A - A^* = x$:

$$\cos A = \cos(A^* + x) = \cos A^* - x \sin A^* - \frac{x^2}{2}\cos A^*$$

$$\cos A - \cos A^* = -\sin A^*\left(x + \frac{x^2}{2}\cot A^*\right).$$

This set into (30) yields, at the same time bearing in mind (29):

$$x + \frac{x^2}{2}\cot A^* = \frac{\triangle}{3r^2} + \frac{1}{90}\frac{\triangle}{r^4}(3\,b^2 + 3\,c^2 - a^2). \tag{30a}$$

The first approximation for x is:

$$x = \frac{\triangle}{3\,r^2} + \frac{1}{r^4} \cdots .$$

To this, we have from the plane triangle:

$$\cos A^* = \frac{b^2 + c^2 - a^2}{2\,b\,c} \qquad \sin A^* = \frac{2\,\triangle}{b\,c};$$

hence from (30a):

$$\frac{x^2}{2}\cot A^* = \frac{\triangle^2}{18\,r^4}\,\frac{b^2 + c^2 - a^2}{4\,\triangle} = \frac{\triangle}{72\,r^4}(b^2 + c^2 - a^2).$$

This set into (30a) yields:

$$x = \frac{\triangle}{3\,r^2} + \frac{\triangle}{360\,r^4}(7\,b^2 + 7\,c^2 + a^2). \tag{30b}$$

The angular reduction $A - A^* = x$ is represented here in radian measure; in order to change to seconds, we must add the factor ρ. If we do this and also write at the same time the other two corresponding formulae for $B - B^*$ and $C - C^*$, then we have:

$$x = A - A^* = \frac{\triangle}{3r^2}\varrho + \frac{\triangle}{360\,r^4}\varrho\,(a^2 + 7\,b^2 + 7\,c^2) \tag{31}$$

$$y = B - B^* = \frac{\triangle}{3\,r^2}\varrho + \frac{\triangle}{360\,r^4}\varrho\,(7\,a^2 + b^2 + 7\,c^2) \tag{32}$$

$$z = C - C^* = \frac{\triangle}{3\,r^2}\varrho + \frac{\triangle}{360\,r^4}\varrho\,(7\,a^2 + 7\,b^2 + c^2) \tag{33}$$

Sum $$\varepsilon = \frac{\triangle}{r^2}\varrho + \frac{\triangle}{24\,r^4}\varrho\,(a^2 + b^2 + c^2). \tag{34}$$

By $\triangle$ there is understood here the area of the *plane* triangle which can be constructed from the three sides a, b, c, and the above formulae are always approximation formulae only, because higher terms are still neglected. If we use, however, the spherical area F of the *spherical* triangle, then we have the rigorous formula, set up already previously in (2a), section 46, p. 89:

$$\varepsilon = \frac{F}{r^2}\rho . \tag{35}$$

By comparison of (35) and (34) we also have a comparison between F and $\triangle$, namely:

$$F = \triangle\left(1 + \frac{a^2 + b^2 + c^2}{24\, r^2}\right). \tag{36}$$

For this, we can also find another form by writing according to (26) or (26a):

$$\sin A = \sin A^* + \frac{\triangle}{3\, r^2}\cos A^* = \sin A^*\left(1 + \frac{\triangle}{3\, r^2}\cot\ A^*\right).$$

If we take to this the simple relations $2\, b\, c\, \cos A^* = b^2 + c^2 - a^2$ and $2\triangle = b\, c\, \sin A^*$, then we find:

$$\frac{\sin A}{\sin A^*} = 1 + \frac{b^2 + c^2 - a^2}{12\, r^2} ;$$

this applied also to the other two angles yields for (36):

$$F = \triangle\sqrt{\frac{\sin A \sin B \sin C}{\sin A^* \sin B^* \sin C^*}} . \tag{37}$$

Of similar significance as (36) is also the equation following from (34):

$$\frac{\triangle}{r^2} = \varepsilon\left(1 - \frac{a^2 + b^2 + c^2}{24\, r^2}\right). \tag{38}$$

If we set this into (31), (32), (33), then $\triangle$ is replaced everywhere by ε, and we have:

$$A - A^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{180}\left(\frac{-2\, a^2 + b^2 + c^2}{r^2}\right) \tag{39a}$$

$$B - B^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{180}\left(\frac{a^2 - 2\, b^2 + c^2}{r^2}\right) \tag{39b}$$

$$C - C^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{180}\left(\frac{a^2 + b^2 - 2\, c^2}{r^2}\right). \tag{39c}$$

Sum $\varepsilon = \varepsilon$ (check). (40)

Finally, we can undertake here a small change of form by introducing the mean value m^2 of a^2, b^2 and c^2, namely:

$$\frac{a^2 + b^2 + c^2}{3} = m^2 . \tag{41}$$

With this, the above formulae will be:

$$A - A^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{60}\,\frac{m^2 - a^2}{r^2} \tag{42a}$$

$$B - B^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{60}\,\frac{m^2 - b^2}{r^2} \tag{42b}$$

$$C - C^* = \frac{\varepsilon}{3} + \frac{\varepsilon}{60}\,\frac{m^2 - c^2}{r^2}\,. \tag{42c}$$

For an equilateral triangle the second terms vanish, which is also clear in itself.

If we take a right isosceles triangle with the legs [Katheten] a and a, hence $c^2 = 2\,a^2$, then we will have:

$$\varepsilon = \frac{a^2}{2\,r^2}\ ,\quad 3\,m^2 = a^2 + a^2 + 2\,a^2\ ,\quad m^2 = \frac{4}{3}\,a^2\,.$$

If we set $a = 100{,}000$ m, then the second terms in (42a), (42b) and (42c), respectively, will be:

$$+\frac{a^4}{360\,r^4}\,\varrho = +\,0.00003''$$

$$+\frac{a^4}{360\,r^4}\,\varrho = +\,0.00003''$$

$$-\frac{a^4}{180\,r^4}\,\varrho = -\,0.00007''\,.$$

For an application of the above formulae to a numerical example we take again the classical triangle Inselsberg, Hohehagen, Brocken, which we already have used repeatedly in sections 46-48, pp. 90-100.

Basing ourselves upon the computations which are to be considered here as preliminary, section 46, pp. 90 and 91, and section 47, p. 96, we obtain:

$$\log r = 6.804\,9621\ ,\quad \log\triangle = 9.467\,2168\ ,\quad \log F = 9.467\,2271$$

and then from (31), (32), (33), p. 110:

$$\begin{aligned} A - A^* &= 4.949\,900'' + 0.000\,136'' = 4.950\,036'' \\ B - B^* &= 4.949\,900 \ + 0.000\,096 \ = 4.949\,996 \\ C - C^* &= 4.949\,900 \ + 0.000\,121 \ = 4.950\,021 \\ \hline \varepsilon &= 14.849\,700'' + 0.000\,353'' = 14.850\,053''\,. \end{aligned}$$

We obtain the same also from the formulae (42a), (42b), (42c), namely:

$$\begin{aligned} A - A^* &= 4.950\,018'' + 0.000\,018'' = 4.950\,036'' \\ B - B^* &= 4.950\,018 \ - 0.000\,021 \ = 4.949\,997 \\ C - C^* &= 4.950\,018 \ + 0.000\,003 \ = 4.950\,021 \\ \hline \varepsilon &= 14.850\,054'' + 0.000\,000'' = 14.850\,054''\,. \end{aligned}$$

With these, we have the following spherical and plane angles:

Inselsberg	$A =$ 40° 39′ 30.380 000″		$A^* =$ 40° 39′ 25.429 964″
Hohehagen	$B =$ 86 13 58.840 000		$B^* =$ 86 13 53.890 004
Brocken	$C =$ 53 6 45.630 053		$C^* =$ 53 6 40.680 032
Sum	180° 0′ 14.850 053″		180° 0′ 0.000 000″.

If we repeat the previous computation (13) to (15), p. 96, with these angles, then we must compute at least to 10 places in order to render the difference still perceptible; nevertheless also in the 10-place logarithms the difference is at the most one in the last place, i.e. = 0·001, and e.g. at the triangle side $a = 69194.105$ m the new, more rigorous, computation yields only a difference of 0.00002 m or 0.02 mm.

Since the triangle used is one of the largest in German geodesy, we can, according to this, neglect unconcernedly the higher terms.

To conclude these developments, let us compute a summary table for the values of the correction term of the fourth order for Legendre's theorem, i.e. according to (42a) for the term:

$$\Delta A_4 = \frac{\varepsilon}{60 r^2}(m^2 - a^2), \text{ where } m^2 = \frac{a^2 + b^2 + c^2}{3}.$$

The two factors ε and $(m^2 - a^2)$ are independent of one another; the excess ε measures the area of the triangle, and the factor $(m^2 - a^2)$ is a measure for the absence of symmetry and inequality of sides of the triangle. If a triangle is very long but narrow, then ε can be small and $(m^2 - a^2)$ large; if a triangle is very large and nearly equilateral, then ε becomes large and $(m^2 - a^2)$ small; and hence, we can comprise best all cases imaginable by a table for ΔA_4 with two independent entries ε and $m^2 - a^2$, as is given in the following:

Angle Correction 4th Order ΔA_4, for Legendre's Theorem

Absence of Symmetry of the Triangle		Spherical Excess ε of the Triangle					
$\sqrt{m^2 - a^2}$	$m^2 - a^2$	$\varepsilon = 10''$	$\varepsilon = 20''$	$\varepsilon = 50''$	$\varepsilon = 100''$	$\varepsilon = 200''$	$\varepsilon = 300''$
km	km²						
10	100	0.00000″	0.00000″	0.00000″	0.00000″	0.00001″	0.00001″
20	400	0.00000	0.00000	0.00001	0.00002	0.00003	0.00005
50	2,500	0.00001	0.00002	0.00005	0.00010	0.00021	0.00031
100	10,000	0.00004	0.00008	0.00021	0.00041	0.00082	0.00123
200	40,000	0.00016	0.00033	0.00081	0.00164	0.00329	0.00493
400	160,000	0.00066	0.00131	0.00325	0.00657	0.01314	0.01972

Undertaking, in addition, a more general study about the error term of Legendre's theorem, we write according to (39a), p. 111, by bringing in the expression $\triangle$ according to (5), section 47, p. 95:

$$\Delta A_4 = \frac{b^2 + c^2 - 2a^2}{720 r^4}\sqrt{a^2(2b^2 + 2c^2 - a^2) - (b - c)^2}.$$

If we take $b = c$ here, i.e. the triangle having equal legs, then $(b - c)^2$ falls out, and the expression becomes a maximum with respect to the relation between b and c. Introducing the angle β adjoining the side a, then, with $c = b$, we can express the error term in two ways:

$$\Delta A_4 = \frac{a^4}{1440 r^4}\frac{1 - 4\cos^2\beta}{\cos^2\beta}\tan\beta \tag{a}$$

or

$$\Delta' A_4 = \frac{b^4}{180 r^4}(1 - 4\cos^2\beta)\sin 2\beta. \tag{b}$$

In the case (a) there results a maximum with $\beta = 45°$, and in the case (b) there result maxima with $\beta = 26°49'$ and $\beta = 73°44'$, and there follows hence:

$$(\beta = 45°), \qquad (\Delta A_4)\ max = 0.001\,389\,\frac{a^4}{r^4}$$

$$(\beta = 26°\,49') \qquad (\Delta' A_4)\ max = 0.009\,759\,\frac{b^4}{r^4}$$

$$(\beta = 73°\,44'), \qquad (\Delta' A_4)\ max = 0.002\,050\,\frac{b^4}{r^4}.$$

If we set here a or, as the case may be, $b = 100{,}000$ m, then the error term in question becomes $= 0.000\,017''$ or, as the case may be, $0.000\,121''$ and $0.000\,026''$.

We see hence that in the case of measurable triangles the correction of fourth order is always neglected.

The simple theorem by Legendre with development to $\frac{1}{r^2}$, inclusive, appeared in the Parisian *Mémoires de l'académie des sciences*, 1787, and meanwhile the proof has been found in numerous forms.

The development as far as terms of the order $\frac{1}{r^4}$ is given first by Buzengeiger in *Zeitschrift für Astronomie und verwandte Wissenschaften*, herausgegeben von Lindenau und Bohnenberger, 6th Volume, pp. 264-270, Tübingen, 1818. This is cited also by Bessel in *Astr. Nachr.*, 19th Volume, 1841, p. 103.

The above manner of treatment (in section 50) is called forth by the corresponding developments for triangles with geodetic lines in the classical treatise by Gauss, *Disquisitiones generales circa superficies curvas*, art. 24-28. This section 50 shall serve as preparation for our later analogous developments for triangles on the ellipsoid.

Baeyer already gives a discussion about the maximum influence of the spherical terms of the order $1{:}r^4$ (*Messen auf d. sphär. Oberfl.*, pp. 73-74).

In the second edition of the *Handbuch*, Vol. II, 1878, p. 131, there was contained such an investigation with the auxiliary condition of constant triangle surface. Helmert investigates in *Math. u. phys. Theorien der höheren Geodäsie*, I, section 16, the maximum influence of the higher terms with the auxiliary condition that the sum of the squares of the sides, i.e. $a^2 + b^2 + c^2 = 3\,m^2$, is constant.

Chapter V

SPHERICAL COORDINATES

Section 51. Summary of Coordinate Systems

In this chapter we consider the earth as a sphere of a given radius.

In this manner of consideration there are found many formulae and computational methods which we can also use at once on the ellipsoid, if we introduce a special spherical radius r which adapts itself to the curvature of the ellipsoid at the point concerned.

Others of the formulae to be developed in this chapter will not permit such an immediate transfer to the ellipsoid and therefore will have to be regarded only as a preparation for later developments.

Now we consider the individual kinds of the point determination by coordinates on the sphere.

I. Geographic coordinates

In Fig. 1 O is the center of the terrestrial sphere, which has the north pole N, the south pole S, therefore the terrestrial axis NS and the equator AA'.

NAS and NBS are the meridians of the two points P and P'. The reciprocal position of the two meridians is determined by the difference of longitude λ, which can be represented either as angle λ at the pole N or as arc AB on the equator.

On the meridian NA, the point P is determined by its geographic latitude φ, which can be represented either as central angle $AOP = \varphi$ or as meridian arc AP.

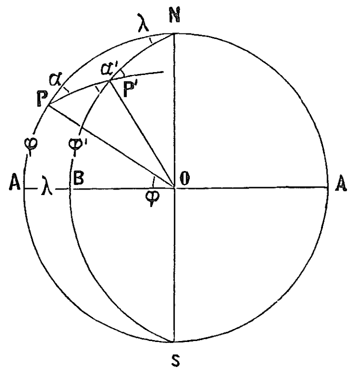

Fig. 1.
Geographic coordinates.

II. Polar coordinates

If P is regarded as a fixed point, then a second point P' can be determined with respect to it by the indication of the arc distance PP' and the azimuth $NPP' = \alpha$. The azimuths are mostly counted from north through east, as is indicated in Fig. 1 by α at P.

The arc PP' has a second azimuth α' at the point P', and in fact, in Fig. 1, the angle α' appears either as the azimuth of PP' from north through east in the extension through P', or as the azimuth of $P'P$ from south through west.

III. Rectangular coordinates

On the sphere as on the plane we can use a rectangular system of coordinates, in which an arbitrary great circle is used as axis of abscissae, while the ordinates are likewise arcs of great circles which lie at right angles to the axis of abscissae.

As a rule, such a system of coordinates is oriented with reference to the north direction, so that the meridian of the zero point forms the axis of abscissae. In this form, rectangular spherical coordinates were

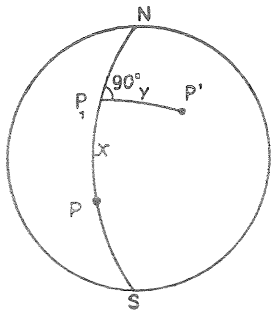

Fig. 2.

used by Soldner for the Bavarian land survey around 1809; in later times, they were also taken as a basis for trigonometric surveys in many other countries.

In Fig. 2 a rectangular spherical system of coordinates in the Soldner arrangement is represented. The meridian SPN of point P used as zero point is the axis of abscissae. To determine a point P' an arc of a great circle $P'P_1$ is laid perpendicularly to SPN, and then $PP_1 = x$ is the abscissa, P_1P' the ordinate of point P'.

We count here the abscissae x in the positive sense to the north and the ordinates y in the positive sense to the east.

Section 52. Rectangular Spherical Coordinates

In Fig. 1 we consider a rectangular system of coordinates in the Soldner arrangement with the zero point O and the meridian ON as axis of abscissae. It is noted, however, that the formulae which we develop in the following also hold for any other rectangular system of coordinates on the sphere with a great circle as axis of abscissae lying in an arbitrary position.

In order to determine a point A by coordinates, we lay an arc of a great circle $Q'A_1AQ$ through point A, perpendicular to the meridian ON, where A_1 on ON is the foot-point of the perpendicular AA_1 and Q' as well as Q are the so-called poles (cross-section poles) of the meridian SON.

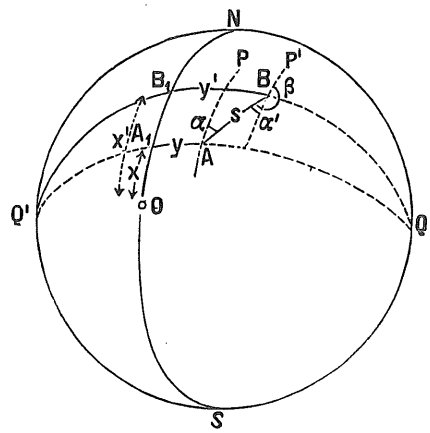

Fig. 1.

By means of the foot-point A_1 there are determined:

$$\left.\begin{aligned} OA_1 &= x, \text{ the abscissa of } A \\ A_1A &= y, \text{ the ordinate of } A. \end{aligned}\right\} \quad (1)$$

If, in addition, a second point B is to be determined by coordinates, then we again lay through it a great circle $Q'B_1BQ$, which furnishes the foot-point B_1 and passes through the same pole-points Q' and Q as the arc for A.

By the foot-point B_1 there are then determined:

$$\left.\begin{aligned} OB_1 &= x', \text{ the abscissa of } B \\ B_1B &= y', \text{ the ordinate of } B. \end{aligned}\right\} \quad (2)$$

Direction angle

Besides the coordinates themselves we have to define the concept of the direction angle. The direction angle α, which is assigned to the arc of a great circle AB at A, is the angle which this arc AB makes with the arc AP drawn parallel to the meridian of O (parallel circle to SON) at the point A.

In the case of the position of the coordinate system previously indicated, with $+x$ to the north and $+y$ to the east, the direction angles α are counted in the positive sense from the northward x to the eastward y, as is entered in Fig. 1.

The angle α, which is called direction angle here, is the same which was already introduced in this sense in the formulae for plane coordinate calculation in our Volume II, 1, 9th edition, 1931, p. 143.*

In addition to the direction angle α of AB at A, the direction angle β of BA at B is also determined as the angle between the arc of the parallel BP', and the ray BA, counted in the positive sense.

Besides β at B we have also entered the angle α' which is smaller than β by 180°, or more generally:

$$\alpha' = \beta \pm 180°. \quad (3)$$

---

* Not translated.

This angle α therefore means essentially the same as β; but it is usually more suitable in the formulae than β itself, since $\alpha - \alpha'$ is a *small* quantity which can be used in a development in series.

Development of the basic formulae

With reference to Fig. 1, p. 116, we set up the following problem:

There are given the coordinates x and y of a point A, further the length s of the arc AB and its direction angle α at A.

There are required the coordinates x' and y' of the point B of the other side and the direction angle β of the arc BA at B, of the other side, or instead of β itself, the difference $\alpha - \alpha'$.

We shall solve this problem with the help of the spherical triangle ABQ of Fig. 1, p. 116, and therefore have drawn especially, once again, this triangle in Fig. 2.

All sides and angles of this triangle are in a simple relation to the coordinates and direction angles discussed, e.g., the angle at A is $= 90° - \alpha$, and the angle at B is $= 90° + \alpha'$, as the comparison with Fig. 1, p. 116, immediately yields; and, moreover, it is to be noted that in Fig. 2 the linear values of Fig. 1, p. 116, are now brought to angles at the earth's center in radian measure by division by the earth's radius r, e.g., the distance s in Fig. 1, p. 116, yields $\frac{s}{r}$ in Fig. 2, and so forth. The value $\frac{x' - x}{r}$ appears in Fig. 2 twice, first as arc $A_1 B_1$, and second as angle Q, because QA_1 and QB_1 are both quadrants, i.e., in radian measure $= \frac{\pi}{2}$.

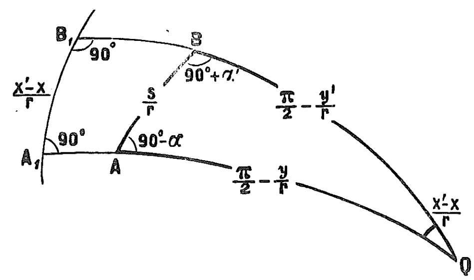

Fig. 2.
Triangle ABQ of Fig. 1, p. 116.

After this preparation we use three formulae of spherical trigonometry, namely:

1. a cosine formula (of p. 16),
2. a sine formula (of p. 16),
3. a Gauss formula (p. 17).

Individually, these three formulae written in greatest detail yield:

1.
$$\cos\left(\frac{\pi}{2} - \frac{y'}{r}\right) = \cos\frac{s}{r}\cos\left(\frac{\pi}{2} - \frac{y}{r}\right) + \sin\frac{s}{r}\sin\left(\frac{\pi}{2} - \frac{y}{r}\right)\cos(90° - \alpha)\,;$$

2.
$$\frac{\sin\frac{x'-x}{r}}{\sin\frac{s}{r}} = \frac{\sin(90° - \alpha)}{\sin\left(\frac{\pi}{2} - \frac{y'}{r}\right)}\,;$$

3.
$$\tan\frac{(90° + \alpha') + (90° - \alpha)}{2} = \frac{\cos\frac{1}{2}\left(\left(\frac{\pi}{2} - \frac{y}{r}\right) - \left(\frac{\pi}{2} - \frac{y'}{r}\right)\right)}{\cos\frac{1}{2}\left(\left(\frac{\pi}{2} - \frac{y}{r}\right) + \left(\frac{\pi}{2} - \frac{y'}{r}\right)\right)}\cot\frac{x'-x}{2r}.$$

If we simplify these three equations, which are written in great detail in order that their mode of origin may remain visible, then we obtain:

1. for y':
$$\sin\frac{y'}{r} = \cos\frac{s}{r}\sin\frac{y}{r} + \sin\frac{s}{r}\cos\frac{y}{r}\sin\alpha\,; \tag{4}$$

2. for x':
$$\sin\frac{x'-x}{r} = \frac{\sin\frac{s}{r}}{\cos\frac{y'}{r}}\cos\alpha\,; \tag{5}$$

3. for α':

$$\cot\frac{\alpha-\alpha'}{2}=\frac{\cos\frac{y'-y}{2r}}{\sin\frac{y'+y}{2r}}\cot\frac{x'-x}{2r}; \qquad (6)$$

and if we invert the numerator and the denominator here, then we have:

$$\tan\frac{\alpha-\alpha'}{2}=\frac{\sin\frac{y'+y}{2r}}{\cos\frac{y'-y}{2r}}\tan\frac{x'-x}{2r}. \qquad (7)$$

At first we take up equation (4) alone:

$$\sin\frac{y'}{r}=\cos\frac{s}{r}\sin\frac{y}{r}+\sin\frac{s}{r}\cos\frac{y}{r}\sin\alpha.$$

The power series for $\sin$ and $\cos$ from p. 23 are applied to the sines and cosines of small magnitudes occurring herein, however with the limitation to terms of third order; this yields:

$$\frac{y'}{r}-\frac{y'^3}{6r^3}=\left(1-\frac{s^2}{2r^2}\right)\left(\frac{y}{r}-\frac{y^3}{6r^3}\right)+\left(\frac{s}{r}-\frac{s^3}{6r^3}\right)\left(1-\frac{y^2}{2r^2}\right)\sin\alpha.$$

If we compute further with the omission of higher terms, then we obtain:

$$y'-\frac{y'^3}{6r^2}=y\left(1-\frac{s^2}{2r^2}-\frac{y^2}{6r^2}\right)+s\sin\alpha\left(1-\frac{s^2}{6r^2}-\frac{y^2}{2r^2}\right). \qquad (8)$$

This equation shall be solved for y'; we thus have to deal with a cubic equation. However, since all terms of an order higher than the third have been neglected from the outset, then, accordingly, the solution of (8) can also be carried out approximately, i.e., we form first the first approximation for y':

$$y'=y+s\sin\alpha+\frac{1}{r^2}\dots.$$

This approximate value of y' is sufficient in order to determine the second term $\frac{y'^3}{6r^2}$ in (8) accurately to terms of the order $\frac{1}{r^2}$ inclusive. We therefore have by introducing the first approximation into that second term:

$$y'-\frac{(y+s\sin\alpha)^3}{6r^2}=y+s\sin\alpha+y\left(-\frac{s^2}{2r^2}-\frac{y^2}{6r^2}\right)+s\sin\alpha\left(-\frac{s^2}{6r^2}-\frac{y^2}{2r^2}\right)$$

$$y'=y+s\sin\alpha-\frac{3s^2y-3s^2y\sin^2\alpha+s^3\sin\alpha-s^3\sin^3\alpha}{6r^2}$$

$$y'=y+s\sin\alpha-\frac{s^2y\cos^2\alpha}{2r^2}-\frac{s^3\sin\alpha\cos^2\alpha}{6r^2}. \qquad (9)$$

With this, the equation for y' is taken care of, and we pass over to the development for x'. For the determination of $x' - x$ we have the equation (5):

$$\sin\frac{x'-x}{r} = \frac{\sin\frac{s}{r}}{\cos\frac{y'}{r}}\cos\alpha .$$

This developed to the third order yields:

$$\frac{x'-x}{r} - \frac{(x'-x)^3}{6r^3} = \frac{\frac{s}{r} - \frac{s^3}{6r^3}}{1 - \frac{y'^2}{2r^2}}\cos\alpha$$

$$\frac{x'-x}{r} - \frac{(x'-x)^3}{6r^3} = \left(\frac{s}{r} - \frac{s^3}{6r^3}\right)\left(1 + \frac{y'^2}{2r^2}\right)\cos\alpha$$

$$x'-x - \frac{(x'-x)^3}{6r^2} = s\cos\alpha\left(1 - \frac{s^2}{6r^2} + \frac{y'^2}{2r^2}\right).$$

First approximation $x' - x = s\cos\alpha + \ldots$;

consequently:

$$x'-x = \frac{(s\cos\alpha)^3}{6r^2} + s\cos\alpha - \frac{s^3\cos\alpha}{6r^2} + \frac{s\cos\alpha\, y'^2}{2r^2}$$

$$x' = x + s\cos\alpha + \frac{s\cos\alpha\, y'^2}{2r^2} - \frac{s^3\cos\alpha\sin^2\alpha}{6r^2}. \tag{10}$$

With this, the second equation for x' is also taken care of, and we pass over to the development for the difference of the direction angles. For the determination of $\alpha - \alpha'$ we have the equation (7), in the development of which we can already stop everywhere at the first term, because on the right-hand side there already results thereby a term of the order $\frac{1}{r^2}$, beyond which we do not go. Therefore, we have in abbreviated form from (7):

$$\frac{\alpha-\alpha'}{2} + \ldots = \frac{\frac{y'+y}{2r} - \ldots}{1 - \ldots}\,\frac{x'-x}{2r} + \ldots$$

$$\alpha - \alpha' = \frac{y'+y}{2r^2}(x'-x). \tag{11}$$

We obtain for this a somewhat different form if we substitute $y' = y + s\sin\alpha + \ldots$, according to (9), namely:

$$\alpha - \alpha' = (x'-x)\frac{y}{r^2} + \frac{(x'-x)\,s\sin\alpha}{2r^2}. \tag{12}$$

For the reduction to seconds, the factor $\rho = 206{,}265''$ is further to be inserted in (11) and (12).

Summary

For an over-all illustration we introduce further an abbreviating notation by setting:

$$s\sin\alpha = v \quad \text{and} \quad s\cos\alpha = u. \tag{13}$$

With this, the formulae (9), (10) and (12) yield, the latter with the insertion of ρ:

$$y' = y + v - \frac{u^2\, y}{2\, r^2} - \frac{u^2\, v}{6\, r^2} \qquad (14)$$

$$x' = x + u + \frac{u\, y'^2}{2\, r^2} - \frac{u\, v^2}{6\, r^2} \qquad (15)$$

$$\alpha - \alpha' = u\, y \frac{\varrho}{r^2} + u\, v \frac{\varrho}{2\, r^2} \quad \text{or} \quad = u \frac{y + y'}{2} \frac{\varrho}{r^2}. \qquad (16)$$

To this, $\beta = \alpha' \pm 180^\circ$,

and hence:

$$\beta = \alpha \pm 180^\circ - u\, y \frac{\varrho}{r^2} - u\, v \frac{\varrho}{2\, r^2}. \qquad (16a)$$

Given the geographic latitude, we can always determine the coefficients of these and some related formulae depending on r according to the auxiliary table on pages [12] to [33] of the Appendix; for the latitudes $\varphi = 45^\circ, 50^\circ, 55^\circ$ the logarithms of these coefficients are the following (with the omission of the -10's, and so forth) and in the case of μ for units of the 7th place:

φ	$\log \frac{1}{2r^2}$	$\log \frac{1}{6r^2}$	$\log \frac{\varrho}{r^2}$	$\log \frac{\varrho}{2r^2}$	$\log \frac{\mu}{2r^2}$	$\log \frac{\mu}{6r^2}$
45°	6.08969	5,61257	1.70514	1.40411	2·72747	2·25035
50°	6.08918	5.61206	1.70464	1.40361	2·72697	2·24985
55°	6.08869	5.61157	1.70415	1.40312	2·72648	2·24936 .

If we set the radius $r = \infty$ in the formulae (14), (15), (16), i.e., if we let the sphere pass over into the plane, then we obtain:

$$y' = y + s \sin \alpha \qquad x' = x + s \cos \alpha \qquad \alpha' = \alpha. \qquad (17)$$

These are the formulae valid for the plane coordinate computation.

We have the same in another form if instead of setting $r = \infty$ we make the distance s and, with it, also u and v very small; we see hence that the spherical formulae pass over by themselves into the formulae of the plane as soon as the distances become so small that the application of the correction terms is not worth while.

The great practical significance of the rectangular spherical coordinates is based essentially on this illustrative and convenient change from the spherical computation to the plane.

Our final formulae (14), (15), (16) are also used, in addition, in some other forms, as was already shown in the case of (11) and (12).

The ordinate formula (14), also, can be written in a transformed manner, thus:

$$y' - y = s \sin \alpha - \frac{2\, y + y'}{3} \frac{u^2}{2\, r^2}. \qquad (18)$$

The circumstance, apparently disturbing from the algebraic point of view, that y' itself, which is to be determined first, occurs in equation (15) for x is unimportant for our applications if y' is determined first in order to have it for the introduction into the correction term for x; if, by way of exception, x alone should have to be determined, then we would have to take an approximate value of $y' = y + s \sin \alpha$ for the computation of the first correction term of x.

Computational auxiliary means for the correction terms of the coordinate formulae

If we have printed forms for frequent application of these coordinate formulae, and the constant logarithms of coefficients $log \frac{1}{2r^2}$, $log \frac{\rho}{2r^2}$, and so forth, are printed here, then the 4-place logarithmic computation proceeds rather quickly according to the formulae (14), (15), (16); however, special auxiliary means have already been used several times.

In our Appendix on page [57] we have given two small tables I and II for the correction terms of formulae (14), (15), (16), as far as all these terms have for the most part the form $\frac{A^2 B}{2r^2}$ or $\frac{A B}{r^2}\rho$; these tables I and II on page [57], however, are not designed for actual computing, but only for an over-all illustration, or as leading values for graphic representations, or else to assist computations with the slide rule and the like.

The work, *Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst* by F. G. Gauss, 4th Edition, 1922, Part II, pp. 54-61, contains a more detailed table of the values $\frac{A^2 B}{2r^2}$, designed for immediate practical use, in the case of which the spherical radius is adapted to the geographic latitude of 51°.

Graphic auxiliaries for Soldner's correction terms have been used in the case of the land survey of Baden. Franke, *Die Grundlehren der trigonometrischen Vermessung*, Leipzig, 1879, Appendix Tables I, II, III gives three specially arranged tables for the purpose under discussion.

But all these auxiliary means are hardly capable of replacing the direct calculation of the correction terms for rigorous computations, in view of the fact that, in order to guard against the many accumulations of rounding-off, we compute the individual terms mostly to 0.001 m in the case of such computations, in order to have 0.01 m still rigorous in the final result, and for this purpose the calculation of the terms in the printed form with 4 (5)-place logarithms is still the best.

Formulae by Zachariae for rectangular spherical coordinates

Equations (14) to (16) of p. 120 can be brought into another form in which they were developed by Zachariae in a different way.

For this, we write for equation (14):

$$y' = y + s \sin \alpha - s \cos \alpha \left(\frac{u\,y}{2\,r^2} + \frac{u\,v}{6\,r^2}\right). \tag{19}$$

With the help of Fig. 1, p. 116, the two terms $\frac{u\,y}{2r^2}$ and $\frac{u\,v}{6r^2}$ admit of a geometric interpretation. In Fig. 1, p. 116, the spherical quadrangle $A_1 B_1 B A$ is divided, by the arc AP, into a spherical rectangle and a rectangular triangle. If the spherical excesses of these two figures are denoted by 2ε and 3γ, then we have

$$2\,\varepsilon = \frac{(x' - x)\,y}{r^2} = \frac{u\,y}{r^2} \qquad 3\,\gamma = \frac{u\,v}{2\,r^2}. \tag{20}$$

Consequently, (19) becomes:

$$y' = y + s \sin \alpha - s \cos \alpha\,(\varepsilon + \gamma)$$

or

$$y' = y + s \sin\,(\alpha - (\varepsilon + \gamma)). \tag{21}$$

For the transformation of (15) we introduce at first $y' = y + v$ into the term with $\frac{1}{r^2}$ and obtain:

$$x' = x + u + \frac{u\,y^2}{2\,r^2} + \frac{u\,v\,y}{r^2} + \frac{u\,v^2}{3\,r^2}, \tag{22}$$

and since the terms with $\frac{1}{r^4}$ are to be neglected, then we can also write for this:

$$x' = x + \left(u + \frac{u\,v\,y}{r^2} + \frac{u\,v^2}{3\,r^2}\right)\left(1 + \frac{y^2}{2\,r^2}\right)$$

or

$$x' = x + \left(s \cos\alpha + s \sin\alpha\,(2\,\varepsilon + 2\,\gamma)\right)\left(1 + \frac{y^2}{2\,r^2}\right)$$

and this is

$$x' = x + s \cos\left(\alpha - (2\,\varepsilon + 2\,\gamma)\right)\left(1 + \frac{y^2}{2\,r^2}\right). \tag{23}$$

With the help of (20), equation (16) of p. 120 passes over immediately into

$$\alpha - \alpha' = 2\,\varepsilon + 3\,\gamma, \tag{24}$$

and from (16a), p. 120, we will have

$$\beta = \alpha \pm 180° - 2\,\varepsilon - 3\,\gamma. \tag{25}$$

We summarize these formulae once again:

$$\left.\begin{aligned}
&\varepsilon = \frac{u\,y}{2\,r^2}\varrho \qquad \gamma = \frac{u\,v}{6\,r^2}\varrho\\
&y' = y + s \sin\left(\alpha - (\varepsilon + \gamma)\right)\\
&x' = x + s \cos\left(\alpha - 2\,(\varepsilon + \gamma)\right)\left(1 + \frac{y^2}{2\,r^2}\right)\\
&\alpha - \alpha' = 2\,\varepsilon + 3\,\gamma\\
&\beta = \alpha \pm 180° - 2\,\varepsilon - 3\,\gamma.
\end{aligned}\right\} \tag{26}$$

The above formulae were developed by G. Zachariae in the work *Die geodätischen Hauptpunkte und ihre Koordinaten*, Kopenhagen, 1876; German by E. Lamp, Berlin, 1878, p. 186.

As a connected numerical example with which the whole course of spherical coordinate computation can be shown, take the northern part of the net of Baden, which is already treated in Volume I, 8th Edition, p. 222.

The triangulation net picture with the indication of the coordinate system is drawn in Fig. 1 in the margin.

From the point Mannheim Observatory, which serves as coordinate zero point, the official survey of Baden counts $+x$ to the south and $+y$ to the west, while we will now count $+x$ to the north and $+y$ to the east, corresponding to the more general usage in Germany.

At the Mannheim Observatory, the azimuth to Speyer was measured astronomically, counted from the north through east:

$$\text{Azimuth Mannheim-Speyer} = 183^\circ\, 40'\, 25.291''. \qquad (1)$$

We will find this *azimuth* Mannheim-Speyer again under the designation *direction angle* in the station adjustment data on p. 124, for at the coordinate zero point through which the meridian passes as x-axis, the azimuth is equal to the direction angle, while at all other points the direction angles vary from the azimuths.

The adjustment of the net was carried out for the angle computations under consideration according to condition adjustments; cf. Volume I, 8th Edition, 1935, p. 222. The spherical excesses of the triangle are formed here, as indicated in section 46, p. 90. The results of the adjustment are thus adjusted angles and distances, which are introduced here according to the computations communicated previously in the 3rd Edition, Volume I, 1888, pp. 202-203.

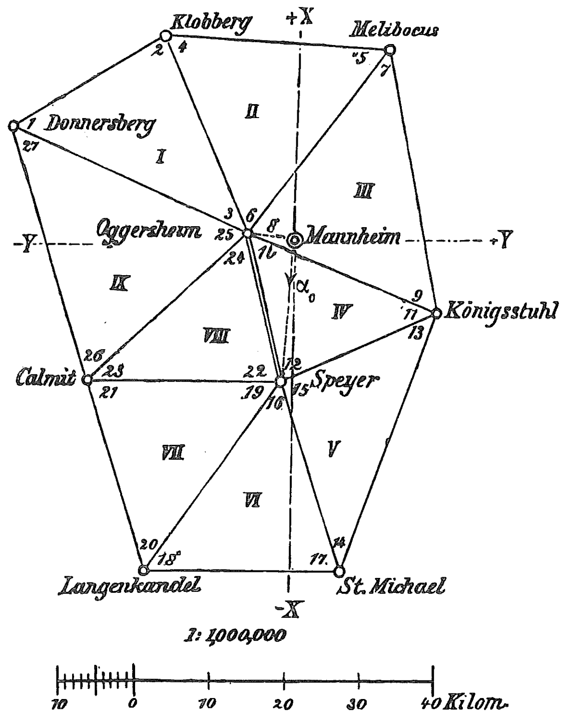

Fig. 1. System of coordinates with the origin Mannheim. $+x$ to the north, $+y$ to the east.

We now give at once the whole data of the net with adjusted direction angles α and adjusted distances $log\ s$, and although, it is true, only all $log\ s$'s can be introduced as results of the net adjustment, the columns of the direction angles α, however, will be filled only gradually in the course of the following coordinate computation.

At first we must form the constants necessary for the terms of second order:

The zero point of coordinates has a geographic latitude of approximately $\varphi = 49^\circ\, 30'$, and with this, we form the constant logarithms of coefficients necessary for us according to page [24] of the Appendix:

$$\left.\begin{array}{lll} log\,\dfrac{1}{2r^2} = 6.089\,23 & log\,\dfrac{1}{6r^2} = 5.612\,11 & log\,\dfrac{1}{3r^2} = 5.913\,14 \\ log\,\dfrac{\varrho}{r^2} = 1.704\,69 & log\,\dfrac{\varrho}{2r^2} = 1.403\,66 & log\,\dfrac{\varrho}{6r^2} = 0.926\,54 \end{array}\right\}. \qquad (2)$$

We begin with the starting value already indicated in (1)

$$\text{Mannheim-Speyer}\quad \alpha = 183^\circ\, 40'\, 25.291'' \quad\text{and}\quad log\ s = 4.275\,4362{\cdot}8. \qquad (3)$$

With this, we can compute at once the coordinates y', x' of Speyer, and in fact, this time the general formulae are simplified because the starting coordinates y, x for Mannheim are both zero. If we thus set

Station Data of the Triangulation of the Net, Fig. 1, p. 123, with Rectangular Spherical Coordinates

Stations and Target Points	Direction Angle: Spherical α	Reduct. $\delta\alpha$	Plane $\alpha_0 = \alpha + \delta\alpha$	Distance: Spherical $\log s$	Reduct. $\delta \log s$	Plane $\log s_0$
1. Mannheim						
Speyer	183° 40′ 25.29″	−0.02″	183° 40′ 25,27″	4.275 4362·8	+ 0·0	4.275 4362·8
Oggersheim	273 42 22.23	+0.00	273 42 22.23	3.779 1890·3	+ 0·3	3.779 1890·6
2. Speyer						
Mannheim	3° 40′ 25.23″	+0.04	3° 40′ 25.27″	4.275 4362·8	+ 0·0	4.275 4362·8
Königsstuhl	65 10 11.04	−0.25	65 10 10.79	4.358 8019·0	+ 1·1	4.358 8020·1
St. Michael	161 20 31.86	+1.13	161 20 32.99	4.430 2529·8	+ 0·8	4.430 2530·6
Langenkandel . . .	215 0 1,15	−0,63	215 0 0.52	4.502 8974·0	+ 5·0	4.502 8979·0
Calmit	270 34 57.86	+0.00	270 34 57.86	4.418 4219·3	+ 0·3	4.418 4219·6
Oggersheim	345 59 7.49	+0.13	345 59 7.62	4.296 5476·9	+ 0·3	4.296 5477·2
3. Oggersheim						
Melibocus	35° 38′ 31,00″	−0.07	35° 38′ 30.93″	4.507 0618·5	+ 1·9	4.507 0620·4
Mannheim	93 42 22.24	−0.01	93 42 22.23	3.779 1890·3	+ 0·3	3.779 1890·6
Königsstuhl	110 37 58.62	+0.14	110 37 58.76	4.435 7945·9	+ 0·9	4.435 7946·8
Speyer	165 59 7.85	−0.23	165 59 7.62	4.296 5476·9	+ 0·3	4.296 5477·2
Calmit	228 30 28,54	−1.02	228 30 27.52	4.456 1549·3	+ 7·7	4.456 1557·0
Donnersberg	294 51 17.21	+1.17	294 51 18.38	4.549 3120·0	+ 5·6	4.549 3125·6
Klobberg	336 22 4.82	+0.75	336 22 5.57	4.479 8976·0	+ 7·2	4.479 8983·2
4. Calmit						
Oggersheim	48° 30′ 26.93″	+0.59	48° 30′ 27.52″	4.456 1549·3	+ 7·7	4.456 1557·0
Speyer	90 34 57.88	−0.02	90 34 57.86	4.418 4219·3	+ 0·3	4.418 4219·6
Langenkandel . . .	163 12 53.76	−1,29	163 12 52.47	4.439 5852·2	+26·9	4.439 5879·1
Donnersberg	342 23 54.18	+3.44	342 23 57.62	4.550 1058·1	+52·3	4.550 1110·4
5. Donnersberg						
Klobberg	57° 29′ 38.98″	+0.08	57° 29′ 39.06″	4.375 9182·8	+13·2	4.375 9196·0
Oggersheim	114 51 18.88	+0.50	114 51 18.38	4.549 3120·0	+ 5·6	4.549 3125·6
Calmit	162 23 59.80	−2.18	162 23 57.62	4.550 1058·1	+52·3	4.550 1110·4
6. Klobberg						
Melibocus	92° 51′ 35.28″	−0.02	92° 51′ 35.26″	4.489 5442·5	+ 0·3	4.489 5442·8
Oggersheim	156 22 6.51	−0.94	156 22 5.57	4.479 8976·0	+ 7·2	4.479 8983·2
Donnersberg	237 29 40.80	−1.74	237 29 39.06	4.375 9182·8	+13·2	4.375 9196·0
7. Melibocus						
Königsstuhl	169° 13′ 40.29″	+1,49	169° 13′ 41.78″	4.560 7787·3	+13·9	4.560 7801·2
Oggersheim	215 38 30.55	+0.38	215 38 30.93	4.507 0618·5	+ 1·9	4.507 0620·4
Klobberg	272 51 35.26	+0.00	272 51 35.26	4.489 5442·5	+ 0·3	4.489 5442·8
8. Königsstuhl						
St. Michael	199° 2′ 30.38″	+1.23	199° 2′ 31.61″	4.569 8613·7	+ 9·2	4.569 8622·9
Speyer	245 10 10.59	+0.20	245 10 10.79	4.358 8019·0	+ 1·1	4.358 8020·1
Oggersheim	290 37 58.96	−0.20	290 37 58.76	4.435 7945·9	+ 0·9	4.435 7946·8
Melibocus	349 13 43.21	−1.43	349 13 41.78	4.560 7787·3	+13·9	4.560 7801·2
9. St. Michael						
Königsstuhl	19° 2′ 32.78″	−1.17	19° 2′ 31.61″	4.569 8613·7	+ 9·2	4.569 8622·9
Langenkandel . . .	268 48 10.93	+0.86	268 48 11.79	4.429 4468·0	+ 0	4.429 4468·0
Speyer	341 20 32.27	−0.27	341 20 32.00	4.430 2529·8	+ 0·8	4.430 2530·6
10. Langenkandel						
Speyer	34° 59′ 59.78″	+0.74	35° 0′ 0.52″	4.502 8974·0	+ 5·0	4.502 8979·0
St. Michael	88 48 10.91	+0.88	88 48 11.79	4.429 4468·0	+ 0·0	4.429 4468·0
Calmit	343 12 50.62	+1.85	343 12 52.47	4.439 5852·2	+26·9	4.439 5879·1

$y = 0$ and $x = 0$ in the formulae (13) to (16), section 52, pp. 119 and 120, then we obtain:

$$\text{Speyer} \quad \left.\begin{aligned} & s \sin \alpha = v \qquad s \cos \alpha = u \\ & y' = v - \frac{u^2 v}{6 r^2}, \qquad x' = u + \frac{u v^2}{3 r^2}, \\ & \alpha' = \alpha - \frac{\varrho}{2 r^2} u v . \end{aligned}\right\} \tag{4}$$

Now we compute, according to this, with 7-place logarithms (with an 8th check place 0·1), in itself uncertain, obtained by interpolation according to Schrön):

$\alpha = 183^\circ\, 40'\, 25{,}291''$	$\log s$	4.275 4362·8	$\log s$	4.275 4362·8
	$\log \sin \alpha$	$8.806\,6825{\cdot}0_n$	$\log \cos \alpha$	$9.999\,1066{\cdot}6_n$
	$\log v$	$3.082\,1187{\cdot}8_n$	$\log u$	$4.274\,5429{\cdot}4_n$
	$v = -1208.144$ m		$u = -18816.678$ m .	

(5)

To this, the correction terms according to (4):

$\log u^2$	8.5491	$\log u$	4.2745_n
$\log v$	3.0821_n	$\log v^2$	6.1642
$\log(-1:6r^2)$	5.6121_n	$\log(1:3r^2)$	5.9131
	7.2433		6.3518_n
	$+0.0017$		-0.0002 .

(6)

Adding these small amounts to (5) we have:

$$\text{Speyer} \quad y' = -1208.142 \text{ m} \qquad x' = -18{,}816.678 \text{ m}. \tag{7}$$

In addition, finally, the difference of the direction angles:

$\log u$	4.2745_n	
$\log v$	3.0821_n	
$\log(-\varrho : 2r^2)$	1.4037_n	
	8.7603_n	$-0.058''$.

(8)

We thus have all together:

Direction angle Mannheim-Speyer	$\alpha = 183^\circ\, 40'\, 25.291''$
to this, according to (8):	-0.058
	$\alpha' = 183^\circ\, 40'\, 25.233''$
	$\pm 180^\circ$.

And hence, direction angle $\alpha_2 = 3^\circ\, 40'\, 25.233''$. (9)

Now we can indicate the direction angles of all rays starting from Speyer, for we only need to combine the angles adjusted at Speyer with α_2. For this, we need the adjusted angles of the triangles of the Station Speyer, which are according to Volume I, 3rd Edition, 1888, pp. 202, 203, as the view of the triangulation net picture, Fig. 1, p. 123, shows:

Mannheim – Königsstuhl	...	$= 61^\circ\, 29'\, 45.804''$	
Königsstuhl – St. Michael	[15]	= 96 10 20.829	
St. Michael – Langenkandel	[16]	= 53 39 29.285	$360^\circ\, 0'\, 0.000''$.
Langenkandel – Calmit	[19]	= 55 34 56.714	
Calmit – Oggersheim	[22]	= 75 24 9.630	
Oggersheim – Mannheim	...	= 17 41 17.738	

If we add these six angles one after the other to the just computed $3^\circ\, 40'\, 25.233''$, then we obtain all direction angles, as indicated for the Station Speyer on p. 124, rounded off to 0.01".

After this first example we can say briefly that the second direction Mannheim-Oggersheim starting from the zero point Mannheim is treated just as Mannheim-Speyer, and then also permits to fill in the data for Station Oggersheim, as is seen on p. 124.

But then there come the general coordinate formulae of (14) to (16), section 52, p. 120, the application of which shall be shown with the example Speyer-Langenkandel:

From the station data on p. 124 thus far filled in we take to this:

Speyer–Langenkandel $\alpha = 215° \, 0' \, 1.150''$ and $\log s = 4.502\,8974{\cdot}0$

$\log s$	4.502 8974·0	$\log s$	4.502 8974·0
$\log \sin \alpha$	$9.758\,5947{\cdot}8_n$	$\log \cos \alpha$	$9.913\,3628{\cdot}3_n$
$\log v$	$4.261\,4921{\cdot}8_n$	$\log u$	$4.416\,2602{\cdot}3_n$
	$v = -18{,}259.639$		$u = -26{,}077.156$
Speyer (7) given:	$y = -\ 1{,}208.142$		$x = -18{,}816.678$
	$y + v = -19{,}467.781$		$x + u = -44{,}893.834$

Langenkandel approximately $= y'$ approximately $= x'$.

Correction terms for y. Correction terms for x.

$\log u^2$	8.8325	$\log u^2$	8.8325	$\log u$	4.4163_n	$\log u$	4.4163_n
$\log y$	3.0821_n	$\log v$	4.2615_n	$\log y'^2$	8.5786	$\log v^2$	8.5230
$\log(-1:2r^2)$	6.0892_n	$\log(-1:6r^2)$	5.6121_n	$\log(1:2r^2)$	6.0892	$\log(-1:6r^2)$	5.6121_n
	8.0038		8.7061		9.0841_n		8.5514
	+ 0.010		+ 0.051		— 0.121		+ 0.036

Summary:

$y + v =$	— 19,467.781 m	$x + u =$	— 44,893.834 m
	+ 0.010		— 0.121
	+ 0.051		+ 0.036
Langenkandel: $y' =$	— 19,467.720	$x' =$	— 44,893.919 .

Correction terms for α:

$\log u$	4.4163_n	$\log u$	4.4163_n
$\log y$	3.0821_n	$\log v$	4.2615_n
$\log(-\varrho : r^2)$	1.7047_n	$\log(-\varrho : 2r^2)$	1.4037_n
	9.2031_n		0.0815_n
	— 0.160		— 1.206

Instead of this, the computation can also be conducted according to the second formula (16), section 52, p. 120.

$$\alpha' - \alpha = u \frac{y + y'}{2} \frac{\varrho}{r^2} .$$

Summary: $\alpha = 215° \ 0' \ 1.150''$
— 0.160
— 1.206

$\alpha' = 214° \, 59' \, 59.784''$
$\pm 180°$
$\alpha_{10} = \ 34° \, 59' \, 59.784''$ = direction angle Langenkandel-Speyer.

With this direction angle α_{10} and with the angles of the triangles adjusted at Langenkandel we can now set up anew an oriented set of data for the Station Langenkandel, as is seen on p. 124.

In this manner, we compute around in the whole net in various ways, whereby numerous checks originate for the direction angles as well as for the coordinates; e.g., after the two Stations Speyer and Oggersheim have been taken care of, we can compute toward Station Calmit from both sides, and we will find:

from Speyer:	Calmit $y_4 = -27{,}414.066$ m	$x_4 = -18{,}550.134$ m
from Oggersheim:	Calmit $y_4 = -27{,}414.065$ m	$x_4 = -18{,}550.135$ m,

which agrees sufficiently. Likewise also the direction angle:

from Speyer:	Direction angle (4.2) = $90° \, 34' \, 57.882''$
from Oggersheim:	Direction angle (4.3) = $48° \, 30' \, 26.932''$.

These two direction angles are introduced into the data of the Station Calmit and yield, with the angles of the triangles adjusted at Calmit, checks which have led in our case to changes of 0.002'' and 0.003'', which, however, can no longer become noticeable on p. 124, where everything is rounded off to 0.01".

We see that the station data of p. 124 with regard to the direction angles are gradually formed. The final values of the coordinates are the following:

Point	y	x	
	m	m	
1. Mannheim . .	0.000	0.000	Rectangular spherical coordinates of all points of the net, Fig. 1, p. 123. Zero point Mannheim with $+x$ to the north, and $+y$ to the east. (10)
2. Speyer	— 1,208.142	— 18,816.676	
3. Oggersheim .	— 6,001.777	+ 388.767	
4. Calmit. . . .	— 27,414.066	— 18,550.134	
5. Donnersberg.	— 38,145.688	+ 15,278.872	
6. Klobberg . .	— 18,104.628	+ 28,049.296	
7. Melibocus . .	+ 12,727.470	+ 26,509.100	
8. Königsstuhl .	+ 19,525.476	— 9,223.075	
9. St. Michael .	+ 7,407.498	— 44,332.386	
10. Langenkandel	— 19,467.721	— 44,893.918	

Comparisons of these coordinates with the official coordinates of Baden, Bavaria, and Hesse and remarks pertaining to it were given in Volume I, 3rd Edition, 1888, pp. 203-204; Volume II, 2nd Edition, 1878, p. 272, and *Astronomische Nachrichten*, 75th Volume, 1870, Nos. 1795-1796, pp. 289-306 and 367.

The station data of p. 124 would be sufficient for the purposes of the treatment of spherical coordinates, if it contained only the spherical direction angles α and the spherical distances $log\ s$. But we have also computed the plane direction angles α_0 and the plane distances s_0 according to the simple formulae:

$$tan\ \alpha_0 = \frac{y' - y}{x' - x} \qquad s_0 = \frac{y' - y}{sin\ \alpha_0} = \frac{x' - x}{cos\ \alpha_0},$$

where the coordinates $x\,y$, $x'\,y'$, even the spherical ones, are contained in the above table (10). The differences $\delta\alpha$ and $\delta\ log\ s$ are then obtained simply from $\alpha_0 - \alpha$ and $log\ s_0 - log\ s$. In the following section 54 it will be shown how we compute these $\delta\alpha$'s and $\delta\ log\ s$'s independently.

Our Volume I, 8th Edition, 1935, gives a further example for the computational procedure, for the city net of Hannover, where the adjustment was carried out with condition equations for direction measurements.

Section 54. Determination of Distance and Direction Angles from Spherical Coordinates

The point in question is the inversion of the problem hitherto treated in sections 52-53, and in order to obtain clarity, we will recall the corresponding simple problems of the plane. As is known, we have on the plane:

$$y' - y = s\ sin\ \alpha \qquad x' - x = s\ cos\ \alpha \tag{a}$$

$$tan\ \alpha = \frac{y' - y}{x' - x} \quad \text{and} \quad s = \frac{y' - y}{sin\ \alpha} = \frac{x' - x}{cos\ \alpha} \tag{b}$$

$$\text{or} \quad s = \sqrt{(y' - y)^2 + (x' - x)^2}. \tag{c}$$

While in section 52 the spherical analogies to the plane formulae (a) have been treated, now it is the question of finding also to the inverse formulae (b) and (c) that which holds, in accordance, on the sphere, i.e. we set up the problem: There are given the spherical coordinates of two points P and P', namely:

$$\left.\begin{array}{l} x \text{ and } y \text{ coordinates of } P \\ x' \text{ and } y' \text{ coordinates of } P'. \end{array}\right\} \tag{1}$$

There are required:

$$\left.\begin{array}{ll} \text{the distance} & PP' = s \\ \text{the direction angle} & (PP') = \alpha \\ \text{the direction angle} & (P'P) = \beta = \alpha' \pm 180°. \end{array}\right\} \tag{2}$$

I. Combined formulae for s and α

We can solve this problem by the inversion of (14), (15), section 52, p. 120, where $u = x' - x$ and $v = y' - y$ are set in the correction terms. In this manner we obtain:

$$s \sin \alpha = (y' - y) + \frac{(x' - x)^2 y}{2 r^2} + \frac{(x' - x)^2 (y' - y)}{6 r^2} = (y' - y) + \delta y \tag{3}$$

$$s \cos \alpha = (x' - x) - \frac{(x' - x) y'^2}{2 r^2} + \frac{(x' - x)(y' - y)^2}{6 r^2} = (x' - x) + \delta x . \tag{4}$$

The symbols δy and δx written here shall only express the combining of the correction terms, for we now have further:

$$\tan \alpha = \frac{(y' - y) + \delta y}{(x' - x) + \delta x} \tag{5}$$

$$s = \frac{(y' - y) + \delta y}{\sin \alpha} \quad \text{or} \quad = \frac{(x' - x) + \delta x}{\cos \alpha}. \tag{6}$$

In order to find also the angle of the counter direction β, we only need to invert the designations for the points P and P' (cf. the following numerical example).

Instead of this, however, we can also apply the formula (16), section 52, p. 120:

$$\alpha' = \alpha - (x' - x)(y' + y) \frac{\varrho}{2 r^2} \tag{7}$$

or

$$\alpha' = \alpha - (x' - x)\, y \frac{\varrho}{r^2} - (x' - x)(y' - y) \frac{\varrho}{2 r^2} \tag{7a}$$

and then

$$\beta = \alpha' \pm 180°. \tag{7b}$$

With this, all requirements are satisfied; it is desirable, however, for many reasons to determine also the distance s without the direction angles or, on the other hand, one or both direction angles without the distance.

II. Individual formula for s

In order to derive the distance s alone from the coordinates, we can use at once equations (3) and (4), for if we square and add these, then we obtain:

$$s^2 = \left((y' - y) + \frac{(x' - x)^2 y}{2 r^2} + \frac{(x' - x)^2 (y' - y)}{6 r^2}\right)^2 + \left((x' - x) - \frac{(x' - x) y'^2}{2 r^2} + \frac{(x' - x)(y' - y)^2}{6 r^2}\right)^2 .$$

If we carry out the squarings and neglect here all terms of the order $1:r^3$, then we obtain:

$$s^2 = (y' - y)^2 + \frac{(x' - x)^2 (y' - y)\, y}{r^2} + \frac{(x' - x)^2 (y' - y)^2}{3\, r^2}$$
$$+ (x' - x)^2 - \frac{(x' - x)^2 y'^2}{r^2} + \frac{(x' - x)^2 (y' - y)^2}{3\, r^2}.$$

Collected and arranged, this yields:

$$s^2 = (y' - y)^2 + (x' - x)^2 + \frac{(x' - x)^2}{3\, r^2}\left(3\, y\, (y' - y) + 2\, (y' - y)^2 - 3\, y'^2\right)$$
$$s^2 = (y' - y)^2 + (x' - x)^2 - \frac{(x' - x)^2}{3\, r^2}\left(y^2 + y\, y' + y'^2\right). \tag{8}$$

We denote here the first terms, which correspond to the computation with plane coordinates, by s_0^2, i.e.:

$$(y' - y)^2 + (x' - x)^2 = s_0^2, \tag{9}$$

and since we can interchange s_0 with s in the correction terms, (8) will yield:

$$s^2 = s_0^2\left(1 - \frac{\cos^2 \alpha}{3\, r^2}\,(y^2 + y\, y' + y'^2)\right)$$
$$s = s_0\left(1 - \frac{\cos^2 \alpha}{6\, r^2}\,(y^2 + y\, y' + y'^2)\right) \quad \text{or} \quad = s_0\left(1 - \frac{\cos^2 \alpha}{6r^2}\,\frac{y'^3 - y^3}{y' - y}\right) \tag{10}$$

or logarithmically:

$$\log s = \log s_0 - \frac{\mu}{6\, r^2}\cos^2 \alpha\, (y^2 + y\, y' + y'^2). \tag{11}$$

This can also be derived directly from Fig. 2, p. 117, to $\frac{s}{r}$ by the use of the cosine formula, as on p. 117.

III. Individual formula for α

In order to obtain a direct formula also for α, we start from the formulae (3) and (4):

$$\left.\begin{aligned} s \sin \alpha &= (y' - y) + \delta y \\ s \cos \alpha &= (x' - x) + \delta x, \end{aligned}\right\} \tag{12}$$

in which therefore:

$$\left.\begin{aligned} \delta y &= \frac{(x' - x)^2\, y}{2\, r^2} + \frac{(x' - x)^2\, (y' - y)}{6\, r^2} \\ \delta x &= -\frac{(x' - x)\, y'^2}{2\, r^2} + \frac{(x' - x)\, (y' - y)^2}{6\, r^2}. \end{aligned}\right\} \tag{13}$$

We will also imagine α itself divided, accordingly, into $\alpha_0 + \delta \alpha$ and have then:

$$\alpha = \alpha_0 + \delta \alpha = \operatorname{arc\ tan} \frac{(y' - y) + \delta y}{(x' - x) + \delta x}. \tag{14}$$

According to Taylor's theorem this yields:

$$\alpha_0 = arc\ tan\ \frac{y'-y}{x'-x} \tag{15}$$

and

$$\delta\alpha = \frac{1}{1+\left(\frac{y'-y}{x'-x}\right)^2}\frac{\delta y}{x'-x} - \frac{1}{1+\left(\frac{y'-y}{x'-x}\right)^2}\frac{y'-y}{(x'-x)^2}\delta x$$

$$\delta\alpha = \frac{x'-x}{(x'-x)^2+(y'-y)^2}\delta y - \frac{y'-y}{(x'-x)^2+(y'-y)^2}\delta x\,. \tag{16}$$

If we substitute the meanings for δy and δx explained above in (13), then we obtain from (16):

$$\delta\alpha = \frac{(x'-x)\,y}{2r^2}\cos^2\alpha + \frac{(x'-x)(y'-y)}{6r^2}\cos^2\alpha + \frac{y'^2}{2r^2}\cos\alpha\sin\alpha - \frac{(x'-x)(y'-y)}{6r^2}\sin^2\alpha\,. \tag{17}$$

This can also be written thusly:

$$\delta\alpha = \frac{(x'-x)\,y}{2r^2}\cos^2\alpha + \frac{y'^2}{4r^2}\sin 2\alpha + \frac{(x'-x)(y'-y)}{6r^2}\cos 2\alpha\,. \tag{18}$$

Another transformation of (17), which produces in the third term of (17) the factor $sin^2\alpha$, is even more useful, namely:

$$\delta\alpha = \frac{x'-x}{6r^2}\cos^2\alpha\,(2y+y') + \frac{x'-x}{6r^2}\sin^2\alpha\left(\frac{3y'^2}{y'-y}-(y'-y)\right);$$

we have here

$$\frac{3y'^2}{y'-y}-(y'-y) = 2y+y' + \frac{y^2+yy'+y'^2}{y'-y}\,,$$

and if we substitute this in the above, then we obtain:

$$\delta\alpha = \frac{x'-x}{6r^2}(2y+y') + \frac{x'-x}{6r^2s^2}(y'^3-y^3)\,. \tag{19}$$

This $\delta\alpha$ is the correction which is to be applied further to the approximate value α_0 of (15); we can thus write all together for the direction angle from a point P (with x, y) to P' (with x', y'), at the same time with the insertion of the necessary ρ's:

$$\alpha = \alpha_0 + \frac{\varrho}{6r^2}(x'-x)(2y+y') + \frac{\varrho}{6r^2}\frac{x'-x}{s^2}(y'^3-y^3) \tag{20}$$

$$\alpha = \alpha_0 + \frac{\varrho}{6r^2}(x'-x)(2y+y') + \frac{\varrho}{6r^2}(y^2+yy'+y'^2)\sin\alpha\cos\alpha\,. \tag{20a}$$

Applied to the point of the other side, this formula yields:

$$\alpha' = \alpha_0 + \frac{\rho}{6r^2}(x - x')(y + 2y') + \text{second term from above.}$$

Subtracted, these two formulae yield:

$$\alpha' - \alpha = -\frac{\rho}{2r^2}(x' - x)(y + y'). \tag{21}$$

This is again formula (11), section 52, p. 119, which can also be seen immediately.

IV. Formulae of Zachariae

Equations (26) in section 52, p. 122, can be inverted likewise. For the two spherical excesses we obtain immediately:

$$\varepsilon = \frac{y(x' - x)}{2r^2}\varrho \qquad \gamma = \frac{(y' - y)(x' - x)}{6r^2}\varrho. \tag{22}$$

We will have further

$$s \sin(\alpha - (\varepsilon + \gamma)) = y' - y$$
$$s \cos(\alpha - 2(\varepsilon + \gamma)) = (x' - x)\left(1 - \frac{y^2}{2r^2}\right).$$

Then we have

$$\frac{\sin(\alpha - (\varepsilon + \gamma))}{\cos(\alpha - 2(\varepsilon + \gamma))} = \frac{\sin(\alpha - \varepsilon - \gamma)}{\cos(\alpha - \varepsilon - \gamma) + \sin(\alpha - \varepsilon - \gamma)(\varepsilon + \gamma)}$$
$$= \frac{\tan(\alpha - \varepsilon - \gamma)}{1 + \tan(\alpha - \varepsilon - \gamma)(\varepsilon + \gamma)}.$$

Therefore,

$$\tan(\alpha - \varepsilon - \gamma) = \frac{y' - y}{x' - x}\left(1 + \frac{y^2}{2r^2}\right)\left(1 + \tan(\alpha - \varepsilon - \gamma)\frac{(\varepsilon + \gamma)}{\varrho}\right). \tag{23}$$

It is true, the last factor already presupposes the knowledge of $\alpha - \varepsilon - \gamma$; the preliminary computation from the first two factors is sufficient for this, however.

V. Numerical examples

In order to apply the above formulae, we can find ample opportunity in the numerical data of section 53, since from the coordinates of (10), p. 127, the direction angles and the distances of the station data on p. 124 must be able to be found backwards again.

We will take the computation for the line Speyer-Langenkandel. The computation of the correction terms δy and δx in (3) and (4) on p. 128 will be in correspondence with that on p. 126 according to (14) and (15), p. 120, for we obtain the following values which are added to the coordinates taken from the station data on p. 127:

	Langenkandel $y' = -19{,}467.721$		$x' = -44{,}893.918$
	Speyer $y = -1{,}208.142$		$x = -18{,}816.676$
	$y' - y = -18{,}259.579$		$x' - x = -26{,}077.242$
according to (3) and (4), p. 128	$\delta y = \begin{cases} - 0.010 \\ - 0.051 \end{cases}$		$\delta x = \begin{cases} + 0.121 \\ - 0.036 \end{cases}$
	$(y'-y) + \delta y = -18{,}259.640$		$(x'-x) + \delta x = -26{,}077.157$
	$\log$ " $= 4.261\,4922_n$		$\log$ " $= 4.416\,2602_n$
with this, according to (5), p. 128	$\log \tan \alpha = 9.845\,2320$	$\alpha = 215° \, 0' \, 01.158''$	
and according to (6), p. 128	$\log \sin \alpha = 9.758\,5948_n$		$\log \cos \alpha = 9.913\,3628_n$
with which:	$\log s = 4.502\,8974$		$\log s = 4.502\,8974$.

In the case of the point adjustment according to the coordinate method, in Vol. I, 8th Edition, 1935, Chapter III, e.g., this computational procedure is considered, if from the given coordinates and the approximate ones, or after the adjustment of the final coordinates the direction angles and distances are to be computed.

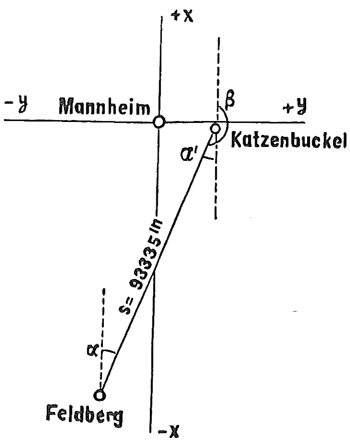

Fig. 1.
Scale 1:5,000,000.

We will now calculate here, in addition, a more extensive numerical example, in which the correction terms amount to more.

According to Fig. 1 in the margin we take the two points Katzenbuckel and Feldberg in the coordinate system Baden (degree-measurement points determined by Jordan in 1870-1871 and computed into the system Baden). The mean latitude is about $= 49°$ and to this,

$$\log \frac{1}{r^2} = 6.39031.$$

The following computations are carried out not only to 7 places but to 10 places in order to bring the checks formally to agree in any case to within 0.001", which is desirable in the case of this school example.

K, Katzenbuckel	$y' = +42{,}176.169$ m	$x' = -1{,}575.546$ m
F, Feldberg	$y = -34{,}075.071$	$x = -179{,}239.479$
	$y' - y = +76{,}251.240$	$x' - x = +177{,}663.933$
To this, the correction terms are computed according to (3) and (4):	$\delta y \begin{cases} -13.210 \\ +9.854 \end{cases}$	$\delta x \begin{cases} -3.881 \\ +4.229 \end{cases}$
	$(y'-y) + \delta y = +76{,}247.884$	$(x'-x) + \delta x = +177{,}664.281$,

with this, the formulae (5) and (6) yield:

$$(F, K) = \alpha = 23° \, 13' \, 38.920'' \qquad \log s = 5.286\,3099{\cdot}9 \qquad s = 193{,}334.779 \text{ m.} \tag{24}$$

The inversion of the denotations yields:

F, Feldberg	$y' = -34{,}075.071$ m	$x' = -179{,}239.479$ m
K, Katzenbuckel	$y = +42{,}176.169$	$x = -1{,}575.546$
	$y' - y = -76{,}251.240$	$x' - x = -177{,}663.933$
Correction terms according to (3) and (4)	$\delta y \begin{cases} +16.350 \\ -9.854 \end{cases}$	$\delta x \begin{cases} +2.534 \\ -4.229 \end{cases}$
	Numerator $= -76{,}244.744$	Denominator $= -177{,}665.628$

with this again according to (5) and (6):

$$(K, F) = \beta = 203^\circ\ 13'\ 35.275'' \qquad log\ s = 5.286\ 3099{\cdot}8 \qquad s = 193{,}334.778 \text{ m.} \tag{25}$$

By (24) and (25) the distance s is thus secured to 0.001 m.

In order to insure also the direction angles α and β, we have, according to (7) and (7a), p. 128, the difference of the two direction angles, and in fact (7a) in two ways, according to how we choose the denotations P and P' in correspondence to F and K, or conversely; we obtain for our example:

$$\text{from (7):} \quad \alpha' - \alpha = -\ 3.646'' \tag{26}$$
$$\text{from (7a):} \quad \alpha' - \alpha = +30.674'' - 34.320'' = -3.646'' \tag{26a}$$
$$\text{or from (7a):} \quad \beta' - \beta = +37.966'' - 34.320'' = +3.646''. \tag{26b}$$

These three values agree with one another, and with the difference of (24) and (25), which amounts to 3.645'', to 0.001''.

Now we still have the formulae (10) and (11), p. 129, which require at first a computation of s_0 or, as the case may be, $log\ s_0$, i.e. a computation which corresponds to plane coordinates; and we also compute here at the same time a plane value α_0:

$$\alpha_0 = 23^\circ\ 13'\ 42.356'' \qquad log\ s_0 = 5.286\ 3122{\cdot}4 \qquad s_0 = 193{,}335.782 \text{ m:} \tag{27}$$

To this, according to (11) and (10):

$$\begin{array}{lrr} & -\ 22{\cdot}6 & -\ 1.004 \\ \hline & log\ s = 5.286\ 3099{\cdot}8 & s = 193{,}334.778\ . \end{array} \tag{28}$$

This agrees sufficiently with (24) and (25).

Finally, we still have various formulae for $\delta\alpha$. The formula (18), p. 130, yields in a twofold application:

$$\begin{array}{rr} \alpha_0 = 23^\circ\ 13'\ 42.356'' & \beta_0 = 203^\circ\ 13'\ 42.356'' \\ -12.951 & -16.030 \\ +\ 1.633 & +\ 1.066 \\ +\ 7.881 & +\ 7.881 \\ \hline \alpha = 23^\circ\ 13'\ 38.919'' & \beta = 203^\circ\ 13'\ 35.273''. \end{array} \tag{29}$$

Finally, the formula (20), p. 130, yields, likewise, twofold:

$$\begin{array}{rr} \alpha_0 = 23^\circ\ 13'\ 42.356'' & \beta_0 = 203^\circ\ 13'\ 42.356'' \\ -3.897 & -7.543 \\ +0.460 & +0.460 \\ \hline \alpha = 23^\circ\ 13'\ 38.919'' & \beta = 203^\circ\ 13'\ 35.273''. \end{array} \tag{30}$$

With this, everything is computed with numerous checks.

The spherical coordinates, which were treated in the above sections 52-54, are first communicated officially by Bohnenberger in the treatise, *De computandis dimensionibus trigonometricis*, etc., Tübingen, 1826, sections 15-16; and Bohnenberger says, in this connection, in section 16: "Formulae [corresponding to our (14), (15), (16), section 52, p. 120], conveniunt cum iis, quibus usus est cel. Soldner in computandis dimensionibus bavaricis." In Württemberg these coordinates have been introduced for the land survey and Bohnenberger's developments have been retained, as is already seen, especially, from Pross, *Lehrbuch der praktischen Geometrie*, Stuttgart, 1838, p. 314, and from the official work by Kohler, *Die Landesvermessung des Königreichs Württemberg*, etc., 1858, pp. 125-146.

From Württemberg this derivation reached also the Bavarian geodetic literature, namely in Bauernfeind's *Elemente der Vermessungskunde*, 1st Edition, Volume II, 1858, pp. 201-206, where (without source indication) an extract from Bohnenberger, *De computandis*, etc., sections 15-17, with three numerical examples are given as "Berechnung einiger Dreiecke der württembergischen Vermessung."

Soldner's developments, originating from 1810, were not published until 1873 in *Bayerische Landesvermessung in ihrer wissenschaftlichen Grundlage*, München, 1873, pp. 263-281, and are newly edited by J. Frischauf: "Theorie der Landesvermessung von Johann Soldner," *Ostwalds Klassiker der exakten Wissenschaften*, No. 184, Leipzig, 1911.

All these treatises give the basic formulae (14) to (16), section 52, p. 120, but with respect to the inversion only the formulae (3) to (7), p. 128. The further formulae (10) to (20), pp. 129-130, are first set up in our 1st Edition, *Taschenbuch der praktischen Geometrie*, 1873, p. 326.

Section 55. Geographic Coordinates φ, λ and Rectangular Coordinates x, y

After having treated in detail the computations within a rectangular spherical coordinate system in the previous sections 52 and 54, we will now develop the relations between geographic coordinates and a rectangular system on the sphere. We begin with the most important case, the Soldner coordinate system, and have represented, in Fig. 1, for a point P, geographic coordinates φ_2, λ as well as rectangular coordinates x, y. For the geographic longitude, there is assumed here as zero meridian the meridian of the coordinate origin O. We have drawn, in Fig. 1, through point P, an additional arc APB parallel to the meridian ON and denote the angle γ, which this arc makes with the meridian of the point P as the *meridian convergence* of point P. Since at the point P the azimuths are counted from the meridian PN, but the direction angles from the arc PB, then the meridian convergence represents directly the difference between the azimuths and the direction angles at point P.

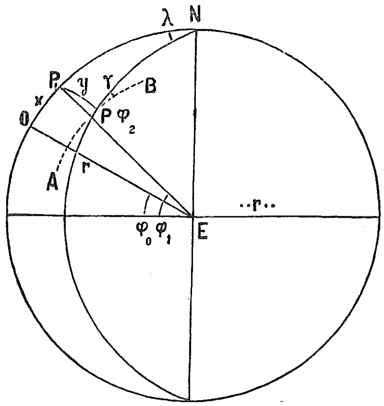

Fig. 1.

Now we pass over to the relations between geographic coordinates and rectangular coordinates.

I. Given x and y. Required φ and λ

According to Fig. 1 and Fig. 2 we take the following problem:

There are given the latitude φ_0 of the coordinate origin O and the rectangular coordinates x, y of a point P.

There are required the latitude φ_2 of point P, the difference of longitude λ between O and P, and the meridian convergence γ for P.

The abscissae x shall be counted in the positive sense to the north, the ordinates y in the positive sense to the east, and the longitudes λ likewise in the positive sense to the east.

From Fig. 1 we take at once the relation between the latitude of the origin φ_0, the latitude of the footpoint φ_1, and the abscissa x, namely:

$$\varphi_1 - \varphi_0 = \frac{x}{r} \quad \text{or, as the case may be} \quad = \frac{x}{r}\rho. \tag{1}$$

All the rest is furnished by the right spherical triangle NP_1P; therefore, we have specially drawn this triangle once again in Fig. 2. This triangle yields at first the cosine equation:

$$\cos(90° - \varphi_2) = \cos(90° - \varphi_1)\cos\frac{y}{r}$$

$$\sin\varphi_2 = \sin\varphi_1\left(1 - \frac{y^2}{2r^2}\right)$$

inversely:

$$\sin\varphi_1 - \sin\varphi_2 = \frac{y^2}{2r^2}\sin\varphi_1 .$$

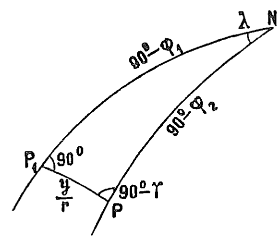

Fig. 2.
Right triangle NP_1P of Fig. 1.

But now we have in the first approximation (e.g. according to p. 31):

$$\sin\varphi_1 - \sin\varphi_2 = (\varphi_1 - \varphi_2)\cos\varphi_1,$$

and hence:
$$\varphi_1 - \varphi_2 = \frac{y^2}{2r^2} \tan \varphi_1. \tag{2}$$

Second, the right triangle, Fig. 2, yields for the determination of λ:

$$\tan \lambda = \frac{\tan \frac{y}{r}}{\sin (90° - \varphi_1)} = \frac{\tan \frac{y}{r}}{\cos \varphi_1} = \frac{1}{\cos \varphi_1}\left(\frac{y}{r} + \frac{y^3}{3r^3}\right).$$

The *arc tan* series, p. 23, yields:

$$\lambda = \operatorname{arc} \tan \lambda = \tan \lambda - \frac{(\tan \lambda)^3}{3}$$
$$\lambda = \frac{1}{\cos \varphi_1}\left(\frac{y}{r} + \frac{y^3}{3r^3}\right) - \frac{y^3}{3r^3 \cos^3 \varphi_1}.$$

If we arrange together the terms in y^3, then we obtain:

$$\lambda = \frac{y}{r \cos \varphi_1} - \frac{y^3}{3r^3} \frac{\tan^2 \varphi_1}{\cos \varphi_1}. \tag{3}$$

Third, the right triangle, Fig. 2, yields for the determination of γ:

$$\tan (90° - \gamma) = \frac{\tan (90° - \varphi_1)}{\sin \frac{y}{r}} \quad \text{or} \quad \tan \gamma = \sin \frac{y}{r} \tan \varphi_1$$
$$\tan \gamma = \left(\frac{y}{r} - \frac{y^3}{6r^3}\right) \tan \varphi_1$$
$$\gamma = \left(\frac{y}{r} - \frac{y^3}{6r^3}\right) \tan \varphi_1 - \frac{y^3}{3r^3} \tan^3 \varphi_1$$
$$\gamma = \frac{y}{r} \tan \varphi_1 - \frac{y^3}{6r^3} \tan \varphi_1 (1 + 2 \tan^2 \varphi_1). \tag{4}$$

By these equations (1) to (4) our problem is solved, but we will in addition form two new equations, which correspond to (3), (4), but which shall contain everywhere the latitude φ_2 instead of the latitude of the foot-point φ_1. For the determination of λ we take then:

$$\sin \lambda = \frac{\sin \frac{y}{r}}{\sin (90° - \varphi_2)} = \frac{\sin \frac{y}{r}}{\cos \varphi_2} = \frac{1}{\cos \varphi_2}\left(\frac{y}{r} - \frac{y^3}{6r^3}\right).$$

The *arc sine* series, p. 23, yields:

$$\lambda = \operatorname{arc} \sin \lambda = \sin \lambda + \frac{(\sin \lambda)^3}{6}$$
$$\lambda = \frac{1}{\cos \varphi_2}\left(\frac{y}{r} - \frac{y^3}{6r^3}\right) + \frac{y^3}{6r^3 \cos^3 \varphi_2}$$
$$\lambda = \frac{y}{r \cos \varphi_2} + \frac{y^3}{6r^3} \frac{\tan^2 \varphi_2}{\cos \varphi_2}. \tag{5}$$

Further, for a second formula for γ from Fig. 2, p. 134:

$$\cos(90^\circ - \gamma) = \frac{\tan \frac{y}{r}}{\tan(90^\circ - \varphi_2)}, \quad \sin\gamma = \tan\frac{y}{r}\tan\varphi_2$$

$$\sin\gamma = \left(\frac{y}{r} + \frac{y^3}{3r^3}\tan\varphi_2\right)$$

$$\gamma = \left(\frac{y}{r} + \frac{y^3}{3r^3}\right)\tan\varphi_2 + \frac{y^3}{6r^3}\tan^3\varphi_2$$

$$\gamma = \frac{y}{r}\tan\varphi_2 + \frac{y^3}{6r^3}\tan\varphi_2(2 + \tan^2\varphi_2). \tag{6}$$

With these formulae (1) to (6) we have reached the purpose intended at the beginning, in fact even twice in the case of λ and γ.

Without any forcing reason for the following we will also form, in addition, for λ a *one*-term formula instead of the two two-term formulae (3) and (5), namely:

$$\lambda = \frac{y}{r}\sec\frac{\varphi_1 + 2\varphi_2}{3} = \frac{y}{r}\sec\left(\varphi_2 + \frac{\varphi_1 - \varphi_2}{3}\right). \tag{7}$$

We can easily prove this convenient formula (7) working backwards by setting:

$$\frac{\varphi_1 + 2\varphi_2}{3} = \varphi_1 - \frac{2}{3}(\varphi_1 - \varphi_2)$$

$$\cos\frac{\varphi_1 + 2\varphi_2}{3} = \cos\varphi_1 + \frac{2}{3}(\varphi_1 - \varphi_2)\sin\varphi_1 = \cos\varphi_1\left(1 + \frac{2}{3}(\varphi_1 - \varphi_2)\tan\varphi_1\right)$$

$$\sec\frac{\varphi_1 + 2\varphi_2}{3} = \frac{1}{\cos\varphi_1}\left(1 - \frac{2}{3}(\varphi_1 - \varphi_2)\tan\varphi_1\right).$$

Because of (2) this yields:

$$\sec\frac{\varphi_1 + 2\varphi_2}{3} = \frac{1}{\cos\varphi_1}\left(1 - \frac{1}{3}\frac{y^2}{r^2}\tan^2\varphi_1\right).$$

Set into (7) this is reduced to (3), with which (7) has been proven.

We can also make a transformation of a similar kind for the meridian convergence, for (4) or (6) can be brought to this form:

$$\gamma = \frac{y}{r}\sin\frac{2\varphi_1 + \varphi_2}{3}\sec\frac{\varphi_1 + 2\varphi_2}{3}. \tag{8}$$

II. Given φ, λ. Required x, y

The inversion of the previous problem means:

There are given the geographic coordinates φ_2, λ of a point P, and in fact the longitude λ referred to the meridian of a given coordinate system whose latitude of origin φ_0 is likewise given.

There are required the rectangular coordinates x, y of the point P, and the meridian convergence γ.

This problem, also, can be solved easily by means of the right triangle, which we have again in Fig. 3 on the following page.

For the determination of $\varphi_1 - \varphi_2$ we have:

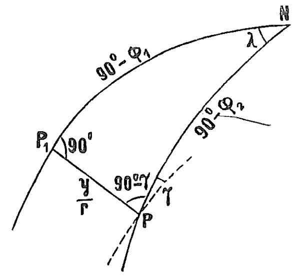

Fig. 3.

$$\cos\lambda = \frac{\tan\ (90^\circ - \varphi_1)}{\tan\ (90^\circ - \varphi_2)} = \frac{\tan\ \varphi_2}{\tan\ \varphi_1}$$

$$\left(1 - \frac{\lambda^2}{2}\right) \tan\ \varphi_1 = \tan\ \varphi_2$$

or

$$\tan\ \varphi_1 - \tan\ \varphi_2 = \frac{\lambda^2}{2} \tan\ \varphi_1 .$$

On the other hand, we have in the first approximation:

$$\tan\ \varphi_1 - \tan\ \varphi_2 = \frac{\varphi_1 - \varphi_2}{\cos\varphi_1 \cos\varphi_2},$$

and hence:

$$\varphi_1 - \varphi_2 = \frac{\lambda^2}{2} \sin\varphi_1 \cos\varphi_2 . \tag{9}$$

In the case of the developments for λ and γ we can again refer either everything to φ_1 or to φ_2 as in the first part; we will set down *both* developments one beside the other, without verbal explanations, which will no longer be necessary after the foregoing remarks.

$\tan\ \lambda = \dfrac{\tan\ \frac{y}{r}}{\cos\varphi_1}$; $\tan\ \dfrac{y}{r} = \left(\lambda + \dfrac{\lambda^3}{3}\right) \cos\varphi_1$	$\sin\lambda = \dfrac{\sin\frac{y}{r}}{\cos\varphi_2}$; $\sin\dfrac{y}{r} = \left(\lambda - \dfrac{\lambda^3}{6}\right) \cos\ \varphi_2$	
$y = r\lambda \cos\varphi_1 + \dfrac{r\lambda^3}{3} \cos\varphi_1 \sin^2\varphi_1$	$y = r\lambda \cos\varphi_2 - \dfrac{r\lambda^3}{6} \cos\varphi_2 \sin^2\varphi_2$	(10)
$\cos(90^\circ - \gamma) = \sin\lambda \cos(90^\circ - \varphi_1)$	$\cos(90^\circ - \varphi_2) = \cot\ \lambda \cot\ (90^\circ - \gamma)$	
$\sin\gamma = \left(\lambda - \dfrac{\lambda^3}{6}\right) \sin\varphi_1$	$\tan\ \gamma = \tan\ \lambda \sin\varphi_2 = \left(\lambda + \dfrac{\lambda^3}{3}\right) \sin\varphi_2$	
$\gamma = \lambda \sin\varphi_1 - \dfrac{\lambda^3}{6} \sin\varphi_1 \cos^2\varphi_1$	$\gamma = \lambda \sin\varphi_2 + \dfrac{\lambda^3}{3} \sin\varphi_2 \cos^2\varphi_2 .$	(11)

The procedure of the solution would now be so that we derive at first, according to (9), from the given latitude φ_2 the latitude of the foot-point φ_1 and compute therefrom, by the difference with respect to the latitude of the origin φ_0, the abscissa x, namely:

$$x = (\varphi_1 - \varphi_0) r \quad \text{or, as the case may be,} \quad = \frac{\varphi_1 - \varphi_0}{\rho} r. \tag{12}$$

After this, we have for y and γ two formulae each, namely (10) for y, and (11) for γ, from which we can choose one or the other under certain circumstances in order to compute y and γ. Or we can also use double computation as a check.

We can also transform the double formulae (10) and (11), which have two terms, into a *single* one-term formula each, in a similar manner as this was shown previously in the case of (7) and (8), p. 136. We find:

$$\text{Conversion of (10):} \quad y = r\lambda \cos \frac{\varphi_1 + 2\varphi_2}{3} \tag{13}$$

$$\text{Conversion of (11):} \quad \gamma = \lambda \sin \frac{2\varphi_1 + \varphi_2}{3} . \tag{14}$$

In the later sections we shall take up further conversions suitable for practical computation.

Section 56. Transformation to the Ellipsoid

Although we occupy ourselves in this chapter only with coordinates on the terrestrial *sphere*, we will in addition show for the case of the Soldner coordinates how we can arrive, at least to some approximation, from the relations on the sphere also to the relations on the ellipsoid without making use of the theory of ellipsoidal coordinates to be treated in the second half-volume.

We begin with the computation of the meridional arc x between the latitudes φ_0 and φ_1, for which in (1), section 55, p. 134, the following equation was found:

$$\varphi_1 - \varphi_0 = x\frac{\rho}{r} \qquad \text{or} \qquad x = (\varphi_1 - \varphi_0)\frac{r}{\rho}. \tag{1}$$

Although the abscissa x is not developed on an arc of a circle with the radius r, but on the arc of a meridian ellipse, then, if x is not very large, we can conduct the computation with an arc of a circle whose radius, however, is then to be taken equal to the radius of curvature of the meridian M for the mean latitude $\frac{\varphi_0 + \varphi_1}{2}$, as we have already shown in detail in section 41 [namely in the case of (11), p. 66, and then specially, once again, in the case of (43), p. 74].

We have found here that the error of this approximate method amounts in our latitudes only to about 5 mm in 1° (cf. the auxiliary table for g on p. 75), so that, especially in the case of the small regions of validity, which, e.g., the 40 Prussian cadastral systems of coordinates hitherto used have, this method is entirely admissible and at the same time very convenient.

Besides, we can also use the auxiliary table on pp. [41] to [44] of the Appendix, about which what was necessary was said on pp. 76 and 77, or we can also calculate according to (38), p. 73, or, as the case may be, (40) to (43), p. 74.

If rectangular coordinates x and y are given, then we are, according to this, in a position to compute the difference of latitude $\varphi_1 - \varphi_0$ and with this also the latitude of the foot-point φ_1.

Now the point in question is further to determine the difference of latitude $\varphi_1 - \varphi_2$ from the ordinate y. The circumstance that the ordinate y deviates only little from the circle of latitude φ_1, because of its perpendicular position to the zero meridian, is helpful here. If we thus choose a sphere which is tangent to the ellipsoid at the circle of latitude φ_1, i.e., a sphere with the radius of curvature in the prime vertical N_1 (cf. section 38, p. 50), then we can assume with close approximation that, in the case of not too great a length, the end-point P_2 of the ordinate lies also on this sphere.

If we make use, in addition, of the point of intersection N of the earth's axis with the sphere, then we obtain a spherical triangle $P_1 P_2 N$, in which we can compute the difference of latitude of the two points P_1 and P_2 according to equation (2), section 55, p. 135.

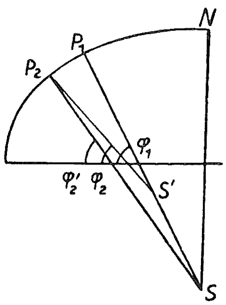

Fig. 1.

However, we are to bear in mind here that the latitude of the point P found on the sphere in this manner does not agree with the *ellipsoidal* latitude. We will therefore denote the latitude of the point P found spherically by φ_2' in distinction from the ellipsoidal latitude φ_2.

For further clarification of these conditions we consider Fig. 1. The normal to the ellipsoid $P_1 S$ is herein equal to the radius of curvature in the prime vertical N_1. If we connect P_2 with S, then $P_2 S$ is normal to the surface of the auxiliary sphere, but not normal to the ellipsoidal surface. The latter proves correct for the straight line $P_2 S'$, and $P_2 S' = P_1 S' = M_1$ is the radius of curvature in the meridian at the point P_1. In Fig. 1 we have represented further the ellipsoidal latitude φ_2 and the spherical latitude φ_2' for the point P_2. With this, we can indicate from two ways the small meridional arc which corresponds to the difference of latitude of the points P_2 and P_1. We have

$$P_2 P_1 = N_1 (\varphi_1 - \varphi_2') = M_1 (\varphi_1 - \varphi_2);$$

therefore, we will have

$$\varphi_1 - \varphi_2 = \frac{N_1}{M_1} (\varphi_1 - \varphi_2'), \tag{2}$$

and since according to (29), section 38, p. 51,

$$\frac{N_1}{M_1} = V_1^2; \tag{3}$$

then we can write our equation (2), section 55, p. 135, in the form:

$$\varphi_1 - \varphi_2 = \frac{V_1^2}{2} \frac{y^2}{N_1^2} \tan \varphi_1. \tag{4}$$

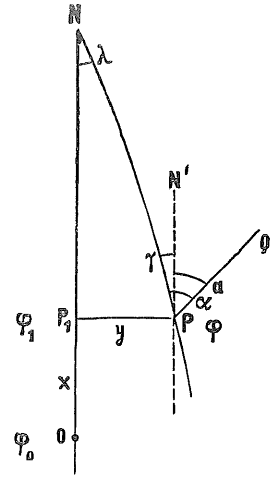

Fig. 2.

With this, the change from the sphere to the ellipsoid for the difference of latitude is carried out.

The values of the difference of longitude λ and the meridian convergence γ computed on the auxiliary sphere also hold for the ellipsoid, as we understand at once.

We will now adapt the formulae of section 55, pp. 134, 135, and 137, to the foregoing conditions, but denote now at the same time the given point by P and its latitude by φ in correspondence with Fig. 2 in the margin. Then we have the following summary of formulae:

$$\left.\begin{array}{ll}\text{Given} & \varphi_0,\ x,\ y \\ \text{Required} & \varphi_1,\ \varphi,\ \lambda \text{ and } \gamma\end{array}\right\}. \tag{5}$$

$$\text{Solution: } \varphi_1 = \varphi_0 + \frac{x}{M}\rho \tag{6}$$

$$\varphi = \varphi_1 - \frac{V_1^2}{2} \frac{y^2}{N_1^2} \varrho \tan \varphi_1 \tag{7}$$

$$\varphi = \varphi_0 + \frac{x}{M}\varrho - \frac{V_1^2}{2} \frac{y^2}{N_1^2} \varrho \tan \varphi_1 \tag{8}$$

(3) p. 135:

$$\lambda = \frac{y}{N_1} \frac{\varrho}{\cos \varphi_1} - \frac{y^3}{3 N_1^3} \frac{\varrho}{\cos \varphi_1} \tan^2 \varphi_1 \tag{9}$$

(5) p. 135 or:

$$\lambda = \frac{y}{N_1} \frac{\varrho}{\cos \varphi} + \frac{y^3}{6 N_1^3} \frac{\varrho}{\cos \varphi} \tan^2 \varphi \tag{10}$$

(4) p. 135:

$$\gamma = \frac{y}{N_1} \varrho \tan \varphi_1 - \frac{y^3}{6 N_1^3} \varrho \tan \varphi_1 (1 + 2 \tan^2 \varphi_1) \tag{11}$$

(6) p. 136 or:

$$\gamma = \frac{y}{N_1} \varrho \tan \varphi + \frac{y^3}{6 N_1^3} \varrho \tan \varphi (2 + \tan^2 \varphi). \tag{12}$$

Now we will in addition introduce the main coefficients [1] and [2] of our auxiliary table of pp. [12] to [33] of the Appendix, namely:

$$\frac{\rho}{M} = [1] \text{ for the mean latitude } \frac{\varphi_0 + \varphi_1}{2} \tag{13}$$

$$\frac{\rho}{N_1} = [2] \text{ for the latitude of the foot-point } \varphi_1. \tag{14}$$

With this, the above formulae will be:

$$\varphi_1 = \varphi_0 + [1]x \tag{6*}$$

$$\varphi = \varphi_1 - \frac{([2]\,y)^2 V_1^2}{2\,\varrho} \tan\ \varphi_1 \tag{7*}$$

$$\varphi = \varphi_0 + [1]\,x - \frac{([2]\,y)^2}{2\,\varrho}\ V_1^2 \tan\ \varphi_1 \tag{8*}$$

$$\lambda = \frac{[2]\,y}{\cos\varphi_1} - \left(\frac{[2]\,y}{\cos\varphi_1}\right)^3 \frac{1}{3\,\varrho^2} \sin^2\varphi_1 \tag{9*}$$

or

$$\lambda = \frac{[2]\,y}{\cos\varphi} + \left(\frac{[2]\,y}{\cos\varphi}\right)^3 \frac{1}{6\,\varrho^2} \sin^2\varphi \tag{10*}$$

$$\gamma = [2]\,y \tan\ \varphi_1 - \frac{([2]\,y)^3}{6\,\varrho^2} \tan\ \varphi_1\,(1 + 2\ \tan^2\varphi_1) \tag{11*}$$

or

$$\gamma = [2]\,y \tan\ \varphi + \frac{([2]\,y)^3}{6\,\varrho^2} \tan\ \varphi\,(2 + \tan^2\varphi)\,. \tag{12*}$$

After we have transformed the first part of the formulae of section 55 from the sphere to the ellipsoid, there cannot exist any difficulty to transform also the second part of those formulae of section 55, p. 137, in the same manner. We write for this at once the results:

Given φ, λ in addition to φ_0 (15)

Required x, y in addition to γ . (16)

(9) p. 137:
$$\varphi_1 = \varphi + \frac{V^2}{2\,\varrho}\,\lambda^2 \sin\varphi \cos\varphi \tag{17}$$

(12) p. 137:
$$x = \frac{\varphi_1 - \varphi_0}{[1]} \tag{18}$$

(10) p. 137:
$$y = \frac{\lambda}{[2]} \cos\varphi_1 + \frac{\lambda^3}{[2]}\frac{1}{3\,\varrho^2} \cos\varphi_1 \sin^2\varphi_1 \tag{19}$$

or
$$y = \frac{\lambda}{[2]} \cos\varphi - \frac{\lambda^3}{[2]}\frac{1}{6\,\varrho^2} \cos\varphi \sin^2\varphi \tag{20}$$

(11) p. 137:
$$\gamma = \lambda \sin\varphi_1 - \lambda^3 \frac{1}{6\,\varrho^2} \sin\varphi_1 \cos^2\varphi_1 \tag{21}$$

or
$$\gamma = \lambda \sin\varphi + \lambda^3 \frac{1}{3\,\varrho^2} \sin\varphi \cos^2\varphi\,. \tag{22}$$

On pp. 141-142 we now give a numerical example for the formulae (7*) to (12*), as well as for their inversion (17) to (22). Since the whole computation is indicated with all individual numbers, nothing further will be necessary for explanation; also a few temporarily introduced intermediate designations (a), (b) and the like are self-explanatory as small transitional aids, with respect to lack of space.

The logarithms of coefficients log [1], log [2], $log\ V^2$, $log\ (1 + 2\,t^2)$, $log\ (2 + t^2)$ are taken from the corresponding auxiliary tables of our Appendix.

Moreover, it is only noted that we must not carry through the sign of y or λ in the whole computation, as was necessary in our case from the theoretical viewpoint; we only need to bear in mind at the end that y, λ and γ have always like signs.

The zero point of coordinates Celle used in the following example on pp. 141-142 is one of the 40 Prussian cadastral points hitherto used, which have been introduced in 1879.

Meridional arcs and differences of latitude

In the case of the small regions of validity of the Prussian cadastral coordinate systems hitherto used, the relation between the abscissa x and the difference of latitude $\varphi_1 - \varphi_0$ is given with sufficient accuracy by the radius of curvature in the meridian M of the mean latitude, namely:

$$x = \frac{\varphi_1 - \varphi_0}{\rho} M \quad \text{or} \quad = \frac{\varphi_1 - \varphi_0}{[1]}\,,$$

where M is the radius of curvature in the meridian for the mean latitude $\frac{\varphi_1 + \varphi_0}{2}$, and [1] is the corresponding coefficient according to (13), p. 139.

In the case of a rather large extent, however, a table of meridional arcs is advisable, as we have added in the Appendix on pp. [41] to [44] for every 1' interval. Such a table is also contained in the Prussian Cadastral Instruction [Katasteranweisung] XI of 11 March 1932.

In the use of these tables we only need to determine, once and for all, the value of the meridional arc B_0 for the zero point of coordinates with given latitude φ_0, in order to find then, for any other latitude φ_1, the value B_1 belonging to it and then $x = B_1 - B_0$.

For the value $\varphi_0 = 52° 37' 32.6709''$, which belongs to the zero point of coordinates Celle, we have already found, in section 41, p. 77, $B_0 = 5{,}832{,}371.046$ m as the meridional arc from the equator to the point Celle.

This leads to the thought that we could set the abscissae x immediately equal to B with the omission of a round number, say, 5,000,000.

Geographic coordinates φ, λ from rectangular coordinates x, y

Zero point of coordinates Celle with $\varphi_0 = 52° 37' 32.6709''$ $\quad L_0 = 27° 44' 54.8477''$

Given Ägidius $x = -28{,}308.394$ m $\quad y = -23{,}271.813$ m

Approximate computation of $\varphi_1 - \varphi_0$ from x according to the auxiliary table on page [47] of the Appendix; with approximately $\varphi = 52\frac{1}{2}°$

20,000 m yields	10' 47.1''
8,000 m yields	4 18.85
300 m yields	9.70
8 m yields	0.24
	$-15' 15.89''$
$\varphi_0 =$	$52° 37' 32.67''$
Approximately $\varphi_1 =$	$52° 22' 16.78''$

$$\varphi_m = \frac{\varphi_1 + \varphi_0}{2} = 52° 29' 54.72''$$

With this, the auxiliary tables of the Appendix give:

with φ_m: Page [37], $\log [1] = 8.509\,9477{\cdot}2$

with φ_1: Page [37], $\log [2] = 8.508\,8706{\cdot}9$; Page [25], $\log V^2 = 0.001\,086$

$$\varphi_1 = \varphi_0 + [1]x \qquad \lambda = \frac{[2]\,y}{\cos\varphi_1} - \left(\frac{[2]\,y}{\cos\varphi_1}\right)^3 \frac{1}{3\varrho^2} \sin^2\varphi_1 \qquad \gamma = [2]y \tan\varphi_1 - \frac{([2]\,y)^3}{6\varrho^2} t_1 (1 + 2t_1^2)$$

or

$$\varphi = \varphi_1 - \frac{([2]\,y)^2}{2\varrho} V^2 \tan\varphi_1 \qquad \lambda = \frac{[2]\,y}{\cos\varphi} + \left(\frac{[2]\,y}{\cos\varphi}\right)^3 \frac{1}{6\varrho^2} \sin^2\varphi \qquad \gamma = [2]y \tan\varphi + \frac{([2]\,y)^3}{6\varrho^3} t\,(2 + t^2)$$

Latitude:

$\log [1]$	$8.509\,9477{\cdot}2$
$\log x$	$4.451\,9152{\cdot}3_n$
$\log [1]x$	$2.961\,8629{\cdot}5_n$
$[1]x =$	$-915.9314''$

$\log ([2]\,y)^2$	$5.751\,402$
$\log \tan \varphi_1$	$0.113\,001$
$\log V^2$	$0.001\,086$
$-(1:2\varrho)$	$4.384\,545_n$
$\log (a)$	$0.250\,034_n$
$(a) =$	$-1.7784''$

Celle $\varphi_0 =$	$52° 37' 32.6709''$
$+[1]x =$	$-0\ 15\ 15.9314$
$\varphi_1 =$	$52° 22' 16.7395''$
$(a) =$	$-\ 1.7784$
$\varphi =$	$52° 22' 14.9611''$

Ägidius (Hannover)

Longitude:

$\log [2]$	$8.508\,8706{\cdot}9$
$\log y$	$4.366\,8302{\cdot}2_n$
$\log [2]\,y$	$2.875\,7009{\cdot}1_n$
$\log \cos \varphi_1$	$9.785\,7153{\cdot}5$
$\log (b)$	$3.089\,9855{\cdot}6_n$
$(b) =$	$-1230.2278''$

$(b)^3$	9.2700_n	$(b')^3$	9.2699_n
$\sin^2\varphi_1$	9.7974	$\sin\varphi$	9.7974
$-(1:3\varrho^2)$	8.8940_n	$(1:6\varrho^2)$	8.5930
$+(c)$	7.9614	$\pm(c)$	7.6603_n
	$+0.0091''$		$-0.0046''$

$\log [2]\,y$	$2.875\,7009{\cdot}1_n$
$\log \cos \varphi$	$9.785\,7202{\cdot}1$
$\log (b')$	$3.089\,9807{\cdot}0_n$
$(b') =$	$-1230.2141''$

$(b) + (c)$	$(b') + (c')$
$-1230.2278''$	$-1230.2141''$
$+\ 0.0091$	$-\ 0.0046$
$-1230.2187''$	$-1230.2187''$

	$\lambda =$	$-0° 20' 30.2187''$
Celle	$L_0 =$	$27\ 44\ 54.8477$
Ägidius	$L =$	$27° 24' 24.6290''$

Meridian convergence:

$\log [2]\,y$	$2.875\,7009{\cdot}1_n$
$\log \tan \varphi_1$	$0.113\,0011{\cdot}1$
$\log (c)$	$2.988\,7020{\cdot}2_n$
$(c) =$	$-974.3209''$

Auxiliary table, p. [59]

$([2]\,y)^3$	8.6271_n	$([2]\,y)^3$	8.6271_n
t_1	0.1130	t	0.1130
$(1+2t_1^2)$	0.6399	$(2+t^2)$	0.5661
$-(1:6\varrho^2)$	8.5930_n	$(1:6\varrho^2)$	8.5930
$+(d)$	7.9730	$+(d')$	7.8992_n
	$+0.0094''$		$-0.0079''$

$\log [2]\,y$	$2.875\,7009{\cdot}1_n$
$\log \tan \varphi$	$0.112\,9933{\cdot}7$
(c')	$2.988\,6942{\cdot}8_n$
$(c') =$	$-974.3035''$

$(c) + (d)$	$(c') + (d')$
$-974.3209''$	$-974.3035''$
$+\ 0.0094$	$-\ 0.0079$
$-974.3115''$	$-974.3114''$

Meridian convergence $\gamma = -16' 14.311''$

Zero point of coordinates Celle	$\varphi_0 = 52°\,37'\,32.6709''$	$L_0 =$	$27°\,44'\,54.8477''$
Given Ägidius	$\varphi = 52\;\;22\;\;14.9611$	$L =$	$27\;\;24\;\;24.6290$
Differences	$\varphi - \varphi_0 = -\;15'\,17.7098''$	$\lambda =$	$-\,0°\,20'\,30.2187''$
		$\lambda =$	$-\;1230.2187''$

$$\varphi_1 = \varphi + \frac{V^2}{2\varrho}\lambda^2 \sin\varphi\cos\varphi$$

$$\frac{\varphi_1 - \varphi_0}{[1]} = x$$

Appendix, p. [25], yields . . . .

$\log\lambda$	3.089 9823·2
$\log\lambda^2$	6.179 965
$\log\sin\varphi$	9.898 714
$\log\cos\varphi$	9.785 720
$\log V^2$	0.001 086
$\log(1:2\varrho)$	4.384 545
$\log(a)$	0.250 030

$(a) = 1.7784''$

$\varphi = 52°\,22'\,14.9611''$
$+(a) \quad + \; 1.7784$
$\varphi_1 = 52°\,22'\,16.7395''$
Celle $\varphi_0 = 52°\,37'\,32.6709''$
$\varphi_1 - \varphi_0 = -\;15'\,15.9314''$
$= -\;915.9314$

$$\varphi_m = \frac{\varphi_1 + \varphi_0}{2} = 52°\,29'\,54.7052''$$

With φ_1 the auxiliary table on p. [37] of the Appendix:

$\log[2] = 8.508\,8706·9$

and with φ_m the same auxiliary table on p. [37]:

$\log[1]$	8.509 9477·2
to this $\log(\varphi_1 - \varphi_0)$	2.961 8629·5
$\log x$	4.451 9152·3

$x = -\,28308.394$ m

$$y = \frac{\lambda}{[2]}\cos\varphi_1 + \frac{\lambda^3}{[2]}\frac{1}{3\varrho^2}\cos\varphi_1\sin^2\varphi_1 \quad \text{or} \quad y = \frac{\lambda}{[2]}\cos\varphi - \frac{\lambda^3}{[2]}\frac{1}{6\varrho^2}\cos\varphi\sin^2\varphi$$

$1:[2]$	1.491 1293·1	(b)	4.3668_n	$1:[2]$	1.491 1293·1	(b)	4.3668_n
λ	$3.089\,9823·2_n$	λ^2	6.1800	λ	$3.089\,9823·2_n$	λ^2	6.1800
$\cos\varphi_1$	9.785 7153·5	$\sin^2\varphi_1$	9.7974	$\cos\varphi$	9.785 7202·1	$\sin^2\varphi$	9.7974
(b)	$4.366\,8269·8_n$	$(1:3\varrho^2)$	8.8940	$(b)'$	$4.366\,8318·4_n$	$-(1:6\varrho^2)$	8.5930_n
		(c)	9.2382_n			(c')	8.9372

$(b) = -\,23{,}271.639$ | $(b') = -\,23{,}271.900$
$(c) = -\,0.173$ | $(c') = +\,0.087$
$y = -\,23{,}271.812$ | $y = -\,23{,}271.813$

$$\gamma = \lambda\sin\varphi_1 - \frac{\lambda^3}{6\varrho^2}\sin\varphi_1\cos^2\varphi_1 \quad \text{or} \quad \gamma = \lambda\sin\varphi + \frac{\lambda^3}{3\varrho^2}\sin\varphi\cos^2\varphi$$

λ	$3.089\,9823·2_n$	$\lambda\sin\varphi_1$	2.9887_n	λ	$3.089\,9823·2_n$	$\lambda\sin\varphi$	2.9887_n
$\sin\varphi_1$	9.898 7164·6	λ^2	6.1800	$\sin\varphi$	9.898 7135·7	λ^2	6.1800
$\lambda\sin\varphi_1$	$2.988\,6987·8_n$	$\cos^2\varphi_1$	9.5714	$\lambda\sin\varphi$	$2.988\,6958·9_n$	$\cos^2\varphi$	9.5714
		$-(1:6\varrho^2)$	8.5930_n			$(1:3\varrho^2)$	8.8940
		(d)	7.3331			(d')	7.6341_n

$\lambda\sin\varphi_1 = -\,974.3136''$ | $\lambda\sin\varphi = -\,974.3072''$
$(d) = +\,0.0022$ | $(d) = -\,0.0043$
$\gamma = -\,974.3114''$ $\quad \gamma = -\,16'\,14.3114''$ | $\gamma = -\,974.3115''$

Final result: Ägidius $y = -\,23{,}271.813$ m $\quad x = -\,28{,}308.394$ m
Meridian convergence $\gamma = -\,16'\,14.311''$.

Computation form of the Prussian Cadastral Instruction IX of 25 October 1881

Although the conversion of geographic into rectangular coordinates no longer has the significance for Prussia it once had, we will in addition discuss "Trig. Form. 6" intended for this in Katasteranweisung IX.

The auxiliary tables necessary for this are not contained in the official instruction IX, but reference is made, for this, to *Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst* by F. G. Gauss, 1876, and 4th Edition, 1922. As source paper for the method of Form. 6 we give: Börsch, *Anleitung zur Berechnung der rechtwinkligen sphärischen Koordinaten*, etc., 1868, 1869, p. 19, and 1885, p. 91, from which the computation with additaments, which correspond to our terms of third order $\frac{\lambda^3 \cos \varphi \sin^2 \varphi}{6}$, etc., is also taken over.

For more detailed explanation, in Fig. 3 there are entered the designations of this form blank, namely:

φ_0 the latitude of the origin of coordinates,
φ_f the latitude of the foot-point of ordinates,
φ the latitude of the point required,
λ_0 the longitude of the origin of coordinates,
λ the longitude of the point required,
x and y the coordinates required.

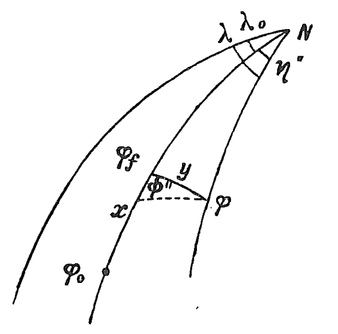

Fig. 3.

x is counted here from φ_0 to φ_f on the meridian to the north in the positive sense, to the south in the negative sense and y perpendicularly to the meridian to the east in the positive sense, to the west in the negative sense.

The problem reads: To compute the coordinates x, y from given φ_0's, φ's, λ_0's, λ's.

The difference $\lambda - \lambda_0$ is converted into seconds, denoted by η'', and further, the difference of latitude $\varphi_f - \varphi = \psi''$, which is computed from η'', is according to the formula:

$$\psi'' = \eta''^2 q. \tag{23}$$

This corresponds to our formula (17), p. 140, for $\varphi_1 - \varphi$, i.e.:

$$\varphi_1 - \varphi = \lambda^2 \frac{V^2}{2\rho} \sin \varphi \cos \varphi. \tag{24}$$

It follows hence that the factor q in the formula (23), converted to our designations of formula (17), has the following meaning:

$$q = \frac{V^2}{2\rho} \sin \varphi \cos \varphi. \tag{25}$$

This value $\log q$ is taken with the argument φ from the table mentioned in "Gauss, Trig. Rechnungen, etc."

Since we add, in the form blank 6 mentioned, the difference ψ'' to φ and with this have obtained φ_f, we can compute the abscissa x from the difference $\varphi_f - \varphi_0$. The form blank uses also for this the auxiliary tables by F. G. Gauss already mentioned and described above, whereby it is noted, however, that the interpolation requires a computation with 7-place logarithms. Cf. also the tables [41] to [44] of our Appendix already mentioned.

Now in order to obtain, in addition, the ordinate y, form blank 6 computes with seconds of longitude L and with additaments, whereby, in another form, the same is obtained as by development in series (19), p. 140.

For if we denote the length of arc of the parallel circle for a second of longitude in the latitude φ_1 by L_1, then we have according to (2), section 42, p. 77, and (10), section 55, p. 137:

$$L_1 = \frac{N_1}{\rho} \cos \varphi_1 \qquad \text{and} \qquad \tan \frac{y}{N_1} = \frac{L_1}{N_1} \rho \tan \lambda.$$

If we introduce then the notation η'' for the difference of longitude in seconds and if $2 A_\eta$ and $2 A_y$ represent logarithmic additaments, then we obtain:

$$\log y = \log \eta'' + \log L_1 + 2 A_\eta - 2 A_y. \tag{26}$$

In the computation in trigonometric form blank 6 of Instruction IX, the tables in F. G. Gauss, *Die trigon. und polygon. Rechnungen*, already mentioned, from which the additaments of tangents $2A_y$ or, as the case may be $2A_\eta$, are to be taken, are presupposed.

With this, we have the following computation for our example on p. 142 with the use of the designations of Instruction IX:

$\varphi_f = 52° 22' 16.7395''$ $\eta = 0° 20' 30.2187''$			log
$\log \eta'' = \log 1230.2187''$	3.089 9823	43.2605	1.6361
Table of additaments for $3.089 + 2A_\eta$	51	$\Delta 1''$	1.4354
Table of $\log L$ for $\varphi = 52° 23'$ $\log L_f$	1.276 7268		3.0715
Interpolation for $-43.2605''$, $\Delta \log L$	1179		
	4.366 8321		
Table of additaments for $4.367 - 2A_y$	-19		
$\log y$	4.366 8302		

$y = 23271.81$ m

The following corresponds to this in the case of our computation on p. 142:

Auxiliary table on p. [37] for $52° 22' 15''$, $\log [2]$	8.508 8707·0	(b')	4.3668
$\log (1:[2])$	1.491 1293·0	λ^2	6.1800
$\lambda = 0° 20' 30.2187'' = 1230.2187''$, $\log \lambda$	3.089 9823·2	$\sin^2 \varphi$	9.7974
$\varphi = 52° 22' 14.9611''$ $\log \cos \varphi$	9.785 7202·1	$(-1:6\varrho^2)$	8.5930_n
[or formula (20), p. 140 with $\cos 52° 22' 15.5539''$] $\log (b')$	4.366 8318·3	$\log (c')$	8.9372_n
$(b') = 23271.900$		$(c') = -0.087$	

$y = 23271.813$ m

Tables of the Reichsamt für Landesaufnahme

Schreiber's tables already mentioned on p. 85, which are in use at the Reichsamt für Landesaufnahme, can also conveniently be used for the computation of rectangular coordinates from longitude and latitude and vice versa.

As already mentioned, in Schreiber's tables the values (1) and (2) are the same as [1] and [2] of our tables of the Appendix on pages [34] to [39]; we have further herein $(3) = \dfrac{V^2}{2\rho}$ (cf. p. 85).

The 7-place tables of order II of the land survey correspond to the accuracy of the computation in the coordinate systems.

With this, we obtain for the summary of formulae (8*), (9*), (11*), as well as (17), (18), (20), (22), p. 140, the following relations:

1. Given: x, y, required: φ, λ, γ

From (8*), p. 140,

$$\varphi = \varphi_0 + (1)\,x - (3)\,((2)\,y \tan \varphi_1)\,(2)\,y\,. \qquad (27)$$

From (9*), p. 140,

$$\lambda = \frac{(2)\,y}{\cos \varphi_1}\left(1 - ((2)\,y \tan \varphi_1)^2 \frac{1}{3\varrho^2}\right)$$

and

$$\log \lambda = \log \frac{(2)\,y}{\cos \varphi_1} - \frac{\mu}{3\,\varrho^2}((2)\,y \tan \varphi_1)^2\,. \qquad (28)$$

In the table of the land survey the constant factor $\dfrac{\mu}{3\rho^2}$ is given with the notation ν, and the addition of the second term can be taken at once in units of the 7th place of logarithm according to the argument: $(2)\, y \tan \varphi_1$, which appears already above in (27).

From (11*), p. 140,
$$\gamma = (2)\, y \tan \varphi_1 \left(1 - \frac{((2)\, y)^2}{6 \varrho^2} (1 + 2 \tan^2 \varphi_1)\right).$$

In this, we have $(1 + 2 \tan^2 \varphi_1) = ((1 + \tan^2 \varphi_1) + \tan^2 \varphi_1) = \left(\frac{1}{\cos^2 \varphi_1} + \tan^2 \varphi_1\right)$,

thus
$$\gamma = (2)\, y \tan \varphi_1 \left(1 - \frac{1}{6 \varrho^2} \left(\frac{(2)\, y}{\cos \varphi_1}\right)^2 - \frac{1}{6 \varrho^2} ((2)\, y \tan \varphi_1)^2\right)$$

and at the same time with the introduction of the above explained value $\nu = \dfrac{\mu}{3 \rho^2}$

$$\log \gamma = \log (2)\, y \tan \varphi_1 - \frac{1}{2} \nu \left(\frac{(2)\, y}{\cos \varphi_1}\right)^2 - \frac{1}{2} \nu ((2)\, y \tan \varphi_1)^2, \tag{29}$$

where
$$\frac{(2)\, y}{\cos \varphi_1} = \lambda' \quad \text{and} \quad (2)\, y \tan \varphi_1 = \gamma' \tag{30}$$

are again arguments for taking immediately the additaments. With this, we can also write in abbreviated form instead of (27), (28), (29):

$$\varphi = \varphi_0 + (1)\, x - (3)\, (2)\, y\, \gamma' \tag{31}$$

$$\log \lambda = \log \lambda' - \nu \gamma'^2 \tag{32}$$

$$\log \gamma = \log \gamma' - \frac{1}{2} \nu \lambda'^2 - \frac{1}{2} \nu \gamma'^2. \tag{33}$$

2. *Given:* φ, λ, *required:* y, x, γ

From (17), (18), p. 140, there follows:

$$x = \frac{\varphi_1 - \varphi_0}{(1)} \quad \text{and} \quad \varphi_1 = \varphi + (3)\, \lambda \sin \varphi\, \lambda \cos \varphi. \tag{34}$$

From (20), p. 140:
$$y = \frac{\lambda}{(2)} \cos \varphi \left(1 - \frac{1}{6 \rho^2} (\lambda \sin \varphi)^2\right)$$

and with the introduction of the above used notation of the land survey

$$\log y = \log \frac{\lambda}{(2)} \cos \varphi - \frac{1}{2} \nu (\lambda \sin \varphi)^2. \tag{35}$$

From (22), p. 140:
$$\gamma = \lambda \sin \varphi \left(1 + \frac{1}{3 \rho^2} (\lambda \cos \varphi)^2\right),$$

and $$\log y = \log \lambda \sin \varphi + \nu (\lambda \cos \varphi)^2, \tag{36}$$

where in (35) and (36) the additaments are again to be taken immediately from the table with the arguments $\log \lambda \sin \varphi$ or, as the case may be, $\log \lambda \cos \varphi$.

Section 57. Transverse-Axis Spherical Coordinates

The Soldner coordinates treated in the previous sections 55-56 are not the only form in which rectangular coordinates can be used on the sphere. We obtain a system of rectangular coordinates connected with the geographic coordinates in an especially simple manner if we choose the equator as axis of abscissae so that the ordinates lie on the meridians. In this case, the abscissae and ordinates of any arbitrary points result by converting the geographic longitudes and latitudes into measure of arc. For a land survey which lies in the neighborhood of the equator, and which extends mainly in the direction of the latter, such a system of coordinates can be of advantage.

For a country whose mean latitude is equal to φ_0, and which has its main extent in the direction of the parallel circle, we can assume as axis of abscissae the great circle which is tangent to the parallel circle φ_0 at the center of the country. Such a system is represented in Fig. 1, in which N is the north pole, while M represents a point lying at the center of the survey territory. Then MC is the great circle of abscissae with the pole Q. For the sake of simplicity, we assume the meridian NM as zero meridian for the counting of longitudes. A point P with geographic coordinates φ and λ has therefore the rectangular coordinates x and y.

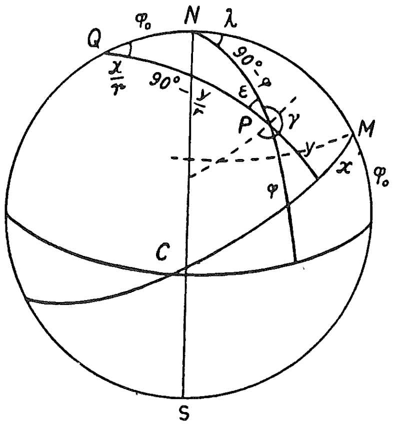

Fig. 1.

For the computation of rectangular from geographic coordinates we need the triangle QNP, in which we have the sides $QN = \varphi_0$ and $NP = 90° - \varphi$. The third side QP is equal to $90° - \frac{y}{r}$ if $\frac{y}{r}$ denotes the angular measure of the arc of the ordinate y. We find further the arc x again as the angle of the triangle $\frac{x}{r}$ at Q.

At first we have for the determination of y

$$\sin \frac{y}{r} = \sin \varphi \cos \varphi_0 - \cos \varphi \sin \varphi_0 \cos \lambda. \tag{1}$$

We will now assume that the coordinates x and y as well as the geographic longitude λ are small quantities. The geographic latitude shall further differ only little from the basic latitude φ_0, so that we can use developments in series for the trigonometric functions of $\frac{x}{r}$, $\frac{y}{r}$, λ and $\varphi - \varphi_0 = \Delta\varphi$. If we develop first $\sin \frac{y}{r}$ and $\cos l$ in (1), then we obtain

$$\frac{y}{r} - \frac{y^3}{6r^3} = \sin\varphi \cos\varphi_0 - \cos\varphi \sin\varphi_0 \left(1 - \frac{\lambda^2}{2} + \frac{\lambda^4}{24}\right)$$

or

$$\frac{y}{r} - \frac{y^3}{6r^3} = \sin \Delta\varphi + \cos\varphi \sin\varphi_0 \left(\frac{\lambda^2}{2} - \frac{\lambda^4}{24}\right) \tag{2}$$

We develop herein

$$\cos\varphi = \cos(\varphi_0 + \Delta\varphi) = \cos\varphi_0 - \Delta\varphi \sin\varphi_0 - \frac{\Delta\varphi^2}{2} \cos \Delta_0,$$

and hence we have

$$\cos\varphi \sin\varphi_0 = \sin\varphi_0 \cos\varphi_0 - \Delta\varphi \sin^2\varphi_0 - \frac{\Delta\varphi^2}{2} \sin\varphi_0 \cos\varphi_0$$

or if we set $\tan\varphi_0 = t_0$

$$\cos\varphi \sin\varphi_0 = t_0 \cos^2\varphi_0 - \Delta\varphi\, t_0^2 \cos^2\varphi_0 - \frac{\Delta\varphi^2}{2} t_0 \cos^2\varphi_0 .$$

If we set this into (2), then we will have

$$\frac{y}{r} - \frac{y^3}{6r^3} = \sin\Delta\varphi + t_0 \cos^2\varphi_0 \left(1 - t_0\Delta\varphi - \frac{\Delta\varphi^2}{2}\right)\left(\frac{\lambda^2}{2} - \frac{\lambda^4}{24}\right)$$

or

$$\frac{y}{r} - \frac{y^3}{6r^3} = \sin\Delta\varphi + t_0 \cos^2\varphi_0 \left(\frac{\lambda^2}{2} - t_0 \frac{\lambda^2\Delta\varphi}{2} - \frac{\lambda^4}{24} - \frac{\lambda^2\Delta\varphi^2}{4}\right). \qquad (3)$$

In the first approximation we can set

$$\frac{y}{r} = \sin\Delta\varphi + t_0 \cos^2\varphi_0 \frac{\lambda^2}{2} + \ldots$$

and hence we have

$$\frac{1}{6}\frac{y^3}{r^3} = \frac{1}{6}\sin^3\Delta\varphi + \frac{1}{2}\sin^2\Delta\varphi\, t_0 \cos^2\varphi_0 \frac{\lambda^2}{2}$$

and we thus obtain from (3)

$$\frac{y}{r} = \sin\Delta\varphi + \frac{1}{2} t_0 \cos^2\varphi_0 \lambda^2 + \frac{1}{6}\Delta\varphi^3 - \frac{1}{2} t^2_0 \cos^2\varphi_0 \lambda^2 \Delta\varphi - \frac{1}{24} t_0^2 \lambda^4 .$$

If we introduce in addition the development in series

$$\sin\Delta\varphi = \Delta\varphi - \frac{1}{6}\Delta\varphi^3,$$

then, finally, we will have

$$\frac{y}{r} = \Delta\varphi + \frac{1}{2} t_0 \cos^2\varphi_0 \lambda^2 - \frac{1}{2} t_0^2 \cos^2\varphi_0 \lambda^2 \Delta\varphi - \frac{1}{24} t_0 \cos^2\varphi_0 \lambda^4 . \qquad (4)$$

We now pass over to the computation of the abscissa x, for which the triangle QNP in Fig. 1 yields:

$$\tan\varphi \sin\varphi_0 = -\cos\varphi_0 \cos\lambda + \sin\lambda \cot\frac{x}{r},$$

therefore

$$\tan\frac{x}{r} = \frac{\sin\lambda}{\cos\varphi_0\,(\tan\varphi\ \ \tan\varphi_0 + \cos\lambda)}. \tag{5}$$

According to our collection of formulae on p. 18 we develop $\tan\varphi = \tan(\varphi_0 + \Delta\varphi)$, where we set again $\tan\varphi_0 = t_0$:

$$\tan\varphi = t_0 + \Delta\varphi\,(1+t_0^2) + \Delta\varphi^2\,t_0\,(1+t_0^2) + \frac{1}{3}\,\Delta\varphi^3\,(1+4\,t_0^2+3\,t_0^4);$$

therefore we have

$$\tan\varphi\ \ \tan\varphi_0 = t_0^2 + \Delta\varphi\,t_0\,(1+t_0^2) + \Delta\varphi^2\,t_0^2\,(1+t_0^2) + \frac{1}{3}\,\Delta\varphi^3\,t_0\,(1+4\,t_0^2+3\,t_0^4).$$

Now since we have $\cos\lambda = 1 - \frac{\lambda^2}{2}$, then the denominator of (5) will be

$$\cos\varphi_0\left\{1 + t_0^2 + \Delta\varphi\,t_0\,(1+t_0^2) + \Delta\varphi^2\,t_0^2\,(1+t_0^2) + \frac{1}{3}\,\Delta\varphi^3\,t_0\,(1+4\,t_0^2+3\,t_0^4) - \frac{1}{2}\,\lambda^2\right\}$$

or, since $1 + t_0^2 = \frac{1}{\cos^2\varphi_0}$ we have

$$\frac{1}{\cos\varphi_0}\left\{1 + \Delta\varphi\,t_0 + \Delta\varphi^2\,t_0^2 + \frac{1}{3}\cos^2\varphi_0\,t_0\,\Delta\varphi^3\,(1+4\,t_0^2+3\,t_0^4) - \frac{1}{2}\cos^2\varphi_0\,\lambda^2\right\}$$

and we will have so far according to (5)

$$\tan\frac{x}{r} = \sin\lambda\,\cos\varphi_0\,\frac{1}{1 + \Delta\varphi\,t_0 + \Delta\varphi^2\,t_0^2 + \frac{1}{3}\cos^2\varphi_0\,t_0\,\Delta\varphi^3\,(1+4\,t_0^2+3\,t_0^4) - \frac{1}{2}\cos^2\varphi_0\,\lambda^2}.$$

If we develop the fraction on the right-hand side according to the series

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + \ldots,$$

then we will have

$$\tan\frac{x}{r} = \sin\lambda\,\cos\varphi_0\left\{1 - \Delta\varphi\,t_0 + \frac{1}{2}\,\lambda^2\cos^2\varphi_0 - \frac{1}{3}\,\Delta\varphi^3\,t_0 - \Delta\varphi\,\lambda^2\cos^2\varphi_0\,t_0\right\}.$$

Now we still have to introduce

$$\tan\frac{x}{r} = \frac{x}{r} + \frac{1}{3}\,\frac{x^3}{r^3} \quad \text{and} \quad \sin\lambda = \lambda - \frac{\lambda^3}{6} = \lambda\left(1 - \frac{\lambda^2}{6}\right)$$

and obtain

$$\frac{x}{r}+\frac{1}{3}\frac{x^3}{r^3}=\lambda\cos\varphi_0\left\{1-\Delta\varphi\, t_0+\frac{1}{6}\lambda^2\left(3\cos^2\varphi_0-1\right)\right.$$
$$\left.-\frac{1}{3}\Delta\varphi^3 t_0-\Delta\varphi\,\lambda^2 t_0\left(\cos^2\varphi_0-\frac{1}{6}\right)\right\}. \tag{6}$$

As a first approximation we can assume

$$\frac{x}{r}=\lambda\cos\varphi_0\,(1-\Delta\varphi\, t_0),$$

and hence we will have $\frac{1}{3}\frac{x^3}{r^3}=\frac{1}{3}\lambda^3\cos^3\varphi_0(1-3\,\Delta\varphi\, t_0)=\lambda\cos\varphi_0\left(\frac{1}{3}\lambda^2\cos^2\varphi_0-\Delta\varphi\,\lambda^2 t_0\cos^2\varphi_0\right)$

and this introduced in (6) yields:

$$\frac{x}{r}=\lambda\cos\varphi_0\left\{1-\Delta\varphi\, t_0-\frac{1}{6}\lambda^2\cos^2\varphi_0\, t_0^2-\frac{1}{3}\Delta\varphi^3 t_0+\frac{1}{6}\Delta\varphi\,\lambda^2 t_0\right\}. \tag{7}$$

In addition to the spherical coordinates x and y we will now compute also a meridian convergence γ according to our former definition of section 55, p. 134. In Fig. 1, p. 146, the meridian convergence γ at the point P is the angle between the north direction and the circle parallel to the great circle of abscissae. If we consider at first the small angle ε in the triangle QNP, then we have

$$\gamma=270^\circ-\varepsilon. \tag{8}$$

For the computation of ε we have the spherical trigonometric equation

$$\cot\varphi_0\cos\varphi=-\sin\varphi\cos\lambda+\sin\lambda\cot\varepsilon$$

or

$$\tan\varepsilon=\frac{\sin\lambda\sin\varphi_0}{\cos\varphi_0\cos\varphi+\sin\varphi_0\sin\varphi\cos\lambda}.$$

If we develop in the denominator $\cos\lambda=1-\frac{\lambda^2}{2}$, then we will have

$$\tan\varepsilon=\frac{\sin\lambda\sin\varphi_0}{\cos\varphi_0\cos\varphi+\sin\varphi_0\sin\varphi-\frac{\lambda^2}{2}\sin\varphi_0\sin\varphi}$$

or

$$\tan\varepsilon=\frac{\sin\lambda\sin\varphi_0}{\cos\Delta\varphi-\frac{1}{2}\lambda^2\sin\varphi_0\sin\varphi}. \tag{9}$$

Now since $\sin\varphi=\sin(\varphi_0+\Delta\varphi)=\sin\varphi_0+\Delta\varphi\cos\varphi_0$

and hence $\sin\varphi_0\sin\varphi=\sin^2\varphi_0+\Delta\varphi\sin\varphi_0\cos\varphi_0$

then (9) passes over into the following, if we develop at the same time $\cos\Delta\varphi$:

$$\tan\varepsilon=\frac{\sin\lambda\sin\varphi_0}{1-\frac{1}{2}\Delta\varphi^2-\frac{1}{2}\lambda^2\sin^2\varphi_0-\frac{1}{2}\lambda^2\Delta\varphi\cos^2\varphi_0\, t_0}.$$

If we form the reciprocal value of the denominator and take into account here only terms of third order, then we will have

$$\tan\ \varepsilon = \sin\lambda\,\sin\varphi_0\left\{1+\frac{1}{2}\,\Delta\,\varphi^2+\frac{1}{2}\,\lambda^2\sin^2\varphi_0+\frac{1}{2}\,\lambda^2\,\Delta\,\varphi\cos^2\varphi_0\,t_0\right\}. \tag{10}$$

Now we develop herein, in addition,

$$\tan\ \varepsilon = \varepsilon+\frac{\varepsilon^3}{3} \qquad \sin\lambda = \lambda\left(1-\frac{1}{6}\,\lambda^2\right)$$

and obtain then

$$\varepsilon+\frac{1}{3}\,\varepsilon^3 = \sin\varphi_0\,\lambda\left\{1+\frac{1}{2}\,\Delta\,\varphi^2+\frac{1}{6}\,\lambda^2\,(3\sin^2\varphi_0-1)+\frac{1}{2}\,\lambda^2\,\Delta\,\varphi\cos^2\varphi_0\,t_0\right\}.$$

If we assume as an approximate value $\varepsilon = \sin\ \varphi_0\lambda$, so that we will have $\frac{1}{3}\,\varepsilon^3 = \frac{1}{3}\sin^3\varphi_0\lambda^3$, then we find finally

$$\varepsilon = \sin\varphi_0\lambda\left\{1+\frac{1}{2}\,\Delta\,\varphi^2-\frac{1}{6}\,\lambda^2\cos^2\varphi_0+\frac{1}{2}\,\lambda^2\,\Delta\,\varphi\cos^2\varphi_0\,t_0\right\}. \tag{11}$$

With this, the meridian convergence will then be according to (8)

$$\gamma = 270^\circ-\sin\varphi_0\,\lambda\left\{1+\frac{1}{2}\,\Delta\,\varphi^2-\frac{1}{6}\,\lambda^2\cos^2\varphi_0+\frac{1}{2}\,\lambda^2\,\Delta\,\varphi\cos^2\varphi_0\,t_0\right\}. \tag{12}$$

Computation of geographic coordinates from spherical rectangular coordinates

We could solve the inverse problem, the computation of the geographic coordinates from the rectangular spherical coordinates, in the same manner by taking first closed spherical trigonometric formulae from Fig. 1, p. 146, and then passing over to developments in series. We arrive at the same result, however, in a shorter way if we invert equations (4) and (7) and determine $\Delta\varphi$ as well as $\lambda\cos\varphi_0$ by gradual approximation. For this, we write down equations (4) and (7) once again in the following form:

$$\Delta\,\varphi = \frac{y}{r}-\frac{1}{2}\,\lambda^2\cos^2\varphi_0\,t_0+\frac{1}{2}\,\Delta\,\varphi\,\lambda^2\cos^2\varphi_0\,t_0^2+\frac{1}{24}\,\lambda^4\cos^2\varphi_0\,t_0 \tag{13}$$

$$\lambda\cos\varphi_0 = \frac{x}{r}+\Delta\,\varphi\,\lambda\cos\varphi_0\,t_0+\frac{1}{6}\,\lambda^3\cos^3\varphi_0\,t_0^2+\frac{1}{3}\,\Delta\,\varphi^3\,\lambda\cos\varphi_0\,t_0$$
$$-\frac{1}{6}\,\Delta\,\varphi\,\lambda^3\cos\varphi_0\,t_0\,. \tag{14}$$

We thus obtain in the first approximation with the omission of terms of second order

$$\Delta\,\varphi = \frac{y}{r}+\ldots \qquad \lambda\cos\varphi_0 = \frac{x}{r}+\ldots$$

and hence we will have, with the omission of the third order,

$$\lambda^2 \cos^2 \varphi_0 = \frac{x^2}{r^2} + \ldots$$

With this, we obtain from (13) and (14) the second approximation

$$\Delta \varphi = \frac{y}{r} - \frac{1}{2}\frac{x^2}{r^2} t_0 + \ldots \qquad \lambda \cos \varphi_0 = \frac{x}{r} + \frac{y\,x}{r^2} t_0 + \ldots$$

and

$$\lambda^2 \cos^2 \varphi_0 = \frac{x^2}{r^2} + 2\frac{y\,x^2}{r^3} t_0 + \ldots,$$

where in the latter equation the terms of fourth order are neglected.

This yields the third approximation

$$\left.\begin{aligned} \Delta \varphi &= \frac{y}{r} - \frac{1}{2}\frac{x^2}{r^2} t_0 - \frac{1}{2}\frac{y\,x^2}{r^3} t_0^2 + \ldots \\ \lambda \cos \varphi_0 &= \frac{x}{r} + \frac{y\,x}{r^2} t_0 + \frac{y^2 x}{r^3} t_0^2 - \frac{1}{3}\frac{x^3}{r^3} t_0^2 + \ldots \end{aligned}\right\} \tag{15}$$

We use these values in order to compute finally:

$$\left.\begin{aligned} \lambda^2 \cos^2 \varphi_0 &= \frac{x^2}{r^2} + \frac{2\,y\,x^2}{r^3} t_0 + \frac{3\,y^2 x^2}{r^4} t_0^2 - \frac{2}{3}\frac{x^4}{r^4} t_0^2 \\ \lambda^3 \cos^3 \varphi_0 &= \frac{x^3}{r^3} + \frac{3\,y\,x^3}{r^4} t_0 \qquad \Delta \varphi^3 = \frac{y^3}{r^3} - \frac{3}{2}\frac{y^2 x^2}{r^4} t_0 . \end{aligned}\right\} \tag{16}$$

If we set the values (15) and (16) into (13) and (14), and take into account that $\dfrac{1}{\cos^2 \varphi_0} = 1 + t^2_0$, then we obtain

$$\Delta \varphi = \frac{y}{r} - \frac{1}{2}\frac{x^2}{r^2} t_0 - \frac{1}{2}\frac{y\,x^2}{r^3} t_0^2 - \frac{1}{2}\frac{y^2 x^2}{r^4} t_0^3 + \frac{1}{24}\frac{x^4}{r^4} t_0 (1 + 3\,t_0^2) \tag{17}$$

$$\begin{aligned} \lambda \cos \varphi_0 = \frac{x}{r} + \frac{y\,x}{r^2} t_0 + \frac{y^2 x}{r^3} t_0^2 - \frac{1}{3}\frac{x^3}{r^3} t_0^2 + \frac{1}{3}\frac{y^3 x}{r^4} t_0 (1 + 3\,t_0^2) \\ - \frac{1}{6}\frac{y\,x^3}{r^4} t_0 (1 + 6\,t_0^2) . \end{aligned} \tag{18}$$

For the computation of the meridian convergence from rectangular spherical coordinates we can replace $\Delta\varphi$ and λ in equation (12) by means of the values (17) and (18) found above. For this, we have already developed $\lambda^2 \cos^2 \varphi_0$ in (16) and form now in addition

$$\Delta \varphi^2 = \frac{y^2}{r^2} - \frac{y\,x^2}{r^3} t_0 .$$

We obtain then from (11)

$$\varepsilon = \lambda \sin \varphi_0 \left\{ 1 + \frac{1}{2}\frac{y^2}{r^2} - \frac{1}{6}\frac{x^2}{r^2} - \frac{1}{3}\frac{y\,x^2}{r^3} t_0 \right\}.$$

We have further

$$\lambda \sin \varphi_0 = \lambda \cos \varphi_0 \, t_0 = \frac{x}{r} t_0 \left\{ 1 + \frac{y}{r} t_0 + \frac{y^2}{r^2} t_0{}^2 - \frac{1}{3} \frac{x^2}{r^2} t_0{}^2 + \frac{1}{3} \frac{y^3}{r^3} t_0 (1 + t_0{}^2) - \frac{1}{6} \frac{y\, x^2}{r^3} t_0 (1 + 6\, t_0{}^2) \right\}$$

and with this, there follows then the meridian convergence

$$\gamma = 270^\circ - \varepsilon$$

or

$$\gamma = 270^\circ - \frac{x}{r} t_0 \left\{ 1 + \frac{y}{r} t_0 + \frac{1}{2} \frac{y^2}{r^2} (1 + 2\, t_0{}^2) - \frac{1}{6} \frac{x^2}{r^2} (1 + 2\, t_0{}^2) + \frac{1}{6} \frac{y^3}{r^3} t_0 (5 + 6\, t_0{}^2) - \frac{1}{3} \frac{y\, x^2}{r^3} t_0 (2 + 3\, t_0{}^2) \right\}. \tag{19}$$

With this, we have set up all relations between rectangular-spherical and geographic coordinates. Moreover, there does not exist any difference between transverse-axis coordinates and Soldner coordinates; for the computations within a transverse-axis system, the same formulae which we have developed in sections 52-54 for the Soldner coordinate system can therefore be used.

The formulae for the computation of transverse-axis spherical coordinates were first developed by Jordan in *Zeitschrift für Vermessungswesen*, 1894, pp. 67-70; however, here – as well as in the former editions of this volume – the coordinates x and y are interchanged for one another. It is also to be noted that the geographic longitudes λ corresponding to the abscissae are counted here from the zero point to the west in the positive sense. A second development of the foregoing equations, in the case of which the terms of fifth order are also taken into account, are contained in the publication: Rosenmund, *Die Änderung des Projektionssystems der schweizerischen Landesvermessung*, Bern, 1903, pp. 84-88. Here, too, the coordinates are interchanged in Jordan's sense opposite our representation.

Section 58. Transformation of Coordinates

At the limits of two spherical systems of coordinates it will occur that we wish to transform the coordinates of one system into those of the other system. Without developing any new formulae, we can always solve this problem by the roundabout way through geographic coordinates. For this, we compute from the given rectangular coordinates of one system, according to the formulae previously derived, the longitude and latitude of the point, and find hence, with the help of the zero point of the second system, the rectangular coordinates in the latter.

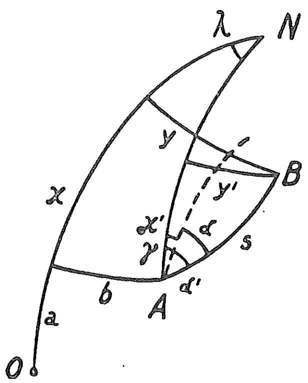

Fig. 1.

Meanwhile the problem also permits a direct treatment, where we will limit ourselves, however, to two Soldner systems of coordinates.

In Fig. 1 in the margin let O and A be the zero points of two Soldner systems of coordinates and γ the meridian convergence of A with regard to the system O. Further let B be an arbitrary point which has the coordinates x, y in the system O and the coordinates x', y' in the system A. The formulae (14) and (15), section 52, p. 120, can now be applied in a twofold manner to the two points A and B. We have

in the system O:

$$\left.\begin{aligned} y - b &= s \sin \alpha - \frac{(x-a)^2\, b}{2\, r^2} - \frac{(x-a)^2 (y-b)}{6\, r^2} \\ x - a &= s \cos \alpha + \frac{(x-a)\, y^2}{2\, r^2} - \frac{(x-a)(y-b)^2}{6\, r^2}; \end{aligned}\right\} \tag{1}$$

in the system A:

$$\left.\begin{aligned} y' &= s\ \sin\alpha' - \frac{x'^2\, y'}{6\, r^2} \\ x' &= s\ \cos\alpha' + \frac{x'\, y'^2}{3\, r^2}\,. \end{aligned}\right\} \tag{2}$$

Here we have $\alpha' = \alpha + \gamma$, therefore

$$\left.\begin{aligned} s\ \sin\alpha' &= s\ \sin\alpha\cos\gamma + s\ \cos\alpha\sin\gamma \\ s\ \cos\alpha' &= s\ \cos\alpha\cos\gamma - s\ \sin\alpha\sin\gamma\,. \end{aligned}\right\} \tag{3}$$

Introduced into (2) this yields:

$$\left.\begin{aligned} y' &= s\ \sin\alpha\cos\gamma + s\ \cos\alpha\sin\gamma - \frac{x'^2\, y'}{6\, r^2} \\ x' &= s\ \cos\alpha\cos\gamma - s\ \sin\alpha\sin\gamma + \frac{x'\, y'^2}{3\, r^2}\,. \end{aligned}\right\} \tag{4}$$

We will now set for simplification of the designations

$$y - b = v \qquad x - a = u \tag{5}$$

and obtain then from (1)

$$\left.\begin{aligned} s\ \sin\alpha &= v + \frac{u^2 b}{2\, r^2} + \frac{u^2 v}{6\, r^2} \\ s\ \cos\alpha &= u - \frac{u\, y^2}{2\, r^2} + \frac{u\, v^2}{6\, r^2}\,. \end{aligned}\right\} \tag{6}$$

We set this into (4) and obtain

$$\left.\begin{aligned} y' &= v\ \cos\gamma + u\ \sin\gamma + \frac{u^2 b}{2\, r^2}\cos\gamma + \frac{u^2 v}{6\, r^2}\cos\gamma - \frac{u\, y^2}{2\, r^2}\sin\gamma + \frac{u\, v^2}{6\, r^2}\sin\gamma - \frac{x'^2 y'}{6\, r^2} \\ x' &= u\ \cos\gamma - v\ \sin\gamma - \frac{u\, y^2}{2\, r^2}\cos\gamma + \frac{u v^2}{6\, r^2}\cos\gamma - \frac{u^2\, b}{2\, r^2}\sin\gamma - \frac{u^2 v}{6\, r^2}\sin\gamma + \frac{x' y'^2}{3\, r^2}\,. \end{aligned}\right\} \tag{7}$$

We can write the last term in the two equations (7) in another form by setting for y' and x'

$$y' = v\ \cos\ \gamma + u\ \sin\ \gamma + \ldots$$

$$x' = u\ \cos\ \gamma - v\ \sin\ \gamma + \ldots$$

and taking into account here that γ is of order $\frac{1}{r}$ according to (4), section 55, p. 135. If we neglect all terms of the order $\frac{1}{r^2}$, then we will have

$$\left.\begin{aligned} +\frac{x' y'^2}{3\, r^2} &= +\frac{u\, v^2}{3\, r^2} - \frac{v^3}{3\, r^2}\sin\gamma + \frac{2\, u^2\, v}{3\, r^2}\sin\gamma \\ -\frac{x'^2 y'}{6\, r^2} &= -\frac{u^2\, v}{6\, r^2} + \frac{u v^2}{3\, r^2}\sin\gamma - \frac{u^3}{6\, r^2}\sin\gamma \end{aligned}\right\} \tag{8}$$

and with this, equation (7) transforms into

$$\left.\begin{aligned} y' &= v\cos\gamma + u\sin\gamma + \frac{u^2 b}{2r^2} - \frac{u\,y^2}{2r^2}\sin\gamma + \frac{u v^2}{2r^2}\sin\gamma - \frac{u^3}{6r^2}\sin\gamma \\ x' &= u\cos\gamma - v\sin\gamma - \frac{u\,y^2}{2r^2} + \frac{u\,v^2}{2r^2} - \frac{u^2 b}{2r^2}\sin\gamma + \frac{u^2 v}{2r^2}\sin\gamma - \frac{v^3}{3r^2}\sin\gamma\,. \end{aligned}\right\} \qquad (9)$$

These two equations become still somewhat more advantageous if we introduce

$$y = (y - b) + b = v + b; \qquad \text{therefore} \qquad y^2 = v^2 + b^2 + 2vb.$$

Then we obtain

$$\left.\begin{aligned} y' &= v\cos\gamma + u\sin\gamma + \frac{u^2 b}{2r^2} - \frac{u b^2}{2r^2}\sin\gamma - \frac{u v b}{r^2}\sin\gamma - \frac{u^3}{6r^2}\sin\gamma \\ x' &= u\cos\gamma - v\sin\gamma - \frac{u b^2}{2r^2} - \frac{u^2 b}{2r^2}\sin\gamma - \frac{u v b}{r^2} + \frac{u^2 v}{2r^2}\sin\gamma - \frac{v^3}{3r^2}\sin\gamma \end{aligned}\right\} \qquad (10)$$

where $$v = y - b \qquad u = x - a.$$

Inversion of the problem

If the coordinates x', y' in the coordinate system A are given, and the coordinates x, y in the system B are required, where, besides, a and b as well as γ are assumed to be known, then a direct application of the formulae (10) is not possible, since the indices cannot forthwith be interchanged herein. We must therefore carry out a new development. For this, we write the equation (1) in the form

$$\left.\begin{aligned} y - b &= s\sin(\alpha' - \gamma) - \frac{u^2 b}{2r^2} - \frac{u^2 v}{6r^2} \\ x - a &= s\cos(\alpha' - \gamma) + \frac{u\,y^2}{2r^2} - \frac{u v^2}{6r^2} \end{aligned}\right\} \qquad (11)$$

or

$$\left.\begin{aligned} y - b &= s\sin\alpha'\cos\gamma - s\cos\alpha'\sin\gamma - \frac{u^2 b}{2r^2} - \frac{u^2 v}{6r^2} \\ x - a &= s\cos\alpha'\cos\gamma + s\sin\alpha'\sin\gamma + \frac{u\,y^2}{2r^2} - \frac{u v^2}{6r^2}\,. \end{aligned}\right\} \qquad (12)$$

Now we have according to (2)

$$s\sin\alpha' = y' + \frac{x'^2 y'}{6r^2} \qquad\qquad s\cos\alpha' = x' - \frac{x' y'^2}{3r^2} \qquad (13)$$

therefore, we will have

$$\left.\begin{aligned} y - b &= y'\cos\gamma - x'\sin\gamma + \frac{x'^2 y'}{6r^2}\cos\gamma + \frac{x' y'^2}{3r^2}\sin\gamma - \frac{u^2 b}{2r^2} - \frac{u^2 v}{6r^2} \\ x - a &= x'\cos\gamma + y'\sin\gamma - \frac{x' y'^2}{3r^2}\cos\gamma + \frac{x'^2 y'}{6r^2}\sin\gamma + \frac{u\,y^2}{2r^2} - \frac{u\,v^2}{6r^2}\,. \end{aligned}\right\} \qquad (14)$$

The two auxiliary quantities u and v had the meaning indicated in (5). But nothing changes in the terms with $\frac{1}{r^2}$ if we set

$$v = s \sin(\alpha' - \gamma) \qquad u = s \cos(\alpha' - \gamma)$$

or

$$\left.\begin{aligned} v &= y' \cos\gamma - x' \sin\gamma \\ u &= x' \cos\gamma + y' \sin\gamma \end{aligned}\right\} \tag{15}$$

so that we can also write equation (14)

$$\left.\begin{aligned} y &= b + v + \frac{x'^2 y'}{6r^2}\cos\gamma + \frac{x' y'^2}{3r^2}\sin\gamma - \frac{u^2 b}{2r^2} - \frac{u^2 v}{6r^2} \\ x &= a + u - \frac{x' y'^2}{3r^2}\cos\gamma + \frac{x'^2 y'}{6r^2}\sin\gamma + \frac{u\, y^2}{2r^2} - \frac{u\, v^2}{6r^2}\,. \end{aligned}\right\} \tag{16}$$

Transformations on a small territory

If a very great number of transformations within a small territory is involved, then the above formulae can in addition be brought into a more convenient form.

For this, we assume at the center of the territory an auxiliary point H for which the coordinates x_h' y_h' are given. According to the above formulae (16) we then compute x_h and y_h in the system O.

This is represented in Fig. 2. There are indicated here, at the same time, the meridian convergences γ and γ' at the point H for the two systems of coordinates, which are likewise to be computed. The two foot-point latitudes, which are required for this, result according to (1), section 55, p. 134, from

$$\varphi_1 = \varphi_0 + \frac{x_h}{r} \qquad \text{and} \qquad \varphi_1' = \varphi_a + \frac{x_h'}{r}, \tag{17}$$

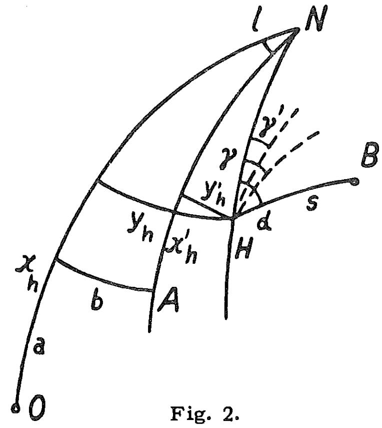

Fig. 2.

and we have then for the computation of the meridian convergences equation (4), section 55, p. 135,

$$\gamma = \frac{y}{r}\tan\varphi_1 - \frac{y^3}{6r^3}\tan\varphi_1\,(1 + 2\tan^2\varphi_1). \tag{18}$$

With this, everything is prepared for the transformation of the points in the neighborhood of the auxiliary point H. We will limit ourselves here to points whose distances from the auxiliary point H are not greater than, say, 6 km.

For an arbitrary point B in Fig. 2 let α be the azimuth of the arc HB. In the system A the direction angle is then equal to $\alpha - \gamma'$, and we have according to (14) and (15), section 52, p. 120,

$$\left.\begin{aligned} y' - y_h' &= \Delta y' = s \sin(\alpha - \gamma') - \frac{u^2}{2r^2} y_h' - \frac{u^2 v}{6r^2} \\ x' - x_h' &= \Delta x' = s \cos(\alpha - \gamma') + \frac{u}{2r^2} y'^2 - \frac{u\, v^2}{6r^2} \end{aligned}\right\} \tag{19}$$

where

$$v = s \sin(\alpha - \gamma') \qquad \text{and} \qquad u = s \cos(\alpha - \gamma'). \tag{20}$$

On the other hand, the direction angle of the arc HB in the system O is equal to $\alpha - \gamma = (\alpha - \gamma') + (\gamma' - \gamma)$. Therefore, for the system O we have

$$\left.\begin{aligned} y - y_h = \Delta y &= s \sin\Big((\alpha - \gamma') + (\gamma' - \gamma)\Big) - \frac{u^2}{2r^2} y_h - \frac{u^2 v}{6r^2} \\ x - x_h = \Delta x &= s \cos\Big((\alpha - \gamma) + (\gamma' - \gamma)\Big) + \frac{u}{2r^2} y^2 - \frac{u v^2}{2r^2}. \end{aligned}\right\} \quad (21)$$

In the terms with $\frac{1}{r^2}$ the difference between the direction angles of the arc HB in the two systems is neglected here.

In the two equations of (21) if we solve the first term on the right-hand side goniometrically and introduce the values of $s \sin(\alpha - \gamma')$ and $s \cos(\alpha - \gamma')$ resulting from (19), then there follows

$$\left.\begin{aligned} \Delta y &= \Delta y' \cos(\gamma' - \gamma) + \frac{u^2}{2r^2} y'_h \cos(\gamma' - \gamma) + \frac{u^2 v}{6r^2} \cos(\gamma' - \gamma) - \frac{u^2}{2r^2} y_h \\ &\quad - \frac{u^2 v}{6r^2} + \Delta x' \sin(\gamma' - \gamma) - \frac{u}{2r^2} y'^2 \sin(\gamma' - \gamma) + \frac{u v^2}{6r^2} \sin(\gamma' - \gamma) \\ \Delta x &= \Delta x' \cos(\gamma' - \gamma) - \frac{u}{2r^2} y'^2 \cos(\gamma' - \gamma) + \frac{u v^2}{6r^2} \cos(\gamma' - \gamma) + \frac{u}{2r^2} y^2 \\ &\quad - \frac{u v^2}{6r^2} - \Delta y' \sin(\gamma' - \gamma) - \frac{u^2}{2r^2} y'_h \sin(\gamma' - \gamma) - \frac{u^2 v}{6r^2} \sin(\gamma' - \gamma). \end{aligned}\right\} \quad (22)$$

In these two equations (22) the last term can forthwith be neglected, since it is no longer considered in the case of the assumed distances. If we collect further in each of the two equations the two terms in $u^2 v$ or, as the case may be, in $u v^2$, then there follows

$$\begin{aligned} + \frac{u^2 v}{6r^2}\Big(\cos(\gamma' - \gamma) - 1\Big) &= - \frac{u^2 v}{3r^2} \sin^2 \frac{\gamma' - \gamma}{2} \\ + \frac{u v^2}{6r^2}\Big(\cos(\gamma' - \gamma) - 1\Big) &= - \frac{u v^2}{3r^2} \sin^2 \frac{\gamma' - \gamma}{2}, \end{aligned}$$

which can likewise be neglected.

In the first equation of (22) we can in addition make use of the approximation

$$y_h - y'_h = b.$$

Then we have

$$\begin{aligned} \frac{u^2}{2r^2} y'_h \cos(\gamma' - \gamma) - \frac{u^2}{2r^2} y_h &= \frac{u^2}{2r^2} y'_h \Big(\cos(\gamma' - \gamma) - 1\Big) - \frac{u^2}{2r^2} b \\ &= - \frac{u^2}{r^2} y'_h \sin^2 \frac{\gamma' - \gamma}{2} - \frac{u^2}{2r^2} b. \end{aligned}$$

Due to the small factor $\sin^2 \frac{\gamma' - \gamma}{2}$ the first term on the right-hand side is likewise no longer considered.

In the second equation of (22) numerical proof can in addition be furnished that the next-to-the-last term cannot reach the amount of 1 mm, and therefore it can also be neglected.

Finally, in the terms with $\frac{1}{r^2}$ we can also set, instead of the expressions (20),

$$\left.\begin{aligned} v &= \Delta y' \cos(\gamma' - \gamma) + \Delta x' \sin(\gamma' - \gamma) \\ u &= \Delta x' \cos(\gamma' - \gamma) - \Delta y' \sin(\gamma' - \gamma) \end{aligned}\right\} \tag{23}$$

or else

$$\left.\begin{aligned} v &= \Delta y' - 2\,\Delta y' \sin^2\frac{\gamma' - \gamma}{2} + \Delta x' \sin(\gamma' - \gamma) \\ u &= \Delta x' - 2\,\Delta x' \sin^2\frac{\gamma' - \gamma}{2} - \Delta y' \sin(\gamma' - \gamma)\,. \end{aligned}\right\} \tag{24}$$

With this, the equations (22) pass over into

$$\left.\begin{aligned} \Delta y &= v - \frac{y_h - y'_h}{2\,r^2}\,u^2 - \frac{u\,y'^2}{2\,r^2}\sin(\gamma' - \gamma) \qquad & y &= y_h + \Delta y \\ \Delta x &= u - \frac{u\,y'^2}{2\,r^2}\cos(\gamma' - \gamma) + \frac{u\,y^2}{2\,r^2} \qquad & x &= x_h + \Delta x\,. \end{aligned}\right\} \tag{25}$$

In the second half-volume, in the case of the treatment of the transformation of coordinates on the ellipsoid, we shall come back once again to the above formulae and, therefore, omit now the presentation of a numerical example for it.

Schmehl gives an extension of the foregoing formulae for application to greater distances in *Zeitschrift für Vermessungswesen*, 1938, pp. 481-485.

Section 59. The Spherical Polar Triangle

After having learned in the previous sections of this chapter the theory of rectangular coordinates on the sphere and the transformation of rectangular into geographic coordinates and vice versa, we will in addition develop the relations between the geographic coordinates of two points on the sphere and their distance and also the two azimuths of their connecting line.

The basis for this investigation is formed by the *polar triangle* between the two points and the north pole, which in geodesy plays the same important role as the nautical triangle (pole-zenith-star) in practical astronomy.

We shall at first set up closed spherical-trigonometric formulae for the two problems occurring here, and then, in the next sections, pass over to developments in series, which shall serve at the same time as a preparation for the developments in series with the geodetic line on the ellipsoid.

Two points P and P' have the geographic latitudes φ and φ' and between them the geographic difference of longitude λ. The connecting arc PP' as the arc of a great circle has the value σ as central angle at the earth's center and the azimuths α and α' at its end-points. The radius of the sphere on which the triangle $PP'N$ is assumed to lie, is not considered.

Our problem will be a twofold one:

either φ, φ', λ is given and σ, α, α' required,

or φ, σ, α is given and φ', λ, α' required.

Since we only deal here with the purely *spherical* solution of the problems in question, and since we understand that in both cases the point in question is only to solve a spherical triangle from two given sides

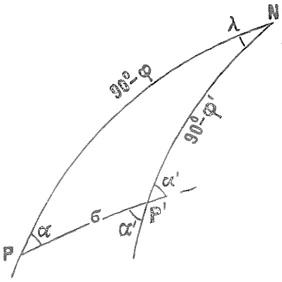

Fig. 1.

and the included angle, in principle there are no difficulties, and the point in question is therefore only to consider the various forms of solution which spherical trigonometry offers for our case and to arrange them for our purposes (for which Gauss has shown the ways in *Untersuchungen über Gegenstände der höheren Geodäsie*, erste Abhandlung, 1843, Art. 16 and 17).

Before we pass over to our developments of formulae and to numerical application of spherical formulae, we will first set forth two examples rigorously computed (with 10-place logarithms), which can serve in a different manner as standard examples:

Small spherical standard example

(Notation according to Fig. 1)

$$\left.\begin{array}{llll} & \varphi = 49^\circ\,30'\,0'' & \varphi' = 50^\circ\,30'\,0'' & \lambda = 1^\circ\;\;0'\,0'' \\ \varphi_0 = \dfrac{\varphi'+\varphi}{2} & = 50\quad 0\;\;0 & \dfrac{\varphi'-\varphi}{2} = \;0\;\;30\;\;0 & \dfrac{\lambda}{2} = 0\;\;30\;\;0 \\ \alpha_0 = \dfrac{\alpha'+\alpha}{2} & = 32^\circ\,44'\;0.2384'' & & \alpha'-\alpha = 0^\circ\,45'\,57.893\,93'' \\ \dfrac{\alpha'-\alpha}{2} & = \;\;0\;\;22\;\;58.9470 & & \dfrac{\sigma}{2} = 0\;\;35\;\;39.740\,93 \\ \hline & \alpha' = 33^\circ\;\;6'\,59.1854'' & & \sigma = 1\;\;11\;\;19.481\,86 \\ & \alpha\; = 32\;\;21\;\;\;1.2914 & & \sigma = 4279.481\,86''. \end{array}\right\} \qquad (1)$$

Large spherical standard example

(Notation according to Fig. 1)

$$\left.\begin{array}{llll} & \varphi = 45^\circ\;\;0'\,0'' & \varphi' = 55^\circ\,0'\,0'' & \lambda = 10^\circ\,0'\,0'' \\ \varphi_0 = \dfrac{\varphi'+\varphi}{2} & = 50\quad 0\;\;0 & \dfrac{\varphi'-\varphi}{2} = \;5\;\;0\;\;0 & \dfrac{\lambda}{2} = \;5\;\;0\;\;0 \\ \alpha_0 = \dfrac{\alpha'+\alpha}{2} & = 32^\circ\,49'\,54.6437'' & & \alpha'-\alpha = \;7^\circ\,41'\,51.671\,00'' \\ \dfrac{\alpha'+\alpha}{2} & = \;\;3\;\;50\;\;55.8355 & & \dfrac{\sigma}{2} = \;5\;\;55\;\;51.321\,53 \\ \hline & \alpha' = 36^\circ\,40'\,50.4792'' & & \sigma = 11\;\;51\;\;42.643\,06 \\ & \alpha\; = 28\;\;58\;\;58.8082 & & \sigma = 42{,}702.643\,06''. \end{array}\right\} \qquad (2)$$

I. Given $\varphi, \varphi', \lambda$. Required σ, α, α'.

I a. The Gauss equations of spherical trigonometry

If we apply Gauss' or, as the case may be, Napier's equations of section 33, p. 17, to our case, then we obtain with the abbreviations

$$\frac{\varphi'+\varphi}{2} = \varphi_0 \quad \text{and} \quad \frac{\alpha'+\alpha}{2} = \alpha_0$$

for the mean values the following:

$$\left.\begin{array}{ll} \sin\dfrac{\sigma}{2}\,\sin\alpha_0 & = \cos\varphi_0\,\sin\dfrac{\lambda}{2} \\ \sin\dfrac{\sigma}{2}\,\cos\alpha_0 & = \sin\dfrac{\varphi'-\varphi}{2}\,\cos\dfrac{\lambda}{2} \\ \cos\dfrac{\sigma}{2}\,\sin\dfrac{\alpha'-\alpha}{2} & = \sin\varphi_0\,\sin\dfrac{\lambda}{2} \\ \cos\dfrac{\sigma}{2}\,\cos\dfrac{\alpha'-\alpha}{2} & = \cos\dfrac{\varphi'-\varphi}{2}\,\cos\dfrac{\lambda}{2}\,. \end{array}\right\} \qquad (3)$$

If we divide the first by the second, then the third by the fourth of these equations and, for abbreviation for the following, introduce the symbols Z and N and Z' and N' for the numerator and denominator of the resulting fractions, then we obtain:

$$\left.\begin{aligned} \tan\ \alpha_0 &= \frac{\cos\varphi_0 \sin\frac{\lambda}{2}}{\sin\frac{\varphi'-\varphi}{2}\cos\frac{\lambda}{2}} = \frac{Z}{N} \\ \sin\frac{\sigma}{2} &= \frac{Z}{\sin\alpha_0} = \frac{N}{\cos\alpha_0} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \tan\ \frac{\alpha'-\alpha}{2} &= \frac{\sin\varphi_0 \sin\frac{\lambda}{2}}{\cos\frac{\varphi'-\varphi}{2}\cos\frac{\lambda}{2}} = \frac{Z'}{N'} \\ \cos\frac{\sigma}{2} &= \frac{Z'}{\sin\frac{\alpha'-\alpha}{2}} = \frac{N'}{\cos\frac{\alpha'-\alpha}{2}} \end{aligned}\right\} \tag{5}$$

As a numerical example we take according to (1):

$$\varphi = 49^\circ\ 30'\ 0'', \qquad \varphi' = 50^\circ\ 30'\ 0'', \qquad \lambda = 1^\circ\ \ 0'\ 0''; \tag{6}$$

therefore $\varphi_0 = 50^\circ\ \ 0'\ 0''$, $\frac{\varphi'-\varphi}{2} = 0^\circ\ 30'\ 0''$, $\frac{\lambda}{2} = 0^\circ\ 30'\ 0''$.

The logarithmic computation yields:

$\log Z$	7.748 9093·6
$\log N$	7.940 8253·2
$\log\tan\ \alpha_0$	9.808 0840·4
$\log\sin\frac{\sigma}{2}$	8.015 9282·7

$\log Z'$	7.825 0958·3
$\log N'$	9.999 9669·3
$\log\tan\frac{\alpha'-\alpha}{2}$	7.825 1289·0
$\log\cos\frac{\sigma}{2}$	9.999 9766·3

$$\begin{aligned} \alpha_0 &= 32^\circ\ 44'\ \ 0.238'' \\ \frac{\alpha'-\alpha}{2} &= \ \ 0\ \ \ 22\ \ 58.947 \\ \hline \alpha' &= 33^\circ\ \ 6'\ 59.185'' \\ \alpha &= 32\ \ \ 21\ \ \ 1.291 \end{aligned} \qquad \left.\begin{aligned} \frac{\sigma}{2} &= 0^\circ\ 35'\ 39.741'' \\ \sigma &= 1\ \ \ 11\ \ 19.482\ . \end{aligned}\right\} \tag{7}$$

Of the two determinations for $\frac{\sigma}{2}$, namely from $\sin\frac{\sigma}{2}$ and from $\cos\frac{\sigma}{2}$, only the first is rigorous in this case, since σ is small, while the second from $\cos$ can only be used as a sum check.

I b. Individual formulae for σ, α and α'

The cosine formula on p. 16 is used for the determination of σ alone:

$$\cos\sigma = \sin\varphi\sin\varphi' + \cos\varphi\cos\varphi'\cos\lambda. \tag{8}$$

But since in our cases σ is always small, we cannot directly compute with respect to $\cos\sigma$;

however, we can easily transform the above formula by setting:

$$\cos\sigma = 1 - 2\sin^2\frac{\sigma}{2} \qquad \text{and} \qquad \cos\lambda = 1 - 2\sin^2\frac{\lambda}{2}.$$

With this, we find easily:

$$\sin\frac{\sigma}{2} = \sqrt{\sin^2\frac{\varphi'-\varphi}{2} + \cos\varphi\cos\varphi'\sin^2\frac{\lambda}{2}}\,. \tag{9}$$

We compute then with an auxiliary angle μ similarly as in the case of the determination of a hypotenuse from two legs:

$$\tan\mu = \frac{\sin\frac{\varphi'-\varphi}{2}}{\sin\frac{\lambda}{2}\sqrt{\cos\varphi\cos\varphi'}}$$

$$\sin\frac{\sigma}{2} = \frac{\sin\frac{\varphi'-\varphi}{2}}{\sin\mu} \quad \text{or} \quad = \frac{\sin\frac{\lambda}{2}\sqrt{\cos\varphi\cos\varphi'}}{\cos\mu}\,.$$

Our small standard example (1), p. 158, yields:

$\log\sin\frac{\varphi'-\varphi}{2}$	7.940 8418·6		(10)
$\log\sin\frac{\lambda}{2}\sqrt{\ldots}$	7.748 8693·3		
$\log\tan\mu$	0.191 9725·3	$\mu = 57°\ 16'\ 11.981''$	
$\log\sin\frac{\sigma}{2}$	8.015 9282·7	$\frac{\sigma}{2} = 0\ \ 35\ \ 39.741$	
		$\sigma = 1\ \ 11\ \ 19.482\,.$	

For the azimuths α and α' spherical trigonometry also gives immediate solutions, namely according to the cotangent formulae of p. 16:

$$\cot\alpha = \frac{\tan\varphi'\cos\varphi}{\sin\lambda} - \sin\varphi\cot\lambda \tag{11}$$

$$\cot(\alpha' \pm 180°) = \frac{\tan\varphi\cos\varphi'}{\sin\lambda} - \sin\varphi'\cot\lambda\,. \tag{12}$$

Our large standard example (2) gives the following application for this:

$$\begin{aligned}
&\varphi = 45°\ 0' \ , \quad \varphi' = 55°\ 0' \ , \quad \lambda = 10°\ 0' \\
&\cot\alpha = 5.815\,512\,455 - 4.010\,201\,831 = 1.805\,310\,624 \\
&\log\cot\alpha = 0.256\,5519{\cdot}4 \qquad \alpha = 28°\ 58'\ 58.808''.
\end{aligned} \tag{13}$$

The small standard example (1), on the other hand, gives:

$$\begin{aligned}
&\varphi = 49°\ 30' \ , \quad \varphi' = 50°\ 30' \ , \quad \lambda = 1°\ 0' \\
&\cot\alpha = 45.142\,3983 - 43.563\,6286 = 1.578\,7697 \\
&\log\cot\alpha = 0.198\,3187{\cdot}8 \qquad \alpha = 32°\ 21'\ 1.290''.
\end{aligned} \tag{13a}$$

If φ and φ' are nearly equal, and λ is small, then the formulae (11) and (12) do not yield rigorous determinations, because a difference of two not very different values is to be calculated here, as (13a) with 45.14 . . . − 43.56 . . . clearly shows.

We can in addition find some other forms of solution for the submitted first problem I, as will result from the analogy with the second problem II, to which we now pass over.

II. Given φ, σ, α. Required φ', λ, α'.

II a. Solution by the Gauss equations

The application of Gauss' or, as the case may be, Napier's equations of p. 17 to our case yields:

$$\left.\begin{aligned} \tan\frac{\alpha'+\lambda}{2} &= \frac{\sin\dfrac{90^\circ-\varphi+\sigma}{2}\,\sin\dfrac{\alpha}{2}}{\sin\dfrac{90^\circ-\varphi-\sigma}{2}\,\cos\dfrac{\alpha}{2}} = \frac{Z}{N} \\ \sin\frac{90^\circ-\varphi'}{2} &= \frac{Z}{\sin\dfrac{\alpha'+\lambda}{2}} = \frac{N}{\cos\dfrac{\alpha'+\lambda}{2}} \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} \tan\frac{\alpha'-\lambda}{2} &= \frac{\cos\dfrac{90^\circ-\varphi+\sigma}{2}\,\sin\dfrac{\alpha}{2}}{\cos\dfrac{90^\circ-\varphi-\sigma}{2}\,\cos\dfrac{\alpha}{2}} = \frac{Z'}{N'} \\ \cos\frac{90^\circ-\varphi'}{2} &= \frac{Z'}{\sin\dfrac{\alpha'-\lambda}{2}} = \frac{N'}{\cos\dfrac{\alpha'-\lambda}{2}} \end{aligned}\right\} \quad (15)$$

In the case of our small standard example (1) we have:

Given $\varphi = 49^\circ\,30'\,0''$ $\quad \sigma = 1^\circ\,11'\,19.482''$ $\quad \alpha = 32^\circ\,21'\,1.291''$.

We therefore have for the application of (14) and (15):

$$\frac{90^\circ-\varphi+\sigma}{2} = 20^\circ\,50'\,39.741'' \qquad \frac{\alpha}{2} = 16^\circ\,10'\,30.646''$$

$$\frac{90^\circ-\varphi-\sigma}{2} = 19\quad 39\quad 20.259$$

$\log Z$	8.996 1858·3
$\log N$	9.509 2708·3
$\log\tan\dfrac{\alpha'+\lambda}{2}$	9.486 9150·0
$\log\sin\dfrac{90^\circ-\varphi'}{2}$	9.528 8096·8

$\log Z'$	9.415 5449·8
$\log N'$	9.956 3857·0
$\log\tan\dfrac{\alpha'-\lambda}{2}$	9.459 1592·8
$\log\cos\dfrac{90^\circ-\varphi'}{2}$	9.973 6708·5

$$\frac{\alpha'+\lambda}{2} = 17^\circ\,3'\,29.592''$$
$$\frac{\alpha'-\lambda}{2} = 16\quad 3\quad 29.592$$
$$\alpha' = 33^\circ\,6'\,59.184''$$
$$\lambda = 1\quad 0\quad 0.000$$

$$\left.\begin{aligned} \frac{90^\circ-\varphi'}{2} &= 19^\circ\,45'\,0.000'' \\ 90^\circ-\varphi' &= 39\quad 30\quad 0.000 \\ \varphi' &= 50\quad 30\quad 0.000\,. \end{aligned}\right\} \quad (15a)$$

II b. Individual formulae for φ', α' and λ

For the determination of φ' from φ, σ and α we have to apply the cosine formula on p. 16, and for α' and λ we have to apply one of the cotangent formulae on p. 16 in each case. We obtain in this way the following three solutions:

$$\sin\varphi' = \sin\varphi\cos\sigma + \cos\varphi\sin\sigma\cos\alpha \tag{16}$$

$$\cot\alpha' = \frac{\cos\sigma\cos\alpha - \sin\sigma\ \tan\varphi}{\sin\alpha} \tag{17}$$

$$\cot\lambda = \frac{\cot\sigma\cos\varphi - \sin\varphi\cos\alpha}{\sin\alpha}\,. \tag{18}$$

For the application to our small standard example we have:

Given $\varphi = 49°\,30'\,0''$, $\sigma = 1°\,11'\,19.482''$, $\alpha = 32°\,21'\,1.291''$.

The calculation according to (16), (17) and (18) yields:

$$\begin{aligned}
\sin\varphi' &= 0.760\,2423 + 0.011\,3823 = 0.771\,6246 \\
\log\sin\varphi' &= 9.887\,4061 \qquad \varphi' = 50°\,30'\,0{,}00'' \\
\cot\alpha' &= 1.578\,4299 - 0.045\,3947 = 1.533\,0352 \\
\log\cot\alpha' &= 0.185\,5521 \qquad \alpha' = 33°\,6'\,59.19'' \\
\cot\lambda &= \frac{31.297\,9570 - 0.642\,3847}{\sin\alpha} = \frac{30.655\,5723}{\sin\alpha} \\
\log\cot\lambda &= 1.758\,0785 \qquad \lambda = 1°\,0'\,0.00''\,.
\end{aligned} \tag{19}$$

Nothing can be said against these three formulae of solution; they yield φ', α' and λ individually with the usual rigorousness. The repeated passage from logarithms to numbers and vice versa, shunned by many computers, can in case of need be avoided by the use of logarithms of addition and subtraction.

II c. Rectangular projection of the north pole on the side σ

In Fig. 2, which is drawn somewhat differently than the previous Fig. 1, but essentially represents the same thing, a perpendicular NP_0 is drawn from the north pole N to PP' extended, by which the length m of this perpendicular itself as well as the length PP_0 is determined, which we will denote by $90° - M$.

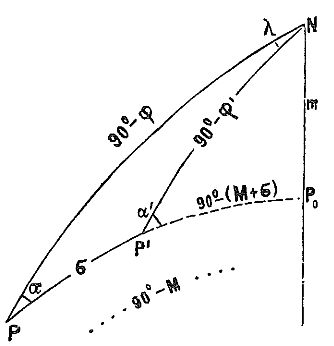

Fig. 2.
Auxiliary angle M and m.

The large right triangle PNP_0 is determined completely by our given φ's and α's, and since by setting off $PP' = \sigma$ on PP_0 the point P' is also determined, then the second smaller right triangle $P'NP_0$ as well as the oblique angle remaining triangle $PP'N$ is thereby determined. Our whole problem is then reduced to the treatment of two right spherical triangles, and therefore we write down here at once the necessary formulae [which we could also derive purely goniometrically from the formulae (16), (17), (18)] in the succession needed for the computation.

We have for the determination of M and m:

$$\left.\begin{aligned}
\tan M &= \frac{\sin\varphi}{\cos\varphi\cos\alpha} \\
\cos m &= \frac{\sin\varphi}{\sin M} \quad \text{or} \quad \cos m = \frac{\cos\varphi\cos\alpha}{\cos M} \\
\sin m &= \sin\alpha\cos\varphi\,.
\end{aligned}\right\} \tag{20}$$

After M and m are thus determined and secured, we have further:

$$\tan \alpha' = \frac{\tan m}{\cos (M + \sigma)} \tag{21}$$

$$\sin \varphi' = \cos m \sin (M + \sigma) \qquad \tan \varphi' = \tan (M + \sigma) \cos \alpha' \tag{22}$$

$$\sin \lambda = \frac{\sin \sigma \sin \alpha'}{\cos \varphi} = \frac{\sin \sigma \sin \alpha}{\cos \varphi'} . \tag{23}$$

The application to our small standard example with the given values φ, α and σ according to (1) leads to the auxiliary angles:

$$M = 54° \, 11' \, 19.61'' \qquad m = 20° \, 20' \, 7.75'',$$

with which the values φ', α' and λ result as previously.

II d. Rectangular coordinates x, y for the point P'

In Fig. 3 the meridians PN and $P'N$ are drawn as straight lines, and an arc of a great circle $P'P_1$ is laid perpendicularly to the meridian PN so that we obtain at $PP_1 = x$ and $P_1P' = y$ rectangular spherical coordinates of the point P'. x and y shall be computed here, just as σ, in angular measure.

The values x and y are determined by the equations

$$\tan x = \tan \sigma \cos \alpha \tag{24}$$

and

$$\sin y = \sin \sigma \sin \alpha, \qquad \tan y = \sin x \tan \alpha. \tag{25}$$

With x we also have the foot-point latitude $\varphi_1 = \varphi + x$, and with this also $P_1N = 90° - \varphi_1 = 90° - (\varphi + x)$, the leg of the large right triangle NP_1P'. In the latter we have then

$$\tan \lambda = \frac{\tan y}{\cos (\varphi + x)} \tag{26}$$

and

$$\left.\begin{aligned} \sin \varphi' &= \sin (\varphi + x) \cos y \\ \tan \varphi' &= \tan (\varphi + x) \cos \lambda . \end{aligned}\right\} \tag{27}$$

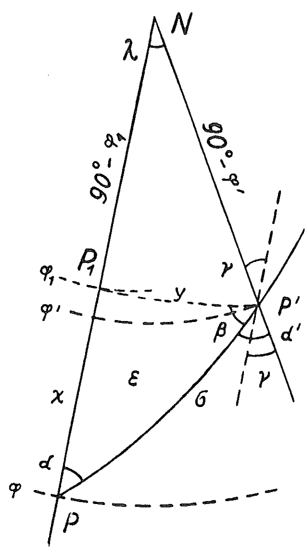

Fig. 3.

Finally according to the law of sines:

$$\sin \alpha' = \frac{\sin \alpha \cos \varphi}{\cos \varphi'} . \tag{28}$$

Gauss further improved this simple and obvious solution (in Art. 16 of *Untersuchungen über Gegenstände der höheren Geodäsie*, erste Abhandlung, Göttingen, 1843) by introducing first the small difference of latitude $\varphi_1 - \varphi'$, which we will denote by δ, second the meridian convergence γ at the point P' and the spherical excess ε of the right triangle PP_1P'.

In order to determine the spherical excess ε, we have the development already used in section 50, p. 104:

$$\cot\alpha \cot\beta = \cos\sigma = 1 - 2\sin^2\frac{\sigma}{2}$$

$$\cos\alpha\cos\beta = \sin\alpha\sin\beta - 2\sin\alpha\sin\beta\sin^2\frac{\sigma}{2}$$

$$\cos(\alpha+\beta) = -\sin\varepsilon = -2\sin\alpha\sin\beta\sin^2\frac{\sigma}{2}$$

$$\sin\varepsilon = 2\sin\alpha\frac{\sin x}{\sin\sigma}\sin^2\frac{\sigma}{2}$$

$$\sin\varepsilon = \tan\frac{\sigma}{2}\sin x\sin\alpha\,. \tag{29}$$

For γ we have from the right triangle NP_1P':

$$\tan(90° - \gamma) = \frac{\tan(90° - (\varphi + x))}{\sin y}$$

$$\tan\gamma = \tan(\varphi + x)\sin y = \tan(\varphi + x)\sin\sigma\sin\alpha\,. \tag{30}$$

In order to determine, in addition, also δ, we have at first according to Fig. 3

$$\delta = (\varphi + x) - \varphi';$$

therefore

$$\sin\delta = \sin((\varphi + x) - \varphi') = \sin(\varphi + x)\cos\varphi' - \cos(\varphi + x)\sin\varphi'$$
$$= \cos(\varphi + x)\cos\varphi'(\tan(\varphi + x) - \tan\varphi')\,.$$

But we have in the right triangle NP_1P':

$$\tan\varphi' = \tan(\varphi + x)\cos\lambda = \tan(\varphi + x)\left(1 - 2\sin^2\frac{\lambda}{2}\right)$$

and with this, we will have:

$$\sin\delta = 2\sin(\varphi + x)\cos\varphi'\sin^2\frac{\lambda}{2}\,.$$

If we bring in, in addition, γ according to (30) and take into account $\cos\varphi'\sin\lambda = \sin y$, then we obtain:

$$\sin\delta = \cos(\varphi + x)\tan\frac{\lambda}{2}\tan\gamma. \tag{31}$$

The course of the computation is now the following:
We determine x and y as well as λ as in the simple case according to (24), (25), (26), then ε and γ follow according to (29) and (30) and δ according to (31).
With this, we obtain then according to Fig. 3:

$$\varphi' = \varphi + x - \delta, \tag{32}$$

further for the azimuths:

$$\alpha' - \gamma + \beta = 90° \quad \text{and} \quad \alpha + \beta = 90° + \varepsilon;$$

therefore

$$\alpha' - \alpha = \gamma - \varepsilon. \tag{33}$$

The application to our small standard example is fashioned thusly:

Given $\varphi = 49° 30' 0''$ $\sigma = 1° 11' 19.482''$ $\alpha = 32° 21' 1.291''$.

According to (24), (25), (26) we find:

$x = 1° 0' 15.420''$ $y = 0° 38' 9.813''$ $\lambda = 1° 0' 0.000''$.

The formulae (29), (30), (31) furnish:

$\varepsilon = 0° 0' 20.0687''$ $\gamma = 0° 46' 17.9616''$ $\delta = 0° 0' 15.4199''$,

and now we combine thusly:

$$\begin{array}{rl} \gamma = & 0° \; 46' \; 17.9616'' \\ \varepsilon = & 0 \quad 0 \;\; 20.0687 \\ \hline \gamma - \varepsilon = & 0° \; 45' \; 57.8929'' = \alpha' - \alpha \,, \\ \alpha = & 32 \;\; 21 \quad 1.291 \\ \hline \alpha' = & 33° \;\; 6' \; 59.184'' \end{array} \qquad \begin{array}{rl} x = & 1° \;\; 0' \; 15.420'' \\ \delta = & 0 \quad 0 \;\; 15.420 \\ \hline x - \delta = & 1° \;\; 0' \;\; 0.000'' \\ \varphi = & 49 \;\; 30 \;\; 0.000 \\ \hline \varphi' = & 50° \; 30' \;\; 0.000'' . \end{array}$$

The advantage of this computation, in comparison to all previously described, consists in the fact that if σ itself is small (which is always the case here), then also all other quantities x, y, γ, ε, affecting the final results, themselves are *small*, and can therefore be computed very rigorously from *sin* or *tan.*

By means of these improved formulae, with regard to computational rigorousness, we can reach nearly the same accuracy with an ordinary 7-place logarithmic table, for which we needed nearly 10-place logarithms with the previous formulae.

Section 60. Differential Equations of the Spherical Polar Triangle

The closed formulae of spherical trigonometry which we have treated in the previous section 59 will in practice be used only rarely; in most cases it is of greater advantage to solve the closed formulae in series, and the first step for this is the setting up of differential formulae.

In Fig. 1, p. 166, we consider again two points P and P' on the sphere whose distance from each other is very small, however. Besides the angular distance $d\sigma$ of the two points, we will now introduce, in addition, the length of arc $PP' = ds$ where $ds = r\,d\sigma$. Let the latitudes of the points be φ and $\varphi + d\varphi$, so that the small difference of latitude between the parallel circles of P and P' appears as $d\varphi$. By means of the two meridians NP and NP' the difference of longitude $d\lambda$ is also expressed, and we assume that the azimuths of the arc ds at P and at P' are α and $\alpha + d\alpha$.

We can now apply the general Gauss formulae (3), section 59, p. 158, to the two points P and P' with their latitudes φ, $\varphi + d\varphi$, their azimuths α, $\alpha + d\alpha$, their distance $d\sigma$ and the difference of longitude $d\lambda$; and if we set here $\sin\frac{d\sigma}{2} = \frac{d\sigma}{2}$, $\cos\frac{d\sigma}{2} = 1$, in the sense of the differential computation, and so forth, then the first three equations (3), section 59, p. 158, yield the following:

$$d\sigma \sin\alpha = d\lambda \cos\varphi \tag{1}$$

$$d\sigma \cos\alpha = d\varphi \tag{2}$$

$$d\alpha = d\lambda \sin\varphi . \tag{3}$$

These are the very important differential equations of the spherical geodetic polar triangle.

These equations (1), (2), (3) can also easily be proved geometrically in Fig. 1, for which the small triangle $P_1 P P'$, treated as a right plane triangle, which is drawn separately in Fig. 2, is first used. It yields:

$$d s \sin \alpha = P_1 P' \tag{1a}$$

$$d s \cos \alpha = P P_1 . \tag{2a}$$

We have herein $ds = r d\sigma$; further $P P_1$ as a meridional arc is equal to $r d \varphi$. Finally, we have in $r \cos(\varphi + d\varphi)$ the radius of the circle of parallel of P', and the arc of the circle of parallel $P_1 P'$ is $r d \lambda \cos(\varphi + d\varphi)$, or $r d \lambda \cos \varphi$ if the product $d \lambda d \varphi$ is neglected. With this, equations (1a) and (2a) transform into (1) and (2).

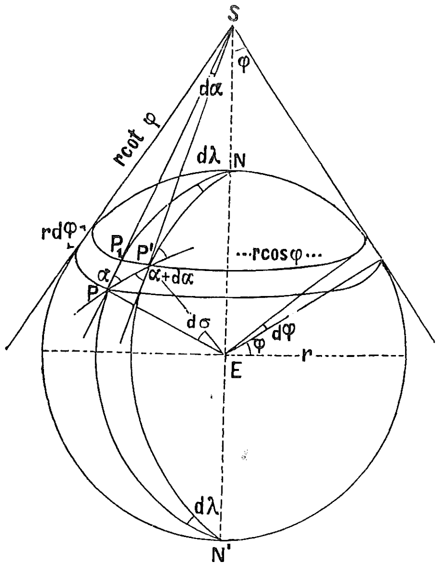

Fig. 1.

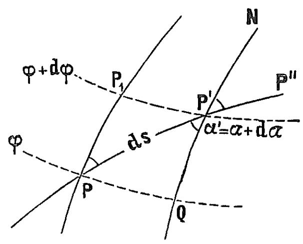

Fig. 2.
Special part of Fig. 1.
$P P' = d s = r d \sigma$.

While the first two equations (1) and (2) can thus directly be derived from the small right triangle $P P_1 P'$ geometrically, the consideration of the meridian tangents PS and $P'S$ is necessary for the geometric proof of the third equation (3). Inasmuch as $P P_1$ is infinitely small, these two tangents intersect at a point S of the earth's axis and make there the angle $d\alpha$, as follows from the straight-line triangle $S P P'$.

In the long isosceles triangle $P_1 P' S$ we have

$$d \alpha = \frac{P_1 P'}{P_1 S}.$$

We have here the arc of the parallel $P_1 P' = r \cos \varphi \, d \lambda$, as already mentioned, and the length of the tangent $P_1 S$ is found $= r \cot \varphi$; therefore:

$$d \alpha = \frac{r \cos \varphi \, d \lambda}{r \cot \varphi} = d \lambda \sin \varphi, \tag{3a}$$

with which equation (3) has been proved.

The correctness of equation (3) follows also from still another consideration. If we draw through P' an arc of the parallel to the meridian of P, then the extension of $P P'$ makes with this arc the angle α at P', and then $d\alpha$ is the meridian convergence at P'. But according to (11), section 55, p. 137, we have for an infinitely small difference of longitude $d \lambda$ the meridian convergence $\gamma = d \lambda \sin \varphi'$ or else $= d \lambda \sin \varphi$. Therefore, $d \alpha = d \lambda \sin \varphi$.

We will now make in addition a practical application of the formulae (1) to (3).

If we apply the differential formulae (1), (2), (3) to finite differences and thereby set instead of the general values φ and α the mean values φ_0 and α_0, then we have from (1), (2), (3):

$$\frac{\varphi + \varphi'}{2} = \varphi_0 \qquad \sigma \sin \alpha_0 = \lambda \cos \varphi_0 \tag{4}$$

$$\frac{\alpha + \alpha'}{2} = \alpha_0 \qquad \sigma \cos \alpha_0 = \varphi' - \varphi \tag{5}$$

$$\alpha' - \alpha = \lambda \sin \varphi_0 \, . \tag{6}$$

From (4) and (5) we find:

$$\tan \alpha_0 = \frac{\lambda \cos \varphi_0}{\varphi' - \varphi} \tag{7}$$

$$\sigma = \frac{\lambda \cos \varphi_0}{\sin \alpha_0} \quad \text{or} \quad = \frac{\varphi' - \varphi}{\cos \alpha_0} \, . \tag{8}$$

We can also find σ directly by squaring and adding (4) and (5):

$$\sigma = \sqrt{(\varphi' - \varphi)^2 + (\lambda \cos \varphi_0)^2} \, . \tag{9}$$

After we have computed $\alpha' + \alpha = 2\,\alpha_0$ from (7) and $(\alpha' - \alpha)$ from (6), we also have α' and α. We will apply these approximate formulae to our small spherical normal example:

Given $\varphi = 49° \, 30'$ $\quad \varphi' = 50° \, 30'$ $\quad \lambda = 1° \, 0' = 3600''$

$\varphi_0 = 59° \, 0'$ $\quad \varphi' - \varphi = 1° \, 0' = 3600''$.

From (7) we find: $\alpha_0 = 32° \, 43' \, 56.67''$.

From (6) we find: $\dfrac{\alpha' - \alpha}{2} = 0° \, 22' \, 58.88''$

$\alpha' = 33° \;\; 6' \, 55.55''$

$\alpha = 32° \, 20' \, 57.79''$.

From (9) we find: $\sigma = 4279.57'' = 1° \, 11' \, 19.57''$.

The more accurate values computed several times in the previous section 59 are:

$\alpha' = 33° \, 6' \, 59.19''$, $\qquad \alpha = 32° \, 21' \, 1.29''$, $\qquad \sigma = 1° \, 11' \, 19.48''$.

In the case of σ the error of the approximate value amounts only to 0.09''. The convenient approximate formulae (4) to (9) are to be used directly for some computations, e.g., in cartography and in general in cases where extreme rigorousness is not required.

However, we shall now pass over to setting up more accurate formulae of this kind.

Section 61. Development in Series with the Mean Latitude

We take up the Gauss equations (3), section 59, p. 158, once again; however, we will now choose the notation somewhat differently, namely according to the indication of Fig. 1 below:

$$\text{Latitudes:} \quad \varphi_1 \text{ and } \varphi_2, \quad \frac{\varphi_2 + \varphi_1}{2} = \varphi, \quad \varphi_2 - \varphi_1 = \beta \tag{1}$$

$$\text{Azimuths:} \quad \alpha_1 \text{ and } \alpha_2, \quad \frac{\alpha_2 + \alpha_1}{2} = \alpha, \quad \alpha_2 - \alpha_1 = \Delta\alpha \tag{2}$$

$$\text{Difference of longitude:} \quad \lambda \tag{3}$$

$$\text{Connecting arc:} \quad \sigma. \tag{4}$$

With this, equations (3), section 59, p. 158, will be:

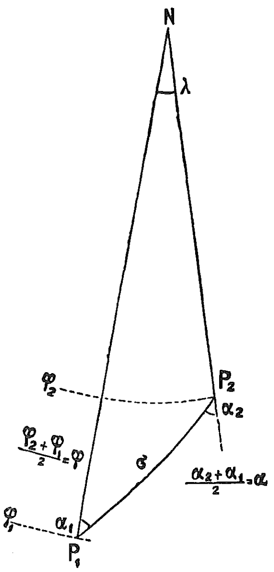

Fig. 1.

$$\sin\frac{\sigma}{2}\sin\alpha = \sin\frac{\lambda}{2}\cos\varphi \tag{5}$$

$$\sin\frac{\sigma}{2}\cos\alpha = \sin\frac{\beta}{2}\cos\frac{\lambda}{2} \tag{6}$$

$$\cos\frac{\sigma}{2}\sin\frac{\Delta\alpha}{2} = \sin\frac{\lambda}{2}\sin\varphi \tag{7}$$

$$\cos\frac{\sigma}{2}\cos\frac{\Delta\alpha}{2} = \cos\frac{\beta}{2}\cos\frac{\lambda}{2}. \tag{8}$$

We take at first (5) and (6) which, developed, yield:

$$\left(\frac{\sigma}{2} - \frac{\sigma^3}{48}\right)\sin\alpha = \left(\frac{\lambda}{2} - \frac{\lambda^3}{48}\right)\cos\varphi$$

$$\left(\frac{\sigma}{2} - \frac{\sigma^3}{48}\right)\cos\alpha = \left(\frac{\beta}{2} - \frac{\beta^3}{48}\right)\left(1 - \frac{\lambda^2}{8}\right).$$

Abbreviating:

$$\sigma\sin\alpha\left(1 - \frac{\sigma^2}{24}\right) = \lambda\cos\varphi\left(1 - \frac{\lambda^2}{24}\right) \tag{9}$$

$$\sigma\cos\alpha\left(1 - \frac{\sigma^2}{24}\right) = \beta\left(1 - \frac{\beta^2}{24}\right)\left(1 - \frac{\lambda^2}{8}\right). \tag{10}$$

We can set here in the correction terms as a first approximation:

$$\sigma^2 = \beta^2 + \lambda^2\cos^2\varphi. \tag{11}$$

If we introduce this into (9) and (10) on the left-hand side, and then solve with respect to $\sigma\sin\alpha$ and $\sigma\cos\alpha$ and collect terms, then we find:

$$\sigma\sin\alpha = \lambda\cos\varphi\left(1 + \frac{\beta^2}{24} - \frac{\lambda^2\sin^2\varphi}{24}\right) \tag{12}$$

$$\sigma \cos \alpha = \beta \left(1 - \frac{\lambda^2}{8} + \frac{\lambda^2 \cos^2 \varphi}{24}\right). \tag{13}$$

By division of these two equations we find $\tan \alpha$, and then σ from each single one. We can however also square and add (12) and (13), and find with this a direct formula for σ^2, namely:

$$\sigma^2 = \beta^2 + \lambda^2 \cos^2 \varphi + \frac{1}{12}(-3\beta^2\lambda^2 + 2\beta^2\lambda^2\cos^2\varphi - \lambda^4 \cos^2\varphi \sin^2\varphi) \tag{14}$$

or

$$\sigma = \sqrt{\beta^2 + \lambda^2 \cos^2 \varphi}\left(1 - \frac{\beta^2\lambda^2}{8\sigma^2} + \frac{\beta^2\lambda^2\cos^2\varphi}{12\sigma^2} - \frac{\lambda^4\cos^2\varphi\sin^2\varphi}{24\sigma^2}\right). \tag{14a}$$

We can also easily derive this formula from (8), section 60, p. 168.

In order to obtain also the azimuth difference $\Delta\alpha$, we form at first from (7) and (8) by division:

$$\tan\frac{\Delta\alpha}{2} = \tan\frac{\lambda}{2}\frac{\sin\varphi}{\cos\frac{\beta}{2}}.$$

Developed just as the previous expression, this yields:

$$\frac{\Delta\alpha}{2} + \frac{\Delta\alpha^3}{24} = \sin\varphi\frac{\frac{\lambda}{2} + \frac{\lambda^3}{24}}{1 - \frac{\beta^2}{8}} = \frac{\sin\varphi}{2}\left(\lambda + \frac{\lambda^3}{12}\right)\left(1 + \frac{\beta^2}{8}\right).$$

First approximation $\Delta\alpha = \lambda \sin\varphi$; therefore $\Delta\alpha^3 = \lambda^3 \sin^3\varphi + \ldots$

$$\Delta\alpha = -\frac{\lambda^3 \sin^3\varphi}{12} + \lambda\sin\varphi\left(1 + \frac{\lambda^2}{12} + \frac{\beta^2}{8}\right)$$

$$\alpha' - \alpha = \Delta\alpha = \lambda\sin\varphi\left(1 + \frac{\beta^2}{8} + \frac{\lambda^2}{12}\cos^2\varphi\right). \tag{15}$$

In these formulae we compute in radian measure, and if we aim to have the small angles in seconds, we must divide all quadratic terms within parentheses by ρ^2. This yields for (12), (13) and (15) the following usable formulae:

$$\sigma\sin\alpha = \lambda\cos\varphi\left(1 + \frac{\beta^2}{24\rho^2} - \frac{\lambda^2\sin^2\varphi}{24\rho^2}\right) \tag{16}$$

$$\sigma\cos\alpha = \beta\left(1 - \frac{\lambda^2}{8\rho^2} + \frac{\lambda^2\cos^2\varphi}{24\rho^2}\right) \tag{17}$$

$$\alpha_2 - \alpha_1 = \Delta\alpha = \lambda\sin\varphi\left(1 + \frac{\beta^2}{8\rho^2} + \frac{\lambda^2\cos^2\varphi}{12\rho^2}\right). \tag{18}$$

By division of (16) and (17) we also find:

$$\tan\alpha = \frac{\lambda\cos\varphi}{\beta}\left(1 + \frac{\beta^2}{24\rho^2} + \frac{\lambda^2}{12\rho^2}\right). \tag{19}$$

The constant coefficients for the above formulae are:

$$\log \frac{1}{8\rho^2} = 8.468\,0597 \qquad \log \frac{1}{12\rho^2} = 8.291\,9685 \qquad \log \frac{1}{24\rho^2} = 7.990\,9385. \tag{20}$$

We can also use the above formulae (16), (17), (18) logarithmically, namely in this form:

$$\log \sigma \sin \alpha = \log \lambda \cos \varphi + \frac{\mu}{24\varrho^2}\beta^2 - \frac{\mu}{24\varrho^2}\lambda^2 \sin^2 \varphi \tag{21}$$

$$\log \sigma \cos \alpha = \log \beta - \frac{\mu}{8\varrho^2}\lambda^2 + \frac{\mu}{24\varrho^2}\lambda^2 \cos^2 \varphi \tag{22}$$

$$\log \Delta\alpha = \log \lambda \sin \varphi + \frac{\mu}{8\varrho^2}\beta^2 + \frac{\mu}{12\varrho^2}\lambda^2 \cos^2 \varphi\,. \tag{23}$$

Then, instead of (20), we need the following constants for the 7th place of logarithms:

$$\log \frac{\mu}{8\rho^2} = 5{\cdot}105\,8441 \qquad \log \frac{\mu}{12\rho^2} = 4{\cdot}929\,7528 \qquad \log \frac{\mu}{24\rho^2} = 4{\cdot}628\,7228. \tag{24}$$

We will apply this computational method to our small normal example (1), section 59, p. 158, and in fact at first with the formulae (16), (17), (18).

Given	$\varphi_1 = 49°\,30'$	$\varphi_2 = 50°\,30'$	$\lambda = 1°\,0'$;
therefore	$\varphi = 50°\;\;0'$	$\beta = 1°\,0' = 3600''$	$\lambda = 3600''$.

The computation according to (16), (17), (18), p. 169, yields:

$\lambda \cos \varphi = 2314.0352''$	$\beta = 3600.0000''$	$\lambda \sin \varphi = 2757.7600''$
$+\,0.0294$	$-\,0.1371$	$+\,0.1050$
$-\,0.0172$	$+\,0.0189$	$+\,0.0289''$
$\sigma \sin \alpha = 2314.0474''$	$\sigma \cos \alpha = 3599.8818''$	$\Delta\alpha = 2757.8939''$
		$\Delta\alpha = 0°\,45'\,57.8939''$

$$\alpha = \frac{\alpha_2 + \alpha_1}{2} = 32°\,44'\;\;0.2385'' \qquad \sigma = 4279.4819''$$

$$\frac{\Delta\alpha}{2} = \frac{\alpha_2 - \alpha_1}{2} = \;\;0\;\;22\;\;58.9470 \qquad \sigma = 1°\,11'\,19.4819''$$

$$\alpha_2 = 33°\;\;6'\,59.1855''$$

$$\alpha_1 = 32\;\;21\;\;\;1.2915\,.$$

Besides, we can also apply the logarithmic formulae (21), (22), (23), whereby we obtain the same as just now, only in another form, namely:

$\log \lambda \cos \varphi$	3.364 3700·0	$\log \beta$	3.556 3025·0	$\log \lambda \sin \varphi$	3.440 5564·7
	+ 55·1		− 165·4		+ 165·4
	− 32·3		+ 22·8		+ 45·5
$\log \sigma \sin \alpha$	3.364 3722·8	$\log \sigma \cos \alpha$	3.556 2882·4	$\log \Delta\alpha$	3.440 5775·6

If we compute further with this, then we obtain the same values α, σ, $\Delta\alpha$, and so forth, as previously.

If, for instance, we do not need σ itself, then we compute $tan\ \alpha$ directly from formula (19), which yields in our case:

$$log\ tan\ \alpha = 9.808\ 0675{\cdot}0 + 165{\cdot}4 = 9.808\ 0840{\cdot}4.$$

This agrees with the difference of $log\ \sigma\ sin\ \alpha$ and $log\ \sigma\ cos\ \alpha$, as it should be.

We have also applied, in addition, formula (14a) to the small normal example with $\beta = 1^\circ$, $\lambda = 1^\circ$, $\varphi = 50^\circ$ and found:

$$\sigma = 4279.5747'' - 0.1153'' - 0.0093'' + 0.0318'' = 4279.4819'' = 1^\circ\ 11'\ 19.4819''.$$

We can also compute here the correction terms in the logarithmic form.

Inversion of the formulae

We can apply the formulae (16), (17), (18) not only for the determination of σ, α_1, α_2 if φ_1, φ_2, λ are given, but also vice versa, in order to compute the missing φ_2's, λ's, α_2's if φ_1, σ, α_1 are given. However, this can only be done indirectly, since approximate values of the unknowns are used and gradually improved.

A few transformations of (16), (17), (18) are then also to be carried out; we form at first by division of (16) and (18):

$$\Delta\alpha = \sigma\ sin\ \alpha\ tan\ \varphi \left(1 + \frac{\beta^2}{8\varrho^2} + \frac{\lambda^2 cos^2\varphi}{12\varrho^2} - \frac{\beta^2}{24\varrho^2} + \frac{\lambda^2 sin^2\varphi}{24\varrho^2}\right).$$

But in the correction terms there holds the first approximation

$$\sigma^2 = \beta^2 + \lambda^2\ cos^2\ \varphi,$$

and with this, the above yields:

$$\Delta\alpha = \sigma\ sin\ \alpha\ tan\ \varphi \left(1 + \frac{\sigma^2}{12\varrho^2} + \frac{\lambda^2 sin^2\varphi}{24\varrho^2}\right); \qquad (25)$$

(17) and (16) inverted yield:

$$\beta = \sigma\ cos\ \alpha \left(1 + \frac{\lambda^2}{8\varrho^2} - \frac{\lambda^2 cos^2\varphi}{24\varrho^2}\right) \qquad (26)$$

$$\lambda = \sigma \frac{sin\ \alpha}{cos\ \varphi}\left(1 - \frac{\beta^2}{24\varrho^2} + \frac{\lambda^2 sin^2\varphi}{24\varrho^2}\right). \qquad (27)$$

We can also use these equations in the logarithmic form, similarly as (21), (22), (23), which we will however no longer write here separately.

As a numerical application we will assume from our small normal example (1), section 59, p. 158:

$$\text{Given} \quad \varphi_1 = 49^\circ\ 30'\ 0'', \quad \sigma = 1^\circ\ 11'\ 19.482'', \quad \alpha_1 = 32^\circ\ 21'\ 1.291''. \qquad (28)$$

For φ_2 and λ we assume to have from somewhere, e.g. from a topographic map, the approximate values:

$$(\varphi_2) = 50^\circ\ 30'\ 10'', \quad (\lambda) = 1^\circ\ 0'\ 10'' = 3610''. \qquad (29)$$

Now we take from φ_1 and (φ_2) the approximate mean value $(\varphi) = 50^\circ\,0'\,5''$ and compute with $(\lambda) = 3610''$, for the first time approximately $(\gamma) = (\lambda)\,sin\,(\varphi) = 2765.48'' = 0^\circ\,46'\,5.48''$; half from this added to α_1 according to (28) yields the first approximation for α:

$$(\alpha) = 32^\circ\,44'\,4''.$$

Now we compute, with $(\varphi) = 50^\circ\,0'\,5''$ and $(\alpha) = 32^\circ\,44'\,4''$ and with the accurately given $\sigma = 4279.482''$, the main terms of the formulae (25), (26), (27), and obtain:

$$(\Delta\,\alpha) = 2757.93'' \qquad (\beta) = 3599.76'' \qquad (\lambda) = 3600.14'' \qquad (30)$$
$$= 0^\circ\,45'\,57.93''.$$

With this $(\Delta\,\alpha)$ we form a new

$$(\alpha) = \alpha_1 + \frac{(\Delta\,\alpha)}{2} = 32^\circ\,44'\,0.25''. \qquad (31)$$

Now the approximations (β) and (λ) in (30) are at any rate fully sufficient for the computation of the *correction terms* in (25), (26), (27), and for the main terms we have, in addition to the given σ, the already very good approximation (31); therefore, we can already make the calculation almost finally according to (25), (26), (27). If $\Delta\,\alpha$ and β are not obtained here in full agreement with the values $\Delta\,\alpha$ and β used in the two main terms $\sigma\,sin\,\alpha\,tan\,\varphi$ and $\sigma\,cos\,\alpha$, then we must repeat the computation with corrected α's and φ's until complete agreement takes place.

Other forms of the correction terms

Since in the main terms only α and φ occur, but not λ, we can limit the gradual correction of the approximate values to these two elements, or to $\Delta\,\alpha$ and β, as the case may be; in the *first* approximation, however, we will not be able to do without λ, because a first approximation for $\Delta\,\alpha$ will hardly be obtained in another way than by $\lambda\,sin\,\varphi$; but once we have such a first approximation for $\Delta\,\alpha$, then we introduce it as directly as possible into the correction terms. This is done by the approximate equations:

$$\left.\begin{aligned} \sigma^2 = \beta^2 + \lambda^2\,cos^2\,\varphi, \qquad \Delta\,\alpha^2 = \lambda^2\,sin^2\,\varphi; \\ \text{therefore, also} \qquad \sigma^2 + \Delta\,\alpha^2 = \beta^2 + \lambda^2. \end{aligned}\right\} \qquad (32)$$

With this, we write equations (26) and (25) for our new purpose thusly:

$$\beta = \sigma\,cos\,\alpha\left(1 + \frac{\Delta\,\alpha^2}{24\,\varrho^2} + \frac{\lambda^2}{12\,\varrho^2}\right) \qquad (33)$$

$$\Delta\,\alpha = \sigma\,sin\,\alpha\,tan\,\varphi\left(1 + \frac{\sigma^2}{12\varrho^2} + \frac{\Delta\,\alpha^2}{24\varrho^2}\right), \qquad (34)$$

or also in the logarithmic form, similar to (21), (22), (23).

These equations (33) and (34) give now an indirect solution for φ_2 and α_2 or, as the case may be, for β and $\Delta\,\alpha$, as was already shown in the case of (25), (26), (27); the third quantity λ occurs in (33) and (34) only in a correction term and is finally determined by the previous equation (27), after β and $\Delta\,\alpha$ are found.

The indirect computing method described above is very accurate; it is also very convenient (contrary to the first opinion).

Gauss himself says about it in Art. 20 of *Untersuchungen über Gegenstände der höheren Geodäsie*, zweite Abhandlung: [translated] "The convenience of this method, however, can be felt in its entirety only when we have made the aids of the small mechanism in the handling of such methods our own. I content myself to indicate here only that what appears as a multiple computation should not be written in the form of several separate computations, but as a single one, as in every new reworking, only the last numbers are supplemented or corrected. At any rate, we only need to preserve always the last computation, and just in this there exists a great advantage, especially in the case of measurements of a considerable extent, that we possess then the whole essential core of the computation for all triangle sides in a space as small as possible and in the clearest form suitable for any desired examination of the correctness."

Section 62. Further Development to the Fifth Order

(Notation according to Fig. 1, p. 168)

We can carry the above developments, which go to the third order, i.e. to terms β^3, λ^3, etc., still one step further, i.e. to β^5, λ^5, and so forth. This has no immediate practical purpose, however, for by doing so the formulae become so detailed that we would have to prefer to compute according to the rigorous closed formulae of spherical trigonometry; however, the development of the terms of fifth order offers the best means for obtaining an opinion about the limits of application of the abbreviated formulae, and these spherical terms of fifth order will also be of importance later in the case of the corresponding spheroidal problem.

Of the rigorous Gauss equations (5) to (8), section 61, p. 168, we take up first, once again, the first two:

$$\sin\frac{\sigma}{2}\sin\alpha = \sin\frac{\lambda}{2}\cos\varphi \tag{1}$$

$$\sin\frac{\sigma}{2}\cos\alpha = \sin\frac{\beta}{2}\cos\frac{\lambda}{2}\,. \tag{2}$$

Now we develop these to the fifth order (cf. p. 23):

$$\left(\frac{\sigma}{2}-\frac{\sigma^3}{48}+\frac{\sigma^5}{3840}\right)\sin\alpha = \left(\frac{\lambda}{2}-\frac{\lambda^3}{48}+\frac{\lambda^5}{3840}\right)\cos\varphi \tag{3}$$

$$\left(\frac{\sigma}{2}-\frac{\sigma^3}{48}+\frac{\sigma^5}{3840}\right)\cos\alpha = \left(\frac{\beta}{2}-\frac{\beta^3}{48}+\frac{\beta^5}{3840}\right)\left(1-\frac{\lambda^2}{8}+\frac{\lambda^4}{384}\right). \tag{4}$$

In order to solve these equations with respect to $\sigma\sin\alpha$ and $\sigma\cos\alpha$, we imagine taken out on the left-hand side:

$$\frac{\sigma}{2}\left(1-\frac{\sigma^2}{24}+\frac{\sigma^4}{1920}\right)=\frac{\sigma}{2}(1-x)\,,\ \text{ i.e. } x=\frac{\sigma^2}{24}-\frac{\sigma^4}{1920}$$

$$\frac{1}{1-x}=1+x+x^2=1+\frac{\sigma^2}{24}+\frac{7\,\sigma^4}{5760}\,. \tag{5}$$

We have here according to (14), section 61, p. 169:

$$\sigma^2=\beta^2+\lambda^2\cos^2\varphi-\frac{\beta^2\lambda^2}{4}-\frac{\lambda^4\sin^2\varphi\cos^2\varphi}{12}+\frac{\beta^2\lambda^2}{6}\cos^2\varphi\,, \tag{6}$$

therefore

$$\sigma^4=\beta^4+\lambda^4\cos^4\varphi+2\,\beta^2\lambda^2\cos^2\varphi\,. \tag{7}$$

If we introduce these equations (6) and (7) into (5) and multiply, by these, the right-hand sides of (3) and (4), then we obtain the desired expressions for $\sigma\sin\alpha$ and $\sigma\cos\alpha$. We can also find, likewise, the series for the azimuth difference $\Delta\alpha$, and through $\sigma^2\sin^2\alpha+\sigma^2\cos^2\alpha$ we also have immediately a series for σ^2.

Since the method of all these developments is sufficiently shown, we write at once the results, and in fact first:

$$\sigma \sin \alpha = \lambda \cos \varphi \left\{1 - \frac{1}{24}(\beta^2 - \lambda^2 + \lambda^2 \cos^2 \varphi) \right.$$
$$\left. + \frac{1}{5760}(7\beta^4 - 70\beta^2\lambda^2 + 3\lambda^4 + 54\beta^2\lambda^2 \cos^2\varphi + 7\lambda^4 \cos^4\varphi - 10\lambda^4 \cos^2\varphi - 20\lambda^4 \sin^2\varphi \cos^2\varphi).\right\} \tag{8}$$

Before we also write down the formulae for $\sigma \cos \alpha$, and so forth, we will introduce a definite order of the terms and write two new symbols for abbreviation. We can in any case express the factors $\cos^2 \varphi$ and $\sin^2 \varphi$ all by $\cos^2 \varphi$, and since they always occur in connection with λ^2, we set forth everywhere equal powers of λ^2 and of $\cos^2 \varphi$ by setting, e.g.:

$$\lambda^4 \cos^2 \varphi = \lambda^4 \cos^4 \varphi \, (1 + \tan^2 \varphi). \tag{9}$$

Let us denote the arc of the parallel $\lambda \cos \varphi$ by setting in particular:

$$\lambda \cos \varphi = p \qquad \text{and} \qquad \tan \varphi = t. \tag{10}$$

With this, we obtain a new transcription of the first formula (8) and insert at once also the remaining formulae of this kind:

$$\sigma \sin \alpha = p \left\{1 + \frac{\beta^2 - p^2 t^2}{24} + \frac{7\beta^4 - 2\beta^2 p^2 (8 + 35 t^2) - 3 p^4 (8 t^2 - t^4)}{5760}\right\} \tag{11}$$

$$\sigma \cos \alpha = \beta \left\{1 - \frac{p^2 (2 + 3 t^2)}{24} + \frac{-4\beta^2 p^2 (4 + 15 t^2) + p^4 (-8 - 20 t^2 + 15 t^4)}{5760}\right\} \tag{12}$$

$$\Delta \alpha = p t \left\{1 + \frac{3\beta^2 + 2 p^2}{24} + \frac{75\beta^4 + 60\beta^2 p^2 (1 - 2 t^2) + 24 p^4 (2 - t^2)}{5760}\right\} \tag{13}$$

$$\sigma^2 = (\beta^2 + p^2) - \frac{\beta^2 p^2 (1 + 3 t^2) + p^4 t^2}{12}$$
$$- \frac{2\beta^4 p^2 (1 + 15 t^2) + 2\beta^2 p^4 (1 + 10 t^2 - 15 t^4) + p^6 (12 t^2 - 4 t^4)}{1440}. \tag{14}$$

We will rather aim to have the formulae (11) to (13) for $\sigma \sin \alpha$, $\sigma \cos \alpha$ and $\Delta \alpha$ in the logarithmic form; we can therefore develop them according to the formula:

$$\log (1 + x) = \mu \left(x - \frac{x^2}{2}\right) \tag{15}$$
$$\log \left(1 + \frac{A^2}{24} + \frac{B^4}{5760}\right) = \mu \left(\frac{A^2}{24} + \frac{1}{2}\,\frac{B^4 - 5 A^4}{2880}\right).$$

In this manner we obtain the following formulae by inserting at the same time the necessary ρ's:

$$\log \sigma \sin \alpha = \log p + \frac{\mu}{24 \rho^2} (\beta^2 - p^2 t^2)$$
$$+ \frac{\mu}{2880 \rho^4} \left(\beta^4 - 2\beta^2 p^2 (4 + 15 t^2) - p^4 (12 t^2 + t^4)\right) \tag{16}$$

$$\log \sigma \cos \alpha = \log \beta - \frac{\mu}{24 \rho^2} p^2 (2 + 3 t^2)$$
$$- \frac{\mu}{2880 \rho^4} \left(2\beta^2 p^2 (4 + 15 t^2) + p^4 (14 + 40 t^2 + 15 t^4)\right). \tag{17}$$

$$\log \Delta\alpha = \log p\, t + \frac{\mu}{24}(3\beta^2+2p^2) + \frac{\mu}{5760}\left(30\beta^4 - 120\beta^2 p^2 t^2 + 4p^4(7-6t^2)\right). \tag{18}$$

We have here the constants for β and p in seconds and for units of the 7th place of logarithm:

$$\log \frac{\mu}{24\varrho^2} = 4{\cdot}628\,7228 \qquad \log \frac{\mu}{2880\varrho^4} = 1{\cdot}920\,691 - 20\,. \tag{19}$$

We can write the higher terms in (16) to (18) in this form:

$$\Delta(\log\sigma\sin\alpha) = I\,\beta^4 - II\,\beta^2\lambda^2 - III\,\lambda^4 \text{ (5th order)} \tag{20}$$
$$\Delta(\log\sigma\cos\alpha) = - IV\,\beta^2\lambda^2 - V\,\lambda^4 \text{ (5th order)} \tag{21}$$
$$\Delta\log\Delta\alpha = VI\,\beta^4 - VII\,\beta^2\lambda^2 - VIII\,\lambda^4 \text{ (5th order).} \tag{22}$$

We have computed the coefficients *I*, *II*, and so forth, for various latitudes φ, as is seen from the following table, where β and λ, however, are not to be taken in seconds, as in (19), but for (20) to (22) in degrees. Besides, the coefficients yield the correction terms in units of the 7th place of logarithm.

φ	log *I*	log *II*	log *III*	log *IV*	log *V*	log *VI*	log *VII*	log *VIII*
40°	6·1459	7·3786	6·6345	7·3786	7.3784	7·3220	7·5402	6·4273
45	6·1459	7·4247	6·6578	7·4247	7·3827	7·3220	7·6230	5·8449
50	6·1459	7·4663	6·6583	7.4663	7.3828	7·3220	7·6926	5·8616ₙ
55	6·1459	7·5031	6·6371	7·5031	7·3789	7·3220	7·7508	6·2004ₙ
60	6·1459	7·5351	6·5950	7.5351	7·3716	7·3220	7·7991	6·2842ₙ

(23)

We have further computed the terms for (20) and (21) for a latitude of $\varphi = 50°$ by setting successively β and $\lambda = 2°$, 4°, 6°, 8°, 10°. The results of this calculation show the following two tables:

Δ (log σ sin α), According to Formula (20) of Fifth Order, for φ = 50°

$\beta =$	$\lambda = 2°$	$\lambda = 4°$	$\lambda = 6°$	$\lambda = 8°$	$\lambda = 10°$
2°	−0·053	−0·304	−1·012	−2·615	−5·724
4	−0·157	−0·831	−2.239	−4·826	−9·198
6	−0·247	−1·621	−4·201	−8·426	−14·902
8	−0·184	−2·540	−6·759	−13·279	−22·708
10	+0·221	−3·399	−9·726	−19·196	−32·416

(24)

Δ (log σ cos α), According to Formula (21) of Fifth Order, for φ = 50°

$\beta =$	$\lambda = 2°$	$\lambda = 4°$	$\lambda = 6°$	$\lambda = 8°$	$\lambda = 10°$
2°	−0·085	−0·805	−3·551	−10·640	−25·314
4	−0·225	−1·368	−4·814	−12·887	−28·824
6	−0·460	−2·303	−6·921	−16·632	−34·677
8	−0·789	−3·614	−9·871	−21·877	−42·871
10	−1·210	−5·299	−13·664	−28·620	−53·405

(25)

Finally, we have also treated, in addition, formula (13) for the azimuth difference; we found:

$$\Delta(\alpha_2 - \alpha_1) = IX\,\lambda\,\beta^4 - X\,\lambda^3\beta^2 + XI\,\lambda^5, \tag{26}$$

where the coefficients have the following values:

φ	$\log IX$	$\log X$	$\log XI$
40°	4·4465	3·7291	3·6012
45	4·4880	4·0901	4·3556
50	4·5227	4·3068	3·0233
55	4·5518	4·4606	$1·6891_n$
60	4·7560	4·5963	$2·8772_n$

(27)

In particular, the following is computed according to this for $\varphi = 50°$:

Fifth Order Correction for $\Delta\alpha$ According to Formula (13) for $\varphi = 50°$

$\beta =$	$\lambda = 2°$	$\lambda = 4°$	$\lambda = 6°$	$\lambda = 8°$	$\lambda = 10°$
2°	0.0000″	−0.0002″	−0.0007″	−0.0003″	+0.0030″
4	+0.0014	+0.0014	−0.0011	−0.0063	−0.0133
6	+0.0080	+0.0127	+0.0110	+0.0006	−0.0192
8	+0.0263	+0.0464	+0.0547	+0.0463	+0.0174
10	+0.0650	+0.1204	+0.1569	+0.1663	+0.1411

(28)

Section 63. Developments in Series According to Powers of σ

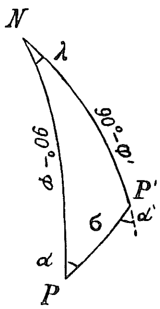

Fig. 1.

We can develop in various ways the very important series which $\varphi' - \varphi$, $\alpha' - \alpha$ and λ express in increasing powers of the distance σ.

We will at first call to mind that if φ, α and σ are given, the other three values φ', α' and λ can be indicated by closed formulae of spherical trigonometry, which we have already indicated in section 59 in (16) to (18), p. 162. We can develop those formulae directly in series, as is done to σ^3 inclusive in our former 3rd Edition, 1890, in section 59; but we will disregard this here and rather pass over at once to the development according to Maclaurin's series, which can be extended as far as desired.

The application of this series to our case yields to the sixth order:

$$\varphi' - \varphi = \left.\frac{d\varphi}{d\sigma}\right]\sigma + \left.\frac{d^2\varphi}{d\sigma^2}\right]\frac{\sigma^2}{2} + \left.\frac{d^3\varphi}{d\sigma^3}\right]\frac{\sigma^3}{6} + \left.\frac{d^4\varphi}{d\sigma^4}\right]\frac{\sigma^4}{24} + \left.\frac{d^5\varphi}{d\sigma^5}\right]\frac{\sigma^5}{120} + \left.\frac{d^6\varphi}{d\sigma^6}\right]\frac{\sigma^6}{720} \qquad (1)$$

$$\lambda = \left.\frac{d\lambda}{d\sigma}\right]\sigma + \left.\frac{d^2\lambda}{d\sigma^2}\right]\frac{\sigma^2}{2} + \left.\frac{d^3\lambda}{d\sigma^3}\right]\frac{\sigma^3}{6} + \left.\frac{d^4\lambda}{d\sigma^4}\right]\frac{\sigma^4}{24} + \left.\frac{d^5\lambda}{d\sigma^5}\right]\frac{\sigma^5}{120} + \left.\frac{d^6\lambda}{d\sigma^6}\right]\frac{\sigma^6}{720} \qquad (2)$$

$$\alpha' - \alpha = \left.\frac{d\alpha}{d\sigma}\right]\sigma + \left.\frac{d^2\alpha}{d\sigma^2}\right]\frac{\sigma^2}{2} + \left.\frac{d^3\alpha}{d\sigma^3}\right]\frac{\sigma^3}{6} + \left.\frac{d^4\alpha}{d\sigma^4}\right]\frac{\sigma^4}{24} + \left.\frac{d^5\alpha}{d\sigma^5}\right]\frac{\sigma^5}{120} + \left.\frac{d^6\alpha}{d\sigma^6}\right]\frac{\sigma^6}{720}\,. \qquad (3)$$

After carrying out the differentiations we are to set the value $\varphi' - \varphi = 0$ in the obtained differential quotients $\frac{d\varphi}{d\sigma}$, $\frac{d^2\varphi}{d\sigma^2}$, and so forth, i.e., the differential quotients are to be computed for the starting value φ and likewise for the starting value α.

We obtain the first differential quotients from (1), (2), (3), section 60, p. 165, namely in the form suitable for us:

$$\frac{d\varphi}{d\sigma} = \cos\alpha \qquad (4)$$

$$\frac{d\lambda}{d\sigma} = \frac{\sin\alpha}{\cos\varphi} \tag{5}$$

$$\frac{d\alpha}{d\sigma} = \sin\alpha\tan\varphi. \tag{6}$$

Now we differentiate (4) further and find by bringing in (6):

$$\frac{d^2\varphi}{d\sigma^2} = -\sin\alpha\frac{d\alpha}{d\sigma} = -\sin^2\alpha\tan\varphi. \tag{7}$$

This differentiated once again yields:

$$\frac{d^3\varphi}{d\sigma^3} = -2\sin\alpha\cos\alpha\frac{d\alpha}{d\sigma}\tan\varphi - \sin^2\alpha(1+\tan^2\varphi)\frac{d\varphi}{d\sigma},$$

therefore, taking into account (6) and (4):

$$\frac{d^3\varphi}{d\sigma^3} = -2\sin^2\alpha\cos\alpha\tan^2\varphi - \sin^2\alpha\cos\alpha(1+\tan^2\varphi)$$

$$\frac{d^3\varphi}{d\sigma^3} = -\sin^2\alpha\cos\alpha(1+3\tan^2\varphi). \tag{8}$$

This is differentiated once again:

$$\frac{d^4\varphi}{d\sigma^4} = (-2\sin\alpha\cos^2\alpha + \sin^3\alpha)\sin\alpha\tan\varphi(1+3\tan^2\varphi) \tag{9}$$

$$- \sin^2\alpha\cos\alpha\, 6\, t(1+t^2)\cos\alpha$$

$$\frac{d^4\varphi}{d\sigma^4} = \sin^4\alpha\tan\varphi(1+3\tan^2\varphi) - 4\sin^2\alpha\cos^2\alpha\tan\varphi(2+3\tan^2\varphi). \tag{10}$$

We can continue in this manner; we write here as usual $\tan\varphi = t$, and with this, we will have:

$$\frac{d^5\varphi}{d\sigma^5} = \sin^4\alpha\cos\alpha(1+30t^2+45t^4) - 4\sin^2\alpha\cos^3\alpha(2+15t^2+15t^4) \tag{11}$$

$$\left.\begin{aligned}\frac{d^6\varphi}{d\sigma^6} = -\sin^6\alpha\, t(1+30t^2+45t^4) + 4\sin^4\alpha\cos^2\alpha\, t(22+135t^2+135t^4)\\ -8\sin^2\alpha\cos^4\alpha\, t(17+60t^2+45t^4).\end{aligned}\right\} \tag{12}$$

Passing over to the differentiations of λ, we have according to (5) above:

$$\frac{d\lambda}{d\sigma} = \frac{\sin\alpha}{\cos\varphi} \tag{13}$$

$$\frac{d^2\lambda}{d\sigma^2} = \frac{\cos\alpha}{\cos\varphi}\frac{d\alpha}{d\sigma} + \sin\alpha\frac{\sin\varphi}{\cos^2\varphi}\frac{d\varphi}{d\sigma}, \tag{14}$$

therefore, taking into account (4) and (6):

$$\frac{d^2\lambda}{d\sigma^2} = \frac{\cos\alpha\sin\alpha}{\cos^2\varphi}\sin\varphi + \sin\alpha\frac{\sin\varphi}{\cos^2\varphi}\cos\alpha$$

$$\frac{d^2\lambda}{d\sigma^2} = 2\sin\alpha\cos\alpha\frac{\tan\varphi}{\cos\varphi} = 2\sin\alpha\cos\alpha\tan\varphi\sec\varphi. \tag{15}$$

Since we notice soon that the denominator $\cos\varphi$, or the factor $\sec\varphi$ appears in all terms of the development of λ, and that the powers of $\tan\varphi$ are found as in the previous case, we also write always the differentiation of $\sec\varphi$ in the form $t\sec\varphi$, and with this, we obtain further (if we set $\tan\varphi = t$ everywhere):

$$\frac{d^3\lambda}{d\sigma^3} = 2(\cos^2\alpha - \sin^2\alpha)\sin\alpha\, t\, t\sec\varphi$$
$$+ 2\sin\alpha\cos\alpha(1+t^2)\sec\varphi\cos\alpha$$
$$+ 2\sin\alpha\cos\alpha\, t\, t\sec\varphi\cos\alpha$$
$$\frac{d^3\lambda}{d\sigma^3} = -\sin^3\alpha\sec\varphi(2t^2) + 2\sin\alpha\cos^2\alpha\sec\varphi(1+3t^2). \qquad (16)$$

In this manner we also find:

$$\frac{d^4\lambda}{d\sigma^4} = 8\sec\varphi\, t\left\{-\sin^3\alpha\cos\alpha(1+3t^2) + \sin\alpha\cos^3\alpha(2+3t^2)\right\} \qquad (17)$$

$$\frac{d^5\lambda}{d\sigma^5} = 8\sec\varphi\left\{\sin^5\alpha(t^2+3t^4) - \sin^3\alpha\cos^2\alpha(1+20t^2+30t^4) + \sin\alpha\cos^4\alpha(2+15t^2+15t^4)\right\} \qquad (18)$$

$$\frac{d^6\lambda}{d\sigma^6} = 16\sec\varphi\, t\left\{\sin^5\alpha\cos\alpha(2+30t^2+45t^4) - \sin^3\alpha\cos^3\alpha(26+150t^2+150t^4) + \sin\alpha\cos^5\alpha(17+60t^2+45t^4)\right\}. \qquad (19)$$

In a similar way we also obtain the differentiations of α with respect to σ:

$$\frac{d\alpha}{d\sigma} = \sin\alpha\, t \qquad (20)$$

$$\frac{d^2\alpha}{d\sigma^2} = \sin\alpha\cos\alpha(1+2t^2) \qquad (21)$$

$$\frac{d^3\alpha}{d\sigma^3} = -\sin^3\alpha\, t(1+2t^2) + \sin\alpha\cos^2\alpha\, t(5+6t^2) \qquad (22)$$

$$\frac{d^4\alpha}{d\sigma^4} = -\sin^3\alpha\cos\alpha(1+20t^2+24t^4) + \sin\alpha\cos^3\alpha(5+28t^2+24t^4) \qquad (23)$$

$$\frac{d^5\alpha}{d\sigma^5} = \sin^5\alpha\, t(1+20t^2+24t^4) - 2\sin^3\alpha\cos^2\alpha\, t(29+140t^2+120t^4) + \sin\alpha\cos^4\alpha\, t(61+180t^2+120t^4) \qquad (24)$$

$$\frac{d^6\alpha}{d\sigma^6} = \sin^5\alpha\cos\alpha(1+182t^2+840t^4+720t^6) - \sin^3\alpha\cos^3\alpha(58+1316t^2+3600t^4+2400t^6) + \sin\alpha\cos^5\alpha(61+662t^2+1320t^4+720t^6). \qquad (25)$$

We can now put together the formulae (1), (2), (3); however, we will do this here only to the fourth order; we set thereby:

$$\sigma\sin\alpha = v \qquad \sigma\cos\alpha = u \qquad \tan\varphi = t. \qquad (26)$$

If we insert everywhere, at the same time, the necessary ρ's, then we obtain:

$$\varphi' - \varphi = u - \frac{1}{2\varrho}v^2 t - \frac{1}{6\varrho^2}v^2 u(1+3t^2) + \frac{1}{24\varrho^3}v^4 t(1+3t^2) - \frac{1}{6\varrho^3}v^2u^2 t(2+3t^2) \qquad (27)$$

$$\left.\begin{aligned}\lambda \cos\varphi = v &+ \frac{1}{\varrho} v\,u\,t \\ &- \frac{1}{3\varrho^2} v^3 t^2 + \frac{1}{3\varrho^2} v\,u^2 (1 + 3\,t^2) \\ &- \frac{1}{3\varrho^3} v^3 u\,t\,(1 + 3\,t^2) + \frac{1}{3\varrho^3} v\,u^3 t\,(2 + 3\,t^2)\end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned}\alpha' - \alpha = v\,t &+ \frac{1}{2\varrho} v\,u\,(1 + 2\,t^2) \\ &- \frac{1}{6\varrho^2} v^3 t\,(1 + 2\,t^2) + \frac{1}{6\varrho^2} v\,u^2 t\,(5 + 6\,t^2) \\ &- \frac{1}{24\varrho^3} v^3 u\,(1 + 20\,t^2 + 24\,t^4) \\ &+ \frac{1}{24\varrho^3} v\,u^3 (5 + 28\,t^2 + 24\,t^4)\,.\end{aligned}\right\} \quad (29)$$

The constant logarithms of coefficients are here:

$$\left.\begin{aligned}&\log\frac{1}{\varrho} = 3.685\,575\,,\ \log\frac{1}{2\varrho} = 4.384\,545\,,\ \log\frac{1}{3\varrho^2} = 8.89403\,,\ \log\frac{1}{6\varrho^2} = 8.59300 \\ &\log\frac{1}{3\varrho^3} = 3.579\,603\,,\ \log\frac{1}{6\varrho^3} = 3.278\,573\,,\ \log\frac{1}{24\varrho^3} = 2.676\,513\,.\end{aligned}\right\} \quad (30)$$

We will compute, according to this, our small normal example, namely:

Given: $\varphi = 49^\circ\,30'\,0''$ $\quad \alpha = 32^\circ\,21'\,1.291''$ $\quad \sigma = 1^\circ\,11'\,19.4819'' = 4279.4819''$.

For the coefficients depending on t, we can use, or at least consult for insurance, the auxiliary table of our Appendix on pages [59] to [61].

We have in our case with $\varphi = 49^\circ\,30'$:

$$\log(1 + 2\,t^2) = 0.573\,078 \quad \log(1 + 3\,t^2) = 0.70865 \quad \log(5 + 6\,t^2) = 1.12141$$
$$\log(1 + 20\,t^2 + 24\,t^4) = 1.86642 \quad \log(5 + 28\,t^2 + 24\,t^4) = 1.94689\,.$$

Besides, the calculation according to formulae (27) to (29), pp. 178-179, yields:

Latitude		Longitude		Azimuth	
$+u =$	$+3615.2710''$	$+v\sec\varphi =$	$+3525.9626''$	$v\,t =$	$+2681.1630''$
$-v^2 t\ldots$	-14.8830	$+v\,u\,t\ldots$	$+72.3593$	$+v\,u\ldots$	$+75.0907$
$-v^2 u\ldots$	-0.3797	$-v^3\ldots$	-0.1986	$-v^3\ldots$	-0.2061
$+v^4\ldots$	$+0.0008$	$+v\,u^2\ldots$	$+1.8460$	$+v\,u^2\ldots$	$+1.8155$
$-v^2 u^2\ldots$	-0.0093	$-v^3 u\ldots$	-0.0152	$-v^3 u\ldots$	-0.0152
		$+v\,u^3\ldots$	$+0.0452$	$+v\,u^3\ldots$	$+0.0455$
$\varphi' - \varphi =$	$+3599.9998''$	$\lambda =$	$+3599.9993''$	$\alpha' - \alpha =$	$+2757.8934''$
	$= 0^\circ\,59'\,59.9998''$		$= +0^\circ\,59'\,59.9993''$		$= +45'\,57.8934''$

We could now examine further what the terms of fifth order amount to in certain cases; but since we have already made it clear to ourselves in another way in section 62, (24), (25), (28), pp. 175-176, we will pass over the terms of fifth order of our new formulae; we will, however, consider the terms of sixth order for *one* more case, namely for the azimuth computation, in the case of which the term of sixth order is the following according to (3) and (25):

$$\Delta \alpha_6 = \frac{\sigma^6}{720 \varrho^5} \Big\{ \sin^5 \alpha \cos \alpha \, (1 + 182\, t^2 + 840\, t^4 + 720\, t^6) - \sin^3 \alpha \cos^3 \alpha \, (58 + 1316\, t^2 + 3600\, t^4 + 2400\, t^6) + \sin \alpha \cos^5 \alpha \, (61 + 662\, t^2 + 1320\, t^4 + 720\, t^6) \Big\}.$$

In order to have a simple case, we set the latitude $\varphi = 45°$; therefore $t = \tan \varphi = 1$, and with this, we will have:

$$\Delta \alpha_6 = \frac{\sigma^6}{720 \varrho^5} \Big\{ 1743 \sin^5 \alpha \cos \alpha - 7374 \sin^3 \alpha \cos^3 \alpha + 2763 \sin \alpha \cos^5 \alpha \Big\}.$$

By a few experiments we find that this function has between $0°$ and $90°$ two maximums, approximately at $\alpha = 16°$ and $\alpha = 77°$, and a minimum at $\alpha = 47°$; the absolute maximum is at $16°$, and yields:

$$(\Delta \alpha_6) \max = \frac{\sigma^6}{720 \rho^5} 491.$$

If we set $\sigma = 2° = 7200''$, then we obtain:

$$(\Delta \alpha_6) \max = 0.00025''.$$

For $\sigma = 3°$, on the other hand, we already obtain $0.0029''$, and for $\sigma = 4°$ we obtain $0.0162''$.

From all this we draw the following conclusions:

For an extent of several degrees, the terms of sixth order become already noticeable, especially in higher latitudes, where the terms in t^2, t^4, t^6 increase very rapidly. Now since the terms of fifth order are already exceedingly difficult, spherical developments in series are advisable at the most up to the fourth order inclusive, e.g. the Gauss formulae of the mean latitude (16) to (19), section 61, p. 169, which apparently contain only terms to the third order, but because of the mean argument are more accurate by one more degree, i.e. accurate to terms of fourth order inclusive.

If we have cases with an extent, say, beyond $2°$, for which the terms of fifth order are already noticeable according to p. 176, then we do better to compute according to the closed formulae of spherical trigonometry with 8- to 10-place logarithms than to have the much greater trouble with the terms of fifth or even sixth order.

We shall also make use of these reflections in the second half of this volume in the case of the spheroidal computations.

Section 64. Transformation of Geographic Coordinates by Means of Rectangular Coordinates

In section 59, p. 163, we have already learned the closed formulae indicated by Gauss for the transformation of geographic coordinates taking as a basis rectangular coordinates, which show great advantages compared to the direct transformation formulae.

In the following we will bring Gauss' solution of the problem by the introduction of logarithmic additaments in a form especially convenient for numerical computation, as has been communicated by L. Krüger in *Zeitschrift für Vermessungskunde*, 1919, pp. 288-289.

We summarize first once again the formulae previously found.

Let φ, α and σ be given and φ', λ and α' are to be computed.

For the computation of rectangular coordinates x and y we have

$$\tan x = \tan \sigma \cos \alpha \qquad (1)$$

$$\sin y = \sin \sigma \sin \alpha \qquad \tan y = \sin x \tan \alpha. \tag{2}$$

From this, we obtain at once λ by means of

$$\tan \lambda = \frac{\tan y}{\cos (\varphi + x)}. \tag{3}$$

The auxiliary quantities ε, γ and δ are then to be computed according to the formulae:

$$\sin \varepsilon = \tan \frac{\sigma}{2} \sin x \sin \alpha \tag{4}$$

$$\tan \gamma = \tan (\varphi + x) \sin \sigma \sin \alpha \tag{5}$$

$$\sin \delta = \cos (\varphi + x) \tan \frac{\lambda}{2} \tan \gamma. \tag{6}$$

Then we have finally

$$\varphi' = \varphi + x - \delta \tag{7}$$

$$\alpha' = \alpha + \gamma - \varepsilon. \tag{8}$$

Under the assumption that the distance $P P' = \sigma$ does not exceed $1.5°$ we will bring the equations (1) to (6) into logarithmic form and use hereby the logarithmic additaments, which we have already learned in section 34, p. 24, for an argument p for the conversion of $\log p$ to $\log \sin p$ or $\log \tan p$. If we set

$$\log \tan p = \log \frac{p}{\rho} + \tau_p \qquad \text{with} \qquad \tau_p = \frac{\mu}{3\rho^2} p^2, \tag{9}$$

then we have

$$\log \sin p = \log \frac{p}{\rho} - \frac{1}{2} \tau_p. \tag{10}$$

With this, we obtain from (1)

$$\log \frac{x}{\rho} + \tau_x = \log \frac{\sigma}{\rho} + \tau_\sigma + \log \cos \alpha$$

or

$$\log x = \log \sigma \cos \alpha + \tau_\sigma - \tau_x. \tag{11}$$

Likewise, we will have according to (4)

$$\log \varepsilon - \frac{1}{2} \tau_\varepsilon = \log \frac{\sigma}{2} + \frac{1}{4} \tau_\sigma + \log x - \frac{1}{2} \tau_x + \log \sin \alpha$$

or, since τ_ε can be neglected,

$$\log \varepsilon = \log \frac{\sigma x}{2\rho} \sin \alpha + \frac{1}{4} \tau_\sigma - \frac{1}{2} \tau_x. \tag{12}$$

If we also transform (5) in the same manner, then we have

$$\log \gamma + \tau_\gamma = \log \tan (\varphi + x) + \log \sin \alpha + \log \sigma - \frac{1}{2} \tau_\sigma,$$

therefore

$$log\ \gamma = log\ \sigma\ sin\ \alpha\ tan\ (\varphi + x) - \tau_\gamma - \frac{1}{2}\tau_\sigma. \tag{13}$$

Before we pass over to the formulae for $log\ \lambda$ and $log\ \delta$, we will express τ_λ by the additaments τ_σ, τ_x and τ_y which occurred already above.

For this, we square the two equations (1) and (2) and have then

$$tan^2\ x = tan^2\ \sigma\ cos^2\ \alpha \qquad sin^2\ y = sin^2\ \sigma\ sin^2\ \alpha.$$

We can also write for this, if we neglect terms of the 4th order,

$$x^2 = \sigma^2\ cos^2\ \alpha + \ldots \qquad y^2 = \sigma^2\ sin^2\ \alpha + \ldots$$

or

$$x^2 + y^2 = \sigma^2 + \ldots \tag{14}$$

We find likewise from the two equations (3), p. 181, and (30), p. 164,

$$\lambda^2 = \frac{y^2}{cos^2\ (\varphi + x)} = y^2 + y^2\ tan^2\ (\varphi + x)$$

$$\gamma^2 = y^2\ tan^2\ (\varphi + x),$$

and hence we have

$$\lambda^2 = y^2 + \gamma^2$$

or with the help of (14)

$$\lambda^2 = \sigma^2 - x^2 + \gamma^2. \tag{15}$$

If we multiply this equation by $\frac{\mu}{3\rho^2}$, then there follows according to p. 24,

$$\tau_\lambda = \tau_\sigma - \tau_x + \tau_\gamma. \tag{16}$$

With this, we can now also set up the equations for the computation of λ and δ. From (3) and the second equation (2) we find

$$log\ \lambda + \tau_\gamma = log\ x - \frac{1}{2}\tau_x + log\ tan\ \alpha\ sec\ (\varphi + x),$$

and if we substitute for $log\ x$ the value from (11),

$$log\ \lambda + \tau_\lambda = log\ \sigma\ cos\ \alpha\ tan\ \ \alpha\ sec\ (\varphi + x) - \frac{3}{2}\tau_x + \tau_\sigma.$$

If we eliminate therefrom τ_λ with the help of (16), then we have

$$log\ \lambda = log\ \frac{\sigma\ sin\ \alpha}{cos\ (\varphi + x)} - \frac{1}{2}\tau_x - \tau_\gamma. \tag{17}$$

We obtain likewise for δ from (6), p. 181, since r_δ can be neglected,

$$\log \delta = \log \frac{\gamma \lambda}{2\varrho} \cos(\varphi + x) + \frac{5}{4}\tau_y + \frac{1}{4}\tau_\sigma - \frac{1}{4}\tau_x$$

and according to (17)

$$\log \delta = \log \frac{\gamma\sigma}{2\varrho} \sin\alpha - \frac{3}{4}\tau_x + \frac{1}{4}\tau_y + \frac{1}{4}\tau_\sigma . \tag{18}$$

In summary, we thus have the following system of formulae for the transformation of geographic coordinates:

$$\left.\begin{aligned}
&\log x = \log \sigma \cos\alpha + \tau_\sigma - \tau_x \\
&\log y = \log \sigma \sin\alpha \tan(\varphi + x) - \tau_y - \frac{1}{2}\tau_\sigma \\
&\log \lambda = \log \frac{\sigma \sin\alpha}{\cos(\varphi + x)} - \frac{1}{2}\tau_x - \tau_y \\
&\log \delta = \log \frac{y}{2\varrho}\sigma \sin\alpha - \frac{3}{4}\tau_x + \frac{1}{4}\tau_y + \frac{1}{4}\tau_\sigma \qquad \varphi' = \varphi + x - \delta \\
&\log \varepsilon = \log \frac{x}{2\varrho}\sigma \sin\alpha - \frac{1}{2}\tau_x + \frac{1}{4}\tau_\sigma \qquad \alpha' = \alpha + y - \varepsilon .
\end{aligned}\right\} \tag{19}$$

We can take the logarithmic additaments τ from the values of the quantities S or T contained in the logarithmic tables, which we need for the computation of the sine and the tangent of small angles. Since, e.g.,

$$\log \tan x = \log x + T_x$$

if x is computed in seconds, then we will have according to (9)

$$\tau_x = T_x + \log \rho .$$

However, the use of the T's as well as that of the S's is not convenient, since these values are tabulated in the logarithmic tables with the argument x in seconds.

A convenient table of the additaments τ_x with the argument $\log x$ is found in *Veröffentlichung des Geodätischen Instituts*, "Tafeln für die Berechnung der geodätischen Linie und der Additamente für den Übergang von *log* auf *log sin* und *log tang*," edited by A. Galle, Berlin, 1920. The table extends to within $\tau = 10{,}000$ units of the eighth place, and hence to about $1^\circ 30' 20''$, which corresponds on the earth's surface to a length of about 167 km in middle latitudes.

In order to show a practical application of the above formulae (19), we will compute with them our small normal example:

Given:

$$\varphi = 49^\circ 30' 0'' \qquad \alpha = 32^\circ 21' 1.2914'' \qquad \sigma = 1^\circ 11' 19.48186'' = 4279.48186''.$$

In order to be able to examine the accuracy of the formulae (19), we carry out the computation with eight-place logarithms; for the additaments we use the tables by Galle just mentioned. With these, we obtain

$\log \sigma$	=	3.631 3911·9
$\log \cos \alpha$	=	9.926 7497·1
		3.558 1409·0
$+\tau_\sigma$	=	+ 623·2
$-\tau_x$	=	− 444·7
$\log x$	=	3.558 1587·5
x	=	3615.4200″
x	=	1° 00′ 15.4200″
φ	=	49° 30′
$\varphi + x$	=	50° 30′ 15.4200″
$\log \sigma \sin \alpha$	=	3.359 8221·4
$\log \cos (\varphi + x)$	=	9.803 4711·3
		3.556 3510·1

$\log \sigma$	=	3.631 3911·9
$\log \sin \alpha$	=	9.728 4309·5
$\log \tan (\varphi + x)$	=	0.083 9616·8
		3.443 7838·2
$-\frac{1}{2}\tau_\sigma$	=	− 311·6
$-\tau_\gamma$	=	− 262·7
$\log \gamma$	=	3.443 7263·9
γ	=	2777.9626″
γ	=	46′ 17.9626″
$\log \frac{1}{2\varrho}$	=	4.384 5448·7
$\log \sigma \sin \alpha$	=	3.359 8221·4
$\log \gamma$	=	3.443 7263·9
		1.188 0934·0

$-\frac{1}{2}\tau_x$	=	− 222·3
$-\tau_\gamma$	=	− 262·7
$\log \lambda$	=	3.556 3025·1
λ	=	3600.0001′
	=	1° 0′ 0.0001″
$\log \frac{1}{2\varrho}\sigma \sin \alpha$	=	7.744 3670·1
$\log x$	=	3.558 1587·5
		1·302 5257·6
$-\frac{1}{2}\tau_x$	=	− 222·4
$+\frac{1}{4}\tau_\sigma$	=	+ 155·8
$\log \varepsilon$	=	1.302 5191·0
ε	=	20.0687″

$-\frac{3}{4}\tau_x$	=	− 333·6
$+\frac{1}{4}\tau_\gamma$	=	+ 65·7
$+\frac{1}{4}\tau_\sigma$	=	+ 155·8
$\log \delta$	=	1.188 0821·9
δ	=	15.4199″
$\varphi + x$	=	50° 30′ 15.4200″
$-\delta$	=	− 15.4199
φ'	=	50° 30′ 0.0001″
α	=	32° 21′ 1.2914″
$+\gamma$	=	+ 46 17.9626
$-\varepsilon$	=	− 20.0687
α'	=	33° 06′ 59.1853″

The comparison with the rigorously computed values of section 59, p. 158, shows an agreement to one unit in the fourth decimal place of a second.

Chapter VI

PROJECTION OF THE SPHERICAL SURFACE ON THE PLANE

Section 65. The Perspective Projection

The end purpose of a land survey is the representation of the country [or land; German word: "Land"] on maps of larger or smaller scale. Whereas in the case of small measurements of the land survey, as have been treated particularly in Volume II first half-volume,* the map yields a reduced image of the terrain's culture, which is to be regarded as plane, the cartographic representation of a land survey meets with difficulties, since the curvature of the earth's surface can no longer be neglected here. It is obvious that a true representation of the curved surface of the earth in the plane is not possible, and that rather only a more or less distorted image can be produced in the plane. The plane projection of the earth's surface can be carried out in a great many different ways, and in each individual case it will be necessary to discuss the question of the most advantageous way; e.g., it may be required that the angles on the map compared with the corresponding angles of the earth's surface show deviations as small as possible, or that the areas of the figures be reproduced without change. On the other hand, a method of constructing the map as conveniently as possible may come into the foreground, among other things.

The map may be obtained by the direct transformation of the earth's surface into the plane. In many cases, however, it is more advantageous to transform at first the earth's surface to a surface which can be developed, e.g., a cylinder or a cone whose unfolding then yields the plane image of the map.

From the vast field of the theory of map projections, in the following we only shall deal with those which are concerned with the purposes of land survey.

In this chapter we hereby take the *spherical* shape of the earth as a basis, whereas the projection of the terrestrial spheroid shall be treated in the second half-volume. The projection of the sphere in itself also plays an important part for geodetic purposes; but its importance is substantially increased by the fact that in some cases we use two separate methods of projection by transforming at first the terrestrial spheroid to the sphere and then projecting the sphere on the plane. The first problem will likewise not occupy us until Volume III, second half.

We begin with a brief summary of the various methods of projection in order to treat then later individually in greater detail those methods which are used for the purposes of land survey.

The perspective projection

The most obvious idea for the projection of the spherical surface on the plane is the application of the perspective projection. On a straight line passing through the center of the sphere, the main ray of the perspective, an arbitrary point is assumed as the perspective center, while an arbitrary plane which lies perpendicular to the straight line serves as the plane of projection or image plane. The rays drawn from the perspective center through the points of the spherical surface intersect the plane in the image points. We shall denote the point of intersection of the main ray with the image plane as the image center. In the case of a stationary perspective center, a change in the position of the image plane, i.e., a parallel displacement of the image plane, causes only a change of the scale of the projection; whereas the form of the projection is changed by the displacement of the perspective center point on the main ray.

According to the position of the main ray, we distinguish three fundamental forms of the perspective projection. If the main ray coincides with the terrestrial axis, then the projection is called a *normal* projection. If the main ray lies in the equatorial plane, then the projection is called a *transverse* projection.

* Not translated.

Finally, if the main ray forms an arbitrary angle with the terrestrial axis or, as the case may be, with the equatorial plane, then we speak of an *oblique-axial* projection. For geodetic purposes, it is primarily a question of the projection of a small part of the surface of the sphere in the middle geographic latitudes, and for this reason the oblique-axial projection is used here.

The position of the main ray is established by the geographic longitude and latitude of the image center, i.e. the center of the map.

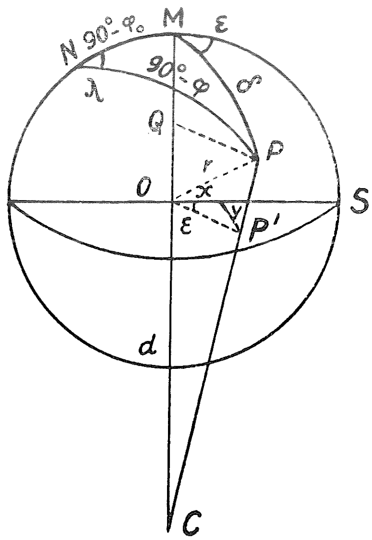

Fig. 1.

For the sake of simplicity, we shall count the geographic longitudes from the meridian of the map center, which has therefore the longitude zero. Let the geographic latitude of the map center be equal to φ_0.

With the help of Fig. 1, we shall derive the relations of the spherical surface to the image plane. In it, N is the north pole of the terrestrial sphere, M the center of the map, so that the angular distance $MN = 90° - \varphi_0$. Let C be the perspective center with the distance d from the center of the sphere O. For the sake of simplicity, we lay the image plane through the center point of the earth, and let a point P of the spherical surface be projected at P'. In the image plane we assume a rectangular coordinate system with the starting point O, whose positive axis of abscissae is to fall in the meridional direction OS. Hence, on the map, the abscissae are counted to the south and the ordinates to the west in the positive sense.

In Fig. 1, the position of the point P is determined by the angle $MNP = \lambda$, the geographic longitude, and by the angle $NP = 90° - \varphi$. On the other hand, let the point P with respect to M be determined by the angle ε and by the distance of arc $MP = \delta$.

In addition, we lay through P a normal PQ to the main ray, which is therefore parallel to $P'O$. If we denote the spherical radius again by r, then we have

$$PQ = r \sin\delta \qquad CQ = d + r\cos\delta$$

$$\frac{OP'}{QP} = \frac{OC}{QC} \qquad \text{hence} \qquad OP' = \frac{QP}{QC} OC$$

or

$$OP' = d\,\frac{r\sin\delta}{d + r\cos\delta}. \tag{1}$$

Since in the image plane the angle $SOP' = \varepsilon$, we obtain for the coordinates x and y the values

$$x = OP'\cos\varepsilon \qquad y = OP'\sin\varepsilon$$

or with (1)

$$x = d\,\frac{r\sin\delta\cos\varepsilon}{d + r\cos\delta} \qquad y = d\,\frac{r\sin\delta\sin\varepsilon}{d + r\cos\delta}. \tag{2}$$

In these, δ and ε are to be replaced by the longitude λ and the latitude φ of the point P as well as by the latitude φ_0 of the image center M. In the triangle NMP the following three relations result:

$$\left.\begin{aligned} \sin\delta\sin\varepsilon &= \sin\lambda\cos\varphi \\ \sin\delta\cos\varepsilon &= \cos\varphi\sin\varphi_0\cos\lambda - \sin\varphi\cos\varphi_0 \\ \cos\delta &= \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos\lambda\,. \end{aligned}\right\} \tag{3}$$

If we set this into (2), then there follows

$$\left.\begin{aligned} x &= d\,\frac{r(\cos\varphi\sin\varphi_0\cos\lambda - \sin\varphi\cos\varphi_0)}{d + r(\cos\varphi\cos\varphi_0\cos\lambda + \sin\varphi\sin\varphi_0)} \\ y &= d\,\frac{r\cos\varphi\sin\lambda}{d + r(\cos\varphi\cos\varphi_0\cos\lambda + \sin\varphi\sin\varphi_0)}\,. \end{aligned}\right\} \tag{4}$$

Accordingly, we can compute for every point φ, λ of the sphere its image point x, y on the plane.

The oblique-axial projection, to which equations (4) correspond, is the most general case of the perspective projection. Now we can derive from (4) the two special cases, the normal projection with $\varphi_0 = 90°$ and the transverse projection with $\varphi_0 = 0°$.

Normal projection. $\varphi_0 = 90°$

$$x = d\frac{r \cos \varphi \cos \lambda}{d + r \sin \varphi} \qquad y = d\frac{r \cos \varphi \sin \lambda}{d + r \sin \varphi}. \tag{5}$$

Transverse projection. $\varphi_0 = 0°$

$$x = -d\frac{r \sin \varphi}{d + r \cos \varphi \cos \lambda} \qquad y = d\frac{r \cos \varphi \sin \lambda}{d + r \cos \varphi \cos \lambda}. \tag{6}$$

For the following considerations it is more convenient to move the image plane to the point M, from which only an enlargement of the image follows. Consequently, we have to multiply the coordinates x and y by the scale factor $\frac{d+r}{d}$. Then we have

$$\left.\begin{aligned} x &= (d+r)\frac{r\,(\cos\varphi \sin\varphi_0 \cos\lambda - \sin\varphi \cos\varphi_0)}{d + r\,(\cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0)} \\ y &= (d+r)\frac{r \cos\varphi \sin\lambda}{d + r\,(\cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0)}. \end{aligned}\right\} \tag{7}$$

Thus far we have assumed arbitrarily the distance d of the perspective center. Now we will introduce three special values for d, and then we have from (7)

1. *Gnomonic Projection.* $d = 0$. The perspective center lies at the center of the sphere.

$$\left.\begin{aligned} x &= r\frac{\cos\varphi \sin\varphi_0 \cos\lambda - \sin\varphi \cos\varphi_0}{\cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0} \\ y &= r\frac{\cos\varphi \sin\lambda}{\cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0}. \end{aligned}\right\} \tag{8}$$

A remarkable property of this method of projection consists in the fact that all great circles of the sphere are projected as straight lines in the image plane. This is illustrated by Fig. 1 if we imagine the perspective center at 0.

2. *Stereographic Projection.* $d = r$.

$$\left.\begin{aligned} x &= 2r\frac{\cos\varphi \sin\varphi_0 \cos\lambda - \sin\varphi \cos\varphi_0}{1 + \cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0} \\ y &= 2r\frac{\cos\varphi \sin\lambda}{1 + \cos\varphi \cos\varphi_0 \cos\lambda + \sin\varphi \sin\varphi_0}. \end{aligned}\right\} \tag{9}$$

The stereographic projection, in which the perspective center of the image center lies opposite on the spherical surface, is the most important case of perspective projections for geodetic purposes. We shall discuss this in greater detail in section 71.

3. *Orthographic Projection.* $d = \infty$. We obtain the basic equations for it if we divide the numerator and the denominator in (7) by d and then set $d = \infty$. Then we obtain

$$\left.\begin{aligned} x &= r\,(\cos\varphi \sin\varphi_0 \cos\lambda - \sin\varphi \cos\varphi_0) \\ y &= r \cos\varphi \sin\lambda. \end{aligned}\right\} \tag{10}$$

Here, the projecting rays run parallel to the principal ray; hence, the projection is the orthogonal projection of the spherical surface onto the image plane.

Above we have given a general summary of all perspective projections, of which the stereographic projection only is involved in problems of national surveys, however. We shall refer to the latter in section 71.

Section 66. Further Methods of Projection

For the land survey operations, those methods of projection are especially important in the case of which the spherical surface is first transformed into a cylindric surface or a conic surface, by the development of which we obtain the plane image.

I. The cylinder projections

We place on a sphere a circular cylinder of the same diameter, which thus touches the terrestrial sphere at a great circle.

According to the position of the cylindric axis, we distinguish the normal, the transverse, and the oblique-axial cylinder projection.

In the case of the *normal* cylinder projection, the axis of the cylinder coincides with the terrestrial axis; the contact of the two surfaces thus takes place at the equator. Consequently, in the case of this projection, the equator is projected at true scale. The transformation of the meridians results directly by the fact that we bring the meridional planes to intersect with the cylindric surface so that after the unfolding of the cylinder, the meridians are straight lines lying parallel in the plane and perpendicular to the equator. The carrying forward of the circles of latitude to the cylinder can be carried out in the simplest form in such a manner that we transform their distances on the sphere to the meridians of the cylinder at true scale. Circles of longitudes and latitudes form then a square net in the plane, and we call a map thus originated a *square flat map*. It is obvious that such a map yields a usable projection of the spherical surface only in the closest vicinity of the equator.

However, the distances of the parallel circles can be chosen also in another form, so that more favorable projections result for definite problems.

The *transverse* cylinder projection, in which case the axis of the cylinder lies in the equatorial plane and the tangency of the two surfaces takes place at a meridian, is considerably more valuable. For a land survey, we can choose the meridian of tangency in such a way that, say, it passes through the center of the survey territory. This projection can therefore be applied to any arbitrary geographic latitudes.

For the case of the transformation of the points from the sphere to the cylinder, we must imagine a transverse graticule which originates by the rotation of the normal graticule on the sphere, namely in such a way that the equator coincides with the meridian of tangency. If we look at, in this connection, Fig. 1, section 52, p. 116, we see that such a transverse graticule is identical with a Soldner coordinate system, in which case in Fig. 1, p. 116, the meridian NOS takes the place of the equator and the two poles Q and Q' of the meridian NOS take the place of the north and south poles. Therefore, if on the sphere points are given by geographic coordinates, we can compute their coordinates in the transverse graticule according to the formulae (10) and (12) in section 55, p. 137. The transformation of the transverse graticule to the cylindric surface can be carried out in the same way as in the case of the normal projection.

The *oblique-axial* cylinder projection, too, can be usable for the purposes of the land survey under special circumstances. We choose hereby as circle of tangency a great circle which touches the parallel circle in the center of the survey territory. In regard to this great circle of the equator, we must then imagine again a graticule – lying this time oblique to the terrestrial axis – and this agrees with an oblique-axial spherical coordinate system, as is shown in Fig. 1, section 52, p. 116. Therefore, for the conversion of the geographic coordinates into oblique-axial coordinates, we again can use the formulae (4) and (7) from section 57, p. 147 and p. 149.

If we place on a terrestrial sphere a cone whose axis coincides with the terrestrial axis and whose convex surface touches the terrestrial sphere at an arbitrary parallel circle with the latitude φ_0, then there is given the possibility of transforming the spherical surface to the conic surface and obtaining, by the development of the latter, a plane projection. This normal conic projection is suited naturally for the projection of a zone of the sphere which runs, in not too great an extent, along both sides of the basic parallel circle. The transformation of the meridians to the cone is done again by the way of bringing the meridional planes to intersect with the conic surface. On the conic surface, all meridians thus run together at the apex of the cone as straight lines.

We shall establish at first how large the angle λ' between two meridians in the plane is which corresponds to a difference of longitude λ on the sphere.

To the difference of longitude λ belongs an arc of the normal parallel circle of length $r \cos \varphi_0 \lambda$, and since the distance between the fundamental parallel and the apex of the cone is, according to Fig. 1,

$$R_0 = r \cot \varphi_0, \tag{1}$$

then we will have

$$R_0 \lambda' = r \cos \varphi_0 \lambda,$$

or

$$\lambda' = \lambda \sin \varphi_0. \tag{2}$$

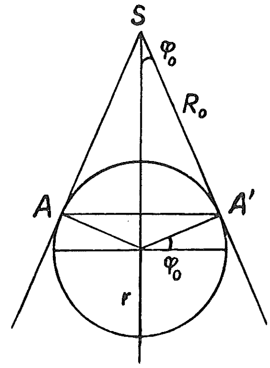

Fig. 1.

If we set the constant $\sin \varphi_0 = n_0$, then we have

$$\lambda' = \lambda n_0. \tag{3}$$

On the cone, the circles of latitude are circles whose planes lie perpendicular to the axis of the cone. The interval of the circles is chosen in different forms in the case of different conic projections. In the case of the simple conic projection which had already been used by Ptolemy, the interval of the parallel circles on the cone is the same as on the sphere. For the latitude φ, the distance of the parallel circle from the apex S thus becomes on the cone

$$R = R_0 - \frac{\varphi - \varphi_0}{\rho} r. \tag{4}$$

After the development of the conic surface, the meridians are projected as rays which start from the center point S, and whose angular distance is $\lambda' = \lambda \sin \varphi_0$. The circles of latitude are concentric circles whose radii are given by the expression (4).

III. The zenithal projections

The zenithal or azimuthal projections serve geodetic purposes for the projection of a spherical cap in the plane. In Fig. 2, p. 190, let M be the center point of the spherical cap with the geographic latitude φ_0, to be projected, so that its polar distance is equal $90° - \varphi_0$. Let an arbitrary point P have, in regard to M, the azimuth α and the angular distance δ. The transformation of the point P into the plane is carried out in such a way that from the meridian, from M, the angle α is set off in true size, and the distance MP' of the image point is made equal to a function $f(\delta)$ of δ to be chosen suitably. The various zenithal projections vary from one another by the choice of this function.

The construction of the zenithal normal projection is the simplest to be fashioned, since the angles α are identical here with the geographic longitudes referred to an arbitrary zero meridian, and the angular distances δ agree with the complements $90° - \varphi$ of the geographic latitudes.

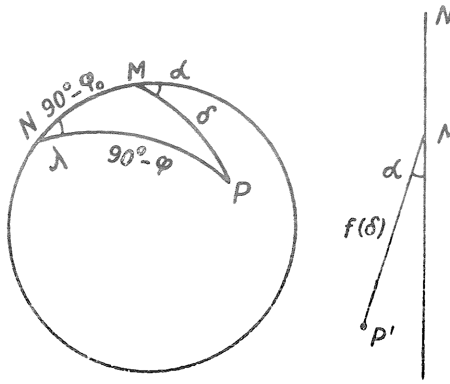

Fig. 2.

For the transverse and the oblique-axial projection, the angles α and δ are to be computed from the geographic coordinates of the center point of the map M and the point P to be projected. According to Fig. 2, we have in the triangle $M\,N\,P$, if we use the meridian of M as zero meridian,

$$\left.\begin{aligned} \cos\delta &= \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos\lambda \\ \sin\alpha &= \sin\lambda\,\frac{\cos\varphi}{\sin\delta}. \end{aligned}\right\} \qquad (5)$$

For the tasks of land survey, the zenithal projection comes into consideration, almost exclusively, as an oblique-axial projection.

Relationship of the three methods of projection

We can regard the cylinder projection and the zenithal projection, explained above, as special cases of the conic projection, which are determined by the values of the conic constant n_0 of p. 189. Thereby we shall apply the conic projection also in transverse and oblique-axial form, where each time a special graticule with the poles in the conic axis must be used. For the sake of simplicity, however, we shall speak of circles of latitude, also in these cases, in the graticule. For $\varphi_0 = 0$ or $n_0 = 0$ we then obtain the cylinder projection whereas for $\varphi_0 = 90°$, hence $n_0 = 1$, the conic projection changes to the zenithal projection. Hence, all three projections are reduced to the fundamental principle of the conic projection.

The conformal projection

We have established, in the case of the cylinder projections as well as in the case of the conic projections and the zenithal projections, that these methods of projection can be carried out in every possible form, in contrast to the perspective projections. Consequently, there is given the possibility here of fashioning these projections in such a way that they are especially suited for definite tasks. For the purposes of the land survey, the *conformal* or *"winkeltreuen"* projections, whose fundamental principles were given by Gauss, are the most advantageous.

In general, we mean by a *conformal* projection such a geometric relation between two surfaces that to each point of the one surface a definite point of the other surface corresponds, and that the projection is *similar* to the original in the smallest parts.

The latter condition is rendered clearer by Figs. 3 and 4 shown here, in this sense:

Let A, B, C be three points of a given surface (original) lying very close to one another and $A'\,B'\,C'$ the corresponding points of another surface (projection); the projection is to be carried out according to such a law that the small triangle $A'\,B'\,C'$ becomes similar to the corresponding small triangle $A\,B\,C$, hence, that the angles α, β, γ of the two triangles are equal to one another, i.e. projected "true," according to which this kind of projection is also called "angle-true," and that there is a constant ratio between the sides:

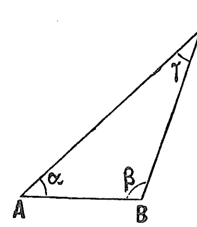

Fig. 3.
Original.

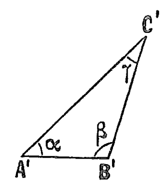

Fig. 4.
Projection.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = m. \qquad (6)$$

In the case of the conformal projection of the sphere on the plane, the requirement would thus have to be established that a small spherical triangle and its plane projection are similar.

Another important property of the conformal projection follows therefrom. We imagine to be drawn on the sphere, around a central point, a small polygon, arbitrarily shaped, and the central point connected with the corner points of the polygon so that a series of triangles results, which contain the central point as a common corner point. In the plane we then obtain, around the image point of the central point, a polygon whose triangles are similar to those of the sphere. It follows therefrom that for all linear elements starting from a point of the sphere to arbitrary directions, the scale factor in the projection is constant.

In the two sections 65 and 66 we have given an over-all view of the various methods of the projection of the terrestrial sphere on the plane, and in the following, we shall treat now more thoroughly those methods which are used in land survey.

Section 67. The Cassini-Soldner Projection

The "Carte géométrique de la France" at the scale 1:86,400 started in 1745 by Cassini de Thury was based on a system of projection which is to be denoted as transverse cylinder projection (section 66, p. 188) with the meridian of the Paris Observatory as meridian of tangency. Since the fundamental principles of computation for the Cassini projection agree with the formulae of the Soldner coordinates, this projection is denoted also as Cassini-Soldner projection.

In the course of the nineteenth century, this method of projection was used in Germany for the cadastral maps of most states; it constituted, up to the year 1927, also the basis of the cadastral maps in Prussia, where the ellipsoidal shape of the earth has been taken into consideration, however.

A map in Cassini-Soldner projection is obtained in the simplest manner by plotting the Soldner coordinates as plane rectangular coordinates.

Since such a map shows distortions compared to the original on the sphere, the distances and direction angles taken from the map or computed from the plane coordinates will not agree with the spherical distances and direction angles. Therefore, if we aim to replace the trigonometric computations on the sphere by computations in the plane, we must reduce first the measured lengths and angles to the plane.

In this connection, we can use the formulae already developed in section 54.

The reduction of length

If s denotes the spherical and s_0 denotes the plane distance of two points A and B in Fig. 1 with the coordinates x, y and x', y', then we have according to (10) and (9), section 54, p. 129,

$$s = s_0\left(1 - \frac{\cos^2 \alpha}{6r^2}(y^2 + y\,y' + y'^2)\right) \qquad (1)$$

$$s_0 = \sqrt{(y' - y)^2 + (x' - x)^2}, \qquad (2)$$

and according to these, we have

$$\frac{s_0}{s} = 1 + \frac{y^2 + y\,y' + y'^2}{6r^2}\cos^2 \alpha. \qquad (3)$$

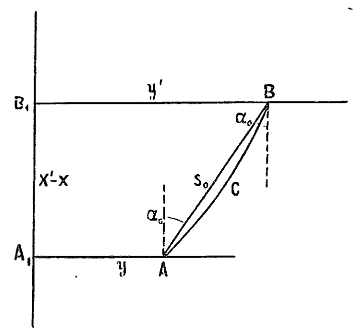

Fig. 1.
Soldner coordinates in plane representation.

For a very short line, we can set, instead of y' and y, a mean value y and then we have

$$\frac{s_0}{s} = 1 + \frac{y^2}{2r^2}\cos^2 \alpha. \qquad (4)$$

We see therefrom that the quotient $\frac{s_0}{s}$ is always larger than 1, and that it is dependent only on the ordinate y and the direction angle α, but not on the abscissa x.

In regard to α, $\frac{s_0}{s}$ reaches its extreme values with $\alpha = 0°$ or $180°$ on the one hand, and with $\alpha = 90°$ or $270°$ on the other, i.e.:

$$\alpha = 0° \text{ yields } \frac{s_0}{s} max = 1 + \frac{y^2}{2r^2} \quad \text{(meridian, } x\text{-axis)} \tag{5}$$

$$\alpha = 90° \text{ yields } \frac{s_0}{s} min = 1 \quad \text{(west-east, } y\text{-axis).} \tag{6}$$

These two results are easily comprehensible in themselves. In the west-east direction, the ordinates are plotted in the *same* manner on the sphere as well as on the plane, i.e. we have $s_0 = s$; in the north direction, however, the plane dimensions must appear too large, because the ordinates y, which actually converge, are parallel in the case of the plane treatment.

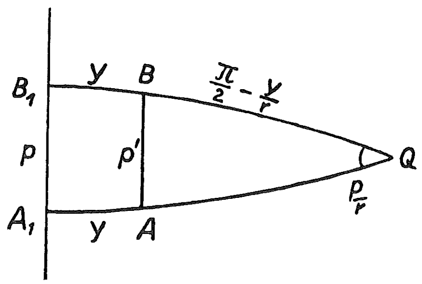

Fig. 2.

In this connection, there is drawn Fig. 2 with the dimension p in the axis of abscissae itself and a dimension p' parallel to the axis of abscissae, at the distance y. But in the plane, the ordinates y are represented parallel, hence p' is *equal* to p, and the scale factor is therefore $= \frac{p}{p'}$. According to Fig. 2, AB is an arc of the parallel of radius $r' = r \cos \frac{y}{r}$, and since at Q there is the angle $\frac{p}{r}$ (Fig. 2, p. 117), we have:

$$p' = AB = \frac{p}{r} r' = p \cos \frac{y}{r}$$

$$\frac{p'}{p} = \cos \frac{y}{r} = 1 - \frac{y^2}{2r^2} \quad \text{or} \quad \frac{p}{p'} = 1 + \frac{y^2}{2r^2}. \tag{7}$$

This is a confirmation of (5).

If we take $y = \frac{1}{100} r$, i.e. approximately equal to 63.7 km, we will have

$$\frac{p}{p'} = 1 + \frac{1}{2}\left(\frac{1}{100}\right)^2 = 1 + \frac{1}{20,000}.$$

For an over-all view of these linear relations of distortion, we have computed, according to (5), the following numerical values with $\log \frac{1}{2r^2} = 6.08918$ for $\varphi = 50°$:

y	$\frac{y^2}{2r^2}$	$\frac{y^2}{2r^2}$ 1000 m	y	$\frac{y^2}{2r^2}$	$\frac{y^2}{2r^2}$ 1000 m
10 km	0.000 0012	0.001 m	70 km	0.000 0602	0.060 m
20	0.000 0048	0.005	80	0.000 0786	0.079
30	0.000 0111	0.011	90	0.000 0995	0.099
40	0.000 0196	0.020	100	0.000 1228	0.123
50	0.000 0307	0.031	150	0.000 2763	0.276
60	0.000 0442	0.044	200	0.000 4912	0.491 .

(8)

In round numbers, the distortion thus amounts to:

$$\left.\begin{aligned} &5 \text{ cm per } 1000 \text{ m or } 0.05 \text{ mm per } 1 \text{ m or } \frac{1}{20{,}000} \text{ for } y = 64 \text{ km} \\ &10 \text{ cm per } 1000 \text{ m or } 0.1 \text{ mm per } 1 \text{ m or } \frac{1}{10{,}000} \text{ for } y = 90 \text{ km.} \end{aligned}\right\} \quad (9)$$

For example, if at the distances $y = 64$ km or 90 km, assumed in the case of (9), a traverse line or a measuring line of 1000-m length goes in the direction of the meridian from one trigonometric point to a second trigonometric point, the traverse or the line will yield, if no measuring errors happen at all, the distance between the two points to be 5 cm or, as the case may be, 10 cm smaller than the coordinates of the points as long as we apply only the computation of plane coordinates.

If, in the case of small surveys, we want to avoid the taking into account of the spherical terms of correction, we must use only small systems of coordinates. For example, this is the reason why the lateral extent of the systems of coordinates in Prussia existing hitherto was so limited that the ordinates would remain smaller than 60,000 m, whereby a distortion of abscissae of 5 cm per 1000 m, as indicated above, thus was permitted.

In Bavaria, where the ordinates in the *single* system of the meridian of Munich amount up to approximately 200 km, special spherical corrections had to be applied in the traverse computation, about which it was reported in *Instruktion für neue Katastermessungen in Bayern*, 1885, paragraph 23, in "Technische Anleitung," etc., by Dr. J. H. Franke, München, 1889, p. 121, and "Geodätische Punktkoordinierung in sphärischen Kleinsystemen" by Dr. J. H. Franke, München, 1898.

But now, in addition, we will consider, according to the relation (4), p. 191, the distortions δs to the various directions α for the maximum values $\dfrac{y^2}{2r^2} = 5$ cm or, as the case may be, 10 cm for a length of 1000 m, introduced in the case of (9).

$\alpha =$	0°	15°	30°	45°	60°	75°	90°
δs in the case of $y = 64$ km	5	4.6	3.7	2.5	1.2	0.3	0 cm
δs in the case of $y = 90$ km	10	9.3	7.5	5.0	2.5	0.7	0 cm

(10)

The average value of the distortion between $\alpha^\circ = 0^\circ$ and $\alpha = 90^\circ$ appears in the case of $\alpha = 45^\circ$, accordingly $\cos^2 45^\circ = \dfrac{1}{2}$.

Now if in the case of the drawing of the map, the spherical coordinates are plotted as has been explained in Fig. 1, p. 191, e.g. for a rectangular map sheet $A\,B\,A'\,B'$ (Fig. 3), which will be treated in the manner described, the south margin $A\,A'$ and the north margin $B\,B'$ appear in the right size, i.e. "true," whereas the west margin $A\,B$ and the east margin $A'\,B'$ become somewhat too large. We will assume that the west margin $A\,B$ has the ordinate $y = 90{,}000$ m and the east margin $A'\,B'$ has $y' = 100{,}000$ m, then, according to the numerical summary (8), the west margin becomes 0.0099% too large and the east margin 0.0123% too large in the drawing; or if the paper size is $A\,B = A'\,B' = 1$ m, then this yields here an error of approximately 0.1 m only, which, in the case of $y = 200$ km, however, increases rapidly to 0.5 m.

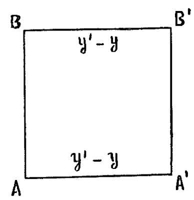

Fig. 3.

Such distortions of 1/20,000 in the case of $y = 64$ km, 1/10,000 in the case of $y = 90$ km, and even yet 1/2000 in the case of $y = 200$ km can be regarded as admissible or harmless in the drawing of the map and on the plane table, and this explains the large lateral extent of the above-mentioned Bavarian system, which had been arranged for the plane-table survey undertaken at the beginning of the nineteenth century.

For the reasons indicated above, the surfaces determined from the plan or from the plane coordinates do not agree with the spherical surfaces.

In Fig. 2, p. 192, we consider a narrow surface strip dF, which is attached to AB at the latitude dy, so that

$$dF = p' \, dy,$$

or, according to (7)

$$dF = p\left(1 - \frac{y^2}{2r^2}\right) dy.$$

If we compute the same surface strip in the plan or from the plane coordinates, then we obtain

$$dF_0 = p \, dy.$$

Hence, we have

$$dF_0 - dF = p \frac{y^2}{2r^2} dy. \tag{11}$$

If we integrate this between the limits y_1 and y_2, then there follows

$$F_0 - F = p \frac{y_2^3 - y_1^3}{6r^2} = p \frac{(y_2 - y_1)(y_2^2 + y_2 y_1 + y_1^2)}{6r^2}$$

or

$$F_0 - F = F_0 \frac{y_2^2 + y_2 y_1 + y_1^2}{6r^2}.$$

If the difference of the two ordinates y_1 and y_2 is small, we can set a mean value y instead of y_1 and y_2, and then we have

$$F_0 - F = F_0 \frac{y^2}{2r^2}. \tag{12}$$

For the two cases assumed in (9), $y = 64$ km and $y = 90$ km, we thus obtain the factor of reduction $\frac{1}{60{,}000}$ or $\frac{1}{30{,}000}$, as the case may be, and with this:

	$y = 64$ km	$y = 90$ km
for $F = 1$ ha $= 10{,}000$ m^2	$\delta_F = 0.5$ m^2	$\delta_F = 1.0$ m^2
for $F = 1$ km$^2 = 1{,}000{,}000$ m^2	$\delta_F = 50.1$ m^2	$\delta_F = 100.0$ m^2.

(13)

Compared with the inaccuracies of the determination of surfaces, these differences have no practical meaning.

The reduction of directions

For the plane computation of triangulations and traverse lines, the measured angles also are to be reduced to the plane. If α_0 is the plane direction angle and α the spherical one, we have from (20a), section 54, p. 130:

$$\alpha - \alpha_0 = \frac{\varrho}{6\,r^2}(x' - x)(2\,y + y') + \frac{\varrho}{6\,r^2}(y^2 + y\,y' + y'^2)\,sin\,\alpha\,cos\,\alpha\,.$$

In most cases it will be sufficient here to introduce a mean value y instead of y and y', so that we obtain:

$$\alpha - \alpha_0 = \frac{\varrho}{2\,r^2}(x' - x)\,y + \frac{\varrho}{4\,r^2}\,y^2\,sin\,2\,\alpha\,. \tag{14}$$

We see at once that in here the first term is very small in comparison to the second term, and that this reaches its maximum for $\alpha = 45°$. For example, if we take in the sense of the small triangulation $x' - x = 1$ km and as in the case of (9), p. 193, $y = 64$ km and $y = 90$ km, then we obtain with the value $\frac{\rho}{2r^2}$ of p. 120 for $\varphi = 50°$

$$\left.\begin{aligned} y &= 64\text{ km} \quad \delta_{max} = 0.16'' + 5.19'' = 5.35'' \\ &= 90\text{ km} \quad \delta_{max} = 0.23 + 10.25 = 10.48\,. \end{aligned}\right\} \tag{15}$$

The first term in (14) is very small in the case of small distances and may not play a role in the case of triangulations of third and fourth order. But the second term is completely independent of the distances; it rather is conditioned, if we disregard the factor $sin\;2\alpha$, only by the distance of the triangulation area from the axis of abscissae of the system of coordinates. Therefore, if we do not want to be satisfied with an accuracy of $\pm\,5''$ in the case of a triangulation of third order, we must not by any means neglect the reductions of direction.

For the correct working up of a triangulation with Soldner coordinates, two ways are thus possible. Either we compute purely spherically, in which case the measured angles can be used directly, or we compute as in the plane system of coordinates, in which case the measured angles are to be reduced to the plane according to the above equation (14). In the latter case, the final direction angles must be carried forward again to the sphere after the adjustment.

Direct use of the angles measured on the sphere for the computation of plane coordinates leads to neglects, under which the accuracy of the measurement of angles becomes illusory.

Projection of arcs of great circles

If Fig. 1, p. 191, is not supposed to represent only a projection of the points A and B of the sphere on the plane but also a projection of the line $A\,B$, i.e. of the arc of great circle $A\,B$ drawn on the sphere, then we are to understand first that in Fig. 1, the straight line $A\,B = s_0$ is *not* the projected image of that spherical arc $A\,B$, but the arc $A\,C\,B$.

This is specially represented once again in Fig. 4, and from a special investigation, which was communicated in *Zeitschr. f. Verm.*, 1891, pp. 289-294, we summarize the following without going into every detail: A and B are two points of projection with the coordinates $x\,y$ and $x'\,y'$. The projection of the spherical arc $A\,B$ is the curve $A\,C\,B$ of Fig. 4, whose tangents have certain direction angles α_1 and α_2, which are not equal, however, either to the hitherto considered α's and α''s or to the α_0's; for α_0 we have

$$tan\;\alpha_0 = \frac{y' - y}{x' - x}\,.$$

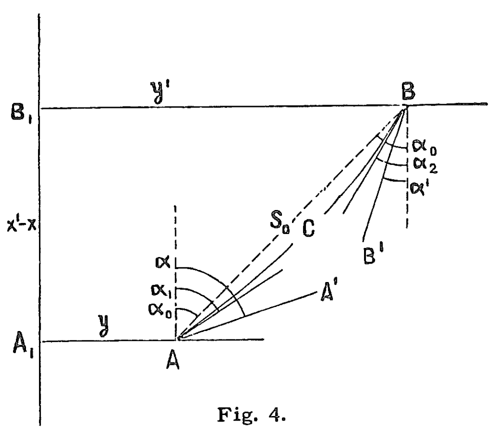

Fig. 4.

In this connection, according to (20a), section 54, p. 130:

$$\alpha - \alpha_0 = \frac{x' - x}{6\,r^2}(2\,y + y') + \frac{\sin\alpha\,\cos\alpha}{6\,r^2}(y^2 + y\,y' + y'^2)\,. \tag{16}$$

Then, according to *Zeitschr. f. Verm.*, 1891, p. 292:

$$\alpha_1 - \alpha_0 = \frac{x' - x}{6\,r^2}(2\,y + y')\,(1 + \sin^2\alpha) \tag{17}$$

$$\alpha - \alpha_1 = \frac{y^2}{2\,r^2}\sin\alpha\,\cos\alpha\,. \tag{18}$$

The oblique-axial cylinder projection

In the above, we have developed the reductions for the case of the transverse cylinder projection. But since it is immaterial for the computations within a spherical rectangular coordinate system which great circle is assumed as axis of abscissae, the above reduction formulae are valid also for every oblique-axial cylinder projection (cf. section 66, *I*, p. 188). This general case is therefore not in need of a special investigation.

Section 68. The Gauss Conformal Projection

In the case of the Cassini-Soldner projection, we have transformed the ordinates of the Soldner coordinate system to the cylinder tangent to the sphere at the zero meridian at true scale.

Now we will undertake another transfer of the Soldner coordinates to the spherical surface,* by leaving the abscissae unchanged, as in the previous section 67, p. 191, but changing the ordinates in such a manner that we obtain in the plane a conformal projection corresponding to section 66, p. 190.

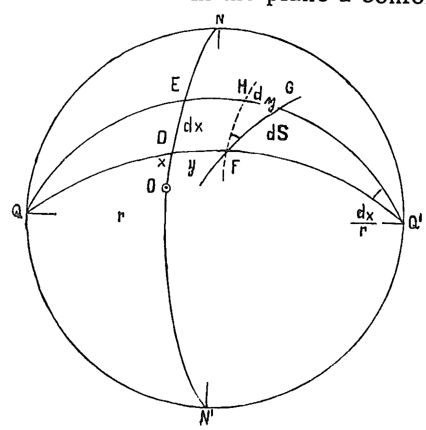

Fig. 1. ($F\,G = d\,S$.)

In Fig. 1, we have represented, on the terrestrial sphere, a Soldner system of coordinates with the meridian $N'ON$ as axis of abscissae and the point O as zero point. Let the coordinates of a point F be x and $\mathfrak{y}$ and those of a neighboring point G be $x + dx$ and $\mathfrak{y} + d\mathfrak{y}$, in which case we use for the ordinates the letter $\mathfrak{y}$ while y is to be reserved for the system of coordinates on the plane.

Fig. 2 shows a plane system of coordinates, in which the two points F and G of the spherical surface are projected as F' and G'. $OD = x$ and $DE = dx$ are equal here in both systems, whereas $DF' = y$ and $EG' = y + dy$ do *not* coincide with the ordinates DF and EG on the sphere, Fig. 1. Let the ordinates in the projection, in comparison with the ordinates in the original, rather be changed in such a manner that the small triangle $F'H'G'$ is similar to the triangle FHG on the sphere. By denoting the hypotenuses in these two triangles by ds and dS, we consider the relation of these hypotenuses, which we shall call m:

$$\frac{d\,s}{d\,S} = m. \tag{1}$$

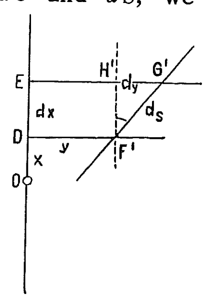

Fig. 2.

According to the fundamental idea of the conformal projection, which is expressed in equation (6), section 66, p. 190, the two triangles must be similar, whence there follows

$$\frac{F'H'}{FH} = \frac{H'G'}{HG} = m. \tag{2}$$

* Translator's note: In the original text "Kugelfläche"; should be "Ebene" [plane].

Here we have $F'H' = dx \qquad H'G' = dy,$

and FH as arc of parallel at the distance $\mathfrak{y}$ from DE has a radius of parallel $r' = r \cos\frac{\mathfrak{y}}{r}$, and since at Q' there is the angle $= \frac{dx}{r}$, we have:

$$FH = r'\frac{dx}{r} = dx\cos\frac{\mathfrak{y}}{r} \text{ and } HG = d\mathfrak{y}. \tag{3}$$

From (2) and (3) we have:

$$m = \frac{1}{\cos\frac{\mathfrak{y}}{r}} = \frac{dy}{d\mathfrak{y}} \tag{4}$$

$$\frac{dy}{r} = \frac{1}{\cos\frac{\mathfrak{y}}{r}}\frac{d\mathfrak{y}}{r}. \tag{5}$$

This equation can be integrated, i.e.:

$$\frac{y}{r} = l\tan\left(\frac{\pi}{4} + \frac{\mathfrak{y}}{2r}\right). \tag{6}$$

First, we will not make use, however, of the rigorous integration but set only the first approximation in (4) and (5):

$$\cos\frac{\mathfrak{y}}{r} = 1 - \frac{\mathfrak{y}^2}{2r^2}, \quad \frac{1}{\cos\frac{\mathfrak{y}}{r}} = 1 + \frac{\mathfrak{y}^2}{2r^2},$$

hence, according to (4):

$$dy = \left(1 + \frac{\mathfrak{y}^2}{2r^2}\right)d\mathfrak{y} \tag{7}$$

$$y = \mathfrak{y} + \frac{\mathfrak{y}^3}{6r^2}. \tag{8}$$

The relation between y and $\mathfrak{y}$ is thereby determined and likewise also the scale factor m; however, we can also interchange here $\mathfrak{y}$ and y in the correction terms, hence:

$$m = 1 + \frac{\mathfrak{y}^2}{2r^2} \quad \text{or} \quad m = 1 + \frac{y^2}{2r^2} \tag{9}$$

$$\text{and} \quad \frac{1}{m} = 1 - \frac{\mathfrak{y}^2}{2r^2} \quad \text{or} \quad \frac{1}{m} = 1 - \frac{y^2}{2r^2}.$$

This holds, in differential sense, at a point in *all* directions, or in an infinitely small right triangle, as e.g. $F'G'H'$, Fig. 2, the same value m is valid for both legs and for the hypothenuse.

Hereby we have found the relation between the Soldner coordinates on the sphere and the conformal Gauss coordinates on the plane. If the geographic coordinates are given from the outset, then we can compute from them the spherical-rectangular coordinates according to section 55, p. 136, and then change to the plane ordinates according to (8).

Now we will determine in the following the reduction of length and the reduction of direction for the Gauss projection corresponding to our procedure in section 67.

We pass over from the infinitely small arc (Figs. 1 and 2, p. 196) to a finite arc AB in Fig. 3, whose end points A and B in the plane have the coordinates $x_1 y_1$ and $x_2 y_2$. In Figs. 1 and 2, p. 196, the arc FG is projected as a line $F'G'$, which holds as a straight line in the case of an infinitely small extent, but must no longer be regarded as a straight line in the case of finite distance, but as a curve, as seen in Fig. 3, in which the curve AB is presented as the projection of a corresponding arc of the sphere, whereas the straight line AB is only an auxiliary line on the plane.

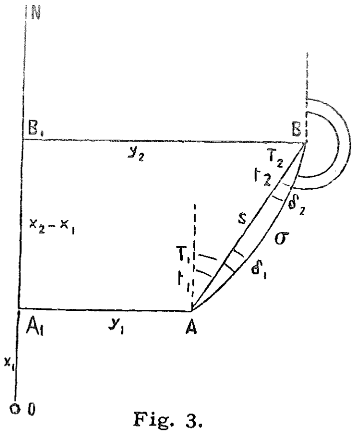

Fig. 3.

We also see immediately that this curve AB in Fig. 3 must be concave to the meridian of abscissae ON, for the rectilinear quadrilateral $A_1 B_1 B A$ has a sum of angles $= 360°$ whereas the corresponding spherical quadrilateral must have a sum of angles of more than $360°$ because of the spherical excess. But since the correct sum of angles must be obtained in the the projection, because of the conformality, the conformal projection of the arc of great circle AB will have to be presented curved in the sense as shown in Fig. 3.

With the denotations of Fig. 3, we have for the rectilinear distance s and the angle of direction t_1 in the plane rectangular system as always:

$$\left.\begin{aligned} \tan t_1 &= \frac{y_2 - y_1}{x_2 - x_1} \\ s = \frac{y_2 - y_1}{\sin t_1} &= \frac{x_2 - x_1}{\cos t_1} = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}\,. \end{aligned}\right\} \tag{10}$$

In the first approximation, the length of chord $AB = s$ can be set equal to the length of arc $AB = \sigma$, or we can denote by ds the differential of the straight line AB as well as that of the arc AB.

On the other hand, let S be the *spherical* distance of the points A and B, which is not represented in Fig. 3; then the following differential equation exists:

$$dS = \frac{1}{m}\,ds = \left(1 - \frac{y^2}{2r^2}\right) ds$$

$$S = s - \int \frac{y^2}{2r^2}\,ds = s - \int \frac{y^2}{2r^2}\frac{dy}{\sin t}$$

$$S = s - \frac{1}{2r^2 \sin t}\frac{y^3}{3} + \text{Integr.-Const.}$$

Between the limits y_1 and y_2, this yields:

$$S = s - \frac{1}{6r^2 \sin t}(y_2^3 - y_1^3)$$

$$S = s - \frac{1}{6r^2}\frac{y_2^3 - y_1^3}{y_2 - y_1}\frac{y_2 - y_1}{\sin t} = s - \frac{1}{6r^2}\frac{y_2^3 - y_1^3}{y^2 - y_1}\,s$$

$$\frac{S}{s} = 1 - \frac{1}{6r^2}\frac{y_2^3 - y_1^3}{y_2 - y_1} = 1 - \frac{1}{6r^2}(y_2^2 + y_2 y_1 + y_1^2)\,. \tag{11}$$

With these, the ratio of the lengths is expressed. But we can suitably transform this relation in different ways. To do so, we introduce the mean ordinate:

$$\frac{y_1 + y_2}{2} = y_0. \tag{12}$$

With $4y_0^2 = y_1^2 + 2y_1y_2 + y_2^2$, (11) is brought to the following form:

$$\frac{S}{s} = 1 - \frac{1}{12r^2}(y_1^2 + 4y_0^2 + y_2^2)\,. \tag{13}$$

The same formula reads in logarithmic form:

$$\log s - \log S = \frac{\mu}{12r^2}(y_1^2 + 4y_0^2 + y_2^2)$$

or

$$= \frac{\mu}{12r^2}(y_1^2 + (y_1 + y_2)^2 + y_2^2)\,. \tag{14}$$

We can form still another form, namely by calculating three values for the beginning, for the center and for the end of the line, in the following manner:

$$m_1 = 1 + \frac{y_1^2}{2r^2} \qquad m_0 = 1 + \frac{(y_1+y_2)^2}{8r^2} \qquad m_2 = 1 + \frac{y_2^2}{2r^2}$$

or

$$\frac{1}{m_1} = 1 - \frac{y_1^2}{2r^2} \qquad \frac{1}{m_0} = 1 - \frac{(y_1+y_2)^2}{8r^2} \qquad \frac{1}{m_2} = 1 - \frac{y_2^2}{2r^2}$$

and then from (13):

$$\frac{S}{s} = \frac{1}{6}\left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2}\right) \tag{15}$$

or else logarithmically from (14):

$$\log s - \log S = \frac{\log m_1 + 4\log m_0 + \log m_2}{6}\,. \tag{16}$$

Instead of (14) we can also write:

$$\log s - \log S = \frac{\mu}{24r^2}(2(y_1+y_2)^2 + 2y_1^2 + 2y_2^2 + 2y_1y_2 - 2y_1y_2)$$

$$\log s - \log S = \frac{\mu}{8r^2}(y_1+y_2)^2 + \frac{\mu}{24r^2}(y_2 - y_1)^2\,. \tag{17}$$

If $(y_2 - y_1)$ is *very* small in comparison with y_1 and y_2, then we can neglect the second term compared to the first one here, which can be done, e.g., in the case of a small triangulation. If we set thereby for small differences of coordinates $y_2 = y_1 = y$, we will have corresponding to (17) and (11):

$$\log s - \log S = \frac{\mu y^2}{2r^2} \tag{17*}$$

$$\frac{s}{S} = 1 - \frac{y^2}{2r^2}\,. \tag{11*}$$

The reduction of area

In accordance with our investigation for the Cassini-Soldner projection in section 67, p. 194, we have for the sphere

$$dF = p\left(1 - \frac{\mathfrak{y}^2}{2r^2}\right)d\mathfrak{y}$$

and for the plane

$$dF_0 = p\,dy.$$

But since we have according to (7), p. 197,

$$dy = d\mathfrak{y}\left(1 + \frac{\mathfrak{y}^2}{2r^2}\right)$$

we will have

$$dF_0 = p\left(1 + \frac{\mathfrak{y}^2}{2r^2}\right) d\mathfrak{y}$$

and we have

$$dF_0 - dF = p\frac{\mathfrak{y}^2}{r^2}\,d\mathfrak{y}\,.$$

Thence we obtain by integration

$$F_0 - F = p\,\frac{\mathfrak{y}_2{}^3 - \mathfrak{y}_1{}^3}{3r^2} = p\,\frac{(\mathfrak{y}_2 - \mathfrak{y}_1)(\mathfrak{y}_2{}^2 + \mathfrak{y}_2\mathfrak{y}_1 + \mathfrak{y}_1{}^2)}{3r^2}\,.$$

In this, we can set with sufficient accuracy on the right side $p\,(\mathfrak{y}_2 - \mathfrak{y}_1) = F_0$; therefore we have

$$F_0 - F = F_0\,\frac{\mathfrak{y}_2{}^2 + \mathfrak{y}_2\mathfrak{y}_1 + \mathfrak{y}_1{}^2}{3r^2}$$

and if the ordinates $\mathfrak{y}_1$ and $\mathfrak{y}_2$ do not differ greatly from one another, we can also write with great approximation

$$F_0 - F = F_0\,\frac{y^2}{r^2} \tag{18}$$

where $\mathfrak{y}$ has been replaced by y at the same time.

The reduction of directions

For the determination of the reduction of directions, we refer once again to the consideration which, in the above, has led to the knowledge that the conformal projection of the arc of great circle FG of Fig. 1, p. 196, must be represented in Fig. 3, p. 198, as a smooth curve, which in Fig. 3 must be convex to the right.

This consideration yields at once the sum of the two small angles δ_1 and δ_2 in Fig. 3, p. 198, for this sum $\delta_1 + \delta_2$ must be equal to the spherical excess of the quadrilateral, i.e. to an accuracy of $\frac{1}{r^2}$, inclusive:

$$\delta_1 + \delta_2 = \frac{(x_2 - x_1)(y_2 + y_1)}{2r^2}\,. \tag{19}$$

If the two points A and B move very closely together, this yields the differential formula:

$$2\delta = \frac{dx \cdot y}{r^2}. \tag{20}$$

Now we consider in Fig. 4 a new rectangular coordinate system, whose origin lies at the point A, and whose direction of abscissae $+\xi$ is to lie from A to B, and whose direction of ordinates $+\eta$ is to lie perpendicularly to AB. If in this system the smooth curve AB is represented by an equation between ξ and η, then the radius of curvature R of this smooth curve can be represented with sufficient accuracy by the equation:

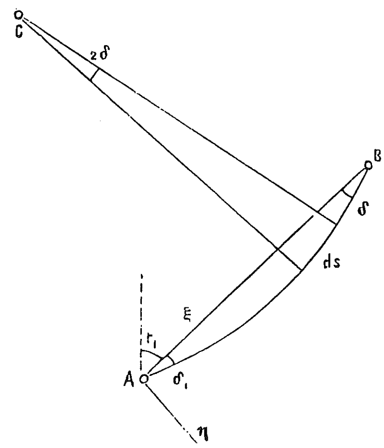

Fig. 4.
($AB = s$.)

$$\frac{1}{R} = \frac{d^2 \eta}{d \xi^2}. \tag{21}$$

If we further denote by 2δ as in the case of (20) the curvature of arc on the extent of the element of arc ds, then we have

$$ds = R \cdot 2\delta, \tag{22}$$

hence from (20) to (22) the differential equation for η:

$$\frac{d^2 \eta}{d \xi^2} = \frac{y}{r^2} \frac{dx}{ds}.$$

But since we can also set with sufficient accuracy $ds = d\xi$, we have from the above the differential equation of the curve AB:

$$\frac{d^2 \eta}{d \xi^2} = \frac{y}{r^2} \frac{dx}{d\xi}.$$

This is developed at first without regard to sign; if the curve, however, lies with its concave side opposite the ξ-axis, as in Fig. 4, then the second derivative must be negative, i.e.:

$$-\frac{d^2 \eta}{d \xi^2} = \frac{y}{r^2} \frac{dx}{d\xi}. \tag{23}$$

According to the view of Fig. 4 we have in the first approximation:

$$x = x_1 + \xi \cos t_1 \quad \text{and} \quad y = y_1 + \xi \sin t_1$$

$$\frac{dx}{d\xi} = \cos t_1,$$

or from (23)

$$-\frac{d^2 \eta}{d \xi^2} = \frac{y_1}{r^2} \cos t_1 + \frac{\xi}{r^2} \sin t_1 \cos t_1 \tag{24}$$

or

$$-\frac{d^2 \eta}{d \xi^2} = A + B\xi, \tag{25}$$

where the meaning of A and B follows from the comparison of (24) and (25):

$$A = \frac{y_1}{r^2} \cos t_1 \qquad B = \frac{\sin t_1 \cos t_1}{r^2}. \tag{26}$$

Equation (25) is integrated twice:

$$-\frac{d\eta}{d\xi} = C_1 + A\,\xi + \frac{B\,\xi^2}{2} \tag{27}$$

$$-\eta = C_1\,\xi + \frac{A\,\xi^2}{2} + \frac{B\,\xi^3}{6}. \tag{28}$$

C_1 is the constant of integration for the integration here, and for the second integration no other constant is added, because there must be for $\xi = 0$ also $\eta = 0$, as we see at a glance at once. In order to determine the constant C_1, we have according to the view of Fig. 4, p. 201, that $\xi = 0$ must yield the value $\frac{d\eta}{d\xi} = +\delta_1$ and $\xi = s$ must yield the value $\frac{d\eta}{d\xi} = -\delta_2$; likewise $\xi = s$ must yield the value $\eta = 0$, i.e.:

$$-\delta_1 = C_1$$

$$+\delta_2 = C_1 + A\,s + \frac{B\,s^2}{2}$$

$$0 = C_1\,s + \frac{A\,s^2}{2} + \frac{B\,s^3}{6}, \qquad \text{or} \qquad 0 = C_1 + \frac{A\,s}{2} + \frac{B\,s^2}{6}.$$

These three equations yield:

$$\delta_1 = \frac{A\,s}{2} + \frac{B\,s^2}{2} \quad \text{and} \quad \delta_2 = \frac{A\,s}{2} + \frac{B\,s^2}{3}. \tag{29}$$

Or if we introduce the meanings of A and B according to (26):

$$\delta_1 = \frac{s\,\cos t_1}{6\,r^2}(3\,y_1 + s\,\sin t_1) \quad \text{and} \quad \delta_2 = \frac{s\,\cos t_1}{6\,r^2}(3\,y_1 + 2\,s\,\sin t_1). \tag{30}$$

Finally, since $s\,\sin t_1 = y_2 - y_1$ and $s\,\cos t_1 = x_2 - x_1$, we can write this also with the denotation of Fig. 3, p. 198, thus, at the same time with the addition of ρ:

$$T_1 - t_1 = \delta_1 = \frac{\varrho}{6\,r^2}(x_2 - x_1)(2\,y_1 + y_2) \tag{31}$$

$$T_2 - t_2 = \delta_2 = \frac{\varrho}{6\,r^2}(x_1 - x_2)(y_1 + 2\,y_2). \tag{32}$$

These formulae can be written also in the following form:

$$T_1 - t_1 = \frac{\varrho}{4\,r^2}(x_2 - x_1)(y_1 + y_2) - \frac{\varrho}{12\,r^2}(x_2 - x_1)(y_2 - y_1) \tag{33}$$

$$T_2 - t_2 = \frac{\varrho}{4\,r^2}(x_1 - x_2)(y_2 + y_1) - \frac{\varrho}{12\,r^2}(x_1 - x_2)(y_1 - y_2). \tag{34}$$

These formulae, which are algebraically identical with the preceding ones, are supposed to express the relative smallness of the second parts for the case in which the coordinate *differences* $(x_2 - x_1)$ and $(y_2 - y_1)$ are relatively small compared to the ordinates y_1 and y_2 themselves, and this occurs in the case of the triangulation of third order at a distance from the axis; and then we can frequently even neglect the second terms in (33) and (34) compared to the first terms, and we have a somewhat more convenient calculation; e.g. for $y_1 + y_2 = 200$ km, $x_1 - x_2 = y_1 - y_2 = 10$ km we have $T - t = 2.5'' - 0.04''$. If we set again

$y_2 = y_1 = y$, then we have from (31) to (34):

$$T - t = \frac{y\,\Delta x}{2\,r^2}\varrho \ldots \qquad (31^*\text{-}34^*)$$

Equation of the curve AB

After the coefficients A, B have been determined in addition to the constant of integration C_1, the equation of the curve can also be written down according to (28) and (29), p. 202:

$$\eta = +\delta_1\,\xi - \frac{A\,\xi^2}{2} - \frac{B\,\xi^3}{6} = \frac{A\,s}{6}\,\xi + \frac{B\,s^2}{2}\,\xi - \frac{A\,\xi^2}{2} - \frac{B\,\xi^3}{6}$$

$$\eta = \frac{A\,\xi}{2}(s-\xi) + \frac{B\,\xi}{6}(s^2-\xi^2) = \frac{y_1\,\xi}{2\,r^2}\cos t_1\,(s-\xi) + \frac{\xi\sin t_1\cos t_1}{6\,r^2}(s^2-\xi^2)\,,$$

or arranged according to powers of ξ with $s \sin t_1 = y_2 - y_1$:

$$\eta = \xi\frac{s\cos t_1}{6\,r^2}(2\,y_1 + y_2) - \frac{\xi^2}{2\,r^2}\,y_1\cos t_1 - \frac{\xi^3}{6\,r^2}\sin t_1\,\cos t_1\,. \qquad (35)$$

According to this, the curve AB of Fig. 4, p. 201, appears represented by a third-degree equation, from which as a proof we can once again determine the δ_1's and δ_2's by differentiation.

Calculation of the constant coefficients

For the mean latitude of Germany we can assume $\varphi = 50°$ and the mean radius of curvature

$$\log r = 6.804\,8936{\cdot}173,$$

and the following coefficients are valid for this:

$\log\frac{1}{2\,r^2} = 6.089\,1828$	$\log\frac{1}{4\,r^2} = 5.788\,1528$	$\log\frac{1}{6\,r^2} = 5.612\,0615$	$\log\frac{1}{12\,r^2} = 5.311\,0315$
$\log\frac{\mu}{2\,r^2} = 2{\cdot}726\,9671$	$\log\frac{\mu}{8\,r^2} = 2{\cdot}124\,9071$	$\log\frac{\mu}{6\,r^2} = 2{\cdot}249\,8458$	$\log\frac{\mu}{24\,r^2} = 1{\cdot}617\,7858$
$\log\frac{\varrho}{2\,r^2} = 1.403\,6079$	$\log\frac{\varrho}{4\,r^2} = 1.102\,5779$	$\log\frac{\varrho}{6\,r^2} = 0.926\,4866$	$\log\frac{\varrho}{12\,r^2} = 0.625\,4567$

Ordinarily, we need these coefficients only to 4-5 places; for all cases, we have put them here to 7 places.

In the auxiliary tables of the Appendix, pages [58] and [59], we have calculated a few functions for the Gauss conformal projection, namely $\log m = \frac{\mu}{2\,r^2}\,y^2$ on page [59], at first as far as $y = 100{,}000$ m with a small interval of 1000 m and below at the end, for a general summary, only to 5 places as far as $y = 690$ km.

In the case of large y's, however (but which in general go beyond the limits of a local system of co-ordinates as far as approximately $y = 100$ km), a fourth-order term is also added so that we have:

$$\log m = \frac{\mu}{2\,r^2}\,y^2 - \frac{\mu}{12\,r^4}\,y^4 = [2{\cdot}726\,6995]\,y^2 - [8{\cdot}33849]\,y^4\,.$$

We shall not treat this until later.

The other table on the upper part of page [58] gives the enlargement of ordinates

$$y - \mathfrak{y} = \frac{\mathfrak{y}^3}{6r^2} = [5.611\,794]\,\mathfrak{y}^3,$$

where $\mathfrak{y}$ is the spherical ordinate. Then the lower part of page [58] gives the differential distortion of ordinates $\frac{y^2}{2r^2}$ or conformal general linear distortion together with the reduction of elevation $\frac{h}{r}$.

Section 69. Example for the Computation of Gauss Conformal Coordinates

In Fig. 1 following, our Palatine net of section 53 is shown once again, essentially as before, only now with a family of parallel lines of constant y, hence parallel to the x-axis, whose meaning will be explained later, because at first only the net itself is needed.

We have now converted this Palatine triangulation net, which has been treated in Soldner coordinates in section 53, to conformal coordinates and, in fact, for the mean latitude $\varphi = 49°\,30'$ with the constants:

$$\log r = 6.804\,8686\ , \quad \log \frac{1}{6r^2} = 5.61211\ , \quad \log \frac{\mu}{2r^2} = 2{\cdot}72702\,. \tag{1}$$

If we denote again the Soldner ordinates by $\mathfrak{y}$ and the conformal ordinates by y, then we have according to (8), section 68, p. 197:

$$y = \mathfrak{y} + \frac{\mathfrak{y}^3}{6r^2} = \mathfrak{y} + [5.612\,11]\,\mathfrak{y}^3 \tag{2}$$

and for a summary of the differences $y - \mathfrak{y}$, we can use the upper part of the table on p. [58] of the Appendix, although it is valid for the latitude 50°, whereas our Palatine net has the mean latitude 49° 30', for this makes almost no difference for small ordinates y.

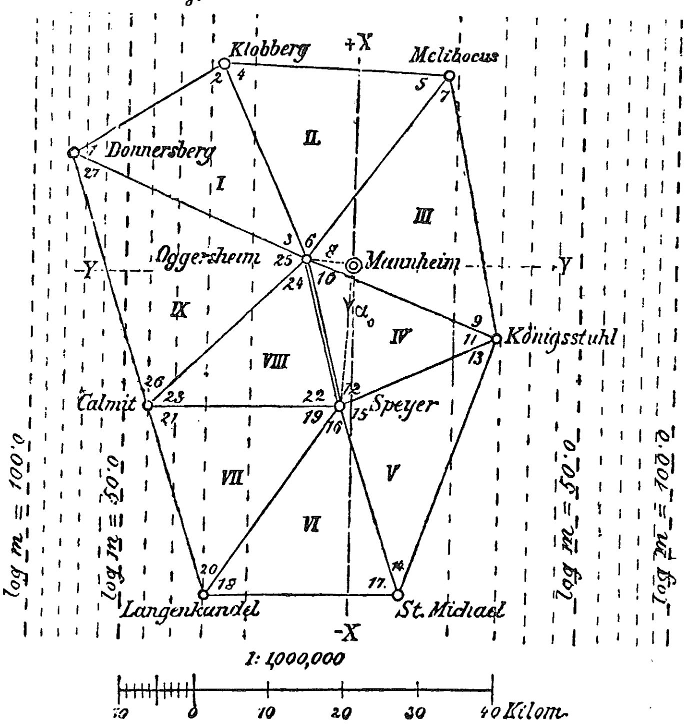

Fig. 1.

According to the above formula (2), the $y - \mathfrak{y}$ are computed in the following table:

Point	$\mathfrak{y}$ Spherical	$\frac{\mathfrak{y}^2}{6r^2} = y - \mathfrak{y}$	y Plane	x	$\log m = \mu \frac{\mathfrak{y}^2}{2r^2}$
1. Mannheim	0.000 m	m	0.000 m	0.000 m	0·0
2. Speyer	— 1208.142	0.000	— 1208.142	— 18,816.676	0·1
3. Oggersheim . . .	— 6001.777	0.001	— 6001.778	+ 388.767	1·9
4. Calmit	— 27,414.066	0.084	— 27,414.150	— 18,550.134	40·1
5. Donnersberg . . .	— 38,145.688	0.227	— 38,145.915	+ 15,278.872	77·6
6. Klobberg.	— 18,104.628	0.024	— 18,104.652	+ 28,049.296	17·5
7. Melibocus	+ 12,727.470	0.008	+ 12,727.478	+ 26,509.100	8·6
8. Königsstuhl . . .	+ 19,525.476	0.030	+ 19,525.506	— 9223.075	20·3
9. St. Michael . . .	+ 7407.498	0.002	+ 7407.500	— 44,332.386	2·9
10. Langenkandel . .	— 19,467.721	0.030	— 19,467.751	— 44,893.918	20·2

$$\log \frac{1}{6r^2} = 5.61211 \qquad \log \frac{\mu}{2r^2} = 2{\cdot}72702\,.$$

We can also compute all distances and direction angles from these coordinates y, x according to the formulae (10), p. 198, (13), p. 199, and (31), p. 202, of section 68, as we will show with an example: Donnersberg-Calmit:

Conformal

Calmit	$y_2 = -27{,}414.150$ m	$x_2 = -18{,}550.134$ m
Donnersberg	$y_1 = -38{,}145.915$	$x_1 = -15{,}278.872$

$y_2 - y_1 = +10{,}731.765 \qquad x_2 - x_1 = -33{,}829.006$

$\log(y_2 - y_1)$	4.030 6711·5		
$\log(x_2 - x_1)$	4.529 2892·4$_n$		
$\log\tan t_1$	9.501 3819·1$_n$	$t_1 = 162°\,23'\,56{,}83''$	
$\log\sin t_1$	9.480 5595·8	$\log\cos t_1$	9.979 1776·6$_n$
$\log(y_2 - y_1)$	4.030 6711·5	$\log(x_2 - x_1)$	4.529 2892·4$_n$
$\log s$	4.550 1115·7	$\log s$	4.550 1115·8 .

This is a purely plane computation, and now there come the corrections with $1 : r^2$:

$y_2 = -27{,}414$, $y_2 = -27{,}414$, $y_1 = -38{,}146$, $y_1 = -38{,}146$

$y_1 + y_2 = -65{,}560 \qquad 2y_1 + y_2 = -103{,}706 \qquad 2y_2 + y_1 = -92{,}974$

$\log(y_1 + y_2)$	4.81 664	$\log(2y_1 + y_2)$	5.01 582$_n$	$\log(2y_2 + y_1)$	4.96 836$_n$
$\log(y_1 + y_2)^2$	9.63 328	$\log(x_2 - x_1)$	4.52 929$_n$	$\log(x_1 - x_2)$	4.52 929
$\log(\mu : 2r^2)$	2.72 701	$\log(\varrho : 6r^2)$	0.92 654	$\log(\varrho : 6r^2)$	0.92 654
$\log \frac{\mu(y_1+y_2)^2}{2r^2}$	2.36 029	$\log(T_1 - t_1)$	0.47 165	$\log(T_2 - t_2)$	0.42 419$_n$

$$\frac{\mu(y_1 + y_2)^2}{2r^2} = 229{\cdot}24 \qquad T_1 - t_1 = +2.962'' \qquad T_2 - t_2 = -2.656''\,.$$

For 229·24, which is $= 4 \log m_0$, we take $\log m_2 = 40{\cdot}1$ and $\log m_1 = 77{\cdot}6$ already listed in the table above for Calmit and Donnersberg and thence for our length:

$$\log s - \log S = \frac{77{\cdot}6 + 229{\cdot}24 + 40{\cdot}1}{6} = 57{\cdot}8\,. \qquad (3)$$

In summary, we thus have:

$$
\begin{array}{lll}
\log s = 4.550\,1115{\cdot}7 & t_1 = 162^\circ\,23'\,56.83'' & t_2 = 342^\circ\,23'\,56.83'' \\
\qquad\quad -\,57{\cdot}8 & \qquad\quad +\,2.96 & \qquad\quad -\,2.66 \\
\hline
\log S = 4.550\,1057{\cdot}9 & T_1 = 162^\circ\,23'\,59.79'' & T_2 = 342^\circ\,23'\,54.17'' \\
 & \text{Donnersberg} & \text{Calmit.}
\end{array} \tag{4}
$$

Thus these values are introduced in the sketch below, and the whole sketch has so originated since we have assumed the coordinates as given. Instead of computing as above, we can also compute according to (17), p. 199, (33) and (34), p. 202, of section 68.

Sketch of the Triangulation of the Net of Fig. 1, p. 204, with Conformal Coordinates

Stations and Target Points	Direction Angle: Spherical T	$t - T$	Plane t	Distance: Spherical $\log S$	$\log s - \log S$	Plane $\log s$
1. Mannheim						
Speyer	183° 40′ 25.29″	− 0.02″	183° 40′ 25.27″	4.275 4362·3	+ 0·0	4.275 4362·3
Oggersheim	273 42 22.23	+ 0.00	273 42 22.23	3.779 1890·8	+ 0·6	3.779 1891·4
2. Speyer						
Mannheim	3° 40′ 25.23″	+ 0.04	3° 40′ 25.27″	4.275 4362·3	+ 0·0	4.275 4362·3
Königsstuhl	65 10 11.05	− 0.14	65 10 10.91	4.358 8019·1	+ 6·2	4.358 8025·3
St. Michael	161 20 31.87	+ 0.11	161 20 31.98	4.430 2529·9	+ 0·8	4.430 2530·7
Langenkandel	215 0 1.16	− 0.48	215 0 0.68	4.502 8974·0	+ 7·4	4.502 8981·4
Calmit	270 34 57.83	+ 0.02	270 34 57.85	4.418 4219·3	+ 14·0	4.418 4233·3
Oggersheim	345 59 7.47	+ 0.14	345 59 7.61	4.296 5476·5	+ 0·8	4.296 5477·3
3. Oggersheim						
Melibocus	35° 38′ 31.00″	− 0.02	35° 38′ 30.98″	4.507 0618·9	+ 2·2	4.507 0621·1
Mannheim	93 42 22.23	− 0.00	93 42 22.23	3.779 1890·8	+ 0·6	3.779 1891·4
Königsstuhl	110 37 58.62	+ 0.06	110 37 58.68	4.435 7946·2	+ 5·3	4.435 7951·5
Speyer	165 59 7.82	− 0.21	165 59 7.61	4.296 5476·5	+ 0·8	4.296 5477·3
Calmit	228 30 28.54	− 0.63	228 30 27.91	4.456 1549·5	+ 16·9	4.456 1566·4
Donnersberg	294 51 17.20	+ 0.63	294 51 17.83	4.549 3120·2	+ 30·6	4.549 3150·8
Klobberg	336 22 4.82	+ 0.70	336 22 5.52	4.479 8976·1	+ 8·4	4.479 8984·5
4. Calmit						
Oggersheim	48° 30′ 26.94″	+ 0.97	48° 30′ 27.91″	4.456 1549·5	+ 16·9	4.456 1566·4
Speyer	90 34 57.90	− 0.05	90 34 57.85	4.418 4219·3	+ 14·0	4.418 4233·3
Langenkandel	163 12 53.74	− 1.65	163 12 52.09	4.439 5851·8	+ 29·8	4.439 5881·6
Donnersberg	342 23 54.17	+ 2.66	342 23 56.83	4.550 1057·9	+ 57·8	4.550 1115·7
5. Donnersberg						
Klobberg	57° 29′ 38.99″	+ 1.02	57° 29′ 40.01″	4.375 9183·2	+ 44·0	4.375 9227·2
Oggersheim	114 51 18.87	− 1.04	114 51 17.83	4.549 3120·2	+ 30·6	4.549 3150·8
Calmit	162 23 59.79	− 2.96	162 23 56.83	4.550 1057·9	+ 57·8	4.550 1115·7
6. Klobberg						
Melibocus	92° 51′ 35.28″	− 0.03	92° 51′ 35.25″	4.489 5442·7	+ 4·6	4.489 5447·3
Oggersheim	156 22 6.51	− 0.99	156 22 5.52	4.479 8976·1	+ 8·4	4.479 8984·5
Donnersberg	237 29 40.81	− 0.80	237 29 40.01	4.375 9183·2	+ 44·0	4 375 9227·2
7. Melibocus						
Königsstuhl	169° 13′ 40.29″	+ 1.36	169° 13′ 41.65″	4.560 7787·6	+ 14·1	4.560 7801.7
Oggersheim	215 38 30.55	+ 0.43	215 38 30.98	4.507 0618·9	+ 2·2	4.507 0621·1
Klobberg	272 51 35.26	− 0.01	272 51 35.25	4.489 5442·7	+ 4·6	4.489 5447·3
8. Königsstuhl						
St. Michael	199° 2′ 30.37″	+ 1.38	199° 2′ 31.75″	4.569 8613·6	+ 10·3	4.569 8623·9
Speyer	245 10 10.60	+ 0.31	245 10 10.91	4.358 8019·1	+ 6·2	4.358 8025·3
Oggersheim	290 37 58.95	− 0.27	290 37 58.68	4.435 7946·2	+ 5·3	4.435 7951·5
Melibocus	349 13 43.21	− 1.56	349 13 41.65	4.560 7787·6	+ 14·1	4.560 7801·7
9. St. Michael						
Königsstuhl	19° 2′ 32.77″	− 1.02	19° 2′ 31.75″	4.569 8613·6	+ 10·3	4.569 8623·9
Langenkandel	268 48 10.93	− 0.00	268 48 10.93	4.429 4468·0	+ 5·1	4.429 4473·1
Speyer	341 20 32.27	− 0.29	341 20 31.98	4.430 2529·9	+ 0·8	4.430 2530·7
10. Langenkandel						
Speyer	34° 59′ 59.80″	+ 0.88	35° 0′ 0.68″	4.502 8974·0	+ 7·4	4 502 8981·4
St. Michael	88 48 10.92	+ 0.01	88 48 10.93	4.429 4468·0	+ 5·1	4.429 4473·1
Calmit	343 12 50.61	+ 1.48	343 12 52.09	4.439 5851·8	+ 29·8	4.439 5881·6

If, vice versa, the whole triangulation is computed with a base line and a starting azimuth or, as the case may be, starting direction angle, we have to do essentially the same. In the most convenient way, we compute preliminary coordinates only to an accuracy of approximately 1 m, which, apart from that, we need in most cases for other purposes; we have the triangle sides S from the net adjustment and net computation. If we compute, to this, all $\log s - \log S$'s and at first only the first $t - T$, then we can carry out the whole computation of coordinates in the plane and we only need to add all $t - T$'s in order to set up the whole sketch of p. 206. Detailed examples are given in this connection in our Vol. I, 8th edition, 1935, namely the city net of Hannover in the conformal system of the land survey in section 71, then section 103, Double Insertion of Points, section 106, Net Insertion, to which we now refer.

Tabular and graphical treatment of the reductions

Since the scale factor m depends only on the ordinate y, we can easily tabulate it, e.g., for the Palatine net with $\varphi = 49° 30'$ and $\log r = 6.804\ 87$ we have the main values

$$\left.\begin{array}{lccccc} y = & 10{,}000\text{ m} & 20{,}000\text{ m} & 30{,}000\text{ m} & 40{,}000\text{ m} & 50{,}000\text{ m} \\ \log m = \dfrac{\mu y^2}{2 r^2} = & 5{\cdot}3 & 21{\cdot}3 & 48{\cdot}0 & 85{\cdot}3 & 133{\cdot}4\ . \end{array}\right\} \tag{5}$$

A detailed table for use would easily be set up. We do not want to go into this here, but note, in addition, the graphical treatment of the matter. We can cover the picture of the net with a family of parallels to the x-axis, hence parallels for constant y's, which correspond to certain round values of m or of $\log m$, and thence we can take for each point its $\log m$.

In our case we have

$$\log m = \frac{\mu}{2 r^2} y^2 \quad \text{with} \quad \log \frac{\mu}{2 r^2} = 2{\cdot}72\ 702 - 10$$

$$y = \sqrt{\frac{2 r^2}{\mu}} \sqrt{\log m} = [3.63\ 619] \sqrt{\log m}\ . \tag{6}$$

With this, the following is computed:

$\log m =$ 10·0	20·0	30·0	40·0	50·0	60·0	70·0	80·0	90·0	100·0
$y =$ 13.7	19.4	23.7	27.4	30.6	33.5	36.2	38.7	41.1	43.3 km.

According to these, the parallels in Fig. 1, p. 204, are drawn; e.g., we can find from this:

Donnersberg	Mean	Calmit
78	57	40

$$\log s - \log S = \frac{78 + 4 \times 57 + 40}{6} = \frac{78 + 228 + 40}{6} = \frac{346}{6}$$

$$\log s - \log S = 57{\cdot}7\ .$$

This is supposed to be the same as the previous 57·8 in (3), p. 205.

In the small net picture of Fig. 1, p. 204, the graphical interpolation for $\log m$ is not sufficiently accurate but, in any case, useful for the control of the computation; if we have removed nets of second and third order from the main axis, where the $\log m$'s become larger and the parallels more equidistant, then the method becomes very good.

We consider, in addition, the reductions of the directions:

$$T_1 - t_1 = \frac{\varrho}{6r^2}(x_2 - x_1)(2y_1 + y_2) = \frac{\varrho}{2r^2}(x_2 - x_1)\,y' \text{ with } y' = \frac{2y_1 + y_2}{3}$$

$$T_2 - t_2 = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \frac{\varrho}{2r^2}(x_1 - x_2)\,y'' \text{ with } y'' = \frac{y_1 + 2y_2}{3}.$$

We can also take off the y''s, y'''s directly with the compasses here, as well as the $x_2 - x_1$'s, if we do not have them already in the computation anyhow, and since $\frac{\rho}{2r^2} = \frac{1}{395}$ for kilometers, we can simply compute with the slide rule:

$$T_1 - t_1 = \frac{x_2 - x_1}{395}\,y' \quad \text{and} \quad T_2 - t_2 = \frac{x_1 - x_2}{395}\,y'',$$

e.g. p. 205,

Donnersberg	Calmit
$x_2 - x_1 = -33.8$ km $y' = -34.6$ km	$x_1 - x_2 = +33.8$ km $y'' = -31.0$ km
$T_1 - t_1 = +2.96''$	$T_2 - t_2 = -2.66''$.

Section 70. Further Development of the Conformal Gauss Projection

Although the formulae found in section 68 will be sufficient in most cases for the direct practical application, we shall carry further, for special purposes, the former developments of section 68 to the terms of fourth order.

We refer to equation (6) found in section 68, p. 197, by rigorous integration, namely

$$\frac{y}{r} = l\,tan\left(\frac{\pi}{4} + \frac{\mathfrak{y}}{2r}\right)$$

or for decadic logarithms, with $\mu = 0.43429\ldots$

$$y = \frac{r}{\mu}\log tan\left(\frac{\pi}{4} + \frac{\mathfrak{y}}{2r}\right). \tag{1}$$

The scale factor, given rigorously at first, is according to (4), section 68, p. 197:

$$m = \frac{dy}{d\mathfrak{y}} = sec\,\frac{\mathfrak{y}}{r}. \tag{2}$$

The function (1) can be developed as a series by converting at first purely goniometrically:

$$tan\left(\frac{\pi}{4} + \frac{\mathfrak{y}}{2r}\right) = \frac{1 + tan\,\frac{\mathfrak{y}}{2r}}{1 - tan\,\frac{\mathfrak{y}}{2r}} = \frac{1+t}{1-t}. \tag{3}$$

The logarithmic series of section 34, p. 21, applied to this yields:

$$\log(1+t) = \mu\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \ldots\right)$$
$$\log(1-t) = \mu\left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \ldots\right)$$
$$\log\frac{1+t}{1-t} = 2\mu\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \ldots\right).$$

The series of tangents, section 34, p. 23, yields:

$$t = \tan\frac{\mathfrak{y}}{2r} = \frac{\mathfrak{y}}{2r} + \frac{\mathfrak{y}^3}{24r^3} + \frac{\mathfrak{y}^5}{240r^5}$$
$$t^3 = +\frac{\mathfrak{y}^3}{8r^3} + \frac{\mathfrak{y}^5}{32r^5}$$
$$t^5 = +\frac{\mathfrak{y}^5}{32r^5}$$
$$t + \frac{t^3}{3} + \frac{t^5}{5} = \frac{\mathfrak{y}}{2r} + \frac{\mathfrak{y}^3}{12r^3} + \frac{\mathfrak{y}^5}{48r^5}$$
$$\log\frac{1+t}{1-t} = \mu\left(\frac{\mathfrak{y}}{r} + \frac{\mathfrak{y}^3}{6r^3} + \frac{\mathfrak{y}^5}{24r^5} + \ldots\right),$$

hence according to (1) and (3):

$$y = \mathfrak{y} + \frac{\mathfrak{y}^3}{6r^2} + \frac{\mathfrak{y}^5}{24r^4} + \ldots \tag{4}$$

This equation must be solved for $\mathfrak{y}$ inversely, and this can be done by approximation carried step by step:

$$\mathfrak{y} = y - \frac{y^3}{6r^2} + \ldots \qquad \mathfrak{y}^3 = y^3 - \frac{3y^5}{6r^2}$$
$$\mathfrak{y} = y - \frac{1}{6}\left(\frac{y^3}{r^2} - \frac{3y^5}{6r^4}\right) - \frac{y^5}{24r^4}$$
$$\mathfrak{y} = y - \frac{y^3}{6r^2} + \frac{y^5}{24r^4}. \tag{5}$$

Instead of series (4) we can also develop a series which yields y as a function of $\sin\mathfrak{y}$. According to section 34, p. 23, the series for the arc sine is

$$\frac{\mathfrak{y}}{r} = \sin\frac{\mathfrak{y}}{r} + \frac{1}{6}\sin^3\frac{\mathfrak{y}}{r} + \frac{3}{40}\sin^5\frac{\mathfrak{y}}{r}$$

hence according to (4):

$$\frac{y}{r} = \left(\sin\frac{\mathfrak{y}}{r} + \frac{1}{6}\sin^3\frac{\mathfrak{y}}{r} + \frac{3}{40}\sin^5\frac{\mathfrak{y}}{r}\right) + \frac{1}{6}\left(\sin^3\frac{\mathfrak{y}}{r} + \frac{1}{2}\sin^5\frac{\mathfrak{y}}{r}\right) + \frac{1}{24}\sin^5\frac{\mathfrak{y}}{r}$$
$$y = r\sin\frac{\mathfrak{y}}{r} + \frac{r}{3}\sin^3\frac{\mathfrak{y}}{r} + \frac{r}{5}\sin^5\frac{\mathfrak{y}}{r}. \tag{6}$$

According to (2) the scale factor m can be developed also as far as $\frac{1}{r^4}$

$$\cos\frac{\mathfrak{y}}{r} = 1 - \frac{\mathfrak{y}^2}{2r^2} + \frac{\mathfrak{y}^4}{24r^4}$$
$$\sec\frac{\mathfrak{y}}{r} = 1 + \left(\frac{\mathfrak{y}^2}{2r^2} - \frac{\mathfrak{y}^4}{24r^4}\right) + \frac{\mathfrak{y}^4}{4r^4} = 1 + \frac{\mathfrak{y}^2}{2r^2} + \frac{5\mathfrak{y}^4}{24r^4}. \tag{7}$$

This agrees with the secant series indicated in section 34, p. 23.

We thus have
$$m = 1 + \frac{\mathfrak{y}^2}{2r^2} + \frac{5\,\mathfrak{y}^4}{24\,r^4},$$

or with the introduction of (5):

$$m = 1 + \frac{1}{2r^2}\left(y - \frac{y^3}{6r^3}\right)^2 + \frac{5\,y^4}{24\,r^4}$$
$$m = 1 + \frac{y^2}{2r^2} + \frac{y^4}{24\,r^4}. \tag{8}$$

To this also the inversion:

$$\frac{1}{m} = 1 - \frac{y^2}{2r^2} + \frac{5\,y^4}{24\,r^4} \tag{9}$$

and in logarithmic form:

$$\log m = \frac{\mu}{2r^2}y^2 - \frac{\mu}{12r^4}y^4.$$

The next thing is the more rigorous computation of the difference of the angles of direction, for which the same consideration as previously in the case of the Soldner coordinates, section 52, is used; and in order not to have to do the same thing twice, we shall change the previous equation into our new denotations in accordance with Fig. 1 by writing $\mathfrak{y}_1$ and $\mathfrak{y}_2$ instead of y and y', then $x_2 - x_1$ instead of $x' - x$ and finally $T_1 - T_2$ instead of $\alpha - \alpha'$. By so doing, (7), section 52, p. 118, changes to this form:

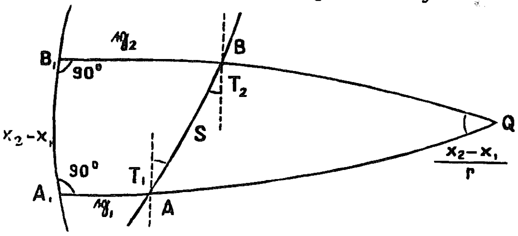

Fig. 1.

$$\tan\frac{T_1 - T_2}{2} = \frac{\sin\frac{\mathfrak{y}_2 + \mathfrak{y}_1}{2r}}{\cos\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{2r}}\tan\frac{x_2 - x_1}{2r}.$$

According to this, we can compute rigorously the difference of the spherical direction angles T_1 and T_2, develop in series as far as desired, etc.; meanwhile, at first we only need the differential therefrom:

$$\tan\frac{dT}{2} = \frac{\sin\frac{\mathfrak{y}}{r}}{\cos\frac{dy}{2r}}\tan\frac{dx}{2r}$$

or to sufficient accuracy:
$$\frac{dT}{2} = \sin\frac{\mathfrak{y}}{r}\,\frac{dx}{2r}; \tag{10}$$

here we have at first:
$$\sin\frac{\mathfrak{y}}{r} = \frac{\mathfrak{y}}{r} - \frac{\mathfrak{y}^3}{6r^3},$$

hence because of (5):

$$\sin\frac{\eta}{r} = \left(\frac{y}{r} - \frac{y^3}{6r^3}\right) - \frac{y^3}{6r^3} = \frac{y}{r} - \frac{y^3}{3r^3},$$

consequently, according to (10):

$$dT = \left(\frac{y}{r} - \frac{y^3}{3r^3}\right)\frac{dx}{r} = \frac{1}{r^2}\left(y\,dx - \frac{y^3}{3r^2}dx\right). \tag{11}$$

Now we have again to consider dT as the differential of curvature of the curve AB, similarly to the previous Fig. 4, section 68, p. 201, which now recurs in Fig. 2.

In the same sense as previously in the case of (23), p. 201, we therefore have for our new case from (11):

$$-\frac{d^2\eta}{d\xi^2} = \frac{dT}{d\xi} = \frac{1}{r^2}\left(y\frac{dx}{d\xi} - \frac{y^3\,dx}{3r^2\,d\xi}\right). \tag{12}$$

This equation is still accurate enough even here, for, it is true, we ought to put $\sqrt{d\xi^2 + d\eta^2}$, instead of $d\xi$, but according to (35), p. 203, $d\eta$ itself is already of the order $\frac{1}{r^2}$, hence $d\eta^2$ is already of the order $\frac{1}{r^4}$, and this would already yield $\frac{1}{r^6}$ with the factor $\frac{1}{r^2}$ existing in (12) already.

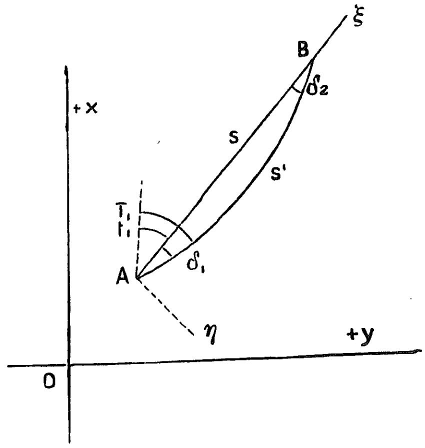

Fig. 2.

In order to carry out further (12), we must express x and y in ξ, and this is done, according to the aspect of Fig. 2, by the following transformation of coordinates:

$$\left.\begin{aligned} x &= x_1 + \xi\cos t_1 - \eta\sin t_1 \\ y &= y_1 + \xi\sin t_1 + \eta\cos t_1 . \end{aligned}\right\} \tag{13}$$

But the η's themselves are functions of ξ, namely according to (35), section 68, p. 203:

$$\eta = \frac{\xi s\cos t_1}{6r^2}(2y_1 + y_2) - \frac{\xi^2}{2r^2}y_1\cos t_1 - \frac{\xi^3}{6r^2}\sin t_1\cos t_1 .$$

This introduced into (13) yields:

$$x = x_1 + \xi\cos t_1 - \frac{\xi s\cos t_1\sin t_1}{6r^2}(2y_1 + y_2) + \frac{\xi^2}{2r^2}y_1\sin t_1\cos t_1 + \frac{\xi^3}{6r^2}\sin^2 t_1\cos t_1 \tag{14}$$

$$y = y_1 + \xi\sin t_1 + \frac{\xi s\cos^2 t_1}{6r^2}(2y_1 + y_2) - \frac{\xi^2}{2r^2}y_1\cos^2 t_1 - \frac{\xi^3}{6r^2}\sin t_1\cos^2 t_1 \tag{15}$$

$$\frac{dx}{d\xi} = \cos t_1 - \frac{s\cos t_1\sin t_1}{6r^2}(2y_1 + y_2) + \frac{\xi}{r^2}y_1\sin t_1\cos t_1 + \frac{\xi^2}{2r^2}\sin^2 t_1\cos t_1 . \tag{16}$$

With these, we can form the first part of (12), namely $y\frac{dx}{d\xi}$, and for the second part of (12), we still need from (13):

to this

$$\left.\begin{aligned} y^3 &= y_1^3 + 3y_1^2\xi\sin t_1 + 3y_1\xi^2\sin^2 t_1 + \xi^3\sin^3 t_1 + \dots \\ \frac{dx}{d\xi} &= \cos t_1 + \dots \end{aligned}\right\} \tag{17}$$

If we work out the multiplication of the two factors (15), (16) and those of (17) and combine the two products according to the direction of equation (12), then we will obtain an expression of the following form, arranged according to powers of ξ:

$$-\frac{d^2 \eta}{d \xi^2} = A + B\xi + C\xi^2 + D\xi^3 \tag{18}$$

where the coefficients A, B, C, D have the following meanings:

$$\left.\begin{aligned} A &= \frac{y_1}{r^2} \cos t_1 - \frac{y_1 s \sin t_1 \cos t_1}{6 r^4} (2 y_1 + y_2) - \frac{y_1^3}{3 r^4} \cos t_1 \\ B &= \frac{\sin t_1 \cos t_1}{r^2} + \frac{s \cos t_1}{6 r^4} (2 y_1 + y_2) (\cos^2 t_1 - \sin^2 t_1) \\ C &= \frac{y_1}{2 r^4} (- \cos^3 t_1 + \sin^2 t_1 \cos t_1) \\ D &= \frac{1}{6 r^4} (- \sin t_1 \cos^3 t_1 + \sin^3 t_1 \cos t_1) . \end{aligned}\right\} \tag{19}$$

Before we make further use of these coefficients, we will treat further the functions (18) by double integration:

$$-\frac{d\eta}{d\xi} = C_1 + A\xi + \frac{B\xi^2}{2} + \frac{C\xi^3}{3} + \frac{D\xi^4}{4} \tag{20}$$

$$-\eta = C_1 \xi + \frac{A\xi^2}{2} + \frac{B\xi^3}{6} + \frac{C\xi^4}{12} + \frac{C\xi^5}{20} . \tag{21}$$

C_1 is thereby the first constant of integration, and the second constant of integration is equal to zero, because $\xi = 0$ must yield also $\eta = 0$. For the determination of the constants C_1 there is used the stipulation that $\xi = 0$ must yield $\frac{d\eta}{d\xi} = +\delta_1$, and $\xi = s$ yields $\frac{d\eta}{d\xi} = -\delta_2$, and we know further that $\xi = s$ must yield also $\eta = 0$, therefore:

$$\begin{aligned} -\delta_1 &= C_1 \\ +\delta_2 &= C_1 + A s + \frac{B s^2}{2} + \frac{C s^3}{3} + \frac{D s^4}{4} \\ 0 &= C_1 + \frac{A s}{2} + \frac{B s^2}{6} + \frac{C s^3}{12} + \frac{D s^4}{20} , \end{aligned}$$

hence there follows:

$$\left.\begin{aligned} \delta_1 &= \frac{A s}{2} + \frac{B s^2}{6} + \frac{C s^3}{12} + \frac{D s^4}{20} \\ \delta_2 &= \frac{A s}{2} + \frac{B s^2}{3} + \frac{C s^3}{4} + \frac{D s^4}{5} . \end{aligned}\right\} \tag{22}$$

The coefficients A, B, C, D are to be introduced from (19) here, and this requires only an algebraic collecting of the homogeneous parts, and will yield after the arranging, if we set at the same time $s \sin t_1 = y_2 - y_1$ and $s \cos t_1 = x_2 - x_1$:

$$\begin{aligned} \delta_1 = \frac{(x_2 - x_1)(2 y_1 + y_2)}{6 r^2} + \frac{(x_2 - x_1)^3}{360 r^4} (8 y_1 + 7 y_2) - \frac{(x_2 - x_1)}{360 r^4} (8 y_1^3 + 21 y_1^2 y_2 \\ + 24 y_1 y_2^2 + 7 y_2^3) \end{aligned} \tag{23}$$

and δ_2 accordingly with interchanged 1 and 2:

$$\delta_2 = \frac{(x_1 - x_2)(y_1 + 2 y_2)}{6 r^2} + \frac{(x_1 - x_2)^3}{360 r^4}(7 y_1 + 8 y_2) - \frac{(x_1 - x_2)}{360 r^4}(8 y_2^3 + 21 y_2^2 y_1 + 24 y_2 y_1^2 + 7 y_1^3) . \tag{23a}$$

Integration for the length S of the spherical arc

We have to distinguish three different lengths: the length of arc S on the sphere, the straight line s = straight line AB of the projection and the curve of length s' = curve AB of the projection (cf. Fig. 2, p. 211).

In the differential sense there exists the equation:

$$m = \frac{d s'}{d S} \quad \text{or} \quad d S = \frac{1}{m} d s' ,$$

hence also:

$$S = \int \frac{1}{m} d s' . \tag{24}$$

We thereby have according to previous development (9), p. 210:

$$\frac{1}{m} = 1 - \frac{y^2}{2 r^2} + \frac{5 y^4}{24 r^4} . \tag{25}$$

The differential ds', which in the case of the previous development of section 68 to an accuracy of $\frac{1}{r^2}$ inclusive could simply be set $= d\xi$, must now be indicated more accurately:

$$d s' = \sqrt{d \xi^2 + d \eta^2} = d \xi \left(1 + \frac{1}{2}\left(\frac{d \eta}{d \xi}\right)^2\right) .$$

Since $\frac{d\eta}{d\xi}$ is already $= \frac{1}{r^2} \ldots$, hence $\left(\frac{d\eta}{d\xi}\right)^2 = \frac{1}{r^4}$, we see immediately that the integral (24) is broken down into two parts:

$$S = \int_0^s \frac{1}{m} d \xi + \int_0^s \frac{1}{2}\left(\frac{d \eta}{d \xi}\right)^2 d \xi = \mathrm{I} + \mathrm{II} . \tag{26}$$

First considering the first integral, we must transform series (25) into a series with increasing powers of ξ.

In this connection, we have from (15) the series of y, which squared yields:

$$\left.\begin{aligned} y^2 = y_1^2 &+ \xi \left(2 y_1 \sin t_1 + \frac{y_1 s \cos^2 t_1}{3 r^2}(2 y_1 + y_2)\right) \\ &+ \xi^2 \left(\sin^2 t_1 - \frac{\cos^2 t_1}{3 r^2}(5 y_1^2 - y_1 y_2 - y_2^2)\right) \\ &- \xi^3 \frac{4}{3} \frac{y_1}{r^2} \sin t_1 \cos^2 t_1 - \xi^4 \frac{1}{3 r^2} \sin^2 t_1 \cos^2 t_1 + \frac{1}{r^4} \ldots \end{aligned}\right\} \tag{27}$$

and further:

$$y^4 = y_1^4 + \xi\, 4\, y_1^3 \sin t_1 + \xi^2\, 6\, y_1^2 \sin^2 t_1 + \xi^3\, 4\, y_1 \sin^3 t_1 + \xi^4 \sin^4 t_1 \,. \tag{28}$$

If we combine the expression (25) with these and arrange according to powers of ξ, there results:

$$\frac{1}{m} = 1 - \frac{y^2}{2\,r^2} + \frac{5}{24}\frac{y^4}{r^4} = \alpha + \beta\,\xi + \gamma\,\xi^2 + \delta\,\xi^3 + \varepsilon\,\xi^4\,,$$

hence the first integral part of (25):

$$\mathrm{I} = \alpha\, s + \beta\,\frac{s^2}{2} + \gamma\,\frac{s^3}{3} + \delta\,\frac{s^4}{4} + \varepsilon\,\frac{s^5}{5}\,.$$

To this, we must collect the parts from (27) and (28), whereby we obtain:

$$\begin{aligned}\frac{\mathrm{I}}{s} = {} & 1 - \frac{y_1^2}{2\,r^2} - \frac{y_1\, s \sin t_1}{2\,r^2} - \frac{s^2 \sin^2 t_1}{6\,r^2}\\ & + \frac{1}{12\,r^4}\left(-\,y_1 \cos^2 t_1\,(2\,y_1 + y_2) + 5\,y_1^3\, s \sin t_1\right)\\ & + \frac{1}{18\,r^4}\, s^2 \cos^2 t_1\,(5\,y_1^2 - y_1\,y_2 - y_2^2) + \frac{5}{12\,r^4}\, y_1^2\, s^2 \sin^2 t_1\\ & + \frac{1}{6\,r^4}\, y_1\, s^3 \sin t_1 \cos^2 t_1 + \frac{5}{24\,r^4}\, y_1\, s^3 \sin^3 t_1\\ & + \frac{1}{30\,r^4}\, s^3 \sin^2 t_1 \cos t_1 + \frac{1}{24\,r^4}\, s^4 \sin^4 t_1\,.\end{aligned}$$

If we set here everywhere $s \sin t_1 = y_2 - y_1$ and $s \cos t_1 = x_2 - x_1$ and collect the homogeneous parts, then we find:

$$\left.\begin{aligned}\frac{\mathrm{I}}{s} = 1 - \frac{y_1^2 + y_1\,y_2 + y_2^2}{6\,r^2} - \frac{(x_2 - x_1)^2}{360\,r^4}\,(8\,y_1^2 + 14\,y_1\,y_2 + 8\,y_2^2)\\ + \frac{1}{24\,r^4}\,(y_1^4 + y_1^3\,y_2 + y_1^2\,y_2^2 + y_1\,y_2^3 + y_2^4)\,.\end{aligned}\right\} \tag{29}$$

In order to determine also the second part of the integral (26), we must refer to (35), p. 203, and take:

$$\begin{aligned}\frac{d\eta}{d\xi} &= \frac{s \cos t_1}{6\,r^2}\,(2\,y_1 + y_2) - \frac{\xi}{r^2}\, y_1 \cos t_1 - \frac{\xi^2}{2\,r^2} \sin t_1 \cos t_1\\ \frac{1}{2}\left(\frac{d\eta}{d\xi}\right)^2 &= \frac{\cos^2 t_1}{72\,r^4}\left\{\begin{aligned}s^2(2y_1+y_2)^2 - \xi\, 12\, s\, y_1\,(2y_1+y_2) + \xi^2\, 36\, y_1^2 - \xi^2\, 6\, s \sin t_1\,(2y_1+y_2)\\ + \xi^3\, 36\, y_1 \sin t_1 + \xi^4\, 9 \sin^2 t_1\end{aligned}\right\}.\end{aligned}$$

This integrated, yields with $s \sin t_1 = y_2 - y_1$:

$$\frac{\mathrm{II}}{s} = -\frac{s^2 \cos^2 t_1}{72\,r^4}\left\{\begin{aligned}(2\,y_1 + y_2)^2 - 6\,y_1\,(2\,y_1 + y_2) + 12\,y_1^2 - 2\,(y_2 - y_1)\,(2\,y_1 + y_2)\\ + 9\,y_1\,(y_2 - y_1) + \frac{9}{5}\,(y_2 - y_1)^2\end{aligned}\right\}.$$

All this assembled is greatly simplified and finally yields:

$$\frac{\mathrm{II}}{s} = \frac{(x_2 - x_1)^2}{360\, r^4} \left\{ 4\, y_1{}^2 + 7\, y_1\, y_2 + 4\, y_2{}^2 \right\}. \tag{30}$$

If we collect the parts I and II of (29) and (30), we have according to (26):

$$\left.\begin{aligned} \frac{S}{s} = \frac{\mathrm{I}}{s} + \frac{\mathrm{II}}{s} = 1 - \frac{y_1{}^2 + y_1\, y_2 + y_2{}^2}{6\, r^2} - \frac{(x_2 - x_1)^2}{360\, r^4} (4\, y_1{}^2 + 7\, y_1\, y_2 + 4\, y_2{}^2) \\ + \frac{1}{24\, r^4} (y_1{}^4 + y_1{}^3\, y_2 + y_1{}^2\, y_2{}^2 + y_1\, y_2{}^2 + y_2{}^4)\,. \end{aligned}\right\} \tag{31}$$

We can bring into another form the first two terms and the last term of the above equation, if we introduce the mean ordinate y_0 according to the equations:

$$\begin{aligned} y_0 &= \frac{y_1 + y_2}{2} \\ y_0{}^2 &= \left(\frac{y_1 + y_2}{2}\right)^2 = \frac{1}{4} (y_1{}^2 + 2\, y_1\, y_2 + y_2{}^2) \\ y_0{}^4 &= \left(\frac{y_1 + y_2}{2}\right)^4 = \frac{1}{16} (y_1{}^4 + 4\, y_1{}^3 y_2 + 6\, y_1{}^2\, y_2{}^2 + 4\, y_1\, y_2{}^3 + y_2{}^4)\,. \end{aligned}$$

In this connection, we write the first two terms of (31) in the form

$$1 - \frac{y_1{}^2 + y_2{}^2 + (y_1 + y_2)^2}{12\, r^2} = 1 - \frac{y_1{}^2 + 4\, y_0{}^2 + y_2{}^2}{12\, r^2}\,. \tag{32}$$

On the other hand, the last term can be broken down into

$$\begin{aligned} &+ \frac{5}{576\, r^4} (5\, y_1{}^4 + 4\, y_1{}^3\, y_2 + 6\, y_1{}^2\, y_2{}^2 + 4\, y_1\, y_2{}^3 + 5\, y_2{}^4) \\ &- \frac{1}{576\, r^4} (y_1{}^4 - 4\, y_1{}^3\, y_2 + 6\, y_1{}^2\, y_2{}^2 - 4\, y_1\, y_2{}^3 + y_2{}^4) \end{aligned}$$

or

$$\begin{aligned} &+ \frac{5}{576\, r^4} (4\, y_1{}^4 + 4\, y_2{}^4 + (y_1 + y_2)^4) - \frac{1}{576\, r^4} (y_2 - y_1)^4 \\ &= \frac{5}{144\, r^4} (y_1{}^4 + 4\, y_0{}^4 + y_2{}^4) - \frac{1}{576\, r^4} (y_2 - y_1)^4\,. \end{aligned} \tag{33}$$

With these, (32) and (33) yield together

$$1 - \frac{y_1{}^2 + 4\, y_0{}^2 + y_2{}^2}{12\, r^2} + \frac{5}{144\, r^4} (y_1{}^4 + 4\, y_0{}^4 + y_2{}^4) - \frac{5}{2880} \frac{(y_2 - y_1)^4}{r^4}\,. \tag{34}$$

Now we will introduce, in addition, the values of $\frac{1}{m}$ corresponding to the starting ordinate, the mean ordinate and the end ordinate, namely

$$\frac{1}{m_1} = 1 - \frac{y_1{}^2}{2\, r^2} + \frac{5}{24} \frac{y_1{}^4}{r^4} \qquad \frac{1}{m_2} = 1 - \frac{y_2{}^2}{2\, r^2} + \frac{5}{24} \frac{y_2{}^4}{r^4} \qquad \frac{1}{m_0} = 1 - \frac{y_0{}^2}{2\, r^2} + \frac{5}{24} \frac{y_0{}^4}{r^4}$$

If we set this into (34), we will have

$$\frac{S}{s} = \frac{1}{6}\left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2}\right) - \frac{(x_2 - x_1)^2}{360\, r^4}(4\, y_1^2 + 7\, y_1\, y_2 + 4\, y_2^2) - \frac{5\,(y_2 - y_1)^4}{2880\, r^4}. \qquad (35)$$

The result of all previous developments and considerations is contained in the two equations (23) and (23a) for the reduction of direction and in the final equation (35) for the reduction of distance. If we omit the terms with $\frac{1}{r^4}$, the formulae go back again to the former formulae (31), (32) and (15) in section 68, pp. 202 and 199.

Introduction of approximations for relatively small $x_2 - x_1$'s and $y_2 - y_1$'s

If in a very extended system the triangulation sides are relatively small compared to the ordinates themselves, we can distinguish the terms of fourth order, i.e. the terms with $\frac{1}{r^4}$, into such in which the powers of y themselves or only powers of $x_2 - x_1$ and $y_2 - y_1$ are predominant, and we can neglect the latter terms compared to the first.

We shall pursue this more closely in connection with a treatise by Lt. Col. von Schmidt, Chief, Trigonometric Division of the Land Survey, in *Zeitschr. f. Verm.*, 1894, pp. 399-400, and changing the denotations, which in part deviate there, to ours (Fig. 3), we have there (7), 1894, p. 399, and (8), p. 400:

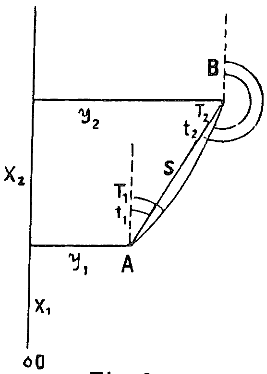

Fig. 3.

$$\log s - \log S = \frac{\mu}{8\, r^2}(y_1 + y_2)^2 - \frac{\mu}{24\, r^2}(y_2 - y_1)^2 - \frac{\mu}{192\, r^4}(y_1 + y_2)^4 \qquad (36)$$

$$T_1 - t_1 = \frac{\varrho}{4\, r^2}(y_1 + y_2)(x_2 - x_1) - \frac{\varrho}{12\, r^2}(y_2 - y_1)(x_2 - x_1) - \frac{\varrho}{48\, r^4}(y_1 + y_2)^3 (x_2 - x_1). \qquad (37)$$

It is not difficult to prove these formulae as simplifications of our formulae (31) and (36). If we take at first (31) neglecting the term $\frac{(x^2 - x_1)^2}{r^4}$ . . . and introducing the mean value $\frac{y_1 + y_2}{2}$ instead of y_1 and y_2 in the last term of (31), then we have from there:

$$\frac{S}{s} = 1 - \frac{y_1^2 + y_1\, y_2 + y_2^2}{6\, r^2} + \frac{1}{24\, r^4}\, 5\left(\frac{y_2 + y_2}{2}\right)^4.$$

We have according to the logarithmic series, p. 21, if we set again in the terms with $\frac{1}{r^4}$ the mean value $\frac{y_1 + y_2}{2}$ for y_1 and y_2:

$$l\left(\frac{S}{s}\right) = -\frac{y_1^2 + y_1\, y_2 + y_2^2}{6\, r^2} + \frac{5}{384\, r^4}(y_1 + y_2)^4 - \frac{1}{2}\left(\frac{3}{6\, r^2}\left(\frac{y_1 + y_2}{2}\right)^2\right)^2$$

$$l\, S - l\, s = -\frac{y_1^2 + y_1\, y_2 + y_2^2}{6\, r^2} + \frac{1}{192\, r^4}(y_1 + y_2)^4\, . \qquad (38)$$

The last term here agrees with the last term of (33), and since the first two terms of (33) prove to be algebraically identical also with the first term of (35) and the logarithmic modulus $l\mu$ in the symbols $\log s$ and $l\, s$, etc., is justified, we have proven now the formula (33) as a simplification of (31).

It is still easier to understand how (37) results from (23), by neglecting the term $\frac{(x_2 - x_1)^3}{360 r^4} \ldots$ in (23) and setting in the last term of (23) the parenthesis $= 60\left(\frac{y_1 + y_2}{2}\right)^3$. Also the fact that the first two terms of (37) are identical with the one first term of (23) was already noted in section 68, p. 202.

Gauss' conformal projection of the spherical surface on the plane, which we have treated above in an elementary way, played an important role at the Prussian Land Survey from 1876 to 1927. There they used a conformal double projection introduced by Schreiber for the adjustment of the triangle points of second to fourth order, as already mentioned briefly on p. 205;* in this case, there was carried out at first a conformal transformation of the terrestrial ellipsoid on the sphere, which we shall treat in Volume III, second half, and then a conformal transformation of the sphere on the plane in accordance with the above developments. The mean radius of curvature of the terrestrial ellipsoid at the latitude 52°42'.2.53251", which is given by $\log r = 6.805\ 0274 \cdot 003$, was valid as spherical radius.

The fundamental work, *Die konforme Doppelprojektion der trigonometrischen Abteilung der Kgl. Preussischen Landesaufnahme*, Formeln und Tafeln, by Dr. O. Schreiber, Berlin, 1897, was used as auxiliary table for these computations.

We shall come once again to the theory of the conformal double projection in Volume III, second half.

The Gauss conformal projection is to be denoted as transversal conformal cylinder projection, as follows from the representations at the beginning of section 68, p. 196. However, the formulae developed above hold true also for any other conformal cylinder projection, where then only the great circle of abscissae, i.e., the great circle at which the cylinder is tangent to the sphere, receives another position. If the tangency takes place at the equator, then we obtain a normal conformal cylinder projection, which is known under the name of *Mercator's projection.*

A treatise which Schols publishes under the title, "Sur l'emploi de la projection de Mercator pour le calcul d'une triangulation dans le voisinage de l'équateur" in the *Annales de l'école polytechnique de Delft,* 1, livraison, Leide, 1884, contains a thorough, mathematically elegant, development of the formulae of the Mercator projection. A detailed report which reproduces the most essential of the theories by Schols is given by Jordan in *Zeitschr. f. Verm.*, 1898, pp. 33-43 and pp. 417-423.

The *oblique-axial* conformal cylinder projection, which forms the basis of the Swiss Land Survey, is treated in detail in *Die Änderung des Projektionssystems der schweizerischen Landesvermessung.* Im Auftrage der Abteilung für Landestopographie des schweizerischen Militärdepartements bearbeitet von Ingenieur M. Rosenmund, Bern, 1903.

A more recent development of the basic formulae of the Gauss conformal representation of the spherical surface is given by Epstein in *Zeitschr. f. Verm.*, 1921, pp. 673-683.

Section 71. The Stereographic Projection

We have already learned the basic formulae for the computation of the rectangular plane coordinates from the geographic coordinates for the oblique-axial stereographic projection in section 65, p. 187. Before we pursue further the results found there, we shall investigate the image distortions which occur in the case of the stereographic projection.

In Fig. 1, there is constructed, at the point M, the image plane as tangent plane to the sphere, while the perspective center lies at C. A point P of the sphere with the angular distance δ from M is projected at P'. If we imagine the angle δ enlarged by $d\delta$, so that P comes to Q, then $P'Q'$ is the projection of the arc PQ.

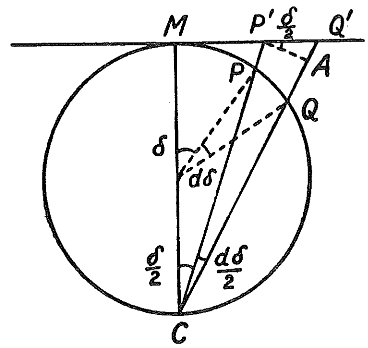

Fig. 1.

From Fig. 1 there follows

$$CP' = \frac{2r}{\cos\frac{\delta}{2}} \qquad P'A = CP'\frac{d\delta}{2} = \frac{r}{\cos\frac{\delta}{2}}\,d\delta$$

$$PQ = r\,d\delta.$$

Then we have further

$$P'Q' = \frac{P'A}{\cos\frac{\delta}{2}} = \frac{r}{\cos^2\frac{\delta}{2}}\,d\delta,$$

* Not translated.

and if we set the scale factor

$$\frac{P'Q'}{PQ} = h$$

then we will have

$$h = \frac{1}{\cos^2 \frac{\delta}{2}}. \tag{1}$$

On the other hand, we imagine on the sphere at the point P an element of arc PR perpendicular to PQ, and let a point R' be the projection of the point R on the plane so that the three points CRR' lie on a straight line. Setting then the scale factor

$$\frac{P'R'}{PR} = k,$$

we have

$$k = \frac{CP'}{CP} = \frac{\frac{2r}{\cos\frac{\delta}{2}}}{2r\cos\frac{\delta}{2}} \quad \text{hence} \quad k = \frac{1}{\cos^2\frac{\delta}{2}}. \tag{2}$$

Hence, there follows for the stereographic projection $h = k$.

On the sphere, the three points PQR form a small right triangle with the right angle at P. The corresponding triangle $P'Q'R'$ on the image plane is likewise rectangular, and it follows from the above that

$$\frac{P'Q'}{P'R'} = \frac{PQ}{PR}; \tag{3}$$

hence, that the two triangles are similar to one another.

We have thus found the important result that the stereographic projection is a conformal or true-angle projection.

We shall investigate this question also from another side, by treating the stereographic projection as a zenithal projection according to section 66, p. 189.

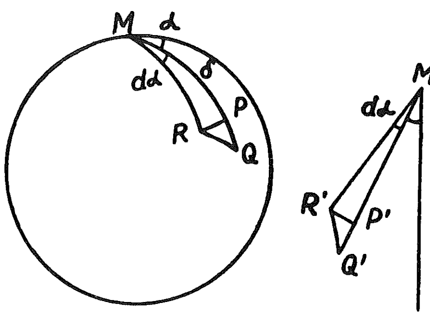

Fig. 2.

If we assume that in Fig. 2 the arc of the great circle $MP = \delta$ has the azimuth α at the point M, then we obtain, by enlarging δ by $d\delta$ the point Q, and by enlarging α by $d\alpha$ the point R, the same points which we have already considered above. There originates hereby on the sphere the small right triangle PQR, whose legs are

$$PQ = r\,d\delta \qquad PR = r \sin\delta\, d\alpha. \tag{4}$$

On the plane we have in the case of the stereographic projection

$$f(\delta) = MP' = 2r \tan\frac{\delta}{2}.$$

We shall denote this by p so that therefore

$$p = 2r \tan\frac{\delta}{2}. \tag{5}$$

If we enlarge p by dp and α by $d\alpha$, then we obtain on the plane likewise a small right triangle $P'Q'R'$ with the legs

$$P'Q' = dp \qquad P'R' = p\,d\alpha. \tag{6}$$

From (5) there results

$$dp = \frac{r}{\cos^2 \frac{\delta}{2}}\, d\delta,$$

hence we have

$$P'Q' = \frac{r}{\cos^2 \frac{\delta}{2}}\, d\delta \qquad P'R' = 2\, r \tan \frac{\delta}{2}\, d\alpha. \tag{7}$$

Then there follow from (4) and (7) the two quotients

$$\frac{P'Q'}{PQ} = \frac{1}{\cos^2 \frac{\delta}{2}} \qquad \frac{P'R'}{PR} = \frac{1}{\cos^2 \frac{\delta}{2}},$$

whence equation (3) results again. Hence, the conformality of the stereographic projection has been proven also in this way.

Projection of a circle

In Fig. 3 in the margin, let P_1 and P_2 be two points on the sphere which have the distance δ_1 and δ_2 from P_0 and the difference of azimuth $\Delta\alpha$. Let the distance P_1P_2 be equal to σ. If we retain the position of the point P_1 and with σ constant, change the two magnitudes $\Delta\alpha$ and δ_2, then P_2 describes on the sphere a circle around P_1 with the radius σ.

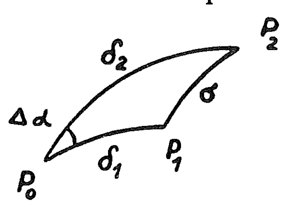

Fig. 3.

Between the four magnitudes denoted in Fig. 3 there exists the equation

$$\cos\delta_1 \cos\delta_2 + \sin\delta_1 \sin\delta_2 \cos\Delta\alpha = \cos\sigma. \tag{8}$$

For simplification, we shall denote here the invariable magnitudes in simple form and set in this connection

$$\cos\delta_1 = c_1 \qquad \sin\delta_1 = s_1 \qquad \cos\sigma = c, \tag{9}$$

with which we obtain

$$c_1 \cos\delta_2 + s_1 \sin\delta_2 \cos\Delta\alpha = c,$$

or else

$$c_1\left(\cos^2\frac{\delta_2}{2} - \sin^2\frac{\delta_2}{2}\right) + 2\, s_1 \sin\frac{\delta_2}{2}\cos\frac{\delta_2}{2}\cos\Delta\alpha = c\,.$$

It follows therefrom

$$c_1\left(1 - \tan^2\frac{\delta_2}{2}\right) + 2\, s_1 \tan\frac{\delta_2}{2}\cos\Delta\alpha = \frac{c}{\cos^2\frac{\delta_2}{2}} = c\left(1 + \tan^2\frac{\delta_2}{2}\right). \tag{10}$$

Now if we change to the plane projection of the points P_1 and P_2, then the angle $\Delta\alpha$ remains unchanged. We have further according to (5)

$$P_0 P'_2 = p_2 = 2r \tan \frac{\delta_2}{2} \qquad \text{and hence} \qquad \tan \frac{\delta_2}{2} = \frac{p_2}{2r}. \tag{11}$$

If we set this into (10), then we obtain

$$c_1 \left(1 - \frac{p_2^2}{4r^2}\right) + s_1 \frac{p_2}{r} \cos \Delta\, \alpha = c\left(1 + \frac{p_2^2}{4r^2}\right)$$

and thence we find immediately

$$\frac{c_1 + c}{4r^2} p_2^2 - \frac{s_1}{r} p_2 \cos \Delta\alpha - (c_1 - c) = 0\,. \tag{12}$$

This is the equation of a circle with the polar coordinates p_2 and $\Delta\alpha$. We have thus proved that any arbitrary circle on the spherical surface is projected on the plane again as a circle.

Conversion of the basic equations of the stereographic projection

For the computation of the rectangular coordinates on the image plane from the spherical geographic coordinates, we have found in (9), section 65, p. 187, the basic equations

$$\left.\begin{aligned} x &= \frac{2r}{N}\left(\cos\varphi \sin\varphi_0 \cos\lambda - \sin\varphi \cos\varphi_0\right) \\ y &= \frac{2r}{N}\cos\varphi \sin\lambda \\ N &= 1 + \cos\varphi\cos\varphi_0\cos\lambda + \sin\varphi\sin\varphi_0\,. \end{aligned}\right\} \tag{13}$$

These expressions are not convenient for numerical application, and for this reason we shall carry out further conversions. First we can write the denominator in the following form, as is easy to understand

$$N = \frac{1}{2}\left(1 + \cos(\varphi - \varphi_0)\right)\left(1 + \cos\lambda\right) + \frac{1}{2}\left(1 - \cos(\varphi - \varphi_0)\right)\left(1 - \cos\lambda\right)$$

or

$$N = 2\cos^2\frac{\varphi - \varphi_0}{2}\cos^2\frac{\lambda}{2} + 2\sin^2\frac{\varphi + \varphi_0}{2}\sin^2\frac{\lambda}{2}\,.$$

Now if we introduce an auxiliary angle ω, by setting

$$\tan\frac{\omega}{2} = \frac{\sin\dfrac{\varphi + \varphi_0}{2}}{\cos\dfrac{\varphi - \varphi_0}{2}}\tan\frac{\lambda}{2} \tag{14}$$

so that we will have therefore

$$\sin^2\frac{\varphi + \varphi_0}{2} = \cos^2\frac{\varphi - \varphi_0}{2}\ \tan^2\frac{\omega}{2}\cot^2\frac{\lambda}{2}$$

then we obtain for the denominator the expression

$$N = 2\,cos^2 \frac{\varphi - \varphi_0}{2} \frac{cos^2 \dfrac{\lambda}{2}}{cos^2 \dfrac{\omega}{2}}$$

and with this, we will have

$$y = r \frac{cos\,\varphi\; sin\,\lambda}{cos^2 \dfrac{\varphi - \varphi_0}{2}\, cos^2 \dfrac{\lambda}{2}}\, cos^2 \frac{\omega}{2} \tag{15}$$

or else

$$y = 2\,r \frac{cos\,\varphi}{sin \dfrac{\varphi + \varphi_0}{2}\, cos \dfrac{\varphi - \varphi_0}{2}}\, tan\, \frac{\omega}{2}\, cos^2 \frac{\omega}{2}\,. \tag{16}$$

For the conversion of the expression for x in (8) we have

$$\begin{aligned} x &= \frac{r}{N} \Big(sin\,(\varphi + \varphi_0)\,(1 - cos\,\lambda) + sin\,(\varphi - \varphi_0)\,(1 + cos\,\lambda) \Big) \\ &= \frac{2\,r}{N} \left(sin\,(\varphi + \varphi_0)\, sin^2 \frac{\lambda}{2} + sin\,(\varphi - \varphi_0)\, cos^2 \frac{\lambda}{2} \right) \\ &= 2\,r \left\{ \frac{sin \dfrac{\varphi + \varphi_0}{2}\, cos \dfrac{\varphi + \varphi_0}{2}\, sin^2 \dfrac{\lambda}{2}\, cos^2 \dfrac{\omega}{2}}{cos^2 \dfrac{\varphi - \varphi_0}{2}\, cos^2 \dfrac{\lambda}{2}} + \frac{sin \dfrac{\varphi - \varphi_0}{2}\, cos^2 \dfrac{\omega}{2}}{cos \dfrac{\varphi - \varphi_0}{2}} \right\} \\ &= 2\,r \left\{ \frac{sin \dfrac{\varphi + \varphi_0}{2}\, cos \dfrac{\varphi + \varphi_0}{2}}{cos^2 \dfrac{\varphi - \varphi_0}{2}}\, tan^2\, \frac{\lambda}{2}\, cos^2 \frac{\omega}{2} = tan\, \frac{\varphi - \varphi_0}{2}\, cos^2 \frac{\omega}{2} \right\} \end{aligned}$$

Now since

$$tan^2\, \frac{\lambda}{2} = \frac{cos^2 \dfrac{\varphi - \varphi_0}{2}}{sin^2 \dfrac{\varphi + \varphi_0}{2}}\, tan^2\, \frac{\omega}{2}$$

we will have

$$\begin{aligned} x &= 2\,r \left(cot\, \frac{\varphi + \varphi_0}{2}\, sin^2 \frac{\omega}{2} + tan\, \frac{\varphi - \varphi_0}{2}\, cos^2 \frac{\omega}{2} \right) \\ &= 2\,r \left(tan\, \frac{\varphi - \varphi_0}{2} + sin^2 \frac{\omega}{2} \left(cot\, \frac{\varphi + \varphi_0}{2} - tan\, \frac{\varphi - \varphi_0}{2} \right) \right) \\ &= 2\,r \left(tan\, \frac{\varphi - \varphi_0}{2} + sin^2 \frac{\omega}{2}\, \frac{cos\,\varphi}{sin\, \dfrac{\varphi + \varphi_0}{2}\, cos \dfrac{\varphi - \varphi_0}{2}} \right). \end{aligned}$$

But this is according to (10)

$$x = 2\,r\,tan\, \frac{\varphi - \varphi_0}{2} + y\,tan\, \frac{\omega}{2}\,. \tag{17}$$

The formulae (15) to (17) with the auxiliary angle ω in (14) were developed by Hk. J. Heuvelink in *Nederlandsche Rijksdriehoeksmeting.* De stereografische kaartprojectie in hare toepassing bij de Rijksdriehoeksmeting. Delft, 1918.

A further direct computation results for x and y, if we compute the angular distance δ of the point P from M in Fig. 1, p. 217, as well as the azimuth of the arc of great circle MP, and for this, e.g. the formulae (3) in section 65, p. 186, can be used. Denoting the azimuth by α, we will have

$$y = 2\,r \tan \frac{\delta}{2} \sin \alpha$$
$$x = 2\,r \tan \frac{\delta}{2} \cos \alpha . \tag{18}$$

Section 72. Development in Series for the Stereographic Projection

Although the above closed formulae are not inconvenient for numerical computation, we still shall pass over to developing series for x and y according to increasing powers of $\varphi - \varphi_0 = \Delta\varphi$ and λ. In this connection, we start best from the equations (14) to (17), section 71, pp. 220-221. If we limit the development in series to the terms of fourth order, then we have first to form for $\tan \frac{\omega}{2}$ according to (14), section 71, p. 220,

$$\sin \frac{\varphi + \varphi_0}{2} = \sin\left(\varphi_0 + \frac{\Delta\varphi}{2}\right) = \sin \varphi_0 + \frac{1}{2} \cos \varphi_0 \,\Delta\varphi - \frac{1}{8} \sin \varphi_0 \,\Delta\varphi^2 - \frac{1}{48} \cos \varphi_0 \,\Delta\varphi^3 + \dots$$

$$\tan \frac{\lambda}{2} = \frac{1}{2}\lambda + \frac{1}{24}\lambda^3 + \dots$$

$$\sec \frac{\varphi - \varphi_0}{2} = 1 + \frac{1}{8}\Delta\varphi^2 + \dots$$

With these, there follows immediately according to equation (14), section 71, p. 220,

$$\tan \frac{\omega}{2} = \frac{1}{2} \sin \varphi_0 \,\lambda + \frac{1}{4} \cos \varphi_0 \,\Delta\varphi\,\lambda + \frac{1}{24} \sin \varphi_0 \,\lambda^3 + \frac{1}{48} \cos \varphi_0 \,\Delta\varphi^3 \lambda + \frac{1}{48} \cos \varphi_0 \,\Delta\varphi\,\lambda^3 + \dots \tag{1}$$

For the development of y we need further $\cos^2 \frac{\omega}{2} = \dfrac{1}{1 + \tan^2 \frac{\omega}{2}}$. According to equation (1) we have

$$1 + \tan^2 \frac{\omega}{2} = 1 + \frac{1}{4} \sin^2 \varphi_0 \,\lambda^2 + \frac{1}{4} \sin \varphi_0 \cos \varphi_0 \,\Delta\varphi\,\lambda^2 + \dots$$

hence

$$\cos^2 \frac{\omega}{2} = 1 + \frac{1}{4} \sin^2 \varphi_0 \,\lambda^2 - \frac{1}{4} \sin \varphi_0 \cos \varphi_0 \,\Delta\varphi\,\lambda^2 + \dots$$

Finally, we form, in addition, the following expressions

$$\cos \varphi = \cos(\varphi_0 + \Delta\varphi) = \cos \varphi_0 - \sin \varphi_0 \,\Delta\varphi - \frac{1}{2} \cos \varphi_0 \,\Delta\varphi^2 + \frac{1}{6} \sin \varphi_0 \,\Delta\varphi^3 + \dots$$

$$\sin \lambda = \lambda - \frac{1}{6}\lambda^3 + \dots$$

$$\sec^2 \frac{\varphi - \varphi_0}{2} = 1 + \frac{1}{4}\Delta\varphi^2 + \dots \qquad \sec^2 \frac{\lambda}{2} = 1 + \frac{1}{4}\lambda^2 + \dots$$

All this yields, introduced into (15), p. 221, after easy collections, if we add at the same time ρ:

$$y = r\left\{\frac{1}{\varrho}\cos\varphi_0\,\lambda - \frac{1}{\varrho^2}\sin\varphi_0\,\Delta\varphi\,\lambda - \frac{1}{4\varrho^3}\cos\varphi_0\,\Delta\varphi^2\lambda + \frac{1}{12\varrho^3}\cos\varphi_0\,(1-3\sin^2\varphi_0)\,\lambda^3 - \frac{1}{12\varrho^4}\sin\varphi_0\,\Delta\varphi^3\lambda + \frac{1}{6\varrho^4}\sin\varphi_0\,(1-3\cos^2\varphi_0)\,\Delta\varphi\,\lambda^3\right\}.$$

If we introduce, in addition, $\tan\varphi_0 = t_0$ instead of $\sin\varphi_0$, then we obtain finally

$$y = \frac{r}{\varrho}\lambda\cos\varphi_0 - \frac{r}{\varrho^2}\Delta\varphi\,\lambda\cos\varphi_0\,t_0 + \frac{1}{12}\frac{r}{\varrho^3}\lambda^3\cos^3\varphi_0\,(1-2t_0^2) - \frac{1}{4}\frac{r}{\varrho^3}\Delta\varphi^2\lambda\cos\varphi_0 - \frac{1}{12}\frac{r}{\varrho^4}\Delta\varphi^3\lambda\cos\varphi_0\,t_0 - \frac{1}{6}\frac{r}{\varrho^4}\Delta\varphi\,\lambda^3\cos^3\varphi_0\,t_0\,(2-t_0^2). \tag{2}$$

For the development of x according to equation (17), section 71, p. 221, we have to form first the product $y\tan\frac{\omega}{2}$ from (1) and (2), for which we find immediately

$$y\tan\frac{\omega}{2} = r\cos\varphi_0\left\{\frac{1}{2}\sin\varphi_0\,\lambda^2 + \frac{1}{4}\cos\varphi_0\,(1-2t_0^2)\,\Delta\varphi\,\lambda^2 - \frac{3}{8}\sin\varphi_0\,\Delta\varphi^2\lambda^2 + \frac{1}{24}\sin\varphi_0\cos^2\varphi_0\,(2-t_0^2)\,\lambda^4\right\}$$

But we have, on the other hand,

$$2r\tan\frac{\varphi-\varphi_0}{2} = r\left(\Delta\varphi + \frac{1}{12}\Delta\varphi^3\right)$$

and this yields together

$$x = r\left\{\Delta\varphi + \frac{1}{2}\sin\varphi_0\cos\varphi_0\,\lambda^2 + \frac{1}{12}\Delta\varphi^3 + \frac{1}{4}\cos^2\varphi_0\,(1-2t_0^2)\,\Delta\varphi\,\lambda^2 - \frac{3}{8}\sin\varphi_0\cos\varphi_0\,\Delta\varphi^2\lambda^2 + \frac{1}{24}\sin\varphi_0\cos^3\varphi_0\,(2-t_0^2)\,\lambda^4\right\}.$$

If we add the necessary ρ's and replace $\sin\varphi_0$ by $\tan\varphi_0 = t_0$, then we obtain finally

$$x = \frac{r}{\varrho}\Delta\varphi + \frac{1}{2}\frac{r}{\varrho^2}\lambda^2\cos^2\varphi_0\,t_0 + \frac{1}{12}\frac{r}{\varrho^3}\Delta\varphi^3 + \frac{1}{4}\frac{r}{\varrho^3}\Delta\varphi\,\lambda^2\cos^2\varphi_0\,(1-2t_0^2) - \frac{3}{8}\frac{r}{\varrho^4}\Delta\varphi^2\lambda^2\cos^2\varphi_0\,t_0 + \frac{1}{24}\frac{r}{\varrho^4}\lambda^4\cos^4\varphi_0\,t_0\,(2-t_0^2). \tag{3}$$

The practical application of the series (2) and (3) is convenient, since the coefficients of $\Delta\varphi$ and λ are to be computed only once for each system of coordinates.

Developments in series for the conformal zenithal projection

After we have recognized in the above section 71, p. 219, that the stereographic projection coincides with the conformal zenithal projection, we shall carry out once again the developments in series for x and y by starting from the basic equation (5), section 71, p. 218, of the conformal zenithal projection. If we apply to this equation the development in series for $\tan\frac{\delta}{2}$, then we obtain by neglecting terms of fifth order

$$p = r\delta + r\frac{\delta^3}{12} + \ldots \qquad (4)$$

For the conversion of the geographic coordinates to rectangular coordinates we start from the end equations of the previous section 63, in which we are to set δ now instead of σ. According to (27) and (28), section 63, pp. 178-179, we have

$$\left.\begin{aligned} \varphi - \varphi_0 &= u - \frac{1}{2} v^2 t_0 - \frac{1}{6} v^2 u (1 + 3 t_0^2) + \frac{1}{24} v^4 t_0 (1 + 3 t_0^2) \\ &\quad - \frac{1}{6} v^2 u^2 t_0 (2 + 3 t_0^2) \\ \lambda \cos\varphi_0 &= v + v u t_0 - \frac{1}{3} v^3 t_0 + \frac{1}{3} v u^2 (1 + 3 t_0^2) \\ &\quad - \frac{1}{3} v^3 u t_0 (1 + 3 t_0^2) + \frac{1}{3} v u^3 t_0 (2 + 3 t_0^2) \end{aligned}\right\} \qquad (5)$$

where we set

$$\delta \sin\alpha = v \qquad \delta \cos\alpha = u \qquad \tan\varphi_0 = t_0. \qquad (6)$$

We shall invert the two equations (5) by the method of successive approximation so that u and v are expressed by $\varphi - \varphi_0$ and $\lambda \cos\varphi_0$. We set here again $\varphi - \varphi_0 = \Delta\varphi$.

As the first approximation we can put down

$$u = \Delta\varphi + \ldots \qquad v = \lambda\cos\varphi_0 + \ldots$$

and have then

$$u^2 = \Delta\varphi^2 + \ldots \qquad v^2 = \lambda^2 \cos^2\varphi_0 + \ldots \qquad u v = \Delta\varphi\,\lambda\cos\varphi_0 + \ldots$$

where only terms of third order are neglected. With the same accuracy we obtain then

$$u = \Delta\varphi + \frac{1}{2}\lambda^2\cos^2\varphi_0\, t_0 + \ldots \qquad v = \lambda\cos\varphi_0 - \Delta\varphi\,\lambda\cos\varphi_0\, t_0 + \ldots$$

We use this for the computation of the terms

$$\begin{aligned} u^2 &= \Delta\varphi^2 + \Delta\varphi\,\lambda^2\cos^2\varphi_0\, t_0 + \ldots \\ u v &= \Delta\varphi\,\lambda\cos\varphi_0 - \Delta\varphi^2\,\lambda\cos\varphi_0\, t_0 + \frac{1}{2}\lambda^3\cos^3\varphi_0\, t_0 + \ldots \\ v^2 &= \lambda^2\cos^2\varphi_0 - 2\,\Delta\varphi\,\lambda^2\cos^2\varphi_0\, t_0 + \ldots \end{aligned}$$

$$u^3 = \Delta\varphi^3 + \ldots \qquad u v^2 = \Delta\varphi\,\lambda^2\cos^2\varphi_0 + \ldots$$

$$v^3 = \lambda^3\cos^3\varphi_0 + \ldots \qquad u^2 v = \Delta\varphi^2\,\lambda\cos\varphi_0 + \ldots$$

where only terms of fourth order are neglected.

If we introduce these expressions into equations (5) again, then we obtain

$$\begin{aligned} u &= \Delta\varphi + \frac{1}{2}\lambda^2\cos^2\varphi_0\, t_0 + \frac{1}{6}\Delta\varphi\,\lambda^2\cos^2\varphi_0 (1 - 3 t_0^2) + \ldots \\ v &= \lambda\cos\varphi_0 - \Delta\varphi\,\lambda\cos\varphi_0\, t_0 - \frac{1}{3}\Delta\varphi^2\,\lambda\cos\varphi_0 - \frac{1}{6}\lambda^3\cos^3\varphi_0\, t_0 + \ldots \end{aligned}$$

With these, we can finally set up, taking into account the terms of fourth order,

$$\left.\begin{aligned}
u^2 &= \Delta\varphi^2 + \Delta\varphi\,\lambda^2 \cos^2\varphi_0\, t_0 + \frac{1}{3}\Delta\varphi^2\lambda^2\cos^2\varphi_0(1-3t_0^2) + \frac{1}{4}\lambda^4\cos^4\varphi_0\, t_0^2\\
u\,v &= \Delta\varphi\,\lambda\cos\varphi_0 - \Delta\varphi^2\lambda\cos\varphi_0 + \frac{1}{2}\lambda^3\cos^3\varphi_0\, t_0 - \frac{1}{3}\Delta\varphi^2\lambda\cos\varphi_0\\
&\qquad + \frac{1}{6}\Delta\varphi\,\lambda^3\cos^3\varphi_0\,(1-7t_0^2)\\
v^2 &= \lambda^2\cos^2\varphi_0 - 2\Delta\varphi\,\lambda^2\cos^2\varphi_0\, t_0 - \frac{1}{3}\Delta\varphi^2\lambda^2\cos^2\varphi_0\,(2-3t_0^2)\\
&\qquad - \frac{1}{3}\lambda^4\cos^4\varphi_0\, t_0^2\\
u^3 &= \Delta\varphi^3 + \frac{3}{2}\Delta\varphi^2\lambda^2\cos^2\varphi_0\, t_0\\
u^2 v &= \Delta\varphi^2\lambda\cos\varphi_0 - \Delta\varphi^3\lambda\cos\varphi_0\, t_0 + \Delta\varphi\,\lambda^3\cos^3\varphi_0\, t_0\\
u\,v^2 &= \Delta\varphi\,\lambda^2\cos^2\varphi_0 - 2\Delta\varphi^2\lambda^2\cos^2\varphi_0\, t_0 + \frac{1}{2}\lambda^4\cos^4\varphi_0\, t_0\\
v^3 &= \lambda^3\cos^3\varphi_0 - 3\Delta\varphi\,\lambda^3\cos^3\varphi_0\, t_0
\end{aligned}\right\} \quad (7)$$

and this yields with the equations (5) the final expressions for u and v

$$\left.\begin{aligned}
u &= \Delta\varphi + \frac{1}{2}\lambda^2\cos^2\varphi_0\, t_0 + \frac{1}{6}\Delta\varphi\,\lambda^2\cos^2\varphi_0\,(1-3t_0^2) - \frac{1}{3}\Delta\varphi^2\lambda^2\cos^2\varphi_0\, t_0\\
&\qquad + \frac{1}{24}\lambda^4\cos^4\varphi_0\, t_0\,(1-t_0^2)\\
v &= \lambda\cos\varphi_0 - \Delta\varphi\,\lambda\cos\varphi_0\, t_0 - \frac{1}{3}\Delta\varphi^2\lambda\cos\varphi_0 - \frac{1}{6}\lambda^3\cos^3\varphi_0\, t_0^2\\
&\qquad - \frac{1}{6}\Delta\varphi\,\lambda^3\cos^3\varphi_0\, t_0\,(1-t_0^2)\,.
\end{aligned}\right\} \quad (8)$$

u and v as well as $\Delta\varphi$ and λ are angular values. If we aim to pass to lengths, then the equations (8) must on both sides be multiplied by the radius r. Instead of the first two equations (6) we have then

$$r\,\delta\,\sin\alpha = r\,v \qquad r\,\delta\,\cos\alpha = r\,u. \quad (9)$$

If we multiply this by the quotients $\frac{p}{r\,\delta}$, then we will have

$$p\,\sin\alpha = \frac{p}{\delta}\,v \qquad p\,\cos\alpha = \frac{p}{\delta}\,u.$$

But the values $p\,\sin\alpha$ and $p\,\cos\alpha$ are nothing else but the rectangular coordinates y and x of the point P on the plane. Since we have, on the other hand, according to (4),

$$\frac{p}{\delta} = r\left(1+\frac{\delta^2}{12}+\ldots\right)$$

we will have

$$\left.\begin{aligned}
y &= r\,v\left(1+\frac{\delta^2}{12}+\ldots\right)\\
x &= r\,u\left(1+\frac{\delta^2}{12}+\ldots\right).
\end{aligned}\right\} \quad (10)$$

But according to (9) we have

$$\delta^2 = u^2 + v^2,$$

hence with the help of the expression (7)

$$\delta^2 = \Delta\varphi^2 + \lambda^2 \cos^2\varphi_0 - \Delta\varphi\lambda^2\cos^2\varphi_0 t_0 . \tag{11}$$

According to (10) we have thus to multiply the equations (8) for u and v by

$$r\left(1+\frac{1}{12}\Delta\varphi^2+\frac{1}{12}\lambda^2\cos^2\varphi_0-\frac{1}{12}\Delta\varphi\lambda^2\cos^2\varphi_0 t_0\right)$$

in order to obtain y and x. There results if we add, at the same time, the necessary ρ's everywhere:

$$\left.\begin{aligned} y &= \frac{r}{\varrho}\lambda\cos\varphi_0 - \frac{r}{\varrho^2}\Delta\varphi\lambda\cos\varphi_0 t_0 + \frac{1}{12}\frac{r}{\varrho^3}\lambda^3\cos^3\varphi_0(1-2t_0^2) \\ &\quad -\frac{1}{4}\frac{r}{\varrho^3}\Delta\varphi^2\lambda\cos\varphi_0 - \frac{1}{12}\frac{r}{\varrho^4}\Delta\varphi^3\lambda\cos\varphi_0 t_0 - \frac{1}{6}\frac{r}{\varrho^4}\Delta\varphi\lambda^3\cos^3\varphi_0 t_0(2-t_0^2) \\ x &= \frac{r}{\varrho}\Delta\varphi + \frac{1}{2}\frac{r}{\varrho^2}\lambda^2\cos^2\varphi_0 t_0 + \frac{1}{4}\frac{r}{\varrho^3}\Delta\varphi\lambda^2\cos^2\varphi_0(1-2t_0^2) \\ &\quad + \frac{1}{12}\frac{r}{\varrho^3}\Delta\varphi^3 - \frac{3}{8}\frac{r}{\varrho^4}\Delta\varphi^2\lambda^2\cos^2\varphi_0 t_0 + \frac{1}{24}\frac{r}{\varrho^4}\lambda^4\cos^4\varphi_0 t_0(2-t_0^2) . \end{aligned}\right\} \tag{12}$$

These equations (12) agree with the equations (2) and (3) found in a different way.

Section 73. Reduction of Length and Reduction of Direction

Let there be given on the sphere two points P_1 and P_2 to which the points P_1' and P_2' correspond on the plane. Then we have on the plane, as the connection of the two points, first the straight line $P_1'P_2'$ and second the arc $P_1'P_2'$, which represents the projection of the arc of great circle P_1P_2 on the sphere, as is shown in Fig. 1. According to section 71, p. 220, the arc $P_1'P_2'$ is likewise an arc of a circle.

Due to the conformality of the stereographic projection, the scale factor m at an arbitrary point is the same in all directions, and we have according to (1) and (2), section 71, p. 218.

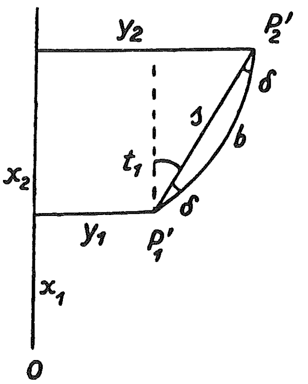

Fig. 1.

$$m = \frac{1}{\cos^2\frac{\delta}{2}} = 1 + \tan^2\frac{\delta}{2} . \tag{1}$$

From Fig. 1, section 71, p. 217, in which $MP' = p$, we see that

$$\tan\frac{\delta}{2} = \frac{p}{2r}, \qquad \text{hence} \qquad m = 1 + \frac{p^2}{4r^2}$$

Since $p^2 = x^2 + y^2$, we have also

$$m = 1 + \frac{x^2+y^2}{4r^2} \qquad \text{and} \qquad \frac{1}{m} = 1 - \frac{x^2+y^2}{4r^2} + \ldots \tag{2}$$

If in Fig. 1 we denote by δ the angle which an arbitrary element of the arc $P_1'P_2'$ forms with the straight line $P_1'P_2'$, then the angle δ is of the second order, as we shall see in the next part, p. 228; consequently, the projection of the element of arc on the straight line will deviate from the element of arc itself only by a quantity of fourth order. Hence, if we aim to neglect terms of the fourth order, then we can neglect also with sufficient accuracy the difference of length between the arc b and the length s.

Then we have, if dS denotes an arbitrary element of the arc of great circle P_1P_2 on the sphere and ds denotes the corresponding element of the straight line P_1P_2,

$$\frac{ds}{dS} = m \qquad \frac{dS}{ds} = \frac{1}{m}, \tag{3}$$

hence according to (2)

$$dS = ds\left(1 - \frac{x^2+y^2}{4r^2}\right) = ds - \frac{x^2}{4r^2}ds - \frac{y^2}{4r^2}ds. \tag{4}$$

Now we have according to Fig. 1

$$ds = \frac{dx}{\cos t_1} = \frac{dy}{\sin t_1}$$

hence

$$dS = ds - \frac{x^2}{4r^2}\frac{dx}{\cos t_1} - \frac{y^2}{4r^2}\frac{dy}{\sin t_1}$$

and integrating

$$S = s - \frac{x^3}{12r^2\cos t_1} - \frac{y^3}{12r^2\sin t_1}.$$

Between the limits x_1, x_2 and y_1, y_2 we will have then

$$S = s - \frac{x_2^3 - x_1^3}{12r^2\cos t_1} - \frac{y_2^3 - y_1^3}{12r^2\sin t_1}.$$

We can write this also in the form

$$S = s - \frac{1}{12r^2}\frac{x_2^3 - x_1^3}{x_2 - x_1}s - \frac{1}{12r^2}\frac{y_2^3 - y_1^3}{y_2 - y_1}s$$

and if the two quotients are cancelled out

$$S = s - \frac{s}{12r^2}(x_2^2 + x_2x_1 + x_1^2 + y_2^2 + y_2y_1 + y_1^2). \tag{5}$$

Now we introduce the mean values

$$x_0 = \frac{x_1+x_2}{2} \qquad y_0 = \frac{y_1+y_2}{2}$$

so that we have

$$x_0^2 = \frac{1}{4}(x_1^2 + 2x_1x_2 + x_2^2)$$
$$y_0^2 = \frac{1}{4}(y_1^2 + 2y_1y_2 + y_2^2)$$

With these we will have

$$2x_1x_2 = 4x_0^2 - x_1^2 - x_2^2 \qquad 2y_1y_2 = 4y_0^2 - y_1^2 - y_2^2,$$

and then we obtain in the place of (5)

$$S = s - \frac{s}{24r^2}\left\{(x_1^2 + y_1^2) + 4(x_0^2 + y_0^2) + (x_2^2 + y_2^2)\right\}$$

or

$$\frac{S}{s} = 1 - \frac{x_1^2 + y_1^2}{24r^2} - \frac{x_0^2 + y_0^2}{6r^2} - \frac{x_2^2 + y_2^2}{24r^2}.$$

According to (2) this is

$$\frac{S}{s} = \frac{1}{6}\left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2}\right). \tag{6}$$

This agrees with the scale factor which we have found in equation (15), section 68, p. 199, for the Gauss conformal projection.

Reduction of direction

In Fig. 2 we have represented again the two points P_1' and P_2' as well as the zero point O on the plane. At the same time there are visible in Fig. 2 the straight-lined connection $P_1'P_2'$ and the circular arc $P_1'P_2'$ with the two equal angles δ. Moreover, OP_1' and OP_2' are the projections of the great circle arcs OP_1 and OP_2 of the spherical surface.

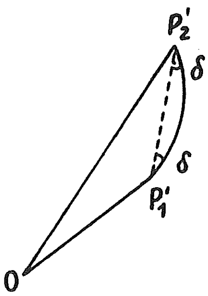

Fig. 2.

Let the spherical triangle OP_1P_2 have the excess ε on the sphere. Since on the plane the angle at O agrees with the angle of the spherical triangle, and due to the conformality, the angles at P_1' and P_2' between the arc $P_1'P_2'$ and the two straight lines $P_1'O$ and $P_2'O$ are equal to the angles in the spherical triangle, the two angles δ together must be equal to the spherical excess ε; hence

$$2\delta = \varepsilon. \tag{7}$$

The computation of ε can be based on the area of the plane triangle, and since O is the starting point of the coordinates, we have, according to Vol. II, 1, 9th edition, 1931, section 39, p. 174,*

$$2F = x_1 y_2 - x_2 y_1,$$

hence,

$$\varepsilon = \frac{x_1 y_2 - x_2 y_1}{2r^2} \quad \text{and} \quad \delta = \frac{x_1 y_2 - x_2 y_1}{4r^2}. \tag{8}$$

We can write this expression for δ also in another form if we introduce the differences of coordinates

$$\Delta y = y_2 - y_1 \qquad \Delta x = x_2 - x_1.$$

Then we will have

$$\delta = \frac{x_1 \Delta y - y_1 \Delta x}{4r^2}. \tag{9}$$

In Fig. 3, p. 229, we have indicated, in addition to the direction angles t_1 and t_2 of the straight line $P_1'P_2'$, also the direction angles T_1 and T_2 of the circular arc $P_1'P_2'$, and now we have

$$t_1 = T_1 - \delta \qquad t_2 = T_2 + \delta. \tag{10}$$

* Not translated.

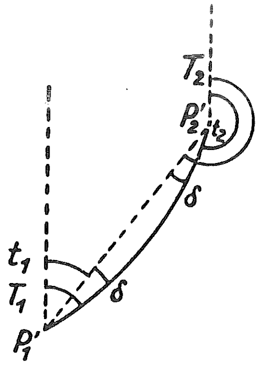

Fig. 3.

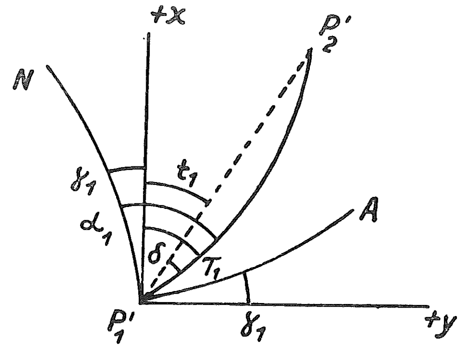

Fig. 4.

The meridian convergence

In order to develop completely the relations between the angles on the sphere and on the plane, we have represented in Fig. 4 at the point P'_1 the projection of the meridian P'_1N and of the parallel P'_1A. We shall call the angle between the meridian and the positive direction of the axis of abscissae or the angle between the parallel and the axis of ordinates the *meridian convergence* and denote it by γ_1 according to our previous section 55, p. 134.

If we consider in the equations (12), section 72, p. 226, the difference of latitude $\Delta\varphi$ as invariable, then these two equations represent the parallel circle passing through P'_1. Then we obtain for the angle γ_1 the equation

$$\tan\gamma_1 = \frac{\partial x}{\partial\lambda} : \frac{\partial y}{\partial\lambda}, \tag{11}$$

and the two differential quotients can easily be indicated from the equations (12), section 72, p. 226. We have

$$\begin{aligned}\frac{\partial y}{\partial\lambda} = r\cos\varphi_0\Big\{1 - \Delta\varphi\, t_0 + \frac{1}{4}\lambda^2\cos^2\varphi_0\,(1-2t_0^2) - \frac{1}{4}\Delta\varphi^2 - \frac{1}{12}\Delta\varphi^3 t_0 \\ - \frac{1}{2}\Delta\varphi\,\lambda^2\cos^2\varphi_0\, t_0\,(2-t_0^2)\Big\}\end{aligned} \tag{12}$$

$$\frac{\partial x}{\partial\lambda} = r\lambda\cos^2\varphi_0\Big\{t_0 + \frac{1}{2}\Delta\varphi\,(1-2t_0^2) - \frac{3}{4}\Delta\varphi^2 t_0 + \frac{1}{6}\lambda^2\cos^2\varphi_0\, t_0\,(2-t_0^2)\Big\}. \tag{13}$$

From equation (12) we form

$$\begin{aligned}1 : \frac{\partial y}{\partial\lambda} = \frac{1}{r\cos\varphi_0}\Big\{1 + \Delta\varphi\, t_0 - \frac{1}{4}\lambda^2\cos^2\varphi_0\,(1-2t_0^2) + \frac{1}{4}\Delta\varphi^2\,(1+4t_0^2) \\ + \frac{1}{12}\Delta\varphi^3 t_0\,(7+12t_0^2) + \frac{1}{2}\Delta\varphi\,\lambda^2\cos^2\varphi_0\, t_0\,(1+t_0^2)\Big\}\end{aligned}$$

and this yields together with (13) according to (11)

$$\begin{aligned}\tan\gamma_1 = \lambda\cos\varphi_0\Big\{t_0 + \frac{1}{2}\Delta\varphi + \frac{1}{12}\lambda^2\cos^2\varphi_0\, t_0\,(1+4t_0^2) + \frac{1}{24}\Delta\varphi^3\,(3+2t_0^2) \\ - \frac{1}{24}\Delta\varphi\,\lambda^2\cos^2\varphi_0\,(3-32t_0^2+4t_0^4)\Big\}.\end{aligned} \tag{14}$$

For the change from $\tan\gamma_1$ to γ_1 we have according to section 34, p. 23,

$$\gamma_1 = \tan\gamma_1 - \frac{1}{3}\tan^3\gamma_1 + \dots,$$

and with this we find easily

$$\gamma_1 = \lambda \cos\varphi_0 \left\{ t_0 + \frac{1}{2} \Delta\varphi + \frac{1}{12}\lambda^2 \cos^2\varphi_0\, t_0 (1 - 8\, t_0^2) + \frac{1}{24} \Delta\varphi^3 (3 + 2\, t_0^2) - \frac{1}{24} \Delta\varphi\, \lambda^2 \cos^2\varphi_0 (3 + 4\, t_0^2 + 4\, t_0^4) \right\} \tag{15}$$

or else in another form, adding ρ at the same time

$$\gamma_1 = \lambda \sin\varphi_0 + \frac{1}{2\varrho}\Delta\varphi\,\lambda \cos\varphi_0 + \frac{1}{12\varrho^2}\lambda^3 \sin\varphi_0 \cos^2\varphi_0 (1 - 8\, t_0^2) + \frac{1}{24\varrho^3}\Delta\varphi^3 \lambda \cos\varphi_0 (3 + 2 t_0^2) - \frac{1}{24\varrho^3}\Delta\varphi\, \lambda^3 \cos^3\varphi_0 (3 + 4\, t_0^2 + 4\, t_0^4)\,. \tag{16}$$

After having found the meridian convergence, now we can also indicate the relation between the azimuths α of the great circle $P_1 P_2$ on the sphere and the direction angles on the plane. Applying Fig. 4 to the two points P_1' and P_2' we have

$$\left.\begin{array}{ll} T_1 = \alpha_1 - \gamma_1 & T_2 = \alpha_2 - \gamma_2 \\ t_1 = T_1 - \delta & t_2 = T_2 + \delta \end{array}\right\} \tag{17}$$

therefore also

$$t_1 = \alpha_1 - \gamma_1 - \delta \qquad t_2 = \alpha_2 - \gamma + \delta\,. \tag{18}$$

Section 74. The Conformal Conic Projection

After we have learned in section 66, p. 189, the simplest form of the conic projection, now we change to the conformal conic projection, in the case of which the projection on the conic surface is again similar in the smallest parts to the original on the sphere.

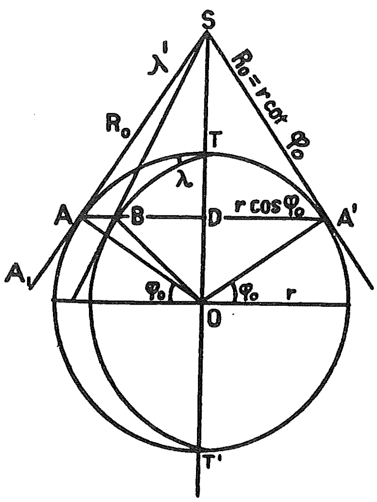

Fig. 1.

In Fig. 1 the cone is tangent to the spherical surface along the normal circle of parallel $A A'$, so that the apex of the cone lies at S. If φ_0 denotes the normal latitude, then we have according to Fig. 1

$$SA = R_0 = r \cot\varphi_0\,. \tag{1}$$

If the difference of longitude of two points A and B of the normal circle of parallel is equal to λ, then the angle between the corresponding meridians on the conic surface is according to (2), section 66, p. 189,

$$\lambda' = \lambda \sin\varphi_0. \tag{2}$$

In section 66, p. 189, we left the question open in which way the parallel circles are to be transformed to the cone, and now we shall carry out this transformation in such a way that the projection becomes conformal.

In this connection, we consider on the sphere a point P at the latitude φ. If we enlarge φ by $d\varphi$, then P moves to Q and by a twisting of the meridian by $d\lambda$, P arrives at R. There is formed then a small right triangle PQR, in which we have $PQ = r\,d\varphi$, $PR = r\cos\varphi\, d\lambda$, as shown in Fig. 2, p. 231.

On the conic surface (Fig. 3, p. 231) we obtain a corresponding triangle $P'Q'R'$. If $P'S = R$, then $P'R' = R\,d\lambda'$ and $P'Q' = -\,dR$, whereby the minus sign must be introduced since R decreases with the increasing latitude φ.

If the two infinitely small triangles, Fig. 2 in the original and Fig. 3 in the projection, are to be similar to one another, then we must have

$$\frac{PQ}{PR} = \frac{P'Q'}{P'R'} \quad \text{or} \quad \frac{r\,d\varphi}{r\cos\varphi\, d\lambda} = -\frac{dR}{R\,d\lambda'}$$

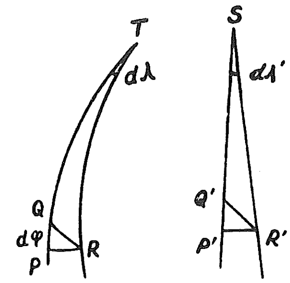

Fig. 2. Fig. 3.

With the introduction of (2), this changes to

$$-\frac{dR}{R} = \sin\varphi_0 \frac{d\varphi}{\cos\varphi}. \tag{3}$$

This is the differential equation for the law of the change of R with increasing latitude φ, if the projection is conformal. The integration yields

$$-\log R = \sin\varphi_0 \log\tan\left(45^0 + \frac{\varphi}{2}\right) + C, \tag{4}$$

where C denotes the constant of integration.

Since for the normal latitude φ_0 the conic ray R assumes the value R_0, then we have

$$-\log R_0 = \sin\varphi_0 \log\tan\left(45^\circ + \frac{\varphi_0}{2}\right) + C \tag{5}$$

and from (4) and (5) together we have

$$\log R - \log R_0 = \sin\varphi_0 \left\{\log\tan\left(45^\circ + \frac{\varphi_0}{2}\right) - \log\tan\left(45^\circ + \frac{\varphi}{2}\right)\right\}$$

or in another form

$$\log\frac{R}{R_0} = \sin\varphi_0 \log\frac{\tan\left(45^\circ + \frac{\varphi^\circ}{2}\right)}{\tan\left(45^\circ + \frac{\varphi}{2}\right)}. \tag{6}$$

We can convert this still further by using the relations

$$1 - \cos x = 2\sin^2\frac{x}{2} \qquad \frac{1-\cos x}{1+\cos x} = \tan^2\frac{x}{2}$$

$$1 + \cos x = 2\cos^2\frac{x}{2}$$

and this yields with $x = 90^\circ + \varphi_0$ or with $x = 90^\circ + \varphi$:

$$\tan^2\left(45^\circ + \frac{\varphi^\circ}{2}\right) = \frac{1+\sin\varphi_0}{1-\sin\varphi_0} \text{ and } \tan^2\left(45^\circ + \frac{\varphi}{2}\right) = \frac{1+\sin\varphi}{1-\sin\varphi},$$

hence (6) becomes

$$\log\frac{R}{R_0} = \frac{1}{2}\sin\varphi_0 \log\frac{(1+\sin\varphi_0)(1-\sin\varphi)}{(1-\sin\varphi_0)(1+\sin\varphi)} \tag{7}$$

or if we change from logarithms to natural numbers

$$\frac{R}{R_0} = \left(\frac{1+\sin\varphi_0}{1-\sin\varphi_0} \quad \frac{1-\sin\varphi}{1+\sin\varphi}\right)^{\frac{\sin\varphi_0}{2}}. \tag{8}$$

According to these formulae (6), (7) or (8) we can compute for each value φ the pertinent R after having obtained R_0, from (1). However, the computation according to these closed formulae is laborious and inaccurate if it is not carried out with 10-place logarithms. Therefore, we shall change now to the development in series from which we obtain the difference $R_0 - R$.

According to Maclaurin's theorem, there will be, in any case, the following series:

$$R = R_0 + \left.\frac{dR}{d\varphi}\right] \Delta\varphi + \left.\frac{d^2R}{d\varphi^2}\right] \frac{\Delta\varphi^2}{2} + \left.\frac{d^3R}{d\varphi^3}\right] \frac{\Delta\varphi^3}{6} + \left.\frac{d^4R}{d\varphi^4}\right] \frac{\Delta\varphi^4}{24} + \dots \tag{9}$$

Now we shall not differentiate directly the function R according to (8) but set for simplification

$$F = \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^{\frac{1}{2}\sin\varphi_0} \qquad F_0 = \left(\frac{1-\sin\varphi_0}{1+\sin\varphi_0}\right)^{\frac{1}{2}\sin\varphi_0} \tag{10}$$

so that we have

$$\frac{R}{R_0} = \frac{F}{F_0}. \tag{11}$$

By differentiation, (11) yields

$$\frac{1}{R_0}\frac{dR}{d\varphi} = \frac{1}{F_0}\frac{dF}{d\varphi}, \tag{12}$$

and with this, we can write Maclaurin's series (9) also thus:

$$\frac{R}{R_0} = 1 + \left.\frac{1}{F_0}\frac{dF}{d\varphi}\right] \Delta\varphi + \left.\frac{1}{F_0}\frac{d^2F}{d\varphi^2}\right] \frac{\Delta\varphi^2}{2} + \left.\frac{1}{F_0}\frac{d^3F}{d\varphi^3}\right] \frac{\Delta\varphi^3}{6} + \left.\frac{1}{F_0}\frac{d^4F}{d\varphi^4}\right] \frac{\Delta\varphi^4}{24} + \dots \tag{13}$$

Now we must form the first four derivatives of the function F according to (10) and have first according to (12), (3) and (11)

$$\frac{dF}{d\varphi} = \frac{F_0}{R_0}\frac{dR}{d\varphi} = -F_0 \frac{R}{R_0}\frac{\sin\varphi_0}{\cos\varphi} = -F\frac{\sin\varphi_0}{\cos\varphi}. \tag{14}$$

Further differentiation of (14) yields:

$$\frac{d^2F}{d\varphi^2} = -\frac{\sin\varphi_0}{\cos^2\varphi}\left(\frac{dF}{d\varphi}\cos\varphi + F\sin\varphi\right)$$

$$\frac{d^2F}{d\varphi^2} = +\sin\varphi_0\, F\frac{\sin\varphi_0 - \sin\varphi}{\cos^2\varphi} \tag{15}$$

$$\frac{d^3F}{d\varphi^3} = \sin\varphi_0\left\{-\frac{F\sin\varphi_0}{\cos\varphi}\frac{\sin\varphi_0 - \sin\varphi}{\cos^2\varphi} + F\frac{-\cos^3\varphi + (\sin\varphi_0 - \sin\varphi)\,2\cos\varphi\sin\varphi}{\cos^4\varphi}\right\}$$

$$\frac{d^3F}{d\varphi^3} = F\sin\varphi_0\left\{\frac{\sin\varphi_0 - \sin\varphi}{\cos^3\varphi}(2\sin\varphi - \sin\varphi_0) - \frac{1}{\cos\varphi}\right\}. \tag{16}$$

Differentiated once again:

$$\left.\begin{aligned}\frac{d^4 F}{d\varphi^4} = \frac{-F\sin^2\varphi_0}{\cos\varphi}\left\{\frac{\sin\varphi_0-\sin\varphi}{\cos^3\varphi}(2\sin\varphi-\sin\varphi_0)-\frac{1}{\cos\varphi}\right\}\\ +F\sin\varphi_0\left\{+\left(\frac{3\sin\varphi_0\sin\varphi}{\cos^4\varphi}-\frac{\cos^4\varphi+3\sin^2\varphi\cos^2\varphi}{\cos^6\varphi}\right)(2\sin\varphi-\sin\varphi_0)\right.\\ \left.+\frac{\sin\varphi_0-\sin\varphi}{\cos^3\varphi}2\cos\varphi-\frac{\sin\varphi}{\cos^2\varphi}\right\}.\end{aligned}\right\} \quad (17)$$

We could carry this still further, but since we break off here, it is not necessary; for we must now determine, according to Maclaurin's scheme (13), the special values of our four derivatives which hold for $\Delta\varphi = 0$, or for $\varphi = \varphi_0$, which is the same. If we begin with (14) and (15), then we have:

$$\left.\frac{dF}{d\varphi}\right] = -F_0\tan\varphi_0 \quad \text{and} \quad \left.\frac{d^2F}{d\varphi^2}\right] = 0. \quad (18)$$

Also (16) and (17) are greatly reduced with $\varphi = \varphi_0$; they yield:

$$\frac{d^3F}{d\varphi^3} = -F_0\tan\varphi_0 \quad \text{and} \quad \left.\frac{d^4F}{d\varphi^4}\right] = -F_0\tan^2\varphi_0. \quad (19)$$

If we set everything into Maclaurin's scheme (13), we obtain:

$$\frac{R}{R_0} = 1 - \Delta\varphi\tan\varphi_0 - \frac{\Delta\varphi^3}{6}\tan\varphi_0 - \frac{\Delta\varphi^4}{24}\tan^2\varphi_0 + \dots \quad (20)$$

We will set for abbreviation:

$$\tan\varphi_0 = t_0 \quad \text{and} \quad R_0 - R = \Delta R. \quad (21)$$

With this, the preceding series (20) becomes:

$$\Delta R = R_0\,\Delta\varphi\,t_0 + R_0\frac{\Delta\varphi^3}{6}t_0 + R_0\frac{\Delta\varphi^4}{24}t_0^2 + \dots \quad (22)$$

With this, we have obtained a convenient formula for the computation of R.

Section 75. Rectangular Coordinates x, y

The central meridian AS of our illustration is taken as x-axis of a rectangular system of coordinates, with a y-axis which is tangent at A to the normal parallel AB, as is drawn in Fig. 1.

Now if an arbitrary point E has the longitude λ referred to the central meridian and the latitude φ, then we know also the angle $ASE = \lambda' = \lambda \sin\varphi_0$ and the meridional length $BE = \Delta R$ according to equation (22) above; consequently, we have according to Fig. 1:

$$x = R_0 - R\cos\lambda' \quad \text{and} \quad y = R\sin\lambda'. \quad (1)$$

We could calculate directly according to these formulae; but

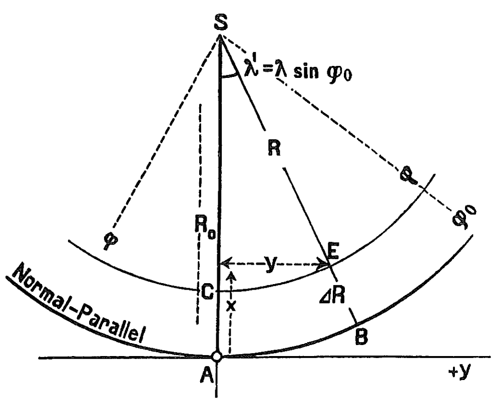

Fig. 1.

developments in series, which result very easily, are better:

$$\frac{x}{R_0} = 1 - \frac{R}{R_0} \cos \lambda' = 1 - \frac{R}{R_0}\left(1 - \frac{\lambda^2 \sin^2 \varphi_0}{2} + \frac{\lambda^4 \sin^4 \varphi_0}{24}\right),$$

and if we introduce $\frac{R}{R_0}$ from (20), p. 233, then we obtain:

$$\frac{x}{R_0} = 1 - \left(1 - \Delta \varphi\, t_0 - \frac{\Delta \varphi^3}{6} t_0 - \frac{\Delta \varphi^4}{24} t_0^2\right)\left(1 - \frac{\lambda^2 \sin^2 \varphi_0}{2} + \frac{\lambda^4 \sin^4 \varphi.}{24}\right).$$

The multiplication with neglect of the terms above the fourth order yields:

$$\frac{x}{R_0} = \Delta \varphi\, t_0 + \frac{\lambda^2 \sin^2 \varphi_0}{2} - \frac{\Delta \varphi\, \lambda^2 \sin^2 \varphi_0}{2} t_0 + \frac{\Delta \varphi^3}{6} t_0 - \frac{\lambda^4 \sin^4 \varphi_0}{24} + \frac{\Delta \varphi^4}{24} t_0^2,$$

but according to (1), p. 230, we have $R_0 t_0 = R_0 \tan \varphi_0 = r$; hence:

$$\frac{x}{r} = \Delta \varphi + \frac{\lambda^2}{2} \sin \varphi_0 \cos \varphi_0 - \frac{\Delta \varphi\, \lambda^2 \sin^2 \varphi_0}{2} + \frac{\Delta \varphi^3}{6} - \frac{\lambda^4 \sin^3 \varphi_0 \cos \varphi_0}{24} + \frac{\Delta \varphi^4 t_0}{24} \qquad (2)$$

or also in different form:

$$\frac{x}{r} = \Delta \varphi + \frac{\lambda^2}{2} \cos^2 \varphi_0\, t_0 - \frac{\Delta \varphi\, \lambda^2}{2} \cos^2 \varphi_0\, t_0^2 + \frac{\Delta \varphi^3}{6} - \frac{\lambda^4}{24} \cos^4 \varphi_0\, t_0^3 + \frac{\Delta \varphi^4}{24} t_0. \qquad (3)$$

In the same way, we obtain also the series for y, for we have first

$$\sin \lambda' = \sin (\lambda \sin \varphi_0) = \lambda \sin \varphi_0 - \frac{\lambda^3 \sin^3 \varphi_0}{6} + \ldots$$

According to (1) we will have

$$\frac{y}{R_0} = \sin \lambda' \frac{R}{R_0},$$

and by means of (20), p. 233, we then have

$$\frac{y}{R_0} = \left(\lambda \sin \varphi_0 - \frac{\lambda^3 \sin^3 \varphi_0}{6}\right)\left(1 - \Delta \varphi\, t_0 - \frac{\Delta \varphi^3}{6} t_0 - \frac{\Delta \varphi^4}{24} t_0^2\right)$$

$$\frac{y}{R_0} = \lambda \sin \varphi_0 - \Delta \varphi\, \lambda \sin \varphi_0\, t_0 - \frac{\lambda^3}{6} \sin^3 \varphi_0 + \frac{\Delta \varphi\, \lambda^3 \sin^3 \varphi_0\, t_0}{6} - \frac{\Delta \varphi^3 \lambda \sin \varphi_0\, t_0}{6}$$

and with $R_0 = r \cot \varphi_0$ according to (1), p. 230,

$$\frac{y}{r} = \lambda \cos \varphi_0 - \Delta \varphi\, \lambda \cos \varphi_0\, t_0 - \frac{\lambda^3}{6} \cos^3 \varphi_0\, t_0^2 + \frac{\Delta \varphi\, \lambda^3}{6} \cos^3 \varphi_0\, t_0^3 - \frac{\Delta \varphi^3 \lambda}{6} \cos \varphi_0\, t_0. \qquad (4)$$

Inversions of series (3) and (4)

Now we have to treat the inverse problem, namely the computation of the geographic coordinates φ, λ

from the rectangular coordinates x, y. In this connection, we have first, according to the view of Fig. 1, the closed formulae:

$$R^2 = (R_0 - x)^2 + y^2 \quad \text{and} \quad \tan \lambda' = \frac{y}{R_0 - x}, \tag{5}$$

hence with

$$R = R_0 - \Delta R$$
$$R_0^2 - 2 R_0 \Delta R + \Delta R^2 = R_0^2 - 2 R_0 x + x^2 + y^2$$
$$\frac{\Delta R}{R_0} - \frac{\Delta R^2}{2 R_0^2} = \frac{x}{R_0} - \frac{x^2}{2 R_0^2} - \frac{y^2}{2 R_0^2}. \tag{6}$$

According to (22), p. 233, we have

$$\frac{\Delta R}{R_0} = \Delta \varphi t_0 + \frac{\Delta \varphi^3}{6} t_0 + \frac{\Delta \varphi^4}{24} t_0^2, \tag{7}$$

hence also with sufficient accuracy:

$$\frac{\Delta R^2}{R_0^2} = \Delta \varphi^2 t_0^2 + \frac{\Delta \varphi^4}{3} t_0^2, \tag{8}$$

hence again with the previous (7) introduced into (6):

$$\Delta \varphi t_0 - \frac{\Delta \varphi^2 t_0^2}{2} + \frac{\Delta \varphi^3}{6} t_0 - \frac{\Delta \varphi^4}{8} t_0^2 = \frac{x}{R_0} - \frac{x^2}{2 R_0^2} - \frac{y^2}{2 R_0^2}. \tag{9}$$

This equation is to be solved, by successive approximation, for $\Delta \varphi$. In any case, we have in the first approximation:

$$\Delta \varphi t_0 = \frac{x}{R_0} + x^2 \ldots, \quad \text{and hence} \quad \Delta \varphi^2 t_0^2 = \frac{x^2}{R_0^2} + x^3 \ldots$$
$$\Delta \varphi t_0 - \frac{x^2}{2 R_0^2} + \ldots \qquad = \frac{x}{R_0} - \frac{x^2}{2 R_0^2} - \frac{y^2}{2 R_0^2}.$$

Second approximation:

$$\Delta \varphi t_0 = \frac{x}{R_0} - \frac{y^2}{2 R_0^2} + , \ldots$$
$$\Delta \varphi^2 t_0^2 = \frac{x^2}{R_0^2} - \frac{x y^2}{R_0^3} \quad , \quad \Delta \varphi^3 t_0^3 = \frac{x^3}{R_0^3}.$$

Hence anew to the third order:

$$\Delta \varphi t_0 - \frac{x^2}{2 R_0^2} + \frac{x y^2}{2 R_0^3} + \frac{x^3}{6 R_0^3 t_0^2} = \frac{x}{R_0} - \frac{x^2}{2 R_0^2} - \frac{y^2}{2 R_0^2}$$
$$\Delta \varphi t_0 = \frac{x}{R_0} - \frac{y^2}{2 R_0^2} - \frac{x y^2}{2 R_0^3} - \frac{x^3}{6 R_0^3 t_0^2}.$$

With these to the fourth order:

$$\Delta \varphi^2 t_0^2 = \frac{x^2}{R_0^2} - \frac{x y^2}{R_0^3} - \frac{x^2 y^2}{R_0^4} - \frac{x^4}{3 R_0^4 t_0^2} + \frac{y^4}{4 R_0^4}$$
$$\Delta \varphi^3 t_0^3 = \frac{x^3}{R_0^3} - \frac{3}{2} \frac{x^2 y^2}{R_0^4} \quad \text{and} \quad \Delta \varphi^4 t_0^4 + \frac{x^4}{R_0^4}.$$

All this introduced into (9) yields the end equation:

$$\Delta\varphi t_0 - \frac{x^2}{2R_0^2} + \frac{xy^2}{2R_0^3} + \frac{x^2y^2}{2R_0^4} + \frac{x^4}{6R_0^4t_0^2} - \frac{y^4}{8R_0^4} + \frac{x^3}{6R_0^3t_0^2} - \frac{x^2y^2}{4R_0^4t_0^2} - \frac{x^4}{8R_0^4t_0^2}$$
$$= \frac{x}{R_0} - \frac{x^2}{2R_0^2} - \frac{y^2}{2R_0^2}.$$

We can solve this end equation directly for $\Delta\varphi t_0$, namely:

$$\Delta\varphi t_0 = \frac{x}{R_0} - \frac{y^2}{2R_0^2} - \frac{xy^2}{2R_0^3} - \frac{x^3}{6R_0^3t_0^2} - \frac{x^2y^2}{4R_0^4t_0^2}(2t_0^2-1) - \frac{x^4}{24R_0^4t_0^2} + \frac{y^4}{8R_0^4} \tag{10}$$

and because $R_0 t_0 = r$, this yields finally:

$$\Delta\varphi = \frac{x}{r} - \frac{y^2}{2r^2}t_0 - \frac{xy^2}{2r^3}t_0^2 - \frac{x^3}{6r^3} - \frac{x^2y^2}{4r^4}t_0(2t_0^2-1) - \frac{x^4}{24r^4}t_0 + \frac{y^4}{8r^4}t_0^3. \tag{11}$$

In order to obtain finally λ according to (5), we need with $\lambda' = \lambda \sin\varphi_0$:

$$\tan(\lambda\sin\varphi_0) = \frac{y}{R_0 - x} = \frac{y}{R_0\left(1-\frac{x}{R_0}\right)} = \frac{y}{R_0}\left(1 + \frac{x}{R_0} + \frac{x^2}{R_0^2} + \frac{x^3}{R_0^3}\right)$$
$$\lambda\sin\varphi_0 + \frac{\lambda^3\sin^3\varphi_0}{3} = \frac{y}{R_0} + \frac{yx}{R_0} + \frac{yx^2}{R_0^3} + \frac{yx^3}{R_0^4}.$$

First approximation:

$$\lambda\sin\varphi_0 = \frac{y}{R_0} + \frac{yx}{R_0^2} + \dots$$
$$\lambda^3\sin^3\varphi_0 = \frac{y^3}{R_0^3} + \frac{3y^3x}{R_0^4}$$
$$\lambda\sin\varphi_0 = \frac{1}{3}\left(\frac{y^3}{R_0^3} + \frac{3y^3x}{R_0^4}\right) = \frac{y}{R_0} + \frac{yx}{R_0^2} + \frac{yx^2}{R_0^3} + \frac{yx^3}{R_0^4}$$
$$\lambda\sin\varphi_0 = \frac{y}{R_0} + \frac{yx}{R_0^2} + \frac{yx^2}{3R_0^3} - \frac{y^3}{3R_0^3} - \frac{y^3x}{R_0^4} + \frac{yx^3}{R_0^4}, \tag{12}$$

or because $R_0 t_0 = R_0 \tan\varphi_0 = r$, this yields

$$\lambda\cos\varphi_0 = \frac{y}{r} + \frac{yx}{r^2}t_0 + \frac{yx^2}{r^3}t_0^2 - \frac{y^3}{3r^3}t_0^2 - \frac{y^3x}{r^4}t_0^3 + \frac{yx^3}{r^4}t_0^3 \tag{13}$$

With this, we have set up all relations required between the geographic and the rectangular coordinates.

The scale factor

The enlargement of the projection in comparison to the original results from the relation of corresponding sides of the two small triangles in Fig. 2 and Fig. 3, p. 231. If we denote the scale factor by m, then we have according to Fig. 2 and Fig. 3, p. 231,

$$m = \frac{P'R'}{PR} = \frac{R\,d\lambda'}{r\cos\varphi\,d\lambda} = \frac{R\sin\varphi_0}{r\cos\varphi}$$

or with $r = R_0 \tan \varphi_0$

$$m = \frac{R \cos \varphi_0}{R_0 \cos \varphi}. \tag{14}$$

To this, we have according to (20), p. 233,

$$\frac{R}{R_0} = 1 - \Delta \varphi \, t_0 - \frac{\Delta \varphi^3}{6} t_0, \tag{15}$$

and because $\varphi = \varphi_0 + \Delta \varphi$, we have according to section 34, p. 18,

$$\cos \varphi = \cos \varphi_0 \left(1 - \Delta \varphi \, t_0 - \frac{\Delta \varphi^2}{2} + \frac{\Delta \varphi^3}{6} t_0\right).$$

The inversion yields, as likewise already indicated on p. 18:

$$\frac{\cos \varphi_0}{\cos \varphi} = 1 + \left(\Delta \varphi \, t_0 + \frac{\Delta \varphi^2}{2} - \frac{\Delta \varphi^3}{6} t_0\right) + \left(\Delta \varphi \, t_0 + \frac{\Delta \varphi^2}{2}\right)^2 + (\Delta \varphi \, t_0)^3$$

$$\frac{\cos \varphi_0}{\cos \varphi} = 1 + \Delta \varphi \, t_0 + \frac{\Delta \varphi^2}{2} (1 + 2 t_0^2) + \frac{\Delta \varphi^3}{6} t_0 (5 + 6 t_0^2). \tag{16}$$

The two series (15) and (16) multiplied together yield:

$$m = 1 + \frac{\Delta \varphi^2}{2} + \frac{\Delta \varphi^3 t_0}{6}. \tag{17}$$

In order to represent this m also as a function of the coordinates x and y, we have from (11), p. 236:

$$\Delta \varphi = \frac{x}{r} - \frac{y^2}{2 r^2} t_0$$

$$\Delta \varphi^2 = \frac{x^2}{r^2} - \frac{x y^2}{r^2} t_0 \qquad \Delta \varphi^3 = \frac{x^3}{r^3}.$$

This $\Delta \varphi^2$ and $\Delta \varphi^3$ introduced into (17) yield:

$$m = 1 + \frac{x^2}{2r^2} - \frac{x y^2}{2 r^3} t_0 + \frac{x^3}{6 r^3} t_0 \tag{18}$$

or

$$\frac{1}{m} = 1 - \frac{x^2}{2 r^2} + \frac{x y^2}{2 r^3} t_0 - \frac{x^3}{6 r^3} t_0. \tag{19}$$

For the practical computation, we will take also $\log m$, which is very simple within this order of magnitude, namely according to (18) and (19):

$$\log m = \frac{\mu x^2}{2 r^2} - \frac{\mu x y^2}{2 r^3} t_0 + \frac{\mu x^3}{6 r^3} t_0 \tag{20}$$

or

$$\log \frac{1}{m} = -\frac{\mu x^2}{2 r^2} + \frac{\mu x y^2}{2 r^3} t_0 - \frac{\mu x^3}{6 r^3} t_0. \tag{21}$$

In the first approximation, i.e. with the limitation to $\frac{1}{r^2}$, the above formulae (18) to (21) correspond to the formulae found previously in section 68, p. 197, equation (9), for the Gauss conformal projection, if we exchange x with y. This corresponds also to the direct view.

Although the formulae (18) to (21) are sufficient for most purposes, we shall develop, in addition, also the fourth order, for which we have according to (20), p. 233:

$$\frac{R}{R_0} = 1 - \Delta\varphi\, t_0 - \frac{\Delta\varphi^3}{6}\, t_0 - \frac{\Delta\varphi^4}{24} t_0^2, \tag{22}$$

and because $\varphi = \varphi_0 + \Delta\varphi$, we have according to section 34, p. 18:

$$\cos\varphi = \cos\varphi_0 \left(1 - \Delta\varphi\, t_0 - \frac{\Delta\varphi^2}{2} + \frac{\Delta\varphi^3}{6}\, t_0 + \frac{\Delta\varphi^4}{24}\right).$$

The inversion yields:

$$\frac{\cos\varphi_0}{\cos\varphi} = 1 + \left(\Delta\varphi\, t_0 + \frac{\Delta\varphi^2}{2} - \frac{\Delta\varphi^3}{6}\, t_0 - \frac{\Delta\varphi^4}{24}\right) + \left(\Delta\varphi\, t_0 + \frac{\Delta\varphi^2}{2} - \frac{\Delta\varphi^3}{6}\, t_0\right)^2 + \left(\Delta\varphi\, t_0 + \frac{\Delta\varphi^2}{2}\right)^3 + (\Delta\varphi\, t_0)^4.$$

The calculation of this, which was also already indicated on p. 18, is to the fourth order:

$$\frac{\cos\varphi_0}{\cos\varphi} = 1 + \Delta\varphi\, t_0 + \frac{\Delta\varphi^2}{2}(1 + 2\, t_0^2) + \frac{\Delta\varphi^3 t_0}{6}(5 + 6\, t_0^2) + \frac{\Delta\varphi^4}{24}(5 + 28\, t_0^2 + 24\, t_0^4). \tag{23}$$

We have to multiply the two series (22) and (23) in order to arrive at the series for m; the execution yields:

$$m = 1 + \frac{\Delta\varphi^2}{2} + \frac{\Delta\varphi^3 t_0}{6} + \frac{\Delta\varphi^4}{24}(5 + 3\, t_0^2). \tag{24}$$

In order to represent m also as a function of x and y, we have according to (11), p. 236:

$$\Delta\varphi = \frac{x}{r} - \frac{y^2}{2r^2} t_0 - \frac{x\, y^2}{2r^3} t_0^2 - \frac{x^3}{6r^3}$$

$$\Delta\varphi^2 = \frac{x^2}{r^2} - \frac{x\, y^2}{r^2} t_0 - \frac{x^2 y^2}{r^4} t_0^2 - \frac{x^4}{3r^4} + \frac{y^4}{4r^4} t_0^2$$

$$\Delta\varphi^3 = \frac{x^3}{r^3} - \frac{3}{2}\frac{x^2 y^2}{r^4} t_0 \quad \text{and} \quad \Delta\varphi^4 = \frac{x^4}{r^4}.$$

This introduced in (24) yields:

$$m = 1 + \frac{x^2}{2r^2} - \frac{x\, y^2}{2r^3} t_0 + \frac{x^3}{6r^3} t_0 - \frac{3x^2 y^2}{4r^4} t_0^2 + \frac{x^4}{24r^4}(1 + 3\, t_0^2) + \frac{y^4}{8r^4} t_0^2. \tag{25}$$

The inversion of this yields:

$$\frac{1}{m} = 1 - \left(\frac{x^2}{2r^2} - \frac{xy^2}{2r^3}t_0 + \ldots\right) + \left(\frac{x^2}{2r^2} - \ldots\right)^2$$

$$\frac{1}{m} = 1 - \frac{x^2}{2r^2} + \frac{xy^2}{2r^3}t_0 - \frac{x^3}{6r^3}t_0 + \frac{3x^2y^2}{4r^4}t_0^2 + \frac{x^4}{24r^4}(5 - 3t^2) - \frac{y^4}{8r^4}t_0^2 . \tag{26}$$

For practical computing, we take also *log m*:

$$l\,m = l\left(1 + \frac{x^2}{2r^2} - \frac{xy^2}{2r^3}t_0 \ldots\right) = \frac{x^2}{2r^2} - \frac{xy^2}{2r^3}t_0 + \ldots - \frac{1}{2}\left(\frac{x^2}{2r^2} - \ldots\right)^2 \tag{27}$$

$$log\; m = \frac{\mu x^2}{2r^2} - \frac{\mu x y^2}{2r^3}t_0 + \frac{\mu x^3}{6r^3}t_0 - \frac{3}{4}\frac{\mu x^2 y^2}{r^4}t_0^2 + \frac{\mu}{24r^4}(-2 + 3t_0^2) + \frac{\mu y^4}{8r^4}t_0^2 . \tag{28}$$

Later we shall be able to use the terms of fourth order occurring here to attach them also to the *spheroidal* formulae, which we shall develop in themselves only to the third order.

Section 76. Reduction of Length and Reduction of Direction

We consider two points P_1 and P_2 on the sphere whose distance is S. These two points are projected on the plane by the image points P_1' and P_2' with the coordinates $x_1 y_1$ and $x_2 y_2$ and the distance s. Let there be further:

$$(P_1' P_2') = t_1 \qquad (P_2' P_1') = t_2 .$$

By neglecting again the difference of the arc b and the length s as in the case of Fig. 1, section 73, p. 226, we have

$$S = \int_0^s \frac{1}{m}\, ds . \tag{1}$$

If we neglect in m the terms of fourth order, then we have to take into account in (25) and (26), section 75, above, one more term of third order, whereas we did not need to go beyond the second order in the case of the Gauss projection and the stereographic projection. Consequently, we take from (26), section 75,

$$\frac{1}{m} = 1 - \frac{x^2}{2r^2} + \frac{xy^2}{2r^3}t_0 - \frac{x^3}{6r^3}t_0 . \tag{2}$$

In order to be able to carry out the integration (1) with this, we introduce a new system of coordinates $\xi\eta$ with P_1' as zero point and with the direction $P_1' P_2'$ as ξ-axis. Then we have

$$x = x_1 + \xi \cos t_1 \qquad y = y_1 + \xi \sin t_1 . \tag{3}$$

If we set this into (2), then there follows, written in a simple form,

$$\frac{1}{m} = a + b\xi + c\xi^2 + d\xi^3 \tag{4}$$

where the coefficients $a,\ b,\ c,\ d$ have the following meaning

$$\left.\begin{aligned}
a &= 1 - \frac{x_1^2}{2r^2} - \frac{x_1 y_1^2}{2r^3} t_0 + \frac{x_1^3}{6r^3} t_0 \\
b &= -\left(\frac{x_1 \cos t_1}{r^2} + \frac{2x_1 y_1 \sin t_1 + y_1^2 \cos t_1 - x_1^2 \cos t_1}{2r^3} t_0\right) \\
c &= -\left(\frac{\cos^2 t_1}{2r^2} + \frac{x_1 \sin^2 t_1 + 2y_1 \sin t_1 \cos t_1 - x_1 \cos^2 t_1}{2r^3} t_0\right) \\
d &= -\frac{3\sin^2 t_1 \cos t_1 + \cos^3 t_1}{6r^3} t_0 .
\end{aligned}\right\} \tag{5}$$

With (4) the integral (1) yields

$$S = \int_0^s \frac{1}{m} d\xi = a\,s + \frac{1}{2} b\,s^2 + \frac{1}{3} c\,s^3 + \frac{1}{4} d\,s^4 . \tag{6}$$

But we can bring this also into another form by introducing again the scale factor m for the starting point, the end point and the central point of the length $P_1'P_2'$.

For $\xi = 0$ we have according to (4) $\dfrac{1}{m_1} = a.$

For $\xi = \dfrac{s}{2}$ we have

$$\frac{1}{m_0} = a + \frac{1}{2} b\,s + \frac{1}{4} c\,s^2 + \frac{1}{8} d\,s^3$$

or

$$\frac{4}{m_0} = 4a + 2b\,s + c\,s^2 + \frac{1}{2} d\,s^3,$$

and for $\xi = s$ we will have

$$\frac{1}{m_2} = a + b\,s + c\,s^2 + d\,s^3.$$

With this, (6) changes to

$$S = \frac{s}{6}\left(\frac{1}{m_1} + \frac{4}{m_0} + \frac{1}{m_2}\right) \tag{7}$$

an expression which we have found also in (15), section 68, p. 199, for the Gauss conformal projection and in (6), section 73, p. 228, for the stereographic projection.

The reduction of direction

In Fig. 1 we have represented again, in the arc $P_1'P_2'$, the projection of the arc of great circle P_1P_2, which includes, with the length $P_1'P_2'$, the two small angles δ_1 and δ_2. For the determination of these two angles, let us consider first an element of arc between the two meridians SM_1 and SM_2. If we denote the change of direction of the arc between the two meridians by $d\delta$, then we have according to Fig. 1

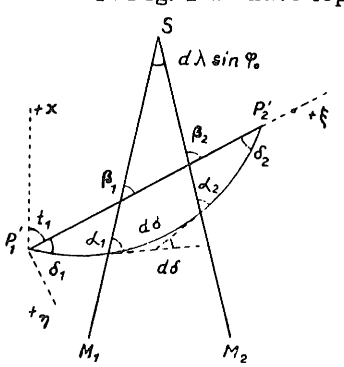

Fig. 1.

$$180° - \beta_1 + \beta_2 + \alpha_1 + 180° - \alpha_2 + 180° - d\delta = 540°;$$

hence

$$d\delta = (\alpha_1 - \alpha_2) - (\beta_1 - \beta_2).$$

Due to the conformality of the projection, the two angles α_1 and α_2

agree with the azimuths of the great circle arc on the sphere. But for the latter we have according to (3), section 60, p. 165,

$$\alpha_1 - \alpha_2 = d\lambda \sin \varphi.$$

On the other hand, we have according to Fig. 1

$$\beta_1 - \beta_2 = d\lambda \sin \varphi_0;$$

hence, we have

$$d\delta = d\lambda(\sin \varphi - \sin \varphi_0). \tag{8}$$

If we denote the curve element lying between the two meridians by $d\sigma$ and the radius of curvature by R, then we have

$$R = \frac{d\sigma}{d\delta}. \tag{9}$$

Now we use again the system of coordinates $\xi\,\eta$ introduced already for the reduction of length, and we can assume for the reciprocal of the radius of curvature, since a very flat curve is involved, the expression

$$\frac{1}{R} = -\frac{d^2\eta}{d\xi^2}.$$

The negative sign is to be introduced here because the curve lies concave toward the ξ-axis. On the other hand, we can replace, for the same reason, in (9) the curve element $d\sigma$, by $d\xi$, so that we have

$$\frac{d^2\eta}{d\xi^2} = -\frac{d\delta}{d\xi}. \tag{10}$$

Now we have to develop in (8) the difference $\sin\varphi - \sin\varphi_0$ as a series and obtain according to section 34, p. 18, with $\varphi - \varphi_0 = \Delta\varphi$

$$\sin\varphi - \sin\varphi_0 = \Delta\varphi \cos\varphi_0 - \frac{1}{2}\Delta\varphi^2 \sin\varphi_0;$$

hence, (8) changes to

$$d\delta = d\lambda\left(\Delta\varphi\cos\varphi_0 - \frac{1}{2}\Delta\varphi^2\sin\varphi_0\right) = d\lambda\left(\Delta\varphi - \frac{1}{2}\Delta\varphi^2 t_0\right)\cos\varphi_0. \tag{11}$$

Now we take from (13), section 75, p. 236,

$$\lambda\cos\varphi_0 = \frac{y}{r} + \frac{xy}{r^2}t_0$$

and obtain

$$d\lambda\cos\varphi_0 = \frac{1}{r}\left(dy + \frac{x\,t_0}{r}dy + \frac{y\,t_0}{r}dx\right). \tag{12}$$

Further we have according to (11), section 75, p. 236,

$$\Delta\varphi = \frac{x}{r} - \frac{y^2}{2r^2}t_0 \qquad \Delta\varphi^2 = \frac{x^2}{r^2}, \tag{13}$$

hence;

$$\Delta\varphi - \frac{\Delta\varphi^2}{2} t_0 = \frac{x}{r} - \frac{x^2}{2r^2} t_0 - \frac{y^2}{2r^2} t_0 .$$

With this, (11) changes to

$$d\delta = \frac{1}{r^2}\left(x\,dy + \frac{x^2}{2r} t_0\,dy + \frac{xy}{r} t_0\,dx - \frac{y^2}{2r} t_0\,dy\right). \tag{14}$$

According to (10) we then have

$$-\frac{d^2\eta}{d\xi^2} = \frac{1}{r^2}\left(x\frac{dy}{d\xi} + \frac{x^2 t_0}{2r}\frac{dy}{d\xi} + \frac{x y t_0}{r}\frac{dx}{d\xi} - \frac{y^2 t_0}{2r}\frac{dy}{d\xi}\right). \tag{15}$$

For the determination of the two differential quotients $\frac{dx}{d\xi}$ and $\frac{dy}{d\xi}$ we use the equations (3), p. 239,

$$x = x_1 + \xi\cos t_1 \qquad y = y_1 + \xi\sin t_1 \tag{16}$$

and have then

$$\frac{dx}{d\xi} = \cos t_1 \qquad \frac{dy}{d\xi} = \sin t_1 . \tag{17}$$

By introducing these expressions into (15) we obtain

$$\begin{aligned} -\frac{d^2\eta}{d\xi^2} = \frac{1}{r^2}\Big\{&(x_1 + \xi\cos t_1)\sin t_1 + \frac{1}{2r}(x_1 + \xi\cos t_1)^2 t_0 \sin t_1 \\ &+ \frac{1}{r}(x_1 + \xi\cos t_1)(y_1 + \xi\sin t_1) t_0 \cos t_1 - \frac{1}{2r}(y_1 + \xi\sin t_1)^2 t_0 \sin t_1\Big\} \end{aligned} \tag{18}$$

and arranged according to powers of ξ

$$\begin{aligned} -\frac{d^2\eta}{d\xi^2} &= \frac{1}{r^2}\left(x_1 \sin t_1 + \frac{t_0}{2r}(x_1^2 \sin t_1 + 2x_1 y_1 \cos t_1 - y_1^2 \sin t_1)\right) \\ &+ \frac{1}{r^2}\left(\sin t_1 \cos t_1 + \frac{t_0}{r}(2x_1 \sin t_1 \cos t_1 + y_1 \cos^2 t_1 - y_1 \sin^2 t_1)\right)\xi \\ &+ \frac{t_0}{2r^3}(3\sin t_1 \cos^2 t_1 - \sin^3 t_1)\,\xi^2 . \end{aligned} \tag{19}$$

We write this with a simple denotation of the coefficients in the following form

$$-\frac{d^2\eta}{d\xi^2} = A + B\xi + C\xi^2 \tag{20}$$

and after integrating twice

$$-\frac{d\eta}{d\xi} = A\xi + \frac{1}{2}B\xi^2 + \frac{1}{3}C\xi^3 + C_1 \tag{21}$$

$$-\eta = \frac{1}{2}A\xi^2 + \frac{1}{6}B\xi^3 + \frac{1}{12}C\xi^4 + C_1\xi . \tag{22}$$

A second constant of integration is not necessary for equation (22) since η for $\xi = 0$ must likewise become equal to zero.

For $\xi = 0$ we will have $\frac{d\eta}{d\xi} = \delta_1$, and for $\xi = s$ we will have $\frac{d\eta}{d\xi} = -\delta_2$ and likewise $\eta = 0$; hence, we have the three equations:

$$\delta_1 = -C_1$$
$$\delta_2 = +C_1 + A\,s + \frac{1}{2}B\,s^2 + \frac{1}{3}C\,s^3$$
$$0 = C_1 + \frac{1}{2}A\,s + \frac{1}{6}B\,s^2 + \frac{1}{12}C\,s^3.$$

We obtain therefrom

$$\left.\begin{aligned} \delta_1 &= \frac{1}{2}A\,s + \frac{1}{6}B\,s^2 + \frac{1}{12}C\,s^3 \\ \delta_2 &= \frac{1}{2}A\,s + \frac{1}{3}B\,s^2 + \frac{1}{4}C\,s^3. \end{aligned}\right\} \tag{23}$$

If we set in here the values of A, B and C from (19) and take into account that

$$s\sin t_1 = y_2 - y_1 \qquad s\cos t_1 = x_2 - x_1,$$

we obtain for δ_1 the value

$$\begin{aligned} \delta_1 = {} & \frac{1}{2r^2}x_1(y_2-y_1) + \frac{t_0}{4r^3}x_1^2(y_2-y_1) + \frac{t_0}{2r^3}x_1y_1(x_2-x_1) - \frac{t_0}{4r^3}y^2(y_2-y_1) \\ & + \frac{1}{6r^2}(x_2-x_1)(y_2-y_1) + \frac{t_0}{3r^3}x_1(x_2-x_1)(y_2-y_1) + \frac{t_0}{6r^3}y_1(x_2-x_1)^2 \\ & - \frac{t_0}{6r^3}y_1(y_2-y_1)^2 + \frac{t_0}{8r^3}(y_2-y_1)(x_2-x_1)^2 - \frac{t_0}{24r^3}(y_2-y_1)^3. \end{aligned} \tag{24}$$

By simple conversion we obtain therefrom, if we add ρ at the same time:

$$\begin{aligned} \delta_1 = {} & \frac{\varrho}{6r^2}(2x_1+x_2)(y_2-y_1) - \frac{t_0\varrho}{12r^3}\Big\{(2y_1^2+y_2^2)(y_2-y_1) - (2x_1^2+x_2^2)(y_2-y_1) \\ & - 2(2x_1y_1+x_2y_2)(x_2-x_1)\Big\} - \frac{t_0\varrho}{24r^3}\Big\{3(x_2-x_1)^2(y_2-y_1) - (y_2-y_1)^3\Big\}. \end{aligned} \tag{25}$$

In the same way, there follows

$$\begin{aligned} \delta_2 = {} & \frac{\varrho}{6r^2}(x_1+2x_2)(y_2-y_1) + \frac{t_0\varrho}{12r^3}\Big\{(2y_2^2+y_1^2)(y_2-y_1) - (2x_2^2+x_1^2)(y_2-y_1) \\ & - 2(2x_2y_2+x_1y_1)(x_2-x_1)\Big\} + \frac{t_0\varrho}{24r^3}\Big\{3(x_2-x_1)^2(y_2-y_1) - (y_2-y_1)^3\Big\}. \end{aligned} \tag{26}$$

With the help of the two angles δ_1 and δ_2, the direction angles t_1 and t_2 of the lengths $P_1'P_2'$ and $P_2'P_1'$ can be computed according to Fig. 1, p. 240.

Conformal conic projection with two parallels projected at true scale

Thus far the conformal conic projection was represented in the form that the circle of tangency of the cone with the sphere was projected at true scale. A somewhat better adaptation of the conic surface to the spherical surface is obtained if the requirement is set up that the scale factors for *two* parallels with the latitudes φ_1 and φ_2 be the same size.

The general equation of projection is according to (4), section 74, p. 231,

$$-\log R = \sin\varphi_0 \log\tan\left(45^\circ + \frac{\varphi}{2}\right) + \log c\,,$$

where the constant of integration may now be denoted by $\log c$. Then we have also

$$R = c\left(\tan\left(45^\circ + \frac{\varphi}{2}\right)\right)^{\sin\varphi_0} \tag{27}$$

Whereas we have assumed thus far a definite parallel of tangency, hence a definite value φ_0, now we shall adhere to the above-mentioned condition and determine the corresponding value of φ_0 by computation.

According to section 75, p. 236, the scale factor for the projection of the parallel is

$$m = \frac{R\sin\varphi_0}{r\cos\varphi}.$$

In accordance with the above condition we thus have

$$\frac{R_1\sin\varphi_0}{r\cos\varphi_1} = \frac{R_2\sin\varphi_0}{r\cos\varphi_2} \quad \text{or} \quad \frac{R_1}{\cos\varphi_1} = \frac{R_2}{\cos\varphi_2}.$$

If we introduce here the value for R from (27), then we obtain

$$\frac{\left(\tan\left(45^\circ + \frac{\varphi_1}{2}\right)\right)^{\sin\varphi_0}}{\cos\varphi_1} = \frac{\left(\tan\left(45^\circ + \frac{\varphi_2}{2}\right)\right)^{\sin\varphi_0}}{\cos\varphi_2}$$

or

$$\left(\frac{\tan\left(45^\circ + \frac{\varphi_1}{2}\right)}{\tan\left(45^\circ + \frac{\varphi_2}{2}\right)}\right)^{\sin\varphi_0} = \frac{\cos\varphi_1}{\cos\varphi_2}. \tag{28}$$

For the computation of $\sin\varphi_0$ we find therefrom by converting to logarithms

$$\sin\varphi_0\left\{\log\tan\left(45^\circ + \frac{\varphi_1}{2}\right) - \log\tan\left(45^\circ + \frac{\varphi_2}{2}\right)\right\} = \log\cos\varphi_1 - \log\cos\varphi_2$$

or

$$\sin\varphi_0 = \frac{\log\cos\varphi_1 - \log\cos\varphi_2}{\log\tan\left(45^\circ + \frac{\varphi_1}{2}\right) - \log\tan\left(45^\circ + \frac{\varphi_2}{2}\right)}. \tag{29}$$

The conformal conic projection was first treated by J. H. Lambert in his *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1772, Nr. VI, pp. 105-109.

In Volume III, second half, we shall come to the subject of geodetic applications of the conformal conic projection, in the case of which the ellipsoidal shape of the earth is taken into account.

The conic projection of Bonne

In conclusion, we mention briefly the projection indicated by the French geographer Bonne in 1752, which was used for various great map works. This is a conic projection in the case of which the parallels are transformed to the cone in the same form as in the case of the simple conic projection in section 66, p. 189. After the development of the cone we thus obtain a system of concentric circles whose radii result from the equation (4), section 66, p. 189. The meridians, however, are projected in a different way in order to diminish the distortions occurring in the case of the simple conic projection. Starting from a central meridian which is projected on the plane as a straight line, the intervals of the meridians are set off at true scale on the individual parallels. If we connect the corresponding points of the individual parallels, then we obtain, in constrast to the simple conic projection, the meridians in the form of curved lines.

In the case of this projection, the meridians and parallels are no longer intersected at right angles; hence, we have no conformal projection here. The graticule consists of oblique trapezoids in which the parallel sides and the altitudes are equal to those on the sphere; hence, the projection is equivalent.

We base Fig. 2 on the computation of the rectangular coordinates of a point P on the plane, which has on the sphere the latitude φ and, with respect to a central meridian, the longitude λ.

M is the central point of the projection and MS is the central meridian here, whereas the projection of the circle of tangency is denoted by φ_0. For the radius of the latter we already have found in equation (1), section 66, p. 189, the expression

$$SM = R_0 = r \cot \varphi_0.$$

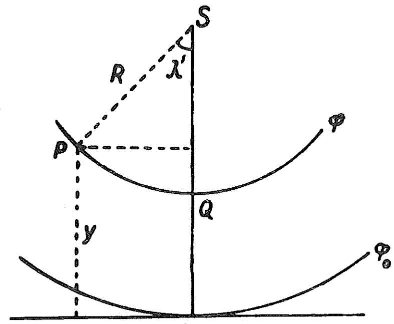

Fig. 2.

A circle of latitude φ is represented according to (4), section 66, p. 189, by a circle with the radius

$$R = R_0 - \frac{\varphi - \varphi_0}{\rho} r.$$

According to Bonne's law of projection we have for the point P the arc $PQ = r \cos \varphi \lambda$; consequently, we will have

$$\lambda' = \frac{r \cos \varphi}{R} \lambda,$$

and with this, we obtain for the coordinates of the point P on the plane

$$x = R \sin \lambda' \qquad y = R_0 - R \cos \lambda'. \tag{30}$$

The Bonne projection forms the basis of most maps in the geographic atlases; in addition, it has also been used several times in the case of large topographic map works, where the ellipsoidal shape of the earth has been taken into account, however.

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