JORDAN'S HANDBOOK OF GEODESY

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by

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FOREWORD

When I handed to my superiors at Army Map Service the completed assignment of the translation of theoretical portions of Jordan-Eggert, *Handbuch der Vermessungskunde*, the work before you, it was with certain thoughts in my mind; and they were understood by those who had entrusted me with the work because it was along this line that I was to undertake the work.

And now, at this place, I speak to the wider circle of those who will use this text as a reference for their research in mathematics and geodesy, submitting the translation to their keen criticism.

While I feel certain that I have always comprehended the various fine shades of the German idea and word, and while I have made a special effort not to lose any of the German concept and important details, I may not always have succeeded in presenting an elegant English reproduction. This should be taken into the user's kind consideration when he finds the texture and the flow of the language at times a little stiff and somewhat rougher than if it had originally been thought and written by an English-speaking author. I have thus sacrificed style for faithfulness. I trust, however, that I have done so precisely to the advantage of those who search for the basic information worked out by this great German geodesist, Professor Jordan, with whose thorough mind I have become so well acquainted in the course of this translation.

As for the technical vocabulary, I was fortunate to work with a geodetic group at Army Map Service so that I was always able to find, with the help of my colleagues and my explanations of the German idea, the literally correct technical equivalent. This deserves mention and special consideration, I feel, since to my knowledge no complete geodetic dictionary exists.

And now I invite all users of this edition to present their valued criticism of the present translation so that it will be possible to publish the following edition in a form as nearly perfect as possible, and to do justice to both style and content.

Special thanks are due Dr. John A. O'Keefe, Mr. Bernard H. Chovitz, and Miss Thelma Cooley, who have reviewed this translation.

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THEORY OF THE PROBABILITY OF ERRORS

APPENDIX, AUXILIARY TABLES

Almost 150 years have now elapsed since the foundation of a theory which deals with the errors of observations and measurements and therefore touches very closely the field and land survey, namely the so-called "method of least squares."

In 1794 this theory was founded by the then only 17 year old mathematician Gauss and tested soon afterward by application to the adjustment of the planet Ceres.

At nearly the same time the Frenchman Legendre also found the fundamental theorem of the sum of least squares and published it in a small treatise in 1806, while the first public treatise by Gauss on the method of least squares originates only from the year 1809, so that Legendre has therefore the precedence of *publication*.

But the precedence of *discovery* doubtlessly is due to Gauss, as is seen from the details to be treated later, and Gauss alone, until the year 1826, created in six classical treatises all essential features of what is called today the method of least squares, so that Gauss unquestionably is the father of the method of least squares.

As the forerunners of the modern method of least squares there are to be considered the theories of Euler, Tob. Mayer, Lambert, Boscovich, Lagrange, and Laplace. The point in question is always to determine, from a number of equations which is larger than the number of the unknowns, a system of the unknowns connected in the best possible manner without unjustly preferring or neglecting individual equations. In the treatise, *Sur les inégalités du mouvement de Saturne et de Jupiter*, in 1748, Euler proceeded in such a way that he sought to form a kind of normal equation for each unknown by the suitably appearing connection of all unknowns. Tob. Mayer also proceeded similarly in his treatise, *Über die Umwälzung des Mondes*; but in his case, also, thin measures had still to replace the lacking sure method. About 1770, Boscovich established the fundamental theorem that, in the case of excessive equations, the unknowns would have to be determined in such a way that the absolute sum of the residual errors becomes a minimum. (Wolf, *Handbuch der Astronomie*, 1891, p. 133, gives the above.)

In *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1765 (2nd Ed., 1792, pp. 424-488), Lambert treats the "theory of the reliability of observations and experiments" by seeking to come to a plausible method of adjustment according to the principle of the center of gravity.

The first theoretical investigation of observational errors based on the calculation of probability was undertaken by Lagrange in 1770 (cf. Encke, *Berl. Astr. Jahrb. f. 1853*, pp. 310-351); this theory, however, fell again into oblivion. The discovery of our modern method of least squares was directly preceded by an experiment of a theory of adjustment by Laplace, who, for the determination of the earth's dimensions from more than two degree-measurements, determined from several equations the unknown elements by means of the two conditions, first that the algebraic sum of the residual errors becomes equal to zero, and second that the absolute sum of the errors becomes a minimum. This theory is developed in *Traité de mécanique céleste*, tome second, an VII (1802), première partie, livre III, art. 40, p. 143.

Further details about the history of origin of the method of least squares are also given by Czuber, *Theorie der Beobachtungsfehler*, Leipzig, 1891, pp. 11 and following, as well as in *Jahresber. d. Deutsch. Mathematiker-Vereinigung*, 7th Vol. (2. Heft), Leipzig (Teubner), 1899, "Entwicklung der Wahrscheinlichkeitstheorien und ihrer Anwendungen."

There is to be mentioned further the American Adrain (1808), according to a communication by Hammer, *Zeitschrift für Vermessungswesen*, 1900, pp. 613-628, "Beitrag zur Geschichte der Ausgleichsrechnung. Ein vergessener Mitbegründer der M.d.kl.Q."

The name "*method of least squares*" originates from Legendre (cf. the following) and this name of our science has also established itself. Later, the designation "method of the sum of least squares" has also been suggested, e.g. in 1841 by Hülsen (cf. Czuber, *Theorie der Beobachtungsfehler*, 1891, p. 234), which, it is true, characterizes the principle more correctly, but compared with the original designation "method of least squares" has not been able to assert itself.

As is already indicated briefly at the beginning, the computation of adjustment according to the method of least squares, generally recognized today, was first treated publicly by Legendre in 1806 in a small treatise at the end of the work, *Nouvelles méthodes pour la détermination des orbites des comètes*, appendice pp. 72-80, "sur la méthode des moindres carrés." (Legendre gave a second publication of it in *Mémoires de la classe des Sciences mathém. et phys. de l'institut de France*, 1810, seconde partie pp. 149 and following.) As justification of his method, Legendre emphasizes the generality and simplicity of its application (*Nouvelles méthodes*, etc., p. 72: "De tous les principes, qu'on peut proposer pour cet objet, je pense qu'il n'en est pas de plus général, de plus exact, ni d'une application plus facile que celui" etc.). As an example of the application, Legendre gives the determination of the earth's dimensions from five latitudes measured in France and the four geodetic meridional arcs lying between.

Independently of Legendre, Gauss had already found the method of least squares in 1794 as a student of mathematics at the University of Göttingen, but did not publish it until the year 1809 in a considerably enlarged and complete treatment in the work, "Theoria motus corporum coelestium in sectionibus conicis solem ambientium." (Liber II, Sectio III, pp. 205-224.)

Gauss says here about the question of priority in Art. 186, p. 220: "Ceterum principium nostrum, quo jam inde ab anno 1795 usi sumus, nuper etiam a clar. Legendre in opere *nouvelles méthodes pour la détermination des orbites des comètes*, Paris 1806, prolatum est" (complete edition of Gauss' works, Vol. VI, pp. 56 and 59).

This indication of the date is in contradiction to a letter by Gauss to Olbers of July 1806, in which he writes with reference to Legendre's work about the determination of the orbits of comets: [Translated] "Thus, for instance, the principle used by me since 1794 that, in order to represent best several magnitudes which we cannot all represent accurately, we must make the sum of the squares a minimum, is also used in Legendre's works and quite well executed." (Gauss' *Werke*, Bd. VIII, Göttingen 1900, pp. 138-139.) In other letters also, Gauss speaks repeatedly of the principle already used since 1794, so that the year 1794 must be regarded as the year of the discovery of the method of least squares.

Sartorius v. Waltershausen reports in his biography of Gauss (*Gauss zum Gedächtnis* 1856, p. 21) on this subject: [Translated] "Such an extraordinary wealth of ideas flew then, day and night, from the mind of this youthful genius that one discovery precipitated, so to say, the other, that he hardly found time and leisure to bring to paper even the mere outlines of the same. The greatest discoveries have thus lain mostly over a decade, even over half a century without their reaching the wider knowledge of the scientific public." And on p. 23: [Translated] "However deep the discoveries by Gauss have then been in the field of pure mathematics, they remained restricted to a very close circle of thinkers for quite some time, even to the present day, as is due to the nature of the material, and yet another discovery from astronomy had to be added to make Gauss' name one of the most celebrated in Europe, even in the wide public."

The circumstances connected with this discovery are the following: On January 1, 1801, Joseph Piazzi in Palermo discovered the planet Ceres (first small planet between Mars and Jupiter), observed it for 41 days and informed several astronomers of his discovery. Since they, however, did not receive Piazzi's news until several months after the first discovery, at a time in which the observation of the planet was no longer possible because of its unfavorable position to the sun, then Piazzi's observations of 41 days remained the only material in order to compute the elements of the orbit and to find the planet next year again. The astronomers occupied themselves with this problem at once.

It was reserved to Gauss to compute, with the help of the method of least squares, an ephemeris of the new planet, which then did justice, in the best possible way, to all individual observations extending only to 9° of the whole orbit. This ephemeris deviated from reality so little that the first rediscoverer of the planet, Zach, expressed his opinion that the ellipse of Dr. Gauss agrees "zur Bewunderung genau" [exact to admiration] with the position of the planet.

Although the method of adjustment by Gauss had achieved a complete triumph by the computation of the elements of the orbit of Ceres, Gauss did not feel as yet induced to bring it before the public. The following point of the introduction to "*Theoria motus*" (praefatio pag. IX) serves as explanation of this delay: [Translated] "Several astronomers wished, immediately after the re-discovery of Ceres, that I should publish the computational methods thereby used; several reasons, however, prevented me from complying with the requests of the friends, first other work, further the desire to treat the matter more thoroughly later, but mainly the hope that continued occupation with this subject will be in the interest of various parts of the solution in regard to generality, simplicity and elegance. And since this hope has not deceived me, I believe that I must not regret the delay."

Gauss' works on the method of least squares are:

1809 *Theoria motus corporum coelestium in sectionibus conicis ambientium.* Auctore Carolo Friderico Gauss. Hamburgi, 1809, Liber II, sectio III.

1810 "Disquisitio de elementis ellipticis Palladis ex oppositionibus." Annorum 1803, 1804, 1805, 1807, 1808, 1809. Auctore Carolo Friderico Gauss, societati regiae tradita, Nov. 25, 1810. (*Commentationes* societatis scientiarum Goettingensis recentiores. Vol. I, 1808-1811.)

1816 "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie und verwandte Wissenschaften*, herausgegeben von Lindenau u. Bohnenberger. Tübingen, 1816, Bd. I, pp. 185-196.

1821 "Theoria combinationis observationum erroribus minimis obnoxiae, pars prior, societati regiae exhibita Febr. 15. 1821." (*Commentationes* societatis regiae scientiarum Goettingensis recentiores. Vol. V, ad a., 1819-1822, pp. 33-62.)

1823 "Theoria combinationis observationum erroribus minimis obnoxiae, pars posterior, societati regiae exhibita Febr. 2. 1823." (*Commentationes* societatis regiae scientiarum Goettingensis recentiores. Vol. V, pp. 63-90.)

1826 "Supplementum theoriae combinationis erroribus minimis obnoxiae, societati regiae exhibita Sept. 16. 1826." (*Commentationes* societatis regiae scientiarum Goettingensis recentiores. Vol. VI, 1823-27, pp. 57-98.)

In the complete edition of Gauss' works (Carl Friedrich Gauss' *Werke*, herausgegeben von der Kgl. Gesellsch. d.

Wissensch. zu Göttingen) the "Theoria motus" forms the VIIth volume. Disquisitio de elementis ellipticis Palladis forms the beginning of volume VI, the remaining papers about the method of least squares are found in volume IV, with supplements in volume VIII.

A summary or, as the case may be, translation of Gauss' publications on the method of least squares is contained in *Abhandlungen zur Methode der kleinsten Quadrate* von Carl Friedrich Gauss, in deutscher Sprache herausgegeben von Dr. A. Börsch und Dr. P. Simon, Assistenten im Kgl. Preuss. Geodätischen Institut, Berlin 1887.

In "Theoria motus" the method of least squares is proven by the calculation of probability; the exponential function $\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}$ is found thereby for the law of the probability of errors. With this, the adjustment and weight determination for indirect observations is treated completely.

In "Disquisitio de elementis ellipticis Palladis" the minimum of the sum of squares of the residual errors is determined by introducing the designations $[bb \cdot 1]$, etc.

The treatise "Über die Genauigkeit der Beobachtungen" determines the probable error of an unknown not only from the sum of squares, but for comparison also from other sums of powers of true observational errors.

After a few general considerations about the laws of errors, "Theoria combinationis" passes over to the independent definition of the mean error and sets up the condition that the unknowns to be determined or functions thereof are affected with mean errors as small as possible.

Finally, "Supplementum theoriae combinationis" applies this condition to conditioned observations and especially to the adjustment of nets of triangles with angle observations or complete direction observations.

These Gaussian theories in addition to a few supplements by Bessel were worked up by Encke in a treatise "Über die Methode der kleinsten Quadrate," which appeared as the Appendix of the three years 1834, 1835, 1836 of *Berliner Astronomisches Jahrbuch.* In the collected works, *J. F. Enckes astronomische Abhandlungen, zusammengestellt aus den Jahrgängen 1830 bis 1862 des Berliner Astronom. Jahrb.*, the above treatise about the method of least squares forms the numbers XII, XIII, XIV of the first volume (1866).

A letter to Schumacher of 25 November 1844, which is reprinted in Vol. VIII of Gauss' works, pp. 147-148, gives information about Gauss' lectures on the method of least squares. Gauss writes in it: [Translated] "In a verbal lecture of the theory of the method of least squares I usually take just the reverse way of that which is reasonable to adopt in a printed treatise. For I teach first the manner of applying it, mixing in more or less of the finer technicalities according to circumstances. Only then, as far as time remains, I take up the various kinds of proofs, whose knowledge can have a lively interest only for one who already understands the use of the method. I generally lecture on three kinds of proofs: 1. one which is based only on principles of appropriateness which, very obviously, can be done easily, 2. the manner of tying to the calculation of probability taught in Th. M. C. C., 3. the manner of tying to the calculation of probability, completely different from it, which is presented in Theoria Comb. Observ., and in my belief is exclusively the only admissible one. For everyone who is still entirely unacquainted with this theory, I consider this succession as the most appropriate one."

Details about the content of a lecture in the winter semester 1850-51 are reported by one of his students, R. Dedekind, in the treatise, *Gauss in seiner Vorlesung über die kleinsten Quadrate* (Festschrift zur Feier des 150 jährigen Bestehens der Kgl. Ges. d. Wiss. zu Göttingen, Beiträge zur Gelehrtengeschichte Göttingens, Berlin, 1901, pp. 45-59).

All this refers to the *theoretical* works by Gauss on the method of least squares. Light has been spread on Gauss' practical applications of this theory to the adjustment of triangulations only in later times by communications by Oberst Schreiber in *Zeitschrift für Vermessungswesen*, 1879, p. 141, and by Hauptmann Gäde, "Beiträge zur Kenntnis von Gauss' praktisch-geodätischen Arbeiten," *Zeitschrift für Vermessungswesen*, 1885, pp. 113-225; cf. also Jordan-Steppe, *Deutsches Vermessungswesen*, Stuttgart, 1882, pp. 5-18.

Among the founders of the method of least squares we are to count also Bessel, who, in a treatise, "Untersuchungen über die Bahn des Olbers' schen Kometen" (*Abhandlungen der Berliner Akademie der Wissenschaften, mathematische Klasse 1812 bis 1813*, p. 119) and in the work, *Fundamenta astronomiae*, 1818, pp. 18-21, investigates the distribution of errors in rather long series of observations.

Hagen, a student of Bessel, in his *Grundzüge der Wahrscheinlichkeitsrechnung*, Berlin, 1837 (2. Aufl. 1867), established a theory of errors which starts from the idea that each observational error is composed of a very great number of very small elementary errors, partly positive, partly negative, but homogeneous in the absolute sense. The application of the calculation of probability to the various cases of possible combinations of these elementary errors leads to the Gaussian law of exponential errors.

In a similar manner, Bessel treated this subject in "Untersuchungen über die Wahrscheinlichkeit der Beobachtungsfehler" (*Astronom. Nachr.* 15. Band Nr. 358 and 359 of October 1838, pp. 369 and following).

In the *Gradmessung in Ostpreussen 1837*, Bessel solved a problem not taken into account by Gauss, namely the adjustment of indirect observations with condition equations and applied it to the adjustment of a net of triangles with incomplete direction observations. Weight and error computations were not discussed here.

The applications of the new science were predominantly *geodetic*.

From Gauss we have from 1823 as the first numerical example a trigonometric adjustment for resection (*Astronomische Nachrichten*, 1. Band 1823, pp. 81-86) and from 1826 already two adjustments of triangulation nets in "Supplementum theoriae combinationis."

The connecting papers by Bessel, Hansen, Andrae and others were essentially caused also by geodetic needs; and if we survey the daily growing literature in the course of the last hundred years on adjustment computation, then we find among it no other subject of instruction represented as much as geodesy, and besides astronomers, physicists and others, we, the field and land surveyors, can pride ourselves to be the most zealous disciples of the master Gauss with respect to methodus quadratorum minimorum.

At the beginning, however, the practicians could not *directly* draw from Gauss' sources; they needed mediating teachers, and these were at first in 1837 the hydraulic architect Hagen and in 1843 G e r l i n g, Professor at Marburg. In 1837 Hagen started from the calculation of probability, and with this, made the subject unnecessarily difficult for the practicians, while Gerling in 1843, as a student of Gauss, put the principle of the sum of least squares simply ahead and on the basis of his experiences in the triangulation of the electorate of Hesse, and adapted the subject, in the most illustrative manner, to the taste of his readers by dividing it into *"seven principal operations"* in each case, so that the concept of the mean error and the basic features of the adjustment of trigonometric measurements generally spread in a few decades, and Gerling's *Ausgleichungsrechnungen der praktischen Geometrie oder die Methode der kleinsten Quadrate* is still valued today.

In his introduction Gerling says: [Translated] "I still recall quite well the time when the surveyor who knew how to deal with logarithmic tables was considered as a scholar among his colleagues. Now one would expect to make himself ridiculous if he only wished to present himself for the examination without this knowledge. In a similar manner, it will soon also go with the adjustment computations."

About 50 years after it was written, this prophecy came true, and now the basic features of adjustment computation are required in all examinations for land surveyors and survey engineers in Germany.

This development of half a century, however, has not always proceeded uniformly and also not without struggles.

The main objection against the geodetic applications of the method of least squares concerned the great work of the numerical computations thereby necessary, especially in the case of the solution of the normal equations. These computations give all beginners great grief at first, and they formed a standing complaint, which we meet most effectively, however, by *turning* the tables, by saying that the method of least squares correctly applied has not brought an increase but a *decrease* in geodetic computational work.

In this connection, we will first mention a remark by the Bavarian geodesist von O r f f, who in 1866 approached the difficult work of recomputing and adjusting the whole old Bavarian triangulation of the beginning of the century; he raises the question here if the various trial computations to be carried out, which are connected with the application of other methods of adjustment without occurring in the final results, do not cause an expenditure of time and trouble similar to the method of least squares.

Of the western neighbor of Bavaria, the excellent B o h n e n b e r g e r in Württemberg, we know that because of the lack of a solid method of adjustment he has never reached a solid closing of his triangulation; and it was likewise in Baden, where up to the forties an innumerable amount of angles had been measured and computed for decades until, finally, the excellent R h e i n e r, chief geometer, learned the method of least squares even in the mature years of his life and, with this, finally brought to a close the triangulation measurements and computations in Baden, lasting for many decades (about 1850).

Nobody can say how triangulations in whole nets were adjusted previously; one only knows this much, that before the method of least squares an eternal rejecting and repeating, trying, moving and pressing was the fashion, and from that time the appropriate expression has remained: If it does not come out right, then one sends out a trigonometer and lets him measure an angle until it becomes 3" larger (or smaller).

In such a confusion the method of least squares worked as a relief, but nevertheless, a long schooling had to be gone through here and many a point in dispute and many purely formal questions had to be clarified.

Now even though such controversial points became gradually silent and the indispensableness of the method of least squares in *higher* geodesy was generally recognized, its applicability for the so-called *lower* geodesy was, on the other hand, long debated.

Nevertheless, the doubts and difficulties of the application of the method of least squares accumulate all the more the further one steps down from the triangulations of Ist order to the IInd - IVth order; but also in this field the method of least squares has proved successful.

General Schreiber as chief of the Prussian land survey has expressed himself on this subject in the report about the Nice conference of the Internationale Erdmessung in 1887 (Annex X b, p. 10): [Translated] "In the triangulations of lower order, the method of least squares serves only the purpose of obtaining plausible results free from discrepancies by a method as free as possible from arbitrariness. But with the help of the method this goal is reached in the *simplest* and most elegant manner imaginable, if only there, where real rigorousness is unattainable anyhow, we even forego the appearance of such."

This forms the transition to approximate methods in general, especially also of a trigonometric kind, which should replace the method of least squares. There is a great number of them, from personal arbitrariness in the individual case to the most stipulated instructions and forced forms. If in the field of adjustment of field and land survey sins have been committed in previous decades, then this has surely happened most in the case of these approximate methods. In general, the method of least squares is recognized as the test for approximate methods, and therefore it must be expected that an approximate method requires considerably *less* work than the actual method of least squares. But this is not the case in many of the numerous proposals of that kind.

A great number of the effects of the method of least squares on our subject can be designated briefly as *moral* advantages; measuring and computing have not only become more reliable by the method of least squares but also more honest. Everyone of us knows what pressures on the mind can result if measurements do not come out right, as they should according to reasonable expectation. Then the prescribed computational procedure for the examination of errors according to the method of least squares steps now helpingly to our side and excludes arbitrariness and doubt.

The report which Captain *Gäde* drew from the files of the degree-measurement by Gauss (*Zeitschrift für Vermessungswesen*, 1885, p. 205; exchange of letters between Gauss and Bessel, p. 423) gives the best insight into the geodetic conditions of fairness of earlier times. In 1830 Gauss wrote to Bessel: [Translated] I have carefully adjusted the system of my main triangles, "*without any arbitrariness, without selecting or excluding*." If Gauss emphasizes this in express terms, then we can conclude conversely from this how things may have gone on previously and elsewhere; and in fact, there are reported details about the Dutch, Bavarian, Austrian and French triangulations, which show clearly how small the objectivity of the measurements of that time was. Let us mention only the one thing that often the diagonal checks (side equations) were already sufficient in order to expose triangulation chains unfairly matched.

All these conditions have now essentially improved, and in our present-day adjustments we have even a certain mathematical measure for the objectivity of the observer, namely in the ratio of the mean error of the unit of weight *after* the adjustment and *before* the adjustment, a ratio which theoretically should be = 1, but frequently turns out 1.5 to 2.0. Undoubtedly, it also depends on entirely unknown elements, but it also expresses, at the same time, the degree of fairness of measurement and computation.

The theory of observational errors has also an important field of application in the accuracy determinations for measurements of any kind, especially for our field and land measurements, determination of official limits of errors and so forth.

In former times, the errors of measurement were mentioned as little as possible; there are whole books on land survey which treat the question of the accuracy of measurement with uncertain terms or even omit it entirely. Many an old statute even obliges the surveyor to measure "*perfectly accurately*" *on his oath.*

Compared with such concepts, the introduction of the mean error in itself is already a great step forward, even though the mean errors were not always computed correctly. The best working up of a survey work no longer consists now in hiding the errors as previously but bringing them out and putting them together, so that we can quickly obtain an opinion about the whole. The extensive literature based upon this about investigations of geodetic accuracy has great significance for the development of the methods of measurement.

The accuracy determinations of earlier times moved almost entirely in percentage data or proportional numbers; one said, for instance, the mean error of a chain measurement is 1:1000, or the admissible error of an area determination is 1/2%, or even, the accuracy of a levelling is 1:500,000 of the length and so forth, all of which is incorrect.

Only with the help of the method of least squares, for most kinds of land survey have there been found correct laws of errors which are of the highest importance for the arrangement of the measurements in advance, for the judgment of the success after the measurement and for the official determination of the limit of error, e.g. also for the errors of the simple chain, tape and rod measurement, which were treated in the seventies of the past century in the German Association of Geometers [Deutscher Geometerverein] and in the first volumes of *Zeitschrift für Vermessungswesen*. This subject was perhaps partly overestimated then by the disputants, but today we find the results of those discussions again recognized indirectly in numerous official and private publications.

Now if the simple question concerning the errors of the chain and rod measurements already could not be solved without the theory of errors, then this is still much more the case with the so important laws of errors of traverse lines, levelling and so forth. With the help of the theory of errors it can thus be derived, for instance, that the mean transverse error of a straight equilateral theodolite polygon line is proportional to the

1-1/2 power of the total length, that the transverse error to be feared of an open polygon is decreased to 1/2 by azimuth attachment and to 1/8 if attachment of coordinates is added, or we know that the transverse error of a theodolite line is inversely proportional to the square root of the range; the error of a compass line, however, is directly proportional to the square root of the range and so forth, and we can make our arrangements accordingly and judge the success.

The general laws of errors mentioned form quite a special gain for our measurements inasmuch as once found and set up they can be benefited, without trouble, by everyone, and it is no exaggeration if we express: In this sense, the method of least squares can be applied with advantage to *all* measurements, even to those with the chain, not at all, however, in such a way that error equations would always be formed and normal equations solved, but so that, while the adjustment is made arbitrarily in the individual case, the influences of errors of all individual operations are taken into account as a whole according to theoretical laws.

Chapter I

GENERAL THEORY OF THE SUM OF THE ERRORS OF LEAST SQUARES

In this first chapter we shall treat, in one phase, the general theory of the method of least squares according to the definition of the mean error and according to the principle of the sum of the errors of the least squares and take up only as much of the applications and examples as is necessary for the explanation of the theories. The actual applications, especially of a geodetic nature, will be treated in particular in the subsequent chapters.

Section 2. Explanations

Anybody who occupies himself with measurements of any kind has thereby the experience that these measurements are subject to errors.

One has mainly two means to examine the correctness and accuracy of measurements; either one repeats a measurement immediately and sees if one obtains again the result of the first measurement, or one measures different quantities which bear a known relationship to one another once each and examines if the measuring results show the relation mentioned; e.g., one measures the three angles of a plane triangle and compares their sum with 180°. If one procures such a check in the case of each measurement and pursues it most rigorously, then one is led to the conclusion that no measurement is completely free from error.

In spite of this unavoidable deficiency of the observations, still the errors will not exceed certain limits if the observer applies the necessary care. If the latter is not the case, then "gross errors" occur (e.g., wrong counting of whole rod lengths in the case of length measurements and others). If the mentioned checks show that such gross errors have in fact occurred, then the faulty measurements must be replaced by remeasurements. Gross errors shall be excluded from the following considerations.

Certain errors always act in the same sense so that they cannot even be discovered by remeasurements, e.g., if a measuring rod is too short, then a measured length is always found too long. To this group there belong generally the measuring errors which result from instrument errors. These errors, too, can be made harmless either by adjusting the measuring instruments, or determining the amount of the instrument errors and taking them into account, or, finally, by applying such methods of measurement by which the errors can be compensated.

In the following we shall only speak of such errors which are unavoidable due to imperfection of the measuring instrument and our organs of sense and, as merely accidental errors, occur with equal probability in the positive or negative sense.

Even merely accidental errors can under certain circumstances act one-sidedly, as, e.g., the slipping away of the measuring rods from the straight line to be measured always leads to too large a measuring result. Such errors shall however not be taken into account in the following if there is no remark to the contrary.

As soon as we have recognized that it is impossible to determine the *true* values of the observed quantities most rigorously, we have to fix a less high aim for ourselves, namely to reach only the values of the unknowns *most probable* under given circumstances which best adapt themselves to the totality of all measurements. According to the kind of the observation and the number of the check measurements applied we will approach more or less the true values of the unknowns; therefore, after determining the most probable values we can still ask the question of what accuracy has been reached.

According to this, it is the problem of the computation of adjustment to draw, from observations which lead to discrepancies due to the unavoidable appendant observational errors, those results which best adapt themselves to the measurements (or let the least errors be feared), further to indicate those amounts by which the found results probably still deviate from the truth.

Or in other words:

The method of least squares deals with the adjustment of observational errors and with the determination of mean errors to be feared.

Section 3. The Average Error

The first considerations of errors always lead to a computation of the average which, it is true, plays only a subordinate role in present-day error theory, but must be communicated here first for an introduction into the understanding of the matter.

If the errors of several homogeneous observations are known, then we can form from these errors (taken absolutely, i.e. without taking into account the signs) an average value which is called the "*average error.*" True, we do not know the true observational errors in general any more than the true values of the observed quantities; yet this does not prevent us from forming the rigorous concept of the average error, i.e., if $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ are a number of true observational errors of the same kind, then the average error is

$$t = \frac{[\pm\varepsilon]}{n}, \tag{1}$$

where the symbol $[\pm\varepsilon]$ shall indicate the absolute sum of the ε's in contrast to the algebraic sum and the brackets serve as the symbol of a sum.

As an example for the computation of the average value of true errors we take the following:

In the case of the "Degree measurement in East Prussia" all angles α, β, γ were measured in 22 triangles and the sums of angles were compared with the theoretical sums 180° + spherical excess. The following 22 discrepancies resulted thereby:

$$\alpha + \beta + \gamma - (180^\circ + \text{spherical excess}) = \varepsilon$$

Number	Error, ε	Number	Error, ε	Number	Error, ε
1	+0.36″	9	+0.56″	17	+1.62″
2	+0.93	10	0.00	18	+1.62
3	−0.51	11	−0.59	19	+1.67
4	−1.46	12	0.00	20	−0.72
5	−0.95	13	−1.36	21	−1.35
6	−1.40	14	+1.86	22	−0.98
7	+1.76	15	−0.42		7.96
8	+0.92	16	+1.68		6.47
	8.29		6.47		8.29
				Total	22.72″ = [±ε]

$$\text{Average} = \frac{[\pm\varepsilon]}{n} = \frac{22.72''}{22} = \pm 1.03''.$$

Now if we consider the sum of the angles $\alpha + \beta + \gamma$ of a triangle as a measured quantity, then we have determined herewith its average error = ±1.03″, and this value as the average triangle error is always of a certain interest (the average error of an individual angle of a triangle was hereby not found, however).

The computation of the average error of a small number of true errors will not be very reliable; only in the case of a large number will the computed average error be suited to serve as a measure of accuracy. In a more precise definition we will therefore say that the average error is the limiting value which the expression

$$t = \frac{[\pm\varepsilon]}{n}$$

obtains in the case of an infinite number of observations.

If we can thus compute only an approximate value of the average error from a finite number of observational errors, then a further difficulty lies in the fact that we very rarely know the *true* errors.

On the other hand, there are known the *apparent* errors, which, e.g., we find if we compare the arithmetic mean of several observations.

This yields the following formulae:

$$\text{observations:} \quad l_1, l_2, l_3, \ldots, l_n$$

$$\text{arithmetic mean:} \quad x = \frac{l_1 + l_2 + l_3 + \ldots l_n}{n}, \tag{2}$$

or with the assumption of the brackets as the sign for algebraic summation

$$x = \frac{[l]}{n}. \tag{3}$$

We form the apparent errors, i.e. the differences

$$v_1 = x - l_1, \quad v_2 = x - l_2, \quad v_3 = x - l_3, \quad \ldots, \quad v_n = x - l_n;$$

we note here that the algebraic sum of these differences is $= 0$, namely

$$\begin{aligned} v_1 + v_2 + v_3 + \ldots &= n x - (l_1 + l_2 + l_3 + \ldots) \\ [v] &= n x - [l] = 0 \quad \text{because of (2) or (3)} \\ [v] &= 0. \end{aligned} \tag{4}$$

This equation (4) can be used as computational check.

We treat these apparent errors v approximately like true observational errors ε, i.e., we compute the average error according to the instruction of equation (1):

$$t = \frac{[\pm v]}{n}.$$

Example. An angle has been measured independently five times:

Observations l	Differences v	Check
35° 26′ 16″	+ 2.8″	
20	− 1.2	
18	+ 0.8	$[+v] = +7.4$
25	− 6.2	$[-v] = -7.4$
15	+ 3.8	$[v] = 0$
Total 94″	14.8″	
Mean $x = 35° 26' 18.8''$	$t = \frac{14.8}{5} = 2.96''$.	

With this, we have a measure for the presumable accuracy of an individual observation. Now it would further be of importance to know what relation the accuracy of an individual observation bears to the accuracy of the arithmetic mean, but these and other pertinent questions *cannot* be solved on the basis of the average error and the above simple considerations.

For this reason, we now drop the average error and pass over to another measure of accuracy, the *mean* error.

Section 4. The Mean Error

If there exists a very large number n of true observational errors $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$, then we form therefrom a mean value m according to the equation

$$m = \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \ldots \varepsilon_n^2}{n}} = \sqrt{\frac{[\varepsilon^2]}{n}}. \tag{1}$$

We call this mean value derived from the squares of the errors the "mean error" or "mean error to be feared" (error medius metuendus).

Here, also, deviating from the rigorous definition, we will have to determine the mean error from a limited number of observational errors and thus obtain only an approximate value.

Compared with the average error, the mean error offers the advantage that in the case of the summation $[\varepsilon^2]$, since all squares are positive, no distinction of the signs is necessary, as well as that the sign of the square root results indeterminately $\pm$ from the computation. Apart from these exterior differences, however, the mean error is a better measure of accuracy than the average error, as is already shown by the fact that in the squares the large errors are weightier.

In general the mean error is larger than the average error, and only if all individual errors ε are equal, then also do the average and the mean error become equal to one another; this can at first easily be conceived for two elements; for if we have only two true errors ε_1 and ε_2, then we have

The Average Error

$$t = \frac{\varepsilon_1 + \varepsilon_2}{2},$$

$$t^2 = \frac{\varepsilon_1^2 + \varepsilon_2^2 + 2\,\varepsilon_1\,\varepsilon_2}{4},$$

The Mean Error

$$m = \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2}{2}},$$

$$m^2 = \frac{\varepsilon_1^2 + \varepsilon_2^2}{2} = \frac{2\,\varepsilon_1^2 + 2\,\varepsilon_2^2}{4}.$$

In order to establish which one of the two has the greater value, we treat the difference of both:

$$m^2 - t^2 = \frac{\varepsilon_1^2 + \varepsilon_2^2 - 2\,\varepsilon_1\,\varepsilon_2}{4}, \quad \text{therefore} \quad = \frac{(\varepsilon_1 - \varepsilon_2)^2}{4}. \tag{2}$$

As square, this is always positive, and hence m^2 is always larger than t^2 (with the exception of the special case $\varepsilon_1 = \varepsilon_2$, for which m and t become equal).

This proof (2) can also easily be extended to an arbitrary number of elements $\varepsilon_1, \varepsilon_2 \ldots$.

For further comparison between the average error and the mean error we consider the following two error series:

											Sum	Sum of Squares
I.	5	6	2	7	3	8	10	9	3	5	58	402
II.	14	3	0	0	6	20	2	2	1	10	58	750 .

The average error becomes equal in both cases, namely = 5.8; the mean error, however, becomes

$$m_{I} = \sqrt{\frac{402}{10}} = \pm 6.34, \qquad m_{II} = \sqrt{\frac{750}{10}} = \pm 8.66.$$

The mean error makes the first series appear better than the second series, and in fact, a rather equal distribution of errors up to the limit 10 is more favorable than the reiterated overstepping of this limit with 14 and even 20 in the case of II, which is not counterbalanced by the reiterated occurrence of 0.

Although we can represent, in this or a similar manner, the introduction of the "mean error" as appropriate, yet it does not become possible to prove this choice of a measure of accuracy as *necessary*. The best justification of the "mean error," however, lies in the fact that an error theory, satisfactory from every point of view and now recognized already for more than 100 years, can be based upon it.

By adhering, according to this, to the definition of the mean error according to equation (1), we apply this equation to the previous small example, section 3, p. 8, first with the assumption as if the apparent errors v could be treated like true observational errors ε.

	Observations	v	v^2
	35° 26′ 16″	+2.8″	7.84
	20	−1.2	1.44
	18	+0.8	0.64
	25	−6.2	38.44
	15	+3.8	14.44
Sum	94″	$[v] = 0.0''$	$[v^2] = 62.80$
Mean	35° 26′ 18.8″		

$$m = \sqrt{\frac{62.80}{5}} = \pm 3.54''. \quad (?) \qquad (3)$$

We have inserted a question mark (?) next to this computation (3), because it is questionable whether the assumption of the v's as true errors, made at first, is admissible. The answer to this will be given in our later section 7. Meanwhile we must turn to another consideration.

The *mean error* according to the definition given in the foregoing is the fundamental concept in the investigations of accuracy; beside this mean error, the average error, which first presented itself in section 3, has only a very subordinate significance.

In addition, there exist two other measures of error, which we will mention here in passing, namely the *probable error* and the *limiting error*.

These two errors, however, will have to be treated in a later chapter.

Section 5. The Law of the Propagation of Errors

The point in question is in what way the mean errors of measured quantities are carried forward to the quantities derived therefrom by computation. This important question must be taken care of before further problems can be solved. We treat at first individual special cases of this question, namely multiplication and addition of measured quantities.

I. *Multiplication.* A measured quantity l is multiplied by a given number a; we thus have a product

$$x = a\,l \qquad (1)$$

Here a is given as a number free from error, but l which is a measured quantity is to be affected by a mean error m, and the question is what will be the mean error M of the product x.

By the multiplication by a the errors with which the quantity l is affected are affected also; therefore, we can quickly see that the result will be

$$M = a\,m. \qquad (2)$$

The measurement of a straight line with unreliable rods may serve as an illustration. If a measuring rod l in itself is uncertain by the amount $\pm m$, i.e., if it is inaccurate compared with the normal measure, and if the handling of the measurement itself is completely free from error, i.e., if we do not speak here at all of the measuring errors themselves, then, if we lay the faulty rod a times, an error $M = a\,m$ is obviously produced, and, in fact, $\pm M$, if $\pm m$ is indeterminate in sign.

II. *Addition.* Two quantities l and l' are measured independently of one another and then added; we thus have a sum

$$x = l + l' ; \tag{3}$$

let l and l' be affected by mean errors $\pm m$ and $\pm m'$, and now the question is what the mean error M of the sum x is.

At first glance it could seem as if there would have to be set simply

$$M = m + m' \text{ , (?)} \tag{3a}$$

but as soon as we examine the matter more closely, we note that this corresponds only to the extreme case of the *accumulation* of the errors m and m', while with irregular signs $\pm m$ and $\pm m'$ also the case of the reciprocal *cancellation* $m - m'$ must be taken into account. Besides, we must bear in mind that the errors of l and l' are not by any means directly m and m' themselves, but that these m's and m''s represent only the *mean* values of the errors with which l and l' are affected, and hence that we cannot speak of such a simple solution as (3a).

In order to arrive at the correct solution of the very important problem under consideration, we must apply the fundamental concept of the mean error, and for this, we imagine the addition $l + l' = x$ undertaken not *once* but repeatedly (about n times), and we take into consideration *all* errors ε, ε' and so forth, which occur in all these cases. In the first case, let l be affected with the error ε, and l' affected with ε', then, indubitably, the error of x in the first case is equal to $\varepsilon + \varepsilon'$, which we shall denote by δ. This applied n times yields

$$\begin{aligned}
\delta_1 &= \varepsilon_1 + \varepsilon_1' \\
\delta_2 &= \varepsilon_2 + \varepsilon_2' \\
\delta_3 &= \varepsilon_3 + \varepsilon_3' \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \\
\delta_n &= \varepsilon_n + \varepsilon_n' .
\end{aligned}$$

In order to pass over to mean errors, we must square

$$\begin{aligned}
\delta_1^2 &= \varepsilon_1^2 + \varepsilon_1'^2 + 2\,\varepsilon_1\,\varepsilon_1' \\
\delta_2^2 &= \varepsilon_2^2 + \varepsilon_2'^2 + 2\,\varepsilon_2\,\varepsilon_2' \\
\delta_3^2 &= \varepsilon_3^2 + \varepsilon_3'^2 + 2\,\varepsilon_3\,\varepsilon_3' \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\delta_n^2 &= \varepsilon_n^2 + \varepsilon_n'^2 + 2\,\varepsilon_n\,\varepsilon_n' \\
\hline
\text{Sum of squares} \quad [\delta^2] &= [\varepsilon^2] + [\varepsilon'^2] + 2\,[\varepsilon\,\varepsilon'] \\
\text{Mean of squares} \quad \frac{[\delta^2]}{n} &= \frac{[\varepsilon^2]}{n} + \frac{[\varepsilon'^2]}{n} + \frac{2\,[\varepsilon\,\varepsilon']}{n} .
\end{aligned} \tag{4}$$

Due to the fundamental explanation of the mean error we have here

$$\frac{[\delta^2]}{n} = M^2 , \qquad \frac{[\varepsilon^2]}{n} = m^2 , \qquad \frac{[\varepsilon'^2]}{n} = m'^2 , \tag{5}$$

and the question is still what $\frac{[\varepsilon\varepsilon']}{n}$ is.

This is the average value of all products $\varepsilon_1\varepsilon_1'$, $\varepsilon_2\varepsilon_2'$ and so forth. While all quadratic terms ε^2 and

ε'^2 in (4) are positive, the products $\varepsilon_1\varepsilon'_1$, $\varepsilon_2\varepsilon'_2$ and so forth will be partly positive, partly negative, so that these products will in part cancel themselves. Since all ε's shall with equal probability be positive or negative, then, if any error sources acting one-sidedly are not existing, we should also not assume that the positive or the negative amounts are predominant. Consequently, the average value of the products is to be set equal to zero. [More accurately stated, the limiting value to which the last term in (4) converges, if the number of n's increases, is equal to zero.]

With this and because of (5), (4) yields now

$$M^2 = m^2 + m'^2 \quad \text{or} \quad M = \sqrt{m^2 + m'^2}\,. \tag{6}$$

This important theorem about the propagation of errors is therefore in conformity with the Pythagorean theorem of geometry, since M belongs to m and m' as the hypotenuse to the legs, and is therefore easily remembered. It reads:

If a measurement with the mean error $\pm m$ is added to a second measurement with the mean error $\pm m'$, then there results a sum whose mean error $\pm M$ is the square root of the sum of squares of m and m', or the sum error M is the hypotenuse to the partial errors m and m' as legs.

This theorem holds for differences as well as for sums, i.e., if from the measurements $l \pm m$ and $l' \pm m'$ the difference $D = (l \pm m) - (l' \pm m')$ is formed, then this difference is affected with *the same* mean error $M = \sqrt{m^2 + m'^2}$, as the sum $l + l'$. We easily convince ourselves of this if we go through once again the above development, for nothing will change with $l - l'$ instead of $l + l'$ except that in (4) the last term becomes negative; but since this last term in (4) is equal to zero, everything still remains the same.

Theorem (6) can also easily be extended to more than two measurements. If we have

$$x = (l_1 \pm m_1) + (l_2 \pm m_2) + (l_3 \pm m_3)\,,$$

then we can first collect the first two elements according to theorem (6) and then add the third, i.e., we have for three elements

$$M^2 = (m_1{}^2 + m_2{}^2) + m_3{}^2\,.$$

Thus we can continue arbitrarily, whereby we obtain

$$M^2 = m_1{}^2 + m_2{}^2 + m_3{}^2 + m_4{}^2 + \ldots\,. \tag{7}$$

If all individual errors $m_1, m_2, m_3 \ldots$ are here *equal* to one another, which often occurs as a special case, then in the case of n errors we will have

$$M^2 = m^2 + m^2 + m^2 \ldots = n\,m^2\,,$$

$$M = m\sqrt{n}\,. \tag{8}$$

For illustration let us use again the example of the rod measurement; however, in contrast to the previous case in (2) it shall now be assumed that the rods in themselves are free from error, but that upon each rod laying l there is committed an error resulting from handling, whose mean value is equal to $\pm m$, where the predominance of positive or negative errors shall be excluded. (The latter is not actually the case, as is known; but let us ignore this at present.)

Under these assumptions we have equation (8) valid for laying the rod n times with $\pm m$ as the mean error of a laying, and according to this, the mean rod measuring error is proportional to the square root of the number n of rod layings, i.e., also proportional to the square root of the measured length L (because L is proportional to n).

III. *Linear function.* By connecting both the theorems of I for multiplication and of II for addition or subtraction, as the case may be, we obtain the general theorem for the determination of the mean error of a linear function of measured quantities:

If $l_1, l_2, l_3 \ldots$ are measured quantities and $m_1, m_2, m_3 \ldots$ are their known mean errors, then we deal with the linear function

$$x = a_1 l_1 + a_2 l_2 + a_3 l_3 + \ldots , \tag{9}$$

whose mean error M results easily from the above:

$$M = \sqrt{(a_1 m_1)^2 + (a_2 m_2)^2 + (a_3 m_3)^2 + \ldots} \tag{10}$$

If we have here $m_1 = m_2 = m_3 \ldots = m$, then we will have

$$M = m \sqrt{[a\,a]} . \tag{11}$$

IV. *General function.* With the above theorems I, II, III, the important law of the propagation of errors in itself is taken care of; anything greater than a linear function (9) cannot in general be treated in the same manner; but we can reduce, at least approximately, any other arbitrary function to a linear function by differentiating according to Taylor's series, with the assumption that the errors are relatively *small* quantities. Although the generalization of the law of propagation of errors based upon it is not needed for our next working with the theory in section 6 to section 11 (and hence, can also first be passed over), we will insert it here and illustrate it by a small geodetic example.

Let us have any function x of the measured quantities $l_1, l_2, l_3 \ldots$, whose mean errors are $m_1, m_2, m_3 \ldots$. If $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$ are the true errors of the measured quantities, then

$$x + \varepsilon = f(l_1 + \varepsilon_1, l_2 + \varepsilon_2, l_3 + \varepsilon_3 \ldots) \tag{12}$$

means the true value of the function.

Under the assumption that these errors are so small that their second and higher powers can be neglected, we can determine the error ε of x with the help of the Taylor series, i.e., we will then have

$$\varepsilon = f(l_1 + \varepsilon_1, l_2 + \varepsilon_2, l_3 + \varepsilon_3 \ldots) - f(l_1, l_2, l_3 \ldots)$$

$$\varepsilon = \frac{\partial f}{\partial l_1}\varepsilon_1 + \frac{\partial f}{\partial l_2}\varepsilon_2 + \frac{\partial f}{\partial l_3}\varepsilon_3 + \ldots$$

By conclusions similar to those applied in the case of I and II we arrive at the result

$$M = \sqrt{\left(\frac{\partial f}{\partial l_1} m_1\right)^2 + \left(\frac{\partial f}{\partial l_2} m_2\right)^2 + \left(\frac{\partial f}{\partial l_3} m_3\right)^2 + \ldots}, \tag{13}$$

where the differential quotients are computed with the help of approximate values of the l_1's, l_2's, l_3's . . . .

A simple example of the application of this theorem (13) is the following (Fig. 1):

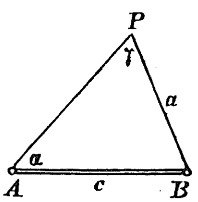


Fig. 1.
Mean error of side a.

A point P is fixed with respect to a fixed base line $AB = c$ by the measurement of the two angles α and γ, whereby the side $BP = a$ is determined:

$$a = \frac{c}{\sin \gamma} \sin \alpha , \tag{14}$$

α and γ are affected here with mean errors $\delta\alpha$ and $\delta\gamma$; the question is what the mean error M of the side a is. The base c is considered here as free from error.

We form the total differential of (14)

$$d\,a = \frac{\partial a}{\partial \alpha}\,d\alpha + \frac{\partial a}{\partial \gamma}\,d\gamma\;,$$

$$d\,a = \frac{c}{\sin\gamma}\,\cos\alpha\,d\alpha - c\,\sin\alpha\,\frac{\cos\gamma}{\sin^2\gamma}\,d\gamma\;.$$

Because of (14) we can also write this thusly:

$$d\,a = a\cot\alpha\,d\alpha - a\cot\gamma\,d\gamma\;.$$

Now we set instead of the differentials $d\alpha$ and $d\gamma$ the mean errors $\pm\delta\alpha$ and $\pm\delta\gamma$, and with this, we have the mean error M according to the rule of (13):

$$M = a\sqrt{\cot^2\alpha\,(\delta\alpha)^2 + \cot^2\gamma\,(\delta\gamma)^2}\;. \tag{15}$$

If both angles α and γ are measured with equal accuracy, then $\delta\alpha = \delta\gamma = \delta$; with this, we will have

$$M = a\,\delta\sqrt{\cot^2\alpha + \cot^2\gamma}\;.$$

δ is understood here in radian measure; if δ'' is the angle error in seconds, then we will have

$$\frac{M}{a} = \frac{\delta''}{\rho''}\sqrt{\cot^2\alpha + \cot^2\gamma}\;.$$

For example, we take $\alpha = \gamma = 60^\circ$ and $\delta = 10''$; this yields

$$\frac{M}{a} = \pm 0.0000396\;,$$

or the error M is about $= 0.004\%$ of a.

If we take here $a = 1000\,m$, then we have therefore $M = \pm 0.04\,m$.

If, for instance, in the triangle (Fig. 1) the third angle at B is also measured, then the computation of the mean error of the side a can no longer be made in such a simple manner; how to proceed then will not be shown until later in the case of the theory of conditioned observations.

Section 6. Combined Effect of Irregular and Regular Errors

Until now we have assumed that no constant errors or errors with a one-sided effect exist; however, the main proposition II of the previous section 5 about propagation of errors is still valid even when a combination of an irregular with a regular or with a constant error is involved, for the mean value $\frac{[\varepsilon\varepsilon']}{n}$, which is the main point in case II, p. 12, still converges with respect to zero even when only ε or ε' is positive and negative with equal probability.

As an example for this we take again the length measurement with measuring rods as in the previous section 5, p. 13. For such a measurement is affected not only with irregular errors, but also with regular errors. Such regular errors, for instance, are the slipping away from the straight line to the left or to the right, upward or downward. These deviations do not cancel themselves reciprocally at all, but they yield only positive errors, i.e., errors which make the length appear too large.

In the case of laying the rod n times, we can assume the error parts acting one-sidedly $= A\,n$ and the parts acting irregularly $= B\sqrt{n}$; then the total mean error is

$$M = \sqrt{(A\,n)^2 + (B\sqrt{n})^2} = \sqrt{A^2 n^2 + B^2 n}\;.$$

The concept of the mean error is thus not bound to the condition of equal probability for positive and negative individual errors; we can also use the mean error as a measure of accuracy for such measurements in the case of which there are error sources with a one-sided effect; however, a mean error which contains constant parts must in general, of course, not be treated further in the same way as a mean error which does not contain such parts.

It is an often-heard objection against the application of the method of least squares to certain measurements that these measurements contain one-sided errors, and that, therefore, the method of least squares is not to be used at all in such cases.

Against this there is to be held that just the method of least squares offers the finest means in order to detect one-sidedly acting or unknown errors, and then to treat the adjustment taking into account such error sources.

Section 7. The Simple Arithmetic Mean

If a measurement has been repeated several times homogeneously and independently with the results $l_1, l_2, l_3, \ldots, l_n$, then we take as the best suited value the arithmetic mean:

$$x = \frac{l_1 + l_2 + l_3 + \ldots l_n}{n} = \frac{[l]}{n} . \tag{1}$$

We have thus far designated the deviations of the individual measurements from the mean x as the apparent errors. But these deviations are at the same time the corrections which we must add to the measurements in order to obtain the value of adjustment, i.e. the arithmetic mean. We will therefore designate them from now on as *corrections* of the observations. The corrections are

$$\left.\begin{array}{l} v_1 = x - l_1 \\ v_2 = x - l_2 \\ v_3 = x - l_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n = x - l_n \end{array}\right\}, \tag{2}$$

whose sum, because of (1), is equal to zero, i.e.

$$[v] = 0 . \tag{3}$$

By squaring the corrections v we find the mean error of an observation, first approximately, by the formula

$$m = \sqrt{\frac{[v\,v]}{n}} . \ (?) \tag{4}$$

The point in question is now the determination of the mean error M of the arithmetic mean itself, for which the instruction of the theorem about propagation of errors (10) and (11), section 5, p. 14, serves, for according to (1) x is a linear function of the observed quantities l, to all of which the mean error m is assigned, as is shown still more clearly if we extend equation (1) and write thusly:

$$x = \frac{1}{n} l_1 + \frac{1}{n} l_2 + \frac{1}{n} l_3 + \ldots \frac{1}{n} l_n . \tag{5}$$

This corresponds to equation (9), section 5, p. 14, namely

$$x = a_1 l_1 + a_2 l_2 + a_3 l_3 + \ldots , \tag{5a}$$

and to this, there belongs the application (10), section 5, p. 14, namely

$$M = \sqrt{(a_1 m_1)^2 + (a_2 m_2)^2 + (a_3 m_3)^2 + \ldots} \quad . \tag{5b}$$

Therefore, the coefficients denoted in (5a) by $a_1, a_2, a_3 \ldots$ are in our case (5) all $= \frac{1}{n}$; the mean errors $m_1, m_2, m_3 \ldots$ are in our case all equal to one another, namely $= m$; and hence, the application of (5b) yields now

$$M = \sqrt{(a_1 m_1)^2 + (a_2 m_2)^2 + (a_3 m_3)^2 + \ldots} = m\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2 + \ldots} ,$$

here we have

$$a_1{}^2 + a_2{}^2 + a_3{}^2 + \ldots = \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \ldots = n\left(\frac{1}{n}\right)^2 = \frac{1}{n},$$

therefore

$$M = m\sqrt{\frac{1}{n}} \quad \text{or} \quad M = \frac{m}{\sqrt{n}}, \tag{6}$$

i.e., the mean error M of the arithmetic mean x is obtained from the mean error m of an individual observation by division by the square root $\sqrt{n}$ of the number n of the observations. We will consider this important and interesting theorem still somewhat more closely later in connection with Fig. 1, p. 14, but before this, we must turn once again to equation (4) in order to treat the question still left unanswered there in what respect the corrections v are suited for the replacement of the true errors ε.

These true errors ε remain unknown, but nevertheless we can take the deviations between the v's and the ε's into account approximately in the following manner:

In contrast to the arithmetic mean x we denote by X the true value of the unknown, and by ε the true observational errors in contrast to the corrections v; then we have the equations

$$\varepsilon = X - l, \qquad v = x - l, \qquad \text{therefore} \qquad \varepsilon = v + (X - x) . \tag{7}$$

If we apply this to n cases, and square $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, then we obtain

$$\begin{aligned}
\varepsilon_1{}^2 &= v_1{}^2 + (X - x)^2 + 2\, v_1 (X - x) \\
\varepsilon_2{}^2 &= v_2{}^2 + (X - x)^2 + 2\, v_2 (X - x) \\
&\ldots\ldots\ldots\ldots \\
\varepsilon_n{}^2 &= v_n{}^2 + (X - x)^2 + 2\, v_n (X - x) \\
\hline
\text{Sum} \quad [\varepsilon^2] &= [v^2] + n\,(X - x)^2 + 2\,(v_1 + v_2 + \ldots v_n)(X - x) .
\end{aligned}$$

But now, according to (3), $v_1 + v_2 + \ldots v_n = [v] = 0$; therefore

$$[\varepsilon^2] = [v^2] + n\,(X - x)^2 . \tag{8}$$

The value $X - x$ is the true error of the arithmetic mean x, and hence a quantity which we can never determine most in all rigor. We must therefore limit ourselves to introduce an average value for $(X - x)^2$. Such an average value of $(X - x)^2$ is M^2, the square of the mean error of x. Then we obtain

from (8) $\quad \dfrac{[\varepsilon^2]}{n} = \dfrac{[v^2]}{n} + M^2 \qquad$ or according to (6) $\quad \dfrac{[\varepsilon^2]}{n} = \dfrac{[v^2]}{n} + \dfrac{m^2}{n}$, and since $\frac{[\varepsilon^2]}{n} = m^2$, then we will have $m^2 = \frac{[v^2]}{n} + \frac{m^2}{n}$ or $m^2 (n-1) = [v^2]$; (9)

there follows hence

$$m^2 = \frac{[v^2]}{n-1} \quad \text{or} \quad m = \sqrt{\frac{[v^2]}{n-1}} \,. \tag{10}$$

This is the required correct formula, which takes the place of the approximate formula (4); we can also substitute formula (10) again in (6), with which we have

$$M = \frac{m}{\sqrt{n}} = \sqrt{\frac{[v^2]}{n(n-1)}} \,. \tag{11}$$

The new formula (10) appears better, even without the mathematical derivation, by mere sight, than the previous formula (4); in particular, in the special case with $n = 1$. For this, (4) would yield: $[v^2] = 0$; therefore $m = 0$. On the other hand, it is true, (10) yields also $[v^2] = 0$, but also in the denominator $n - 1 = 0$; therefore altogether $m = \sqrt{\frac{0}{0}}$, i.e. it is indeterminate, and in fact, if only *one* observation exists, the accuracy must remain undetermined. [As a last extreme case we can also set, in addition, $n = 0$, i.e., ask for the accuracy of an observation which has not been made at all; for this, also, (10) gives the correct answer, since because $n - 1 = 0 - 1 = -1$ the mean error m becomes imaginary.]

For the application of the theory of the simple arithmetic mean developed above, we take again the small numerical example at the end of section 3, p. 9:

	Observations l	v	v^2
1.	35° 26′ 16″	+ 2.8″	7.84
2.	20	− 1.2	1.44
3.	18	+ 0.8	0.64
4.	25	− 6.2	38.44
5.	15	+ 3.8	14.44
Sum	= 94″	0.0″	62.80 = $[v^2]$.
Mean	= 35° 26′ 18.8″ .	(Check)	

Mean error of an individual observation:

$$m = \sqrt{\frac{[v^2]}{n-1}} = \sqrt{\frac{62.80}{4}} = \pm 3.96'' .$$

Mean error of the arithmetic mean:

$$M = \frac{m}{\sqrt{n}} = \sqrt{\frac{[v^2]}{n(n-1)}} = \frac{3.96}{\sqrt{5}} = \pm 1.77'' .$$

Altogether we write now, rounding off:

$$x = 35°26'18.8'' \pm 1.8'' .$$

A second larger numerical example

In the *Gradmessung in Ostpreussen* (p. 78) Bessel gives 18 independent measurements for the angle Mednicken-Fuchsberg at the station Trenk, as shown in the following summary, which contains at the same time the error computation:

No.	Observations l	v	v^2
1	83° 30′ 36,25″	− 1,38″	1,90
2	7,50	− 2,63	6,92
3	6,00	− 1,13	1,28
4	4.77	+ 0,10	0.01
5	3,75	+ 1,12	1,25
6	0,25	+ 4,62	21,34
7	3,70	+ 1,17	1,37
8	6.14	− 1,27	1,61
9	4.04	+ 0,83	0,69
10	6,96	− 2,09	4,37
11	3.16	+ 1,71	2,92
12	4,57	+ 0,30	0,09
13	4,75	+ 0,12	0.01
14	6,50	− 1,63	2,66
15	5,00	− 0,13	0,02
16	4.75	+ 0.12	0,01
17	4.25	+ 0.62	0,38
18	5.25	− 0.38	0,14
Sum	87,59″	+10,71″	46.97
Mean	83° 30′ 34,87″	−10,64″	$= [v^2]$

Mean error of an individual observation:

$$m = \sqrt{\frac{[v^2]}{n-1}} = \sqrt{\frac{46{,}97}{17}} = \pm 1{,}66'' \,.$$

Mean error of the arithmetic mean:

$$M = \frac{m}{\sqrt{n}} = \frac{1{,}66}{\sqrt{18}} = \pm 0{,}39'' \,.$$

Main result $= 83° \; 30' \; 34{,}87'' \pm 0{,}39''$.

Graphical representation of equation (6)

Equation (6) found on p. 17 is very important in more than one respect; it says in words that the mean error of the arithmetic mean decreases in proportion to the *square root* of the number of the measurements. For illustration of this law we can plot M as the ordinate to the abscissa n whereby we obtain a hyperbolic curve of the third degree, which is represented in the following figure. The numerical values for this are:

$$\left.\begin{array}{rccccccccccc} n = & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 & 20 & 50 & 100 \\ \dfrac{1}{\sqrt{n}} = & 1{,}00 & 0.71 & 0.58 & 0{,}50 & 0.45 & 0.41 & 0.35 & 0.32 & 0.22 & 0.14 & 0.10 \end{array}\right\} . \qquad (12)$$

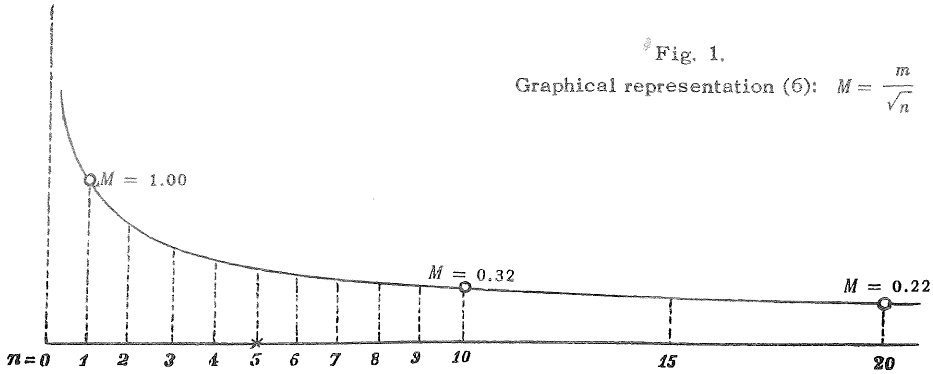


Fig. 1.
Graphical representation (6): $M = \frac{m}{\sqrt{n}}$

The curve runs asymptotically, i.e., the mean error M approaches zero infinitely close, without ever reaching the value zero itself. The first 5-10 repetitions yield rapidly a decrease of M, i.e. an increase of accuracy; but then further repetition is not very successful, and in order to bring down the mean error of *one* first measurement to a tenth of its value we would have to make 100 repetitions.

But usually one does not do this; one and the same measurement is usually repeated at the most 5-10 times, and in fact not just because from there on the value M decreases but slowly, but for still a much more important reason: For the equation (6) and the numerical values (12) computed according to it, in addition to the pertinent curve, assume that the measurements are affected *only* with irregular errors, positive or negative with equal probability, and this is in reality hardly ever the case. On the contrary, the finer the measurements become and the oftener we repeat them, all the more we come to the conclusion that constant error sources act nearly everywhere. Therefore, in the case of repetitions we shall also change, as much as possible, the accidental circumstances, e.g. use, in the case of angle measurements, gradually different marks of the circular subdivision and so forth.

Introduction of weight

For later application, e.g., in the following section 8 we consider once again the important equation (6) in another respect:

$$M = \frac{m}{\sqrt{n}} \qquad \text{or} \qquad M^2 = \frac{m^2}{n} \,. \tag{13}$$

It follows, hence, that we can consider the arithmetic mean as the result of *one* observation whose mean error is M; therefore, the ratio of accuracy of this imaginary observation and an original observation is determined by the values M and m.

This leads to the concept of *weight*.

The ratio of m^2 to M^2 is determined by the number n, which indicates how many observations of one kind must be combined into an arithmetic mean, in order that the latter has the accuracy of an observation of the other kind. n is called, in this case, the *weight* of the latter observation (where we set the weight of an observation of the first kind $= 1$).

Section 8. The General Arithmetic Mean

If for the determination of an unknown there exist several observations having *unequal values*, i.e., such to which equal accuracy is not attributed from the outset, then the simple arithmetic mean treated in the previous section 7 is not the most probable value of the unknown. How to proceed in this case will be shown by a consideration, which we introduce with a simple case:

Let us have five homogeneous observations $l'_1, l'_2, l'_3, l'_4, l'_5$ and their total simple arithmetic mean:

$$x = \frac{l'_1 + l'_2 + l'_3 + l'_4 + l'_5}{5} . \tag{1}$$

Besides, we consider two partial means from two and three observations:

$$l_1 = \frac{l'_1 + l'_2}{2}, \qquad l_2 = \frac{l'_3 + l'_4 + l'_5}{3}, \tag{2}$$

then it is easy to see that from the partial means l_1 and l_2 we can again produce the total mean x without returning to the original observations l'; for there results from (1) and (2)

$$x = \frac{2l_1 + 3l_2}{2 + 3} . \tag{3}$$

The basic idea shown by this simple case with $2 + 3 = 5$ observations can also easily be carried out more generally; we will assume group means $l_1, l_2, l_3 \ldots$ from $p_1, p_2, p_3 \ldots$ original, equally accurate observations, and compute therefrom the total mean by analogy to (3):

$$x = \frac{p_1 l_1 + p_2 l_2 + p_3 l_3 + \ldots}{p_1 + p_2 + p_3 \ldots} = \frac{[p\,l]}{[p]} . \tag{4}$$

The numbers p have here the meaning of weights according to the explanation at the end of the previous section 7, i.e., the p's are the same as the numbers of repetition denoted there by n, to which the partial means $l_1, l_2, l_3 \ldots$ belong.

We go one step further by assuming that these l_1's, l_2's, l_3's ... are not partial means but themselves *direct* observations of different accuracy; then we would have to proceed just as in the previous case according to equation (4), after having determined first those numbers p which correspond to the individual accuracies, and this is achieved if we take for p the weights according to the definition given at the end of the previous section 7.

Since, moreover, the value of x in (4) is not changed if we multiply all weights p by an arbitrary number, then it follows that it is sufficient for the solution of the problem if there are used such weights p which are *proportional* to the numbers defined previously as weights, and therefore we shall henceforth understand the weights generally only as proportional numbers.

It can further easily be shown that the weights p must be inversely proportional to the squares of the mean errors, because according to (6), section 7, p. 17, the mean error is inversely proportional to the square root of the number of repetitions, i.e. of the weight number p. In order to show this more clearly, we will apply the mentioned equation (6), section 7, p. 17, namely

$$M = \frac{m}{\sqrt{n}},$$

in which n occurs as the weight, to two cases with $n = p$ and $n = p'$:

$$M = \frac{m}{\sqrt{p}}, \qquad M' = \frac{m}{\sqrt{p'}} .$$

There follows hence the theorem: If M, M' are the mean errors, and p, p' the weights of two observations, then

$$\frac{p}{p'} = \frac{M'^2}{M^2} \quad \text{or} \quad \frac{M}{M'} = \frac{\sqrt{p'}}{\sqrt{p}}, \tag{5}$$

i.e., the weights are in inverse ratio to the squares of the mean errors, or the mean errors are in inverse ratio to the roots of the weights.

A concept related to the concept weight is that of *accuracy*; as the mean error decreases, the weight as well as the accuracy increases, however in a different measure. While the weight is inversely proportional to the *square* of the mean error, we understand by accuracy a quantity which is only inversely proportional to the mean error itself. Moreover, no need exists for introducing especially accurate numbers into our computations.

In what manner the weights are determined cannot generally be indicated. The determination of the weights is simplest, if we do not have to deal with direct observations, but, according to the investigation at the beginning of this section, with partial means from individual groups of observations, which at times happens.

After the meaning of the weights p has been clarified, all the rest results in a similar manner as in the case of the simple arithmetic mean. Thus far we have the following:

Given the observations: $l_1, l_2, l_3, \ldots, l_n$

with the weights: $p_1, p_2, p_3, \ldots, p_n$.

The most probable value of the unknown is the arithmetic mean:

$$x = \frac{p_1 l_1 + p_2 l_2 + p_3 l_3 + \ldots p_n l_n}{p_1 + p_2 + p_3 + \ldots p_n} = \frac{[p\,l]}{[p]}. \tag{6}$$

We form, as previously, the corrections

$$\left.\begin{array}{llll} v_1 = x - l_1 & \text{with the weight} & & p_1 \\ v_2 = x - l_2 & \text{"} \quad \text{"} & \text{"} & p_2 \\ v_3 = x - l_3 & \text{"} \quad \text{"} & \text{"} & p_3 \\ \ldots\ldots\ldots & \ldots\ldots & \ldots & \ldots \\ v_n = x - l_n & \text{"} \quad \text{"} & \text{"} & p_n \end{array}\right\}. \tag{7}$$

If we multiply these v's by their respective p's and then add them, then we obtain because of (6)

$$[p\,v] = 0. \tag{8}$$

Now we consider that mean error m, which belongs to a weight $p = 1$; whether among the series $p_1, p_2, p_3 \ldots$, a value $p = 1$ actually occurs is immaterial here; at any rate, we can imagine such a value $p = 1$ and consider the mean error m corresponding to it, which we call briefly also the *error of the unit of weight*. From this and from the weights $p_1, p_2, p_3 \ldots$, according to the relations (5), there also follow the mean errors of the l_1's, l_2's, l_3's $\ldots$, i.e., respectively,

$$\frac{m}{\sqrt{p_1}}, \frac{m}{\sqrt{p_2}}, \frac{m}{\sqrt{p_3}}, \ldots, \frac{m}{\sqrt{p_n}}. \tag{9}$$

In order to apply now the law of propagation of errors, we write first (6) in the extended form

$$x = \frac{p_1}{[p]} l_1 + \frac{p_2}{[p]} l_2 + \frac{p_3}{[p]} l_3 + \ldots \frac{p_n}{[p]} l_n \,. \tag{10}$$

This equation (10) corresponds again to the linear function (9) of the general law of propagation of errors in section 5, p. 14, namely

$$x = a_1 l_1 + a_2 l_2 + a_3 l_3 + \ldots \quad . \tag{11}$$

In (10) the a_1's, a_2's . . . are replaced by $\frac{p_1}{[p]}, \frac{p_2}{[p]} \ldots$, and since we also have the mean errors of the individual sum terms in (9), the application of (11) yields

$$M^2 = \left(\frac{p_1}{[p]} \frac{m}{\sqrt{p_1}}\right)^2 + \left(\frac{p_2}{[p]} \frac{m}{\sqrt{p_2}}\right)^2 + \left(\frac{p_3}{[p]} \frac{m}{\sqrt{p_3}}\right)^2 + \ldots .$$

$$M^2 = \left(\frac{m}{[p]}\right)^2 \left((\sqrt{p_1})^2 + (\sqrt{p_2})^2 + (\sqrt{p_3})^2 + \ldots\right).$$

$$M^2 = m^2 \frac{p_1 + p_2 + p_3 + \ldots}{[p]^2} = m^2 \frac{[p]}{[p]^2} = \frac{m^2}{[p]},$$

$$M = \frac{m}{\sqrt{[p]}} . \tag{12}$$

In addition, we can also indicate the weight P of the arithmetic mean x. Since the weights are in inverse ratio to the squares of the mean errors, and the weight 1 corresponds to the mean error m, then we have

$$P:1 = m^2:M^2 \qquad \text{or} \qquad P = \frac{m^2}{M^2} \,,$$

and hence, according to (12) $$P = [p] \,.$$

We pass over to the determination of the mean error m of an observation with the weight 1 (error of unit of weight) by adopting, for this, the same way as in section 7, p. 16. According to equation (7) there, p. 17, we have $\varepsilon = v + (X - x)$ or after multiplication by $\sqrt{p}$

$$\varepsilon \sqrt{p} = v \sqrt{p} + (X - x) \sqrt{p} \,.$$

If this is applied to n cases and squared, then we obtain

$$\begin{array}{rl} & \varepsilon_1^2 p_1 = v_1^2 p_1 + (X - x)^2 p_1 + 2\, v_1 p_1 (X - x) \\ & \varepsilon_2^2 p_2 = v_2^2 p_2 + (X - x)^2 p_2 + 2\, v_2 p_2 (X - x) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \varepsilon_n^2 p_n = v_n^2 p_n + (X - x)^2 p_n + 2\, v_n p_n (X - x) \\ \hline \text{Sum} & [\varepsilon^2 p] = [v^2 p] + (X - x)^2 [p] + 2 (X - x) [v p] \end{array}$$

Now we have according to (8), p. 22, $[v p] = 0$; and hence, there remains

$$[\varepsilon^2 p] = [v^2 p] + (X - x)^2 [p] \,.$$

Because neither the ε's nor the difference $X - x$ are known herein, the average values of the first and the last term can only be introduced by estimation. First we have for $(X - x)^2$ the average value M^2, and since according to (12) $M^2 [p] = m^2$, then we can replace the last term by m^2. Therefore, we have

$$\varepsilon_1^2 p_1 + \varepsilon_2^2 p_2 + \ldots + \varepsilon_n^2 p_n = [v^2 p] + m^2 .$$

The average values for the individual squares $\varepsilon_1^2, \varepsilon_2^2 \ldots \varepsilon_n^2$ are $m_1^2, m_2^2 \ldots m_n^2$. Since the weight $p = 1$ corresponds to the mean error m, then we have further

$$m_1^2 = \frac{m^2}{p_1}, \qquad m_2^2 = \frac{m^2}{p_2} \quad \text{and so forth;}$$

consequently, the above equation changes now to

$$m^2 + m^2 + \ldots + m^2 = [v^2 p] + m^2 \qquad \text{or} \qquad (n-1)\, m^2 = [v^2 p] ,$$

and there follows hence

$$m = \sqrt{\frac{[p\, v^2]}{n-1}}. \tag{13}$$

In the two equations (12) and (13) we have now the whole error theory of the general arithmetic mean; from (12) and (13) together we can in addition also form:

$$M = \sqrt{\frac{[p\, v^2]}{[p]\,(n-1)}}. \tag{14}$$

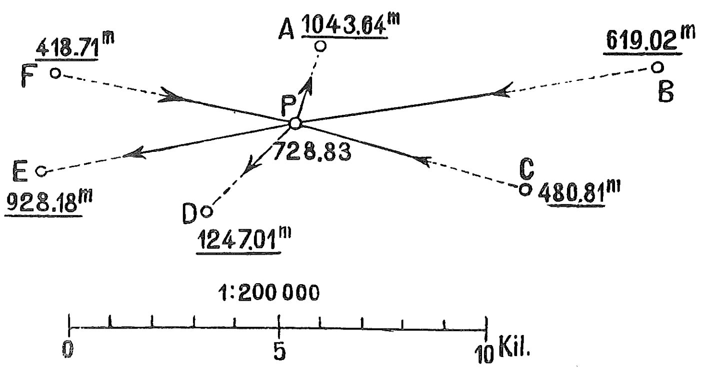


Fig. 1.

As a numerical example, in Fig. 1 we will take an elevation computation in which the weights, for instance, are not determined by heterogeneous repetitions of measurement, but result from the nature of the problem itself.

We assume that A, B, C, D, E, F are elevation points of the land survey whose elevations above N.N. [Normal Null] are fixed; e.g., A has the elevation 1043.64 m above N.N., and so forth. We also assume that these six elevation data are free from error, or that their errors shall not be considered beside the errors of the sixfold elevation determination from a point P at which the elevation angles have been measured to A, B and so forth and used with the known distances PA, PB and so forth for elevation computations. In this manner, the following has been obtained:

Aiming Range s	Given Elevations Above N.N.	Measured Elevation Differences	Computed Elevations of P
$AP = 2010\ m$	A 1043.64 m	$h_1 = -314.73\ m$	728.91 m
$BP = 8903$	B 619.02	$h_2 = +109.20$	728.22
$CP = 5820$	C 480.81	$h_3 = +248.24$	729.05
$DP = 3002$	D 1247.01	$h_4 = -518.43$	728.58
$EP = 6197$	E 928.18	$h_5 = -199.16$	729.02
$FP = 5800$	F 418.71	$h_6 = +310.13$	728.84

(Simple Mean = 728.77 m) (15)

The simple arithmetic mean of the six elevation determinations for the point would be 728.77 m; but we cannot assume this simple mean as the result, because in view of the inequality of the distances s the six determinations are not equivalent. From the theory of trigonometric elevation measurement we know that the errors of the elevation differences h are (nearly) proportional to the aiming ranges s, and consequently, the weights p must be inversely proportional to the squares of the aiming ranges s, or

$$p = \frac{1}{s^2} .$$

The unit of measure is arbitrary here; we take s in kilometers, and in fact rounded off according to (15):

$$s = 2.0 \quad 8.9 \quad 5.8 \quad 3.0 \quad 6.2 \quad 5.8 \text{ km} .$$

We compute therefrom

$$p = \frac{1}{s^2} = 0.25 \quad 0.01 \quad 0.03 \quad 0.11 \quad 0.03 \quad 0.03 . \tag{16}$$

We have intentionally rounded off rather strongly, because it has no practical value in such cases where the measurement of weight itself rests on certain, nonrigorous surveys, to compute with many decimals.

With the elevations of (15) and the weights (16) we obtain now the following computation, which follows the formulae (6), (12), and (13):

l	p	pl	$v = 0.83 - l$	pv	pv^2
728 + 0.91	0.25	0.2275	− 0.08	− 0.0200	0.0016
+ 0.22	0.01	0.0022	+ 0.61	+ 0.0061	0.0037
+ 1.05	0.03	0.0315	− 0.22	− 0.0066	0.0015
+ 0.58	0.11	0.0638	+ 0.25	+ 0.0275	0.0070
+ 1.02	0.03	0.0306	− 0.19	− 0.0057	0.0011
+ 0.84	0.03	0.0252	− 0.01	− 0.0003	0.0000
Sum:	0.46	0.3808		+ 0.0336	0.0149
				− 0.0326	
				Check $[pv]$.	

$$x = \frac{[pl]}{[p]} = \frac{0.3808}{0.46} = 0.83 ,$$

$$m = \sqrt{\frac{[pv^2]}{n-1}} = \sqrt{\frac{0.0149}{5}} = \pm 0.055\, m . \tag{17}$$

$$M = \frac{m}{\sqrt{[p]}} = \frac{0.055}{\sqrt{0.46}} = \pm 0.080\, m .$$

And hence, all together: $H = 728 \text{ m} + x = 728.83 \text{ m}$ with the mean error: ± 0.08 m .

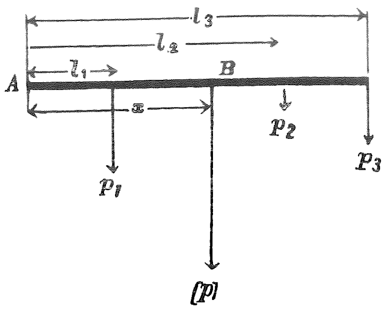


Fig. 2.

The formulae treated on p. 25 permit in part simple mechanical interpretations.

If we imagine, according to Fig. 2, different weights $p_1, p_2, p_3 \ldots$ acting at an axis of rotation A with lever arms $l_1, l_2, l_3 \ldots$, then the static moments of these weights are $p_1 l_1, p_2 l_2, p_3 l_3 \ldots$, respectively, and the equation

$$x = \frac{[p\,l]}{[p]}$$

yields that lever arm x, which, charged with the sum of all weights $[p]$, yields the same static moment as the sum of the individual moments.

If we imagine now the axis of rotation A displaced to B by the amount x, then the weights p_1, p_2, p_3 have the lever arms v_1, v_2, v_3, with reference to the new axis B, where

$$v_1 = x - l_1\,, \quad v_2 = x - l_2\,, \quad v_3 = x - l_3\,,$$

and we have

$$p_1 v_1 + p_2 v_2 + p_3 v_3 = [p\,v] = 0\,,$$

or B is the center of gravity of a system of masses corresponding to the weights p_1, p_2, p_3.

The sum $[p\,v^2]$, also, has a mechanical interpretation; this is the moment of inertia of the just mentioned system of masses with reference to the axis B, and the condition $[p\,v^2]$ = Minimum, which determines the mean value x in the sense of the adjustment computation, means, in the sense of mechanics, that B shall be an axis of the smallest moment of inertia.

Section 9. Special Case of Two Observations

To practice the theory of the arithmetic mean we take the case of two observations, which in addition is important in some respects.

If there exist two equally accurate observations which deviate from one another by the amount d, then we assume that the first observation is l and the second observation $l + d$; therefore the mean

$$x = l + \frac{d}{2}\,,$$

then the corrections are

$$v_1 = l + \frac{d}{2} - l = +\frac{d}{2} \quad \text{and} \quad v_2 = l + \frac{d}{2} - (l + d) = -\frac{d}{2}\,,$$

and hence, the mean error of a measurement itself according to (10), section 7, p. 18, is

$$m = \sqrt{\frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}{2 - 1}} = \frac{d}{\sqrt{2}} = 0.707\,d \tag{1}$$

and the mean error of the mean itself according to (11), section 7, p. 18, is

$$M = \frac{m}{\sqrt{2}} = \frac{d}{2} = 0.5\,d\,. \tag{2}$$

This case was simple and will occupy us once again in section 11 in the wider sense.

In addition, we also pass over to the case of two unequally accurate measurements of the same magnitude.

If there exist two heterogeneous observations with the weights p and q, which deviate from one another by the amount d, while, for instance, the first has furnished the value l and the second the value $l + d$, then the mean value will be

$$x = \frac{pl + q\,(l + d)}{p + q} = l + \frac{q}{p + q}\,d\,. \tag{3}$$

The deviations from the mean value, i.e. the corrections, will be

$$v_1 = + \frac{q}{p + q}\,d\,, \qquad v_2 = - \frac{p}{p + q}\,d\,, \tag{4}$$

and (without taking into account the signs) there exists the ratio

$$v_1 : v_2 = \frac{1}{p} : \frac{1}{q}\,, \tag{5}$$

i.e., the discrepancy d is divided among the two observations in inverse proportion to the weights. The mean error of an observation of weight 1 will be according to (13), section 8, p. 24

$$m = \sqrt{\frac{p\,v_1^2 + q\,v_2^2}{2 - 1}} = d\sqrt{\frac{p\,q}{p + q}} \tag{6}$$

and the mean error of the mean itself, according to (14), section 8, p. 24

$$M = \frac{m}{\sqrt{p + q}} = \frac{d}{p + q}\,\sqrt{p\,q}\,. \tag{7}$$

The mean errors to be feared m_1 and m_2 of the two observations *before* the adjustment are

$$m_1 = \frac{m}{\sqrt{p}}\,, \qquad m_2 = \frac{m}{\sqrt{q}}\,. \tag{8}$$

Therefore, there exists the ratio

$$m_1 : m_2 = \frac{1}{\sqrt{p}} : \frac{1}{\sqrt{q}}\,. \tag{9}$$

We shall make further use of these considerations in the later section 11.

Section 10. Angle Adjustment in a Triangle

A simple problem, which first does not appear as the formation of the mean with weights, but can be reduced to it, is the adjustment of the three angles in a triangle.

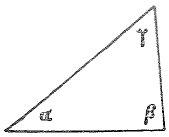


Fig. 1.
$\alpha + \beta + \gamma - 180^\circ = w$

If in a plane triangle (Fig. 1) the three angles are measured with the same accuracy, with the results α, β, γ, then, because of the measuring errors, there will occur a discrepancy of the sum $\alpha + \beta + \gamma$ with respect to 180°, which we divide among the three measured angles in equal parts. In order to prove this known method by the principle of the arithmetic mean, we consider first only *one* of the three angles as unknown.

For the first angle there exist two independent observational results:

first $\quad x_1 = \alpha \qquad$ with the weight $= p_1 = 1$, (1)

second $\quad x_2 = 180^\circ - (\beta + \gamma) \qquad$ with the weight $= p_2 = \frac{1}{2}$, (2)

because in x_2 the errors of β *and* of γ produce a combined effect, or in greater detail:

mean error of x_1 is $m_1 = \pm m$,

mean error of x_2 is $m_2 = \pm m \pm m = m\sqrt{2}$,

therefore

$$m_1 : m_2 = 1 : \sqrt{2}$$

$$p_1 : p_2 = \frac{1}{m_1^2} : \frac{1}{m_2^2} = 1 : \frac{1}{2} .$$

Taking into account these weights, the mean value from (1) and (2) is therefore:

$$x = \frac{1 \times \alpha + \frac{1}{2} \times [180^\circ - (\beta + \gamma)]}{1 + \frac{1}{2}} . \tag{3}$$

If we introduce herein the closure error of the triangle w:

$$\alpha + \beta + \gamma - 180^\circ = w ,$$

therefore

$$180^\circ - (\beta + \gamma) = \alpha - w , \tag{4}$$

then the introduction into (3) yields:

$$x = \frac{2\alpha + (\alpha - w)}{3} = \alpha - \frac{w}{3} . \tag{5}$$

The same also holds true for the two other angles of the triangle, which we assume to be denoted by y and z, i.e.:

$$x = \alpha - \frac{w}{3}$$

$$y = \beta - \frac{w}{3}$$

$$z = \gamma - \frac{w}{3}$$

Sum: $x + y + z = \alpha + \beta + \gamma - w$, i.e., the discrepancy w is divided equally among the three angles.

If, in addition, we aim to compute also the mean errors, then we must remember that in view of the chosen method of treatment, in (1) and (2) we have to deal with *two* observations of an unknown x.

The correction v_1 of the first observation x_1 according to (1) is

$$v_1 = -\frac{w}{3} \tag{6}$$

and the correction v_2 of the second observation x_2 according to (2) is

$$v_2 = +\frac{w}{3} + \frac{w}{3} = \frac{2w}{3}; \tag{7}$$

the mean error of an observation with weight 1 becomes [according to formula (13), section 8, p. 24]

$$m = \pm\sqrt{\frac{p_1 v_1^2 + p_2 v_2^2}{2-1}} = \pm\sqrt{\left(\frac{w}{3}\right)^2 + \frac{1}{2}\left(\frac{2w}{3}\right)^2} = \pm\frac{w}{\sqrt{3}}. \tag{8}$$

Since the weight 1 was assigned to the angle α before the adjustment (as well as to the angles β and γ), then this value m is at the same time the mean error of an angle *before* the adjustment. *After* the adjustment the corrected angle value x (as well as y and z) obtains a larger weight, namely the sum

$$p_1 + p_2 = \frac{3}{2},$$

and with this, the anticipated mean error of the angle x (or also y or z) after the adjustment becomes [according to the formula (12), section 8, p. 23]

$$M = \pm\frac{m}{\sqrt{\frac{3}{2}}} = \pm\frac{w\sqrt{2}}{3}. \tag{9}$$

The mean error according to (8), $m = \pm\frac{w}{\sqrt{3}}$, can be interpreted directly in a very illustrative manner: i.e., the closure error w is the irregular combined effect of three errors $\pm m$; therefore

$$\pm m \pm m \pm m = \pm w,$$

whence

$$m^2 + m^2 + m^2 = w^2,$$

which agrees with (8).

The mean error m before the adjustment and the mean error M after the adjustment illustrate, by their ratio, the gain of accuracy achieved by the adjustment. According to (8) and (9) we have

$$M : m = \sqrt{\frac{2}{3}} : 1 = 0.816 : 1. \tag{10}$$

On account of the adjustment, the mean error has decreased at the ratio of approximately 0.8:1, and hence, the accuracy has increased at the ratio 1:0.8.

Or we can also say: By means of the adjustment of the three angles in a triangle there has been reached an increase in accuracy of 20% for an individual angle.

The above example, which is well suited for an introduction into the theory of weights, can in addition also be extended so that the three measured angles are not observed with equal weights, as previously, but themselves with unequal weights.

We will carry out also this case:

Let there be measured the three angles $\alpha, \ \beta, \ \gamma$ (11)

with the weights $p_\alpha, \ p_\beta, \ p_\gamma$; (12)

the triangle discrepancy is $\alpha + \beta + \gamma - 180° = w$. (13)

If m is the mean error of the unit of weight, then the mean angle errors are according to the weights (12) before the adjustment

$$m_\alpha = \frac{m}{\sqrt{p_\alpha}}, \qquad m_\beta = \frac{m}{\sqrt{p_\beta}}, \qquad m_\gamma = \frac{m}{\sqrt{p_\gamma}}; \tag{14}$$

therefore

$$m_\alpha^2 = m^2\left(\frac{1}{p_\alpha}\right), \qquad m_\beta^2 + m_\gamma^2 = m^2\left(\frac{1}{p_\beta} + \frac{1}{p_\gamma}\right). \tag{15}$$

If we now denote by p_1 the weight of the measured angle α and by p_2 the weight of the sum of the two other angles $\beta + \gamma$, then we will have according to (15)

$$p_1 = p_\alpha = \frac{1}{\frac{1}{p_\alpha}}, \qquad p_2 = \frac{1}{\frac{1}{p_\beta} + \frac{1}{p_\gamma}}. \tag{16}$$

For abbreviation we set the sum $\frac{1}{p_\alpha} + \frac{1}{p_\beta} + \frac{1}{p_\gamma} = \left[\frac{1}{p}\right]$, and with this, (16) yields

$$p_1 + p_2 = \frac{\left[\frac{1}{p}\right]}{\frac{1}{p_\alpha}\left(\frac{1}{p_\beta} + \frac{1}{p_\gamma}\right)}. \tag{17}$$

After these preparations with regard to the weights, we have now as in the case of (3)

$$x = \frac{p_1 \alpha + p_2 [180° - (\beta + \gamma)]}{p_1 + p_2} = \frac{p_1 \alpha + p_2 (\alpha - w)}{p_1 + p_2},$$

$$x = \alpha - \frac{p_2}{p_1 + p_2} w .$$

If we introduce here (16) and (17), then we will have

$$x = \alpha - \frac{\frac{1}{p_\alpha}}{\left[\frac{1}{p}\right]} w . \tag{18}$$

Since similar equations hold for the two other angles, we have the theorem that the discrepancy w is divided among the three angles α, β, γ in inverse proportion to the weights or in proportion to the squares of the mean errors.

Now we can also develop, for unequal weights, formulae similar to (8) and (9), which hold for equal weights. By submitting this development as an example, we only write the results of it

mean error of unit of weight:
$$m = \frac{w}{\sqrt{\left[\frac{1}{p}\right]}}, \tag{19}$$

mean error of the angle α before the adjustment:

$$m_\alpha = \frac{m}{\sqrt{p_\alpha}} = w\frac{\sqrt{\frac{1}{p_\alpha}}}{\sqrt{\left[\frac{1}{p}\right]}}, \tag{20}$$

mean error of the angle α or, as the case may be, x after the adjustment:

$$M_\alpha = w\frac{\sqrt{\frac{1}{p_\alpha}}\sqrt{\frac{1}{p_\beta}+\frac{1}{p_\gamma}}}{\left[\frac{1}{p}\right]}. \tag{21}$$

Let us put here also a numerical example for a triangle with three angles of unequal weights, which we have assembled from *Astronomische Nachrichten*, vol. 75, 1870, p. 293, from angles of Baden and Hesse of divers origins, with the note that such weight distinctions are in practice always doubtful, but this submitted example is certainly suited for a formal computation.

We have in the triangle Oggersheim-Mannheim-Speyer, which we have represented in Fig. 2,

	Measured Angles	Weights
	$\alpha =$ 72° 16′ 44.86″	$p_\alpha = 27$
	$\beta =$ 90 1 56.46	$p_\beta = 42$
	$\gamma =$ 17 41 17.43	$p_\gamma = 65$
Sum	$\alpha+\beta+\gamma =$ 179° 59′ 58.75″	
Theoretical value	180° 0′ 0.29″	
Discrepancy	$w =$ − 1.54″;	

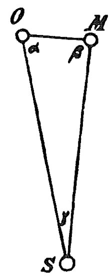


Fig. 2.

Reciprocal values of weight: $\frac{1}{p_\alpha} = 0.037$, $\frac{1}{p_\beta} = 0.024$, $\frac{1}{p_\gamma} = 0.015$, $\left[\frac{1}{p}\right] = 0.076$;

Angle corrections: $v_\alpha = \frac{0.037}{0.076}1.54'' = +0.75''$, $v_\beta = +0.49''$, $v_\gamma = +0.30''$;

Measured	Corrections	Adjusted
72° 16′ 44.86″	+ 0.75″	72° 16′ 45.61″
90 1 56.46	+ 0.49	90 1 56.95
17 41 17.43	+ 0.30	17 41 17.73
179° 59′ 58.75″	+ 1.54″	180° 0′ 0.29″

According to (19) we determine the mean error for the unit of weight:

$$m = \pm 5.59'',$$

and according to (21) we compute the mean errors of the adjusted angles and, with this, have the final result

$$\begin{aligned} \alpha &= 72^\circ 16' 45.61'' \pm 0.77'' \\ \beta &= 90 \quad 1 \quad 56.95 \quad \pm 0.72 \\ \gamma &= 17 \quad 41 \quad 17.73 \quad \pm 0.61 . \end{aligned}$$

In a similar manner, as has happened here with the problem of triangle adjustment in a triangle, a few other problems, which at first sight seem to require an adjustment with several unknowns, can be reduced to the determination of *one* unknown by means of the arithmetic mean; at any rate, an adjustment with *one* sum check for several directly measured quantities can be treated entirely according to the previous pattern of a sum check for three elements.

In a similar way, every adjustment with a condition equation to be satisfied rigorously can also be reduced to the case of the arithmetic mean.

Section 11. Observational Differences

If we make a measurement twice, e.g., measure a straight line forward and back or make a leveling forward and back and so forth, then we can apply the theorems of the arithmetic mean to every two such measurements, as we have already done in section 9, p. 26; and whenever observational differences are under discussion, we can always produce the two measurements involved independently; meanwhile, the observational differences also permit a very useful, more general consideration, for whose introduction, however, we shall make the tie once again to the arithmetic mean.

If l_1 and l_2 are the two measured results with the difference $l_1 - l_2 = d$, then we have the arithmetic mean

$$x = \frac{l_1 + l_2}{2}$$

and the corrections

$$v_1 = x - l_1 = -\frac{d}{2} \quad \text{and} \quad v_2 = x - l_2 = +\frac{d}{2}.$$

Therefore, the mean error of a measurement according to (10), section 7, p. 18, with $n = 2$ is

$$m = \sqrt{\frac{[v^2]}{n-1}} = \sqrt{\frac{d^2}{4} + \frac{d^2}{4}} = \frac{d}{\sqrt{2}} \tag{1}$$

and the mean error of the arithmetic mean according to (11), section 7, p. 18, is

$$M = \frac{m}{\sqrt{n}} = \frac{d}{2}. \tag{2}$$

Of this, we bear in mind, in particular, equation (1), namely $d = m\sqrt{2}$, i.e., the difference of two measurements to be anticipated is equal to $\sqrt{2}$ times the mean error of a measurement.

After clarifying these very simple relations we go one step further by taking several such repetition measurements together; e.g., we assume that we have measured different lines, each back and forth, or different leveling lines, each independently back and forth, and obtained in each of these cases a difference d.

But we will first assume here that the different lengths or leveling lengths are nearly *equally* long, or more generally, the differences d belong to measurements of only equal weights.

We denote the individual observation pairs by l and l' and their true errors by ε and ε'. Then we have

$$\left.\begin{array}{lcl} l_1+\varepsilon_1=l_1'+\varepsilon_1' & \text{or} & l_1-l_1'=d_1=\varepsilon_1'-\varepsilon_1 \\ l_2+\varepsilon_2=l_2'+\varepsilon_2' & & l_2-l_2'=d_2=\varepsilon_2'-\varepsilon_2 \\ \dots\dots\dots\dots & & \dots\dots\dots\dots\dots \\ l_r+\varepsilon_r=l_r'=\varepsilon_r' & & l_r-l_r'=d_r=\varepsilon_r'-\varepsilon_r \end{array}\right\}; \tag{3}$$

and there follows therefrom the sum of squares of the differences

$$[d^2]=[\varepsilon^2]+[\varepsilon'^2]-2[\varepsilon\varepsilon']$$

and

$$\frac{[d^2]}{r}=\frac{[\varepsilon^2]}{r}+\frac{[\varepsilon'^2]}{r}-\frac{2[\varepsilon\varepsilon']}{r}. \tag{4}$$

For the first two terms on the right-hand side we can introduce at once m^2, the square of the mean error of observations which are regarded as equally accurate. For the last term $\frac{2[\varepsilon\varepsilon']}{r}$, as done in section 5, p. 12, we can only estimate an average value which it would assume if the observational series in question were repeated infinitely often. But this average value is to be assumed equal to zero, as was explained on p. 13. Therefore, (4) changes to

$$\frac{[d^2]}{r}=m^2+m^2$$

or the mean error of a measurement
$$m=\sqrt{\frac{[d^2]}{2r}}, \tag{5}$$

mean error of a double measurement
$$M=\frac{1}{2}\sqrt{\frac{[d^2]}{r}}. \tag{6}$$

Now if there exist observation pairs which do not have equal weights, e.g., back and forth levelings of several stretches of different length, then we must take into account these different weights of the measured quantities l and l' in (3), in order to determine then the mean error for a measurement of weight 1.

If we have an observation l and form hence the quantity $\sqrt{p}\,l$, then, according to (1) and (2), section 5, p. 11, the mean errors of these two quantities bear the relation as $1:\sqrt{p}$ and their weights as $p:1$. Since the observation l has the weight p, then, accordingly, the quantity $\sqrt{p}\,l$ has the weight 1. Therefore, if we multiply the equations (3) by $\sqrt{p_1}, \sqrt{p_2}\ldots$, then we have to deal only just with quantities of equal weight $p=1$, whose true errors are $\varepsilon_1\sqrt{p_1}, \varepsilon_2\sqrt{p_2}\ldots$. Instead of equation (4) we obtain

$$\frac{[d^2p]}{r}=\frac{[\varepsilon^2p]}{r}+\frac{[\varepsilon'^2p]}{r}-\frac{2[\varepsilon\varepsilon'\sqrt{p}\sqrt{p'}]}{r}. \tag{7}$$

For the last term of this equation we can introduce again the average value zero, and there follows then, if m denotes the mean error of an observation with weight 1,

$$\frac{[d^2p]}{r}=m^2+m^2$$

or

$$m = \sqrt{\frac{[p\,d^2]}{2\,r}} \tag{8}$$

and the mean error M of a mean from two measurements of weight 1 or the mean error of a double measurement is

$$M = \frac{1}{2}\sqrt{\frac{[p\,d^2]}{r}}. \tag{9}$$

As one of the most important applications of these latter formulae we will take repetition measurements of lengths, or back and forth levelings. All these measurements have mean errors, which increase with the square roots of the distances s; consequently, the weights are to be set in inverse proportion to $(\sqrt{s})^2$, i.e., in inverse proportion to the s's, and with this, (8) and (9) become

$$m = \sqrt{\frac{1}{2r}\left[\frac{d^2}{s}\right]}, \tag{10}$$

$$M = \frac{1}{2}\sqrt{\frac{1}{r}\left[\frac{d^2}{s}\right]}. \tag{11}$$

As an example for this we take a part of a railroad leveling (which is described more closely in our volume II, 2*, 1933, 9th ed., p. 84):

Point	Leveling I	Leveling II	I—II $=d$	d^2	Distance s	$\frac{d^2}{s}$
(1)						
(2)	−0.1853 m	−0.1859 m	+0.6 mm	0,36	0.72 km	0.50
(3)	+1.6258	+1,6262	−0.4	0.16	0,42	0.38
(4)	+1,4329	+1,4323	+0.6	0.36	0.47	0.77
(5)	+0.5106	+0.5094	+1.2	1,44	0.48	3.00
(6)	−0.0073	−0,0049	−2.4	5.76	0.51	11.30
$r=5$	+3,5693	+3.5679	+2.4		2.60	15.95
	−0.1926	−0.1908	−2.8			
	+3.3767	+3.3771	−0.4			

We have therefrom the mean error of a double leveling of 1 km

$$M = \frac{1}{2}\sqrt{\frac{15.95}{5}} = \pm\,0.89 \text{ mm} \tag{12}$$

and the mean error of a simple leveling of 1 km

$$m = M\sqrt{2} = \pm 1.26 \text{ mm}\ . \tag{13}$$

Such an accuracy determination is more or less reliable the more or less comparison lengths there are available; on the other hand, the lengths of the stretches are not a point in question here, but only their number. (However, *very small* lengths, e.g. below 100 m, will not count, for various reasons, as fully justified.)

In addition, we will apply the formulae (10) and (11) also to a length measurement and, in fact, to the classical example of Bessel's degree-measurement in East Prussia, which, in itself already interesting, will also give us occasion to remark about the origin of the computation of differences.

* Not translated.

In his degree-measurement in East Prussia, Bessel measured a base line in two parts twice each, i.e.

	1st part:	2nd part:	
First measurement	441.1852 m	1381.1571 m	total mean
Second measurement	441.1839	1381.1632	1822.3447 m
Differences	$d_1 = +1.3$ mm	$d_2 = -6.1$ mm	

(14)

By counting now the differences d in millimeters, and the distances in kilometers, therefore $s_1 = 0.441$ and $s_2 = 1.381$, where $r = 2$, we have according to (10) the mean square of error of a measurement of the unit of length:

$$m^2 = \frac{1}{4}\left(\frac{d_1^2}{s_1} + \frac{d_2^2}{s_2}\right), \tag{15}$$

$$m^2 = \frac{1}{4}\left(\frac{1.3^2}{0.441} + \frac{6.1^2}{1.381}\right) = 7.70 , \qquad m = \pm 2.78 \text{ mm (for 1 km)} . \tag{15a}$$

This is the mean error of *one* measurement of the unit of length of 1 km. The mean error of a mean from every *two* (back and forth) measurements of a length of 1 km belonging together, i.e., the mean error of a double measurement of 1 km becomes, according to (11),

$$M = \frac{m}{\sqrt{2}} = \frac{2.78}{\sqrt{2}} = \pm 1.96 \text{ mm (for 1 km)} . \tag{16}$$

But, finally, we can also compute the mean error of the total mean $s_1 + s_2 = 1822.3447$ m; i.e., its square is

$$\left.\begin{aligned} M'^2 &= M^2 (s_1 + s_2) = \frac{1}{8}\left(\frac{d_1^2}{s_1} + \frac{d_2^2}{s_2}\right)(s_1 + s_2) \\ \text{or} \quad M' &= \frac{1}{2\sqrt{2}}\sqrt{\frac{s_1 + s_2}{s_1} d_1^2 + \frac{s_1 + s_2}{s_2} d_2^2} \end{aligned}\right\}, \tag{17}$$

whereas in *Gradmessung in Ostpreussen*, p. 55, there is indicated instead of this:

$$\frac{1}{2}\sqrt{\frac{s_1 + s_2}{s_1} d_1^2 + \frac{s_1 + s_2}{s_2} d_2^2} . \tag{17a}$$

(Cf. in this connection the notes in small print in the following on pp. 36 and 37.)

The calculation according to (17) yields

$$M' = M\sqrt{s_1 + s_2} = 1.96\sqrt{1.822} = \pm 2.65 \text{ mm} . \tag{17b}$$

Therefore, from (14) and (17b) we will now form the final result:

$$\text{Base length} = 1822.3447 \text{ m} \pm 0.0027 \text{ m} . \tag{18}$$

Finally, the theory of differences offers, in addition, the possibility of a second, very illustrative, proof of the theorem that in the case of the accuracy computation from the arithmetic mean we must not divide the sum of squares $[v^2]$ by n, but by $n - 1$, i.e. theorem (10), section 7, p. 18.

If an unknown is observed n times, then, for obtaining a judgment about the accuracy of the individual observation, we can compare the individual results *with one another* and, in fact, we can combine the n observations into $n\frac{n-1}{2}$ different differences. If l_1 and l_2 mean the first two observations, then their difference is $d = l_2 - l_1$, or if we introduce the corrections $v_1 = x - l_1$ and $v_2 = x - l_2$, then we also have $d = (x - v_2) - (x - v_1) = v_1 - v_2$.

All $n\frac{n-1}{2}$ differences thus to be formed are the following:

$$\begin{array}{llll}
v_1 - v_2 & & & \\
v_1 - v_3 & v_2 - v_3 & & \\
v_1 - v_4 & v_2 - v_4 & v_3 - v_4 & \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \\
v_1 - v_n, & v_2 - v_n, & v_3 - v_n, \ \ldots, & v_{n-1} - v_n.
\end{array}$$

Now these differences, however, are not independent, and therefore we do not have the right to treat them like independent observational errors, but they are at least homogeneous and therefore permit the formation of a mean value. Their sum of squares will be, if each individual v^2 is found $(n - 1)$ times,

$$[d^2] = (n - 1)\,[v^2] - 2\,[v_i v_k]\,, \tag{19}$$

if $[v_i v_k]$ means the sum of all products occurring upon squaring of the differences.

In order to eliminate again this unknown sum $[v_i v_k]$, we use the identical equation

$$(v_1 + v_2 + v_3 + \ldots v_n)^2 = [v]^2 = [v^2] + 2\,[v_i v_k]\,; \tag{20}$$

(19) and (20) yield together

$$[d^2] = (n - 1)\,[v^2] + [v^2] - [v]^2\,.$$

But the corrections v yield the algebraic sum $[v] = 0$,

therefore

$$[d^2] = n\,[v^2]\,. \tag{21}$$

The number of the differences d is $n\frac{n-1}{2}$, therefore the mean error of a measurement according to (5), p. 33, is

$$m = \sqrt{\frac{[v^2]}{n-1}}\,; \tag{22}$$

this is in agreement with (10), section 7, p. 18.

About the origin of the computation with observational differences Czuber writes in *Theorie der Beobachtungsfehler*, Leipzig, 1891, p. 174: [Translated] "The idea of using, for the judgment of the accuracy of an observational series, instead of the deviations of the individual observations from the arithmetic mean, their deviations from one another, in other words, instead of the apparent errors, the observational *differences*, started from Jordan in 1869, and was developed further by Andrae and Helmert. Probably independently of Jordan, in 1881, Brégет suggested the application of the observational differences for the determination of the precision of a series of observations (*Comptes rendus* 93, 1881, pp. 119-1121, "Sur les différences successives des observations") and also examined, by experiment, the correctness of the basic equation (5) (see p. 33).

To this report by Czuber, we will give, in addition, a few explanations from our own experience: The computation of differences originated as a preparation of the lengthy discussions which are conducted in the first two volumes of *Zeitschrift für Vermessungswesen* 1872-1873. The point in question was the computation of mean length measuring errors from series of double measurements. For the determination of the mean error of a measurement of the unit of length from the differences of a number of double measurements of different lengths has previously, for the most part, not been carried out correctly, and Dienger had first called our attention, in *Grunerts Archiv*, 31st Part, 1858, p. 225, to the error generally committed here (cf. *Zeitschrift für Vermessungswesen* 1872, p. 19).

In 1869 I had noticed a mistake which even Bessel committed in *Gradmessung in Ostpreussen*, pp. 54-55, by the computation of the mean error of his base measurement by dividing the sum of squares of two differences not by 2, but by $2 - 1 = 1$, and set the matter right in my mind, in the best way, by the introduction of the observational *differences* as independent error elements, which was then further worked out in a treatise, "Über die Genauigkeit mehrfach wiederholter Beobachtungen einer Unbekannten" in *Astronomische Nachrichten*, vol. 74, 1869, pp. 209-226, where on p. 226 the mentioned error, $2 - 1$ instead of 2, in *Gradmessung in Ostpreussen*, p. 54, is considered and set right. For Bessel does not compute in the same way as we correctly indicated in the foregoing (15) to (18), p. 35, but he computes $m^2 = \frac{1}{2}\left(\frac{d_1^2}{s_1} + \frac{d_2^2}{s_2}\right)$ and so forth, whereby m and all the following is obtained too large, namely $\sqrt{2}$ times as large as the correct value, as we have already noted above in the case of (17) and (17a), p. 35.

Soon after that, on pp. 283-284 of *Astronomische Nachrichten*, vol. 74, Andrae came with a brief reply, written in Danish, to my formulae of pp. 209-226 in regard to the theory that I had missed the simple relation between the sum of squares $[d^2]$ of all differences and the sum of squares $[v^2]$ of the apparent errors, namely $[d^2] = n[v^2]$, i.e., Andrae's Danish development gives what we have communicated in the foregoing in (19) to (22), p. 36. There followed further in this matter *Astronomische Nachrichten*, vol. 79, 1872, pp. 219-222 and pp. 257-272.

In connection with the above, there were published, in addition, in *Astronom. Nachr.*, vol. 80, 1872, pp. 67-70, Zachariae; pp. 189-190, Jordan; vol. 81, 1873, pp. 49-52, Helmert; pp. 51-56, Jordan; pp. 225-267, Zachariae; and vol. 88, 1876, pp. 127-131, Helmert.

Since it is difficult for those without access to the above to find, still today, the actual course from all those treatises, we will briefly indicate that course thusly: My first treatise of 1869 had only the defect that the simple relation $[d^2] = n[v^2]$ was not found, which Andrae corrected soon afterwards. But, then, Zachariae in 1872 brought in *another* debatable question by assigning (*Astronom. Nachr.*; vol. 80, p. 68), according to the Danish degree-measurement, the differences in length measurements, different weights in the error computation, according to the differences belonging to great or small lengths s, i.e., not only in so far as the $\frac{d^2}{s}$'s are to be reduced to the unit s, but he gives his $\frac{d^2}{s}$'s themselves the weight s again afterwards, starting from the assumption that a difference d from great lengths contributes more to the determination of accuracy than a d from small lengths; the mean square of differences for the unit of length would be, according to this,

$$D^2 = \frac{s_1 \frac{d_1^2}{s_1} + s_2 \frac{d_2^2}{s_2}}{s_1 + s_2} = \frac{d_1^2 + d_2^2}{s_1 + s_2} .$$

This conception is somewhat tempting, yet it is not the correct one, as rigorously proven first by Helmert in *Astronom. Nachr.*, vol. 81, pp. 49-52. (The remark which we have made above in the case of (13), p. 34, about the lengths of the leveling stretches refers to this.)

Now if today every land surveyor or water leveler thus computes his levelings, with respect to mean errors, according to formula (11), p. 34, namely $M = \frac{1}{2}\sqrt{\frac{1}{r}\left[\frac{d^2}{s}\right]}$, then he will probably accept it as a matter of course, while this formula, however, had to run through, from 1868 to 1873, the long and often-winding path, which we have described in the above citations.

Section 12. Example of an Adjustment of Indirect Observations

The arithmetic mean is only sufficient for the adjustment of observations of *one* unknown. If for the simultaneous determination of *several* unknowns there are observations in excessive number, than a more general principle of adjustment must be sought for. For the basic problem of such an adjustment, we will first consider a simple example.

For the determination of the lengths of four measuring rods, there was available a comparison apparatus similar to that illustrated in Volume II, section 1, 9th Ed., 1931, p. 64, which, however, had a length of 10 m, so that an individual 5-m rod could not offhand be examined. Therefore, two rods at a time were laid in different combinations on the comparison apparatus, where each time the total length of the two rods was determined.

Four such combinations, e.g. (1) + (2), (1) + (3), (1) + (4), (2) + (3), would then have been sufficient in order to compute the four individual rod lengths. Because of symmetry, however, the two combinations (2) + (4) and (3) + (4) were included also, so that for the determination of the four unknowns six measurements are now available.

The lengths measured on the comparison apparatus are

(1) + (2) = 10,002.26 mm	(2) + (3) = 10,002.33 mm
(1) + (3) = 10,003.09 mm	(2) + (4) = 10,001.98 mm
(1) + (4) = 10,002.29 mm	(3) + (4) = 10,002.62 mm

Since the rod lengths deviate very little from 5 m, then we will determine only these deviations and denote them for the four rods by x, y, z, t. Then we must also reduce the measured quantities by 10 m and obtain herewith the six equations

$$\left.\begin{array}{llllll} x+y & . & . & = 2.26 \text{ mm} \\ x & . & +z & . & = 3.09 \text{ mm} \\ x & . & . & +t & = 2.29 \text{ mm} \\ . & y & +z & . & = 2.33 \text{ mm} \\ . & y & . & +t & = 1.98 \text{ mm} \\ . & . & z & +t & = 2.62 \text{ mm} \end{array}\right\} . \tag{1}$$

The computation of the four unknowns from the six equations meets with difficulties, since the numerical values on the right-hand side have resulted from measurements and, consequently, will show small discrepancies; e.g., if we use the first four equations, then there follow hence other values for the unknowns than, say, from the last four equations or from any other combination. In order to avoid this difficulty, then nothing remains but to correct the measured quantities in such a way that they form a uniform system, where we will choose the corrections as small as possible.

Instead of the above equations we thus have

$$\left.\begin{array}{llllll} x+y & . & . & = 2.26+v_1 \\ x & . & +z & . & = 3.09+v_2 \\ x & . & . & +t & = 2.29+v_3 \\ . & y & +z & . & = 2.33+v_4 \\ . & y & . & +t & = 1.98+v_5 \\ . & . & z & +t & = 2.62+v_6 \end{array}\right\} . \tag{2}$$

We will express the requirement that the corrections be *as small as possible* in the form: The *sum of squares* of the corrections v shall be as small as possible, or in a formula:

$$[v^2] = v_1^{\,2} + v_2^{\,2} + \ldots = \text{minimum} . \tag{3}$$

With this, we have reached a mathematical formulation of the problem, which can conveniently be treated further. For this, we write the equations (2) in the form

$$\left.\begin{array}{l} v_1 = x+y \quad . \quad . -2.26 \\ v_2 = x \quad . +z \quad . -3.09 \\ v_3 = x \quad . \quad . +t-2.29 \\ v_4 = . \quad y+z \quad . -2.33 \\ v_5 = . \quad y \quad . +t-1.98 \\ v_6 = . \quad . \quad z+t-2.62 \end{array}\right\} , \tag{4}$$

square both sides and form the sum of all equations. We have

$$\left.\begin{array}{rl} [v^2] = & 3x^2 + 3y^2 + 3z^2 + 3t^2 \\ & -15.28\,x - 13.14\,y - 16.08\,z - 13.78\,t \\ & +2xy + 2xz + 2xt + 2yz + 2yt + 2zt \end{array}\right\} . \tag{5}$$

This expression becomes a minimum if we set

$$\frac{\partial[v^2]}{\partial x} = 0 , \quad \frac{\partial[v^2]}{\partial y} = 0 , \quad \frac{\partial[v^2]}{\partial z} = 0 , \quad \frac{\partial[v^2]}{\partial t} = 0 .$$

This yields the equations

$$\left.\begin{aligned} 6x+2y+2z+2t-15.28=0 \\ 2x+6y+2z+2t-13.14=0 \\ 2x+2y+6z+2t-16.08=0 \\ 2x+2y+2z+6t-13.78=0 \end{aligned}\right\}, \tag{6}$$

and we obtain therefrom

$$\left.\begin{aligned} x &= +1.39 \qquad & z &= 1.59 \\ y &= +0.86 \qquad & t &= 1.02 \end{aligned}\right\}. \tag{7}$$

The four measuring rods have therefore the lengths

$$\left.\begin{aligned} 5001.39\ \text{mm} \qquad 5001.59\ \text{mm} \\ 5000.86\ \text{mm} \qquad 5001.02\ \text{mm} \end{aligned}\right\}. \tag{8}$$

Besides, it is of interest to see what corrections were required for the measurements in order to render the latter into a uniform system by applying the principle of the sum of least squares. From (4) and (7) we obtain

$$\begin{aligned} v_1 &= -0.01\ \text{mm} \qquad & v_4 &= +0.12\ \text{mm} \\ v_2 &= -0.11\ \text{mm} \qquad & v_5 &= -0.10\ \text{mm} \\ v_3 &= +0.12\ \text{mm} \qquad & v_6 &= -0.01\ \text{mm}\ . \end{aligned}$$

With this, $[v^2] = 0.0511$ is the smallest possible sum of squares for the corrections v.

Section 13. General Principle of Adjustment and the Arithmetic Mean

In the above example, for the elimination of the discrepancies we have set up the requirements that the sum of squares of the corrections of the observations shall be as small as possible, or

$$[v^2] = v_1{}^2 + v_2{}^2 + v_3{}^2 + \ldots + v_n{}^2 = \text{minimum}\ . \tag{1}$$

But this simple condition holds only under the assumption that the observations l, to which the corrections v belong, are to be considered *equally* accurate a priori. If this is not the case, and the mean errors $m_1, m_2, m_3, \ldots, m_n$ to be feared are assigned the observations $l_1, l_2, l_3, \ldots, l_n$ a priori, then, instead of (1), there holds the changed condition

$$\left[\frac{v^2}{m^2}\right] = \left(\frac{v_1}{m_1}\right)^2 + \left(\frac{v_2}{m_2}\right)^2 + \left(\frac{v_3}{m_3}\right)^2 + \ldots + \left(\frac{v_n}{m_n}\right)^2 = \text{minimum}\ . \tag{2}$$

Since the weights $p_1, p_2, p_3, \ldots, p_n$ are inversely proportional to the squares of the mean errors, then, instead of (2), we can also write

$$[p\,v^2] = p_1 v_1{}^2 + p_2 v_2{}^2 + p_3 v_3{}^2 + \ldots + p_n v_n{}^2 = \text{minimum}\ . \tag{3}$$

This fundamental theorem of the sum of least squares $[v^2]$ or $[p\,v^2]$ as the case may be, is first just as arbitrary as the concept of the mean error itself, as we have already noted in the case of the mean error, section 4, p. 10. The usefulness of the principle of the sum of squares proves best from its results, since the entire present-day method of least squares could be based on this fundamental theorem. But we will enter on the subject of discussion more closely later as to what theoretical advantages the method of least squares offers.

Before we make further use of the fundamental theorem of equation (1) or, as the case may be, (2) or (3), we convince ourselves that this fundamental theorem is in agreement with the arithmetic mean already treated previously.

We assume that we have observed repeatedly a quantity x

with the results $l_1, l_2, l_3, \ldots, l_n$,

and with the weights $p_1, p_2, p_3, \ldots, p_n$.

The most probable corrections v of the following observations correspond to the assumption that x is the most probable value:

$$v_1 = x - l_1 , \quad v_2 = x - l_2 , \quad v_3 = x - l_3 , \quad \ldots , \quad v_n = x - l_n .$$

According to (3) there shall be:

$$p_1 (x - l_1)^2 + p_2 (x - l_2)^2 + p_3 (x - l_3)^2 + \ldots = \text{minimum} ,$$

from which there is obtained by differentiation with respect to the variable x:

$$2p_1(x - l_1) + 2p_2(x - l_2) + 2p_3(x - l_3) + \ldots = 0 ,$$

therefore, solving for x:

$$x = \frac{p_1 l_1 + p_2 l_2 + p_3 l_3 + \ldots}{p_1 + p_2 + p_3 + \ldots} = \frac{[p\,l]}{[p]}$$

in agreement with (4), section 8, p. 21.

We can also prove in yet another way the arithmetic mean as a special case of our more general problem of adjustment.

For the determination of the unknown quantity x we assume that not the latter itself but different multiples $a_1 x, a_2 x, a_3 x \ldots$ are measured. If $l_1, l_2, l_3 \ldots$ are the observations and we denote again by x the most probable value of the unknown, then the corrections of the observations are

$$\left.\begin{array}{l} v_1 = a_1\,x - l_1 \\ v_2 = a_2\,x - l_2 \\ v_3 = a_3\,x - l_3 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \end{array}\right\} . \qquad (4)$$

The fundamental theorem of the sum of least squares requires that we will have

$$[v^2] = (a_1 x - l_1)^2 + (a_2 x - l_2)^2 + (a_3 x - l_3)^2 + \ldots = \text{minimum} ,$$

from which we obtain by differentiation:

$$2a_1 (a_1 x - l_1) + 2a_2 (a_2 x - l_2) + 2a_3 (a_3 x - l_3) + \ldots = 0 .$$

This yields for x the value

$$x = \frac{[a\,l]}{[a\,a]} . \qquad (5)$$

We also reach the same value, however, if we apply only the theorem of the arithmetic mean, which we will show now.

From the individual observations we obtain for the unknowns the values

$$\frac{l_1}{a_1}, \quad \frac{l_2}{a_2}, \quad \frac{l_3}{a_3} \dots . \tag{6}$$

The weights of the original observations $l_1, l_2, l_3 \dots$ shall now all be equal, i.e. = 1, and we assume that m is the mean error of an observation with the weight 1; then the mean errors of the various determinations of x contained in (6) are

$$\frac{m}{a_1}, \quad \frac{m}{a_2}, \quad \frac{m}{a_3} \dots , \tag{7}$$

respectively; consequently, the weights of the values x are inversely proportional to their squares, i.e.

$$\text{weights:} \quad a_1^2, \quad a_2^2, \quad a_3^2 \dots . \tag{8}$$

After the weights (8) are determined, we obtain the most probable value of x from (6) and (8), according to the fundamental theorem of equation (4) section 8, p. 21:

$$x = \frac{a_1^2 \frac{l_1}{a_1} + a_2^2 \frac{l_2}{a_2} + a_3^2 \frac{l_3}{a_3} + \dots}{a_1^2 + a_2^2 + a_3^2 + \dots},$$

$$x = \frac{a_1 l_1 + a_2 l_2 + a_3 l_3 + \dots}{a_1^2 + a_2^2 + a_3^2 + \dots},$$

$$x = \frac{[a\,l]}{[a\,a]} \tag{9}$$

in agreement with the above equation (5).

After, according to this, the general principle of adjustment $[v^2]$ = minimum has been recognized to be in agreement with the arithmetic mean, and thereby deserving all the more of confidence, we will now turn to the more general treatment of the problem.

Section 14. Indirect Observations with Two Unknowns

We begin with the special case of only *two* unknowns and have two reasons for this slow progression; first, in the case of only two unknowns all developments are shaped much simpler and are more readily conceivable than if we begin at once with an arbitrary number of unknowns, and second, the special case of two unknowns x and y occurs so often, especially in the case of adjustments with coordinates x, y, that it is certainly worth while to treat this case separately.

By generalizing equations (1), section 12, p. 38, we assume that n different linear functions, $a_1x + b_1y,\ a_2x + b_2y,\ \ldots,\ a_nx + b_ny$, have been observed. Let the observations be $l_1, l_2, \ldots, l_n$. In order to obtain from all n observations the same values of the unknowns x and y, we add small corrections v to the observations l as on p. 38, and obtain the equations

$$\left.\begin{aligned} l_1 + v_1 &= a_1 x + b_1 y \\ l_2 + v_2 &= a_2 x + b_2 y \\ &\cdots\cdots \\ l_n + v_n &= a_n x + b_n y \end{aligned}\right\}, \qquad (1)$$

in which the corrections are to be determined in such a way that the sum of their squares becomes as small as possible. For further treatment, we write the equations (1) in the form

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y - l_1 \\ v_2 &= a_2 x + b_2 y - l_2 \\ &\cdots\cdots \\ v_n &= a_n x + b_n y - l_n \end{aligned}\right\}, \qquad (2)$$

in which they are designated as *error equations*.

An individual v squared yields

$$\begin{aligned} v^2 = a^2x^2 &+ 2abxy - 2alx \\ &+ b^2y^2 - 2bly \\ &+ l^2 . \end{aligned}$$

If we imagine all individual v's so squared and added, then we obtain

$$\left.\begin{aligned} [v^2] = [aa]x^2 &+ 2[ab]xy - 2[al]x \\ &+ [bb]y^2 - 2[bl]y \\ &+ [ll] \end{aligned}\right\}. \qquad (3)$$

Our principle of a minimum of the squares requires

$$\frac{\partial [vv]}{\partial x} = 0 , \qquad \frac{\partial [vv]}{\partial y} = 0 .$$

This yields

$$\begin{aligned} 2[aa]x + 2[ab]y - 2[al] &= 0 \\ 2[ab]x + 2[bb]y - 2[bl] &= 0 , \end{aligned}$$

or with the omission of the common factor 2 we have two equations which we call *normal equations*.

Normal equations:

$$\left.\begin{aligned}[a\,a]\,x + [a\,b]\,y - [a\,l] = 0\\ [a\,b]\,x + [b\,b]\,y - [b\,l] = 0\end{aligned}\right\} . \tag{4}$$

We solve these normal equations for x and y, which, with only *two* unknowns, cannot offer any difficulties. We find

$$x = \frac{[b\,b]\,[a\,l] - [a\,b]\,[b\,l]}{[a\,a]\,[b\,b] - [a\,b]\,[a\,b]}, \qquad y = \frac{[a\,a]\,[b\,l] - [a\,b]\,[a\,l]}{[a\,a]\,[b\,b] - [a\,b]\,[a\,b]} . \tag{5}$$

The equations (4) or (5) contain the complete directions of adjustment for the submitted error equations (2). In words, we have, according to this, the following directions:

If the coefficients a, b and the absolute terms l of the error equations (2) are given, then we form hence all squares and products and their sums:

$$\begin{aligned} &a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = [a\,a]\\ &a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots a_n b_n = [a\,b]\\ &b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2 = [b\,b]\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

We form from these sum coefficients the normal equations (4) or at once their solutions (5).

We can also write the normal equations (4) in this abbreviated form:

$$\left.\begin{aligned}[a\,v] = 0\\ [b\,v] = 0\end{aligned}\right\} , \tag{6}$$

for we have

$$[a\,v] = a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots ,$$

which, carried out further according to (2), p. 42, yields:

$$[a\,v] = [a\,a]\,x + [a\,b]\,y - [a\,l] .$$

These forms (6) correspond to equation $[v] = 0$ in the case of the arithmetic mean, p. 16.

For this, we take at once a numerical example:

In *Württembergische Naturwissenschaftliche Jahreshefte*, Jahrgang XXIV, 1868, p. 260, there are communicated, by Professor S c h o d e r, the elevations above sea level h and the mean values of barometer readings B for twelve years of nine meteorological stations, namely

	h	B
1. Bruchsal	120.2 m	751.18 mm
2. Cannstatt	225.1	742.37
3. Stuttgart	270.6	738.50
4. Calw	347.6	731.27
5. Friedrichshafen	406.7	726.99
6. Heidenheim	492.4	718.16
7. Isny	708.1	700.48
8. Freudenstadt	733.5	697.64
9. Schopfloch	768.9	695.23

(7)

We will assume that the theory of the barometric elevation measurement is entirely unknown to us; but we note that as elevations h increase the barometer readings B decrease rather regularly; and in order to examine the law of this decrease, we begin with representing graphically the barometer readings B as a function of the elevations h, as Fig. 1 shows.

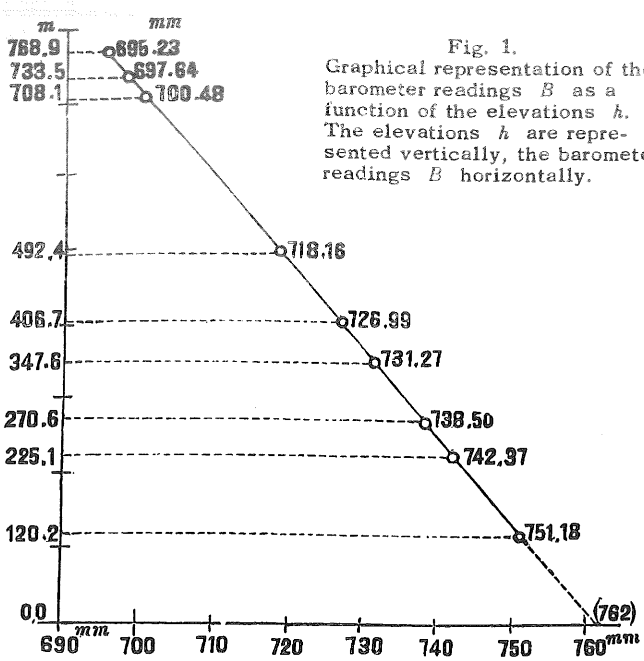


Fig. 1.
Graphical representation of the barometer readings B as a function of the elevations h. The elevations h are represented vertically, the barometer readings B horizontally.

For many purposes it will now be sufficient to lay through the points obtained a straight line or a continuous curve for a best possible fit, and to assume the ordinates of the adjustment curve as adjusted values.

Even if we aim to undertake an adjustment according to the method of least squares, such a plotting before beginning the computation is always advisable, in particular, if the relation of the variables is theoretically unclear; the graphical representation has the purpose of illustrating the type of dependence, of determining, under certain circumstances, the form of the function of adjustment, of detecting large observational errors, and of determining approximate values of the unknowns.

The view of our figure lets a linear function between h and B appear acceptable, i.e. we set

$$B = x + h y . \tag{8}$$

Any two of the nine observations (7) would be sufficient to determine the two unknowns x and y of function (8). But in order to do justice as much as possible to all nine observations, we must proceed according to the above developed equations (1) to (5), and have to form first the error equations. The elevations h given under (7) shall be assumed here as free from error, but the barometer readings B as observed incorrectly so that all occurring discrepancies are to be charged to the errors of the B's.

Each of the nine observations yields an equation of the form (8); but the corresponding nine equations will in general not agree; therefore, each B must be assigned a correction v, and so instead of (8) we will have

$$B + v = x + h y$$

$$\text{or} \quad v = -B + x + h y . \tag{9}$$

This equation (9) is already the form of the error equations in our special case, and for comparison we have the general form of error equation (2):

$$v = a x + b y - l . \tag{10}$$

The comparison between (9) and (10) yields

$$a = 1 , \quad b = h , \quad l = B , \tag{11}$$

i.e. in our case, all coefficients $a = 1$; the coefficients b are the given heights and the absolute terms l are the observations B.

With this, we obtain the following computation:

No.	a	b	$-l$	b^2	$-b\,l$
1	1.0	120.2	−751.18	14,448	−90,291.836
2	1.0	225.1	−742.37	50,670	−167,107.487
3	1.0	270.6	−738.50	73,224	−199,838.100
4	1.0	347.6	−731.27	120,826	−254,189.452
5	1.0	406.7	−726.99	165,405	−295,666.833
6	1.0	492.4	−718.16	242,458	−353,621.984
7	1.0	708.1	−700.48	501,406	−496,009.888
8	1.0	733.5	−697.64	538,022	−511,718.940
9	1.0	768.9	−695.23	591,207	−534,562.347
	9.0	4,073.1	−6,501.82	2,297,666	−2,903,006.867.

Since all coefficients $a = 1$, also all $a^2 = 1$, hence $[a a] = 9$, and also $[a b]$ and $[a l]$ do not need to be computed especially, for because of $a = 1$, $[a b] = [b] = 4073.1$, and $-[a l] = -[l] = -6501.82$, and altogether we have the coefficients of the normal equations:

$$\left.\begin{array}{lll} [a a] = +9.0 & [a b] = + \quad 4,073.1 & -[a l] = - \quad 6,501.82 \\ & [b b] = +2,297,666 & -[b l] = -2,903,006.867 \end{array}\right\} . \tag{12}$$

If we set these coefficients into (5), then the calculation yields

$$[a a][b b] = +20,678,994.00$$
$$[a b][a b] = +16,590,143.61$$
$$[a a][b b] - [a b][a b] = + \quad 4,088,850.39$$

$$[b b][a l] = +14,939,010,752.12 \qquad [a a][b l] = +26,127,061.803$$
$$[a b][b l] = +11,824,237,811.70 \qquad [a b][a l] = +26,482,563.042$$
$$[b b][a l] - [a b][b l] = + \quad 3,114,772,940.42 \quad [a a][b l] - [a b][a l] = - \quad 355,501.239 .$$

The unknowns themselves become

$$x = \frac{+3,114,772,940}{+ \quad 4,088,850} = +761.77$$
$$y = \frac{- \quad 355,501}{+ \quad 4,088,850} = -0.08695 .$$

The function of adjustment (8) is therefore

$$B = 761.77 - 0.08695\, h ,$$

or solved for h

$$h = 11.50\,(761.77 - B) . \tag{13}$$

Remarks

The sum of squares $[v^2]$ is frequently written $[v v]$ and then also $[p v^2] = [p v v]$, which is obviously equally justified, just as $a^2 = a a$, and so forth. Because of the connection to $[a b]$, $[a c]$, and so forth, it is of course also suitable to write $[a a]$, not $[a^2]$ as well as $[v v]$, while the single $[v^2]$ is more natural.

The nomenclature "error equation" [Fehlergleichung] of the type (4), section 12, p. 38, or type (2), section 14, p. 42, was introduced by Helmert, *Ausgleichungsrechnung* (1st Edition, Leipzig, 1872); moreover, it had already been used by Gauss (cf. *Gesammelte Werke*, vol. V, p. 632). The right-hand side of our error equation (2), section 14, p. 42, i.e. the expression $a x + b y - l$, can be called "error expression" [Fehlerausdruck] according to Schreiber. The designation "correction equation" [Verbesserungsgleichung] suggested from several sides, which most likely proves more correct, has not found a more general acceptance.

To the letter designation, also, we will make a few remarks: To denote the corrections of the adjustments by v is generally customary since Gauss, Gerling and Encke. Instead of v we also find frequently the symbol δ, which we used in the first two editions of this work (or, as the case may be, *Taschenbuch der praktischen Geometrie*), but then dropped in favor of the more customary v.

The error equations were written originally (Gauss, Theoria combinationis, Art. 20) in the form $v = a x + b y + l$. The absolute term of an error equation has always the character of an observation, as was shown in the example on p. 37, and in the general treatment on p. 41; in the original manner of writing, the absolute term presents therefore a negative observation. This circumstance occasioned Helmert to choose, in his *Ausgleichungsrechnung nach der Methode der kleinsten Quadrate*, Leipzig, 1872, the form

$$\lambda = a x + b y - l$$

for the error equations. This new manner of expressing the absolute term has displaced to a great extent the older one. With the 6th edition of this volume we have assumed, likewise, Helmert's manner of writing. For the case of direct observations of one unknown, the observations were denoted anyhow, thus far also, by l, while in the error equations with several unknowns the quantity l had the meaning of negative observations.

Encke denotes the absolute term by n. The designation f for the absolute term of the error equation occurs also, and leads then to the nomenclature of error term. In its innermost meaning, however, whether it is denoted by l or f, the absolute term has the character of an observation, not of an error. In our system, f always refers to a function of measurements or adjusted elements.

Section 15. Introduction of Approximate Values in the Case of Two Unknowns

We have inserted the calculation of the numerical example in the previous section 14, which has first presented itself, not for imitation, but, so to speak, as a warning example; for we can considerably simplify the computation by the introduction of *approximate values* of the unknowns x and y. Just as, in the case of the arithmetic mean, section 7, p. 16, we did not bring the degrees and minutes (35°26') into the summation, and as we introduced, in our first adjustment example in section 12, p. 38, only the deviations of the rod lengths of 5 m, we can also in the present case separate first approximations.

We will not enter here on the more general treatment of the introduction of approximate values, but only show the matter in our present special case:

If we consider the barometer curve (Fig. 1, p. 44) by extending it downward to the right as far as the elevation $h = 0$, then we find approximately the value 762 mm, and since it is known that at the elevation zero above sea level the barometer reading is about 760 mm, however, then we carry the graphically found approximate value 762. This is an approximate value for x, which we will denote by (x), and hence

$$(x) = 762.00 \, . \tag{1}$$

In order to obtain also an approximate value for y, we take best the first and the last observation from the group (7), p. 43, i.e.

1.	$h = 120.2$	$B \doteq 751.18$
9.	768.9	695.23
Differences:	$\Delta h = 648.7$	$\Delta B = -55.95$

$$(y) = \frac{55.95}{648.7} = 0.08625 \, . \tag{2}$$

The approximate function therefore is now according to (1) and (2):

$$\text{Approximate} \quad B = 762.00 - 0.08625\,h \, . \tag{3}$$

Incidentally, it is noted to this: It is indifferent in what way we procure the first approximations; very often we have them from somewhere else, from previous preliminary computations, and so forth. At any rate, we can procure usable approximations by choosing, in the case of two unknowns, two observations as different as possible and solving the error equations corresponding to them with $v = 0$.

The approximations (x) and (y) must now in addition be corrected, which shall be done by the adding of corrections, which shall be δx and $\delta y'$, respectively; we do not write here δy, but $\delta y'$, because δy is reserved for something else [later in (7)]. Therefore:

$$x = (x) + \delta x \, , \qquad y = (y) + \delta y' \, . \tag{4}$$

Numerically, this corresponds to and we should have:

$$B = (762 + \delta x) - (0.08625 + \delta y')\,h \, . \tag{5}$$

Because of the observational errors, the equation (5) is in general not satisfied, because the B's represent observed values. A correction v must be assigned to every B, i.e. in contrast to (5) we have:

$$B + v = (762 + \delta x) - (0.08625 + \delta y')\,h \, .$$

This yields the error equation

$$v = -B + \delta x - h\,\delta y' + (762 - 0.08625\,h)\ . \tag{6}$$

Meanwhile, a small matter is in addition involved: By assuming (6), the coefficients of δx and $\delta y'$ become very unequal. Namely, the coefficients of δx are all $= 1$, and the coefficients of $\delta y' = h$, from 120 to 769, i.e. much larger than the coefficients of δx. But such an inequality is formally very disturbing, as any computational test will show at once; but we can always make the coefficients nearly equal by introducing new unknowns, namely instead of (6):

$$v = -B + \delta x - \frac{h}{100}\,(100\ \delta y') + (762 - 0.08625\,h)\ ,$$

or for abbreviation

$$100\ \delta y' = \delta y \tag{7}$$

yields:

$$v = -B + \delta x - \frac{h}{100}\ \delta y + (762 - 0.08625\,h)\ . \tag{8}$$

In this error equation, besides the observation B there occurs an additional absolute term, namely $762 - 0.08625\,h$. As we shall see later, this will occur in all those cases in which approximate values of the unknowns are introduced. But that does not preclude combining this additional absolute term with the observation and denoting the sum by l.

We have then again an error equation of the general form

where

$$\left.\begin{aligned} v &= a\,\delta x + b\,\delta y - l\ , \\ a = 1 \qquad b &= -\frac{h}{100} \\ -l &= (762 - 0.08625\,h) - B \end{aligned}\right\}. \tag{9}$$

To illustrate the meaning of l we will in addition introduce a new symbol (B), namely

$$-l = (B) - B\ , \qquad (B) = 762 - 0.08625\,h\ . \tag{10}$$

For (B) is that value of the observational quantity B, which B would assume if the approximate values 762 and 0.08625 held rigorously. With respect to the sign, we remember once for all:

$$-l = (B) - B = \text{approximation} - \text{observation}\ . \tag{11}$$

According to the formulae (11), (10), and (9) we compute the following:

h	(B)	B	$-l$	a	b	b^2	l^2	$-bl$
120.2	751.63	751.18	+ 0.45	+ 1.0	— 1.20	1.44	0.20	— 0.54
225.1	742.59	742.37	+ 0.22	+ 1.0	— 2.25	5.06	0.05	— 0.50
270.6	738.66	738.50	+ 0.16	+ 1.0	— 2.71	7.34	0.03	— 0.43
347.6	732.02	731.27	+ 0.75	+ 1.0	— 3.48	12.11	0.56	— 2.61
406.7	726.92	726.99	— 0.07	+ 1.0	— 4.07	16.56	0.00	+ 0.28
492.4	719.53	718.16	+ 1.37	+ 1.0	— 4.92	24.21	1.88	— 6.74
708.1	700.93	700.48	+ 0.45	+ 1.0	— 7.08	50.13	0.20	— 3.19
733.5	698.74	697.64	+ 1.10	+ 1.0	— 7.34	53.88	1.21	— 8.07
768.9	695.68	695.23	+ 0.45	+ 1.0	— 7.69	59.14	0.20	— 3.46
			+ 4.88	+ 9.0	— 40.74	229.87	4.33	— 25.26

$$\left.\begin{aligned} [a\,a] = +\,9.00 \qquad [a\,b] &= -\ 40.74 \qquad -[a\,l] = +\ \ 4.88 \\ [b\,b] &= +\ 229.87 \qquad -[b\,l] = -\ 25.26 \\ & \qquad\qquad\qquad\ \ [l\,l] = +\ \ 4.33 \end{aligned}\right\}. \tag{12}$$

We have computed the sum $[l\,l]$ at the same time here, although it does not appear necessary as yet according to the development hitherto presented.

Now we compute again as previously:

$$\delta x = \frac{[b\,b]\,[a\,l] - [a\,b]\,[b\,l]}{[a\,a]\,[b\,b] - [a\,b]\,[a\,b]} = \frac{-1121.7 + 1029.1}{2068.8 - 1659.7} = -\frac{92.6}{409.1} = -0.226\,,$$

$$\delta y = \frac{[a\,a]\,[b\,l] - [a\,b]\,[a\,l]}{[a\,a]\,[b\,b] - [a\,b]\,[a\,b]} = \frac{+227.34 - 198.81}{2068.8 - 1659.7} = +\frac{28.53}{409.1} = +0.06974\,.$$

The δy thus found is only an auxiliary unknown; the actual correction $\delta y'$ is, according to (7), equal to the hundredth part of δy, i.e.

$$\delta y' = \frac{\delta y}{100} = +0.000697\ .$$

Now we have according to (1) and (2) the approximations, and we have just computed the corrections, therefore together

Approximations	$(x) = 762.00$	$(y) = 0.086250$
Corrections	$\delta x = -0.23$	$\delta y' = +0.000697$
Solutions	$x = 761.77$	$y = 0.086947$. (13)

With this, the adjusted function is

$$B = x - y\,h = 761.77 - 0.086947\,h\ . \tag{14}$$

This agrees sufficiently with the previous (13), section 14, p. 45. We have thus obtained again, by a second more convenient way, the result of the previous section 14, and if a small deviation in y occurs, since the new coefficient yields $y = 0.086947$ compared with 0.08695 of the previous calculation, then without investigating further in detail, we shall give preference to the new result 0.086947, because in spite of the smaller set of numbers in the calculation, it must be obtained more rigorously.

If we calculate, according to (14), the individual B's for the given elevations h and compare them with the observed B's, then we obtain the following:

	h	B Observed	B, According to (14) Adjusted	v	v^2
1.	120.2 m	751.18 mm	751.32 mm	+ 0.14 mm	0.0196
2.	225.1	742.37	742.20	— 0.17	0.0289
3.	270.6	738.50	738.24	— 0.26	0.0676
4.	347.6	731.27	731.55	+ 0.28	0.0784
5.	406.7	726.99	726.41	— 0.58	0.3364
6.	492.4	718.16	718.96	+ 0.80	0.6400
7.	708.1	700.48	700.21	— 0.27	0.0729
8.	733.5	697.64	698.00	+ 0.36	0.1296
9.	768.9	695.23	694.92	— 0.31	0.0961
					$1.4695 = [v^2]$. (15)

This sum $[v^2]$ is needed for the computation of the mean error.

If the v's were true errors, then we would simply compute

$$m = \sqrt{\frac{[v^2]}{n}} = \sqrt{\frac{1.4695}{9}} = \pm 0.404 \text{ mm}\ . \ (?) \tag{16}$$

But this is not correct for the same reason for which also the first computation (3), section 4, p. 11, was not correct, namely, because the v's are not true errors but only corrections. The more rigorous computation of the mean error m will be taught in the later section 17.

Introduction of an auxiliary unknown

For the setting up of the error equations we have to remember the rule that the measured quantities, after the addition of their corrections, hence, the quantities $l + v$, are to be expressed in terms of the unknowns of the problem. According to this fundamental theorem, we have obtained, in the first example in section 12, the error equations (2), p. 38, and, likewise, the error equations (9) in section 14, p. 44, as well as the equation (6) in section 15, p. 47, have originated in this way.

Sometimes, however, it is not possible to express the corrected observations directly in terms of the present unknowns. In such cases, the introduction of a further unknown is necessary, the knowledge of which in itself does not offer any interest.

We will explain this by a simple example.

For a scale divided into rheinländische Zoll, the length of a Zoll [inch] shall be determined in millimeters. For this, as Fig. 1 shows, the Zoll scale was arbitrarily laid on a millimeter scale, and on the latter, the position of several Zoll marks was read off. The result was

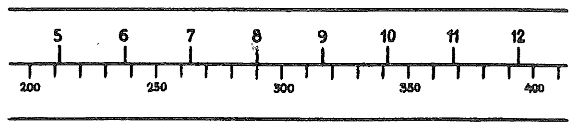


Fig. 1.

$$\left.\begin{array}{lcccccc} \text{Zoll} & 6 & 7 & 8 & 9 & 10 & 11 \\ \text{mm} & 237.9 & 264.1 & 290.3 & 316.5 & 342.6 & 368.9 \end{array}\right\} . \tag{17}$$

The single unknown of the problem is the quantity of a Zoll interval in millimeters; but it is not possible to set up the error equations according to the above rule with this unknown alone. We will therefore introduce still a second unknown, namely the reading for the zero mark of the Zoll scale (not represented in Fig. 1). If we denote this reading by x, the magnitude of a Zoll in millimeters by y, then we obtain, according to the above principle, the error equations

$$\left.\begin{aligned} 237.9 + v_1 &= x + 6y \\ 264.1 + v_2 &= x + 7y \\ 290.3 + v_3 &= x + 8y \\ 316.5 + v_4 &= x + 9y \\ 342.6 + v_5 &= x + 10y \\ 368.9 + v_6 &= x + 11y \end{aligned}\right\} . \tag{18}$$

For further treatment of the problem we will use approximate values of the unknowns, which we find easily if we neglect the corrections v in the first and last equation. In round numbers we obtain

$$(x) = 82.0 \,, \qquad (y) = 26.0 \,, \tag{19}$$

hence, the final unknowns are

$$x = 82.0 + \delta x \,, \qquad y = 26.0 + \delta y \,, \tag{20}$$

and if we introduce this in the above equations, then there follow the new error equations

$$\left.\begin{aligned} v_1 &= +\delta x + 6\,\delta y + 0.1 \\ v_2 &= +\delta x + 7\,\delta y - 0.1 \\ v_3 &= +\delta x + 8\,\delta y - 0.3 \\ v_4 &= +\delta x + 9\,\delta y - 0.5 \\ v_5 &= +\delta x + 10\,\delta y - 0.6 \\ v_6 &= +\delta x + 11\,\delta y - 0.9 \end{aligned}\right\} . \tag{21}$$

From these error equations, which have the form (2), section 14, p. 42, we pass over at once to the normal equations (4), section 14, p. 43, namely

$$\left.\begin{aligned} 6\,\delta x + \;\;51\,\delta y - \;\;2.3 &= 0 \\ 51\,\delta x + 451\,\delta y - 22.9 &= 0 \end{aligned}\right\}, \tag{22}$$

and we obtain therefrom

$$\delta x = -1.240\,, \qquad \delta y = +0.191\,. \tag{23}$$

The final values of the unknowns are thus

$$x = 80.760 \text{ mm}\,, \qquad y = 26.191 \text{ mm}\,. \tag{24}$$

For the calculation of the corrections v we can start from the equations (18) as well as from the equations (21). We do best to use both systems of equations in order to check at the same time the introduction of the approximate values (x) and (y). We find the following corrections v, to which we add at the same time the squares v^2 for later use:

v	v^2
+ 0.01 mm	0.0001
0.00	0.0000
— 0.01	0.0001
— 0.02	0.0004
+ 0.07	0.0049
— 0.04	0.0016
	0.0071 = $[v^2]$. (25)

We shall return also to this example once again later.

Section 16. Gauss' Elimination and Sum of Squares of the Corrections for Two Unknowns

Instead of the direct solution of the normal equations, which is given in (5), section 14, p. 43, there is recommended, in most cases, the gradual elimination indicated by Gauss with a characteristic, readily conceivable method of notation.

We take once more the normal equations (4), section 14, p. 43:

$$[a\,a]\,x + [a\,b]\,y - [a\,l] = 0\,, \tag{1}$$

$$[a\,b]\,x + [b\,b]\,y - [b\,l] = 0\,. \tag{2}$$

We multiply the first normal equation (1) by $-\frac{[a\,b]}{[a\,a]}$ and add it to the second, whereby x vanishes and the following equation remains:

$$\left([b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b]\right) y - \left([b\,l] - \frac{[a\,b]}{[a\,a]}[a\,l]\right) = 0\,. \tag{3}$$

This gives rise to introducing an abbreviated notation, namely

$$[b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b] = [b\,b \cdot 1]\,, \qquad [b\,l] - \frac{[a\,b]}{[a\,a]}[a\,l] = [b\,l \cdot 1]\,, \tag{4}$$

with this, (3) becomes

$$[b\,b\cdot 1]\,y - [b\,l\cdot 1] = 0\,, \qquad y = \frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}. \tag{5}$$

If we make the elimination in reverse order, then we obtain

$$[a\,a] - \frac{[a\,b]}{[b\,b]}[a\,b] = [a\,a\cdot 1]\,, \qquad [a\,l] - \frac{[a\,b]}{[b\,b]}[b\,l] = [a\,l\cdot 1]\,; \tag{6}$$

$$[a\,a\cdot 1]\,x - [a\,l\cdot 1] = 0\,, \qquad x = \frac{[a\,l\cdot 1]}{[a\,a\cdot 1]}. \tag{7}$$

The brackets $[b\,b\cdot 1]$, $[b\,l\cdot 1]$, and so forth, are a symbolic notation of a similar kind, as also, for instance, the notation of determinants.

In order to impress the construction of our bracket coefficients on our memory, let us remember first that each such value becomes = zero as soon as we conceive the symbolic notation algebraically, e.g.:

$$[b\,b\cdot 1] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b] = b\,b - \frac{a\,b}{a\,a}a\,b = b\,b - \frac{b}{a}a\,b = b\,b - b\,b = 0\,. \tag{8}$$

Sum of squares of the corrections

We can represent also the sum of squares $[v\,v]$ of the corrections v by a similarly constructed expression, as $[b\,b\cdot 1]$, and so forth.

According to (3), section 14, p. 42, this sum is first

$$\left.\begin{aligned}[v\,v] = [a\,a]\,x^2 + 2\,[a\,b]\,x\,y &- 2\,[a\,l]\,x \\ + [b\,b]\,y^2 &- 2\,[b\,l]\,y \\ &+ [l\,l]\end{aligned}\right\}. \tag{9}$$

We can also write this in the form

$$\begin{aligned}[v\,v] = &\,([a\,a]\,x + [a\,b]\,y - [a\,l])\,x \\ &+ ([a\,b]\,x + [b\,b]\,y - [b\,l])\,y \\ &- [a\,l]\,x - [b\,l]\,y + [l\,l]\end{aligned}$$

and according to (1) and (2), this is

$$[v\,v] = [l\,l] - [a\,l]\,x - [b\,l]\,y\,. \tag{10}$$

We can transform this expression still further by eliminating x and y. First, we have according to (1)

$$x = \frac{[a\,l]}{[a\,a]} - \frac{[a\,b]}{[a\,a]}\,y\,,$$

with which (10) passes over into

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \left([b\,l] - \frac{[a\,l]\,[a\,b]}{[a\,a]}\right)y$$

or according to (4)

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - [b\,l \cdot 1]\,y\,.$$

If we introduce, in addition, the value of y from (5), then we will have

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \frac{[b\,l \cdot 1]^2}{[b\,b \cdot 1]}\,. \tag{11}$$

We can also apply, in addition, to this equation the symbolic notation (4) or, as the case may be, (6) in the corresponding form. If we set

$$[l\,l] - \frac{[a\,l]}{[a\,a]}[a\,l] = [l\,l \cdot 1]\,, \tag{12}$$

then (11) changes into

$$[v\,v] = [l\,l \cdot 1] - \frac{[b\,l \cdot 1]^2}{[b\,b \cdot 1]}\,,$$

and if we set, accordingly,

$$[l\,l \cdot 1] - \frac{[b\,l \cdot 1]}{[b\,b \cdot 1]}[b\,l \cdot 1] = [l\,l \cdot 2]$$

then

$$[v\,v] = [l\,l \cdot 2]\,. \tag{13}$$

The newly introduced terms $[l\,l \cdot 1]$ and $[l\,l \cdot 2]$ in (11) and (13), which were not needed for the elimination itself, are computed following the elimination.

We will show this in the second example of section 15, p. 49, and take from there the normal equations (22), in which we replace now δx and δy by x and y.

	$-\dfrac{[a\,b]}{[a\,a]}$	$+\dfrac{[a\,l]}{[a\,a]}$
$6\,x + 51\,y - 2.3 = 0$	$-\dfrac{51}{6}$	$+\dfrac{2.3}{6}$
$51\,x + 451\,y - 22.9 = 0$		
$[l\,l] = +1.53$		

The multiplication of the first equation by the quotient $-\dfrac{51}{6}$ yields

in addition to this,

$$\begin{aligned} -51\,x - 433.5\,y + 19.55 &= 0 \\ +51\,x + 451.0\,y - 22.90 &= 0 \\ \hline +\ 17.5\,y - 3.35 &= 0\,. \end{aligned}$$

If we further multiply the absolute term of the first normal equation by the quotient $+\dfrac{2.3}{6}$ and add to $[l\,l]$, then there follows

$$\begin{aligned} &-0.882 \\ [l\,l] = &+1.530 \\ \hline [l\,l \cdot 1] = &+0.648\,. \end{aligned}$$

The result of the single reduction is thus

$$+17.5\,y - 3.35 = 0 \qquad \left| \quad + \frac{[b\,l \cdot 1]}{[b\,b \cdot 1]} \quad + \frac{3.35}{17.5} \right.$$

$$[l\,l \cdot 1] = +0.648$$

The second reduction is limited to the multiplication of the absolute term -3.35 by the quotient $+\frac{3.35}{17.5}$ and yields

$$\begin{aligned} & -0.641 \\ [l\,l \cdot 1] &= +0.648 \\ [l\,l \cdot 2] &= +0.007 = [v\,v]\,. \end{aligned}$$

This agrees with the value (25) of $[v\,v]$ found directly in section 15, p. 50.

The abbreviated notation $[b\,b \cdot 1]$, $[b\,c \cdot 1]$, $[l\,l \cdot 1]$, and so forth, with the corresponding elimination was first introduced by Gauss in 1810 in the treatise, "Disquisitio de elementis ellipticis Palladis etc." The symbols are written there $[b\,b \cdot 1]$, $[b\,c \cdot 1]$, etc., and while now $[b\,b \cdot 1]$ is usually written, others also use $[b\,b_1]$. Apart from these small changes, this classical notation has been adopted since that time in all well-known publications about the method of least squares and is retained, so to speak, as a sacred notation. The attempt of replacing these symbols, fortunately now established, by others would have to be termed as unfortunate and not tenable for any length of time.

Section 17. Mean Error of the Unit of Weight in the Case of Two Unknowns

From the sum of squares $[v\,v]$, with which we have dealt just now, we can also compute the mean error of an observation (or, as the case may be, the mean error of an observation of weight = 1). In the first approximation we can write

$$m = \sqrt{\frac{[v\,v]}{n}}\,. \quad (?) \tag{1}$$

This formula, however, is not sufficient because the v's are not true errors, but only corrections, as has already been considered in section 15, p. 48.

We must change the formula (1) in a similar manner, as has already been done with the arithmetic mean in (10), section 7, p. 18, where the denominator n was converted to $n-1$ (for *one* unknown x).

We shall have to distinguish again:

Adjustment results	$x\;y$,	true unknowns	$X\,Y$
Corrections	$v_1, v_2, v_3 \ldots,$	true errors	$\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots.$

Then there exist the equations

$$\left.\begin{aligned} v &= a\,x + b\,y - l \\ \varepsilon &= a\,X + b\,Y - l \end{aligned}\right\} \tag{2}$$

hence, the difference $\quad v - \varepsilon = a\,(x - X) + b\,(y - Y)$

or

$$v = \varepsilon + a\,(x - X) + b\,(y - Y)\,. \tag{3}$$

If we form therefrom the quantities $[a v] = 0$ and $[b v] = 0$, then we also obtain a type of normal equations, namely

$$\left.\begin{aligned} [a v] &= [a a] (x - X) + [a b] (y - Y) + [a\varepsilon] = 0 \\ [b v] &= [a b] (x - X) + [b b] (y - Y) + [b\varepsilon] = 0 \end{aligned}\right\} \qquad (4)$$

Also the sum $[v v]$ can be formed in a similar form as previously, namely

$$\begin{aligned} [v v] = [a a] (x - X)^2 &+ 2 [a b] (x - X) (y - Y) + 2 [a\varepsilon] (x - X) \\ &+ [b b] (y - Y)^2 \qquad + 2 [b\varepsilon] (y - Y) \\ &\qquad\qquad\qquad\qquad + [\varepsilon\varepsilon] , \end{aligned}$$

and with this, the same transformation can be made as in the case of the previous $[v v]$ in (11), section 16, p. 52, namely

$$[v v] = [\varepsilon\varepsilon] - \frac{[a\varepsilon]^2}{[a a]} - \frac{[b\varepsilon \cdot 1]^2}{[b b \cdot 1]} . \qquad (5)$$

We see hence that $[v v]$ is smaller than $[\varepsilon\varepsilon]$, and the point in question is to determine the difference between $[v v]$ and $[\varepsilon\varepsilon]$, i.e. the two final terms of (5) as accurately as is possible in view of the indeterminacy of the true errors ε, i.e. we aim at determining the average values of the last two terms in (5). We have first

$$\begin{aligned} [a\varepsilon]^2 &= (a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 + \ldots)^2 \\ &= a_1{}^2\varepsilon_1{}^2 + a_2{}^2\varepsilon_2{}^2 + a_3{}^2\varepsilon_3{}^2 + \ldots + 2\, a_1 a_2 \varepsilon_1\varepsilon_2 + \ldots \end{aligned} \qquad (6)$$

In place of the squares $\varepsilon_1{}^2$, $\varepsilon_2{}^2 \ldots$ we set their common average value m^2, and the products $\varepsilon_1, \varepsilon_2 \ldots$ must vanish, because of the irregular change of sign $\pm\varepsilon$ in the average value; consequently, (6) yields

$$[a\varepsilon]^2 = (a_1{}^2 + a_2{}^2 + \ldots) m^2 + 0 ,$$

$$[a\varepsilon]^2 = [a a] m^2 \quad \text{and} \quad \frac{[a\varepsilon]^2}{[a a]} = m^2 . \qquad (7)$$

In the same manner, we can also treat the last term of (5), namely

$$\begin{aligned} [b\,\varepsilon \cdot 1] &= [b\,\varepsilon] - \frac{[a\,b]}{[a\,a]} [a\,\varepsilon] \\ &= (b_1\,\varepsilon_1 + b_2\,\varepsilon_2 + \ldots) - \frac{[a\,b]}{[a\,a]} (a_1\,\varepsilon_1 + a_2\,\varepsilon_2 + \ldots) \\ &= \left(b_1 - \frac{[a\,b]}{[a\,a]} a_1\right) \varepsilon_1 + \left(b_2 - \frac{[a\,b]}{[a\,a]} a_2\right) \varepsilon_2 + \ldots \\ [b\,\varepsilon \cdot 1]^2 &= \left(b_1 - \frac{[a\,b]}{[a\,a]} a_1\right)^2 \varepsilon_1{}^2 + \left(b_2 - \frac{[a\,b]}{[a\,a]} a_2\right)^2 \varepsilon_2{}^2 + \ldots \\ &\qquad + 2 \left(b_1 - \frac{[a\,b]}{[a\,a]} a_1\right) \left(b_2 - \frac{[a\,b]}{[a\,a]} a_2\right) \varepsilon_1\,\varepsilon_2 + \ldots \end{aligned}$$

We shall form hence the average value, taking into account the different ε_1's, ε_2's . . . , and here we make again the same considerations as in the case of (6), namely that the average value of the various *squares* $\varepsilon_1{}^2$, $\varepsilon_2{}^2 \ldots$ is generally set $= m^2$, and that the average value of the *products* $\varepsilon_1, \varepsilon_2$ vanishes because of the irregularly varying signs $\pm$ of the individual ε's. We thus find from the above:

$$[b\varepsilon\cdot 1]^2 = \left((b_1 - \frac{[ab]}{[aa]}a_1)^2 + (b_2 - \frac{[ab]}{[aa]}a_2)^2 + \ldots\right) m^2$$

$$\frac{[b\varepsilon\cdot 1]^2}{m^2} = \left(b_1^2 + \frac{[ab]^2}{[aa]^2}a_1^2 - 2a_1 b_1 \frac{[ab]}{[aa]}\right) + \left(b_2^2 + \frac{[ab]^2}{[aa]^2}a_2^2 - 2a_2 b_2 \frac{[ab]}{[aa]}\right) + \ldots$$

$$= (b_1^2 + b_2^2 + \ldots) + \frac{[ab]^2}{[aa]^2}(a_1^2 + a_2^2 + \ldots) - 2\frac{[ab]}{[aa]}(a_1 b_1 + a_2 b_2 + \ldots)$$

$$= [bb] + \frac{[ab]^2}{[aa]^2}[aa] - 2\frac{[ab]}{[aa]}[ab]$$

$$\frac{[b\varepsilon\cdot 1]^2}{m^2} = [bb] - \frac{[ab]}{[aa]}[ab] = [bb\cdot 1]. \tag{8}$$

If we set these equations (8) besides (7) into (5), then we obtain

$$[vv] = [\varepsilon\varepsilon] - m^2 - m^2 ,$$

$$[vv] = [\varepsilon\varepsilon] - 2\, m^2 .$$

Now $\frac{[\varepsilon\varepsilon]}{n} = m^2$ is the true determination of m^2, which, combined with the above, yields:

$$[vv] = n\, m^2 - 2\, m^2 ,$$

therefore

$$m^2 = \frac{[vv]}{n-2}, \qquad m = \sqrt{\frac{[vv]}{n-2}}. \tag{9}$$

This is the correct formula instead of the dubious formula (1). Formula (9) reminds us in its construction of the earlier formula (10), section 7, p. 18, for the arithmetic mean. Just as for *one* unknown in the case of the arithmetic mean the denominator n changed to $n-1$, for *two* unknowns z and y the denominator must now become $n-2$.

If we apply the formula (9) to our scale comparison computed in section 15, p. 49, then we have according to (25), p. 50, $[vv] = 0.0071$. Since $n = 6$ and there are two unknowns, then we obtain

$$m^2 = \frac{0.0071}{4} = 0.00178 ,$$

$$m = \pm 0.04 \text{ mm} .$$

With this, the accuracy of the reading on the millimeter scale is unobjectionably computed.

We can probably assume that this will now continue so that with three unknowns x, y, z the denominator will be $n-3$ and generally with u unknowns the denominator will be $n-u$; but before we shall have proved this, we cannot yet accept it as valid.

The distinction of true errors ε and corrections v in the computation of the mean error is one of the finest considerations of the method of least squares, a real flower of the Gauss genius. The general theorem, whose special case for two unknowns we have just treated, was developed first by Gauss in article 38 of "Theoria combinationis" (of the year 1823).

With what we have taught in sections 14 to 17 we can already make small adjustments, and in many cases nothing further is done (and in the sense of the gradual learning of the whole theory it might be advisable to take up now immediately once again the first numerical example of section 15 and to continue it according to section 16 and section 17).

However, just as in the case of the arithmetic mean the mean error of the mean value itself, i.e. of the adjusted x, had to be determined, the continuation of our adjustment with two unknowns still requires now also the computation of the mean errors of the adjusted x's and y's.

In order to reach this, we take again our general law of propagation of errors (10), section 5, p. 14, and assume that x and y are connected with measured quantities l by the following linear equations:

$$x = \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \ldots + \alpha_n l_n , \tag{1}$$

$$y = \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \ldots + \beta_n l_n . \tag{2}$$

If m is here the mean error of an individual l, then, according to the mentioned law of error propagation, the mean squares of the errors of x and y are

$$m_x^2 = \alpha_1^2 m^2 + \alpha_2^2 m^2 + \ldots = (\alpha_1^2 + \alpha_2^2 + \ldots)\, m^2 = [\alpha\alpha]\, m^2 , \tag{3}$$

$$m_y^2 = \beta_1^2 m^2 + \beta_2^2 m^2 + \ldots = (\beta_1^2 + \beta_2^2 + \ldots)\, m^2 = [\beta\beta]\, m^2 , \tag{4}$$

or in weight form, if the weight 1 belongs to the mean error m,

$$p_x = \frac{1}{[\alpha\alpha]} , \qquad p_y = \frac{1}{[\beta\beta]} . \tag{5}$$

This holds first for arbitrary values α and β; but we will now understand by y the unknown which is derived from the solution of our normal equations, namely according to (5), section 16, p. 51

$$y = \frac{[bl \cdot 1]}{[bb \cdot 1]} . \tag{6}$$

In order to bring this equation (6) into agreement with (2), we must solve the numerator of (6) in such a way that all l_1's, l_2's . . . occurring in it appear alone; therefore, we develop

$$\begin{aligned}
[bl \cdot 1] &= [bl] - \frac{[ab]}{[aa]}[al] \\
&= (b_1 l_1 + b_2 l_2 \ldots) - \frac{[ab]}{[aa]}(a_1 l_1 + a_2 l_2 + \ldots) \\
&= \left(b_1 - \frac{[ab]}{[aa]} a_1\right) l_1 + \left(b_2 - \frac{[ab]}{[aa]} a_2\right) l_2 + \ldots
\end{aligned}$$

This taken into account in (6) yields

$$y = \frac{b_1 - \frac{[ab]}{[aa]} a_1}{[bb \cdot 1]} l_1 + \frac{b_2 - \frac{[ab]}{[aa]} a_2}{[bb \cdot 1]} l_2 + \ldots$$

The comparison with (2) shows that the β's have the following meanings:

$$\beta_1 = \frac{b_1 - \frac{[a\,b]}{[a\,a]} a_1}{[b\,b:1]}, \quad \beta_2 = \frac{b_2 - \frac{[a\,b]}{[a\,a]} a_2}{[b\,b:1]} \cdots \tag{7}$$

therefore,

$$\beta_1^2 = \frac{1}{[b\,b:1]^2}\left(b_1^2 - 2\frac{[a\,b]}{[a\,a]} a_1 b_1 + \frac{[a\,b]^2}{[a\,a]^2} a_1^2\right),$$

likewise,

$$\beta_2^2 = \frac{1}{[b\,b\cdot 1]^2}\left(b_2^2 - 2\frac{[a\,b]}{[a\,a]} a_2 b_2 + \frac{[a\,b]^2}{[a\,a]^2} a_2^2\right).$$

$$\cdots\cdots\cdots\cdots$$

Total

$$[\beta^2] = \frac{1}{[b\,b\cdot 1]^2}\left([b^2] - 2\frac{[a\,b]}{[a\,a]}[a\,b] + \frac{[a\,b]^2}{[a\,a]^2}[a^2]\right)$$

or with

$[b^2] = [b\,b]$, $[a^2] = [a\,a]$ etc.:

$$[\beta\,\beta] = \frac{1}{[b\,b\cdot 1]^2}\left([b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b]\right) = \frac{1}{[b\,b\cdot 1]^2}[b\,b\cdot 1],$$

therefore,

$$[\beta\,\beta] = \frac{1}{[b\,b:1]} \quad \text{or} \quad p_y = [b\,b:1]. \tag{8}$$

This holds for the unknown y, and accordingly we have for x, just by interchanging everywhere b and a,

$$[\alpha\,\alpha] = \frac{1}{[a\,a\cdot 1]}, \qquad p_x = [a\,a\cdot 1]. \tag{9}$$

We call the sums of squares $[\alpha\,\alpha]$ and $[\beta\,\beta]$ weight coefficients. With this, we have also the mean error of y and of x for the error of the unit of weight m:

$$m_y = \frac{m}{\sqrt{p_y}} = \frac{m}{\sqrt{[b\,b:1]}}, \tag{10}$$

$$m_x = \frac{m}{\sqrt{p_x}} = \frac{m}{\sqrt{[a\,a\cdot 1]}}; \tag{11}$$

m itself is determined here according to (9), section 17, p. 55, and (13), section 16, p. 52, namely

$$m = \sqrt{\frac{[v\,v]}{n-2}} = \sqrt{\frac{[l\,l:2]}{n-2}}. \tag{12}$$

Determinant of coefficients. If we introduce the determinant of the coefficients of normal equations $[a\,a]$, $[a\,b]$, and so forth, namely

$$[a\,a]\,[b\,b] - [a\,b]\,[a\,b] = D, \tag{13}$$

then we can also construct the following forms, as is shown at once by solving:

$$[b\,b \cdot 1] = \frac{D}{[a\,a]}, \qquad [a\,a \cdot 1] = \frac{D}{[b\,b]}. \tag{14}$$

With this, all that is needed for the determination of errors of x and y itself is found. But we will find later that, in addition, there is also a problem which is not included in the material thus far treated, namely the determination of the mean error of a function of the adjusted x's and y's.

It is not advisable to occupy ourselves with this before the mean errors of x and y themselves are fully understood according to the formulae (9) and (10) on p. 57 and also practiced by numerical examples; for the sake of the connection, however, we will in addition give the formulae $[\alpha\beta]$, and so forth. Namely, we need later (in section 26) not only the sums of squares $[\alpha\alpha]$ and $[\beta\beta]$, but also the sum of products $[\alpha\beta]$ of the coefficients α and β of (1) and (2).

A single value β_1 or α_1 results from (7), with the use of D according to (14):

$$\beta_1 = \frac{[a\,a]}{D}\left(b_1 - \frac{[a\,b]}{[a\,a]}a_1\right), \qquad \alpha_1 = \frac{[b\,b]}{D}\left(a_1 - \frac{[a\,b]}{[b\,b]}b_1\right),$$

$$\alpha_1\,\beta_1 = +\frac{1}{D^2}([a\,a]\,b_1 - [a\,b]\,a_1)([b\,b]\,a_1 - [a\,b]\,b_1)$$

$$= \frac{1}{D^2}([a\,a]\,[b\,b]\,a_1\,b_1 - [a\,a]\,[a\,b]\,b_1\,b_1 - [a\,b]\,[b\,b]\,a_1\,a_1 + [a\,b]\,[a\,b]\,a_1\,b_1),$$

then the sum of all such products is:

$$[\alpha\beta] = \frac{1}{D^2}([a\,a]\,[b\,b]\,[a\,b] - [a\,a]\,[a\,b]\,[b\,b] - [a\,b]\,[b\,b]\,[a\,a] + [a\,b]\,[a\,b]\,[a\,b]).$$

The first two terms cancel each other, and if we take again into account the meaning of D, then we obtain

$$[\alpha\beta] = \frac{-[a\,b]}{D} = \frac{-[a\,b]}{[a\,a]\,[b\,b] - [a\,b]\,[a\,b]}, \tag{15}$$

and because of analogy with $[b\,b \cdot 1]$ we will introduce in addition:

$$[a\,b] - \frac{[a\,a]}{[a\,b]}[b\,b] = [a\,b \cdot 1], \tag{16}$$

whereby we will have:

$$[\alpha\beta] = \frac{1}{[a\,b \cdot 1]}. \tag{17}$$

Summary:

$$[\alpha\,\alpha] = \frac{1}{[a\,a \cdot 1]} = \frac{[b\,b]}{D} = \frac{1}{p_x} \qquad \left.\begin{aligned}[\alpha\,\beta] &= \frac{1}{[a\,b \cdot 1]} = \frac{-[a\,b]}{D} \\ [\beta\,\beta] &= \frac{1}{[b\,b \cdot 1]} = \frac{[a\,a]}{D} = \frac{1}{p_y}\end{aligned}\right\} \tag{18}$$

$$(m_x)^2 = \frac{m^2}{p_x} = [\alpha\,\alpha]\,m^2 = \frac{[b\,b]}{D}m^2 \qquad (m_y)^2 = \frac{m^2}{p_y} = [\beta\,\beta]\,m^2 = \frac{[a\,a]}{D}m^2 \tag{19}$$

$$D = [a\,a]\,[b\,b] - [a\,b]\,[a\,b]. \tag{20}$$

There corresponds to this determinant of coefficients D also a weight coefficient determinant

$$\Delta = [\alpha\alpha]\,[\beta\beta] - [\alpha\beta]\,[\alpha\beta]\ , \tag{21}$$

and between both there exists the relation

$$D\,\Delta = 1\ . \tag{22}$$

In this connection, however, we repeat the remark that we do *not* need all the latter, from (13) to (22), for the following problems.

Section 19. Computation of Coefficients and Sum Checks

Although the adjustment with two unknowns x and y is completely clarified by the preceding sections 14 to 18, yet before beginning a numerical example, we will in addition make a few remarks about the calculation and insurance of the sums of squares and sums of products $[a\,a]$, $[a\,b]$, and so forth.

According to whether the numbers are indicated simply or with many places the calculation of the squares $a\,a$ and the products $a\,b$, and so forth, can be done differently. There are mentioned, in this connection, the mechanical auxiliary means for computation, which we have described in our Volume II, 1, 9th Edition, pp. 111 to 142; slide rule, circular slide rule, calculating machines, etc.

In any case, we form the squares with a table of squares, like the one, e.g. on pp. [2] to [6] of our Appendix with three-place argument. For greater accuracy (which, however, is seldom needed if the computational form is well prepared) we have more detailed square tables as supplements to numerous logarithmic tables and other tabular works.

We can multiply the products $a\,b$, etc., out directly, as has been done in the example on p. 47, with $[b\,l]$. In the case of numbers having many places we can use a table of products, of which there are several, e.g.:

Dr. A. L. Crelle's *Rechentafeln, welche alles Multiplizieren und Dividieren mit Zahlen unter Tausend ganz ersparen*, with a Foreword by Bremiker; New Edition, prepared by O. Seeliger, Berlin, 1907. (Three- and three-place.)

Rechentafel nebst Sammlung häufig gebrauchter Zahlenwerte, designed and edited by Dr. H. Zimmermann, Wirklicher Geheimer Oberbaurat; Tenth Edition, Berlin, 1930. (Three- and two-place.)

Rechentafeln zum Gebrauch für Schulen und Praxis, by L. Zimmermann, Liebenwerda, 1895. (Two- and three-place.)

Rechentafeln, welche die Produkte aller Zahlen unter 10,000 in alle Zahlen bis 100 enthalten, Great Edition, L. Zimmermann, Liebenwerda, 1896. (Two- and four-place.)

Kühtmanns Rechentafeln, ein handliches Zahlenwerk mit zwei Millionen Lösungen, Dresden, 1911. (Three- and three-place.)

But there is also a very good method to determine the products with the square table. Namely, we have

$$(a+b)^2 = a^2 + b^2 + 2\,a\,b \qquad \text{or} \qquad a\,b = \frac{(a+b)^2 - (a^2+b^2)}{2}\ ,$$

from which we find:

$$[a\,b] = \frac{[(a+b)^2] - ([a\,a] + [b\,b])}{2}\ , \tag{1}$$

and since we need $[a\,a]$ as well as $[b\,b]$ anyhow, then, only $[(a+b)^2]$, i.e. the sum of the squares $(a+b)^2$ has to be computed in addition.

Instead of $(a+b)^2$ we can also use $(a-b)^2$; in this manner:

$$[a\,b] = \frac{-[(a-b)^2] + [a\,a] + [b\,b]}{2}\ . \tag{2}$$

We will show this method by an example, which belongs to section 15 and section 20:

b	$-l$	$-bl$		b^2	l^2	$(b-l)$	$(b-l)^2$
1.20	0.45	+ 0.54		1.44	0.20	1.65	2.72
2.25	0.22	0.50		5.06	0.05	2.47	6.10
2.71	0.16	0.43		7.34	0.03	2.87	8.24
3.48	0.75	2.61		12.11	0.56	4.23	17.89
4.07	−0.07		− 0.28	16.56	0.00	4.00	16.00
4.92	1.37	6.74		24.21	1.88	6.29	39.56
7.08	0.45	3.19		50.13	0.20	7.53	56.70
7.34	1.10	8.07		53.88	1.21	8.44	71.23
7.69	0.45	3.46		59.14	0.20	8.14	66.26
40.74	4.88	+ 25.54	− 0.28	229.87	4.33	45.62	284.70
45.62		+ 25.26		$=[bb]$	$=[ll]$		− 234.20
		$=-[bl]$		$[bb]+[ll]=234.20$			$-2[bl]=+50.50$
							$-[bl]=+25.25$.

We have thus found $-[b\,l]$ directly $=+25.26$ and by way of $(b-l)^2$ and $[b\,b]$ and $[l\,l]$ indirectly $-[b\,l]=+25.25$, which agrees sufficiently. With this, not only $-[b\,l]$ itself but also $[b\,b]$ and $[l\,l]$ are insured.

We can make such checks in every possible connection; e.g., we can check $[a\,a]$ and $[b\,b]$ besides $[a\,b]$ by $[(a+b)^2]$ and then, in addition, form the sum of products of $-(a+b)\,l$, whereby also $-[a\,l]$ and $-[b\,l]$ are insured in addition.

Further, we can also determine a product by squares in another way, according to the equation

$$a\,b = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2 .$$

Referring to this, there is computed: *Tafel der Viertelquadrate aller ganzen Zahlen von 1 bis 200,000*, etc., by Joseph Blater, Wien, 1887 (reviewed in *Zeitschrift für Vermessungswesen*, 1888, p. 485). For multiplication of numbers having many places, this table may be of advantage; for products of small factors, as mostly occur in the method of least squares, the table of fourths of a square is hardly worth while. (Cf. *Handbuch*, Volume II, 1, 9th Edition, 1931, p. 174.) We mention further: Plassmann, *Tafeln der Viertelquadrate aller Zahlen von 1 bis 20,009 zur Erleichterung des Multiplizierens vierstelliger Zahlen*, 1933.

Sum checks

We can also check the computation of the coefficients $[a\,a]$, $[a\,b]$, and so forth, by sums, which always extend over the whole series a, b, $-l$.

In the case of only *two* elements x, y or, as the case may be, a, b the great advantage of such sum checks does not stand out so clearly as in the case of many unknowns, where the sum checks, especially effective also in the elimination from line to line, are of excellent use through the whole computation. Meanwhile, in order to practice the method gradually, we must already learn the whole check-sum apparatus also in the case of two unknowns. Here it is convenient not to take into account the sums themselves, but the *negative* sums so that everything must always end with zero, i.e. we set

$$\left.\begin{aligned} a_1 + b_1 - l_1 + s_1 &= 0 \\ a_2 + b_2 - l_2 + s_2 &= 0 \text{ etc.} \end{aligned}\right\}, \qquad (3)$$

then we have also:

$$\left.\begin{aligned} [a\,a] + [a\,b] - [a\,l] + [a\,s] &= 0 \\ [a\,b] + [b\,b] - [b\,l] + [b\,s] &= 0 \\ -[a\,l] - [b\,l] + [l\,l] - [l\,s] &= 0 \\ [a\,s] + [b\,s] - [l\,s] + [s\,s] &= 0 \end{aligned}\right\}. \qquad (4)$$

We will indicate this by lines in the following manner:

$$\begin{array}{cccc}
[a\,a] & [a\,b] & -[a\,l] & [a\,s] \\
 & [b\,b] & -[b\,l] & [b\,s] \\
 & & [l\,l] & -[l\,s] \\
 & & & [s\,s]\,.
\end{array}$$

We thus obtain a system of coordinates as if instead of two unknowns there were three. Accordingly, we also compute further, not only $[b\,b\cdot 1]$, $-[b\,l\cdot 1]$, but also $[b\,s\cdot 1]$, $-[l\,s\cdot 1]$, etc., i.e.

$$[bb\cdot 1]=[bb]-\frac{[a\,b]}{[a\,a]}[ab]$$

$$-[bl\cdot 1]=-\left([bl]-\frac{[a\,b]}{[a\,a]}[al]\right) \qquad [ll\cdot 1]=[ll]-\frac{[a\,l]}{[a\,a]}[a\,l]$$

$$[bs\cdot 1]=[bs]-\frac{[a\,s]}{[a\,a]}[as] \qquad -[ls\cdot 1]=-\left([ls]-\frac{[a\,l]}{[a\,a]}[as]\right) \qquad [ss\cdot 1]=[ss]-\frac{[a\,s]}{[a\,a]}[as]\,.$$

With this, we have again checks, namely

$$\begin{array}{ccc}
[b\,b\cdot 1] & -[b\,l\cdot 1] & [b\,s\cdot 1] \\
 & [l\,l\cdot 1] & -[l\,s\cdot 1] \\
 & & [s\,s\cdot 1]\,.
\end{array}$$

The correctness of the check equations hereby indicated can be easily understood, e.g.:

$$\begin{aligned}
[b\,b\cdot 1] &= [b\,b]-\frac{[a\,b]}{[a\,a]}[a\,b] \\
-[b\,l\cdot 1] &= -\left([b\,l]-\frac{[a\,b]}{[a\,a]}[a\,l]\right) \\
0 &= [a\,b]-\frac{[a\,b]}{[a\,a]}[a\,a] \\
\hline
[b\,b\cdot 1]-[bl\cdot 1] &= -[b\,s]-\frac{[a\,b]}{[a\,a]}(-[a\,s]) = -[b\,s\cdot 1]
\end{aligned}$$

$$[b\,b\cdot 1]-[b\,l\cdot 1]+[b\,s\cdot 1]=0$$

and likewise:

$$\begin{aligned}
-[b\,l\cdot 1]+[l\,l\cdot 1]-[l\,s\cdot 1] &= 0 \\
[b\,s\cdot 1]-[l\,s\cdot 1]+[s\,s\cdot 1] &= 0\,.
\end{aligned}$$

So it also goes further:

$$[l\,l\cdot 2]=[l\,l\cdot 1]-\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[b\,l\cdot 1]$$

$$-[l\,s\cdot 2]=-\left([l\,s\cdot 1]-\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[b\,s\cdot 1]\right) \qquad [s\,s\cdot 2]=[s\,s\cdot 1]-\frac{[b\,s\cdot 1]}{[b\,b\cdot 1]}[b\,s\cdot 1]\,.$$

The checks valid here are indicated by

$$\begin{array}{cc}
[l\,l\cdot 2] & -[l\,s\cdot 2] \\
 & [s\,s\cdot 2]\,,
\end{array}$$

i.e., the absolute values of these last three terms must become equal to one another, as can be easily proved in a similar way as on p. 61 in the case of the coefficients $[\ldots 1]$.

Now, in addition, there comes the important check that we must have $[ll \cdot 2] = [vv]$ according to (13), section 16, p. 52, since $[vv]$ is determined by squaring of the individual v's to be computed.

The very numerous checks communicated are partly superfluous with some practice, if there are only two unknowns, and as far as the elimination is involved, we can probably be satisfied here with the check that $[ll \cdot 2]$, whose determination occurs in the case of y and in the case of x, must be obtained here by agreement on both sides. We can often omit the *quadratic* final terms $[ss]$, $[ss \cdot 1]$, $[ss \cdot 2]$; we have taken up these last check terms more because of the symmetry of the formulae than because of the practical need.

On the other hand, a computer who otherwise is sure of himself can probably be satisfied with *one* check:

$$\begin{aligned}[ss] = [aa] + 2\,[ab] - 2\,[al] \\ + \quad [bb] - 2\,[bl] \\ + \quad [ll]\end{aligned} \qquad \text{or} \qquad \begin{aligned}&= [aa] + [ab] - [al] \\ &+ [ab] + [bb] - [bl] \\ &- [al] - [bl] + [ll] .\end{aligned}$$

If all these checks agree, then we can assert the faultlessness of the computation with a probability bordering on absolute certainty.

Our numerical examples of sections 14 and 15 had the particularity that the coefficients are $a = 1$. This case happens frequently and simplifies greatly the calculation of the sum coefficients, for we have then, in the case of n observations,

$$[aa] = n\,, \qquad [ab] = [b]\,, \qquad -[al] = -[l]\,,$$

and only $[bb]$, $-[bl]$, and $[ll]$ must be computed specially, whereby hardly any sum terms s need be taken, while the elimination can well be made with sum terms.

Another form of the sum check. We can carry out the sum check in still another way by introducing only the sum of the coefficients of the unknowns into the computation, hence, not taking along the absolute terms l. In the case of two unknowns we therefore set

$$\left.\begin{aligned} a_1 + b_1 &= s_1 \\ a_2 + b_2 &= s_2 \\ \ldots\ldots\ldots \end{aligned}\right\} . \tag{5}$$

If, in addition, we form then, besides the coefficients of the normal equations, the sum terms $[as]$, $[bs]$, and $-[ls]$, then it follows from (5) that

$$\left.\begin{aligned} [aa] + [ab] &= [as] \\ [ab] + [bb] &= [bs] \\ -[al] - [bl] &= -[ls] \end{aligned}\right\} , \tag{6}$$

and with this, we obtain then, in addition, a sum equation to the normal equations; we thus have altogether

$$\begin{aligned} [aa]\,x + [ab]\,y - [al] &= 0 \\ [ab]\,x + [bb]\,y - [bl] &= 0 \\ \hline [as]\,x + [bs]\,y - [ls] &= 0\,, \end{aligned}$$

where, according to (6), the sums of the columns of the normal equations must yield the terms of the sum equation.

In addition, we only have to show that this also holds for the reduced equations.

The once reduced normal equation and sum equation read

$$[b\,b\cdot 1]\,y-[b\,l\cdot 1]=0$$
$$[b\,s\cdot 1]\,y-[l\,s\cdot 1]=0\,.$$

Here we have

$$[b\,s\cdot 1]=[b\,s]-\frac{[a\,b]}{[a\,a]}[a\,s]\,,\qquad [l\,s\cdot 1]=[l\,s]-\frac{[a\,l]}{[a\,a]}[a\,s]\,.$$

If we substitute in $[b\,s\cdot 1]$ for $[b\,s]$ and $[a\,s]$ the values from (6), then we obtain

$$[b\,s\cdot 1]=[a\,b]+[b\,b]-\frac{[a\,b]}{[a\,a]}([a\,a]+[a\,b])\,,$$
$$[b\,s\cdot 1]=[b\,b]-\frac{[a\,b]}{[a\,a]}[a\,b]=[b\,b\cdot 1]\,.$$

It can be shown likewise that $[l\,s\cdot 1]=[b\,l\cdot 1]$, i.e. the check given by the sum equation exists also for the reduced equations. By means of this form of sum check, the solution of the normal equations can also be thoroughly checked step by step.

This computation, however, does not contain a check for the reduction of $[l\,l]$, so that the value $[l\,l\cdot 2]$ cannot be examined. In this case, we may check $[v\,v]$ by means of equation (10) or (11) in section 16, pp. 51 and 52.

Section 20. Example of Elimination with Two Unknowns

We take once again the first example of section 15, which we have treated there according to the usual algebraic methods.

The system of coefficients of p. 47 besides sum terms is:

	a	b	$-l$	s
a	$+9.00$	-40.74	$+4.88$	$+26.86$
b		$+229.87$	-25.26	-163.87
$-l$			$+4.33$	$+16.05$

(1)

By leaving, for the present, undecided whether we aim to calculate the values $\frac{[a\,b]}{[a\,a]}[a\,b]$, $-\frac{[a\,b]}{[a\,a]}[a\,l]$, etc., directly or with the slide rule or with logarithms or otherwise, we obtain the following arrangement, where we may assume for the present that the *numbers in small print* have been computed, in case of need, on a supplementary sheet of paper:

				Check
$[a\,a]=+9.00$	$[a\,b]=-40.74$	$-[a\,l]=+4.88$	$[a\,s]=+26.86$	0.00
	$[b\,b]=+229.87$	$-[b\,l]=-25.26$	$[b\,s]=-163.87$	0.00
	$-\frac{[a\,b]}{[a\,a]}[a\,b]=-184.42$	$+\frac{[a\,b]}{[a\,a]}[a\,l]=+22.09$	$-\frac{[a\,b]}{[a\,a]}[a\,s]=+121.58$	
		$[l\,l]=+4.33$	$-[l\,s]=+16.05$	0.00
		$-\frac{[a\,l]}{[a\,a]}[a\,l]=-2.65$	$+\frac{[a\,l]}{[a\,a]}[a\,s]=-14.56$	
	$[b\,b\cdot 1]=+45.45$	$-[b\,l\cdot 1]=-3.17$	$[b\,s\cdot 1]=+42.29$	-0.01
		$[l\,l\cdot 1]=+1.68$	$-[l\,s\cdot 1]=+1.49$	0.00
		$-\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[b\,l\cdot 1]=-0.22$	$+\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[b\,s\cdot 1]=-2.95$	
		$[l\,l\cdot 2]=+1.46$	$-[l\,s\cdot 2]=-1.46$	0.00

$$\left.\begin{array}{l} y=\dfrac{+3.17}{+45.45}=+0.06975 \\ p_y=45.45 \qquad [v\,v]=1.46 \end{array}\right\}. \qquad (2)$$

For the inverse computation, namely the elimination of y and determination of x besides p_x, we only write the numbers, and that with one place less than previously

b	a	$-l$	s	Check
+229.9	−40.7	−25.3	−163.9	0.0
	+9.0	+4.9	+26.8	0.0
	−7.2	−4.5	−29.0	
		+4.3	+16.1	0.0
		−2.8	−18.0	
	+1.8	+0.4	−2.2	0.0
		+1.5	−1.9	0.0
		−0.1	+0.5	
		+1.4	−1.4	0.0

$$\left.\begin{array}{l} x = \dfrac{-0.4}{+1.8} = -0.22 \text{ (more accurately } = -2.226) \\ p_x = 1.8 \qquad [v\,v] = 1.4 \end{array}\right\}. \tag{3}$$

If a computation with so few places as here is involved, then we make the computation best with the *slide rule*. For the first line we set up the quotient $\dfrac{40.7}{229.9}$, and multiply by it in succession:

40.7, 25.3, 169.9, i.e. we read with *one* setting the three points

$$\frac{40.7}{229.9}\,40.7 = 7.2 \qquad \frac{40.7}{229.9}\,25.3 = 4.5 \qquad \frac{40.7}{229.9}\,163.9 = 29.0\,.$$

In order to set the − or + signs of the quantities $-\dfrac{[a\,b]}{[a\,a]}\,[a\,b]$, etc., correctly, we can count, in each individual case, the different +'s and −'s which affect the result; but we arrive soon at an easily seen, mechanical rule, which we will form on the basis of the above example:

a)	+	−	+	+
b)		+	−	−
		$b \cdot 1)$ −	+	+
$-l$)			+	+
			$-l \cdot 1)$ −	−

1. The signs of a line $b \cdot 1)$ or $-l \cdot 1)$ have, in any case, the same sequence as the signs of the first line *a*) above it.

2. The signs of a line $b \cdot 1)$ or $-l \cdot 1)$ begin always with −; there follows hence:

3. If over $b \cdot 1)$ in the line *a*) there is a − sign, then the signs of *a*) pass over directly to $b \cdot 1)$; if, on the other hand, above $b \cdot 1)$ in the line *a*) there is a + sign, then the signs of *a*) all change oppositely to $b \cdot 1)$.

We will illustrate this rule further by an example with five elements:

+	+	−	+	−
	−	−	−	+
	−	+	−	+
		+	−	−
		−	+	−
			+	−
			−	+

Now in order to summarize all which the preceding elimination has furnished for the final purpose, we take from (2) and (3)

$$\left.\begin{array}{llll} y = +0.06975 & x = -0.22 & \text{more accurately} & = -0.226 \\ p_y = 45.45 & p_x = 1.8 & \text{more accurately} & = 1.78 \\ [ll\cdot 2] = [vv] = 1.46 & [ll\cdot 2] = [vv] = 1.4 & & \end{array}\right\} . \tag{4}$$

$x = -0.22$ and $p_x = 1.8$ indicated here are obtained with the slide rule, limiting ourselves to the very least number of places; if we compute a little bit more rigorously, then we obtain $x = -0.226$ and $p_x = 1.78$, with which we will compute further.

The final sum $[ll \cdot 2] = 1.46$ must agree with the direct calculation of the v's and the v^2's according to (15), section 15, p. 48, which is sufficiently the case with $[v^2] = 1.4695$; and after this important check agrees, we have the mean error of the unit of weight:

$$m = \sqrt{\frac{[vv]}{n-2}} = \sqrt{\frac{1.47}{9-2}} = \sqrt{\frac{1.47}{7}} = \pm 0.46 \text{ mm} . \tag{5}$$

Then:

$$m_y = \frac{m}{\sqrt{p_y}} = \frac{0.46}{\sqrt{45.45}} = \pm 0,06797 , \quad m_x = \frac{m}{\sqrt{p_x}} = \frac{0.46}{\sqrt{1.78}} = \pm 0.34 ; \tag{6}$$

therefore

$$y = +0.06975 \pm 0.06797 , \qquad x = -0.23 \pm 0.34 ,$$

$$\frac{y}{100} = +0.000697 \pm 0.000680 .$$

$\frac{y}{100}$ and x have this time the same meaning as $\delta y'$ and δx had previously in (13), section 15, p. 48, and therefore, we have now as there:

Approximation	$(y) = 0.086250$	
Correction	$+ 0.000697 \pm 0.000680$	(7)
Adjusted	0.086947 ± 0.000680 .	

Likewise,

Approximation	$(x) = 762.00$ mm	
Correction	$-$ 0.23 $\pm$ 0.34 mm	(8)
Adjusted	761.77 mm $\pm$ 0.34 mm .	

The adjusted function is therefore now:

$$\left.\begin{array}{l} B = 761.77 \text{ mm} - 0.086947\, h \\ \pm \quad 0.34 \text{ mm} \pm 0.000680 \\ \quad \text{with } m = \pm 0.46 \text{ mm} \end{array}\right\} . \tag{9}$$

We thus have again the same function of adjustment as previously in (14), section 15, p. 48; however, the new calculation has the advantage that it permits us to estimate the uncertainty of the formula obtained, e.g. 761.77 is for $h = 0$ the annual barometer reading valid for that region (Württemberg) reduced to sea level, and this value was obtained with a mean error of ± 0.34 mm. Similarly, we know now of the coefficient 0.086947 that it was determined approximately to an accuracy of about 1:128 or about 1% of its value.

In addition, we will also treat this example with the second form of the sum check of section 19, p. 62, and have then to compute to the coefficients of the normal equations (12), section 15, p. 47, also the coefficients of the sum equation.

From the table indicated there, there result the values of the s's, namely

$$s_1 = a_1 + b_1 = -0.20$$
$$s_2 = a_2 + b_2 = -1.25, \text{ etc.},$$

and with this, we will have

$$[a\,s] = -31.74\,, \qquad [b\,s] = +189.13\,, \qquad -[l\,s] = -20.38\,.$$

We put together only the coefficients of the two normal equations and the sum equation and add the reduction terms in small print.

$$\begin{array}{lll}
[a\,a] = +\ 9.00 & [a\,b] = -\ 40.74 & -[a\,l] = +\ 4.88 \\
[a\,b] = -40.74 & [b\,b] = +229.87 & -[b\,l] = -25.26 \\
 & -\frac{[a\,b]}{[a\,a]}[a\,b] = -\ 184.42 & +\frac{[a\,b]}{[a\,a]}[a\,l] = +\ 22.09 \\
\hline
[a\,s] = -31.74 & [b\,s] = +189.13 & -[l\,s] = -20.38 \\
 & -\frac{[a\,s]}{[a\,a]}[a\,b] = -\ 143.68 & +\frac{[a\,s]}{[a\,a]}[a\,l] = +\ 17.21 \\
 & [bb\cdot 1] = +45.45 & -[b\,l\cdot 1] = -3.17 \\
 & [bs\cdot 1] = +45.45 & -[l\,s\cdot 1] = -3.17
\end{array}$$

Therefore, there follows the same result as previously in (2), p. 63.

For the computation of $[v\,v]$ we use best the formula (11), section 16, p. 52, and have to return for this, once again, to the summary of the coefficients in section 15, p. 47, from which we obtain

$$[l\,l] = 4.33\,.$$

By using further the above numerical values, we find according to (11), section 16, p. 52

$$[v\,v] = 4.33 - \frac{4.88^2}{9.00} - \frac{3.17^2}{45.45} = 4.33 - 2.65 - 0.22$$

or

$$[v\,v] = 1.46\,,$$

which agrees again with the direct computation of p. 48.

With only two unknowns x and y the elimination with the slide rule is nearly always sufficient; however, if the coefficients have several places and, in general, if we have cause to compute more accurately, we pass over to logarithmic computation, which probably is self-evident in the case of only two unknowns, by following simply the formulae of section 16, p. 50, e.g. $[b\,b \cdot 1] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b]$, for which we determine first $\log \frac{[a\,b]}{[a\,a]}$, which we need repeatedly, and so forth.

However, once we decide to compute logarithmically, then it is proper to set up a fixed lined scheme, and here we can make the continued additions of the logarithms with *sliding slips* and thereby save much writing, as is seen from the following table.

The computation according to this scheme, below, takes the following course:

The coefficients $[a\,a]$, $[a\,b]$, etc., previously computed, are written at the places fixed by the line and column designation, e.g., the line $[b$ and the column $-l]$ determine by their coincidence the column for $-[b\,l] = -25.26$.

After this, the 4 logarithms are put in one line:

$$\log [a\,a] = 0.95424\ , \qquad \log [a\,b] = 1.61002\ , \qquad \log(-[a\,l]) = 0.68842\ , \qquad \log [a\,s] = 1.42911\ .$$

The further logarithmic calculation is carried out with the help of *slips* (strips of paper), which are indicated below. (We can, of course, write in practice that which the *two* slips a and b contain also on *one* strip of paper; in the description, however, we speak of *two* separate slips.)

Slip a is obtained as follows: We lay a strip of paper over the line $\log [a$ and write over 0.95424 the decadic supplement 9.04576. This is $\log \frac{1}{[a\,a]}$ and is needed for the computation of $\log \frac{[a\,b]}{[a\,a]}$ and of $\log\left(-\frac{[a\,l]}{[a\,a]}\right)$. For this, we slide the slip by one column to the right, so that 9.04576 comes over 1.61002;

Logarithmic Solution of the Normal Equations with Sliding Slips

	$a]$	$b]$	$-l]$	$s]$	Checks
$[a$	$+9.00$	-40.74	$+4.88$	$+26.86$	0.00
$\log [a$	0.95424	1.61002	0.68842	1.42911	
$\log\left(\frac{[a\,b]}{[a\,a]}\right)[a$		2.26580	1.34420	2.08489	
$\log\left(-\frac{[a\,l]}{[a\,a]}\right)[a$			0.42260	1.16329	
$[b$		$+229.87$	-25.26	-163.87	0.00
$-\left(\frac{[a\,b]}{[a\,a]}\right)[a$		-184.42	$+22.09$	$+121.58$	
$-[l$			$+4.33$	$+16.05$	0.00
$+\left(\frac{[a\,l]}{[a\,a]}\right)[a$			-2.65	-14.56	
		$b \cdot 1]$	$-l \cdot 1]$	$s \cdot 1]$	
$y = \frac{+3.17}{+45.45} = +0.06975$	$[b$	$+45.45$	-3.17	-42.29	-0.01
	$\log [b$	1.65753	0.50106	1.62624	
	$\log\left(-\frac{[b\,l \cdot 1]}{[b\,b \cdot 1]}\right)[b$		9.34459	0.46977	
	$-[l$		$+1.68$	$+1.49$	0.00
	$+\left(\frac{[b\,l \cdot 1]}{[b\,b \cdot 1]}\right)[b$		-0.22	-2.95	
		$-[l$	$-l \cdot 2]$ $+1.46$	$s \cdot 2]$ -1.46	0.00
Slip a	$\log \frac{1}{[a\,a]}$ 9.04576	$\log \frac{[a\,b]}{[a\,a]}$ 0.65578	$\log\left(-\frac{[a\,l]}{[a\,a]}\right)$ 9.73418		
Slip b		$\log \frac{1}{[b\,b \cdot 1]}$ 8.34247	$\log\left(-\frac{[b\,l \cdot 1]}{[b\,b \cdot 1]}\right)$ 8.84353		

$8.84353 = \log y$

$0{,}06975 = y\,.$

the sum of the two 0.65578 is written on the slip on the right. The slip is moved again by one column to the right and yields 9.04576 over 0.68842, which are together 9.73418.

With this, the slip a in itself is finished and can be used for the computation of $\frac{[a\,b]}{[a\,a]}[a\,b]$, $-\frac{[a\,b]}{[a\,a]}[a\,l]$, etc. For this purpose, the slip comes again into the normal position and yields the following by gradual sliding to the right, adding 0.65578 downward:

(slip)	0.65578	0.65578	0.65578
	1.61002	0.68842	1.42911
	2.26580	1.34420	2.08489 ;

similarly, we also obtain in addition:

(slip)	9.73418	9.73418
	0.68842	1.42911
	0.42260	1.16329 .

To the thus obtained five logarithms we add the antilogarithms and put them into the prepared places below:

−184.42	+22.09	+121.58
	− 2.65	− 14.56 .

We determine the − or + signs here according to the rule already indicated on p. 64.

If we add these amounts algebraically to the coefficients $[b\,b] = +229.87$, etc., above them, then we obtain

$$[b\,b\cdot 1] = +45.45 \qquad -[b\,l\cdot 1] = -3.17 \qquad [b\,s\cdot 1] = -42.29$$
$$[l\,l\cdot 1] = +1.68 \qquad -[l\,s\cdot 1] = +\ 1.49\,.$$

To this system reduced for the first time there belongs now the second slip b.

If we proceed with this just as previously with the slip a, then the elimination is completed, and on the slip b itself we have

$$\log\left(-\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}\right) = \log y \quad \text{(except the sign)}.$$

In order to obtain also x, we can, it is true, introduce the obtained y, in the usual manner, into one of the normal equations

$$+\ 9.00\,x - \ 40.74\,y + \ 4.88 = 0$$
$$-40.74\,x + 229.87\,y - 25.26 = 0$$

and obtain with this $x = -0.226$. But if we aim to have also the weight of x, then we reverse the whole elimination, so that y becomes the first and x the second unknown, as is already indicated on p. 64. In this manner, we obtain

$$\left.\begin{matrix} x = -0.226 & [l\,l\cdot 2] = 1.46 & y = +0.06975 \\ [a\,a\cdot 1] = +1.78 & & [b\,b\cdot 1] = +45.45 \end{matrix}\right\}. \qquad (1)$$

$[l\,l\cdot 2] = 1.46$ agrees sufficiently with $[v\,v] = 1.4695$, computed already previously [at the end of section 15, (15), p. 48; we thus have now the mean error of the unit of weight

$$m = \sqrt{\frac{1.47}{9-2}} = \pm 0.46 \text{ mm} \tag{2}$$

and with this, also the mean errors of x and y:

$$m_x = \frac{m}{\sqrt{1.78}} = \pm 0.34 \text{ mm} \qquad m_y = \frac{m}{\sqrt{45.45}} = \pm 0.06797$$

$$x = -0.23 \text{ mm} \pm 0.34 \text{ mm} \qquad y = +0.06975 \pm 0.06797 .$$

This is thus the result of pure elimination; the further utilization of the obtained y and x is then just as has already been shown in the previous section 20, p. 65.

Section 22. Unequal Weights

So far it was assumed that all observations l are equally accurate from the outset. If this is not the case, the individual l's have different weights. We will assume:

$$\left.\begin{array}{lllll} \text{Observations} & l_1 & l_2 & l_3 \ldots l_n \\ \text{with weights} & p_1 & p_2 & p_3 \ldots p_n \end{array}\right\} . \tag{1}$$

Let the error equations be the same as previously:

$$\left.\begin{array}{ll} v_1 = a_1 x + b_1 y - l_1 & \text{weights } = p_1 \\ v_2 = a_2 x + b_2 y - l_2 & = p_2 \\ v_3 = a_3 x + b_3 y - l_3 & = p_3 \\ \cdots & \cdots \\ v_n = a_n x + b_n y - l_n & = p_n \end{array}\right\} . \tag{2}$$

Then we have no longer to make $[v\,v]$ a minimum, but

$$[p\,v\,v] = \text{minimum} . \tag{3}$$

This yields the normal equations:

$$\left.\begin{array}{l} [p\,a\,a]\,x + [p\,a\,b]\,y - [p\,a\,l] = 0 \\ [p\,a\,b]\,x + [p\,b\,b]\,y - [p\,b\,l] = 0 \end{array}\right\} . \tag{4}$$

These equations vary from (4), section 14, p. 43, only by the fact that p is inserted in each expression in brackets; in comparison to before there is no further change. In the elimination, also, $[p\,b\,b \cdot 1]$, e.g. takes the place of $[b\,b \cdot 1]$, etc. The mean error of the unit of weight becomes

$$m = \sqrt{\frac{[p\,v\,v]}{n-2}} \quad \text{or} \quad = \sqrt{\frac{[p\,l\,l \cdot 2]}{n-2}} . \tag{5}$$

This error m does in general not belong to a real observation, but to a fictitious observation, which has the weight $p = 1$.

If the roots of the weights, $\sqrt{p}$, are convenient numbers, then it is often useful, instead of calculating directly $[p\,a\,a]$, $[p\,a\,b]$, etc., to proceed as follows:

We multiply all coefficients a, b and the absolute terms $-l$ of the error equations by $\sqrt{p}$ and imagine now the error equations (2) written as follows:

$$\left.\begin{array}{l} v_1\sqrt{p_1} = a_1\sqrt{p_1}\,x + b_1\sqrt{p_1}\,y - l_1\sqrt{p_1} \\ v_2\sqrt{p_2} = a_2\sqrt{p_2}\,x + b_2\sqrt{p_2}\,y - l_2\sqrt{p_2} \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\}, \tag{6}$$

then the squaring yields by itself the sum $[p\,v\,v]$, according to (3) and the normal equations (4).

We can also understand this matter in the following way: The sums may have originated directly according to (4) or according to (6); we can always attribute to the symbol $[a\,a]$ the meaning:

$$p_1 a_1^{\,2} + p_2 a_2^{\,2} + p_3 a_3^{\,2} + \ldots = [a\,a], \text{ and so on }, \tag{7}$$

and with this, all formulae hitherto established hold also for unequal weights.

The coefficients α, β in section 18 obtain here, also, changed meanings; for we have now according to (1), p. 56,

$$x = \frac{\alpha_1}{\sqrt{p_1}}\left(l_1\sqrt{p_1}\right) + \frac{\alpha_2}{\sqrt{p_2}}\left(l_2\sqrt{p_2}\right) + \ldots, \tag{8}$$

$$\frac{1}{p_x} = \left[\frac{\alpha\,\alpha}{p}\right], \quad \frac{1}{p_y} = \left[\frac{\beta\,\beta}{p}\right]. \tag{9}$$

The further computation according to section 18 leads again to the formula

$$p_y = [p\,b\,b \cdot 1], \tag{10}$$

so that, therefore, we do not have to concern ourselves further with the weights altogether as soon as the sum coefficients are available either directly according to (4) or computed according to (6).

With this, all that is necessary for the adjustment with unequal weights is already treated; but we will still add a few remarks (which can be passed over at first).

Mean errors assumed a priori

If the proportions of accuracy of the observations and the error equations have been assumed directly by mean errors (by a priori estimation or otherwise), then the weights p are taken in inverse proportion to the squares of these mean errors, and the computation is carried according to formulae (1) to (5) or (6) to (10); we can however write the formulae also in such a way that we are not speaking of weights p, but only of mean errors m, and this more illustrative form is advisable in many cases. We have thus:

$$\left.\begin{array}{lc} \text{Error Equations} & \begin{array}{c}\text{Mean Errors of the } l\text{'s} \\ \textit{Before} \text{ the Adjustment}\end{array} \\ v_1 = a_1 x + b_1 y - l_1 & \pm m_1 \\ v_2 = a_2 x + b_2 y - l_2 & \pm m_2 \\ \cdots\cdots & \cdots \\ v_n = a_n x + b_n y - l_n & \pm m_n \end{array}\right\}. \tag{11}$$

Principle of adjustment:

$$\left[\frac{v\,v}{m\,m}\right] = \left(\frac{v_1}{m_1}\right)^2 + \left(\frac{v_2}{m_2}\right)^2 + \ldots \left(\frac{v_n}{m_n}\right)^2 = \text{minimum} . \tag{12}$$

Normal equations:

$$\left.\begin{aligned} \left[\frac{a\,a}{m\,m}\right] x + \left[\frac{a\,b}{m\,m}\right] y - \left[\frac{a\,l}{m\,m}\right] = 0 \\ \left[\frac{a\,b}{m\,m}\right] x + \left[\frac{b\,b}{m\,m}\right] y - \left[\frac{b\,l}{m\,m}\right] = 0 \end{aligned}\right\} . \tag{13}$$

Mean error of the unit of weight *after* the adjustment:

$$m = \sqrt{\frac{1}{n-2}\left[\frac{v\,v}{m\,m}\right]} \quad \text{or} \quad = \sqrt{\frac{1}{n-2}\left[\frac{l\,l}{m\,m}\cdot 2\right]} . \tag{14}$$

The mean errors of the values of l *after* the adjustment are, respectively,

$$m'_1 = \frac{m}{1} m_1 , \quad m'_2 = \frac{m}{1} m_2 , \quad \ldots , \quad m'_n = \frac{m}{1} m_n . \tag{15}$$

If the m_1's, m_2's, . . . , m_n's were already apportioned correctly *before* the adjustment, then we will have

$$m = 1 \quad \text{and} \quad m'_1 = m_1 , \quad m'_2 = m_2 \ldots .$$

In order to reach a further consideration with regard to weights, we start from the simple case in which an error equation has the weight 2 and all others have the weight 1, hence, for instance,

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y - l_1 & p &= 1 \\ v_2 &= a_2 x + b_2 y - l_2 & p &= 1 \\ v_3 &= a_3 x + b_3 y - l_3 & p &= 2 \end{aligned}\right\} ; \tag{16}$$

then the first normal equation is

$$(a_1{}^2 + a_2{}^2 + 2\,a_3{}^2)x + (a_1 b_1 + a_2 b_2 + 2\,a_3 b_3)y - (a_1 l_1 + a_2 l_2 + 2\,a_3 l_3) = 0 .$$

We would also obtain the same if we introduced the third error equation twice, and then computed further with the following system:

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y - l_1 & p &= 1 \\ v_2 &= a_2 x + b_2 y - l_2 & p &= 1 \\ \Big\{ v_3 &= a_3 x + b_3 y - l_3 & p &= 1 \\ v_4 &= a_3 x + b_3 y - l_3 & p &= 1 \end{aligned}\right\} . \tag{17}$$

From the two systems (16) and (17) we obtain the same unknowns x, y with the same weights and the same sum $[l\,l \cdot 2] = [v\,v]$.

However, if we compute the mean errors, then, in (17), we must *not* carry in the computation four equations, but only three, i.e. we have

$$m^2 = \frac{[v\,v]}{3-2} \quad \textit{not} \quad = \frac{[v\,v]}{4-2} . \tag{17a}$$

If, conversely, equations (17) correspond to the observations, then, it is true, we may use, for the adjustment itself, a system of the form (16) instead of (17), but in the error computation the original number of the error equations, i.e. the observations, is decisive:

If there exist two error equations with the same coefficients $a, b, \ldots$, but with unequal absolute terms $-l$, and with unequal weights:

$$\left.\begin{array}{ll} v_1 = a x + b y + c z + \ldots - l_1 & \text{weight} = p_1 \\ v_2 = a x + b y + c z + \ldots - l_2 & \text{weight} = p_2 \end{array}\right\}, \tag{18}$$

then these yield, according to the usual method, the following contributions to the coefficients of the normal equations:

$$p_1 a^2 + p_2 a^2 = (p_1 + p_2) a^2,\ (p_1 + p_2) a b,\ - p_1 a l_1 - p_2 a l_2 = - a\,(p_1 l_1 + p_2 l_2)\,. \tag{19}$$

Contribution to the error squared term:

$$p_1 l_1^{\,2} + p_2 l_2^{\,2}\,. \tag{20}$$

Instead of the *two* equations (18) we will now write the following *one*:

$$v' = a x + b y + c z + \ldots - \frac{p_1 l_1 + p_2 l_2}{p_1 + p_2}\,, \qquad \text{weight} = p_1 + p_2\,. \tag{18'}$$

This *one* equation gives the following contributions to the normal equations:

$$(p_1 + p_2) a^2,\ (p_1 + p_2) a b \ldots - a (p_1 l_1 + p_2 l_2) \tag{19'}$$

$$\ldots\ldots\ldots\ldots\ldots \frac{(p_1 l_1 + p_2 l_2)^2}{p_1 + p_2}\,. \tag{20'}$$

The coefficients (19) and (19') are identical; however, the contributions to the sum of the squares $[p\,l\,l]$ are not identical in (20) and in (20'), and only if $l_1 = l_2$, (20') changes to (20).

This result means in words: We can replace two error equations of the form (18), i.e. with the same coefficients but unequal absolute terms, by *one* equation (18') so far as only the unknowns $x, y, z \ldots$ themselves and their weights are involved; however, for the computation of mean errors the equation (18') does not give the right substitute of the two original equations, but only a perhaps approximately admissible determination of accuracy.

Section 23. Nonlinear Functions

Until now we have assumed that the relations between the observations and the unknowns can be represented by linear equations. In many cases, however, we will first obtain nonlinear equations, which can, however, be reduced to linear error equations in the following manner.

Deviating from previous practice, we will denote the observations by $L_1, L_2, \ldots, L_n$ and the unknowns by X and Y, while we reserve the denotations l and x, y for the final linear error equations.

From the nature of the problem, let first be derived the following error equations:

$$\left.\begin{array}{l} L_1 + v_1 = F_1(X, Y) \\ L_2 + v_2 = F_2(X, Y) \\ \cdots\cdots\cdots\cdots \\ L_n + v_n = F_n(X, Y) \end{array}\right\}, \tag{1}$$

in which arbitrary functions shall be indicated by the symbols F.

If we understand by (X) and (Y) approximate values of X and Y, and by x and y their corrections, hence

$$X = (X) + x, \qquad Y = (Y) + y; \tag{2}$$

then, with the help of Taylor's theorem, we can develop each of the functions F, limiting ourselves to the first powers of x and y, as follows:

$$F(X, Y) = F((X) + x, (Y) + y),$$

$$F(X, Y) = F((X), (Y)) + \frac{\partial F}{\partial X} x + \frac{\partial F}{\partial Y} y,$$

and with this, we obtain from equations (1)

$$\left.\begin{array}{l} v_1 = a_1 x + b_1 y - l_1 \\ v_2 = a_2 x + b_2 y - l_2 \\ v_3 = a_3 x + b_3 y - l_3 \\ \cdots\cdots\cdots\cdots \\ v_n = a_n x + b_n y - l_n \end{array}\right\}, \tag{3}$$

where the coefficients a, b and the absolute terms $-l$ have the following meaning:

$$a = \frac{\partial F}{\partial X}, \qquad b = \frac{\partial F}{\partial Y}, \tag{4}$$

$$-l = F((X), (Y)) - L \qquad \text{or} \qquad -l = (L) - L. \tag{5}$$

Quite generally, (L) is that value which an observation L would assume if the approximations (X), (Y) were valid.

Equations (3) take the place of the original error equations (1); these equations (3) themselves are error equations in regard to the new unknowns x, y and in regard to the values l, which take the place of the observations L.

We have made use of approximate values in the example, section 15, p. 46, where they only served, however, to simplify numerical computation and, besides, remained without influence on the results. In the case of the transformation of the nonlinear error equations, however, the approximations (X) and (Y) must in any case be so accurate that the second and higher powers of their corrections x and y can be neglected.

We obtain a check whether the approximate values are sufficient by introducing, after the completed adjustment, the approximate values (X) and (Y) as well as their corrections x and y into the equation

$$F\left((X)+x,\ (Y)+y\right)=F\left((X),(Y)\right)+\frac{\partial F}{\partial X}x+\frac{\partial F}{\partial Y}y\,. \tag{6}$$

But this is also synonymous with computing, from the error equations (3), the corrections v and forming, with the latter and the final values of the unknowns, the equations (1). If a discrepancy shows up here, then this is caused by the neglected higher terms. Then, nothing is left but to repeat the adjustment by introducing now the values $(X)+x$ and $(Y)+y$ as approximate values.

Section 24. Adjustment of a Multiple Section of Arc

As a simple example for the treatment of nonlinear error equations we consider the determination of a point by the section of arc of several measured lengths.

In Fig. 1 let the coordinates of the points P_1, P_2, P_3 and P_4 be given:

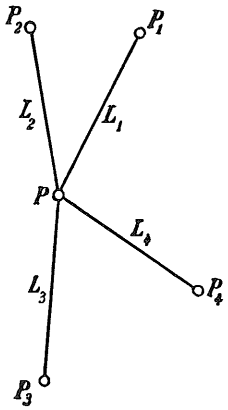


Fig. 1.

$$\left.\begin{aligned} X_1 &= +551.78\text{ m} & Y_1 &= +899.06\text{ m}\\ X_2 &= +548.11 & Y_2 &= +824.95\\ X_3 &= +307.63 & Y_3 &= +850.40\\ X_4 &= +377.57 & Y_4 &= +955.51 \end{aligned}\right\}. \tag{1}$$

The four distances to point P to be newly determined are measured:

$$\left.\begin{aligned} L_1 &= 123.81\text{ m}\\ L_2 &= 114.59\\ L_3 &= 129.25\\ L_4 &= 118.78 \end{aligned}\right\}, \tag{2}$$

from which the coordinates X and Y of the point are to be computed.

Since the measured distances must receive, by the adjustment, small corrections, then there follow at once the relations

$$\left.\begin{aligned} L_1+v_1 &= \sqrt{(X_1-X)^2+(Y_1-Y)^2}\\ L_2+v_2 &= \sqrt{(X_2-X)^2+(Y_2-Y)^2}\\ L_3+v_3 &= \sqrt{(X_3-X)^2+(Y_3-Y)^2}\\ L_4+v_4 &= \sqrt{(X_4-X)^2+(Y_4-Y)^2} \end{aligned}\right\} \tag{3}$$

or the error equations

$$\left.\begin{aligned} v_1 &= \sqrt{(X_1-X)^2+(Y_1-Y)^2}-L_1\\ v_2 &= \sqrt{(X_2-X)^2+(Y_2-Y)^2}-L_2\\ v_3 &= \sqrt{(X_3-X)^2+(Y_3-Y)^2}-L_3\\ v_4 &= \sqrt{(X_4-X)^2+(Y_4-Y)^2}-L_4 \end{aligned}\right\} \tag{4}$$

which correspond to the general equations (1), section 23, p. 73.

For the application of the method indicated in the previous section, we determine, from the section of arc of two arbitrary lengths, e.g. L_1 and L_4, approximate coordinates (X) and (Y) of point P and set the final coordinates

$$\left.\begin{aligned} X &= (X) + x \\ Y &= (Y) + y \end{aligned}\right\} . \qquad (5)$$

The first of the error equations (4) changes into

$$v_1 = \sqrt{(X_1 - (X) - x)^2 + (Y_1 - (Y) - y)^2} - L_1 ,$$

and if we develop the radical, according to Taylor's theorem, as a series in which we only take into account the first powers of x and y, then we obtain

$$v_1 = -\frac{X_1 - (X)}{(L_1)} x - \frac{Y_1 - (Y)}{(L_1)} y - L_1 + (L_1) ,$$

where

$$\sqrt{(X_1 - (X))^2 + (Y_1 - (Y))^2} = (L_1) \qquad (6)$$

is set. If we further introduce the designations

$$-L_1 + (L_1) = -l_1 , \qquad -\frac{X_1 - (X)}{(L_1)} = a_1 , \qquad -\frac{Y_1 - (Y)}{(L_1)} = b_1 \qquad (7)$$

and treat the remaining error equations (4) in the same manner, then we obtain the linear error equations

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y - l_1 \\ v_2 &= a_2 x + b_2 y - l_2 \\ v_3 &= a_3 x + b_3 y - l_3 \\ v_4 &= a_4 x + b_4 y - l_4 \end{aligned}\right\} . \qquad (8)$$

For numerical computation we assume the approximate coordinates

$$(X) = +436.95 \text{ m} , \qquad (Y) = +852.66 \text{ m} \qquad (9)$$

with which we compute according to (6)

$$(L_1) = 123.850 \text{ m} , \ (L_2) = 114.562 \text{ m} , \ (L_3) = 129.340 \text{ m} , \ (L_4) = 118.761 \text{ m} .$$

With this, we find according to (7)

$$\begin{aligned} a_1 &= -0.928 & b_1 &= -0.375 & -l_1 &= +0.040 \\ a_2 &= -0.970 & b_2 &= +0.242 & -l_2 &= -0.028 \\ a_3 &= +1.000 & b_3 &= +0.017 & -l_3 &= +0.090 \\ a_4 &= +0.500 & b_4 &= -0.866 & -l_4 &= -0.019 \end{aligned}$$

and, finally, the error equations

$$\left.\begin{aligned} v_1 &= -0.928\,x - 0.375\,y + 0.040 \\ v_2 &= -0.970\,x + 0.242\,y - 0.028 \\ v_3 &= +1.000\,x + 0.017\,y + 0.090 \\ v_4 &= +0.500\,x - 0.866\,y - 0.019 \end{aligned}\right\} . \qquad (10)$$

In this example, we will forego details which do not directly belong to the special treatment of nonlinear error equations, and pass over at once to the normal equations

$$\left.\begin{aligned} 3.0521\,x - 0.3028\,y + 0.0705 &= 0 \\ -0.3028\,x + 0.9494\,y - 0.0039 &= 0 \end{aligned}\right\}, \qquad (11)$$

from which there follows

$$x = -0.0234 \text{ m}\,, \qquad y = -0.0034 \text{ m}\,,$$

and according to (5) and (9)

$$X = +436.927 \text{ m}\,, \qquad Y = +852.657 \text{ m}\,. \qquad (12)$$

In connection with this, we will in addition examine the transformation of the original nonlinear error equations (4) into the linear form (10) according to the method indicated at the end of section 23, p. 74. For this, we compute the corrections v from equations (10) and introduce them as well as the final coordinates X and Y into equations (3) or (4). Upon correct computation, these equations must be satisfied.

We find for the corrections the values

$$\left.\begin{aligned} v_1 &= +0.062 \text{ m} \qquad & v_3 &= +0.067 \text{ m} \\ v_2 &= -0.007 & v_4 &= -0.028 \end{aligned}\right\}, \qquad (13)$$

hence, the corrected distances are

$$\left.\begin{aligned} L_1 + v_1 &= 123.872 \text{ m} \qquad & L_3 + v_3 &= 129.317 \text{ m} \\ L_2 + v_2 &= 114.583 & L_4 + v_4 &= 118.752 \end{aligned}\right\}. \qquad (14)$$

On the other hand, we compute from the final coordinates of P and from the coordinates of the given points:

$$\left.\begin{aligned} PP_1 &= 123.873 \text{ m} \qquad & PP_3 &= 129.317 \text{ m} \\ PP_2 &= 114.583 & PP_4 &= 118.751 \end{aligned}\right\}. \qquad (15)$$

The deviations between the values (14) and (15) do not exceed 1 mm, and hence, can well be explained by the computational inaccuracy produced in rounding off.

Section 25. Adjustment of Barometer Readings

As a major numerical example of adjustment with nonlinear functions, which we will now treat with all details, checks and so on, we take up once again the barometer measurements of section 14, p. 43, namely

	h	B
1. Bruchsal	120.2 m	751.18 mm
2. Cannstatt	225.1	742.37
3. Stuttgart	270.6	738.50
4. Calw	347.6	731.27
5. Friedrichshafen .	406.7	726.99
6. Heidenheim . . .	492.4	718.16
7. Isny	708.1	700.48
8. Freudenstadt . .	733.5	697.64
9. Schopfloch . . .	768.9	695.23

(1)

If we denote by X the barometer reading at sea level, hence at the elevation $h = 0$, and by Y a constant, then the following equation holds to a certain approximation according to the theory of barometric elevation measurement

$$h = Y \log \frac{X}{B}. \tag{2}$$

We set ourselves the problem to determine the values of X and Y from the above numerical data, where the trigonometrically determined elevations above sea level h shall be treated as free from error, and the barometer readings B as equally accurate observations.

In order to obtain again error equations, we must represent the observations B as functions of the unknowns X and Y. We have

$$\frac{h}{Y} = \log \frac{X}{B}, \quad \text{hence} \quad \frac{X}{B} = 10^{\frac{h}{Y}}$$

or

$$B = X\,10^{-\frac{h}{Y}}$$

The two unknowns will result without discrepancy from the new observations only if the latter receive the corrections $v_1, v_2, \ldots, v_9$. Then there result the equations

$$\begin{aligned} B_1 + v_1 &= X\,10^{-\frac{h_1}{Y}} \\ B_2 + v_2 &= X\,10^{-\frac{h_2}{Y}} \\ &\cdots\cdots \\ B_9 + v_9 &= X\,10^{-\frac{h_9}{Y}}. \end{aligned} \tag{3}$$

First we deal now with the determination of approximate values (X) and (Y). For this, we write (2) in the form

$$\log X - \log B = \frac{h}{Y}$$

and apply this equation to the first and to the last observation; this yields

$$\log (X) - \log 751.18 = \frac{120.2}{(Y)}$$

$$\log (X) - \log 695.23 = \frac{768.9}{(Y)}.$$

We can solve these two equations for (X) and (Y), and we find

$$(X) = 762.03\,, \qquad (Y) = 19298\,. \tag{4}$$

With these approximate values we have to bring equations (3) into the linear form according to the directions of section 23. The function denoted there on p. 73 by $F(X, Y)$ is in our case

$$F(X, Y) = X\,10^{-\frac{h}{Y}}$$

and with this, we compute according to the instruction of (4) and (5), p. 73:

$$\left.\begin{aligned} a &= \frac{\partial\left(X\,10^{-\frac{h}{Y}}\right)}{\partial X} = 10^{-\frac{h}{Y}} \\ b &= \frac{\partial\left(X\,10^{-\frac{h}{Y}}\right)}{\partial Y} = X\,10^{-\frac{h}{Y}}\frac{h}{Y^2}\frac{1}{M} \\ -l &= (X)\,10^{-\frac{h}{(Y)}} - B \quad \text{or} \quad = (B) - B \end{aligned}\right\} \tag{5}$$

In the calculation of a and b, (X) and (Y) is set everywhere in place of X and Y. The calculation of formulae (5) is carried out best in the logarithmic form, i.e.

$$\left.\begin{aligned} \log a &= -\frac{h}{(Y)} \quad \text{or} \quad \log\frac{1}{a} = \frac{h}{(Y)} \\ \log b &= -\frac{h}{(Y)} + \log\frac{(X)\,h}{M\,(Y)^2} = \log a + \log\frac{(X)\,h}{(Y)^2\,M} \\ \log(-l+B) &= \log(X) - \frac{h}{(Y)} = \log(X) + \log a \end{aligned}\right\}. \tag{6}$$

If we introduce here the numerical values according to (1) and (4), then we obtain

No.	a	b	$-l$
1	+ 0.986	+ 0.00056	0.00
2	+ 0.973	+ 0.00103	— 0.53
3	+ 0.968	+ 0.00123	— 0.68
4	+ 0.959	+ 0.00157	— 0.20
5	+ 0.953	+ 0.00182	— 1.06
6	+ 0.943	+ 0.00219	+ 0.38
7	+ 0.919	+ 0.00307	— 0.20
8	+ 0.916	+ 0.00317	+ 0.53
9	+ 0.912	+ 0.00331	0.00

The circumstance that the first and the last values $-l$ become zero is not accidental; this comes from the fact that the first and the last observations B were used for the determination of the approximate values. If the approximate values of none of the observations at all are rigorously sufficient, then we will not have for any value $-l = 0$.

Hence, the error equations will now be

$$\left.\begin{aligned} v_1 &= +0.986\,x + 0.00056\,y' + 0.00 \\ v_2 &= +0.973\,x + 0.00103\,y' - 0.53 \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\}. \tag{7}$$

The second unknown is called here y', because we want to change it once more. For the coefficients are still too unequal, which is inconvenient in the numerical computation. Therefore, instead of the error equations (7) we will rather write the following:

$$\left.\begin{aligned} v_1 &= +0.986\,x + 0.056\left(\frac{y'}{100}\right) + 0.00 \\ v_2 &= +0.973\,x + 0.103\left(\frac{y'}{100}\right) - 0.53 \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\}, \tag{8}$$

i.e., we introduce, instead of y', the new unknown:

$$\frac{y'}{100} = y \quad \text{(therefore, } y' = 100\,y\text{)}\,, \tag{9}$$

where y' is the correction of the approximate value (Y) and y is the unknown to be determined from the normal equations. With this, we obtain the following table of the coefficients, besides the sums s, where

$$a + b - l + s = 0\,. \tag{10}$$

No.	a	b	$-l$	s
1	+ 0.986	+ 0.056	0.000	− 1.042
2	+ 0.973	+ 0.103	− 0.530	− 0.546
3	+ 0.968	+ 0.123	− 0.680	− 0.411
4	+ 0.959	+ 0.157	− 0.200	− 0.916
5	+ 0.953	+ 0.182	− 1.060	− 0.075
6	+ 0.943	+ 0.219	+ 0.380	− 1.542
7	+ 0.919	+ 0.307	− 0.200	− 1.026
8	+ 0.916	+ 0.317	+ 0.530	− 1.763
9	+ 0.912	+ 0.331	0.000	− 1.243
Sums	+ 8.529	+ 1.795	− 1.760	− 8.564 .

We will make the calculation of the sum coefficients $[a\,a]$, $[b\,b]$, and so on, with the table of squares according to the method of section 19, p. 59.

With only two unknowns we also have a simplification here insofar as $a + s$, $b + s$, $-l + s$ do not have to be computed specially, for because of (10) we have

$$(a + s) = -(b - l)\,, \quad (b + s) = -(a - l)\,, \quad (-l + s) = -(a + b)$$

a	b	$-l$	s	$a+b$ $=(l-s)$	$a-l$ $=-(b+s)$	$b-l$ $=-(a+s)$	Checks	
+ 0.986	+ 0.056	0.000	− 1.042	+ 1.042	+ 0.986	+ 0.056	+ 8.529	+ 8.529
+ 0.973	+ 0.103	− 0.530	− 0.546	+ 1.076	+ 0.443	− 0.427	+ 1.795	+ 1.795
+ 0.968	+ 0.123	− 0.680	− 0.411	+ 1.091	+ 0.288	− 0.557	− 1.760	+ 10.324
+ 0.959	+ 0.157	− 0.200	− 0.916	+ 1.116	+ 0.759	− 0.043	− 8.564	+ 8.529
+ 0.953	+ 0.182	− 1.060	− 0.075	+ 1.135	− 0.107	− 0.878	0.000	− 1.760
+ 0.943	+ 0.219	+ 0.380	− 1.542	+ 1.162	+ 1.323	+ 0.599		+ 6.769
+ 0.919	+ 0.307	− 0.200	− 1.026	+ 1.226	+ 0.719	+ 0.107		+ 1.795
+ 0.916	+ 0.317	+ 0.530	− 1.763	+ 1.233	+ 1.446	+ 0.847		− 1.760
+ 0.912	+ 0.331	0.000	− 1.243	+ 1.243	+ 0.912	+ 0.331		
+ 8.529	+ 1.795	− 1.760	− 8.564	+ 10.324	+ 6.769	+ 0.035		+ 0.035

a^2	b^2	l^2	s^2	$(a+b)^2$ $=(l-s)^2$	$(a-l)^2$ $=(b+s)^2$	$(b-l)^2$ $=(a+s)^2$
0.9722	0.0031	0.0000	1.0858	1.0858	0.9722	0.0031
0.9467	0.0106	0.2809	0.2981	1.1578	0.1962	0.1823
0.9370	0.0151	0.4624	0.1689	1.1903	0.0829	0.3102
0.9197	0.0246	0.0400	0.8391	1.2455	0.5761	0.0018
0.9082	0.0331	1.1236	0.0056	1.2882	0.0114	0.7709
0.8892	0.0475	0.1444	2.3778	1.3502	1.7503	0.3588
0.8446	0.0942	0.0400	1.0527	1.5031	0.5170	0.0114
0.8391	0.1005	0.2809	3.1082	1.5203	2.0909	0.7174
0.8317	0.1096	0.0000	1.5450	1.5450	0.8317	0.1096
8.0884	0.4383	2.3722	10.4812	11.8862	7.0287	2.4655

[a a] = 8.0884		[a a] = 8.0884		[a a] = 8.0884	
[b b] = 0.4383		[l l] = 2.3722		[s s] = 10.4812	
— 8.5267		— 10.4606		— 18.5696	
+ 11.8862		+ 7.0287		+ 2.4655	
+ 3.3595		— 3.4319		— 16.1041	
[a b] = + 1.6798		— [a l] = — 1.7160		[a s] = — 8.0520	
		[b b] = 0.4383		[b b] = 0.4383	
		[l l] = 2.3722		[s s] = 10.4812	
		— 2.8105		— 10.9195	
		+ 2.4655		+ 7.0287	
		— 0.3450		— 3.8908	
		— [b l] = — 0.1725		[b s] = — 1.9454	
				[l l] = 2.3722	
				[s s] = 10.4812	
				— 12.8534	
				+ 11.8862	
				— 0.9672	
				— [l s] = — 0.4836	

These computations are treated in greater detail and also more accurately than it would be necessary for the immediate purpose of our example; by means of this thorough treatment, however, the procedure shall in general be made clear in all respects.

We summarize these coefficients in the usual manner:

	a	b	$-l$	s	Check
a	+ 8,0884	+ 1,6798	— 1,7160	— 8,0520	+ 2
b		+ 0,4383	— 0,1725	— 1,9454	+ 2
$-l$			+ 2,3722	— 0,4836	+ 1
s				+ 10,4812	+ 2 .

Written in detail, the third check, e.g., reads:

$$-1.7160 - 0.1725 + 2.3722 - 0.4836 = +0.0001 \text{ .}$$

The residues +2, +2, +1, +2 remaining in the case of the checks originate only from rounding off, and we let them remain as they are.

If we solve now the normal equations logarithmically according to the pattern of section 21, p. 67, then we obtain:

	a	b	$-l$	s	Check
Logarithmic solution of the normal equations according to the pattern of p. 67.	+ 8.088 0.90784	+ 1.680 0.22531 9.54278	— 1.716 0.23452 9.55199 9.56120	— 8.052 0.90590 0.22337 0.23258	0.000
		+ 0.438 — 0.349	— 0.172 + 0.356	— 1.946 + 1.673	0.000
			+ 2.372 — 0.364	— 0.484 — 1.708	0.000
$y = \frac{-0.184}{+0.089} = -2.07$		+ 0.089 8.94939	+ 0.184 9.26482 9.58025	— 0.273 9.43616 9.75159	0.000
			+ 2.008 — 0.380	— 2.192 + 0.564	0.000
			+ 1.628	— 1.628	0.000

Now we reverse the coefficients as follows:

b	a	$-l$	s
+ 0.438	+ 1.680	— 0.172	— 1.946
	+ 8.088	— 1.716	— 8.052
		+ 2.372	— 0.484

Further computation yields then:

$$x = \frac{+1{,}056}{+1{,}644} = +0{,}642$$

+ 1,644	— 1,056	— 0,588
	+ 2,304	— 1,248
	— 1,626	+ 1,626

Therefore, we have now altogether:

$$x = +0{,}642\,, \qquad [v\,v] = [l\,l\cdot 2]\,, \qquad y = -2{,}07\,,$$
$$p_x = 1{,}644\,, \qquad = 1.627 \qquad p_y = 0{,}089\,,$$

$$m = \sqrt{\frac{1{,}627}{9-2}} = \pm 0{,}48\,,$$

$$m_x = \frac{m}{\sqrt{p_x}} = \pm 0{,}38 \qquad m_y = \frac{m}{\sqrt{p_y}} = \pm 1{,}62$$

hence: $x = +0{,}64 \pm 0{,}38$ $\qquad y = -2{,}07 \pm 1{,}62$

$y' = -\ 207 \pm 162$

Approximations $(X) = 762{,}03$ $\qquad (Y) = 19298$

Results $X = 762.67 \pm 0{,}38$ $\qquad Y = 19091 \pm 162\,.$ (11)

Hence, the formula sought for is:

$$h = 19091 \log \frac{762.67}{B} \quad \text{or} \quad \log B = \log 762.67 - \frac{h}{19091}\,. \tag{12}$$

In order to examine the error distribution and to obtain, at the same time, a thorough computational check, we compute according to formula (12), from the given elevations above sea level h, the barometer readings B, and compare them with the observations:

No.	Elevation Above Sea Level, h	Barometer Reading B		v	v^2
		Adjusted	Observed		
1	120,2 m	751,69 mm	751,18 mm	+ 0,51 mm	0,2601 mm
2	225,1	742,24	742,37	— 0,13	0,0169
3	270,6	738,18	738,50	— 0,32	0,1024
4	347,6	731,36	731,27	+ 0,09	0,0081
5	406,7	726,26	726,99	— 0,83	0,6889
6	492,4	718,69	718,16	+ 0,53	0,2809
7	708,1	700,24	700,48	— 0,24	0,0576
8	733,4	698,10	697,64	+ 0,46	0,2116
9	768,9	695,12	695,23	— 0,11	0,0121
					1,6386 = $[v\,v]$.

The sum obtained therefrom, $[v\,v] = 1.6386$, agrees sufficiently with the value obtained from the elimination, $[l\,l\cdot 2] = 1.628$ or 1.626.

We can now compare this adjustment made according to the logarithmic law with the previous adjustment made according to the linear law. According to (15), section 15, p. 48, the earlier adjustment yielded for the sum of the squares of the remaining corrections $[v\,v] = 1.4695$, hence, somewhat *smaller* than the new adjustment with 1.6386.

In such cases we consider the remaining sum $[v\,v]$ as the criterion of the quality of the function applied, and in this case, the linear function thus appears *better* than the theoretical logarithmic function, which is not usually the case, and can only be explained by accident.

As a material result we set forth: The mean barometer reading in Württemberg reduced to sea level is

$$B_0 = 762.7 \text{ mm} \pm 0.4 \text{ mm} .$$

A more extensive adjustment by interpolation of this kind was published by us in *Zeitschrift der österreichischen Gesellschaft für Meteorologie*, 1880, pp. 162 to 167.

Twenty-six stations of Baden and Württemberg between the elevations 111 m and 1009 m yielded the formula

$$h = 18517 \log \frac{762.56}{B} (1 + 0.003665\, t) ,$$

where t is the mean annual temperature. We have here

$$B_0 = 762.56 \text{ mm} \pm 0.10 \text{ mm} .$$

Section 26. Weight of a Function of the Adjusted x and y

Already at the end of section 18, p. 57, when we were speaking of the weights and mean errors of the adjusted x and y, with the weight coefficients $[\alpha\alpha]$ and $[\beta\beta]$, we have made the remark that, in addition, there also is a similar coefficient $[\alpha\beta]$, which is needed when a *function* of the adjusted x and y is involved.

While we leave it up to the reader's judgment whether to become acquainted at once with this non-urgent matter, we must here, for the sake of completeness, also take care of the weight of a function of two elements before we pass over to an arbitrary number of unknowns.

At any rate, already before the beginning of the development of the formula we must caution against an error, which can easily occur, namely, to continue to treat the mean errors m_x and m_y, which were computed according to section 18, as *independent*, which is not admissible.

Often it is of a lesser interest to know the accuracy of the unknowns x and y themselves than the accuracy of a function of the same; we take the linear function

$$F = f_1 x + f_2 y . \tag{1}$$

It could appear at first sight as if we could offhand compute the mean error of F from the mean errors of x and y according to formula (10), section 5, p. 14

$$M^2 = (f_1 m_x)^2 + (f_2 m_y)^2 . \quad (?) \tag{2}$$

But this is not correct, because x and y are *not* by any means independent observations with the likewise independent mean errors m_x and m_y; x and y rather depend on the *same* measurements $l_1, l_2, \ldots, l_n$. If, e.g., the errors of all l's would accidentally be positive, then, according to (1) and (2), section 18, p. 56, the errors of x *and* of y both would also be positive.

We arrive at a theoretically very simple solution of the problem, if we separate one of the two unknowns x or y from the error equations with the help of equation (1), p. 56, and place in their stead the quantity F as a new unknown. If, e.g., we take from (1)

$$y = \frac{1}{f_2} F - \frac{f_1}{f_2} x \tag{3}$$

and introduce this in the error equations (1), section 14, p. 42, then the latter assumes the form

$$l + v = \left(a - b\frac{f_1}{f_2}\right) x + \frac{b}{f_2} F , \tag{4}$$

or with simplified notation of the coefficients

$$l + v = a' x + b' F , \tag{5}$$

if we set

$$a - b \frac{f_1}{f_2} = a' \qquad \text{and} \qquad \frac{b}{f_2} = b' . \tag{6}$$

With the error equation (5) we set up the normal equations

$$\left.\begin{array}{l}[a' a'] x + [a' b'] F - [a' l] = 0 \\ [a' b'] x + [b' b'] F - [b' l] = 0\end{array}\right\}, \tag{7}$$

and obtain then according to section 18, p. 56

$$F = \frac{[b' l \cdot 1]}{[b' b' \cdot 1]} \tag{8}$$

and

$$M^2 = \frac{m^2}{[b' b' \cdot 1]} , \qquad P = [b' b' \cdot 1] . \tag{9}$$

This kind of calculation will in practice rarely be considered, since, as a rule, we need the unknowns x and y anyhow. Therefore, we will now show how we can connect the weight computation for F to the computation of the weights for the unknowns x and y.

If we assume first definite real errors $\varepsilon_1, \varepsilon_2 \ldots$ of the l's, then there follow therefrom the errors ε_x and ε_y determined according to (1) and (2), section 18, p. 56

$$\varepsilon_x = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_3 \varepsilon_3 + \ldots , \tag{10}$$

$$\varepsilon_y = \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 + \beta_3 \varepsilon_3 + \ldots , \tag{11}$$

and further there follows hence the error of F determined according to (1)

$$\varepsilon_F = f_1 \varepsilon_x + f_2 \varepsilon_y . \tag{12}$$

Squared, this yields

$$\varepsilon_F^2 = f_1^2 \varepsilon_x^2 + 2 f_1 f_2 \varepsilon_x \varepsilon_y + f_2^2 \varepsilon_y^2 . \tag{13}$$

For the individual terms of this expression we introduce average values, which they would assume on an average from an infinite number of observations. On the left-hand side there results as the average value, M^2, the square of the mean error of F; the first and the last term on the right-hand side are to be replaced by m_x^2 and m_y^2.

The average value of $\varepsilon_x \varepsilon_y$ is not to be assumed equal to zero, which would be the most obvious thing, but we have according to (10) and (11)

$$\left.\begin{array}{l}\varepsilon_x \varepsilon_y = \alpha_1 \beta_1 \varepsilon_1^2 + \alpha_2 \beta_2 \varepsilon_2^2 + \ldots \\ \qquad + \alpha_1 \beta_2 \varepsilon_1 \varepsilon_2 + \alpha_1 \beta_3 \varepsilon_1 \varepsilon_3 + \ldots\end{array}\right\}, \tag{14}$$

and herein the average value of the expressions $\varepsilon_1^2, \varepsilon_2^2 \ldots$ is equal to m^2, and that of the terms $\varepsilon_1 \varepsilon_2, \varepsilon_1 \varepsilon_3 \ldots$ equal to zero.

Therefore, we are to set for (13), p. 83:

$$M^2 = (f_1 m_x)^2 + (f_2 m_y)^2 + 2 f_1 f_2 [\alpha\beta] m^2 . \tag{15}$$

Since

$$m_x^{\ 2} = [\alpha\alpha] m^2 \qquad \text{and} \qquad m_y^{\ 2} = [\beta\beta] m^2 \tag{16}$$

then we can also set

$$M^2 = m^2 \{ f_1^{\ 2} [\alpha\alpha] + f_2^{\ 2} [\beta\beta] + 2 f_1 f_2 [\alpha\beta] \} . \tag{17}$$

Hence, if we had applied the incorrect formula (2), then the term $2 f_1 f_2 [\alpha\beta] m^2$ in (17), which expresses the combined effect of the two errors m_x and m_y, would have been lost.

Instead of (17) we can also write [because of (18), section 18, p. 58]:

$$M^2 = m^2 \left\{ \frac{f_1^2}{[aa \cdot 1]} + 2 \frac{f_1 f_2}{[ab \cdot 1]} + \frac{f_2^2}{[bb \cdot 1]} \right\} . \tag{18}$$

In addition, we can bring equation (17) into another form with the use of the relations which are summarized at the end of section 18, p. 58, namely

$$[\alpha\alpha] = \frac{[b\,b]}{D}, \qquad [\beta\beta] = \frac{[a\,a]}{D}, \qquad [\alpha\beta] = \frac{-[a\,b]}{D} ;$$

with this, (17) becomes

$$\frac{M^2}{m^2} = \frac{1}{P} = \frac{1}{D} \{ f_1^{\ 2} [b\,b] + f_2^{\ 2} [a\,a] - 2 f_1 f_2 [a\,b] \} , \tag{19}$$

and this can also be transformed as follows:

$$\frac{1}{P} = \frac{f_1^{\ 2}}{[a\,a]} + \frac{[f_2 \cdot 1]^2}{[b\,b \cdot 1]} , \tag{20}$$

where

$$[f_2 \cdot 1] = f_2 - \frac{[a\,b]}{[a\,a]} f_1 . \tag{21}$$

Example of a weight of a function

In order to have also an application of the foregoing formulae, we must above all compute the coefficient $[a\,b \cdot 1] = \dfrac{1}{[\alpha\beta]}$ for which formula (16), section 18, p. 58, is used:

$$[a\,b \cdot 1] = [a\,b] - \frac{[a\,a]}{[a\,b]} [b\,b] \qquad \text{or} \qquad = [a\,b] - \frac{[b\,b]}{[a\,b]} [a\,a] .$$

We have written this in a twofold form, because both these forms are suited for the eliminations of x and y.

In order to show this clearly, we insert once more the beginnings of that elimination of section 20, p. 63, and add the computation of $[a\,b \cdot 1]$ both times:

	a]	b]	−l]		b]	a]	−l]
[a	+ 9.0	− 40.7	+ 4.9	[b	+ 229.9	− 40.7	− 25.3
[b	− 40.7	+ 229.9	− 25.3	[a	− 40.7	+ 9.0	+ 4.9
	+ 50.8	− 184.4	+ 22.1		+ 50.8	− 7.2	− 4.5
	+ 10.1	+ 45.5	− 3.2		+ 10.1	+ 1.8	+ 0.4
	= [ab·1]	[bb·1]			= [ba·1]	[aa·1] .	

For the elevation $h = 1000$ m we shall now compute the mean barometer reading B and its mean error or, as the case may be, weight. Therefore, the point in question is the function

$$B = 761.77 - 0.08695\, h \qquad \text{or} \qquad = 761.77 - 8.695 \left(\frac{h}{100}\right)$$

with $h = 1000$:

$$B = 761.77 - 10 \times 8.695 = X - 10\, Y = 674.82\,. \tag{22}$$

This corresponds to function (1):

$$F = f_1 x + f_2 y\,, \qquad \text{i.e.} \qquad f_1 = 1\,,\ f_2 = -10\,.$$

For the computation of the mean error M of function (1) we have found above the formula (18), p. 84, which yields with the insertion of all numerical values:

$$M^2 = 0.46^2 \left\{\frac{1}{1.8} - \frac{20}{10.1} + \frac{100}{45.5}\right\} = 0.46^2 \times 0.77\,,$$

$$M = \pm 0.40 \text{ mm}\,,$$

therefore: $$B_{1000} = 674.82 \text{ mm} \pm 0.40 \text{ mm}\,.$$

Hence, this means: At the elevation 1000 m above sea level, according to the method of observation (1), section 25, p. 76, an annual mean barometer reading $= 674.82$ mm will have to be expected and we can conclude that from the existing observations with a mean error to be anticipated of ± 0.40 mm.

Section 27. Change to an Arbitrary Number of Unknowns

In sections 14 to 26 we have treated in particular the adjustment of indirect observations with *two* unknowns, because this case could be represented in a very readily conceivable form, and because it is useful for the first study of the method of least squares not to discuss at once the voluminous general formulae for an arbitrary number of unknowns. In addition, the case of *two* unknowns occurs so often, e.g., in trigonometric point determinations with coordinates x and y that it is well worth while to take care of the case specially.

We can even permit ourselves a few analogous conclusions from the formulae for two unknowns to the formulae for more unknowns; e.g., after it has been proved that in the case of two unknowns x and y, we have the weight $p_y = [b\,b \cdot 1]$ we can well *assume* that with three unknowns x, y, z there will hold: $p_z = [c\,c \cdot 2]$, and so on.

In the case of the error formula with the denominator $n - 2$ (section 17, p. 55), also, we have already inserted the conclusion by analogy that with u unknowns the denominator $n - u$ will have to be expected; however, this was no proof as yet for an arbitrary number u.

We change now from the special case of two unknowns x and y to the general case of an arbitrary number of unknowns $x, y, z \ldots$, and convince ourselves at once that all which was said earlier about the introduction of approximate values (section 15), about the general course of the Gaussian elimination with $[b\,b \cdot 1]$, $[c\,c \cdot 1] \ldots$ (section 16), the computation of coefficients and sum checks (section 19), about unequal weights and nonlinear functions (sections 22-23), among other things, does not hold only for two, but for an arbitrary number of unknowns.

For example, with four unknowns x, y, z, t we have the following computational operation:

Error equations:

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y + c_1 z + d_1 t - l_1 \\ v_2 &= a_2 x + b_2 y + c_2 z + d_2 t - l_2 \\ v_3 &= a_3 x + b_3 y + c_3 z + d_3 t - l_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}. \tag{1}$$

Normal equations written in detail:

$$\left.\begin{aligned}
[a\,a]\,x + [a\,b]\,y + [a\,c]\,z + [a\,d]\,t - [a\,l] &= 0\\
[a\,b]\,x + [b\,b]\,y + [b\,c]\,z + [b\,d]\,t - [b\,l] &= 0\\
[a\,c]\,x + [b\,c]\,y + [c\,c]\,z + [c\,d]\,t - [c\,l] &= 0\\
[a\,d]\,x + [b\,d]\,y + [c\,d]\,z + [d\,d]\,t - [d\,l] &= 0
\end{aligned}\right\} . \qquad (2)$$

Since all nonquadratic coefficients $[a\,b]$, $[a\,c]$ . . . occur here *doubly*, in order not to have to write them doubly, we apply the following abbreviation, where we *underline* the quadratic terms in the diagonal and then simply omit the repetitions of $[a\,b]$, $[a\,c]$ . . . .

Normal equations written in the abbreviated form, besides $[l\,l]$:

$$\left.\begin{aligned}
\underline{[a\,a]}\,x + [a\,b]\,y + [a\,c]\,z + [a\,d]\,t - [a\,l] &= 0\\
\underline{[b\,b]}\,y + [b\,c]\,z + [b\,d]\,t - [b\,l] &= 0\\
\underline{[c\,c]}\,z + [c\,d]\,t - [c\,l] &= 0\\
\underline{[d\,d]}\,t - [d\,l] &= 0\\
\underline{[l\,l]}\qquad &
\end{aligned}\right\} . \qquad (3)$$

Reduced normal equations written in the abbreviated form:

$$\text{1. Reduction: }\left.\begin{aligned}
\underline{[b\,b\cdot 1]}\,y + [b\,c\cdot 1]\,z + [b\,d\cdot 1]\,t - [b\,l\cdot 1] &= 0\\
\underline{[c\,c\cdot 1]}\,z + [c\,d\cdot 1]\,t - [c\,l\cdot 1] &= 0\\
\underline{[d\,d\cdot 1]}\,t - [d\,l\cdot 1] &= 0\\
\underline{[l\,l\cdot 1]}\qquad &
\end{aligned}\right\} ; \qquad (4)$$

$$\text{2. Reduction: }\left.\begin{aligned}
\underline{[c\,c\cdot 2]}\,z + [c\,d\cdot 2]\,t - [c\,l\cdot 2] &= 0\\
\underline{[d\,d\cdot 2]}\,t - [d\,l\cdot 2] &= 0\\
\underline{[l\,l\cdot 2]}\qquad &
\end{aligned}\right\} ; \qquad (5)$$

$$\text{3. Reduction: }\left.\begin{aligned}
\underline{[d\,d\cdot 3]}\,t - [d\,l\cdot 3] &= 0\\
\underline{[l\,l\cdot 3]}\qquad &
\end{aligned}\right\} ; \qquad (6)$$

$$\text{4. Reduction: } \underline{[l\,l\cdot 4]}\,. \qquad (7)$$

The construction of the coefficients of elimination can be easily impressed on the memory if we note, in addition to the immediately obvious properties, the following:

1. Every bracket coefficient becomes $= 0$, if we interpret the symbolic signs algebraically, e.g.

$$-[b\,l\cdot 1] = -[b\,l] + \frac{[a\,b]}{[a\,a]}[a\,l] = -b\,l + \frac{a\,b\,a\,l}{a\,a} = -b\,l + b\,l = 0\,.$$

2. If we have 1, 2, 3 . . . in the brackets, then we have in the denominator of the subtrahend $[a\,a]$, $[b\,b\cdot 1]$, $[c\,c\cdot 2]$ . . . , respectively.

In order to have a numerical example, we will solve the following system of equations:

$$\left.\begin{aligned}
+\underline{459}\,x - 308\,y - 389\,z + 244\,t - 507 &= 0\\
+\underline{464}\,y + 408\,z - 269\,t + 695 &= 0\\
+\underline{676}\,z - 331\,t + 653 &= 0\\
+\underline{469}\,t - 283 &= 0\\
+\underline{1129}\qquad &
\end{aligned}\right\} . \qquad (8)$$

The solution with the slide rule shows the following table on p. 87, which has a similar arrangement as the table for two unknowns x, y in section 20, p. 63. In the last two columns, the sum terms and the checks are indicated.

The initial equations of all groups taken together are called completely reduced normal equations or end equations:

$$\left.\begin{aligned} A &= [aa]\,x + [ab]\,y + [ac]\,z + [ad]\,t - [al] = 0 \\ B' &= [bb\cdot 1]\,y + [bc\cdot 1]\,z + [bd\cdot 1]\,t - [bl\cdot 1] = 0 \\ C'' &= [cc\cdot 2]\,z + [cd\cdot 2]\,t - [cl\cdot 2] = 0 \\ D''' &= [dd\cdot 3]\,t - [dl\cdot 3] = 0 \\ & [ll\cdot 4] = [vv] \end{aligned}\right\}. \qquad (9)$$

These are real equations, each of which contains always one unknown less than the preceding, while (3) or (8) with the underlinings are only abbreviations of equations (2), each of which contains still *all* the unknowns.

The foregoing numerical example yields the end equations:

$$\left.\begin{aligned} +459\,x - 308\,y - 389\,z + 244\,t - 507 &= 0 \\ +256\,y + 146\,z - 105\,t + 354 &= 0 \\ +263\,z - 64\,t + 21 &= 0 \\ +281\,t + 137 &= 0 \\ 11 &= [vv] \end{aligned}\right\}. \qquad (10)$$

The fourth equation determines t, and if we set this backward into the third equation, then we also have z, and so on. In short, we can gradually find all unknowns in the succession t, z, y, x by backward insertion.

Solution of a System of Four Normal Equations

	a	b	c	d	$-l$	s	Check
A	+ 459	− 308	− 389	+ 244	− 507	− 501	0
		+ 464	+ 408	− 269	+ 695	− 990	0
		− 208	− 262	+ 164	− 341	+ 337	
			+ 676	− 331	+ 653	− 1017	0
			− 330	+ 207	− 430	+ 425	
				+ 469	− 283	+ 170	0
				− 130	+ 270	− 267	
					+ 1129	− 1687	0
					− 560	+ 554	
B'		+ 256	+ 146	− 105	+ 354	− 653	− 2
			+ 346	− 124	+ 223	− 592	− 1
			− 83	+ 60	− 202	+ 373	
				+ 339	− 13	− 97	0
				− 43	+ 145	− 268	
					+ 569	− 1133	0
					− 490	+ 902	
C''			+ 263	− 64	+ 21	− 219	+ 1
				+ 296	+ 132	− 365	− 1
				− 15	+ 5	− 53	
					+ 79	− 231	+ 1
					− 2	+ 17	
D'''				+ 281	+ 137	− 418	0
					+ 77	− 214	0
					− 66	+ 203	
					+ 11	− 11	

$$t = \frac{-137}{+281} = -0.488$$

The calculation of the deductions -208, -262, $+164$, and so on, is done with the slide rule. We set here first the quotient $\frac{308}{459}$, and read off, with this *single* setting, in succession:

$$\frac{308}{459}308 = 208\ , \quad \frac{308}{459}389 = 262\ , \quad \frac{308}{459}244 = 164\ , \quad \frac{308}{459}507 = 341\ , \quad \frac{308}{459}501 = 337\ .$$

Then in the second line with *one* setting:

$$\frac{389}{459}389 = 330\ , \quad \frac{389}{459}244 = 207\ , \quad \frac{389}{459}507 = 430\ , \quad \frac{389}{459}501 = 425\ .$$

These results 208, 262, etc., are inserted with correct signs, i.e. according to the rules of section 20, p. 64, under 464, 408, etc.

The eliminations and the computations related to them are done only with the slide rule, like most smaller numerical computations in this book; the hereby resulting small inaccuracies are insignificant in the above example.

For more rigorous computation, four- or five-place logarithms are used to advantage; but the computing machine remains the most important auxiliary means. V. Gruber gives a simple computing scheme for the computing machine in *Zeitschrift für Vermessungswesen*, 1925, pp. 133 to 140.

We shall treat another method for the solution of normal equations in the later section 50.

Section 28. Reduced Error Equations

After the reduced normal equations, we can also form reduced error equations, which are very useful for many purposes..

With the limitation to three unknowns x, y, z we have the general form of an error equation

$$v = a x + b y + c z - l\ , \tag{1}$$

and, to this, the first normal equation

$$[a\,a]x + [a\,b]y + [a\,c]z - [a\,l] = 0\ , \tag{2}$$

whence

$$x = -\frac{[a\,b]}{[a\,a]}y - \frac{[a\,c]}{[a\,a]}z + \frac{[a\,l]}{[a\,a]}\ . \tag{3}$$

We set this x into (1) and have

$$v = \left(b - \frac{[a\,b]}{[a\,a]}a\right)y + \left(c - \frac{[a\,c]}{[a\,a]}a\right)z - \left(l - \frac{[a\,l]}{[a\,a]}a\right) \tag{4}$$

or

$$v = b'y + c'z - l'\ , \tag{5}$$

where

$$b' = b - \frac{[a\,b]}{[a\,a]}a\ , \quad c' = c - \frac{[a\,c]}{[a\,a]}a\ , \quad -l' = -l + \frac{[a\,l]}{[a\,a]}a\ . \tag{6}$$

We call (4) or (5) a "reduced error equation," and we can call $b', c', -l'$, according to (6), reduced coefficients.

We can easily convince ourselves that the following is correct:

$$\left.\begin{array}{lll} [b'b'] = [bb\cdot1] & [b'c'] = [bc\cdot1] & -[b'l'] = -[bl\cdot1] \\ & [c'c'] = [cc\cdot1] & -[c'l'] = -[cl\cdot1] \\ & & [l'l'] = [ll\cdot1] \end{array}\right\}, \tag{7}$$

for, e.g., we have

$$b_1' = b_1 - \frac{[a\,b]}{[a\,a]}a_1 \qquad b_1'^2 = b_1^2 - 2\,a_1\,b_1\frac{[a\,b]}{[a\,a]} + \frac{[a\,b]^2}{[a\,a]^2}a_1^2\,,$$

$$[b'\,b'] = [b\,b] - 2\,[a\,b]\frac{[a\,b]}{[a\,a]} + \frac{[a\,b]^2}{[a\,a]^2}[a\,a]\,,$$

$$[b'\,b'] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b] = [b\,b\cdot 1]\,, \text{etc.}$$

Now we can reduce the error equation (5) once again. For if we take up the first reduced normal equation

$$[b\,b\cdot 1]\,y + [b\,c\cdot 1]\,z - [b\,l\cdot 1] = 0 \quad \text{or} \quad [b'\,b']\,y + [b'\,c']\,z - [b'\,l'] = 0\,,$$

and determine therefrom

$$y = -\frac{[b'\,c']}{[b'\,b']}z + \frac{[b'\,l']}{[b'\,b']}\,,$$

and if we set this into (5), then we will have

$$v = \left(c' - \frac{[b'\,c']}{[b'\,b']}b'\right)z - \left(l' - \frac{[b'\,l']}{[b'\,b']}b'\right), \tag{8}$$

$$v = c''\,z - l''\,, \tag{9}$$

where

$$c'' = c' - \frac{[b'\,c']}{[b'\,b']}b'\,, \qquad -l'' = -l' + \frac{[b'\,l']}{[b'\,b']}b'\,, \tag{10}$$

and there hold the relations, easily proved, similarly to (7),

$$[c''\,c''] = [c\,c\cdot 2] \qquad \left.\begin{aligned} -[c''\,l''] &= -[c\,l\cdot 2] \\ [l''\,l''] &= [l\,l\cdot 2] \end{aligned}\right\}. \tag{11}$$

Thus we can also continue until we have

$$v = -l''' = -l'' + \frac{[c''\,l'']}{[c''\,c'']}c''\,, \tag{12}$$

or by returning to (10) and (6)

$$v = -l''' = -l + \frac{[a\,l]}{[a\,a]}a + \frac{[b'\,l']}{[b'\,b']}b' + \frac{[c''\,l'']}{[c''\,c'']}c''\,. \tag{13}$$

Therefore, we have now proved that all coefficients of elimination $[b\,b\cdot 1]$, $[b\,c\cdot 1]$, . . . , $-[c\,l\cdot 2]$, and so on, are not only in form, but also in reality, sums of squares and sums of products, whose elements b', c', . . . , c'' etc., can be indicated.

In particular, $[b\,b\cdot 1]$, $[c\,c\cdot 2]$ . . . are sums of *squares* and therefore always positive.

The development of $[v\,v]$, which was made in section 16 (9) to (11), pp. 51 and 52, for two unknowns can be generally carried further:

For three elements x, y, z we have

$$\left.\begin{aligned}[v\,v] = [a\,a]\,x^2 + 2\,[a\,b]\,x\,y + 2\,[a\,c]\,x\,z - 2\,[a\,l]\,x \\ + [b\,b]\,y^2 + 2\,[b\,c]\,y\,z - 2\,[b\,l]\,y \\ + [c\,c]\,z^2 - 2\,[c\,l]\,z \\ + [l\,l]\end{aligned}\right\} . \tag{1}$$

We write this in the form:

$$\begin{aligned}[v\,v] = x\,([a\,a]\,x + [a\,b]\,y + [a\,c]\,z - [a\,l]) - [a\,l]\,x \\ + y\,([a\,b]\,x + [b\,b]\,y + [b\,c]\,z - [b\,l]) - [b\,l]\,y \\ + z\,([a\,c]\,x + [b\,c]\,y + [c\,c]\,z - [c\,l]) - [c\,l]\,z \\ + [l\,l]\,.\end{aligned}$$

According to (2), section 27, p. 86, the quantities in brackets are equal to zero; hence, there remains

$$[v\,v] = [l\,l] - [a\,l]\,x - [b\,l]\,y - [c\,l]\,z\,. \tag{1a}$$

For further conversion, we can bring the left-hand sides A, B', C'' . . . of the end equations (9), section 27, p. 87, into close relations with the sum of the squares $[v\,v]$. First we have

$$A = [a\,a]\,x + [a\,b]\,y + [a\,c]\,z - [a\,l]\,, \tag{2}$$

hence

$$\begin{aligned}\frac{A\,A}{[a\,a]} = [a\,a]\,x^2 + 2\,[a\,b]\,x\,y + 2\,[a\,c]\,x\,z \quad - 2\,[a\,l]\,x \\ + \frac{[a\,b]^2}{[a\,a]}\,y^2 + 2\,\frac{[a\,b]}{[a\,a]}\,[a\,c]\,y\,z - 2\,\frac{[a\,b]}{[a\,a]}\,[a\,l]\,y \\ + \frac{[a\,c]^2}{[a\,a]}\,z^2 \quad - 2\,\frac{[a\,c]}{[a\,a]}\,[a\,l]\,z \\ + \frac{[a\,l]^2}{[a\,a]}\,.\end{aligned}$$

This expression can be subtracted term by term from the sum $[v\,v]$ in (1); this yields

$$\left.\begin{aligned}[v\,v] - \frac{A\,A}{[a\,a]} = [b\,b\cdot 1]\,y^2 + 2\,[b\,c\cdot 1]\,y\,z - 2\,[b\,l\cdot 1]\,y \\ + [c\,c\cdot 1]\,z^2 - 2\,[c\,l\cdot 1]\,z \\ + [l\,l\cdot 1]\end{aligned}\right\} . \tag{3}$$

We can continue with the second end equation in exactly the same manner:

$$B' = [b\,b\cdot 1]\,y + [b\,c\cdot 1]\,z - [b\,l\cdot 1] \tag{4}$$

$$\begin{aligned}\frac{B'\,B'}{[b\,b\cdot 1]} = [b\,b\cdot 1]\,y^2 + 2\,[b\,c\cdot 1]\,y\,z - 2\,[b\,l\cdot 1]\,y \\ + \frac{[b\,c\cdot 1]^2}{[b\,b\cdot 1]}\,z^2 - 2\,\frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}\,[b\,l\cdot 1]\,z \\ + \frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}\,[b\,l\cdot 1]\,.\end{aligned}$$

This can be subtracted again term by term from (3), whereby we obtain

$$\left.[v\,v]-\frac{A\,A}{[a\,a]}-\frac{B'\,B'}{[b\,b\cdot 1]}=[c\,c\cdot 2]\,z^2-2\,[c\,l\cdot 2]\,z+[l\,l\cdot 2]\right\}. \tag{5}$$

The continuation of this procedure yields

$$\begin{aligned} C'' &= [c\,c\cdot 2]\,z - [c\,l\cdot 2] \\ \frac{C''\,C''}{[c\,c\cdot 2]} &= [c\,c\cdot 2]\,z^2 - 2\,[c\,l\cdot 2]\,z + \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}[c\,l\cdot 2]\,. \end{aligned} \tag{6}$$

In addition, if we subtract this, also, term by term, from (5), then we obtain

$$[v\,v]-\frac{A\,A}{[a\,a]}-\frac{B'\,B'}{[b\,b\cdot 1]}-\frac{C''\,C''}{[c\,c\cdot 2]}=[l\,l\cdot 3]\,. \tag{7}$$

But now the A's, B''s, C'''s, according to (2), (4), and (6), i.e. the left-hand sides of the end equations (9), section 27, p. 87, are all *zero*, and (7), above, becomes very simply

$$[v\,v]=[l\,l\cdot 3]\,. \tag{7a}$$

If we break this remaining term $[l\,l\cdot 3]$ again into its parts, then we have

$$\begin{aligned} [l\,l\cdot 3] &= [l\,l\cdot 2]-\frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}[c\,l\cdot 2] \\ [l\,l\cdot 2] &= [l\,l\cdot 1]-\frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[b\,l\cdot 1] \\ [l\,l\cdot 1] &= [l\,l]-\frac{[a\,l]}{[a\,a]}[a\,l]\,, \end{aligned}$$

consequently:

$$[v\,v]=[l\,l\cdot 3]=[l\,l]-\frac{[a\,l]^2}{[a\,a]}-\frac{[b\,l\cdot 1]^2}{[b\,b\cdot 1]}-\frac{[c\,l\cdot 2]^2}{[c\,c\cdot 2]}\,. \tag{8}$$

At the end of section 28 it was shown that all denominators $[a\,a]$, $[b\,b\cdot 1]$, $[c\,c\cdot 2]$, etc., are positive; we thus see clearly from (8) how the sum $[l\,l]$ gradually decreases.

From $[v\,v]$ we also compute the mean error m of an individual observation (with the weight 1). If the v's were the true observational errors, then we would have, in the case of n error equations,

$$m^2=\frac{[v\,v]}{n}\,;\ (?) \tag{9}$$

however, we obtain from the true observational errors ε correctly

$$m^2=\frac{[\varepsilon\,\varepsilon]}{n}\,. \tag{10}$$

If we denote the true unknowns by X, Y, Z, then we have

$$\begin{aligned} v &= -\,l+a\,x+b\,y+c\,z \\ \varepsilon &= -\,l+a\,X+b\,Y+c\,Z \\ \hline \text{Difference}\quad v &= +\,\varepsilon+a\,(x-X)+b\,(y-Y)+c\,(z-Z)\,. \end{aligned} \tag{11}$$

If we form from n equations (11) the expressions $[a\,v] = 0$, $[b\,v] = 0$, $[c\,v] = 0$, then we obtain three equations which agree with the normal equations, if we imagine x, y, z replaced by $x - X$, $y - Y$, $z - Z$ and $-l$ by $+\varepsilon$. Therefore, we can also apply to equation (11) all transformations undertaken with the error equations; in particular, we have according to (8)

$$[v\,v] = [\varepsilon\,\varepsilon] - \frac{[a\,\varepsilon]^2}{[a\,a]} - \frac{[b\,\varepsilon \cdot 1]^2}{[b\,b \cdot 1]} - \frac{[c\,\varepsilon \cdot 2]^2}{[c\,c \cdot 2]}\,, \tag{12}$$

and with the reduced coefficients b', c'' of section 28, pp. 88 and 89:

$$[v\,v] = [\varepsilon\,\varepsilon] - \frac{[a\,\varepsilon]^2}{[a\,a]} - \frac{[b'\,\varepsilon']^2}{[b'\,b']} - \frac{[c''\,\varepsilon'']^2}{[c''\,c'']}\,, \tag{13}$$

where ε', ε'' . . . have the following meanings:

$$\varepsilon' = \varepsilon - \frac{[a\,\varepsilon]}{[a\,a]}a\,, \qquad \varepsilon'' = \varepsilon' - \frac{[b\,\varepsilon \cdot 1]}{[b\,b \cdot 1]}b' = \varepsilon' - \frac{[b'\,\varepsilon']}{[b'\,b']}b'\,. \tag{14}$$

From (13) we recognize, because of the quadratic form of the terms, that $[\varepsilon\,\varepsilon]$ is in any case larger than $[v\,v]$. The difference $[\varepsilon\,\varepsilon] - [v\,v]$ itself is a function of the ε's, and hence, can never be determined most rigorously; but we can at least determine the average values, with respect to which the terms of (13) converge. We consider first:

$$\begin{aligned}[a\,\varepsilon]^2 &= (a_1\,\varepsilon_1 + a_2\,\varepsilon_2 + a_3\,\varepsilon_3 + \ldots)^2 \\ &= a_1^2\,\varepsilon_1^2 + a_2^2\,\varepsilon_2^2 + a_3^2\,\varepsilon_3^2 + \ldots \\ &\quad + 2\,a_1\,a_2\,\varepsilon_1\,\varepsilon_2 + 2\,a_1\,a_3\,\varepsilon_1\,\varepsilon_3 + \ldots\end{aligned} \tag{15}$$

The average value of the products $\varepsilon_1\,\varepsilon_2$, $\varepsilon_1\,\varepsilon_3$ . . . vanishes because of the equally probable $+$ and $-$ signs, and the average value of the squares ε_1^2, ε_2^2 . . . is to be set $= m^2$. Hence, as the average value of function (15) we are to take

$$[a\,\varepsilon]^2 = (a_1^2 + a_2^2 + a_3^2 + \ldots)\,m^2 = [a\,a]\,m^2 \quad \text{and} \quad \frac{[a\,\varepsilon]^2}{[a\,a]} = m^2\,. \tag{16}$$

If we change to the second term of (13), then we have according to the first equation of (14)

$$[b'\varepsilon'] = [b'\varepsilon] - \frac{[a\,\varepsilon]}{[a\,a]}[a\,b']\,.$$

We have further according to (6), section 28, p. 88:

$$b' = b - \frac{[a\,b]}{[a\,a]}a\,;$$

therefore,

$$a\,b' = a\,b - \frac{[a\,b]}{[a\,a]}a\,a\,,$$

and consequently

$$[a\,b'] = [a\,b] - \frac{[a\,b]}{[a\,a]}[a\,a] = 0\,. \tag{16a}$$

We thus have $[b' \varepsilon'] = [b' \varepsilon]$,

and according to the same conclusions as in the case of (15) and (16) we have to set

$$\frac{[b' \varepsilon']^2}{[b' b']} = \frac{[b' \varepsilon]^2}{[b' b']} = m^2 .$$

We can likewise prove that $[c'' \varepsilon''] = [c'' \varepsilon]$ and that the average value of $\frac{[c'' \varepsilon'']^2}{[c'' c'']}$ is to be set equal to m^2. We have therefore from (13)

$$[v\,v] - [\varepsilon\,\varepsilon] = -\,m^2 - m^2 - m^2 \qquad \text{or} \qquad [\varepsilon\,\varepsilon] = [v\,v] + 3\,m^2 . \tag{17}$$

The rigorous value of the mean square of error is, according to (10),

$$m^2 = \frac{[\varepsilon\,\varepsilon]}{n} ,$$

and this gives in connection with the above equation (17)

$$m^2 = \frac{[v\,v]}{n - 3} . \tag{18}$$

This holds for three unknowns, since, because of brevity, we have only written three symbols x, y, z; the consideration, however, holds generally, and, therefore, gives for n error equations with u unknowns

$$m^2 = \frac{[v\,v]}{n - u} . \tag{19}$$

This equation takes the place of (9).

Appendix to Section 29

Similar developments as the foregoing development of $[v\,v]$ occur repeatedly in the method of least squares; we therefore form, from this development, a general theorem by setting, in (1) and (7), pp. 90 and 91, $l = 0$ and, then, setting (1) and (7) equal to one another. If we take into account here the meanings of A according to (2), B' according to (4), and C'' according to (6), then (1) and (7) yield

$$\left\{\begin{array}{l} [a\,a]\,x^2 + 2\,[a\,b]\,x\,y + 2\,[a\,c]\,x\,z \\ \qquad + \;[b\,b]\,y^2 + 2\,[b\,c]\,y\,z \\ \qquad\qquad + \;[c\,c]\,z^2 \end{array}\right\} = \left\{\begin{array}{l} \dfrac{([a\,a]\,x + [a\,b]\,y + [a\,c]\,z)^2}{[a\,a]} \\ \quad + \dfrac{([b\,b\cdot 1]\,y + [b\,c\cdot 1]\,z)^2}{[b\,b\cdot 1]} \\ \quad + \dfrac{([c\,c\cdot 2]\,z)^2}{[c\,c\cdot 2]} \end{array}\right\} . \tag{20}$$

This is an algebraic identity, which is not related at all to the condition that the sum of the squares $[v\,v]$ = minimum. It only is assumed here that the coefficients $[b\,b \cdot 1]$, etc., are formed according to the general law of elimination.

According to what has been mentioned hitherto, we have, with the limitation to $n = 4$, $u = 3$, the following:

Error equations:

$$\text{Number} \quad n = 4 \left\{ \underbrace{\begin{array}{l} v_1 = a_1 x + b_1 y + c_1 z - l_1 \\ v_2 = a_2 x + b_2 y + c_2 z - l_2 \\ v_3 = a_3 x + b_3 y + c_3 z - l_3 \\ v_4 = a_4 x + b_4 y + c_4 z - l_4 \end{array}}_{\text{Number} \quad u = 3} \right\}, \tag{1}$$

Normal equations:

$$\left.\begin{array}{l} [a a]\, x + [a b]\, y + [a c]\, z - [a l] = 0 \\ [a b]\, x + [b b]\, y + [b c]\, z - [b l] = 0 \\ [a c]\, x + [b c]\, y + [c c]\, z - [c l] = 0 \end{array}\right\}. \tag{2}$$

Reduced normal equations:

$$\left.\begin{array}{l} [bb \cdot 1]\, y + [bc \cdot 1]\, z - [b l \cdot 1] = 0 \\ [bc \cdot 1]\, y + [cc \cdot 1]\, z - [c l \cdot 1] = 0 \end{array}\right\}, \tag{3}$$

$$[cc \cdot 2]\, z - [c l \cdot 2] = 0 \,. \tag{4}$$

Normal equations in abbreviated manner of writing:

$$\left.\begin{array}{r} \underline{[a a]}\, x + [a b]\, y + [a c]\, z - [a l] = 0 \\ \underline{[b b]}\, y + [b c]\, z - [b l] = 0 \\ \underline{[c c]}\, z - [c l] = 0 \\ \underline{[l l]} \end{array}\right\} \tag{2*}$$

Reduced normal equations in abbreviated manner of writing:

$$\left.\begin{array}{r} \underline{[b b \cdot 1]}\, y + [b c \cdot 1]\, z - [b l \cdot 1] = 0 \\ \underline{[c c \cdot 1]}\, z - [c l \cdot 1] = 0 \\ \underline{[l l \cdot 1]} \end{array}\right\}, \tag{3*}$$

$$\left.\begin{array}{r} \underline{[c c \cdot 2]}\, z - [c l \cdot 2] = 0 \\ \underline{[l l \cdot 2]} \end{array}\right\}, \tag{4*}$$

$$[\, l l \cdot 3] \,. \tag{5}$$

We insert here once again the meaning of the coefficients of elimination:

$$\left.\begin{array}{lll} [bb \cdot 1] = [bb] - \frac{[ab]}{[aa]}[ab] & [bc \cdot 1] = [bc] - \frac{[a b]}{[aa]}[ac] & -[bl \cdot 1] = -[bl] + \frac{[a b]}{[aa]}[al] \\ & [cc \cdot 1] = [cc] - \frac{[a c[}{[a a]}[ac] & -[cl \cdot 1] = -[cl] + \frac{[a c]}{[aa]}[al] \\ & & [ll \cdot 1] = \quad [ll] - \frac{[a l]}{[aa]}[al] \end{array}\right\} \tag{3**}$$

$$[cc\cdot 2] = [cc\cdot 1] - \frac{[bc\cdot 1]}{[bb\cdot 1]}[bc\cdot 1] \quad -[cl\cdot 2] = -[cl\cdot 1] + \frac{[bc\cdot 1]}{[bb\cdot 1]}[bl\cdot 1] \left.\begin{matrix} \\ \end{matrix}\right\} \tag{4**}$$
$$[ll\cdot 2] = \quad [ll\cdot 1] - \frac{[cl\cdot 1]}{[cc\cdot 1]}[cl\cdot 1]$$

$$[ll\cdot 3] = \quad [ll\cdot 2] - \frac{[cl\cdot 2]}{[cc\cdot 2]}[cl\cdot 2] \; . \tag{5*}$$

Now we will imagine the normal equations (2) solved in such a way that x, y, z are shown as linear functions of the l's

$$\left.\begin{aligned} x &= \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 \\ y &= \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \beta_4 l_4 \\ z &= \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3 + \gamma_4 l_4 \end{aligned}\right\} . \tag{6}$$

Of these, we consider first the third equation, namely the one for z, and apply to it the general law of the propagation of error of (9) and (11), section 5, p. 14, namely by denoting the mean error of the l's by m and the mean error of z by m_z:

$$\left.\begin{aligned} m_z^2 &= \gamma_1^2 m^2 + \gamma_2^2 m^2 + \gamma_3^2 m^2 + \gamma_4^2 m^2 \\ m_z^2 &= (\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2) m^2 = [\gamma\gamma] m^2 \end{aligned}\right\} ; \tag{7}$$

or the same in weight form:

$$\frac{1}{p_z} = [\gamma\gamma] \; . \tag{8}$$

Now we must, on the other hand, also determine z from the normal equations (2) by elimination of x and y, which we will do according to the method of undetermined coefficients, i.e., we multiply equations (2) by the coefficients, left first still undetermined, Q_1, Q_2, Q_3, whereby we obtain

$$\left.\begin{aligned} Q_1 [aa] x + Q_1 [ab] y + Q_1 [ac] z - Q_1 [al] &= 0 \\ Q_2 [ab] x + Q_2 [bb] y + Q_2 [bc] z - Q_2 [bl] &= 0 \\ Q_3 [ac] x + Q_3 [bc[y + Q_3 [cc[z - Q_3 [cl] &= 0 \end{aligned}\right\} . \tag{9}$$

If we add these three equations, then x and y will vanish and z will obtain a coefficient $= 1$, i.e., we dispose of the coefficients Q_1, Q_2, Q_3 in such a way that we will have

$$\left.\begin{aligned} Q_1 [aa] + Q_2 [ab[+ Q_3 [ac] &= 0 \\ Q_1 [ab] + Q_2 [bb] + Q_3 [bc] &= 0 \\ Q_1 [ac] + Q_2 [bc] + Q_3 [cc] &= 0 \end{aligned}\right\} . \tag{10}$$

With this, the addition of the three equations (9) yields

$$z - Q_1 [al] - Q_2 [bl] - Q_3 [cl] = 0 \; . \tag{11}$$

This is compared with the original assumption, i.e. with the third equation of (6), and for this it is necessary that we remove the brackets $[al]$, $[bl]$, $[cl]$ in (11) and arrange all according to l_1, l_2, l_3, l_4, i.e.

$$
\begin{aligned}
z = {} & Q_1 (a_1 l_1 + a_2 l_2 + a_3 l_3 + a_4 l_4) \\
& + Q_2 (b_1 l_1 + b_2 l_2 + b_3 l_3 + b_4 b_4) \\
& + Q_3 (c_1 l_1 + c_2 l_2 + c_3 l_3 + c_4 l_4)
\end{aligned}
$$

$$
\left.
\begin{aligned}
z = {} & (Q_1 a_1 + Q_2 b_1 + Q_3 c_1) l_1 + (Q_1 a_2 + Q_2 b_2 + Q_3 c_2) l_2 \\
& + (Q_1 a_3 + Q_2 b_3 + Q_3 c_3) l_3 + (Q_1 a_4 + Q_2 b_4 + Q_3 c_4) l_4
\end{aligned}
\right\} . \tag{12}
$$

For comparison, we insert here the third equation of (6), namely

$$z = \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3 + \gamma_4 l_4 .$$

This compared with (12) yields

$$
\left.
\begin{aligned}
\gamma_1 &= Q_1 a_1 + Q_2 b_1 + Q_3 c_1 \\
\gamma_2 &= Q_1 a_2 + Q_2 b_2 + Q_3 c_2 \\
\gamma_3 &= Q_1 a_3 + Q_2 b_3 + Q_3 c_3 \\
\gamma_4 &= Q_1 a_4 + Q_2 b_4 + Q_3 c_4
\end{aligned}
\right\} . \tag{13}
$$

If we multiply these equations in succession by a_1, a_2, a_3, a_4 and take into account (10), then we obtain

$$[a\gamma] = [a\,a]\,Q_1 + [a\,b]\,Q_2 + [a\,c]\,Q_3 = 0 . \tag{14a}$$

If we do the same also with b and with c, then we obtain also

$$[b\,\gamma] = [a\,b]\,Q_1 + [b\,b]\,Q_2 + [b\,c]\,Q_3 = 0 , \tag{14b}$$

$$[c\,\gamma] = [a\,c]\,Q_1 + [b\,c]\,Q_2 + [c\,c]\,Q_3 = 0 . \tag{14c}$$

These three equations $[a\,\gamma] = 0$, $[b\,\gamma] = 0$, $[c\,\gamma] = 1$ have been obtained in the elimination of x and y, i.e. in the determination of z; if we reversed the elimination (whereby other coefficients Q would occur again), then we would obtain analogous equations and the totality of all such formulae is

$$
\left.
\begin{matrix}
[a\,\alpha] = 1 & [b\,\alpha] = 0 & [c\,\alpha] = 0 \\
[a\,\beta] = 0 & [b\,\beta] = 1 & [c\,\beta] = 0 \\
[a\,\gamma] = 0 , & [b\,\gamma] = 0 , & [c\,\gamma] = 1
\end{matrix}
\right\} . \tag{15}
$$

We go one step further, and multiply equations (13) in succession by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$; this yields

$$[\alpha\gamma] = Q_1\,[a\alpha] + Q_2\,[b\,\alpha] + Q_3\,[c\,\alpha] ,$$

i.e. because of (15)

$$[\alpha\gamma] = Q_1 . \tag{16a}$$

Similarly, i.e., if we multiply equations (13) by $\beta_1, \beta_2, \beta_3, \beta_4$ and then also by $\gamma_1, \gamma_2, \gamma_3; \gamma_4$, and add them, we obtain also

$$[\beta\,\gamma] = Q_2 , \qquad [\gamma\,\gamma] = Q_3 . \qquad \text{(16b) and (16c)}$$

If we introduce these equations (16) into (11), then we obtain

$$z = [\alpha\gamma]\,[a\,l] + [\beta\,\gamma]\,[b\,l] + [\gamma\,\gamma]\,[c\,l] . \tag{17}$$

If we introduce these equations also into (10), then we obtain

$$\left.\begin{aligned}[a a]\,[\alpha\gamma]+[a b]\,[\beta\gamma]+[a c]\,[\gamma\gamma]&=0\\ [a b]\,[\alpha\gamma]+[b b]\,[\beta\gamma]+[b c]\,[\gamma\gamma]&=0\\ [a c]\,[\alpha\gamma]+[b c]\,[\beta\gamma]+[c c]\,[\gamma\gamma]&=1\end{aligned}\right\}. \tag{18}$$

In these equations (17) and (18) the third unknown z is favored; by cyclic permutation, from (17) and (18) we obtain the complete system:

Undetermined solution of the normal equations

$$\left.\begin{aligned}x&=[\alpha\alpha]\,[a l]+[\alpha\beta]\,[b l]+[\alpha\gamma]\,[c l]\\ y&=[\alpha\beta]\,[a l]+[\beta\beta]\,[b l]+[\beta\gamma]\,[c l]\\ z&=[\alpha\gamma]\,[a l]+[\beta\gamma]\,[b l]+[\gamma\gamma]\,[c l]\end{aligned}\right\}, \tag{19}$$

Weight equations

$$\left.\begin{aligned}&[a a]\,[\alpha\alpha]+[a b]\,[\alpha\beta]+[a c]\,[\alpha\gamma]=1\quad [a a]\,[\alpha\beta]+[a b]\,[\beta\beta]+[a c]\,[\beta\gamma]=0\\ &[a b]\,[\alpha\alpha]+[b b]\,[\alpha\beta]+[b c]\,[\alpha\gamma]=0\quad [a b]\,[\alpha\beta]+[b b]\,[\beta\beta]+[b c]\,[\beta\gamma]=1\\ &[a c]\,[\alpha\alpha]+[b c]\,[\alpha\beta]+[c c]\,[\alpha\gamma]=0\quad [a c]\,[\alpha\beta]+[b c]\,[\beta\beta]+[c c]\,[\beta\gamma]=0\\ &\qquad\qquad [a a]\,[\alpha\gamma]+[a b]\,[\beta\gamma]+[a c]\,[\gamma\gamma]=0\\ &\qquad\qquad [a b]\,[\alpha\gamma]+[b b]\,[\beta\gamma]+[b c]\,[\gamma\gamma]=0\\ &\qquad\qquad [a c]\,[\alpha\gamma]+[b c]\,[\beta\gamma]+[c c]\,[\gamma\gamma]=1\end{aligned}\right\}. \tag{20}$$

By solving the weight equations (20) we can determine all weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc., in particular the nonquadratic, e.g. $[\alpha\beta]$, $[\beta\gamma]$, etc., each twice, which serves as a computational check.

If we set the thus obtained coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc., into (19), then we have the so-called "undetermined solution of the normal equations," i.e. the development of the x's, y's, z's as linear functions of the absolute terms $[a l]$, $[b l]$, $[c l]$.

The weight equations (20) have the same coefficients $[a a]$, $[a b]$, etc., as the original normal equations (2); therefore, the solution of the weight equations fits the solution (2*) (3*) (4*).

Either 1 or 0 have now taken here the place of the previous $[a l]$'s, $[b l]$'s, $[c l]$'s, and we will soon find that also all following $[b l \cdot 1]$'s, $[c l \cdot 1]$'s, $[c l \cdot 2]$'s can become only 1 or 0; e.g., the third group of (20) results from (2*), p. 94, with regard to the absolute terms, by setting

$$[a l]=0\,,\qquad [b l]=0\,,\qquad [c l]=1\,;$$

with this, we will have according to (3**), p. 94

$$[b l\cdot 1]=[b l]-\frac{[a b]}{[a a]}[a l]=0-0=0\,,$$

$$[c l\cdot 1]=[c l]-\frac{[a c]}{[a a]}[a l]=1-0=1\,.$$

In this way we obtain for the third group of (20) the following system, which corresponds to (3*), p. 94, and (4*), p. 94:

$$\left.\begin{aligned}\underline{[a a]\,[\alpha\gamma]}+[a b]\,[\beta\gamma]+[a c]\,[\gamma\gamma]&=0\\ \underline{[b b]\,[\beta\gamma]}+[b c]\,[\gamma\gamma]&=0\\ \underline{[c c]\,[\gamma\gamma]}&=1\end{aligned}\right\}, \tag{21a}$$

$$\left.\begin{aligned}\underline{[b b\cdot 1]\,[\beta\gamma]}+[b c\cdot 1]\,[\gamma\gamma]&=0\\ \underline{[c c\cdot 1]\,[\gamma\gamma]}&=1\end{aligned}\right\}, \tag{21b}$$

$$[c c\cdot 2]\,[\gamma\gamma]=1\,. \tag{21c}$$

We take this end equation of (21c) together with (8), p. 95, and have

$$p_z = \frac{1}{[\gamma\gamma]} = [c\,c\cdot 2]\,. \tag{22}$$

This is the generalization of the proposition which we have already had in the form $p_y = [b\,b\cdot 1]$, for two unknowns in (8), section 18, p. 57.

Since our consideration is carried, it is true, with *three* elements x, y, z, but in the idea is not bound to *three*, this theorem according to (22), above, means generally:

If we reduce the normal equations gradually according to the Gaussian method, i.e. form $[b\,b\cdot 1]$, $[c\,c\cdot 2]$, etc., until only one equation with one unknown remains, then in this last equation the coefficient of the unknown is at the same time the weight of the unknown.

This method of weight determination is very customary.

In order to arrive at the independent determination of the weights of all unknowns according to this method, we must invert the elimination at least once completely, hence, for instance, determine first the unknown z and p_z in the order x, y, z, then carry out the determination of x and p_x in the order z, y, x whereupon y, besides p_y, is found by inversion, either from the two equations which have remained after the elimination of x, or from the two equations which have resulted after the elimination of z.

Each time we can also determine here, at the same time, a nonquadratic coefficient $[\alpha\beta]$, etc., for, if, e.g., z is eliminated, so that we have

$$\begin{aligned} \underline{[b\,b\cdot 1]}\,y + [b\,c\cdot 1]\,z - [b\,l\cdot 1] &= 0 \\ \underline{[c\,c\cdot 1]}\,z - [c\,l\cdot 1] &= 0\,, \end{aligned}$$

then we can also compute

$$[b\,c\cdot 2] = [b\,c\cdot 1] - \frac{[b\,b\cdot 1]\,[c\,c\cdot 1]}{[b\,c\cdot 1]} = \frac{1}{[\beta\gamma]}\,. \tag{23}$$

We prove this just as (22) was proved, i.e., we consider $[\beta\gamma]$ as that special value of z, which arises in (19) if we set $[a\,l] = 0$, $[b\,l] = 1$, and $[c\,l] = 0$.

We could in this way find gradually all weight coefficients by repeated inversion of the elimination computation, and with only two or three unknowns this is done, under certain circumstances, quite conveniently.

With two unknowns, everything is in general simple, as is shown in the example, section 21, p. 67.

With three unknowns, we can proceed as follows:

1)	a	b	c	2)	c	b	a
		b	c			b	a
			c				a

gives z and $[\gamma\gamma]$ besides $[\beta\gamma]$, gives x and $[\alpha\alpha]$ besides $[\alpha\beta]$.

At any rate, we must now invert once more in order to obtain y and $[\beta\beta]$; hence

3)	c	b	or	4)	a	b
		b				b

gives y and $[\beta\beta]$ besides $[\beta\gamma]$, gives y and $[\beta\beta]$ besides $[\alpha\beta]$.

Since with 1), 2), and 3) we have already all unknowns and their weights, we can, instead of forming, in addition, 4), make also use of the first of equations (20) itself for the last coefficient $[\alpha\beta]$ still missing.

With more than three unknowns, however, the method of equation (23) is advisable at best if we need only *single* ones of the coefficients $[\beta\gamma]$, and so on, according to which we can then arrange the order of elimination. If we need all weight coefficients (which, e.g., is the case in the Bessel triangulation adjustment), then we cannot do without the general weight equations (20) or (21), as the case may be, whose joint solution we shall treat separately in section 36.

Although the individual coefficients α, β, γ will hardly ever have to be computed, we will also indicate, in addition, the required formulae, of which we shall make use later. After the weight coefficients

$[\alpha\alpha]$, $[\alpha\beta]$, etc., have been computed, we have at once from (13) and (16a), p. 96,

$$\left.\begin{array}{l} \gamma_1 = a_1[\alpha\gamma] + b_1[\beta\gamma] + c_1[\gamma\gamma] \\ \gamma_2 = a_2[\alpha\gamma] + b_2[\beta\gamma] + c_2[\gamma\gamma] \\ \cdots\cdots\cdots\cdots \end{array}\right\}. \tag{24}$$

Without repeating the whole development of formulae, we can also set up the following expressions:

$$\left.\begin{array}{l} \alpha_1 = a_1[\alpha\alpha] + b_1[\alpha\beta] + c_1[\alpha\gamma] \\ \alpha_2 = a_2[\alpha\alpha] + b_2[\alpha\beta] + c_2[\alpha\gamma] \\ \cdots\cdots\cdots\cdots \end{array}\right\} \tag{25}$$

and

$$\left.\begin{array}{l} \beta_1 = a_1[\alpha\beta] + b_1[\beta\beta] + c_1[\beta\gamma] \\ \beta_2 = a_2[\alpha\beta] + b_2[\beta\beta] + c_2[\beta\gamma] \\ \cdots\cdots\cdots\cdots \end{array}\right\}. \tag{26}$$

We have further

$$\left.\begin{array}{l} a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 = a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 = a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 \\ \cdots\cdots\cdots\cdots \end{array}\right\}, \tag{27}$$

which becomes visible at once if we introduce for α, β, γ the values (24) to (26).

Unequal weights

Although the treatment of observations with unequal accuracy is easily carried out according to the pattern of the previous section 22, p. 69, valid for two unknowns, we will in addition assemble, in a readily conceivable manner, the most important formulae for it.

Normal equations:

$$\left.\begin{array}{l} [aap]x + [abp]y + [acp]z - [alp] = 0 \\ [abp]x + [bbp]y + [bcp]z - [blp] = 0 \\ [acp]x + [bcp]y + [ccp]z - [clp] = 0 \end{array}\right\}; \tag{28}$$

Reduced normal equations:

$$\left.\begin{array}{l} [bbp\cdot 1]y + [bcp\cdot 1]z - [blp\cdot 1] = 0 \\ [bcp\cdot 1]y + [ccp\cdot 1]z - [clp\cdot 1] = 0 \end{array}\right\} \tag{29}$$

$$[ccp\cdot 2]z - [clp\cdot 2] = 0; \tag{30}$$

Weight equations for the unknown z:

$$\left.\begin{array}{l} [aap]\ [\alpha\gamma] + [abp]\ [\beta\gamma] + [acp]\ [\gamma\gamma] = 0 \\ [abp]\ [\alpha\gamma] + [bbp]\ [\beta\gamma] + [bcp]\ [\gamma\gamma] = 0 \\ [acp]\ [\alpha\gamma] + [bcp]\ [\beta\gamma] + [ccp]\ [\gamma\gamma] = 1 \end{array}\right\}; \tag{31}$$

Coefficients γ:

$$\left.\begin{array}{l} \gamma_1 = a_1 p_1[\alpha\gamma] + b_1 p_1[\beta\gamma] + c_1 p_1[\gamma\gamma] \\ \gamma_2 = a_2 p_2[\alpha\gamma] + b_2 p_2[\beta\gamma] + c_2 p_2[\gamma\gamma] \\ \cdots\cdots\cdots\cdots \end{array}\right\}. \tag{32}$$

According to equations (31) and (32), the weight equations for the unknowns x and y and the equations for the coefficients α and β are also to be set up.

Section 31. Weight of a Function of x, y, z According to the Adjustment

We consider the linear function:

$$F = f_1 x + f_2 y + f_3 z . \tag{1}$$

We cannot directly determine the weight of F from the *single* weights of the elements x, y, z adjusted in common, because the weights of x, y, z are not independent; we must rather return to the observations themselves, and represent F as a function of the same, as in the case of section 18.

We use again equations (6), section 30, p. 95:

$$\left.\begin{array}{l} x = \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 \\ y = \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \beta_4 l_4 \\ z = \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3 + \gamma_4 l_4 \end{array}\right\} . \tag{2}$$

If we introduce these values into (1), then we obtain:

$$F = \left\{\begin{array}{l} + (f_1 \alpha_1 + f_2 \beta_1 + f_3 \gamma_1) l_1 \\ + (f_1 \alpha_2 + f_2 \beta_2 + f_3 \gamma_2) l_2 \\ + (f_1 \alpha_3 + f_2 \beta_3 + f_3 \gamma_3) l_3 \\ + (f_1 \alpha_4 + f_2 \beta_4 + f_3 \gamma_4) l_4 \end{array}\right\} .$$

Since the l's hold in weight computations as direct observations, we obtain the weight P according to the general law of the propagation of error, just as in the case of (7) and (8) of the previous section 30, p. 95.

$$\frac{1}{P} = \left\{\begin{array}{c} (f_1 \alpha_1 + f_2 \beta_1 + f_3 \gamma_1)^2 \\ + (f_1 \alpha_2 + f_2 \beta_2 + f_3 \gamma_2)^2 \\ \cdots\cdots\cdots \end{array}\right\} ,$$

or by carrying out the squares:

$$\frac{1}{P} = \left\{\begin{array}{r} f_1 f_1 [\alpha\alpha] + 2 f_1 f_2 [\alpha\beta] + 2 f_1 f_3 [\alpha\gamma] \\ + \ f_2 f_2 [\beta\beta] + 2 f_2 f_3 [\beta\gamma] \\ + \ f_3 f_3 [\gamma\gamma] \end{array}\right\} . \tag{3}$$

We can write equation (3) also as follows:

$$\left.\begin{array}{l} \frac{1}{P} = f_1 (f_1 [\alpha\alpha] + f_2 [\alpha\beta] + f_3 [\alpha\gamma]) \\ + f_2 (f_1 [\alpha\beta] + f_2 [\beta\beta] + f_3 [\beta\gamma]) \\ + f_3 (f_1 [\alpha\gamma] + f_2 [\beta\gamma] + f_3 [\gamma\gamma]) \end{array}\right\} , \tag{4}$$

or by arranging according to vertical series and collecting the coefficients of f_1, f_2, f_3:

$$\frac{1}{P} = q_1 f_1 + q_2 f_2 + q_3 f_3 = [qf] , \tag{5}$$

where q_1, q_2, q_3 have the following meanings:

$$\left.\begin{aligned} q_1 &= [\alpha\alpha] f_1 + [\alpha\beta] f_2 + [\alpha\gamma] f_3 \\ q_2 &= [\alpha\beta] f_1 + [\beta\beta] f_2 + [\beta\gamma] f_3 \\ q_3 &= [\alpha\gamma] f_1 + [\beta\gamma] f_2 + [\gamma\gamma] f_3 \end{aligned}\right\} . \tag{6}$$

Now it will be evident that there exists the equation

$$[a\,a]\, q_1 + [a\,b]\, q_2 + [a\,c]\, q_3 = f_1 , \tag{7}$$

for the working out of this function according to (6) yields

$$\left.\begin{aligned} [a\,a]\, q_1 + [a\,b]\, q_2 + [a\,c]\, q_3 = &([a\,a]\,[\alpha\,\alpha] + [a\,b]\,[\alpha\,\beta] + [a\,c]\,[\alpha\,\gamma])\, f_1 \\ &([a\,a]\,[\alpha\,\beta] + [a\,b]\,[\beta\,\beta] + [a\,c]\,[\beta\,\gamma])\, f_2 \\ &([a\,a]\,[\alpha\,\gamma] + [a\,b]\,[\beta\,\gamma] + [a\,c]\,[\gamma\,\gamma])\, f_3 \end{aligned}\right\} . \tag{8}$$

But the coefficients occurring here $([a\,a]\,[\alpha\alpha] + \ldots)$ are partly = 1, partly 0, as was shown in the preceding section 30 (20), p. 97. Development (8) thus gives, in fact, the very simple relation (7), and by extending the same consideration also to f_2 and f_3, we have, according to (7), the whole system:

$$\left.\begin{aligned} [a\,a]\, q_1 + [a\,b]\, q_2 + [a\,c]\, q_3 &= f_1 \\ [a\,b]\, q_1 + [b\,b]\, q_2 + [b\,c]\, q_3 &= f_2 \\ [a\,c]\, q_1 + [b\,c]\, q_2 + [c\,c]\, q_3 &= f_3 \end{aligned}\right\} . \tag{9}$$

These are equations of the same form as the original normal equations; therefore, we can also treat them further just like those:

$$\left.\begin{aligned} \underline{[a\,a]\, q_1} + [a\,b]\, q_2 \quad + [a\,c]\, q_3 \quad - f_1 &= 0 \\ \underline{[b\,b]\, q_2} \quad + [b\,c]\, q_3 \quad - f_2 &= 0 \\ \underline{[c\,c]\, q_3} \quad - f_3 &= 0 \end{aligned}\right\} , \tag{10a}$$

$$\left.\begin{aligned} \underline{[b\,b\cdot 1]\, q_2} + [b\,c\cdot 1]\, q_3 - [f_2\cdot 1] &= 0 \\ \underline{[c\,c\cdot 1]\, q_3} - [f_3\cdot 1] &= 0 \end{aligned}\right\} , \tag{10b}$$

$$\underline{[c\,c\cdot 2]\, q_3} - [f_3\cdot 2] = 0 . \tag{10c}$$

The end terms have the following meanings:

$$\left.\begin{aligned} [f_2\cdot 1] = f_2 - \frac{[a\,b]}{[a\,a]} f_1 \qquad & [f_3\cdot 1] = f_3 - \frac{[a\,c]}{[a\,a]} f_1 \\ & [f_3\cdot 2] = [f_3\cdot 1] - \frac{[b\,c\cdot 1]}{[b\,b\cdot 1]} [f_2\cdot 1] \end{aligned}\right\} . \tag{11}$$

We can annex these end terms to the usual elimination (2*) to (5), section 30, p. 94.

If we imagine the q_1's, q_2's, q_3's from equations (9) and (10) numerically determined and introduced in (5), then we have the value $\frac{1}{P}$ sought for.

But still a simpler method is suited here: Let us set the f_1's, f_2's, f_3's determined through (9) into

(5), and obtain with this

$$\left.\begin{aligned}\frac{1}{P} = [a\,a]\,q_1^2 + 2\,[a\,b]\,q_1\,q_2 + 2\,[a\,c]\,q_1\,q_3 \\ + \quad [b\,b]\,q_2^2 \;\; + 2\,[b\,c]\,q_2\,q_3 \\ + \quad [c\,c]\,q_3^2\end{aligned}\right\} . \tag{12}$$

We can apply to this expression (12) the general development (20), section 29, p. 93, namely

$$\frac{1}{P} = \frac{([a\,a]\,q_1 + [a\,b]\,q_2 + [a\,c]\,q_3)^2}{[a\,a]} + \frac{([b\,b\cdot 1]\,q_2 + [b\,c\cdot 1]\,q_3)^2}{[b\,b\cdot 1]} + \frac{([c\,c\cdot 2]\,q_3)^2}{[c\,c\cdot 2]} ,$$

i.e. with the insertion of (10a), (10b), (10c) this yields

$$\frac{1}{P} = \frac{f_1^{\,2}}{[a\,a]} + \frac{[f_2\cdot 1]^2}{[b\,b\cdot 1]} + \frac{[f_3\cdot 2]^2}{[c\,c\cdot 2]} . \tag{13}$$

Theoretically, this is the most readily conceivable formula for the weight of a function.

It will depend on the circumstances whether, in the individual case, we will compute numerically according to this, i.e., according to (10a), (10b), (10c), and (13), or whether we will use the original formulae (3) or (5).

Section 32. Weight of a Function of Functions

A consideration, further deviating and therefore, at first, to be passed over, refers further to the weight of a function of functions of the adjusted x's, y's, z's.

Let us have two functions:

$$X = f_1 x + f_2 y + f_3 z , \qquad Y = f_1' x + f_2' y + f_3' z . \tag{1}$$

These two functions are supposed to have been treated according to the previous section 31 (13), and have obtained the following weights:

$$\frac{1}{P_x} = \frac{f_1^2}{[aa]} + \frac{[f_2\cdot 1]^2}{[b\,b\cdot 1]} + \frac{[f_3\cdot 2]^2}{[c\,c\cdot 2]} , \qquad \frac{1}{P_y} = \frac{f_1'^2}{[a\,a]} + \frac{[f_2'\cdot 1]^2}{[b\,b\cdot 1]} + \frac{[f_3'\cdot 2]^2}{[c\,c\cdot 2]} . \tag{2}$$

Now we set up further a function of X and of Y:

$$(F) = r\,X + r'\,Y , \tag{3}$$

whose weight shall likewise be determined.

For this, we have in any case the way of expressing (F) in terms of x and y according to (1) and (3) and producing a function of x and y, namely

$$(F) = (rf_1 + r'f_1')\,x + (rf_2 + r'f_2')\,y + (rf_3 + r'f_3')\,z . \tag{4}$$

The weight of this function of x, y, and z is determined by

$$\frac{1}{(P)} = \frac{(r\,f_1 + r'\,f_1')^2}{[a\,a]} + \frac{[(r\,f_2 + r'\,f_2')\cdot 1]^2}{[b\,b\cdot 1]} + \frac{[(r\,f_3 + r'\,f_3')\cdot 2]^2}{[c\,c\cdot 2]} . \tag{5}$$

Besides this way which presents itself directly for the computation of the weight (P) of function (3), there is however still a second way by means of the weights P_x and P_y, which we already have according to (2).

We consider the constituent parts of (2) and (5) according to the original law (11), section 31, p. 101:

$$f_1 = f_1 \qquad\qquad f_1' = f_1'$$

$$[f_2 \cdot 1] = f_2 - \frac{[ab]}{[aa]} f_1 \qquad\qquad [f_2' \cdot 1] = f_2' - \frac{[ab]}{[aa]} f_1'$$

$$[f_3 \cdot 2] = f_3 - \frac{[ac]}{[aa]} f_1 - \frac{[bc \cdot 1]}{[bb \cdot 1]} [f_2 \cdot 1], \quad [f_3' \cdot 2] = f_3' - \frac{[ac]}{[aa]} f_1' - \frac{[bc \cdot 1]}{[bb \cdot 1]} [f_2' \cdot 1]$$

$$r f_1 + r' f_1' = r f_1 + r' f_1'$$

$$[(r f_2 + r' f_2') \cdot 1] = (r f_2 + r' f_2') - \frac{[ab]}{[aa]} (r f_1 + r' f_1')$$

$$[(r f_3 + r' f_3') \cdot 2] = (r f_3 + r' f_3') - \frac{[ac]}{[aa]} (r f_1 + r' f_1') - \frac{[bc \cdot 1]}{[bb \cdot 1]} [(r f_2 + r' f_2') \cdot 1] .$$

We thus have the very simple relations

$$\left.\begin{aligned} r f_1 + r' f_1' &= r f_1 + r' f_1' \\ [(r f_2 + r' f_2') \cdot 1] &= r [f_2 \cdot 1] + r' [f_2' \cdot 1] \\ [(r f_3 + r' f_3') \cdot 2] &= r [f_3 \cdot 2] + r' [f_3' \cdot 2] \end{aligned}\right\} . \tag{6}$$

If we set this into (5) and square it, then we find

$$\left.\begin{aligned} \frac{1}{(P)} = & \, r^2 \left\{ \frac{f_1^2}{[aa]} + \frac{[f_2 \cdot 1]^2}{[bb \cdot 1]} + \frac{[f_3 \cdot 2]^2}{[cc \cdot 2]} \right\} \\ & + r'^2 \left\{ \frac{f_1'^2}{[aa]} + \frac{[f_2' \cdot 1]^2}{[bb \cdot 1]} + \frac{[f_3' \cdot 2]^2}{[cc \cdot 2]} \right\} \\ & + 2 r r' \left\{ \frac{f_1 f_1'}{[aa]} + \frac{[f_2 \cdot 1][f_2' \cdot 1]}{[bb \cdot 1]} + \frac{[f_3 \cdot 2][f_3' \cdot 2]}{[cc \cdot 2]} \right\} \end{aligned}\right\} \tag{7}$$

or

$$\frac{1}{(P)} = r^2 \frac{1}{P_x} + r'^2 \frac{1}{P_y} + 2 r r' \frac{1}{P_{xy}} . \tag{8}$$

Here $\frac{1}{P_x}$ and $\frac{1}{P_y}$ have the meanings already indicated in (2), and $\frac{1}{P_{xy}}$ can be easily derived from (7), above:

$$\frac{1}{P_{xy}} = \frac{f_1 f_1'}{[aa]} + \frac{[f_2 \cdot 1][f_2' \cdot 1]}{[bb \cdot 1]} + \frac{[f_3 \cdot 2][f_3' \cdot 2]}{[cc \cdot 2]} . \tag{9}$$

In words, this can be expressed as follows:

If for two functions X and Y according to (1) the weights are determined in formulae (2), and if, then, a function (F) of functions X and Y is involved again, then we need not to compute the weight (P) of (F) anew, but we can derive it, according to (9) and (8), from the weights P_x and P_y already computed.

Section 33. Partial Elimination

The theory of partial elimination, also, is not an essential component of our course of development; partial elimination, however, may be of advantage later.

First we will in part repeat and summarize the developments of section 28:

Error equations

$$\text{Number} = n \left\{ \begin{array}{l} v_1 = a_1 x + b_1 y + c_1 z - l_1 \\ v_2 = a_2 x + b_2 y + c_2 z - l_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n = \underbrace{a_n x + b_n y + c_n z}_{\text{Number} = u} - l_n \end{array} \right\} \tag{1}$$

Normal equations

$$\text{Number} = u \left\{ \begin{array}{r} \underline{[a\,a]}\,x + [a\,b]\,y + [a\,c]\,z - [a\,l] = 0 \\ \underline{[b\,b]}\,y + [b\,c]\,z - [b\,l] = 0 \\ \underline{[c\,c]}\,z - [c\,l] = 0 \\ \underline{[l\,l]} \end{array} \right\} . \tag{2}$$

Normal equations reduced for the first time

$$\begin{array}{l} \text{Number} \\ = u - 1 \end{array} \left\{ \begin{array}{r} \underline{[bb\cdot 1]}\,y + [b\,c\cdot 1]\,z - [b\,l\cdot 1] = 0 \\ \underline{[c\,c\cdot 1]}\,z - [c\,l\cdot 1] = 0 \\ \underline{[l\,l\cdot 1]} \end{array} \quad \text{or} \quad \begin{array}{r} \underline{[b'\,b']}\,y + [b'\,c']\,z - [b'\,l'] = 0 \\ \underline{[c'\,c']}\,z - [c'\,l'] = 0 \\ \underline{[l'\,l']} \end{array} \right\} . \tag{3}$$

Here

$$b' = b - \frac{[a\,b]}{[a\,a]}\,a \qquad c' = c - \frac{[a\,c]}{[a\,a]}\,a \qquad -l' = -l + \frac{[a\,l]}{[a\,a]}\,a\,. \tag{4}$$

We can reach system (3) also by writing n reduced (fictitious) error equations:

Reduced error equations

$$\text{Number} = n \left\{ \begin{array}{l} v_1 = b_1'\,y + c_1'\,z - l_1' \\ v_2 = b_2'\,y + c_2'\,z - l_2' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n = \underbrace{b_n'\,y + c_n'\,z}_{\text{Number} = u - 1} - l_n' \end{array} \right\} \tag{5}$$

All this expressed in words gives the theorem:

If we write down, instead of the n error equations (1), the n reduced error equations (5), and form from them in the usual manner, with equal weights as in (1), normal equations (3), then we obtain hence the same unknowns y and z, and also all weights and mean errors just like from the original error equations (1). In the computation of the mean square of error

$$m^2 = \frac{[v\,v]}{n - u} \tag{6}$$

we must however include in counting the first eliminated x in the denominator $n - u$ in the number u of the unknowns, or we must in (6) not take, for instance, the denominator $= n - (u - 1)$, according to (5).

From the above, we can also derive, in addition, the following generally valid theorem:

If we already eliminate an unknown x in the error equations with the help of relations (4) and with this, set up reduced error equations (5), then we obtain hence *the same* system of normal equations (3), which we would have obtained in the usual way by elimination of x from the normal equations.

We consider further the following case:

$$\begin{array}{ll} \textit{Error equations} & \textit{Normal equations} \\ v_1 = c_1 z + a_1 x + b_1 y - l_1 & \underline{[c\,c]}\, z + [c\,a]\, x + [c\,b]\, y - [c\,l] = 0 \\ v_2 = c_2 z + a_2 x + b_2 y - l_2 & \underline{[a\,a]}\, x + [a\,b]\, y - [a\,l] = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \underline{[b\,b]}\, y - [b\,l] = 0 \\ v_n = c_n z + a_n x + b_n y - l_n & \underline{[l\,l]} \end{array} \qquad (7)$$

Let a second group of observations give:

$$\begin{array}{ll} v_{n+1} = d_{n+1} t + a_{n+1} x + b_{n+1} y - l_{n+1} & \underline{(d\,d)}\, t + (d\,a)\, x + (d\,b)\, y - (d\,l) = 0 \\ v_{n+2} = d_{n+2} t + a_{n+2} x + b_{n+2} y - l_{n+2} & \underline{(a\,a)}\, x + (a\,b)\, y - (a\,l) = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \underline{(b\,b)}\, y - (b\,l) = 0 \\ & \underline{(l\,l)} \end{array} \qquad (8)$$

The unknowns x and y occur in both groups, z however only in the first, and t only in the second group.

Now z is to be eliminated from the normal equations of (7), and t from the normal equations of (8). This yields

$$\text{from (7):} \quad [a\,a \cdot 1]\, x + [a\,b \cdot 1]\, y - [a\,l \cdot 1] = 0 \qquad (9)$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\text{from (8):} \quad (a\,a \cdot 1)\, x + (a\,b \cdot 1)\, y - (a\,l \cdot 1) = 0 \qquad (10)$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

We take these two equations together as follows:

$$([a\,a \cdot 1] + (a\,a \cdot 1))\, x + ([a\,b \cdot 1] + (a\,b \cdot 1))\, y - ([a\,l \cdot 1] + (a\,l \cdot 1)) = 0 \qquad (11)$$
$$\cdot\ \cdot$$

This is exactly the same first normal equation which we would have obtained if we had formed normal equations *assembled* from all the error equations of (7) and (8) and from this eliminated z and t.

In order to prove this, we form immediately the total system of normal equations for the error equations (7) and (8), above:

$$\begin{array}{l} \underline{[c\,c]}\, z\ .\ .\ .\ + [c\,a]\, x \qquad + [c\,b]\, y \qquad - [c\,l] \qquad = 0 \\ \qquad \underline{(d\,d)}\, t + (d\,a)\, x \qquad + (d\,b)\, y \qquad - (d\,l) \qquad = 0 \\ \qquad\qquad ([a\,a] + (a\,a))\, x + ([a\,b] + (a\,b))\, y - ([a\,l] + (a\,l)) = 0 \\ \qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

reduced for the first time:

$$\left((d\,d) - \frac{0}{[c\,c]}\, 0\right) t + \left((d\,a) - \frac{0}{[c\,c]}\, [c\,a]\right) x + \ldots$$
$$\left([a\,a] + (a\,a) - \frac{[c\,a]}{[c\,c]}\, [c\,a]\right) x + \ldots$$

reduced once more:

$$\left([a\,a] + (a\,a) - \frac{[c\,a]}{[c\,c]}\, [c\,a] - \frac{(d\,a)}{(d\,d)}\, (d\,a)\right) x + \ldots\ . \qquad (12)$$

The coefficient of x in (11) contains in fact the same components as the coefficient of x in (12), and since it is similar to all other coefficients, then, with this, the theorem expressed in (11) has been proved.

Now we can also go one step further and come to the following conclusion by means of the theorem expressed in (5) about reduced error equations:

If individual unknowns z and t, as in (7) and (8), occur only in a *part* of the error equations, then, for these partial groups of error equations we can form reduced error equations of the form (5) and then compute further with the totality of all reduced error equations as if they were original error equations.

In the computation of the mean error, however, the unknowns (z and t) eliminated at the beginning are to be included in the number u of all unknowns.

Equivalent observations

If two systems of error equations lead to the same normal equations, then we obtain from both systems the same values for the unknowns as well as for their weights; the two systems are therefore completely equivalent.

We consider now the two systems of error equations

$$\left.\begin{array}{ll} v_1 = a_1 x + b_1 y + c_1 z - l_1 & \text{weight} = 1 \\ v_2 = a_2 x + b_2 y + c_2 z - l_2 & \text{weight} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ v_n = a_n x + b_n y + c_n z - l_n & \text{weight} = 1 \end{array}\right\} \tag{13}$$

and

$$\left.\begin{array}{ll} \lambda_1 = x + \frac{[a\,b]}{[a\,a]} y + \frac{[a\,c]}{[a\,a]} z - u_1 & \text{weight } [a\,a] \\ \lambda_2 = \qquad\quad y + \frac{[b\,c\cdot 1]}{[b\,b\cdot 1]} z - u_2 & \text{weight } [b\,b\cdot 1] \\ \lambda_3 = \qquad\qquad\qquad\quad z - u_3 & \text{weight } [c\,c\cdot 2] \end{array}\right\}, \tag{14}$$

where

$$u_1 = \frac{[a\,l]}{[a\,a]}, \qquad u_2 = \frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}, \qquad u_3 = \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]} . \tag{15}$$

Although, in system (14), there exist now only three error equations, which are just sufficient for the determination of the three unknowns, we can likewise set up, from them, normal equations and obtain then as the first normal equation

$$[a\,a]\,x + [a\,b]\,y + [a\,c]\,z - [a\,l] = 0 . \tag{16}$$

For the second normal equation there follows first

$$[a\,b]\,x + \left(\frac{[a\,b]^2}{[a\,a]} + [b\,b\cdot 1]\right) y + \left(\frac{[a\,b]\,[a\,c]}{[a\,a]} + [b\,c\cdot 1]\right) z - \left(\frac{[a\,b]\,[a\,l]}{[a\,a]} + [b\,l\cdot 1]\right) = 0 .$$

If, however, we consider herein the meaning of the coefficients $[b\,b\cdot 1]$, $[b\,c\cdot 1]$, and $[b\,l\cdot 1]$ according to (3**), section 30, p. 94, then the equation changes into

$$[a\,b]\,x + [b\,b]\,y + [b\,c]\,z - [b\,l] = 0 . \tag{17}$$

Finally, the third normal equation has the following form, if we now take over the coefficients of x and y previously obtained from (16) and (17),

$$[ac]\,x + [bc]\,y + \left(\frac{[ac]^2}{[aa]} + \frac{[bc\cdot 1]^2}{[bb\cdot 1]} + [cc\cdot 2]\right) z$$
$$- \left(\frac{[ac]\,[al]}{[aa]} + \frac{[bc\cdot 1]\,[bl\cdot 1]}{[bb\cdot 1]} + [cl\cdot 2]\right) = 0\,.$$

If we make use again of the summary (3**) and (4**) in section 30, pp. 94 and 95, then this equation can easily be transformed into

$$[ac]\,x + [bc]\,y + [cc]\,z - [cl] = 0\,. \tag{18}$$

We thus obtain the same normal equations from the error equations (14) as from the error equations (13). Therefore, the fictitious error equations (14) are equivalent to the original error equations (13).

We can use the error equations (14), in which the quantities u_1, u_2, u_3 to be regarded as independent observations, for any further adjustments. If, for instance, new error equations exist, then equations (14) can be submitted, together with them, to a new adjustment, which substitutes a total adjustment of the new error equations and the original error equations.

On the other hand, we can also easily express the unknowns x, y, z by the independent observations u_1, u_2, u_3 and their corrections. We obtain from (14) the expressions

$$\left.\begin{aligned} x &= u_1 + \lambda_1 - \frac{[ab]}{[aa]}(u_2 + \lambda_2) - \left(\frac{[ac]}{[aa]} - \frac{[ab]}{[aa]}\frac{[bc\cdot 1]}{[bb\cdot 1]}\right)(u_3 + \lambda_3) \\ y &= u_2 + \lambda_2 - \frac{[bc\cdot 1]}{[bb\cdot 1]}(u_1 + \lambda_1) \\ z &= u_3 + \lambda_3 \end{aligned}\right\}, \tag{19}$$

to the importance of which we shall refer later.

Section 34. Elimination According to Schreiber

We will now carry out the elimination of the unknowns once more in another way by pursuing a method indicated by General Schreiber. This method consists in setting up, instead of the given error equations, fictitious error equations with definite weights. We will limit ourselves here to three unknowns for which n observations exist.

Let the error equations be

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y + c_1 z - l_1 \\ v_2 &= a_2 x + b_2 y + c_2 z - l_2 \\ &\cdots\cdots\cdots \\ v_n &= a_n x + b_n y + c_n z - l_n \end{aligned}\right\}, \tag{1}$$

from which we obtain as the first normal equation

$$0 = [aa]\,x + [ab]\,y + [ac]\,z - [al]\,. \tag{2}$$

We form now a new error equation

$$v' = [ab]\,y + [ac]\,z - [al] \tag{3}$$

with the weight $-\dfrac{1}{[aa]}$. We combine this equation with the error equations (1), after the term in x is omitted in the latter. We thus have the new system

$$\left.\begin{array}{ll} v_1 = b_1 y + c_1 z - l_1 & \text{weight} = 1 \\ v_2 = b_2 y + c_2 z - l_2 & \text{weight} = 1 \\ \cdots & \cdots \\ v_n = b_n y + c_n z - l_n & \text{weight} = 1 \\ v' = [a b] y + [a c] z - [a l] & \text{weight} = -\dfrac{1}{[a a]} \end{array}\right\} \tag{4}$$

It is evident at once that these error equations (4) completely replace the original ones with respect to the unknowns y and z, if we form from them the two normal equations. These read

$$\left.\begin{array}{l} \left([b b]-\dfrac{[a b]\,[a b]}{[a a]}\right) y+\left([b c]-\dfrac{[a b]\,[a c]}{[a a]}\right) z-\left([b l]-\dfrac{[a b]\,[a l]}{[a a]}\right)=0 \\ \left([b c]-\dfrac{[a b]\,[a c]}{[a a]}\right) y+\left([c c]-\dfrac{[a c]\,[a c]}{[a a]}\right) z-\left([c l]-\dfrac{[a c]\,[a l]}{[a a]}\right)=0 \end{array}\right\} \tag{4a}$$

or with the use of the Gaussian notation

$$\begin{array}{l} [b b \cdot 1]\, y + [b c \cdot 1]\, z - [b l \cdot 1] = 0 \\ [b c \cdot 1]\, y + [c c \cdot 1]\, z - [c l \cdot 1] = 0\,. \end{array} \tag{5}$$

The fictitious error equations (4) thus lead directly to the normal equations reduced once, which we would have obtained from the original error equations (1).

We go one step further and form again, from the first normal equation (5), a new error equation, which we take together with equations (4) after omitting the terms in y. The new system is then

$$\left.\begin{array}{ll} v_1 = c_1 z - l_1 & \text{weight} = 1 \\ v_2 = c_2 z - l_2 & \text{weight} = 1 \\ \cdots & \cdots \\ v_n = c_n z - l_n & \text{weight} = 1 \\ v' = [a c] z - [a l] & \text{weight} = -\dfrac{1}{[a a]} \\ v'' = [b c \cdot 1] z - [b l \cdot 1] & \text{weight} = -\dfrac{1}{[b b \cdot 1]} \end{array}\right\}, \tag{6}$$

and these error equations are equivalent to the original equations (1) with respect to the unknown z. For from the error equations (6) there follows the normal equation

$$\left([c c]-\frac{[a c]^2}{[a a]}-\frac{[b c \cdot 1]^2}{[b b \cdot 1]}\right) z-\left([c l]-\frac{[a c]\,[a l]}{[a a]}-\frac{[b c \cdot 1]\,[b l \cdot 1]}{[b b \cdot 1]}\right)=0, \tag{6a}$$

for which we can also write

$$\left([c c \cdot 1]-\frac{[b c \cdot 1]^2}{[b b \cdot 1]}\right) z-\left([c l \cdot 1]-\frac{[b c \cdot 1]\,[b l \cdot 1]}{[b b \cdot 1]}\right)=0,$$

or also

$$[c c \cdot 2]\, z - [c l \cdot 2] = 0\,. \tag{7}$$

It is, besides, easy to understand that with this method of the elimination of the unknowns, nothing is saved in computing work compared to the Gauss method. All individual terms, which must be computed in the setting up and reducing of the normal equations, occur also in the foregoing treatment of the error equations. Nevertheless there may occur cases in which the method is advantageous.

We will use equation (7) once more for the setting up of a fictitious error equation and form a new system, in which also the unknown z is omitted. We have then

$$\left.\begin{array}{ll} v_1 = -l_1 & \text{weight} = 1 \\ v_2 = -l_2 & \text{weight} = 1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ v_n = -l_n & \text{weight} = 1 \\ v' = -[a\,l] & \text{weight} = -\dfrac{1}{[a\,a]} \\ v'' = -[b\,l\cdot 1] & \text{weight} = -\dfrac{1}{[b\,b\cdot 1]} \\ v''' = -[c\,l\cdot 2] & \text{weight} = -\dfrac{1}{[c\,c\cdot 2]} \end{array}\right\}, \tag{8}$$

from which we will form the sum of the squares of the corrections. This reads

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \frac{[b\,l\cdot 1]^2}{[b\,b\cdot 1]} - \frac{[c\,l\cdot 2]^2}{[c\,c\cdot 2]}, \tag{9}$$

in agreement with equation (8), section 29, p. 91.

The result is therefore that the systems of the fictitious error equations are equivalent to the original error equations with regard to the determination of the unknowns and the sum of the squares $[v\,v]$, but not with regard to the individual corrections v, which cannot be computed from system (4), (6), or (8). In these three systems of equations, the quantities v are only computational quantities, which receive other values in every individual system.

Section 35. Formation of End Equations Without Intermediate Terms

The method of the preceding section 34 amounts to setting up directly, from the error equations, the first normal equation as well as the first equation of each reduced system of normal equations, hence, the total end equations. Even though nothing is saved here in computing work compared with the gradual elimination according to section 27, equations (4) to (7), p. 86, as we have already mentioned, we avoid the writing down of the many intermediate terms. With some practice, we will also not have to set up, for this, the fictitious systems of error equations (4) and (6) of the previous section 34, but only compute the different supplemental amounts $-\dfrac{[a\,b]\,[a\,b]}{[a\,a]}$, $-\dfrac{[b\,c\cdot 1]\,[b\,c\cdot 1]}{[b\,b\cdot 1]}$, and so on, of the first equation of (4a) and the equation (6a), which are to be added to the coefficients of the first normal equation.

According to this, the following scheme is formed for four unknowns:

A	$[a\,a]$	$[a\,b]$	$[a\,c]$	$[a\,d]$	$-[a\,l]$
a_2		$[b\,b]$	$[b\,c]$	$[b\,d]$	$-[b\,l]$
α_1		$-\frac{[a\,b]}{[a\,a]}[a\,b]$	$-\frac{[a\,b]}{[a\,a]}[a\,c]$	$-\frac{[a\,b]}{[a\,a]}[a\,d]$	$+\frac{[a\,b]}{[a\,a]}[a\,l]$
B'		$[b\,b\cdot 1]$	$[b\,c\cdot 1]$	$[b\,d\cdot 1]$	$-[b\,l\cdot 1]$
a_3			$[c\,c]$	$[c\,d]$	$-[c\,l]$
α_2			$-\frac{[a\,c]}{[a\,a]}[a\,c]$	$-\frac{[a\,c]}{[a\,a]}[a\,d]$	$+\frac{[a\,c]}{[a\,a]}[a\,l]$
β_1			$-\frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}[b\,c\cdot 1]$	$-\frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}[b\,d\cdot 1]$	$+\frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}[b\,l\cdot 1]$
C''			$[c\,c\cdot 2]$	$[c\,d\cdot 2]$	$-[c\,l\cdot 2]$
a_4				$[d\,d]$	$-[d\,l]$
α_3				$-\frac{[a\,d]}{[a\,a]}[a\,d]$	$+\frac{[a\,d]}{[a\,a]}[a\,l]$
β_2				$-\frac{[b\,d\cdot 1]}{[b\,b\cdot 1]}[b\,d\cdot 1]$	$+\frac{[b\,d\cdot 1]}{[b\,b\cdot 1]}[b\,l\cdot 1]$
γ_1				$-\frac{[c\,d\cdot 2]}{[c\,c\cdot 2]}[c\,d\cdot 2]$	$+\frac{[c\,d\cdot 2]}{[c\,c\cdot 2]}[c\,l\cdot 2]$
D'''				$[d\,d\cdot 3]$	$-[d\,l\cdot 3]$.

In the following, below, the numerical example of section 27, p. 87, of four equations besides sum checks and terms of sums of errors $[l\,l]$, etc., is treated according to this scheme once more, and, in fact, all amounts $\frac{[a\,b]}{[a\,a]}[a\,b]$, etc., are computed with the slide rule, so that not a number more than is inserted here has to be written.

The numbers between horizontal lines give the system of end equations (10), section 27, p. 87.

Whether by means of the above elimination arrangement a computational gain in comparison to section 27, p. 87, is achieved, depends, especially, on the number of unknowns. With a few unknowns it is not the case, but with numerous unknowns this arrangement is useful, if the additions are conveniently arranged with changing signs.

This arrangement is in use (with logarithms) at the Trigonometric Section of the Land Survey, but with subtraction in the form of decadic supplements throughout, so that the end of the following table is written thus:

+ 1129	× 8313
× 9440	554
× 9510	902
× 9998	17
× 9934	203
40011	× 9989
= 11	= − 11

The small oblique cross × is here the sign for decadic supplement; for instance: × 8313 = 8313 − 10,000 = − 1687.

	x a	y b	z c	t d	$-l$	s	Check
A	+ 459	− 308	− 389	+ 244	− 507	+ 501	
		+ 464	+ 408	− 269	+ 695	− 990	
		− 208	− 262	+ 164	− 341	+ 337	
B'		+ 256	+ 146	− 105	+ 354	− 653	− 2
			+ 676	− 331	+ 653	− 1017	
			− 330	+ 207	− 430	+ 425	
			− 83	+ 60	− 202	+ 373	
C''			+ 263	− 64	+ 21	− 219	+ 1
				+ 469	− 283	+ 170	
				− 130	+ 270	− 267	
				− 43	+ 145	− 268	
				− 15	+ 5	− 53	
D'''				+ 281	+ 137	− 418	0
					+ 1129	− 1687	
					− 560	+ 554	
					− 490	+ 902	
					− 2	+ 17	
					− 66	+ 203	
					+ 11	− 11	0

$$t = \frac{-137}{+281} = 0.486$$

(Note. The above elimination, just as on p. 87, is done only with an ordinary slide rule; therefore, the last place is not rigorous everywhere.)

Section 36. Simultaneous Determination of All Unknowns $x, y, z \ldots$ and the Weight Coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc.

In continuation to the tabulation on p. 110, we have formed a further method, not necessary for ordinary purposes, by which *all* unknowns $x, y, z \ldots$ are obtained at *one* time.

We take up the end equations (9), section 27, p. 87, once again:

End equations

$$\left.\begin{array}{lllllll} A &= [a\,a]\,x + [a\,b]\,y &+ [a\,c]\,z &+ [a\,d]\,t &- [a\,l] &= 0 \\ B' &= & [b\,b\cdot 1]\,y + [b\,c\cdot 1]\,z &+ [b\,d\cdot 1]\,t &- [b\,l\cdot 1] &= 0 \\ C' &= & & [c\,c\cdot 2]\,z + [c\,d\cdot 2]\,t &- [c\,l\cdot 2] &= 0 \\ D''' &= & & & [d\,d\cdot 3]\,t - [d\,l\cdot 3] &= 0 \\ & & & & [l\,l\cdot 4] &= [v\,v] \end{array}\right\}. \tag{1}$$

The fourth equation determines t, and if we insert this back into the third equation, then we have also z, and so on, from the second and first equation also y and x. This is a method which is often used numerically.

But now we can carry out the backward insertion in question also generally algebraically, upon which a readily conceivable scheme for numerical computation can be based.

If we introduce, in addition to the brackets $[b\,b\cdot 1]$, etc., already occurring in (1), a few other coefficients, constructed similarly, which we will distinguish, in the following, from the previous ones by parentheses $(a\,c\cdot 1)$, etc., then we obtain the following system of equations, of whose correctness we best convince ourselves in a backward manner by setting the expressions (3) into (2) and comparing the results with (1):

$$\left.\begin{array}{ll} [a\,a]\,x &= [a\,l] - \frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[a\,b] - \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}(a\,c\cdot 1) - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}(a\,d\cdot 2) \\ [b\,b\cdot 1]\,y &= [b\,l\cdot 1] - \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}[b\,c\cdot 1] - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}(b\,d\cdot 2) \\ [c\,c\cdot 2]\,z &= [c\,l\cdot 2] - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}[c\,d\cdot 2] \\ [d\,d\cdot 3]\,t &= [d\,l\cdot 3] \end{array}\right\}. \tag{2}$$

The newly introduced coefficients distinguished by parentheses have the following meanings here:

$$\left.\begin{array}{ll} (a\,c\cdot 1) = & [a\,c] - \frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}[a\,b] \\ (a\,d\cdot 1) = & [a\,d] - \frac{[b\,d\cdot 1]}{[b\,b\cdot 1]}[a\,b] \\ (a\,l\cdot 1) = & [a\,l] - \frac{[b\,l\cdot 1]}{[b\,b\cdot 1]}[a\,b] \end{array}\right\}, \tag{3}$$

$$\left.\begin{array}{ll} (a\,d\cdot 2) = (a\,d\cdot 1) - \frac{[c\,d\cdot 2]}{[c\,c\cdot 2]}(a\,c\cdot 1), & (b\,d\cdot 2) = [b\,d\cdot 1] - \frac{[c\,d\cdot 2]}{[c\,c\cdot 2]}[b\,c\cdot 1] \\ (a\,l\cdot 2) = (a\,l\cdot 1) - \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}(a\,c\cdot 1), & (b\,l\cdot 2) = [b\,l\cdot 1] - \frac{[c\,l\cdot 2]}{[c\,c\cdot 2]}[b\,c\cdot 1] \end{array}\right\}. \tag{4}$$

Now we go on further and set

$$(a\,l\cdot 3) = (a\,l\cdot 2) - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}(a\,d\cdot 2),\quad (b\,l\cdot 3) = (b\,l\cdot 2) - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}(b\,d\cdot 2),\quad (c\,l\cdot 3) = [c\,l\cdot 2] - \frac{[d\,l\cdot 3]}{[d\,d\cdot 3]}[c\,d\cdot 2] \tag{5}$$

and with this, we can express x, y, z, t briefly as follows:

$$x = \frac{(a\,l \cdot 3)}{[a\,a]}, \quad y = \frac{(b\,l \cdot 3)}{[b\,b \cdot 1]}, \quad z = \frac{(c\,l \cdot 3)}{[c\,c \cdot 2]}, \quad t = \frac{[d\,l \cdot 3]}{[d\,d \cdot 3]}. \tag{6}$$

It would be of little value to set up all these formulae if there were not a simple scheme according to which they can be computed mechanically. But this is the case, as shown in the following:

$$\left.\begin{array}{ccccc|ccccc}
[a\,a]_3 & [a\,b]_0 & [a\,c]_0 & [a\,d]_0 & -[a\,l]_0 & & & & & \\
 & [b\,b] & [b\,c] & [b\,d] & -[b\,l] & & & & & \\
A & & [c\,c] & [c\,d] & -[c\,l] & & & & & \\
 & & & [d\,d] & -[d\,l] & & & & & \\
 & & & & [l\,l] & & & & & \\ \hline
 & [b\,b \cdot 1]_3 & [b\,c \cdot 1]_1 & [b\,d \cdot 1]_1 & -[b\,l \cdot 1]_1 & [a\,b]_0 & & & & \\
B & & [c\,c \cdot 1] & [c\,d \cdot 1] & -[c\,l \cdot 1] & [a\,c]_0 & B' & & & \\
 & & & [d\,d \cdot 1] & -[d\,l \cdot 1] & [a\,d]_0 & & & & \\
 & & & & [l\,l \cdot 1] & -[a\,l]_0 & & & & \\ \hline
 & & [c\,c \cdot 2]_3 & [c\,d \cdot 2]_2 & -[c\,l \cdot 2]_2 & (a\,c \cdot 1) & [b\,c \cdot 1]_1 & & & \\
 & & C & [d\,d \cdot 2] & -[d\,l \cdot 2] & (a\,d \cdot 1) & [b\,d \cdot 1]_1 & C' & & \\
 & & & & [l\,l \cdot 2] & -(a\,l \cdot 1) & -[b\,l \cdot 1]_1 & & & \\ \hline
 & & D & [d\,d \cdot 3]_3 & -[d\,l \cdot 3]_3 & (a\,d \cdot 2) & (b\,d \cdot 2) & [c\,d \cdot 2]_2 & & D' \\
 & & & & [l\,l \cdot 3] & -(a\,l \cdot 2) & -(b\,l \cdot 2) & -[c\,l \cdot 2]_2 & & \\ \hline
 & & & L & [l\,l \cdot 4] & -(a\,l \cdot 3) & -(b\,l \cdot 3) & -(c\,l \cdot 3) & -[d\,l \cdot 3]_3 & L' \\
 & & & & & [a\,a]_3 & [b\,b \cdot 1]_3 & [c\,c \cdot 2]_3 & [d\,d \cdot 3]_3 & \\ \cline{6-9}
 & & & & & x & y & z & t &
\end{array}\right\} . \tag{7}$$

Following is the course of the computation:

1. Section A is filled in with the given coefficients.
2. The whole section B is computed in the usual manner according to the rule $[b\,b \cdot 1] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b]$, and so on.
3. We fill in section B' by setting down the quantities of A denoted by $_0$.
4. We compute C and the first column of C' entirely again according to the old rule, with which also the three new quantities are connected according to the previously indicated formulae (3).
5. We complete section C' by setting down the brackets of B denoted by $_1$.
6. We form again D, in addition to the first two columns of D', according to the old rule, which holds in *common* for $[l\,l \cdot 3]$ as well as for $(a\,l \cdot 2)$, and so on [cf. (4)].
7. We set again the $[\ldots]_2$'s from C down to D'.
8. L and the upper numbers of L' are formed according to the old rule [see above (5)].
9. The lower numbers $[\ldots]_3$ are set down from A, B, C, D to L'.
10. The formation of quotients in L' yields the unknowns according to the formulae (6).

Numerical example with three unknowns

$$\left.\begin{array}{rrrr}
+\underline{17.50}\,x - 6.50\,y - & 6.50\,z - & 2.14 = 0 \\
+\underline{17.50}\,y - & 6.50\,z - & 13.96 = 0 \\
 & +\underline{20.50}\,z + & 5.40 = 0 \\
 & & +100.34 &
\end{array}\right\} . \tag{8}$$

a	b	c	$-l$				
+ 17.50	− 6.50 + 17.50 − 2.41	− 6.50 − 6.50 − 2.41 + 20.50 − 2.41	− 2.14 − 13.96 − 0.79 + 5.40 − 0.79 + 100.34 − 0.26				
	+ 15.09	− 8.91 + 18.09 − 5.26	− 14.75 + 4.61 − 8.71 + 100.08 − 14.42	− 6.50* − 6.50* − 3.84 − 2.14* − 6.35			
		+ 12.83	− 4.10 + 85.66 − 1.31	− 10.34 − 8.49 − 3.30	− 8.91* − 14.75* − 2.85		
			+ 84.35 = $[v\,v]$	− 11.79 − 17.50*	− 17.60 − 15.09*	− 4.10* − 12.83*	denominator
				+ 0.67 $= x$	+ 1.17 $= y$	+ 0.32 $= z$	

(9)

The subtrahends of the form $\frac{[a\,b]}{[a\,a]}[a\,b]$, etc., are printed here in *smaller* numbers for distinction; they are computed throughout with the slide rule.

All numbers transferred from the left side to the right side are denoted by * on the right.

Now if the point in question is further to determine also all weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc., then we have to set the values 1, 0, 0 instead of the absolute terms $[a\,l]$, $[b\,l]$, $[c\,l]$ according to (20), section 30, p. 97, etc., and to repeat the elimination in the relative parts, i.e. we obtain the following:

a	b	c				
+ 17.50	− 6.50 + 17.50 − 2.41	− 6.50 − 6.50 − 2.41 + 20.50 − 2.41	− 1.00 0.00 − 0.37 0.00 − 0.37			
	+ 15.09	− 8.91 + 18.09 − 5.26	− 0.37 − 0.37 − 0.22	− 6.50* − 6.50* − 3.84 − 1.00* − 0.16		
		+ 12.83	− 0.59	− 10.34 − 1.16 − 0.48	− 8.91* − 0.37* − 0.41	
				− 1.64 − 17.50*	− 0.78 − 15.09*	− 0.59* − 12.83
				+ 0.094 $= [\alpha\,\alpha]$	+ 0.052 $= [\alpha\,\beta]$	+ 0.046 $= [\alpha\,\gamma]$

(10)

Note to the fourth column: Instead of the absolute terms −1.00, −1.00, −1.00 we can also write −10.00, −10.00, −10.00, and then we obtain 10 times the values of $[\alpha\alpha]$, $[\alpha\beta]$, etc. Such a change of measure is often useful to obtain a more rigorous computation.

Note to the fourth column: Instead of the absolute terms −1.00, −1.00, −1.00 we can also write −10.00, −10.00, −10.00, and then we obtain 10 times the values of [αα], [αβ], etc. Such a change of measure is often useful to obtain a more rigorous computation.

b	c				
+ 15.09	− 8.91 + 18.09 − 5.26	− 1.00 0.00 − 0.59			
	+ 12.83	− 0.59	− 8.91* − 1.00* − 0.41		
			− 1.41 − 15.09	− 0.59* − 12.83	
			+ 0.093 = [β β]	+ 0.046 = [β γ]	(11)

	c			
	+ 12.83	− 1.00	− 1.00* − 12.83*	
			+ 0.078 = [γ γ]	(12)

We have continued the scheme consistently until $[\gamma\gamma]$ although the reciprocal value of $[\gamma\gamma]$, namely 12.83, is already contained in the form $[c\,c\cdot 2]$ in the computation of x, y, z in (9).

With this, we have obtained *everything* which is to be derived at all from the normal equations (8) for any adjustment purposes, namely in summary

$$x = +0.67\,,\quad y = +1.17\,,\quad z = +0.32\,,\quad [v\,v] = 84.35\,. \tag{13}$$

$$\left.\begin{array}{lll} [\alpha\,\alpha] = +0.094 & [\alpha\,\beta] = +0.052 & [\alpha\,\gamma] = +0.046 \\ & [\beta\,\beta] = +0.093 & [\beta\,\gamma] = +0.046 \\ & & [\gamma\,\gamma] = +0.078 \end{array}\right\}. \tag{14}$$

Example with six unknowns

In order to test the above described method also by a larger example, we have taken six normal equations which originate from our 2nd edition, 1877, pp. 53 to 64 (station adjustment at Kandel with 7 rays and 16 angles), namely, written in detail:

$$\begin{array}{lccccccl} & a & b & c & d & e & f & -l \\ a & +\underline{18\,x_1} & +13\,x_2 & +\;8\,x_3 & +\;8\,x_4 & +\;6\,x_5 & +5\,x_6 & -\;\;91.8 = 0 \\ b & +13\,x_1 & +\underline{29\,x_2} & +12\,x_3 & +11\,x_4 & +\;7\,x_5 & +5\,x_6 & -\;\;95.4 = 0 \\ c & +\;8\,x_1 & +12\,x_2 & +\underline{27\,x_3} & +22\,x_4 & +12\,x_5 & +5\,x_6 & -\;118.9 = 0 \\ d & +\;8\,x_1 & +11\,x_2 & +22\,x_3 & +\underline{31\,x_4} & +18\,x_5 & +5\,x_6 & -\;168.4 = 0 \\ e & +\;6\,x_1 & +\;7\,x_2 & +12\,x_3 & +18\,x_4 & +\underline{22\,x_5} & +5\,x_6 & -\;139.4 = 0 \\ f & +\;5\,x_1 & +\;5\,x_2 & +\;5\,x_3 & +\;5\,x_4 & +\;5\,x_5 & +\underline{8\,x_6} & -\;\;75.5 = 0 \\ & & & & & & & +\underline{1856.1} = [l\,l]\,. \end{array}$$

These six equations are found in the following table solved for the six unknowns x_1, x_2, x_3, x_4, x_5, x_6, and in addition, there is determined the sum of the squares term $[l\,l\cdot 6] = 499 = [v\,v]$.

The elimination below is carried out with an ordinary slide rule (only exceptionally, if in the case of the terms with l larger numbers occur, other means are taken as a help). Not *one* more number or figure is required than is written below, and the subtractions, e.g. 29.0 − 9.4 = 19.6, could all easily be made mentally.

	a	b	c	d	e	f	$-l$	$-s$	Check
a	+ 18.0	+ 13.0	+ 8.0	+ 8.0	+ 6.0	+ 5.0	− 91.8	+ 33.8	0.0
		+ 29.0	+ 12.0	+ 11.0	+ 7.0	+ 5.0	− 95.4	+ 18.4	0.0
		− 9.4	− 5.8	− 5.8	− 4.3	− 3.6	+ 66.3	− 24.4	
			+ 27.0	+ 22.0	+ 12.0	+ 5.0	− 118.9	+ 32.9	0.0
			− 3.6	− 3.6	− 2.7	− 2.2	+ 40.8	− 15.0	
				+ 31.0	+ 18.0	+ 5.0	− 168.4	+ 73.4	0.0
				− 3.6	− 2.7	− 2.2	+ 40.8	− 15.0	
					+ 22.0	+ 5.0	− 139.4	+ 69.4	0.0
					− 2.0	− 1.7	+ 30.6	− 11.3	
						+ 8.0	− 75.5	+ 42.5	0.0
						− 1.4	+ 25.5	− 9.4	
							+ 1856.1	− 1166.7	0.0
							− 468.2	+ 172.4	
b	+ 19.6	+ 6.2	+ 5.2	+ 2.7	+ 1.4	− 29.1	+ 13.0*	− 19.0	0.0
		+ 23.4	+ 18.4	+ 9.3	+ 2.8	− 78.1	+ 8.0*	+ 9.9	− 0.1
		− 2.0	− 1.6	− 0.9	− 0.4	+ 9.2	− 4.1	+ 6.1	
			+ 27.4	+ 15.3	+ 2.8	− 127.6	+ 8.0*	+ 50.4	− 0.1
			− 1.4	− 0.7	− 0.4	+ 7.7	− 3.4	+ 5.0	
				+ 20.0	+ 3.3	− 108.8	+ 6.0*	+ 52.1	− 0.1
				− 0.4	− 0.2	+ 4.0	− 1.8	+ 2.6	
					+ 6.6	− 50.0	+ 5.0*	+ 28.1	0.0
					− 0.1	+ 2.1	− 0.9	+ 1.4	
						+ 1387.9	− 91.8*	− 902.5	0.0
						− 43.2	+ 19.3	− 28.2	
c	+ 21.4	+ 16.8	+ 8.4	+ 2.4	− 68.9	+ 3.9	+ 6.2*	+ 9.8	− 0.1
		+ 26.0	+ 14.6	+ 2.4	− 119.9	+ 4.6	+ 5.2*	+ 50.2	− 0.1
		− 13.2	− 6.6	− 1.9	+ 54.1	− 3.1	− 4.9	− 7.7	
			+ 19.6	+ 3.1	− 104.8	+ 4.2	+ 2.7*	+ 52.0	− 0.2
			− 3.3	− 0.9	+ 27.0	− 1.5	− 2.4	− 3.8	
				+ 6.5	− 47.9	+ 4.1	+ 1.4*	+ 28.1	+ 0.1
				− 0.3	+ 7.7	− 0.4	− 0.7	− 1.1	
					+ 1344.7	− 72.5	− 29.1*	− 901.6	0.0
					− 221.8	+ 12.5	+ 20.0	+ 31.6	
d	+ 12.8	+ 8.0	+ 0.5	− 65.8	+ 1.5	+ 0.3	+ 16.8*	+ 25.7	− 0.2
		+ 16.3	+ 2.2	− 77.8	+ 2.7	+ 0.3	+ 8.4*	+ 39.8	− 0.1
		− 5.0	− 0.3	+ 41.2	− 0.9	− 0.2	− 10.5	− 16.1	
			+ 6.2	− 40.2	+ 3.7	+ 0.7	+ 2.4*	+ 24.6	+ 0.1
			− 0.0	+ 2.6	− 0.1	− 0.0	− 0.7	− 1.0	
				+ 1122.9	− 60.0	− 9.1	− 68.9*	− 801.1	0.0
				− 338.3	+ 7.7	+ 1.5	+ 86.4	+ 133.0	
e	+ 11.3	+ 1.9	− 36.6	+ 1.8	+ 0.1	− 2.1	+ 8.0*	+ 15.8	+ 0.2
		+ 6.2	− 37.6	+ 3.6	+ 0.7	+ 1.7	+ 0.5*	+ 23.2	+ 0.2
		− 0.3	+ 6.2	− 0.3	− 0.0	+ 0.4	− 1.3	− 2.6	
			+ 784.6	− 52.3	− 7.6	+ 17.5	− 65.8*	− 602.3	− 0.1
			− 118.5	+ 5.8	+ 0.3	− 6.8	+ 25.9	+ 50.5	
f	+ 5.9	− 31.4	+ 3.3	+ 0.7	+ 2.1	− 0.8	+ 1.9*	+ 18.7	+ 0.4
		+ 666.1	− 46.5	− 7.3	+ 10.7	− 39.9	− 36.6*	− 515.2	− 0.1
		− 167.1	+ 17.5	+ 3.7	+ 11.2	− 4.2	+ 10.1	+ 97.0	
$-l$	+ 499.0	− 29.0	− 3.6	+ 21.9	− 44.1	− 26.5	− 31.4*	− 386.8	− 0.5
	= [ll·6]	+ 18.0	+ 19.6*	+ 21.4*	+ 12.8*	+ 11.3*	+ 5.9*		
		+ 1.6	+ 0.2	− 1.0	+ 3.4	+ 2.3	+ 5.3		
		$= x_1$	$= x_2$	$= x_3$	$= x_4$	$= x_5$	$= x_6$		

Each individual line is also insured by a sum check, whose correct outcome is set out in the outermost column to the right. The term with s itself at the end of each line to the right must however be inserted by a small auxiliary computation, where each time the term denoted by *, set down from the left above, is to be taken in the negative sense to the preceding sum term; e.g. -19.0 results from

$$(+18.4 - 24.4) - 13.0^* = -6.0 - 13.0 = -19.0$$

or

$$(+32.9 - 15.0) - 8.0^* = 17.9 - 8.0 = +9.9\,,$$

and only *then* there comes the usual check of the sum total of the digits

$$+6.2 + 23.4 + 18.4 + 9.3 + 2.8 - 78.1 + 8.0 + 9.9 = -0.1\,.$$

This -0.1, which should be 0.0, is set out in the last column to the right in order to show how the line has come out. These terms of the outcome of the checks increase more and more downward, finally until 0.5, which is to be explained by the gradual accumulation of the unavoidable uncertainties of rounding off. Besides, in the computation itself we can always cast a glance at the check terms and add a part of them to the term s, which has happened in our case and is to be considered in the eventual recalculation, since the total check has thereby been improved somewhat. Whoever wants to apply such a scheme as on p. 115 must, obviously, follow it slowly with the slide rule by hand, and only when he has practiced the whole procedure mechanically, take up a new example of his own.

Section 37. Approximation Methods for the Solution of Normal Equations

If the number of normal equations is very large, then, in some cases, instead of the Gaussian method of elimination we will use an approximation method for the determination of the unknowns. The different methods, which have been indicated for this, start from the assumption that the quadratic terms of the normal equations outweigh the remaining ones in size; in this case, the successive approximation leads quickly to the values of the unknowns.

We will assume a system of error equations with three unknowns and assume for the unknowns the approximate values x_1, y_1, z_1 instead of the final values x, y, z. Then there follow the corrections of the observations:

$$\left.\begin{array}{l} v_1 = a_1 x_1 + b_1 y_1 + c_1 z_1 - l_1 \\ v_2 = a_2 x_1 + b_2 y_1 + c_2 z_1 - l_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}. \qquad (1)$$

After multiplication by $a_1, a_2 \ldots$ and summation we obtain therefrom

$$[a\,v] = [a\,a]\,x_1 + [a\,b]\,y_1 + [a\,c]\,z_1 - [a\,l] = r_1\,, \qquad (2)$$

i.e., the first normal equation is not satisfied by the values x_1, y_1, z_1, but yields a remainder which we denote by r_1.

If we replace then x_1 by $x_1 + x_2$ and retain the values y_1 and z_1, then we obtain, according to (1), new corrections:

$$\left.\begin{array}{l} v_1' = v_1 + a_1 x_2 \\ v_2' = v_2 + a_2 x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\}. \qquad (3)$$

In order to arrive at the sums of the squares $[v'\,v']$ and $[v\,v]$, we square the individual equations (3) and obtain the sum

$$[v'\,v'] = [v\,v] + 2\,[a\,v]\,x_2 + [a\,a]\,x_2^2$$

and according to (2)

$$[v'\,v'] = [v\,v] + 2\,r_1\,x_2 + [a\,a]\,x_2^2\,. \qquad (4)$$

We will choose the correction x_2 in such a way that $[v'\,v']$ is decreased with respect to $[v\,v]$ as much as possible. This is true, as follows from (4), if we set

$$x_2 = -\frac{r_1}{[a\,a]}\,. \qquad (5)$$

Then we have

$$[v' v'] = [v v] - \frac{r_1^2}{[a a]} . \tag{6}$$

Now if we introduce, for instance, the values $x_1 + x_2$, y_1 and z_1 into the second normal equation, then there results a remainder r_2 and there can now be determined a correction

$$y_2 = - \frac{r_2}{[b b]} \tag{7}$$

for y_1, by which the sum of the squares of the corrections is again decreased. If, finally, a correction z_2 for z_1 has been determined, then we start anew with the unknown x. Besides, the order of succession is indifferent; we will always prefer that unknown by which the sum of the squares of the corrections is decreased most.

If, after repeated use of the method, the remainders have become so small that they do not yield noteworthy corrections of the unknowns, then the final values are

$$\left. \begin{aligned} x &= x_1 + x_2 + x_3 + \dots \\ y &= y_1 + y_2 + y_3 + \dots \\ z &= z_1 + z_2 + z_3 + \dots \end{aligned} \right\} . \tag{8}$$

Sometimes the convergence can be accelerated by the introduction of an auxiliary unknown t. If we set

$$x = x' - t , \quad y = y' - t , \quad z = z' - t , \tag{9}$$

then the error equations change to

$$\left. \begin{aligned} v_1 &= a_1 x' + b_1 y' + c_1 z' - (a_1 + b_1 + c_1) t - l_1 \\ v_2 &= a_2 x' + b_2 y' + c_2 z' - (a_2 + b_2 + c_2) t - l_2 \\ &\dots\dots\dots\dots\dots\dots \end{aligned} \right\} . \tag{10}$$

There follow hence, it is true, four normal equations, which, however, are not independent of one another and, therefore, are not sufficient for the determination of the four unknowns; any arbitrary value can rather be assumed for t. In the above approximation method, the unknown t is treated just like the remaining unknowns.

According to this method we will solve the normal equations (8) of p. 112.

If we introduce the auxiliary quantity t according to (9), then to each normal equation there is to be added a term in t, whose coefficient is equal to the negative sum of the coefficients of x, y, z of the same equation. The terms of the additional fourth normal equation are equal to the negative sum of the corresponding terms of first three equations. According to this, we have

$$\begin{array}{rrrrrl} \underline{+17.50\,x'} & -\;6.50\,y' & -\;6.50\,z' & -\;4.50\,t & -\;2.14 & = 0 \\ -\;6.50\,x' & \underline{+17.50\,y'} & -\;6.50\,z' & -\;4.50\,t & -13.96 & = 0 \\ -\;6.50\,x' & -\;6.50\,y' & \underline{+20.50\,z'} & -\;7.00\,t & +\;5.40 & = 0 \\ -\;4.50\,x' & -\;4.50\,y' & -\;7.00\,z' & \underline{+16.00\,t} & +10.70 & = 0 \; . \end{array}$$

The successive determination of the approximate values yields the following table:

$y' = +0.8$	Residue	$t = -0.44$	Residue	$x' = +0.30$	Residue	$t = +0.08$	Residue
− 5.20	− 7.34	+ 1.98	− 5.36	+ 5.25	− 0.11	− 0.36	− 0.47
+ 14.00	+ 0.04	+ 1.98	+ 2.02	− 1.95	+ 0.07	− 0.36	− 0.29
− 5.20	+ 0.20	+ 3.08	+ 3.28	− 1.95	+ 1.33	− 0.56	+ 0.77
− 3.60	+ 7.10	− 7.04	+ 0.06	− 1.35	− 1.29	+ 1.28	− 0.01

$z' = -0.04$	Residue	$t = -0.02$	Residue	$x' = +0.01$	Residue	$t = +0.01$	Residue
+ 0.26	− 0.21	+ 0.09	− 0.12	+ 0.18	+ 0.06	− 0.04	+ 0.02
+ 0.26	− 0.03	+ 0.09	+ 0.06	− 0.06	0.00	− 0.04	− 0.04
− 0.82	− 0.05	+ 0.14	+ 0.09	− 0.06	+ 0.03	− 0.07	− 0.04
+ 0.28	+ 0.27	− 0.32	− 0.05	− 0.04	− 0.09	+ 0.16	+ 0.07

The result is:

$$x' = +0.30 + 0.01 = +0.31 \qquad x = +0.68$$
$$y' = +0.80 \qquad y = +1.17$$
$$z' = -0.04 \qquad z = +0.34$$
$$t = -0.44 + 0.08 - 0.02 + 0.01 = -0.37\ .$$

With the introduction of the auxiliary unknown there is at the same time connected the advantage that the sum of the remainders must at any time be equal to zero, wherein we have a valuable computational check.

Sometimes there happens the case that two successive substitutions are directly contradictory, so that the effect of one is canceled again by that of the other and the convergence is much delayed. If, for instance, the unknowns x and z show this property, then it is of advantage to determine these two unknowns simultaneously from the equations

$$[a\,a]\,x + [a\,c]\,z - r_1 = 0\ ,$$
$$[a\,c]\,x + [c\,c]\,z - r_2 = 0\ ,$$

where r_1 and r_2 denote the remainders existing now. By so doing, the delay of the convergence will in most cases be eliminated. Under certain circumstances, we will also have to collect more than two unknowns for the simultaneous determination.

The above described approximation method with the introduction of an auxiliary unknown originates from C. F. Gauss, as his pupil, Ch. L. Gerling, indicates in *Die Ausgleichungsrechnungen der praktischen Geometrie*, Hamburg and Gotha, 1843, p. 386; in section 111, pp. 386-393. The method for indirect and conditioned observations is treated in detail by Gerling.

Other approximation methods for the solution of normal equations are reported in Czuber, "Die Entwicklung der Wahrscheinlichkeitstheorie und ihrer Anwendungen," *Jahresber. d. Deutsch. Mathematiker-Vereinigung*, 7. Band, 1897-1898, pp. 199-200, as well as in Helmert, *Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate*, 2nd Edition, Leipzig and Berlin, 1907, pp. 175-180.

Section 38. Possibility or Impossibility of the Solution of Normal Equations

To the auxiliary considerations, which can in addition be connected to the main theory of the adjustment of indirect observations, there also belongs the question of whether the normal equations will always yield a solution at all.

In practical cases, we will already know from the nature of the problem itself, whether or not a solution in the usual sense is possible; for instance, if a point on the plane is determined by the intersection of several straight lines, which, in a single case, become all, by chance, nearly parallel, then the view of the figure already says that there will not be expected a good or, as the case may be, no solution at all from an adjustment.

The following investigation is to be understood in this and a similar sense.

We consider first the special case in which the number of the error equations is *equal* to the number of unknowns, and assume only two elements:

Error equations	*Normal equations*	
$v_1 = a_1 x + b_1 y - l_1$ $v_2 = a_2 x + b_2 y - l_2$	$\left.\begin{array}{l}[a\,a]\,x + [a\,b]\,y - [a\,l] = 0\\ [a\,b]\,x + [b\,b]\,y - [b\,l] = 0\end{array}\right\}.$	(1)

Reduced normal equations: $[bb\cdot 1]\,y - [b\,l\cdot 1] = 0\ .$ (2)

Sum of squares of errors: $[vv] = [l\,l\cdot 2] = [l\,l] - \dfrac{[a\,l]^2}{[a\,a]} - \dfrac{[b\,l\cdot 1]^2}{[b\,b\cdot 1]}\ .$ (3)

But in this simple case we have

$$[a\,a] = a_1^2 + a_2^2\,, \qquad [a\,b] = a_1\,b_1 + a_2\,b_2 \qquad [a\,l] = a_1\,l_1 + a_2\,l_2$$
$$[b\,b] = b_1^2 + b_2^2\,, \qquad [b\,l] = b_1\,l_1 + b_2\,l_2$$
$$[l\,l] = l_1^2 + l_2^2\,.$$

If we set this into (2), then we obtain after a brief transformation

$$[b\,b \cdot 1] = \frac{(a_1 b_2 - a_2 b_1)^2}{a_1^{\,2} + a_2^{\,2}}\,, \qquad [b\,l \cdot 1] = \frac{(a_1 b_2 - a_2 b_1)(a_1 l_2 - a_2 l_1)}{a_1^{\,2} + a_2^{\,2}}\,, \tag{4}$$

consequently,

$$y = \frac{[b\,l \cdot 1]}{[b\,b \cdot 1]} = \frac{(a_1 l_2 - a_2 l_1)}{(a_1 b_2 - a_2 b_1)}\,, \tag{5}$$

This is *the same* value y, which we would also obtain directly from the error equations with $v_1 = 0$ and $v_2 = 0$. Therefore, the normal equations form here a roundabout way, which we take, however, under certain circumstances, because there is obtained thereby the weight of y, $p_y = [b\,b \cdot 1]$ which the error equations in themselves would not yield.

If we go now further to the sum of the squares of the errors $[v\,v]$ according to (3), then we soon see that the latter will be equal to zero, for the calculation yields

$$[v\,v] = [l\,l \cdot 2] = l_1^{\,2} + l_2^{\,2} - \frac{(a_1 l_1 + a_2 l_2)^2}{a_1^{\,2} + a_2^{\,2}} - \frac{(a_1 l_2 - a_2 l_1)^2}{a_1^{\,2} + a_2^{\,2}} = 0\,. \tag{6}$$

If we wanted to compute further a mean error of the unit of weight, then we would obtain

$$m^2 = \frac{[l\,l \cdot 2]}{2 - 2} = \frac{0}{0}\,. \tag{7}$$

If we imagine, for illustration, for the two error equations, say, two linear determinations of a point with the coordinates x and y, then the results (4) to (7) have the following interpretations:

In the case of two rays, an adjustment for the point determination is not necessary; but if we use the adjustment formulae nevertheless, then they yield the same point of intersection xy, which we would have obtained also without the adjustment, with definite weights $p_y = [b\,b \cdot 1]$ and $p_x = [a\,a \cdot 1]$, but without a definite mean error of the unit of weight m.

But if we have the knowledge of such a value m from elsewhere, then we can also compute mean errors of coordinates:

$$m_y = \frac{m}{\sqrt{[b\,b \cdot 1]}}\,, \qquad m_x = \frac{m}{\sqrt{[a\,a \cdot 1]}}\,.$$

The determination of y and x becomes impossible when the coefficients $a_1 b_1$, $a_2 b_2$ (1) have the same ratio:

$$\frac{b_1}{a_1} = \frac{b_2}{a_2}\,; \qquad \text{therefore} \qquad a_1 b_2 - a_2 b_1 = 0\,,$$

i.e. if the rays in question become parallel.

On the other hand, the determination of x and y becomes possible again, if to two error equations with a constant ratio $b{:}a$ a third with another ratio $b_3{:}a_3$ is added, or generally, if in the case of an arbitrary number of error equations, with two unknowns, there are at least two error equations whose coefficient ratios $b{:}a$ are *not* equal. For, if *all* ratios $b{:}a$ were equal to one another, then we would also have

$$\frac{[a\,b]}{[a\,a]} = \frac{b}{a}\,, \qquad \text{therefore} \qquad [b\,b \cdot 1] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b] = [b\,b] - \frac{b}{a}[a\,b] = [b\,b] - [b\,b] = 0\,.$$

With only two unknowns x and y all these proportions can be easily seen at a glance; for an arbitrary number of unknowns, Gauss has already reflected on the possibility or impossibility of the solution in "Theoria Combinationis," art. 23.

This question can, theoretically, be made still clearer by the theory of determinants in the following manner:

The necessary and sufficient condition that the normal equations admit one and only one solution rests in the fact that the determinant composed of the coefficients of the unknowns occurring in these equations is different from zero.

This determinant results from the system of the coefficients of the unknowns in the error equations by composition with itself. It is therefore equal (according to the theorem of multiplication of determinants) to the sum of the squares of that determinant which we obtain by picking out from the system of error equations, by all possible kinds of ways, as many equations each time as there are unknowns to be determined, and summarizing the coefficients of the unknowns in the chosen equations as a determinant. Even if only one of these latter determinants is different from zero, then the determinant of the coefficients of the unknowns in the normal equations is likewise different from zero.

Therefore, the system of normal equations has one and only one solution each time if, from the error equations, only one single group of equations can be separated which contains as many equations as unknowns, and has the property of possessing one and only one solution.

Section 39. Determinant Formulae for Three Elements

(Without connection with our general course of development)

With three elements x, y, z we can still indicate the solution formulae for x, y, z and the weight coefficients $[\alpha\alpha], [\alpha\beta]$ . . . in a somewhat closed form, similarly to the case of two unknowns, especially if we use *determinants*, which, even with three elements, still permit a clear calculation.

The following formulae have hardly *practical* significance, for the numerical evaluation of the unknowns and their weights is always done best by the way of gradual elimination.

If there exist the error equations

$$\left.\begin{array}{l} v_1 = a_1 x + b_1 y + c_1 z - l_1 \\ v_2 = a_2 x + b_2 y + c_2 z - l_2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ v_n = a_n x + b_n y + c_n z - l_n \end{array}\right\}, \tag{1}$$

then the normal equations belonging to them are

$$\left.\begin{array}{l} [aa]x + [ab]y + [ac]z - [al] = 0 \\ [ab]x + [bb]y + [bc]z - [bl] = 0 \\ [ac]x + [bc]y + [cc]z - [cl] = 0 \end{array}\right\}. \tag{2}$$

By gradual elimination we find therefrom

$$\left.\begin{array}{l} [bb\cdot 1]y + [bc\cdot 1]z - [bl\cdot 1] = 0 \\ [bc\cdot 1]y + [cc\cdot 1]z - [cl\cdot 1] = 0 \end{array}\right\} \tag{3}$$

$$[cc\cdot 2]z - [cl\cdot 2] = 0\,. \tag{4}$$

On the other hand, we can use the theory of determinants:

$$D = \begin{vmatrix} [aa] & [ab] & [ac] \\ [ab] & [bb] & [bc] \\ [ac] & [bc] & [cc] \end{vmatrix} = \left.\begin{array}{l} +[aa][bb][cc] - [ac][bb][ac] \\ +[ab][bc][ac] - [aa][bc][bc] \\ +[ac][ab][bc] - [ab][ab][cc] \end{array}\right\}. \tag{5}$$

Accordingly, let there be

$$D_x = \begin{vmatrix} [al] & [ab] & [ac] \\ [bl] & [bb] & [bc] \\ [cl] & [bc] & [cc] \end{vmatrix} \qquad D_y = \begin{vmatrix} [aa] & [al] & [ac] \\ [ab] & [bl] & [bc] \\ [ac] & [cl] & [cc] \end{vmatrix} \qquad D_z = \begin{vmatrix} [aa] & [ab] & [al] \\ [ab] & [bb] & [bl] \\ [ac] & [bc] & [cl] \end{vmatrix} \Bigg\} . \tag{6}$$

Then x, y, z are determined as follows:

$$x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D}, \qquad z = \frac{D_z}{D}. \tag{7}$$

Between (4) and (5) and (6) there exist simple relations; for we have

$$[cc \cdot 2] = \frac{D}{[aa]\,[bb] - [ab]\,[ab]}, \qquad [cl \cdot 2] = \frac{D_z}{[aa]\,[bb] - [ab]\,[ab]}, \tag{8}$$

where the denominator $[aa]\,[bb] - [ab]\,[ab]$ is the determinant of coefficients of second order already used previously in section 18.

$[cc \cdot 2]$ is at the same time the reciprocal of the weight coefficient $[\gamma\gamma]$. All weight coefficients can be represented similarly:

$$\left.\begin{array}{lll} [\alpha\alpha] = \dfrac{[bb]\,[cc] - [bc]\,[bc]}{D} & [\alpha\beta] = -\dfrac{[ab]\,[cc] - [ac]\,[bc]}{D} & [\alpha\gamma] = -\dfrac{[ac]\,[bb] - [ab]\,[bc]}{D} \\ & [\beta\beta] = \dfrac{[aa]\,[cc] - [ac]\,[ac]}{D} & [\beta\gamma] = -\dfrac{[bc]\,[aa] - [ab]\,[ac]}{D} \\ & & [\gamma\gamma] = \dfrac{[aa]\,[bb] - [ab]\,[ab]}{D} \end{array}\right\} . \tag{9}$$

The sum of the squares of the residual errors, $[ll \cdot 2]$, also, can be represented in a threefold manner by determinants:

$$[ll \cdot 2] = \frac{\begin{vmatrix} [ll] & [lb] & [lc] \\ [lb] & [bb] & [bc] \\ [lc] & [bc] & [cc] \end{vmatrix}}{\begin{vmatrix} [bb] & [bc] \\ [bc] & [cc] \end{vmatrix}} = \frac{\begin{vmatrix} [aa] & [al] & [ac] \\ [al] & [ll] & [lc] \\ [ac] & [lc] & [cc] \end{vmatrix}}{\begin{vmatrix} [aa] & [ac] \\ [ac] & [cc] \end{vmatrix}} = \frac{\begin{vmatrix} [aa] & [ab] & [al] \\ [ab] & [bb] & [bl] \\ [al] & [bl] & [ll] \end{vmatrix}}{\begin{vmatrix} [aa] & [ab] \\ [ab] & [bb] \end{vmatrix}} \Bigg\} . \tag{10}$$

We can understand at once the correctness of all the above formulae, if we develop the determinants of second and third degree in the usual manner. About the validity of analagous formulae for more than three elements we refer to V o g l e r, *Lehrbuch der praktischen Geometrie*, 1 Teil, pp. 253 and following, and take from there, in addition, that the determinant of the coefficients D has, besides the original form (5), the other following forms, which is easily proved:

$$D = \begin{vmatrix} [aa] & [ab] & [ac] \\ 0 & [bb \cdot 1] & [bc \cdot 1] \\ 0 & [bc \cdot 1] & [cc \cdot 1] \end{vmatrix} = \begin{vmatrix} [aa] & [ab] & [ac \\ 0 & [bb \cdot 1] & [bc \cdot 1] \\ 0 & 0 & [cc \cdot 2] \end{vmatrix} = [aa] \cdot [bb \cdot 1] \cdot [cc \cdot 2] .$$

For the weight coefficients, also, there exist still some other forms:

$$[\gamma\gamma] = \frac{\begin{vmatrix} [aa] & [ab] & 0 \\ [ab] & [bb] & 0 \\ [ac] & [bc] & 1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} [aa] & [ab] & 0 \\ 0 & [bb \cdot 1] & 0 \\ 0 & [bc \cdot 1] & 1 \end{vmatrix}}{D} = \frac{[aa] \begin{vmatrix} [bb \cdot 1] & 0 \\ [bc \cdot 1] & 1 \end{vmatrix}}{D} .$$

The first of these three formulae, when solved, yields the same as (9).

We can use such formulae occasionally, perhaps, for computational checks.

The determinants are symbolic designations, similar to our $[bb \cdot 1]$'s, $[cc \cdot 2]$'s, etc., and as long as this symbolism, especially suited for the method of least squares, is sufficient, there is no reason for using another symbolism less suited for numerical computation.

The use of general theorems on determinants for considerations of such kind as at the end of section 38 is however very useful.

Section 40. Accuracy of Different Methods of Adjustment

Until now we have always based the adjustment of indirect observations on the principle of the sum of least squares. We will now disregard this principle and express, first, the unknowns in a general form by the observations. We will limit ourselves here to such solutions of the problem in which the unknowns are found as linear functions of the observations; all other solutions can be reduced to this with the use of approximate values and developments in series, similarly as in section 23.

Let there be n error equations

$$\left.\begin{aligned} l_1 + \varepsilon_1 &= a_1 X + b_1 Y + c_1 Z \\ l_2 + \varepsilon_2 &= a_2 X + b_2 Y + c_2 Z \\ &\cdots\cdots\cdots \\ l_n + \varepsilon_n &= a_n X + b_n Y + c_n Z \end{aligned}\right\} \tag{1}$$

in which the quantities l are the observations, the ε's their true errors, and X, Y, Z the true values of the unknowns. If we multiply the equations (1) successively by the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$, which satisfy the conditions

$$[a\alpha] = 1 \,, \qquad [b\alpha] = 0 \,, \qquad [c\alpha] = 0 \tag{2}$$

but, besides, can be assumed completely arbitrarily, then we obtain as the sum of equations (1)

$$[\alpha l] + [\alpha \varepsilon] = X \,. \tag{3}$$

The term $[\alpha\varepsilon]$ is unknown to us; we neglect it and arrive then at a value of the unknown X, which deviates from the true value and which we let be denoted by x. We have

$$x = \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_n l_n \,. \tag{4}$$

As we introduce herein arbitrary value systems $\alpha_1, \alpha_2, \ldots, \alpha_n$, which must only satisfy equations (2), we obtain each time another value of x. The mean errors of these values result, according to (11) section 5, from the equation:

$$m_x^{\,2} = (\alpha_1^{\,2} + \alpha_2^{\,2} + \ldots + \alpha_n^{\,2})\, m^2 = [\alpha\alpha]\, m^2 \,, \tag{5}$$

if m denotes the mean error of the observations assumed as equally accurate.

We find corresponding values y and z for the unknowns Y and Z, by introducing further coefficients $\beta_1, \beta_2 \ldots$ and $\gamma_1, \gamma_2 \ldots$ with the conditions

$$[a\beta] = 0 \,, \qquad [b\beta] = 1 \,, \qquad [c\beta] = 0 \tag{6}$$

$$[a\gamma] = 0 \,, \qquad [b\gamma] = 0 \,, \qquad [c\gamma] = 1 \,. \tag{7}$$

Then we will have

$$y = \beta_1 l_1 + \beta_2 l_2 + \ldots + \beta_n l_n \,, \tag{8}$$

$$z = \gamma_1 l_1 + \gamma_2 l_2 + \ldots + \gamma_n l_n \,. \tag{9}$$

If we set these values of the three unknowns into the error equations (1), then, on the left-hand side we no longer obtain the true errors ε, but certain quantities $v_1, v_2, \ldots, v_n$. With this, equations (1) change to

$$\left.\begin{array}{l} l_1 + v_1 = a_1 x + b_1 y + c_1 z \\ l_2 + v_2 = a_2 x + b_2 y + c_2 z \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ l_n + v_n = a_n x + b_n y + c_n z \end{array}\right\} . \tag{10}$$

Now, if we deal first with the unknown X alone, then we can set up the requirement that the coefficients α shall be chosen in such a way that the mean error of the unknown becomes as small as possible. To the conditions (2) there is then added, according to (5), the further condition that $[\alpha\alpha]$ shall be a minimum.

We multiply the three condition equations (2) by the still undetermined factors $-2\,Q_1, -2\,Q_2, -2\,Q_3$ and have then to find the absolute minimum for the function

$$F = [\alpha\alpha] - 2\,Q_1\,([a\alpha] - 1) - 2\,Q_2\,[b\,\alpha] - 2\,Q_3\,[c\,\alpha]\,.$$

This happens if we have

$$\begin{array}{l} \dfrac{\partial F}{\partial \alpha_1} = 2\,\alpha_1 - 2\,Q_1\,a_1 - 2\,Q_2\,b_1 - 2\,Q_3\,c_1 = 0 \\ \dfrac{\partial F}{\partial \alpha_2} = 2\,\alpha_2 - 2\,Q_1\,a_2 - 2\,Q_2\,b_2 - 2\,Q_3\,c_2 = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

and we obtain therefrom

$$\left.\begin{array}{l} \alpha_1 = a_1\,Q_1 + b_1\,Q_2 + c_1\,Q_3 \\ \alpha_2 = a_2\,Q_1 + b_2\,Q_2 + c_2\,Q_3 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right\} . \tag{11}$$

By introducing these values into equations (2) we obtain three equations for the determination of the factors Q_1, Q_2, Q_3, namely

$$\left.\begin{array}{l} [a\,a]\,Q_1 + [a\,b]\,Q_2 + [a\,c]\,Q_3 = 1 \\ [a\,b]\,Q_1 + [b\,b]\,Q_2 + [b\,c]\,Q_3 = 0 \\ [a\,c]\,Q_1 + [b\,c]\,Q_2 + [b\,c]\,Q_3 = 0 \end{array}\right\} , \tag{11*}$$

with which the problem is solved.

In addition to this, we will derive a further relation. Equations (10) yield after multiplication with $\alpha_1, \alpha_2 \ldots$ and summation

$$[\alpha\,l] + [\alpha\,v] = [a\,\alpha]\,x + [b\,\alpha]\,y + [c\,\alpha]\,z\,;$$

therefore, by taking into account (2) and (4)

$$[\alpha\,v] = 0\,. \tag{12}$$

We further multiply equations (11) by $v_1, v_2 \ldots$ and form their sum, from which we obtain according to (12):

$$0 = [a\,v]\,Q_1 + [b\,v]\,Q_2 + [c\,v]\,Q_3\,. \tag{13}$$

Since the quantities Q_1, Q_2, Q_3 depend only on the coefficients a, b, c and hence, can assume quite arbitrary values, then equation (13) can exist only if

$$[a\,v] = 0\,, \qquad [b\,v] = 0\,, \qquad [c\,v] = 0\,. \tag{14}$$

But these are the basic equations of the method of least squares; for, if we multiply equations (10) by $a_1, a_2, \ldots,$ then by $b_1, b_2 \ldots$ and, finally, by $c_1, c_2 \ldots$ and sum them up each time, then, bearing in mind (14), there follow the equations

$$\left.\begin{aligned} [a\,l] &= [a\,a]\,x + [a\,b]\,y + [a\,c]\,z \\ [b\,l] &= [a\,b]\,x + [b\,b]\,y + [b\,c]\,z \\ [c\,l] &= [a\,c]\,x + [b\,c]\,y + [c\,c]\,z \end{aligned}\right\}, \qquad (15)$$

which agree with the normal equations found according to the method of least squares. We arrive at the same result if we repeat the development for the other two unknowns with equations (6) to (9). With this, it is proved that the method of least squares leads to those values of the unknowns which have the smallest mean errors.

After we have recognized that the method of least squares yields the most accurate values of the unknowns, the additional question of what accuracy we reach if the adjustment is carried out according to any approximation method is of interest. Even in the case of such an approximate adjustment, we will finally obtain each of the unknowns as a linear function of the observations, so that, e.g., for the unknown x there is found a value

$$x' = \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_n l_n \,;$$

the coefficients α, however, will now be different from those of the rigorous adjustment.

Between the two computations of the unknowns there exists yet another essential difference. For, whereas the most accurate value x is found only by means of a single system of coefficients α, we can indicate an infinite number of systems of coefficients α, which yield the value x', namely all systems which satisfy the four equations

$$[a\alpha] - 1 = 0\,, \qquad [b\,\alpha] = 0\,, \qquad [c\,\alpha] = 0\,, \qquad [\alpha\, l] - x' = 0\,. \qquad (16)$$

In other words, an infinite number of approximation methods of adjustment can lead to the same value x'.

If we apply the law of the propagation of errors (11), section 5, p. 14, to these different systems of coefficients α, then we will obtain, each time, another mean error, i.e., the same value x' is found with a different accuracy according to the different methods. We see hence that in the case of a redundant determination of an unknown, the law of propagation of errors yields a unique value of the mean error only when the adjustment is carried out according to the method of least squares.

Now, among the infinite number of adjustment methods, all of which yield the same value x', there must be one which yields for x' the smallest mean error. Let the system suited for this have the coefficients $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$.

If we multiply the conditions (16) again by the still unknown factors $-2\,k_1, -2\,k_2, -2\,k_3,$ and $-2\,k$, then we have to find the absolute minimum for the function

$$F' = [\alpha'\,\alpha'] - 2\,k_1\,([a\,\alpha'] - 1) - 2\,k_2\,[b\,\alpha'] - 2\,k_3\,[c\,\alpha'] - 2\,k\,([\alpha'\,l] - x')\,.$$

If we set the differential quotients

$$\frac{\partial F'}{\partial \alpha_1'} = \frac{\partial F'}{\partial \alpha_2'} = \ldots = \frac{\partial F'}{\partial \alpha_n'} = 0\,,$$

then we obtain, for the system of coefficients sought for, the equations

$$\left.\begin{aligned} \alpha_1' &= a_1\,k_1 + b_1\,k_2 + c_1\,k_3 + k\,l_1 \\ \alpha_2' &= a_2\,k_1 + b_2\,k_2 + c_2\,k_3 + k\,l_2 \\ &\;\;\vdots \\ \alpha_n' &= a_n\,k_1 + b_n\,k_2 + c_n\,k_3 + k\,l_n \end{aligned}\right\}. \qquad (17)$$

For the determination of the factors k_1, k_2, k_3 we multiply equations (17), successively, by $a_1, a_2, \ldots, a_n$, then by $b_1, b_2, \ldots, b_n$, and finally, by $c_1, c_2, \ldots, c_n$ and form each time the sum. Taking into account equations (16) we have then

$$\left.\begin{aligned}[a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 &= 1 - [a\,l]\,k \\ [a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 &= \quad - [b\,l]\,k \\ [a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 &= \quad - [c\,l]\,k\end{aligned}\right\}. \tag{18}$$

In order to establish a relation between the coefficients α', β', γ' and the coefficients α, β, γ, we set

$$\left.\begin{aligned}k_1 &= Q_1 + \Delta_1 \\ k_2 &= Q_2 + \Delta_2 \\ k_3 &= Q_3 + \Delta_3\end{aligned}\right\} \tag{19}$$

and obtain from equations (11*) and (18)

$$\left.\begin{aligned}[a\,a]\,\Delta_1 + [a\,b]\,\Delta_2 + [a\,c]\,\Delta_3 &= - [a\,l]\,k \\ [a\,b]\,\Delta_1 + [b\,b]\,\Delta_2 + [b\,c]\,\Delta_3 &= - [b\,l]\,k \\ [a\,c]\,\Delta_1 + [b\,c]\,\Delta_2 + [c\,c]\,\Delta_3 &= - [c\,l]\,k\end{aligned}\right\}. \tag{20}$$

These equations are identical with the normal equations (15) with the exception of the factor $-k$, if we replace $\Delta_1, \Delta_2, \Delta_3$ by x, y, z. Consequently, we have

$$\left.\begin{aligned}\Delta_1 &= -k\,x \\ \Delta_2 &= -k\,y \\ \Delta_3 &= -k\,z\end{aligned}\right\} (16) \qquad \text{and} \quad \left.\begin{aligned}k_1 &= Q_1 - k\,x \\ k_2 &= Q_2 - k\,y \\ k_3 &= Q_3 - k\,z\end{aligned}\right\}. \tag{21}$$

For the coefficients $\alpha_1', \alpha_2', \ldots, \alpha_n'$ there thus result the values

$$\left.\begin{aligned}\alpha_1' &= a_1\,Q_1 + b_1\,Q_2 + c_1\,Q_3 - k\,(-l_1 + a_1\,x + b_1\,y + c_1\,z) \\ \alpha_2' &= a_2\,Q_1 + b_2\,Q_2 + c_2\,Q_3 - k\,(-l_2 + a_2\,x + b_2\,y + c_2\,z) \\ &\cdots\cdots\cdots\cdots \\ \alpha_n' &= a_n\,Q_1 + b_n\,Q_2 + c_n\,Q_3 - k\,(-l_n + a_n\,x + b_n\,y + c_n\,z)\end{aligned}\right\}, \tag{22}$$

and from equations (10) and (11) there follows

$$\left.\begin{aligned}\alpha_1' &= \alpha_1 - k\,v_1 \\ \alpha_2' &= \alpha_2 - k\,v_2 \\ &\cdots\cdots\end{aligned}\right\}. \tag{23}$$

With this, we obtain for x' the value

$$x' = [\alpha l] - k\,[v\,l] \;. \tag{24}$$

If we multiply equations (10) by v_1, $v_2 \ldots$ and add them up, then, taking into account (14) we have $[v\,l] = -[v\,v]$;

therefore,

$$x' = [\alpha\,l] + k\,[v\,v] = x + k\,[v\,v] \;. \tag{25}$$

For the computation of the mean error of x' we have to form the expression

$$m_{x'}^2 = \left\{ \left(\frac{\partial x'}{\partial l_1}\right)^2 + \left(\frac{\partial x'}{\partial l_2}\right)^2 + \ldots + \left(\frac{\partial x'}{\partial l_n}\right)^2 \right\} m^2 . \tag{26}$$

According to (24) we have, since the v's are constant,

$$\frac{\partial x'}{\partial l_1} = \alpha_1 - k\, v_1 . \tag{27}$$

Likewise, we will have

$$\begin{aligned} \frac{\partial x'}{\partial l_2} &= \alpha_2 - k\, v_2 \\ \vdots \quad & \quad \vdots \quad \vdots \\ \frac{\partial x'}{\partial l_n} &= \alpha_n - k\, v_n . \end{aligned}$$

By introducing these values into (26) we find

$$m_{x'}^2 = ([\alpha\alpha] - 2\, k\, [\alpha v] + k^2\, [v\, v])\, m^2$$

or, since according to (12) $[\alpha v] = 0$ and according to (5) $[\alpha\alpha]\, m^2 = m_x^2$,

$$m_{x'}^2 = m_x^2 + k^2\, [v\, v]\, m^2 .$$

Herein we can set, in addition, $m^2 = \dfrac{[v\, v]}{n-3}$, and according to (25) $k\, [v\, v] = x' - x$ and obtain

$$m_{x'}^2 = m_x^2 + \frac{(x' - x)^2}{n - 3} . \tag{28}$$

We can indicate immediately corresponding expressions for the other two unknowns.

We will summarize once more the result of the development:

If from n error equations there are found, for three unknowns, the values x, y, z with the mean errors m_x, m_y, m_z, according to the method of least squares and the values x', y', z' according to another arbitrary method of adjustment, then the mean errors of the latter are to be computed according to the equations

$$\left. \begin{aligned} m_{x'}^2 &= m_x^2 + \frac{(x' - x)^2}{n - 3} \\ m_{y'}^2 &= m_y^2 + \frac{(y' - y)^2}{n - 3} \\ m_{z'}^2 &= m_z^2 + \frac{(z' - z)^2}{n - 3} \end{aligned} \right\} . \tag{29}$$

It is evident that the above holds also for more than three, e.g. for u unknowns, if in the denominator of the second term on the right-hand side, we replace $n - 3$ by $n - u$.

Example

In order to apply the theory on p. 126, also numerically, we consider, in Fig. 1, the measurement of angles at a station between four rays. For the six angles indicated in Fig. 1, the following measured values are found:

$$\left.\begin{array}{llll} l_1 = 112°25'37'' & & l_4 = & 75°40'41'' \\ l_2 = 84\;19\;48 & & l_5 = & 196\;45\;15 \\ l_3 = 87\;34\;24 & & l_6 = & 163\;14\;17 \end{array}\right\} . \tag{30}$$

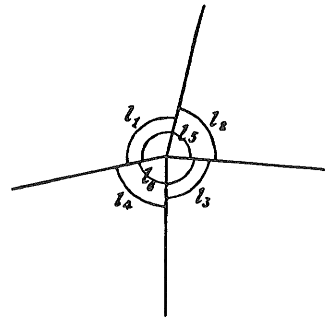


Fig. 1.

First we carry out the rigorous adjustment according to the method of least squares by introducing the adjusted values of the angles l_1, l_2, l_3 as unknowns x, y, z. According to the view of Fig. 1 there result the following error equations:

$$\begin{array}{l} l_1 + v_1 = +x \quad . \quad . \quad . \\ l_2 + v_2 = \quad . \quad +y \quad . \quad . \\ l_3 + v_3 = \quad . \quad . \quad +z \quad . \\ l_4 + v_4 = -x - y - z + 360° \\ l_5 + v_5 = +x + y \quad . \quad . \\ l_6 + v_6 = -x - y \quad . \quad +360° . \end{array}$$

To simplify the numerical computation we set

$$\left.\begin{array}{l} x = 112°25'40'' + \delta x \\ y = \;\;84\;19\;50 + \delta y \\ z = \;\;87\;34\;20 + \delta z \end{array}\right\} , \tag{31}$$

and if we set this as well as the measured values into the error equations, then we obtain

$$\left.\begin{array}{llll} v_1 = +\;\;3'' & +\delta x & . & . \\ v_2 = +\;\;2 & . & +\delta y & . \\ v_3 = -\;\;4 & . & . & +\delta z \\ v_4 = -31 & -\delta x & -\delta y & -\delta z \\ v_5 = +15 & +\delta x & +\delta y & . \\ v_6 = +13 & -\delta x & -\delta y & . \end{array}\right\} . \tag{32}$$

The normal equations belonging to this are:

$$\left.\begin{array}{l} +4\,\delta x + 3\,\delta y + 1\,\delta z + 36 = 0 \\ +3\,\delta x + 4\,\delta y + 1\,\delta z + 35 = 0 \\ +1\,\delta x + 1\,\delta y + 2\,\delta z + 27 = 0 \end{array}\right\} . \tag{33}$$

By solving the normal equations and the weight equations we find

$$\begin{array}{lll} \delta x = -4.2'' & \delta y = -3.2'' & \delta z = -9.9'' \\ p_x = 1.714 & p_y = 1.714 & p_z = 1.714 \end{array}$$

and the final values of the unknowns are then according to (31)

$$\left.\begin{array}{l} x = 112°25'35.8'' \\ y = \;\;84\;19\;46.8 \\ z = \;\;87\;34\;10.1 \end{array}\right\} . \tag{34}$$

After determination of the corrections v we have, finally, the following computation of the mean errors from the error equations (32)

v	v^2
$-1.2''$	1.4
-1.2	1.4
-13.9	193.2
-13.7	187.7
$+7.6$	57.8
$+20.4$	416.2
	857.7

$$m^2 = \frac{857.7}{6-3} = 285.9 \qquad m = \pm 16.91''$$

$$m_x = m_y = m_z = \frac{16.91''}{\sqrt{1.714}} = \pm 12.91''. \tag{35}$$

All this refers to the rigorous adjustment, which we have represented here as briefly as possible, especially by omitting all computational checks.

We will now take up a second approximate adjustment by making first the sum of the two angles l_5 and l_6 come out to 360° and then making the angles l_1 and l_2 or, as the case may be, l_3 and l_4 come out to the found values of the angles l_5 and l_6. The adjustment assumes then the following form:

Angle	Measured	Corr. v	Adjusted
l_5	196° 45′ 15″	+14″	196° 45′ 29″
l_6	163 14 17	+14	163 14 31
	359 59 32	+28	360 00 00
l_1	112° 25′ 37″	+2″	112° 25′ 39″
l_2	84 19 48	+2	84 19 50
	196 45 25	+4	196 45 29
l_3	87° 34′ 24″	−17″	87° 34′ 07″
l_4	75 40 41	−17	75 40 24
	163 15 05	−34	163 14 31

(36)

First we state that the sum of the squares of the corrections of this approximate adjustment

$$2 \times 14^2 + 2 \times 2^2 + 2 \times 17^2 = 978$$

is considerably larger than the sum of least squares, 858, of the rigorous adjustment, which was to be expected.

Passing over to the computation of the mean errors of the unknowns, we will first use directly the law of propagation of errors and, for this, must express the unknowns in terms of the measured quantities. We have

$$v_5 = v_6 = \frac{1}{2}(360° - l_5 - l_6),$$

$$l_1 + v_1 + l_2 + v_2 = l_5 + v_5 \qquad v_1 = v_2,$$

hence,

$$v_1 = v_2 = \frac{1}{2}\left\{-l_1 - l_2 + l_5 + \frac{1}{2}(360° - l_5 - l_6)\right\}$$

or

$$v_1 = v_2 = 90° - \frac{1}{2}l_1 - \frac{1}{2}l_2 + \frac{1}{4}l_5 - \frac{1}{4}l_6$$

and accordingly,

$$v_3 = v_4 = 90° - \frac{1}{2}l_3 - \frac{1}{2}l_4 - \frac{1}{4}l_5 + \frac{1}{4}l_6.$$

Now since $x' = l_1 + v_1$, $y' = l_2 + v_2$, $z' = l_3 + v_3$,

then we obtain

$$\left.\begin{aligned} x' &= 90^\circ + \frac{1}{2} l_1 - \frac{1}{2} l_2 + \frac{1}{4} l_5 - \frac{1}{4} l_6 = 112^\circ\ 25'\ 39'' \\ y' &= 90^\circ - \frac{1}{2} l_1 + \frac{1}{2} l_2 + \frac{1}{4} l_5 - \frac{1}{4} l_6 = \ \ 84\ \ 19\ \ 50 \\ z' &= 90^\circ + \frac{1}{2} l_3 - \frac{1}{2} l_4 - \frac{1}{4} l_5 + \frac{1}{4} l_6 = \ \ 87\ \ 34\ \ 07 \end{aligned}\right\} \tag{37}$$

and, for this, the law of propagation of errors (11), section 5, p. 14, yields

$$m_{x'}^2 = \left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2\right) m^2 = \frac{5}{8}\, m^2 .$$

Since $m_{y'}^2$ and $m_{z'}^2$ obtain the same values, then we have

$$m_{x'}^2 = m_{y'}^2 = m_{z'}^2 = 0.625\ m^2 ,$$

and if we use the value found by rigorous adjustment

$$m = \pm 16.91''$$

then we will have

$$m'_x = m'_y = m'_z = \pm 13.37'' . \tag{38}$$

We can however, from the outset, regard these mean errors as too large, since the values (37) of the unknowns can be found by an infinite number of ways, each of which leads to other mean errors according to the law of propagation of errors. We find the smallest mean errors for the values (37) with the help of equations (29). For this, we have now from (34) and (37)

$$\begin{aligned} x' - x &= +3.2'' \\ y' - y &= +3.2 \\ z' - z &= -3.1 \ ; \end{aligned}$$

therefore, according to (29) we will have

$$m_{x'}^2 = m_{y'}^2 = m_{z'}^2 = 166.67 + \frac{10.24}{3} = 170.08 ,$$

$$m'_x = m'_y = m'_z = \pm 13.04'' . \tag{39}$$

We see hence that the accuracy of the approximate values (37) is only a little smaller than that of the rigorous values (34).

Now, after we have taken care of the general theory of adjustment of indirect observations, it may be appropriate for the conclusion of this theory to take up, and insert here, a practical example, which is not of a geodetic nature, as nearly all of our other later applications are.

We consider an adjustment of observations, which by nature have a *periodic* behavior, as, e.g., meteorological observations, water-gauge observations on the seashore, etc.

As a periodic function we take the following:

$$F = (F) + r_1 \sin(\alpha_1 + \varphi) + r_2 \sin(\alpha_2 + 2\varphi) + r_3 \sin(\alpha_3 + 3\varphi) \ldots, \tag{1}$$

where φ is an independent variable, for instance, corresponding to the time. The constants $(F)\ r_1, r_2, r_3 \ldots, \alpha_1, \alpha_2, \alpha_3 \ldots$ shall be determined in such a way that the function values fit, as well as possible, the given observations, i.e., that the sum of the squares of the deviations of the function values F from the observations becomes a minimum.

For the purpose of the adjustment we must make function (1) linear with respect to the unknowns to be determined, and this is done simply by the solution:

$$\begin{aligned} \sin(\alpha_1 + \varphi) &= \sin\alpha_1 \cos\varphi + \cos\alpha_1 \sin\varphi \\ \sin(\alpha_2 + 2\varphi) &= \sin\alpha_2 \cos 2\varphi + \cos\alpha_2 \sin 2\varphi \\ &\ldots\ldots\ldots \end{aligned}$$

We set

$$\left.\begin{matrix} r_1 \sin\alpha_1 = y_1 & r_1 \cos\alpha_1 = x_1 \\ r_2 \sin\alpha_2 = y_2 & r_2 \cos\alpha_2 = x_2 \\ \ldots\ldots & \ldots\ldots \end{matrix}\right\}, \tag{2}$$

and with this, function (1) assumes the following form:

$$\left.\begin{aligned} F = (F) &+ y_1 \cos\varphi + y_2 \cos 2\varphi + y_3 \cos 3\varphi + \ldots \\ &+ x_1 \sin\varphi + x_2 \sin 2\varphi + x_3 \sin 3\varphi + \ldots \end{aligned}\right\}. \tag{3}$$

Instead of the unknowns r and α we thus introduce the new unknowns y and x, and when these are determined, we can at any time return again to r and α by solving equations (2) for the unknowns r and α.

We presuppose the following observations:

$$\left.\begin{matrix} \text{Amplitude } \varphi = & 0 & i & 2i & 3i\ldots & (n-1)i \\ \text{Observation} & F_0 & F_1 & F_2 & F_3\ldots & F_{n-1} \end{matrix}\right\}. \tag{4}$$

Here i is the interval of the amplitude φ, and the observations are distributed uniformly over *one* period, i.e.

$$n\,i = 360^\circ. \tag{5}$$

From (3) we form now the n error equations corresponding to the n observations F, whose general form is the following:

$$v = -F + (F) + y_1 \cos\varphi + x_1 \sin\varphi + y_2 \cos 2\varphi + x_2 \sin 2\varphi + \ldots. \tag{6}$$

The n error equations result therefrom if we substitute for φ, successively, the values $0, i, 2i \ldots$

and for F, accordingly, the values $F_0, F_1, F_2 \ldots$. We form the table of the coefficients of the error equations, and in order to fix the designations, we write in addition to (6) the corresponding equation

$$v = -l + a(F) + b y_1 + c x_1 + d y_2 + e x_2 + \ldots \tag{7}$$

The table of the coefficients $a\, b\, c \ldots$ and the absolute terms l is:

a	b	c	d	e	$\ldots$	$-l$
$+1$	$\cos 0$	$\sin 0$	$\cos 0$	$\sin 0$	$\ldots$	$-F_0$
$+1$	$\cos i$	$\sin i$	$\cos 2i$	$\sin 2i$	$(3i)$	$-F_1$
$+1$	$\cos 2i$	$\sin 2i$	$\cos 4i$	$\sin 4i$	$(6i)$	$-F_2$
$+1$	$\cos 3i$	$\sin 3i$	$\cos 6i$	$\sin 6i$	$(9i)$	$-F_3$
$+1$	$\cos 4i$	$\sin 4i$	$\cos 8i$	$\sin 8i$	$\ldots$	$-F_4$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

(8)

If we now form the sum coefficients $[aa], [ab] \ldots$, then we soon notice that, because of the symmetrical arrangement of the observations, they become extremely simple. For the coefficients are

$$\left.\begin{array}{lllll} [aa]=n & [ab]=0 & [ac]=0 & [ad]=0 & [ae]=0 \\ & [bb]=\frac{n}{2} & [bc]=0 & [bd]=0 & [be]=0 \\ & & [cc]=\frac{n}{2} & [cd]=0 & [ce]=0 \\ & & & [dd]=\frac{n}{2} & [de]=0 \\ & & & & [ee]=\frac{n}{2} \end{array}\right\} . \tag{9}$$

The proof of these formulae can be carried out by a geometrical consideration according to Fig. 1 in the margin: Equation (5), $n i = 360°$, corresponds to the construction of a regular polygon of n sides, whose projections are on two axes:

Projection on x:

$$s + s \cos i + s \cos 2i + s \cos 3i + \ldots$$

Projection on y:

$$0 + s \sin i + s \sin 2i + s \sin 3i + \ldots$$

The projections of a closed polygon on two axes, however, algebraically = zero; therefore, the two equations of group (9) are proved by these projections:

$$[ab] = 0 \quad \text{and} \quad [ac] = 0 .$$

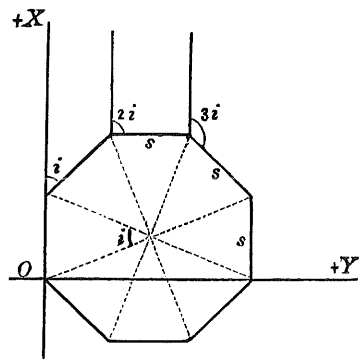


Fig. 1.
Geometrical representation of equations
$[ab] = 0$ and $[ac] = 0$

It is quite similar with the remaining sums of products $[ad] = 0$, $[ae] = 0$, etc., for here it is only a question of a *multiple* running through of polygons with central angles $2i$, $3i$, etc., and it is also similar with $[bc]$, $[bd]$, etc.; we convince ourselves that there always corresponds to each term $\sin(\ldots)\cos(\ldots)$ a term of the form $\sin(\ldots \pm 180°)\cos(\ldots \pm 180°)$. in the case of the sums of squares $[aa]$, $[bb]$, etc., we always find a grouping $\sin^2(\ldots) + \cos^2(\ldots)$.

By considering, according to this, the coefficients (9) as taken care of, we pass over to the solution of the normal equations. The latter are:

$$\left.\begin{array}{llll} [a\,a]\,(F) & \ldots\ldots\ldots & & -[a\,l]=0 \\ & [b\,b]\,y_1 & \ldots\ldots & -[b\,l]=0 \\ & & [c\,c]\,x_1 & -[c\,l]=0 \\ & & \ldots\ldots\ldots\ldots & \end{array}\right\} . \tag{9a}$$

Every normal equation thus contains only *one* unknown, and elimination is not necessary at all. By forming now also the absolute terms $[a\,l]$, $[b\,l]$ . . . according to (8), we obtain the following solution of the normal equations:

$$\left.\begin{array}{ll} n\,(F) = [a\,l] = [F] & \\ \frac{n}{2}\,y_1 = [b\,l] = (F_0 + F_1 \cos i + F_2 \cos 2\,i + F_3 \cos 3\,i + . .) & \\ \frac{n}{2}\,x_1 = [c\,l] = (. . \quad F_1 \sin i + F_2 \sin 2\,i + F_3 \sin 3\,i + . .) & \\ \frac{n}{2}\,y_2 = [d\,l] = (F_0 + F_1 \cos 2i + F_2 \cos 4\,i + F_3 \cos 6\,i + . .) & \\ \frac{n}{2}\,x_2 = [e\,l] = (. . \quad F_1 \sin 2i + F_2 \sin 4\,i + F_3 \sin 6i + . .) & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}\right\} . \tag{10}$$

The first equation $n\,(F) = [F]$ states that (F) is the arithmetic mean of all observations F, and it appears now suitable to count all observations from this mean value, and to set

$$F_0 = (F) + f_0, \quad F_1 = (F) + f_1, \quad F_2 = (F) + f_2, \text{ etc. ;} \tag{11}$$

here

$$[f] = 0 . \tag{12}$$

Now we can also write in (10) everywhere f instead of F, for, e.g., we will have by substitution of (11) in y_1:

$$y_1 = \frac{2}{n}\,(f_0 + f_1 \cos i + f_2 \cos 2\,i + \ldots) + \frac{2}{n}\,(F)\,(1 + \cos i + \cos 2\,i + \ldots) .$$

But the expression in parentheses of the second term is $= [a\,b] = 0$, consequently

$$y_1 = \frac{2}{n}\,(f_0 + f_1 \cos i + f_2 \cos 2\,i + f_3 \cos 3\,i + \ldots) , \tag{13}$$

and similarly with x_1, y_2, x_2 . . . .

We can in addition indicate the sum $[v\,v]$ and the mean error m of one observation. Since, according to (9), all sums of products $[a\,b]$, $[a\,c]$ . . . are equal to zero, then all $[b\,c \cdot 1]$'s, $[c\,d \cdot 2]$'s, etc., become also equal to zero and with this, formula (8), section 29, p. 91, is reduced to the simple form

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \frac{[b\,l]^2}{[b\,b]} - \frac{[c\,l]^2}{[c\,c]} \ldots .$$

Because of (10) and (9) we have here:

$$\frac{[a\,l]^2}{[a\,a]} = n\,(F)^2, \qquad \frac{[b\,l]^2}{[b\,b]} = \frac{n}{2}\,y_1^2, \qquad \frac{[c\,l]^2}{[c\,c]} = \frac{n}{2}\,x_1^2, \text{ etc.}$$

$$[v\,v] = [l\,l] - n\,(F)^2 - \frac{n}{2}\,(y_1^2 + x_1^2) - \frac{n}{2}\,(y_2^2 + x_2^2) - \dots \quad . \tag{14}$$

Because of (2) we have:

$$y_1^{\;2} + x_1^{\;2} = r_1^{\;2}\,, \qquad y_2^{\;2} + x_2^{\;2} = r_2^{\;2} \;\dots\; ;$$

therefore:

$$[v\,v] = [l\,l] - n\,(F)^2 - \frac{n}{2}\,[r\,r]\,. \tag{15}$$

The initial term $[l\,l]$ is first, according to (8):

$$[l\,l] = [F\,F]\,.$$

From (11) there follows:

$$[F\,F] = [f\,f] + n\,(F)^2 + 2\,(F)\,[f]\,, \tag{16}$$

or because, according to (12), $[f] = 0$:

$$[l\,l] = [F\,F] = [f\,f] + n\,(F)^2\,.$$

This set into (15) yields:

$$[v\,v] = [f\,f] - \frac{n}{2}\,[r\,r]\,. \tag{17}$$

The mean error of one observation will be

$$m = \sqrt{\frac{[v\,v]}{n-u}}\,, \tag{18}$$

where u is the number of the constants in the interpolation formula, e.g. in (1) we have $u = 7$; however, if we already break off at the terms in $2\,\varphi$, as in (14), then $u = 5$.

We set up, in addition, special formulae for the case $n = 12$ occurring frequently; therefore according to (5) $i = 30°$. In this case we will have according to (13)

$$\begin{aligned} 6\,y_1 = {} & f_0 + f_1 \cos 30° + f_2 \cos 60° + f_3 \cos 90° + f_4 \cos 120° + f_5 \cos 150° + f_6 \cos 180° \\ & + f_7 \cos 210° + f_8 \cos 240° + f_9 \cos 270° + f_{10} \cos 300° + f_{11} \cos 330°\,. \end{aligned}$$

But since all goniometric functions occurring here can be reduced to $\sin 30° = 0.5$ and $\cos 30° = 0.8660$, then we obtain

$$6\,y_1 = (f_0 - f_6) + (f_2 - f_4 - f_8 + f_{10}) \sin 30° + (f_1 - f_5 - f_7 + f_{11}) \cos 30°\,.$$

All such formulae for $n = 12$, corresponding to a function (1), which is continued to $4\,\varphi$, are obtained from (10), taking into account the substitution of the F's by f, as in (13). The results are:

$$\left.\begin{aligned} 6\,y_1 &= (f_0 - f_6) + (f_2 - f_4 - f_8 + f_{10}) \sin 30° + (f_1 - f_5 - f_7 + f_{11}) \cos 30° \\ 6\,x_1 &= (f_3 - f_9) + (f_1 + f_5 - f_7 - f_{11}) \sin 30° + (f_2 + f_4 - f_8 - f_{10}) \cos 30° \\ 6\,y_2 &= (f_0 - f_3 + f_6 - f_9) + (f_1 - f_2 - f_4 + f_5 + f_7 - f_8 - f_{10} + f_{11}) \sin 30° \\ 6\,x_2 &= (f_1 + f_2 - f_4 - f_5 + f_7 + f_8 - f_{10} - f_{11}) \cos 30° \\ 6\,y_3 &= (f_0 - f_2 + f_4 - f_6 + f_8 - f_{10}) \\ 6\,x_3 &= (f_1 - f_3 + f_5 - f_7 + f_9 - f_{11}) \\ 6\,y_4 &= (f_0 + f_3 + f_6 + f_9) - (f_1 + f_2 + f_4 + f_5 + f_7 + f_8 + f_{10} + f_{11}) \sin 30° \\ 6\,x_4 &= (f_1 - f_2 + f_4 - f_5 + f_7 - f_8 + f_{10} - f_{11}) \cos 30° \end{aligned}\right\}. \tag{19}$$

As a numerical example for the formulae (19) we take the 6 years means of barometer readings from 1868 to 1873 in Cairo (Jordan, *Physische Geogr. und Meteorol. der Lyb. Wüste*, p. 144). These mean values for the individual months enter into our computation as observations F, as the following table shows:

φ	Month	Observed F	Observed $f = F - 758.26$	Observed f^2	Adjusted f'	Adjusted F'	Discrepancies v	Discrepancies v^2
		mm	*mm*		*mm*	*mm*	*mm*	
0°	January . . . . . 0.	761.70	+ 3.44	11.83	+ 3.95	762.21	+0.51	0.26
30	February . . . . 1.	761.74	+ 3.48	12.11	+ 2.80	761.06	−0.68	0.46
60	March . . . . . . 2.	757.62	− 0.64	0.41	+ 0.09	758.35	+0.73	0.53
90	April . . . . . . 3.	758.14	− 0.12	0.01	− 0.72	757.54	−0.60	0.36
120	May . . . . . . 4.	757.15	− 1.11	1.23	− 0.77	757.49	+0.34	0.12
150	June . . . . . . 5.	755.75	− 2.51	6.30	− 2.52	755.74	−0.01	0.00
180	July . . . . . . 6.	754,51	− 3.75	14.06	− 4.00	754.26	−0.25	0.06
210	August . . . . . 7.	754,40	− 3.86	14.90	− 3.46	754.80	+0.40	0.16
240	September . . . . 8.	757.10	− 1.16	1.35	− 1.52	756.74	−0.36	0.13
270	October . . . . . 9.	758.90	+ 0.64	0.41	+ 0.84	759.10	+0.20	0.04
300	November . . . . 10.	760.51	+ 2.25	5.06	+ 2.27	760.53	+0.02	0.00
330	December . . . . 11.	761.61	+ 3.35	11.22	+ 3.08	761.34	−0.27	0.07
		Mean (F) = 758,26	+ 13.16 − 13.15	78.89	+ 13.03 − 12.99	758.26.	+ 2.20 − 2.17	2.19

According to (19) we compute:

$$6y_1 = +20.561 \qquad 6y_2 = -0.270 \qquad 6y_3 = +3.31 \qquad 6y_4 = +0.11$$

$$6x_1 = -\ 2.479\,, \qquad 6x_2 = -3.603\,, \qquad 6x_3 = +2.24\,, \qquad 6x_4 = +1.49\,.$$

According to (2) we have $\tan\alpha_1 = \dfrac{+20.561}{-2.479}\,, \qquad \alpha_1 = 96°53'$

$$r_1 = \frac{1}{6}\,\frac{20.561}{\sin\alpha_1} = \frac{1}{6}\,\frac{-2.479}{\cos\alpha_1}\,, \qquad r_1 = 3.452\,,$$

and in the same manner, we also find the remaining α's and r's, so that we have:

$$\begin{aligned} \alpha_1 &= \ \ 96°53' & r_1 &= 3.452 \\ \alpha_2 &= 184\ \ 17 & r_2 &= 0.602 \\ \alpha_3 &= \ \ 55\ \ 55 & r_3 &= 0.666 \\ \alpha_4 &= \ \ \ \ 4\ \ 13\,, & r_4 &= 0.249\,. \end{aligned}$$

The interpolation formula reads now

$$\left.\begin{aligned} f' = 3.452 \sin(96°\,53' + \varphi) + 0.602 \sin(184°\,17' + 2\varphi) \\ + 0.666 \sin(55°\,55' + 3\varphi) + 0.249 \sin(4°\,13' + 4\varphi) \end{aligned}\right\}, \tag{20}$$

or the whole formula according to (1)

$$F' = 758.26 + f'\,.$$

The adjusted values f' and F' of the above table are computed according to this.

By comparison of the observed and of the adjusted function values F or f we obtain the discrepancies v, whose sum of squares resulting from the preceding table (above) by direct computation is $[v\,v] = 2.19$.

As a thorough check of the whole adjustment computation, we determine this sum also according to (17):

$$\left.\begin{array}{ll} r_1 = 3.452 & r_1^2 = 11.916 \\ r_2 = 0.602 & r_2^2 = 0.362 \\ r_3 = 0.666 & r_3^2 = 0.444 \\ r_4 = 0.249 & r_4^2 = 0.062 \end{array}\right\} \quad \begin{array}{r} [r\,r] = 12.784 \\ 6\,[r\,r] = 76.704 \end{array}$$

$$[v\,v] = [f\,f] - \frac{n}{2}\,[r\,r] = 78.89 - 76.70 = 2.19\,,$$

which agrees perfectly with the direct calculation of $[v\,v] \doteq 2.19$. Finally, we have in addition the mean error of an observation according to (18):

$$m = \sqrt{\frac{2.19}{12-9}} = \pm\,0.85 \text{ mm}\,. \tag{21}$$

If we want to go only as far as the terms in 3φ, then we have only to omit in (20) the last term, and conversely, if we are not satisfied with the agreement between the computation and the observation, and therefore, take into the formula an additional term, then we add additional terms, retaining all results obtained the first time.

It cannot of course be generally established how many terms we shall give the interpolation formula (1); we can only require that the mean error (18), after the adjustment, corresponds as well as possible to the deviations which the observations showed among themselves before the adjustment.

The computation by interpolation treated here was first treated by Bessel in *Astronom. Nachr.*, Vol. 6, 1828, pp. 333-356: "Über die Bestimmung des Gesetzes einer periodischen Erscheinung." Our above representation gives Bessel's method, only in a somewhat more illustrative preparation.

The coefficients $\sin i$, $\sin 2i$, $\sin 3i \ldots$, $\cos i$, $\cos 2i$, $\cos 3i \ldots$, which we have calculated in (19) for $n = 12$, hence $i = 30°$, can also be computed for other cases once for all, e.g., for $n = 73$, $i = 4°55'53''$ (five days mean values, $n = 365:5$) the logarithms of such coefficients are communicated in: *Des anomalies de la température observées à Genève*, by Plantamour, Geneva and Basel, 1867, pp. 20 and 21, where the amplitude φ, however, is counted differently so that the formulae and coefficients belonging to them do not directly agree with ours.

In this connection, there is also mentioned "Ein graphisches Verfahren zur Herstellung der Koeffizienten der Besselschen Reihe," by Prof. Dr. Paul Schreiber in *Meteorologische Zeitschrift*, June 1891, p. 237.

Section 42. Conditioned Observations, Reduced to Indirect Observations

After having treated in detail the adjustment of indirect observations in the foregoing sections 12 to 41, we turn now to a second problem of adjustment calculation, in the case of which definite conditions exist from the outset between the observed quantities. Since the measurements will not satisfy these conditions due to the unavoidable observational errors, then they are to be corrected in such a way that the corrected measurements satisfy the conditions and that, at the same time, the sum of the squares of the corrections becomes a minimum.

The problem is therefore the following:

There are observed directly n quantities, between which there exist condition equations rigorously to be satisfied, where n is larger than r; the observations are to be corrected in such a way that they satisfy the condition equations and that the sum of the squares of the corrections is a minimum.

A very simple case of this kind is, for instance, the measurement of three angles x_1, x_2, x_3 in a triangle, where a condition equation $x_1 + x_2 + x_3 - 180° = 0$ exists (or, as the case may be, 180° + spherical excess), hence $n = 3$, $r = 1$.

In section 10, pp. 28 to 32, we have already reduced this problem to an adjustment according to the arithmetic mean.

We can always reduce the adjustment of conditioned observations to indirect observations, which we shall now show.

We assume that there are to be determined n unknown quantities $x_1, x_2, x_3, \ldots, x_n$, and we have made, for this, equally accurate measurements $l_1, l_2, l_3, \ldots, l_n$. Between the unknowns x there exist the following r condition equations rigorously to be satisfied.

$$\text{Number} = r \left\{ \begin{array}{l} a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n = 0 \\ b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots + b_n x_n = 0 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots + c_n x_n = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \underbrace{r_0 + r_1 x_1 + r_2 x_2 + r_3 x_3 + \ldots + r_n x_n}_{\text{Number } = n.} = 0 \end{array} \right\} \tag{1}$$

If we set in the place of the unknowns x the observational values l, then equations (1) are no longer satisfied, but they give

$$\left. \begin{array}{l} a_0 + a_1 l_1 + a_2 l_2 + a_3 l_3 + \ldots + a_n l_n = w_1 \\ b_0 + b_1 l_1 + b_2 l_2 + b_3 l_3 + \ldots + b_n l_n = w_2 \\ c_0 + c_1 l_1 + c_2 l_2 + c_3 l_3 + \ldots + c_n l_n = w_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r_0 + r_1 l_1 + r_2 l_2 + r_3 l_3 + \ldots + r_n l_n = w_r \end{array} \right\} . \tag{2}$$

To the l's there are to be applied such corrections v that the discrepancies w vanish again, e.g. for the first equation of (2):

$$a_0 + a_1 (l_1 + v_1) + a_2 (l_2 + v_2) + \ldots + a_n (l_n + v_n) = 0 .$$

This compared with the first equation of (2) yields

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + w_1 = 0 ,$$

hence, for the whole system of this kind;

$$\text{Number} = r \left\{ \begin{array}{l} a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n + w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + \ldots + b_n v_n + w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n + w_3 = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \underbrace{r_1 v_1 + r_2 v_2 + r_3 v_3 + \ldots + r_n v_n}_{\text{Number } = n.} + w_r = 0 \end{array} \right\} \tag{3}$$

To this, we have in addition the condition of adjustment

$$[v v] = \text{minimum} . \tag{4}$$

For the number n of the unknowns and the number r of the condition equations it is essential that n is larger than r, i.e.

$$n > r , \tag{5}$$

for it will not be possible to determine the v's already from equations (3) without the use of (4).

Besides, the condition equations (1) should be independent of one another, and the same holds good of the derived condition equations (3), which must likewise be independent of one another.

We can now reduce the adjustment to indirect observations, and that by expressing, with the help of the condition equations (3), an arbitrary choice of r unknowns by the remaining $n - r$. Let us obtain:

$$\text{Number} = r \left\{ \begin{array}{l} v_1 = A_1 v_{r+1} + B_1 v_{r+2} + \ldots + H_1 v_n \\ v_2 = A_2 v_{r+1} + B_2 v_{r+2} + \ldots + H_2 v_n \\ \vdots \\ v_r = \underbrace{A_r v_{r+1} + B_r v_{r+2} + \ldots + H_r v_r}_{\text{Number } = n - r.} \end{array} \right\} \tag{6}$$

To this, we also write the $v_{r+1} v_{r+2} \ldots v_n$ themselves chosen as independent:

$$\text{Number} = n - r \left\{ \begin{array}{l} v_{r+1} = v_{r+1} \quad \cdots \quad \cdots \\ v_{r+2} = \quad \cdots \quad v_{r+2} \quad \cdots \\ \vdots \\ v_n \quad = \underbrace{\quad \cdots \cdots \cdots \quad v_n}_{\text{Number } = n - r.} \end{array} \right\} \tag{7}$$

Now (6) and (7) represent a system of $r + (n - r)$, i.e. of n error equations with $n - r$ independent unknowns, which together are to be treated according to section 27 and following. The mean error of the unit of weight m is determined, for this, according to (19), section 29, p. 93:

$$m = \sqrt{\frac{[v\,v]}{(r + (n - r)) - (n - r)}} = \sqrt{\frac{[v\,v]}{r}}. \tag{8}$$

Whether this reduction to indirect observations, indicated in general equations, is useful depends on the individual case. If the condition equations are of a simple nature, and if their number is not large, then this method may be advisable. We shall treat an example for it in the later section 76.

First, the important thing for us to do was only to survey the conditions as a whole and to find, at the same time, the error formula (8), which holds always, whether we may carry out the adjustment by reduction to indirect observations or according to the correlate method to be treated later.

About the *sign of the absolute terms* w, which, according to (1) and (2), represent the discrepancies between observation and theory, we note further, in general, that it is determined, in any case, according to the formula:

Discrepancy w = observation – theory

or

$$w = \textit{observational value} - \textit{theoretical value}\,. \tag{9}$$

For the problem set up in the previous section 42 we will now set up an independent solution.

We take up again the equations set up there on p. 136 but, for the sake of over-all clearness, we write here everywhere only four symbols x_1, x_2, x_3, x_4, where there are in the general case n, and three equations with a, b, c instead of, generally, r equations.

Between the unknowns x there exist the following condition equations rigorously to be satisfied:

$$\left.\begin{aligned} a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 &= 0 \\ b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 &= 0 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 &= 0 \end{aligned}\right\} . \tag{1}$$

If instead of the unknowns x we set the observational values l_1, l_2, l_3, l_4, then equations (1) are not satisfied, but we obtain

$$\left.\begin{aligned} a_0 + a_1 l_1 + a_2 l_2 + a_3 l_3 + a_4 l_4 &= w_1 \\ b_0 + b_1 l_1 + b_2 l_2 + b_3 l_3 + b_4 l_4 &= w_2 \\ c_0 + c_1 l_1 + c_2 l_2 + c_3 l_3 + c_4 l_4 &= w_3 \end{aligned}\right\} . \tag{2}$$

Therefore, we are to apply to the l's such corrections that the discrepancies w vanish, i.e. we set instead of the l's

$$l_1 + v_1 , \quad l_2 + v_2 , \quad l_3 + v_3 , \quad l_4 + v_4 , \tag{3}$$

and with this, the first equation (2) yields

$$a_0 + a_1 (l_1 + v_1) + a_2 (l_2 + v_2) + a_3 (l_3 + v_3) + a_4 (l_4 + v_4) = 0 .$$

The comparison with the first equation of (2) yields the following first equation, to which we add, at the same time, the other two:

$$\begin{matrix}\text{General} \\ \text{number} = r\end{matrix} \left\{\left.\underbrace{\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 &+ w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 &+ w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 &+ w_3 = 0 \end{aligned}}_{\text{General number} = n .}\right\}\right. \tag{4}$$

With the use of sum brackets, we can also write (2) and (4) as follows:

$$\left.\begin{aligned} a_0 + [a l] &= w_1 , \qquad & [a v] &= - w_1 \\ b_0 + [b l] &= w_2 , & [b v] &= - w_2 \\ c_0 + [c l] &= w_3 , & [c v] &= - w_3 \end{aligned}\right\} . \tag{5}$$

We designate again the r equations (4), likewise, as condition equations, for they indicate the conditions which the v's must satisfy.

Besides equations (4), we have for the determination of the v's the basic condition

$$[v v] = v_1^2 + v_2^2 + v_3^2 + v_4^2 = \text{minimum} . \tag{6}$$

In order to obtain this minimum, taking into account the side equations (4), analysis teaches us to proceed as follows:

We multiply the condition equations (4) by undetermined coefficients $-2\,k_1, -2\,k_2, -2\,k_3$ (which are chosen in negative and twofold form, because -2 cancels again by itself later), hence:

$$\left.\begin{aligned} -2\,a_1 k_1 v_1 - 2\,a_2 k_1 v_2 - 2\,a_3 k_1 v_3 - 2\,a_4 k_1 v_4 - 2\,w_1 k_1 &= 0 \\ -2\,b_1 k_2 v_1 - 2\,b_2 k_2 v_2 - 2\,b_3 k_2 v_3 - 2\,b_4 k_2 v_4 - 2\,w_2 k_2 &= 0 \\ -2\,c_1 k_3 v_1 - 2\,c_2 k_3 v_2 - 2\,c_3 k_3 v_3 - 2\,c_4 k_3 v_4 - 2\,w_3 k_3 &= 0 \end{aligned}\right\}. \tag{7}$$

These equations are added to the minimum condition (6), and then we collect terms everywhere with respect to v:

$$\left.\begin{aligned} \Omega = v_1^2 &- 2\,v_1\,(a_1 k_1 + b_1 k_2 + c_1 k_3) \\ + v_2^2 &- 2\,v_2\,(a_2 k_1 + b_2 k_2 + c_2 k_3) \\ + v_3^2 &- 2\,v_3\,(a_3 k_1 + b_3 k_2 + c_3 k_3) \\ + v_4^2 &- 2\,v_4\,(a_4 k_1 + b_4 k_2 + c_4 k_3) \\ &- 2\,\quad (w_1 k_1 + w_2 k_2 + w_3 k_3) \end{aligned}\right\}. \tag{8}$$

Now if we determine the absolute minimum of function Ω, then this is equivalent to the minimum of $[v\,v]$ and the taking into account of the side conditions (4).

The minimum of Ω and the differentiation with respect to v_1, v_2, v_3, v_4, necessary for this, yields

$$0 = 2\,v_1 - 2\,(a_1 k_1 + b_1 k_2 + c_1 k_3),\ \text{etc.},$$

i.e., as a whole, the *correlate equations*

$$\left.\begin{aligned} v_1 &= a_1 k_1 + b_1 k_2 + c_1 k_3 \\ v_2 &= a_2 k_1 + b_2 k_2 + c_2 k_3 \\ v_3 &= a_3 k_1 + b_3 k_2 + c_3 k_3 \\ v_4 &= a_4 k_1 + b_4 k_2 + c_4 k_3 \end{aligned}\right\}. \tag{9}$$

If we set these expressions again into the condition equations (4) and collect terms with respect to k_1, k_2, k_3, then we obtain

$$\left.\begin{aligned} [a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 + w_1 &= 0 \\ [a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 + w_2 &= 0 \\ [a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 + w_3 &= 0 \end{aligned}\right\}. \tag{10}$$

These equations are called *normal equations*; their number is equal to the number of condition equations.

Computation procedure. After the coefficients a, b, c and the absolute terms w of the condition equations (4) are determined, we compute the coefficients $[a\,a]$, $[a\,b]$, etc., of the normal equations (10), solve these for k_1, k_2, k_3 and compute then the corrections v according to the correlate equations (9).

Addition of the v's to the observations l yields then, finally, the adjusted values of the unknowns.

After all individual v's are computed, we find also the sum of their squares and the *mean error of an observation* according to (8), section 42, p. 137, are:

$$m = \sqrt{\frac{[v\,v]}{r}}, \tag{11}$$

where r is the number of condition equations. (In the above formulae we have everywhere $r = 3$.)

Section 44. Unequal Weights

If the observations have unequal weights $p_1, p_2, p_3, p_4,$ then the new condition

$$[p\,v\,v] = p_1 v_1^2 + p_2 v_2^2 + p_3 v_3^2 + p_4 v_4^2 = \text{minimum}\,, \tag{1}$$

takes the place of $[v\,v]$ = minimum, and if we continue the computation with this, then we see soon that the values $\frac{a}{p}, \frac{b}{p}, \frac{c}{p}$ take the place of a, b, c everywhere, and the most important systems (10) and (9) of the previous section 43 change then into

Normal equations

$$\left.\begin{aligned} \left[\frac{a\,a}{p}\right]k_1 + \left[\frac{a\,b}{p}\right]k_2 + \left[\frac{a\,c}{p}\right]k_3 + w_1 &= 0 \\ \left[\frac{a\,b}{p}\right]k_1 + \left[\frac{b\,b}{p}\right]k_2 + \left[\frac{b\,c}{p}\right]k_3 + w_2 &= 0 \\ \left[\frac{a\,c}{p}\right]k_1 + \left[\frac{b\,c}{p}\right]k_2 + \left[\frac{c\,c}{p}\right]k_3 + w_3 &= 0 \end{aligned}\right\}, \tag{2}$$

Correlate equations

$$\left.\begin{aligned} v_1 &= \frac{a_1}{p_1}k_1 + \frac{b_1}{p_1}k_2 + \frac{c_1}{p_1}k_3 \\ v_2 &= \frac{a_2}{p_2}k_1 + \frac{b_2}{p_2}k_2 + \frac{c_2}{p_2}k_3 \\ v_3 &= \frac{a_3}{p_3}k_1 + \frac{b_3}{p_3}k_2 + \frac{c_3}{p_3}k_3 \\ v_4 &= \frac{a_4}{p_4}k_1 + \frac{b_4}{p_4}k_2 + \frac{c_4}{p_4}k_3 \end{aligned}\right\}. \tag{3}$$

If we do not want to carry along the weights p everywhere, then we can divide all coefficients a, b, c, from the beginning, by $\sqrt{p}$, respectively, and then compute further as if all weights = 1.

The first condition equation reads then as follows:

$$\frac{a_1}{\sqrt{p_1}} v_1\sqrt{p_1} + \frac{a_2}{\sqrt{p_2}} v_2\sqrt{p_2} + \frac{a_3}{\sqrt{p_3}} v_3\sqrt{p_3} + \frac{a_4}{\sqrt{p_4}} v_4\sqrt{p_4} + w_1 = 0\,,$$

or

$$a'_1 v'_1 + a'_2 v'_2 + a'_3 v'_3 + a'_4 v'_4 + w_1 = 0\,.$$

The first normal equation:

$$[a'\,a']k_1 + [a'\,b']k_2 + [a'\,c']k_3 + w_1 = 0\,.$$

The first correlate equation:

$$v_1' = v_1\sqrt{p_1} = a_1' k_1 + b_1' k_2 + c_1' k_3\,;$$

therefrom

$$v_1 = \frac{v_1'}{\sqrt{p_1}}\,.$$

$[p\,v\,v]$ results directly by squaring the v''s, namely

$$[p\,v\,v] = [(v\sqrt{p})^2] = [v'\,v'] \,. \tag{4}$$

Unequal weights by mean errors a priori

If the mean errors m_1, m_2, m_3, m_4 are estimated a priori, which often happens, then we can assume the weights directly:

$$p_1 = \frac{1}{m_1^2}, \qquad p_2 = \frac{1}{m_2^2}, \qquad p_3 = \frac{1}{m_3^2}, \text{ and so on }, \tag{5}$$

and with this, the above equations assume the following forms:

$$\left[\frac{v\,v}{m\,m}\right] = \left(\frac{v_1}{m_1}\right)^2 + \left(\frac{v_2}{m_2}\right)^2 + \left(\frac{v_3}{m_3}\right)^2 + \ldots = \text{minimum} \,. \tag{6}$$

Condition equations:

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + w_1 &= 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + \ldots + w_2 &= 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + w_3 &= 0 \end{aligned}\right\} . \tag{7}$$

Mean errors: $$m_1, \quad m_2, \quad m_3 \ldots \tag{8}$$

We form from (7) and (8) the new coefficients:

$$\left.\begin{aligned} m_1 a_1 &= a_1' & m_2 a_2 &= a_2' & m_3 a_3 &= a_3' \\ m_1 b_1 &= b_1' & m_2 b_2 &= b_2' & m_3 b_3 &= b_3' \\ m_1 c_1 &= c_1' & m_2 c_2 &= c_2' & m_3 c_3 &= c_3' \end{aligned}\right\} , \tag{9}$$

then the *normal equations* become:

$$\left.\begin{aligned} &[m^2aa]k_1+[m^2ab]k_2+[m^2ac]k_3+w_1=0 & &[a'a']k_1+[a'b']k_2+[a'c']k_3+w_1=0 \\ &[m^2ab]k_1+[m^2bb]k_2+[m^2bc]k_3+w_2=0 \quad \text{or} & &[a'b']k_1+[b'b']k_2+[b'c']k_3+w_2=0 \\ &[m^2ac]k_1+[m^2bc]k_2+[m^2cc]k_3+w_3=0 & &[a'c']k_1+[b'c']k_2+[c'c']k_3+w_3=0 \end{aligned}\right\} . \tag{10}$$

Correlate equations

$$\left.\begin{aligned} v_1' &= \frac{v_1}{m_1} = a_1' k_1 + b_1' k_2 + c_1' k_3 \\ v_2' &= \frac{v_2}{m_2} = a_2' k_1 + b_2' k_2 + c_2' k_3 \\ &\ldots\ldots\ldots\ldots \\ v_1 &= v_1' m_1 , \quad v_2 = v_2' m_2 , \quad v_3 = v_3' m_3 \ldots \end{aligned}\right\} . \tag{11}$$

Mean error of the unit of weight after the adjustment:

$$m = \sqrt{\frac{[v'\,v']}{r}} \quad \text{or} \quad = \sqrt{\frac{1}{r}\left[\frac{v\,v}{m\,m}\right]} \,. \tag{12}$$

If the mean errors m_1, m_2, m_3 [see p. 141 in (8)] were apportioned correctly *before* the adjustment, then, *after* the adjustment there must be $m = 1$, according to (12).

The values $v'_1, v'_2, v'_3 \ldots$ are pure ratios whose more or less considerable deviation from 1 gives a convenient insight into the distribution of errors and into the usability of the m_1's, m_2's, m_3's ... estimated a priori.

If the adjustment does not yield the expected $m = 1$, but if, say, $m = M$ results from formula (12), then it is proper to change all mean errors estimated a priori by the ratio $M : m$.

Section 45. Sum of Squares $[v\,v]$ of the Corrections

The computation of the mean error of the unit of weight is carried out according to (8), section 42, p. 137, namely:

$$m = \sqrt{\frac{[v\,v]}{r}} \qquad \text{or, as the case may be,} \qquad \sqrt{\frac{[p\,v\,v]}{r}}\,. \tag{1}$$

We shall speak hereafter, as a rule, only of the sum $[v\,v]$, because in the case of unequal weights the sum $[p\,v\,v]$ can be set easily in place of $[v\,v]$ according to section 44.

After completion of the adjustment we can square and add the individual v's, and with this, we have directly $[v\,v]$.

Instead of forming $[v\,v]$ from the individual v's, we can adopt yet other ways, as also previously in the case of indirect observations.

For this, in particular, we take up once more the expressions for the individual v's according to (9), section 43, p. 139:

$$\left.\begin{aligned} v_1 &= a_1\,k_1 + b_1\,k_2 + c_1\,k_3 \\ v_2 &= a_2\,k_1 + b_2\,k_2 + c_2\,k_3 \\ v_3 &= a_3\,k_1 + b_3\,k_2 + c_3\,k_3 \\ v_4 &= a_4\,k_1 + b_4\,k_2 + c_4\,k_3 \end{aligned}\right\}. \tag{2}$$

We find therefrom

$$\left.\begin{aligned} [v\,v] = [a\,a]\,k_1\,k_1 + 2\,[a\,b]\,k_1\,k_2 &+ 2\,[a\,c]\,k_1\,k_3 \\ + \quad [b\,b]\,k_2\,k_2 &+ 2\,[b\,c]\,k_2\,k_3 \\ &+ \quad [c\,c]\,k_3\,k_3 \end{aligned}\right\}, \tag{3}$$

and if we compare this with the normal equations (10), section 43, p. 139, then we have at once

$$[v\,v] = -k_1 w_1 - k_2 w_2 - k_3 w_3 = -[w\,k]\,. \tag{4}$$

Besides, we can apply the general transformation (20) at the end of section 29, p. 93, which yields for (3):

$$[v\,v] = \frac{([a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3)^2}{[a\,a]} + \frac{([b\,b\cdot 1]\,k_2 + [b\,c\cdot 1]\,k_3)^2}{[b\,b\cdot 1]} + \frac{([c\,c\cdot 2]\,k_3)^2}{[c\,c\cdot 2]}\,,$$

i.e. with the insertion of the w's according to (10), section 43, p. 139, and corresponding to $[w_2 \cdot 1]$, etc.:

$$[v\,v] = \frac{w_1{}^2}{[a\,a]} + \frac{[w_2\cdot 1]^2}{[b\,b\cdot 1]} + \frac{[w_3\cdot 2]^2}{[c\,c\cdot 2]}\,. \tag{5}$$

We can connect this computation, likewise, to the elimination of the normal equations, as was done with $[vv] = [ll \cdot u]$ in the case of indirect observations, i.e., we add to the normal equations the end term 0, and obtain then the following system of stepwise reduced normal equations:

$$\begin{array}{ll} \underline{[aa]}\,k_1 + [ab]\,k_2 + [ac]\,k_3 + w_1 & \\ \underline{[bb]}\,k_2 + [bc]\,k_3 + w_2 & \underline{[bb\cdot 1]}\,k_2 + [bc\cdot 1]\,k_3 + [w_2\cdot 1] \\ \underline{[cc]}\,k_3 + w_3 & \underline{[cc\cdot 1]}\,k_3 + [w_3\cdot 1] \\ \underline{0} & \underline{[0\cdot 1]} \\ & \underline{[cc\cdot 2]}\,k_3 + [w_3\cdot 2] \\ & \underline{[0\cdot 2]} \\ & \underline{[0\cdot 3]}\,. \end{array}$$

The end term $[0.3]$ is then $= -[vv]$.

Section 46. Weight of a Function of the Adjusted Observations

With the theory of section 43 and section 45 we can already make the usual adjustment of conditioned observations, to which the first part (1) to (8) of the summary of formulae of section 47 belongs. Especially for the study of geodesy it is advisable, after consideration of the first part of section 47 (1) to (8), to pass over now at once to the adjustment of triangulations of our chapter II. However, the deductive course of our total theory leads us to the weight of a function.

We consider an arbitrary quantity F, which is to be computed from the observations. If we use directly the observations for this, then we assume that the value

$$(F) = f_0 + f_1 l_1 + f_2 l_2 + f_3 l_3 + f_4 l_4 \tag{1}$$

results, and the weight becomes, if all observations have the weight 1,

$$\frac{1}{P} = f_1^{\,2} + f_2^{\,2} + f_3^{\,2} + f_4^{\,2} = [ff]\,. \tag{2}$$

The quantity (F) (which, for instance, represents a triangle side in the case of a triangulation) does not need to contain all observations; besides, we can compute it by different ways (for instance, a triangle side from different triangles), where, in general, disagreeing values for (F) result because of the measuring errors.

It is different if we use the *adjusted* observations, with which we obtain the value

$$F = f_0 + f_1\,(l_1 + v_1) + f_2\,(l_2 + v_2) + f_3\,(l_3 + v_3) + f_4\,(l_4 + v_4)\ . \tag{3}$$

Since the adjusted observations no longer contain any discrepancies, we always find the same value for F no matter what way we adopt in the computation.

The corrections v are computed from the adjustment from all observations l and are linear functions of the observations, so that we obtain

$$F = F_0 + F_1 l_1 + F_2 l_2 + F_3 l_3 + F_4 l_4\,, \tag{4}$$

and the weight of F is

$$\frac{1}{P} = F_1^{\,2} + F_2^{\,2} + F_3^{\,2} + F_4^{\,2} = [FF]\,. \tag{5}$$

For the determination of the coefficients $F_0, F_1, \ldots, F_4$ we start from the correlate equations (9), section 43, p. 139, which, if we assume three condition equations, read

$$\left.\begin{aligned} v_1 &= a_1 k_1 + b_1 k_2 + c_1 k_3 \\ v_2 &= a_2 k_1 + b_2 k_2 + c_2 k_3 \\ v_3 &= a_3 k_1 + b_3 k_2 + c_3 k_3 \\ v_4 &= a_4 k_1 + b_4 k_2 + c_4 k_3 \end{aligned}\right\} . \tag{6}$$

If we introduce this into (3), and collect all homogeneous terms into sum brackets, then we will have

$$\left.\begin{aligned} F = \quad & f_0 + [f\,l] \\ & + [a\,f]\,k_1 + [b\,f]\,k_2 + [c\,f]\,k_3 \end{aligned}\right\} . \tag{7}$$

The correlates k result from the normal equations (10), section 43, p. 139, which we write as follows, taking into account (5), section 43, p. 138:

$$\left.\begin{aligned} r_1) \qquad & [a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 + [a\,l] + a_0 = 0 \\ r_2) \qquad & [a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 + [b\,l] + b_0 = 0 \\ r_3) \qquad & [a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 + [c\,l] + c_0 = 0 \end{aligned}\right\} . \tag{8}$$

We multiply the three normal equations by the coefficients r_1, r_2, r_3 left undetermined for the present and add them to (7), so that we obtain

$$\left.\begin{aligned} F = \quad & f_0 + [f\,l] + [a\,f]\,k_1 + [b\,f]\,k_2 + [c\,f]\,k_3 \\ & + a_0 r_1 + [a\,l]\,r_1 + [a\,a]\,k_1 r_1 + [a\,b]\,k_2 r_1 + [a\,c]\,k_3 r_1 \\ & + b_0 r_2 + [b\,l]\,r_2 + [a\,b]\,k_1 r_2 + [b\,b]\,k_2 r_2 + [b\,c]\,k_3 r_2 \\ & + c_0 r_3 + [c\,l]\,r_3 + [a\,c]\,k_1 r_3 + [b\,c]\,k_2 r_3 + [c\,c]\,k_3 r_3 \end{aligned}\right\} . \tag{9}$$

We dispose of the r_1's, r_2's, r_3's in such a way that the terms multiplied by k_1, k_2, k_3 vanish. This happens if the r_1's, r_2's, r_3's satisfy the

Transformation equations

$$\left.\begin{aligned} [a\,a]\,r_1 + [a\,b]\,r_2 + [a\,c]\,r_3 + [a\,f] = 0 \\ [a\,b]\,r_1 + [b\,b]\,r_2 + [b\,c]\,r_3 + [b\,f] = 0 \\ [a\,c]\,r_1 + [b\,c]\,r_2 + [c\,c]\,r_3 + [c\,f] = 0 \end{aligned}\right\} . \tag{10}$$

Of expression (9) there remains then

$$F = f_0 + [f\,l] + (a_0 + [a\,l])\,r_1 + (b_0 + [b\,l])\,r_2 + (c_0 + [c\,l])\,r_3 \,, \tag{11}$$

and if we resolve this into its individual terms and collect terms according to l, then we obtain

$$\left.\begin{aligned} F = \quad & f_0 + a_0 r_1 + b_0 r_2 + c_0 r_3 \\ & + (f_1 + a_1 r_1 + b_1 r_2 + c_1 r_3) l_1 \\ & + (f_2 + a_2 r_1 + b_2 r_2 + c_2 r_3) l_2 \\ & + (f_3 + a_3 r_1 + b_3 r_2 + c_3 r_3) l_3 \\ & + (f_4 + a_4 r_1 + b_4 r_2 + c_4 r_3) l_4 \end{aligned}\right\} . \tag{12}$$

With this, there is obtained what was tried for in (4), namely the development of F as a linear function of the l's, where the coefficients are

$$\left.\begin{aligned} F_0 &= f_0 + a_0 r_1 + b_0 r_2 + c_0 r_3 \\ F_1 &= f_1 + a_1 r_1 + b_1 r_2 + c_1 r_3 \\ F_2 &= f_2 + a_2 r_1 + b_2 r_2 + c_2 r_3 \\ F_3 &= f_3 + a_3 r_1 + b_3 r_2 + c_3 r_3 \\ F_4 &= f_4 + a_4 r_1 + b_4 r_2 + c_4 r_3 \end{aligned}\right\} . \tag{13}$$

Therefore, we have now the following procedure:

After the coefficients f_1, f_2, f_3, f_4 of function (1), whose weight is to be determined, are established, we form the sums $[af]$, $[bf]$, $[cf]$, which, connected to the normal equations as absolute terms, yield the transformation equations (10). We solve these for r_1, r_2, r_3, set these r's into (13), and can calculate, with these, all F's, squaring which yields the required weight reciprocal according to (5).

In individual cases, this method may be appropriate; in any case, it is insofar illustrative as it follows exactly the way of the adjustment itself, for the coefficients of transformation r take the place of the correlates k, and the F's play the role of the v's.

However, we are not at all required to calculate the individual F's in order to arrive at $[FF]$ any more than it was necessary in section 30 to find the single α's, β's . . . in order to determine $[\alpha\alpha], [\alpha\beta]$ . . . , or in section 29 to have the individual v's in order to obtain $[vv]$. Exactly by analogy to this last transformation (8), section 29, p. 91, we can derive from (13) and (10) the following:

$$\frac{1}{P} = [FF] = [ff] - \left\{ \frac{[af]^2}{[aa]} + \frac{[bf \cdot 1]^2}{[bb \cdot 1]} + \frac{[cf \cdot 2]^2}{[cc \cdot 2]} \right\} . \tag{14}$$

The calculation thereof is simply connected to the elimination of correlates, just as previously in the case of (10), section 31, p. 101.

Equation (4) shows clearly how the weight P has increased by the entry of redundant observations. For $[ff]$ is, according to (2), the reciprocal value of the weight of a function of *nonadjusted* observations, which would be just sufficient for the computation of the function; all terms of the brackets $\{\ldots\}$ are positive; hence, by the entry of these brackets, $\frac{1}{P}$ is decreased, or P increased.

If the observational weights are *not* all $= 1$, as is assumed until now, then the changes already mentioned in section 44 occur, and the final formula becomes then, instead of (14)

$$\frac{1}{P} = \left[\frac{FF}{p}\right] = \left[\frac{ff}{p}\right] - \left\{ \frac{\left[\frac{af}{p}\right]^2}{\left[\frac{aa}{p}\right]} + \frac{\left[\frac{bf}{p} \cdot 1\right]^2}{\left[\frac{bb}{p} \cdot 1\right]} + \frac{\left[\frac{cf}{p} \cdot 2\right]^2}{\left[\frac{cc}{p} \cdot 2\right]} \right\} . \tag{15}$$

All that formula (14) or, as the case may be, (15) requires can also be comprised into a mechanical rule, which reads as follows:

We add to the system of the condition equations an imaginary equation with the coefficients f, therefore

$$\left.\begin{array}{ll} k_1) & a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \ldots + w_1 = 0 \\ k_2) & b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 \ldots + w_2 = 0 \\ k_3) & c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \ldots + w_3 = 0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{Supplement} & f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 \ldots\ldots = 0 \end{array}\right\} . \tag{16}$$

Now we compute further in just the same way as if we had instead of the absolute terms w the coefficients $[af]$, $[bf]$, . . . , $[ff]$ as end terms:

$$\left.\begin{array}{r} \underline{[aa]}\,k_1 + [ab]\,k_2 + [ac]\,k_3 + \ldots [af] = 0 \\ \underline{[bb]}\,k_2 + [bc]\,k_3 + \ldots [bf] = 0 \\ \underline{[cc]}\,k_3 + \ldots [cf] = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \underline{[ff]} \end{array}\right\} . \qquad (17)$$

Now if we eliminate step by step k_1, k_2, k_3, then there remains an end term, which is $= \frac{1}{P}$ according to (14).

For the function F, after the adjustment we can make, in addition, a new form by connecting (11), p. 144, with (5), section 43, p. 138, whereby we obtain

$$\left.\begin{array}{l} F = f_0 \quad + f_1\,l_1 + f_2\,l_2 + f_3\,l_3 + f_4\,l_4 \\ \qquad\quad + r_1\,w_1 + r_2\,w_2 + r_3\,w_3 \end{array}\right\} , \qquad (18)$$

or

$$\begin{array}{l} F = (F) + r_1\,w_1 + r_2\,w_2 + r_3\,w_3 \\ F = (F) + [r\,w] \,. \end{array} \qquad (19)$$

Hence, if (F) was the value of the function *before* the adjustment, then we obtain F *after* the adjustment by the addition of $r_1w_1 + r_2w_2 + r_3w_3$, i.e., the addition of such parts which vanish or, as the case may be, become unnecessary if no discrepancies w are to be eliminated.

Section 47. Summary of Formulae for Adjustment of Conditioned Observations

$$\left.\begin{array}{lccccc} \text{Measured quantities} & l_1 & l_2 & l_3 & l_4 \;\ldots & l_n \\ \text{Weights} & p_1 & p_2 & p_3 & p_4 \;\ldots & p_n \\ \text{Corrections} & v_1 & v_2 & v_3 & v_4 \;\ldots & v_n \\ \text{Results} & l_1 + v_1 & l_2 + v_2 & l_3 + v_3 & l_4 + v_4 \ldots & l_n + v_n \end{array}\right\} . \qquad (1)$$

Condition equations referred to the unknowns x:

$$\text{Number} = r \left\{ \begin{array}{l} a_0 + a_1\,x_1 + a_2\,x_2 + a_3\,x_3 + a_4\,x_4 + \ldots a_n\,x_n = 0 \\ b_0 + b_1\,x_1 + b_2\,x_2 + b_3\,x_3 + b_4\,x_4 + \ldots b_n\,x_n = 0 \\ c_0 + \underbrace{c_1\,x_1 + c_2\,x_2 + c_3\,x_3 + c_4\,x_4 + \ldots c_n\,x_n}_{\text{Number} = n\,.} = 0 \end{array} \right\} \qquad (2)$$

From here on, we always write $n = 4$, $r = 3$ and have with this:

Discrepancies

$$\left.\begin{array}{l} a_0 + a_1\,l_1 + a_2\,l_2 + a_3\,l_3 + a_4\,l_4 = w_1 \\ b_0 + b_1\,l_1 + b_2\,l_2 + b_3\,l_3 + b_4\,l_4 = w_2 \\ c_0 + c_1\,l_1 + c_2\,l_2 + c_3\,l_3 + c_4\,l_4 = w_3 \end{array}\right\} . \qquad (3)$$

The sign of the discrepancies w is determined by the formula

$$w = \textit{observation} - \textit{theoretical value}\,. \qquad (3^*)$$

Condition equations referred to the corrections v:

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 + w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + w_3 = 0 \end{aligned}\right\} . \tag{4}$$

Normal equations:

$$\left.\begin{aligned} \left[\frac{a\,a}{p}\right] k_1 + \left[\frac{a\,b}{p}\right] k_2 + \left[\frac{a\,c}{p}\right] k_3 + w_1 = 0 \\ \left[\frac{a\,b}{p}\right] k_1 + \left[\frac{b\,b}{p}\right] k_2 + \left[\frac{b\,c}{p}\right] k_3 + w_2 = 0 \\ \left[\frac{a\,c}{p}\right] k_1 + \left[\frac{b\,c}{p}\right] k_2 + \left[\frac{c\,c}{p}\right] k_3 + w_3 = 0 \end{aligned}\right\} . \tag{5}$$

These normal equations are solved for k_1, k_2, k_3, then the v's are computed by means of the correlate equations

$$\left.\begin{aligned} p_1 v_1 = a_1 k_1 + b_1 k_2 + c_1 k_3 \\ p_2 v_2 = a_2 k_1 + b_2 k_2 + c_2 k_3 \\ p_3 v_3 = a_3 k_1 + b_3 k_2 + c_3 k_3 \end{aligned}\right\} . \tag{6}$$

When writing down these equations, we follow the condition equations (4) according to a vertical series.

By adding the v's to the observations l, we obtain the adjusted observations

$$l_1 + v_1 , \quad l_2 + v_2 , \quad l_3 + v_3 , \quad l_4 + v_4 . \tag{7}$$

Mean error of an observation of weight 1:

$$m = \sqrt{\frac{[p\,v\,v]}{r}} . \tag{8}$$

We can compute the sum $[p\,v\,v]$ necessary for this directly from the individual v's; besides, we have the check formulae:

$$[p\,v\,v] = -[w\,k] \tag{8*}$$

or

$$[p\,v\,v] = \frac{w_1{}^2}{[a\,a]} + \frac{[w_2 \cdot 1]^2}{[b\,b \cdot 1]} + \frac{[w_3 \cdot 2]^2}{[c\,c \cdot 2]} . \tag{8**}$$

We consider now a function of the adjusted observations:

$$F = f_1 (l_1 + v_1) + f_2 (l_2 + v_2) + f_3 (l_3 + v_3) + f_4 (l_4 + v_4) , \tag{9}$$

which, however, does not need to contain all $l + v$'s.

The weight P of this function is to be determined.

For this purpose, we compute the sums

$$\left[\frac{a\,f}{p}\right] \quad \left[\frac{b\,f}{p}\right] \quad \left[\frac{c\,f}{p}\right] \tag{10}$$

and

$$\left[\frac{f\,f}{p}\right] . \tag{11}$$

For the further computation we have now two ways, a more circumstantial one from the practical viewpoint, with the transformation coefficients, and a shorter one by extending the elimination.

The method of the transformation coefficients, which is clearer from the theoretical viewpoint, does not need the square term (11), but forms from the original coefficients of the normal equations (5) and the newly computed coefficients (10) the transformation equations

$$\left.\begin{array}{l} \left[\frac{a\,a}{p}\right] r_1 + \left[\frac{a\,b}{p}\right] r_2 + \left[\frac{a\,c}{p}\right] r_3 + \left[\frac{a\,f}{p}\right] = 0 \\ \left[\frac{a\,b}{p}\right] r_1 + \left[\frac{b\,b}{p}\right] r_2 + \left[\frac{b\,c}{p}\right] r_3 + \left[\frac{b\,f}{p}\right] = 0 \\ \left[\frac{a\,c}{p}\right] r_1 + \left[\frac{b\,c}{p}\right] r_2 + \left[\frac{c\,c}{p}\right] r_3 + \left[\frac{c\,f}{p}\right] = 0 \end{array}\right\} . \tag{12}$$

These equations are solved for r_1, r_2, r_3, and then there follows by analogy to the v's in (6)

$$\left.\begin{array}{l} F_1 = f_1 + a_1\,r_1 + b_1\,r_2 + c_1\,r_3 \\ F_2 = f_2 + a_2\,r_1 + b_2\,r_2 + c_2\,r_3 \\ F_3 = f_3 + a_3\,r_1 + b_3\,r_2 + c_3\,r_3 \\ F_4 = f_4 + a_4\,r_1 + b_4\,r_2 + c_4\,r_3 \end{array}\right\} . \tag{13}$$

From these individual F's we form

$$\frac{F_1^2}{p_1} + \frac{F_2^2}{p_2} + \frac{F_3^2}{p_3} + \frac{F_4^2}{p_4} = \left[\frac{F\,F}{p}\right] = \frac{1}{P}; \tag{14}$$

with this, we have the reciprocal of the required weight P and the mean error M of the considered function F:

$$M = m\sqrt{\frac{1}{P}} \; . \tag{15}$$

In the second above-mentioned course of computation we annex the terms (10) and (11) to the elimination of the normal equations and form, successively, the following systems:

$$\left.\begin{array}{llllllllll} \left[\frac{a\,a}{p}\right] & \left[\frac{a\,b}{p}\right] & \left[\frac{a\,c}{p}\right] & \left[\frac{a\,f}{p}\right] & & & & & & \\ & \left[\frac{b\,b}{p}\right] & \left[\frac{b\,c}{p}\right] & \left[\frac{b\,f}{p}\right] & \left[\frac{b\,b}{p}\cdot 1\right] & \left[\frac{b\,c}{p}\cdot 1\right] & \left[\frac{b\,f}{p}\cdot 1\right] & & & \\ & & \left[\frac{c\,c}{p}\right] & \left[\frac{c\,f}{p}\right] & & \left[\frac{c\,c}{p}\cdot 1\right] & \left[\frac{c\,f}{p}\cdot 1\right] & \left[\frac{c\,c}{p}\cdot 2\right] & \left[\frac{c\,f}{p}\cdot 2\right] & \\ & & & \left[\frac{f\,f}{p}\right] & & & \left[\frac{f\,f}{p}\cdot 1\right] & & \left[\frac{f\,f}{p}\cdot 2\right] & \\ & & & & & & \left[\frac{f\,f}{p}\cdot 3\right] = \left[\frac{F\,F}{p}\right] = \frac{1}{P} & & & \end{array}\right\} . \tag{16}$$

All terms in which f does not occur are the same here as those already used in the elimination of the k's from (5). If we proceed according to this scheme, then we do nothing else but that which formula (15), section 46, p. 145, prescribes.

If the weights p are given a priori in the form of mean errors, then the formulae (5) to (12), section 44, p. 141, are applied.

We always compute the functional value F after the adjustment directly according to (9) by introducing the adjusted observations; however, there also exists, for this, formula (18) or (19) of section 46, p. 146:

$$F = f_0 + f_1 l_1 + f_2 l_2 + f_3 l_3 + f_4 l_4 + r_1 w_1 + r_2 w_2 + r_3 w_3 \,. \tag{17}$$

Section 48. Adjustment of the Three Angles of a Plane Triangle

As a simple example for the explanation of the formulae summarized in the previous section 47 we take the adjustment of the angles of a plane triangle (which in section 10, pp. 28 to 32, has already been treated according to the principle of the arithmetic mean).

The numbers of the equations correspond to those in the previous section 47.

Measured angles	l_1	l_2	l_3	(1)
Weights . . .	1	1	1	
Corrections .	v_1	v_2	v_3	
Results . . .	$l_1 + v_1$	$l_2 + v_2$	$l_3 + v_3$	

Condition equation referred to the unknowns x:

$$-180^\circ + x_1 + x_2 + x_3 = 0 \,. \tag{2}$$

Discrepancy:

$$-180^\circ + l_1 + l_2 + l_3 = w \,. \tag{3}$$

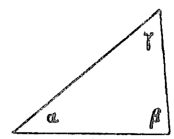


Fig. 1
$\alpha + \beta + \gamma - 180^\circ = w$

Condition equation referred to the v's:

$$v_1 + v_2 + v_3 + w = 0 \,. \tag{4}$$

Coefficients of the condition equation:

$$a_1 = +1 \,, \qquad a_2 = +1 \,, \qquad a_3 = +1 \qquad \text{and} \qquad w_1 = w \,.$$

Coefficients of the normal equation:

$$\left[\frac{a\,a}{p}\right] = 3 \,, \qquad \left[\frac{a\,b}{p}\right] = 0 \,, \qquad \left[\frac{a\,c}{p}\right] = 0 \,, \ldots$$

Normal equation: $3k + w = 0$

Solution of the normal equation: $k = -\dfrac{w}{3}$ (5)

Corrections:

$$v_1 = -\frac{w}{3} \,, \qquad v_2 = -\frac{w}{3} \,, \qquad v_3 = -\frac{w}{3} \,. \tag{6}$$

Adjusted angles of the triangle:

$$x_1 = l_1 - \frac{w}{3} \,, \qquad x_2 = l_2 - \frac{w}{3} \,, \qquad x_3 = l_3 - \frac{w}{3} \,. \tag{7}$$

Mean error of a measurement of weight 1, i.e. of a measured angle before the adjustment:

$$m = \sqrt{\frac{[v\,v]}{1}} = \frac{w}{\sqrt{3}} . \tag{8}$$

Let the function, whose weight is to be determined, be

$$F = l_1 + v_1 ,$$

i.e., the weight of an adjusted angle shall be determined; hence, the function coefficients are very simple: (9)

$$f_1 = 1 , \qquad f_2 = 0 , \qquad f_3 = 0 ,$$

$$\left[\frac{a\,f}{p}\right] = 1, \qquad \left[\frac{b\,f}{p}\right] = 0, \qquad \left[\frac{c\,f}{p}\right] = 0 . \tag{10}$$

The transformation equation is:

$$3\,r + 1 = 0 .$$

Hence, the solution of the transformation equation yields very simply (11)

$$r = -\frac{1}{3} .$$

The computation of the F's, also, becomes very short in our case:

$$F_1 = +\frac{2}{3} , \qquad F_2 = -\frac{1}{3} , \qquad F_3 = -\frac{1}{3} , \tag{12}$$

$$\left[\frac{F\,F}{p}\right] = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{6}{9} = \frac{2}{3} . \tag{13}$$

$$M = \frac{w}{\sqrt{3}} \sqrt{\frac{2}{3}} = \frac{w}{3} \sqrt{2} . \tag{14}$$

But now we could have computed function F also by *another* way; for x_1 by way of x_2 and x_3 is

$$x_1 = F = 180^\circ - x_2 - x_3 ,$$

with which: $$f_1 = 0 , \qquad f_2 = -1 , \qquad f_3 = -1 , \tag{9*}$$

$$3\,r - 2 = 0 , \qquad r = +\frac{2}{3} , \tag{11*}$$

$$F_1 = +\frac{2}{3} , \qquad F_2 = -\frac{1}{3} , \qquad F_3 = -\frac{1}{3} . \tag{12*}$$

These F_1's, F_2's, F_3's are thus the same as in the first case in (12), although function F itself has become a different one.

According to the second method of weight computation, we have, with application to (9), first to compute the term:

$$\left[\frac{f\,f}{p}\right] = 1^2 + 0^2 + 0^2 . \tag{15}$$

Formula (15), section 46, p. 145, is reduced in our case, since there is only *one* condition equation, to

$$\left.\begin{aligned} \frac{1}{P} &= \left[\frac{ff}{p}\cdot 1\right] = \left[\frac{ff}{p}\right] - \frac{\left[\frac{af}{p}\right]}{\left[\frac{aa}{p}\right]}\left[\frac{af}{p}\right]. \\ &\text{This yields with the coefficients of (5) and of (10)} \\ &\left[\frac{ff}{p}\cdot 1\right] = 1 - \frac{1}{3}1 = \frac{2}{3}. \end{aligned}\right\} \qquad (16)$$

This agrees with (13), whereupon also M results, just as in (14).

Section 49. Adjustment of the Angles and Sides of a Plane Triangle

In order to be able to show also the adjustment with different weights by an example, in which, at the same time, also condition equations of a nonlinear form occur, we will treat the triangle represented in Fig. 1, in which, besides the 3 angles, the sides are also measured. The measured values are the following:

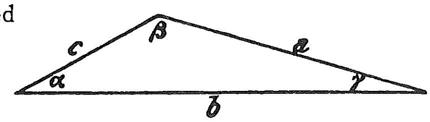


Fig. 1.

$$\left.\begin{aligned} \alpha &= 28^\circ 12' 52'' & a &= 79.306 \text{ m} \\ \beta &= 136\ 03\ 05 & b &= 116.406 \\ \gamma &= 15\ 43\ 55 & c &= 45.501 \end{aligned}\right\}. \qquad (1)$$

For the determination of the weights, the mean errors of the measured quantities must be known, for which the following values were determined:

$$\left.\begin{aligned} &m_\alpha = m_\beta = m_\gamma = \pm 7'', \\ &m_a = \pm 8 \text{ mm}, \qquad m_b = \pm 12 \text{ mm}, \qquad m_c = \pm 5 \text{ mm} \end{aligned}\right\}. \qquad (2)$$

As was already indicated on p. 140, we can then simply set the weights equal to the reciprocal squares of the mean errors. But since in the determination of the mean errors a definite uncertainty is not avoidable, then we will undertake a rounding off of the mean squares of the errors and set now

$$\left.\begin{aligned} &m_\alpha^2 = m_\beta^2 = m_\gamma^2 = 50, \\ &m_a^2 = 75, \qquad m_b^2 = 150, \qquad m_c^2 = 25 \end{aligned}\right\}. \qquad (3)$$

With this, we can then assume the following values for the weights, which have only a relative significance:

$$\left.\begin{aligned} &p_\alpha = p_\beta = p_\gamma = \frac{1}{2}, \\ &p_a = \frac{1}{3}, \qquad p_b = \frac{1}{6}, \qquad p_c = 1 \end{aligned}\right\}. \qquad (4)$$

We will remember here that the mean error ±5 corresponds to the weight 1.

In the weight determination of quantities of various kinds, as the angles and lengths in the present case, we are to take care of the appropriate choice of the units of the mean errors. We have chosen here the second and the millimeter, which in itself would not absolutely be necessary. However, if we had expressed the mean errors, for instance, in minutes and millimeters, then the weights would have become so unequal that they would have led to an inconvenient numerical computation.

Passing over to the setting up of the condition equations, we have first the sum equation for the 3 angles

$$\alpha + \beta + \gamma - 180° = 0 \,. \tag{5}$$

Since for the construction of the triangle only one side is required, and in the present case the other two sides are likewise measured, then there must in addition exist two condition equations for redundant sides. These equations can be set up in different forms; it is simplest to use the relations given by the law of sines. We have then

$$\left.\begin{array}{l} a \sin\beta - b \sin\alpha = 0 \\ a \sin\gamma - c \sin\alpha = 0 \end{array}\right\}. \tag{6}$$

Further independent condition equations can no longer be set up, for all further equations can be derived from the above three equations.

While equation (5) is linear with regard to the angles, the two equations (6) in the above form are not yet usable for the adjustment and must still undergo a transformation. If we add corrections to the angles and sides, then we obtain from (6)

$$\left.\begin{array}{l} (a + v_a) \sin(\beta + v_\beta) - (b + v_b) \sin(\alpha + v_\alpha) = 0 \\ (a + v_a) \sin(\gamma + v_\gamma) - (c + v_c) \sin(\alpha + v_\alpha) = 0 \end{array}\right\}. \tag{7}$$

Herein we can develop the individual sines according to Taylor's theorem and limit the development in view of the small size of the corrections v to terms of first order with respect to the v's so that we obtain

$$\begin{array}{l} (a + v_a)(\sin\beta + \cos\beta\, v_\beta) - (b + v_b)(\sin\alpha + \cos\alpha\, v_\alpha) = 0 \\ (a + v_a)(\sin\gamma + \cos\gamma\, v_\gamma) - (c + v_c)(\sin\alpha + \cos\alpha\, v_\alpha) = 0 \end{array}$$

or, if the products of the v's are likewise neglected,

$$\left.\begin{array}{l} a \sin\beta - b \sin\alpha + \sin\beta\, v_a + a \cos\beta\, v_\beta - \sin\alpha\, v_b - b \cos\alpha\, v_\alpha = 0 \\ a \sin\gamma - c \sin\alpha + \sin\gamma\, v_a + a \cos\gamma\, v_\gamma - \sin\alpha\, v_c - c \cos\alpha\, v_\alpha = 0 \end{array}\right\}. \tag{8}$$

With this, we have obtained linear forms also for the two side equations; it is only to be borne in mind further that in (8) the corrections v of the angles are not to be understood as seconds but in radian measure. If these corrections are computed in seconds, then in (8) we have to divide the v's by ρ. Therefore, we have the final three equations

$$\left.\begin{array}{l} v_\alpha + v_\beta + v_\gamma + w_1 = 0 \\ -\dfrac{b}{\rho} \cos\alpha\, v_\alpha + \dfrac{a}{\rho} \cos\beta\, v_\beta + \sin\beta\, v_a - \sin\alpha\, v_b + w_2 = 0 \\ -\dfrac{c}{\rho} \cos\alpha\, v_\alpha + \dfrac{a}{\rho} \cos\gamma\, v_\gamma + \sin\gamma\, v_a - \sin\alpha\, v_c + w_3 = 0 \end{array}\right\}, \tag{9}$$

where we set for abbreviation:

$$\left.\begin{array}{l} w_1 = \alpha + \beta + \gamma - 180° \\ w_2 = a \sin\beta - b \sin\alpha \\ w_3 = a \sin\gamma - c \sin\alpha \end{array}\right\}. \tag{10}$$

Since the corrections of the triangle sides are computed in millimeters, then we must introduce the triangle sides themselves as likewise in millimeters for the computation of the coefficients of (9) and (10). We insert further the logarithmic computation for the first side equation in order to present the example in all completeness.

a	4.899 306	b	5.065 975	a	4.8993	b	5.0660	
$\sin\beta$	9.841 368	$\sin\alpha$	9.674 653	$\cos\beta$	9.8573 n	$\cos\alpha$	9.9451	$\sin\alpha = +0.473$
	4.740 674		4.740 628	$1/\varrho$	4.6856	$1/\varrho$	4.6856	$\sin\beta = +0.694$
					9.4422 n		9.6967	
+55 039.4 mm		−55 033.6 mm		−0.277		+0.497		

$w_2 = +5.8$ mm .

Therefore, the first side equation reads

$$-0.50\,v_\alpha - 0.28\,v_\beta + 0.69\,v_a - 0.47\,v_b + 5.8 = 0\ .$$

The second side equation is to be treated likewise, so that we can now summarize the coefficients of the three condition equations:

	v_α	v_β	v_γ	v_a	v_b	v_c	w
a	+1.00	+1.00	+1.00	..	..	..	−8.0
b	−0.50	−0.28	..	+0.69	−0.47	..	+5.8
c	−0.19	..	+0.37	+0.27	..	−0.47	−8.8
$1/p$	2	2	2	3	6	1	

(11)

With these coefficients and the weight reciprocals we have to form, according to (5), section 47, p. 147, the expressions $\frac{a_1 a_1}{p_1} + \frac{a_2 a_2}{p_2} + \ldots$, further $\frac{a_1 b_1}{p_1} + \frac{a_2 b_2}{p_2} + \ldots$, etc., and obtain then the following abbreviated normal equations:

$$\left.\begin{aligned} \underline{6.00}\,k_1 - 1.56\,k_2 + 0.36\,k_3 - 8.0 &= 0 \\ + \underline{3.42}\,k_2 + 0.75\,k_3 + 5.8 &= 0 \\ + \underline{0.78}\,k_3 - 8.8 &= 0 \end{aligned}\right\} . \tag{12}$$

The solution of the normal equations yields for the correlates the values

$$k_1 = -1.33\ ,\qquad k_2 = -6.17\ ,\qquad k_3 = +17.67\ ,$$

and, with this, we obtain the corrections

$$\left.\begin{aligned} v_\alpha &= -\ \ 3.22'' & v_a &= +\ \ 1.54 \text{ mm} \\ v_\beta &= +\ \ 0.80 & v_b &= +17.40 \\ v_\gamma &= +10.42 & v_c &= -\ \ 8.30 \end{aligned}\right\} . \tag{13}$$

We compute now at the same time also the final values of the angles and sides by adding the corrections (13) to equations (1). Then we have

$$\left.\begin{aligned} \alpha + v_\alpha &= \ \ 28^\circ\ 12'\ 48.8'' & a + v_a &= \ \ 79\,307.5 \text{ mm} \\ \beta + v_\beta &= 136\ \ 03\ \ 05.8 & b + v_b &= 116\,423.4 \\ \gamma + v_\gamma &= \ \ 15\ \ 44\ \ 05.4 & c + v_c &= \ \ 45\,492.7 \\ &\overline{180^\circ\ 00'\ 00.0''} \end{aligned}\right\} . \tag{14}$$

After the adjustment is thus concluded and the condition equation (5) is satisfied by the final angles, we have to examine further the two condition equations (6). We will have

a	4.899 314	b	5.066 041	a	4.899 314	c	4.657 942
$\sin\beta$	9.841 366	$\sin\alpha$	9.674 640	$\sin\gamma$	9.433 266	$\sin\alpha$	9.674 640
	4.740 680		4.740 681		4.332 580		4.332 582 .

These checks, also, agree sufficiently so that all is checked adequately.

Now there still remains the determination of the mean error, for which we have the following computation:

vv	vvp	$-wk$
10.37	5.18	
0.64	0.32	
108.58	54.29	$-$ 10.64
2.37	0.79	$+$ 35.79
302.76	50.46	$+$ 155.50
68.89	68.89	$+$ 180.65
	179.93	

Both values show good agreement; we have therefore as the mean error of the unit of weight

$$m = \pm\sqrt{\frac{179.93}{3}} = \pm 7.75\,. \tag{15}$$

In fixing the weights (4) we had started from the theory that the mean error ± 5 corresponds to the weight 1. The value of m, just found, should agree with this in the case of the correct assumption of the mean errors (2); however, the difference is not considerable, since not too great a certainty can be attributed to the computation of m in view of the small number of the v's, on the other hand, also, the mean errors (2) assumed a priori are not very accurate.

In order to treat also a function of the adjusted observations, we will determine the weight of the adjusted angle α. We have for this

$$F = \alpha + v_\alpha\,,$$

hence

$$\left.\begin{array}{l} F = \alpha + v_\alpha\,, \\ f_1 = 1\,, \quad f_2 = f_3 = \ldots = 0 \end{array}\right\}. \tag{16}$$

With this, we have to set up the transformation equations (12), section 47, p. 148, and obtain

$$\left[\frac{af}{p}\right] = +2.00\,, \quad \left[\frac{bf}{p}\right] = -1.00\,, \quad \left[\frac{cf}{p}\right] = -0.38\,. \tag{17}$$

Since the remaining coefficients of the transformation equations are to be taken over from the normal equations, then we find

$$\left.\begin{array}{r} +\underline{6.00}\,r_1 - 1.56\,r_2 + 0.36\,r_3 + 2.00 = 0 \\ +\underline{3.42}\,r_2 + 0.75\,r_3 - 1.00 = 0 \\ +\underline{0.78}\,r_3 - 0.38 = 0 \end{array}\right\}. \tag{18}$$

The solution of these equations is carried out in the same manner as that of the normal equations and yields

$$r_1 = -0.39\,, \qquad r_2 = -0.04\,, \qquad r_3 = +0.70\,. \tag{19}$$

With this, we compute the factors F according to equations (13), section 47, p. 148. We have, for instance,

$$F_1 = +1.00 - 0.39 + 0.02 - 0.13 = +0.50$$
$$F_2 = \quad .. \quad -0.39 + 0.01 \quad .. \quad = -0.38$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Altogether we obtain

$$F_1 = +0.50 \qquad F_4 = +0.16$$
$$F_2 = -0.38 \qquad F_5 = +0.02$$
$$F_3 = -0.13 \qquad F_6 = -0.33$$

and, with this, we will have

$$\frac{1}{P} = \left[\frac{FF}{p}\right] = 0.50 + 0.29 + 0.03 + 0.08 + 0.00 + 0.11 \, ,$$

or

$$\frac{1}{P} = 1.01 \, , \qquad P = 0.99 \, .$$

Before the adjustment we had $p_\alpha = \frac{1}{2}$; the weight of the angle α has thus been doubled by the adjustment.

Section 50. Special Methods of Adjustment According to Conditioned Observations

In some cases it may be useful not to satisfy the condition equations of an adjustment all at the same time, but to form several groups of condition equations and to adjust them one after the other. By repetition of the process we obtain the same results which the simultaneous adjustment yields. Before we enter on the subject more closely, we summarize once more the course of computation for the complete adjustment without separation of the condition equations.

We take a case with three equations and four observations. Let the corrections, whose sum of squares is to become $[\delta\delta]$ = minimum, be $\delta_1, \delta_2, \delta_3, \delta_4$.

Condition equations:

$$\left.\begin{aligned} a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 + w_1 &= 0 \\ b_1\delta_1 + b_2\delta_2 + b_3\delta_3 + b_4\delta_4 + w_2 &= 0 \\ c_1\delta_1 + c_2\delta_2 + c_3\delta_3 + c_4\delta_4 + w_3 &= 0 \end{aligned}\right\} . \tag{1}$$

Normal equations:

$$\left.\begin{aligned} [aa]k_1 + [ab]k_2 + [ac]k_3 + w_1 &= 0 \\ [ab]k_1 + [bb]k_2 + [bc]k_3 + w_2 &= 0 \\ [ac]k_1 + [bc]k_2 + [cc]k_3 + w_3 &= 0 \end{aligned}\right\} . \tag{2}$$

Reduced normal equations:

$$\left.\begin{aligned} [bb \cdot 1]k_2 + [bc \cdot 1]k_3 + [w_2 \cdot 1] &= 0 \\ [bc \cdot 1]k_2 + [cc \cdot 1]k_3 + [w_3 \cdot 1] &= 0 \end{aligned}\right\} . \tag{3}$$

Correlate equations:

$$\left.\begin{aligned}\delta_1 &= a_1 k_1 + b_1 k_2 + c_1 k_3\\ \delta_2 &= a_2 k_1 + b_2 k_2 + c_2 k_3\\ \delta_3 &= a_3 k_1 + b_3 k_2 + c_3 k_3\\ \delta_4 &= a_4 k_1 + b_4 k_2 + c_4 k_3\end{aligned}\right\}. \qquad (4)$$

Adjustment by groups. We will now assume that for some reason we are made to treat the first condition equation first by itself.

With one correlate k'_1, the first condition equation yields one normal equation

$$[a\,a]\,k'_1 + w_1 = 0\,, \qquad k'_1 = -\frac{1}{[a\,a]}\,w_1\,, \qquad (5)$$

and the corresponding corrections, which, in distinction from the complete corrections δ, we will denote by u, are

$$u_1 = -\frac{a_1}{[a\,a]}\,w_1\,, \qquad u_2 = -\frac{a_2}{[a\,a]}\,w_1\,, \qquad u_3 = -\frac{a_3}{[a\,a]}\,w_1\,, \qquad u_4 = -\frac{a_4}{[a\,a]}\,w_1\,. \qquad (6)$$

We introduce at once also the second corrections v so that we have

$$\delta_1 = u_1 + v_1\,, \qquad \delta_2 = u_2 + v_2\,, \qquad \delta_3 = u_3 + v_3\,, \qquad \delta_4 = u_4 + v_4\,. \qquad (7)$$

If we set these expressions (6) and (7) into the original condition equations (1), then there result new condition equations, which refer only just to the v's:

$$\left.\begin{aligned}a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + 0 \qquad &= 0\\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 + [w_2\cdot 1] &= 0\\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + [w_3\cdot 1] &= 0\end{aligned}\right\}. \qquad (8)$$

The computation with the u's according to (6) yields here

$$\left.\begin{aligned}[][w_2\cdot 1] &= w_2 - b_1\frac{a_1}{[a\,a]}\,w_1 - b_2\frac{a_2}{[a\,a]}\,w_1 - \ldots = w_2 - \frac{[a\,b]}{[a\,a]}\,w_1\\ [w_3\cdot 1] &= w_3 - c_1\frac{a_1}{[a\,a]}\,w_1 - c_2\frac{a_2}{[a\,a]}\,w_1 - \ldots = w_3 - \frac{[a\,c]}{[a\,a]}\,w_1\end{aligned}\right\}, \qquad (9)$$

i.e., these terms are the same as in (3).

System (8) treated by itself would yield the following adjustment:

$$\left.\begin{aligned}[][a\,a]\,k_1'' + [a\,b]\,k_2 + [a\,c]\,k_3 + 0 \qquad &= 0\\ [a\,b]\,k_1'' + [b\,b]\,k_2 + [b\,c]\,k_3 + [w_2\cdot 1] &= 0\\ [a\,c]\,k_1'' + [b\,c]\,k_2 + [c\,c]\,k_3 + [w_3\cdot 1] &= 0\end{aligned}\right\}, \qquad (10)$$

$$\left.\begin{aligned}[][b\,b\cdot 1]\,k_2 + [b\,c\cdot 1]\,k_3 + [w_2\cdot 1] &= 0\\ [b\,c\cdot 1]\,k_2 + [c\,c\cdot 1]\,k_3 + [w_3\cdot 1] &= 0\end{aligned}\right\}. \qquad (10^*)$$

k_2 and k_3 are the same correlates as in (3); k''_1, however, is changed compared to before. Further computation yields according to (8) for v_1:

$$v_1 = a_1 k''_1 + b_1 k_2 + c_1 k_3\,. \qquad (11)$$

To this, we take according to (6) and (4)

$$u_1 = -\frac{a_1}{[a\,a]}\, w_1 , \tag{12}$$

$$\delta_1 = a_1 k_1 + b_1 k_2 + c_1 k_3 . \tag{13}$$

(11) and (12) together yield

$$u_1 + v_1 = a_1 \left(k''_1 - \frac{1}{[a\,a]}\, w_1\right) + b_1 k_2 + c_1 k_3 . \tag{14}$$

But the comparison of (2) and (10) yields

$$[a\,a]\, k_1 + w_1 = [a\,a]\, k''_1 ,$$

and, with this, (14) changes into (13):

$$u_1 + v_1 = a_1 k_1 + b_1 k_2 + c_1 k_3 = \delta_1 . \tag{15}$$

The same also holds for δ_2, δ_3, δ_4, and we have now the following theorem:

If we treat a condition equation a of (1) first by itself, and derive therefrom first corrections u, and if we adjust then, once again, the observations corrected for the first time as if they were original measurements, taking into account all condition equations, then we obtain the same final values as if we had treated all equations together in *one* adjustment.

The method can also be generalized: We adjust first a group of several condition equations independently, introduce the found corrections in all equations and obtain new discrepancies, and after this, we adjust a further arbitrary group and add the new corrections to those already found. The procedure is repeated until the discrepancies of all condition equations vanish (cf. Gauss, Suppl. theoriae combinationis, art. 18-20).

At this point we will mention further the method which Schleiermacher used for the angle adjustment in a triangulation net (cf. the report by Nell, *Zeitschrift f. Verm.*, 1881, pp. 1 and following), and which can be of use in some cases in the solution of the normal equations. Let there be, for instance, a system of five normal equations in which the coefficients of the first three correlates are so simple that these three correlates can be easily expressed in terms of the remaining ones. We introduce the values hereby found into the last two equations and determine then the last two correlates according to the usual method of elimination. With this, the first three correlates are also known.

Reduced condition equations. We can obtain further a refinement of the adjustment by groups by the introduction of *reduced condition equations*, to which we pass over now.

Condition equations can be arbitrarily transformed; by combination of a different kind, new equations can also be formed, if only their number and their independence are not disturbed. In this manner, we will proceed with equations (1) by multiplying the first equation by $-\frac{[a\,b]}{[a\,a]}$ and, after this, by $-\frac{[a\,c]}{[a\,a]}$ and adding it to the second or, as the case may be, third equation. If we denote the new coefficients of the second and third equation by b' and c' and the new absolute terms by $[w_2 \cdot 1]$ and $[w_3 \cdot 1]$, then the three condition equations read

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 \qquad &= 0 \\ b_1' v_1 + b_2' v_2 + b_3' v_3 + b_4' v_4 + [w_2 \cdot 1] &= 0 \\ c_1' v_1 + c_2' v_2 + c_3' v_3 + c_4' v_4 + [w_3 \cdot 1] &= 0 \end{aligned}\right\} . \tag{16}$$

The coefficients b', c' are here the same values which we have already learned in section 28 as coefficients of the reduced error equations, namely

$$b' = b - \frac{[a\,b]}{[a\,a]}\, a , \qquad c' = c - \frac{[a\,c]}{[a\,a]}\, a . \tag{17}$$

The absolute terms of (16) are the same as in (3) and (10), namely

$$[w_2 \cdot 1] = w_2 - \frac{[a\,b]}{[a\,a]} w_1 , \qquad [w_3 \cdot 1] = w_3 - \frac{[a\,c]}{[a\,a]} w_1 . \tag{18}$$

If we form normal equations from equations (16) and take into account that according to section 29, equation (16a), p. 92, $[a\,b'] = 0$ and, accordingly, $[a\,c'] = 0$, then we obtain

$$\begin{array}{l} [a\,a]\,k_1 \ldots\ldots\ldots\ldots + w_1 = 0 \\ \ldots\ldots [b'\,b']\,k_2 + [b'\,c']\,k_3 + [w_2 \cdot 1] = 0 \\ \ldots\ldots [b'\,c']\,k_2 + [c'\,c']\,k_3 + [w_3 \cdot 1] = 0 \, , \end{array}$$

or according to (7), section 28, p. 88,

$$\left.\begin{array}{l} [a\,a]\,k_1 \ldots\ldots\ldots\ldots + w_1 = 0 \\ \ldots\ldots [b\,b \cdot 1]\,k_2 + [b\,c \cdot 1]\,k_3 + [w_2 \cdot 1] = 0 \\ \ldots\ldots [b\,c \cdot 1]\,k_2 + [c\,c \cdot 1]\,k_3 + [w_3 \cdot 1] = 0 \end{array}\right\} . \tag{19}$$

We see hence that k_1 becomes independent of the two correlates k_2 and k_3, and that, consequently, the corrections δ can also be broken into two parts:

$$\left.\begin{array}{lll} u_1 = a_1\,k_1 & v_1 = b_1'\,k_2 + c_1'\,k_3 & \delta_1 = u_1 + v_1 \\ u_2 = a_2\,k_1 & v_2 = b_2'\,k_2 + c_2'\,k_3 & \delta_2 = u_2 + v_2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \end{array}\right\} . \tag{20}$$

We can therefore also replace the total adjustment of the condition equations (16) by two separate adjustments by adjusting first the first equation and, after this, the two other equations together.

We will arrange the found result further for the simple case in which all coefficients are $a = 1$. (Application to polygon lines.)

Condition equations

$$\left.\begin{array}{l} \delta_1 + \delta_2 + \delta_3 + \ldots + \delta_n + w_1 = 0 \\ b_1\,\delta_1 + b_2\,\delta_2 + b_3\,\delta_3 + \ldots + b_n\,\delta_n + w_2 = 0 \\ c_1\,\delta_1 + c_2\,\delta_2 + c_3\,\delta_3 + \ldots + c_n\,\delta_n + w_3 = 0 \end{array}\right\} . \tag{21}$$

The first equation alone yields

$$\text{First corrections} \quad u_1 = u_2 = u_3 = \ldots = u_n = -\frac{w_1}{n} . \tag{22}$$

Reduced condition equations

$$\left.\begin{array}{l} b_1'\,v_1 + b_2'\,v_2 + b_3'\,v_3 + \ldots + b_n'\,v_n + w_2' = 0 \\ c_1'\,v_1 + c_2'\,v_2 + c_3'\,v_3 + \ldots + c_n'\,v_n + w_3' = 0 \end{array}\right\} , \tag{23}$$

where

$$b_1' = b_1 - \frac{[b]}{n} , \quad b_2' = b_2 - \frac{[b]}{n} \ldots , \quad w_2' = w_2 - \frac{[b]}{n} w_1 ,$$

$$c_1' = c_1 - \frac{[c]}{n} , \quad c_2' = c_2 - \frac{[c]}{n} \ldots , \quad w_3' = w_3 - \frac{[c]}{n} w_1 ,$$

$$\text{Check:} \quad [b'] = 0 , \quad [c'] = 0 .$$

The two equations (23) are treated further like original equations with v's having equal weight and yield the second corrections v.

We can also go one step further on our adopted way by reducing the third equation (16) once more. With the notation of section 28 we obtain

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 &= 0 \\ b_1' v_1 + b_2' v_2 + b_3' v_3 + b_4' v_4 + [w_2 \cdot 1] &= 0 \\ c_1'' v_1 + c_2'' v_2 + c_3'' v_3 + c_4'' v_4 + [w_3 \cdot 2] &= 0 \end{aligned}\right\}, \tag{24}$$

where

$$c'' = c' - \frac{[b' c']}{[b' b']} b' \quad \text{and} \quad [w_3 \cdot 2] = [w_3 \cdot 1] - \frac{[b' c']}{[b' b']} [w_2 \cdot 1].$$

Just as we have found in equation (16a), section 29, p. 92, that $[a b'] = 0$ we also find that $[b' c''] = 0$ and obtain from (24) the normal equations

$$\left.\begin{aligned} [a\ a]\ k_1 + w_1 &= 0 \\ [b' b']\ k_2 + [w_2 \cdot 1] &= 0 \\ [c'' c'']\ k_3 + [w_3 \cdot 2] &= 0 \end{aligned}\right\}, \tag{25}$$

or

$$\left.\begin{aligned} [a\ a]\ k_1 + w_1 &= 0 \\ [bb \cdot 1]\ k_2 + [w_2 \cdot 1] &= 0 \\ [cc \cdot 2]\ k_3 + [w_3 \cdot 2] &= 0 \end{aligned}\right\}. \tag{26}$$

The three condition equations (24) are therefore completely independent of one another and yield, adjusted individually, the same result as in the case of simultaneous adjustment.

Adjustment by groups with transformed condition equations. – The property of equations (24) suggests the idea of forming, from the given condition equations by transformation, two or more groups which are independent of one another and, therefore, can be adjusted individually. This means a perfecting of the adjustment by groups, indicated previously on p. 156, in the case of which the result was obtained only after repeated adjustment of the individual groups.

The setting up of two groups independent of one another, through which it becomes possible, in the adjustment of a triangulation net, to separate the angle equations from the side equations, is especially important. L. Krüger treats this problem in the treatise "Über die Ausgleichung von bedingten Beobachtungen in zwei Gruppen," *Veröff. des Kgl. Preuss. Geodätischen Instituts*, Neue Folge, Nr. 18, Potsdam, 1905, according to which we arrive at the following representation.

Let there be given the following two groups of condition equations:

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 &= 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 + w_2 &= 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + w_3 &= 0 \end{aligned}\right\}, \tag{27}$$

$$\left.\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 + \varphi_1 &= 0 \\ \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \beta_4 v_4 + \varphi_2 &= 0 \end{aligned}\right\}. \tag{28}$$

While group (1) remains unchanged, group (2) is to be transformed in such a way that it is independent of equations (1).

For this, we multiply equations (27) by the three still undetermined factors ξ_1, η_1, ζ_1 and add them to the first equation (28); after this, we multiply equations (27) once again by the factors ξ_2, η_2, ζ_2 and add them to the second equation (28). There follows hereby:

$$\left.\begin{array}{r}(\alpha_1 + a_1\,\xi_1 + b_1\,\eta_1 + c_1\,\zeta_1)\,v_1 + (\alpha_2 + a_2\,\xi_1 + b_2\,\eta_1 + c_2\,\zeta_1)\,v_2 + \ldots \\ + (\varphi_1 + w_1\,\xi_1 + w_2\,\eta_1 + w_3\,\zeta_1) = 0 \\ (\beta_1 + a_1\,\xi_2 + b_1\,\eta_2 + c_1\,\zeta_2)\,v_1 + (\beta_2 + a_2\,\xi_2 + b_2\,\eta_2 + c_2\,\zeta_2)v_2 + \ldots \\ + (\varphi_2 + w_1\,\xi_2 + w_2\,\eta_2 + w_3\,\zeta_2) = 0\end{array}\right\} \qquad (29)$$

or with simplified notation for the coefficients

$$\left.\begin{array}{l}A_1 v_1 + A_2 v_2 + A_3 v_3 + A_4 v_4 + W_1 = 0 \\ B_1 v_1 + B_2 v_2 + B_3 v_3 + B_4 v_4 + W_2 = 0\end{array}\right\}, \qquad (30)$$

where we set

$$\left.\begin{array}{ll}A_1 = \alpha_1 + a_1\,\xi_1 + b_1\,\eta_1 + c_1\,\zeta_1 & B_1 = \beta_1 + a_1\,\xi_2 + b_1\,\eta_2 + c_1\,\zeta_2 \\ A_2 = \alpha_2 + a_2\,\xi_1 + b_2\,\eta_1 + c_2\,\zeta_1 & B_2 = \beta_2 + a_2\,\xi_2 + b_2\,\eta_2 + c_2\,\zeta_2 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ W_1 = \varphi_1 + w_1\,\xi_1 + w_2\,\eta_1 + w_3\,\zeta_1 & W_2 = \varphi_2 + w_1\,\xi_2 + w_2\,\eta_2 + w_3\,\zeta_2\end{array}\right\}. \qquad (31)$$

The requirement of the independence of equations (27) and (30) is expressed by the six conditions

$$\left.\begin{array}{l}[a\,A] = [b\,A] = [c\,A] = 0 \\ [a\,B] = [b\,B] = [c\,B] = 0\end{array}\right\}. \qquad (32)$$

If we substitute herein the values of the coefficients A and B, as they result from (31), then we obtain the six equations

$$\left.\begin{array}{l}[a\,\alpha] + [a\,a]\,\xi_1 + [a\,b]\,\eta_1 + [a\,c]\,\zeta_1 = 0 \\ [b\,\alpha] + [a\,b]\,\xi_1 + [b\,b]\,\eta_1 + [b\,c]\,\zeta_1 = 0 \\ [c\,\alpha] + [a\,c]\,\xi_1 + [b\,c]\,\eta_1 + [c\,c]\,\zeta_1 = 0 \\ \text{and} \\ [a\,\beta] + [a\,a]\,\xi_2 + [a\,b]\,\eta_2 + [a\,c]\,\zeta_2 = 0 \\ [b\,\beta] + [a\,b]\,\xi_2 + [b\,b]\,\eta_2 + [b\,c]\,\zeta_2 = 0 \\ [c\,\beta] + [a\,c]\,\xi_2 + [b\,c]\,\eta_2 + [c\,c]\,\zeta_2 = 0\end{array}\right\}. \qquad (33)$$

From equations (33) the six factors ξ, η, ζ are computed; with these, the coefficients A and B as well as the absolute terms W can be determined according to (31), and equations (30) can then be set up. The course of computation will turn out most conveniently if we set up first the normal equations for equations (27) and connect with the solution of these normal equations at the same time the determination of the factors ξ, η, ζ according to (33).

This adjustment by groups will be preferred to the simultaneous adjustment of all condition equations if the equations of the first group are of a very simple nature, so that their normal equations can be easily solved.

The above method is already indicated by Gauss, however only for the special case in which the second group consists only of a single equation (cf. Gerling, *Die Ausgleichungsrechnungen der praktischen Geometrie*, Hamburg and Gotha, 1843, pp. 400 to 403).

The above-indicated condition equations (24) with reduced coefficients are contained likewise in the general theory.

Solution of the normal equations by Cholesky's method. Let four condition equations be given, from which the normal equations

$$\left.\begin{aligned}
[a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 + [a\,d]\,k_4 + w_1 &= 0\\
[a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 + [b\,d]\,k_4 + w_2 &= 0\\
[a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 + [c\,d]\,k_4 + w_3 &= 0\\
[a\,d]\,k_1 + [b\,d]\,k_2 + [c\,d]\,k_3 + [d\,d]\,k_4 + w_4 &= 0
\end{aligned}\right\} \tag{34}$$

are derived.

For the solution of these equations we introduce a fictitious system of condition equations between the four auxiliary variables x_1, x_2, x_3, x_4 which have the same absolute terms as the original condition equations. We assume that these fictitious equations are

$$\left.\begin{aligned}
\alpha_1\,x_1 \quad . \quad . \quad . \quad + w_1 &= 0\\
\beta_1\,x_1 + \beta_2\,x_2 \quad . \quad . \quad + w_2 &= 0\\
\gamma_1\,x_1 + \gamma_2\,x_2 + \gamma_3\,x_3 \quad . \quad + w_3 &= 0\\
\delta_1\,x_1 + \delta_2\,x_2 + \delta_3\,x_3 + \delta_4\,x_4 + w_4 &= 0
\end{aligned}\right\}. \tag{35}$$

Here the coefficients $\alpha, \beta, \gamma, \delta$ are to be calculated so that the normal equations, which belong to the equations (35), are identical with the given normal equations (34). Therefore, it must be true that

$$\left.\begin{aligned}
&[\alpha\,\alpha] = [a\,a], \quad [\alpha\,\beta] = [a\,b], \quad [\alpha\,\gamma] = [a\,c], \quad [\alpha\,\delta] = [a\,d],\\
&\qquad\qquad\qquad\; [\beta\,\beta] = [b\,b], \quad [\beta\,\gamma] = [b\,c], \quad [\beta\,\delta] = [b\,d],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\;\; [\gamma\,\gamma] = [c\,c], \quad [\gamma\,\delta] = [c\,d],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; [\delta\,\delta] = [d\,d].
\end{aligned}\right\} \tag{36}$$

We have therefore 10 equations in all, from which the 10 coefficients $\alpha, \beta, \gamma, \delta$ may be calculated. In detailed form, these 10 equations may be written out

$$\left.\begin{aligned}
&\alpha_1^2 = [a\,a], \quad \alpha_1\,\beta_1 = [a\,b], \quad \alpha_1\,\gamma_1 = [a\,c], \quad \alpha_1\,\delta_1 = [a\,d],\\
&\beta_1^2 + \beta_2^2 = [b\,b], \quad \beta_1\,\gamma_1 + \beta_2\,\gamma_2 = [b\,c], \quad \beta_1\,\delta_1 + \beta_2\,\delta_2 = [b\,d],\\
&\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = [c\,c], \quad \gamma_1\,\delta_1 + \gamma_2\,\delta_2 + \gamma_3\,\delta_3 = [c\,d],\\
&\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = [d\,d].
\end{aligned}\right\} \tag{37}$$

From the first row there follows $\alpha_1, \beta_1, \gamma_1, \delta_1$. Using these, we obtain from the second row $\beta_2, \gamma_2, \delta_2$; further from the third row γ_3, δ_3; and finally, from the fourth row δ_4.

After the 10 coefficients have thus been computed, we determine the four auxiliary quantities x_1, x_2, x_3, x_4 from the equations (35).

After this, we can now set up the correlation equations which belong to the condition equations (35), namely the equations

$$\left.\begin{aligned}
x_1 &= \alpha_1\,k_1 + \beta_1\,k_2 + \gamma_1\,k_3 + \delta_1\,k_4\\
x_2 &= \quad . \quad\;\; \beta_2\,k_2 + \gamma_2\,k_3 + \delta_2\,k_4\\
x_3 &= \quad . \quad\;\; . \quad\;\; \gamma_3\,k_3 + \delta_3\,k_4\\
x_4 &= \quad . \quad\;\; . \quad\;\; . \quad\;\; \delta_4\,k_4
\end{aligned}\right\}. \tag{38}$$

From these equations we can determine, in succession, the correlates k_4, k_3, k_2, k_1, whereby the solution of the normal equations (34) is complete.

This method can also be applied to the adjustment of indirect observations, when in place of the discrepancies w the absolute terms $-[a\,l]$, $-[b\,l]$, etc., are substituted.

We will use this method to solve the normal equations (8) of section 36, p. 112, once more, where we will designate now the three unknowns by k_1, k_2, k_3.

Normal equations:

$$
\begin{aligned}
+17.50\,k_1 - 6.50\,k_2 - 6.50\,k_3 - 2.14 &= 0\\
-6.50\,k_1 + 17.50\,k_2 - 6.50\,k_3 - 13.96 &= 0\\
-6.50\,k_1 - 6.50\,k_2 + 20.50\,k_3 + 5.40 &= 0\,.
\end{aligned}
$$

Auxiliary coefficients:

$$\alpha_1{}^2 = 17.50\,, \qquad \beta_1 = -\frac{6.50}{4.18}\,, \qquad \gamma_1 = -\frac{6.50}{4.18}\,,$$

$$\alpha_1 = +4.18\,, \qquad \beta_1 = -1.55\,, \qquad \gamma_1 = -1.55\,,$$

$$\beta_2{}^2 = 17.50 - 2.40 = 15.10\,, \qquad \gamma_2 = \frac{-6.50 - 2.40}{3.89} = -\frac{8.90}{3.89}\,,$$

$$\beta_2 = +3.89\,, \qquad \gamma_2 = -2.29\,.$$

$$\gamma_3{}^2 = 20.50 - 5.24 - 2.40 = +12.86\,, \qquad \gamma_3 = +3.59\,.$$

Auxiliary quantities:

$$
\begin{aligned}
+4.18\,x_1 &= +2.14 & x_1 &= +0.51\\
+3.89\,x_2 &= +13.96 + 1.55 \times 0.51 & &\\
&= +13.96 + 0.79 = +14.75 & x_2 &= +3.79\\
+3.59\,x_3 &= -5.40 + 1.55 \times 0.51 + 2.29 \times 3.79 & &\\
&= -5.40 + 0.79 + 8.68 = +4.07 & x_3 &= +1.15\,.
\end{aligned}
$$

Unknowns:

$$
\begin{aligned}
+0.51 &= 4.18\,k_1 - 1.55\,k_2 - 1.55\,k_3 & &\\
+3.79 &= \quad . \quad + 3.89\,k_2 - 2.29\,k_3 & &\\
+1.15 &= \quad . \qquad . \qquad + 3.59\,k_3 & k_3 &= +0.31\\
3.89\,k_2 &= 3.79 + 0.71 = +4.50 & k_2 &= +1.16\\
4.18\,k_1 &= 0.51 + 0.48 + 1.80 = +2.79 & k_1 &= +0.67\,.
\end{aligned}
$$

These values show sufficient agreement with the earlier results of section 36, p. 113.

The preceding method was derived by the French Army Major Cholesky, who died in the World War in 1918. It was published in the *Bulletin Géodésique*, 1924, pp. 67-77.

As a case very seldom to be applied we consider the problem in which we have computed the weights for two functions and will determine hence also the weight of a new function of these functions.

The most important case of this kind consists in assuming that we have computed the coordinates X and Y and the weights of the coordinates P_x and P_y of a triangulation point, and with this, we shall answer any further question of accuracy, e.g. position and size of an error ellipse, or distance and direction angle.

Let the two functions originally considered be:

$$X = f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4 + \dots , \qquad Y = f_1' x_1 + f_2' x_2 + f_3' x_3 + f_4' x_4 \dots \tag{1}$$

and let the function weights belonging to them be computed according to section 46, p. 143:

$$\frac{1}{P_x} = [f f] - \frac{[a f]^2}{[a a]} - \frac{[b f \cdot 1]^2}{[b b \cdot 1]} - \dots , \qquad \frac{1}{P_y} = [f' f'] - \frac{[a f']^2}{[a a]} - \frac{[b f' \cdot 1]^2}{[b b \cdot 1]} - \dots . \tag{2}$$

Now a new function is involved:

$$(F) = r X + r' Y . \tag{3}$$

We can imagine this function represented as a function of the x's according to (1):

$$(F) = (r f_1 + r' f_1') x_1 + (r f_2 + r' f_2') x_2 + (r f_3 + r' f_3') x_3 + \dots , \tag{4}$$

and then the weight (P) is determined from it according to the same rule as (2), namely

$$\frac{1}{(P)} = [(r f + r' f')^2] - \frac{[a r f + a r' f']^2}{[a a]} - \frac{[(b r f + b r' f') \cdot 1]^2}{[b b \cdot 1]} - \dots . \tag{5}$$

Now the sum is

$$\begin{aligned} [(r f + r' f')^2] &= [r^2 f^2 + 2 r r' f f' + r'^2 f'^2] \\ &= r^2 [f f] + 2 r r' [f f'] + r'^2 [f' f'] . \end{aligned} \tag{6}$$

The same law of formation holds also directly for the second term of (5), and this law holds also for all following terms of (5), which we prove just as before for indirect observations, section 32, p. 103.

We will carry this out in greater detail only for $[\dots 1]$:

$$[b f \cdot 1] = [b f] - \frac{[a b]}{[a a]} [a f] , \qquad [b f' \cdot 1] = [b f'] - \frac{[a b]}{[a a]} [a f'] ,$$

$$[(b r f + b r' f') \cdot 1] = [b r f + b r' f'] - \frac{[a b]}{[a a]} [a r f + a r' f'] ,$$

$$[(b r f + b r' f') \cdot 1] = r [b f \cdot 1] + r' [b f' \cdot 1] .$$

This corresponds entirely to the second equation of group (6), section 32, p. 103, and all the rest turns out also as there. Therefore, we have the general rule:

If for two functions X and Y according to (1) the weights are developed in the following form:

$$\frac{1}{P_x} = [f f] - \left\{ \frac{[a f]^2}{[a a]} + \frac{[b f \cdot 1]^2}{[b b \cdot 1]} + \frac{[c f \cdot 2]^2}{[c c \cdot 2]} + \dots \right\}, \tag{7}$$

$$\frac{1}{P_y} = [f' f'] - \left\{ \frac{[a f']^2}{[a a]} + \frac{[b f' \cdot 1]^2}{[b b \cdot 1]} + \frac{[c f' \cdot 2]}{[c c \cdot 2]} + \dots \right\}, \tag{8}$$

then for a function

$$(F) = r X + r' Y \tag{9}$$

we determine the weight (P) in the following form. We compute first:

$$\frac{1}{P_{xy}} = [f f'] - \left\{ \frac{[a f][a f']}{[a a]} + \frac{[b f \cdot 1][b f' \cdot 1]}{[b b \cdot 1]} + \frac{[c f \cdot 2][c f' \cdot 2]}{[c c \cdot 2]} + \dots \right\}, \tag{10}$$

and then we have

$$\frac{1}{(P)} = r^2 \frac{1}{P_x} + 2 r r' \frac{1}{P_{xy}} + r'^2 \frac{1}{P_y}. \tag{11}$$

Section 52. Various Secondary Considerations

The conditions which the adjustment is to satisfy rigorously can be expressed in various forms, and the condition equations can be transformed several times if only that which they express is essentially retained.

Therefore, we can multiply or divide the condition equations by arbitrary numbers (which we cannot do in the case of *error* equations without changing the proportions of weight); e.g., we can also form from two condition equations a and b with arbitrary factors $m\,n$, $m'\,n'$ two new equations $m\,a + n\,b$, $m'\,a + n'\,b$ and use these in place of the first.

In the case of all these transformations, however, the main results, namely the corrections v of the observations l, remain always the same, for the problem to make $[v\,v]$ a minimum with certain side conditions is essentially a clearly defined one, and therefore cannot be influenced by the algebraic *form* of the solution.

A further question concerns the *independence and completeness* of the condition equations.

For the answer to this question we take up once more equations (4) and (10) of section 43, pp. 138 and 139:

Condition equations

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 &= 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 + w_2 &= 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + w_3 &= 0 \end{aligned}\right\}. \tag{1}$$

Normal equations

$$\left.\begin{aligned} [a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 + w_1 &= 0 \\ [a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 + w_2 &= 0 \\ [a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 + w_3 &= 0 \end{aligned}\right\}. \tag{2}$$

Until now we have assumed that the condition equations (1) are independent of one another, i.e. that none is contained in the remaining ones. But we will now examine what consequences arise if this is no longer the case, and for this, we set:

$$c_1 = \alpha a_1 + \beta b_1, \qquad c_2 = \alpha a_2 + \beta b_2 \dots, \qquad w_3 = \alpha w_1 + \beta w_2. \tag{3}$$

Then the coefficients of the normal equations, also, are no longer independent, for we will have

$$\left.\begin{array}{l}[a c] = \alpha\,[a a] + \beta\,[a b], \quad [b c] = \alpha\,[a b] + \beta\,[b b], \quad w_3 = \alpha\,w_1 + \beta\,w_2 \\ {[c c]} = \alpha^2[a a] + \beta^2[b b] + 2\,\alpha\,\beta\,[a b] \quad \text{oder} \quad = \alpha\,[a c] + \beta\,[b c]\end{array}\right\} . \tag{4}$$

The theory of determinants shows in elementary theorems that the unknowns k can then no longer be determined, but are obtained in the indeterminate form $\frac{0}{0}$.

We shall also pursue this with our symbols $[b\,b \cdot 1]$, $[b\,c \cdot 1]$ . . . .

The reduced normal equations according to (2) are

$$\left.\begin{array}{l}[b b \cdot 1]\,k_2 + [b c \cdot 1]\,k_3 + [w_2 \cdot 1] = 0 \\ {[b c \cdot 1]}\,k_2 + [c c \cdot 1]\,k_3 + [w_3 \cdot 1] = 0\end{array}\right\} . \tag{5}$$

Because of (4) we have here

$$[b c \cdot 1] = [b c] - \frac{[a b]}{[a a]}[a c] = \alpha\,[a b] + \beta\,[b b] - \frac{[a b]}{[a a]}(\alpha\,[a a] + \beta\,[a b])$$

$$[b c \cdot 1] = \beta\,([b b] - \frac{[a b]}{[a a]}[a b]) = \beta\,[b b \cdot 1] .$$

Similarly, we find $[c\,c \cdot 1] = \beta^2\,[b\,b \cdot 1]$ and $[w_3 \cdot 1] = \beta\,[w_2 \cdot 1]$; the system (5) thus reads

$$\begin{array}{l}[b b \cdot 1]\,k_2 + \beta\,[b b \cdot 1]\,k_3 + [w_2 \cdot 1] = 0 \\ \beta\,[b b \cdot 1]\,k_2 + \beta^2[b b \cdot 1]\,k_3 + \beta[w_2 \cdot 1] = 0 .\end{array}$$

The elimination of k_2 leads to

$$[c\,c \cdot 2]\,k_3 + [w_3 \cdot 2] = 0 ,$$

and we have

$$[c c \cdot 2] = \beta^2[b b \cdot 1] - \frac{\beta\,[b b \cdot 1]}{[b b \cdot 1]}\,\beta\,[b b \cdot 1] = 0$$

$$[w_3 \cdot 2] = \beta\,[w_2 \cdot 1] - \frac{\beta\,[b b \cdot 1]}{[b b \cdot 1]}\,[w_2 \cdot 1] = 0 ,$$

therefore,

$$-k_3 = \frac{[w_3 \cdot 2]}{[c c \cdot 2]} = \frac{0}{0} . \tag{6}$$

The same holds also of the other two unknowns k_1 and k_2.

The considered case is in practice not unimportant, because it happens sometimes that, *by mistake*, a condition equation which is already contained in the remaining ones is set up. For instance, if in a leveling adjustment there were set up also, as a condition, the closure of the total circumference of the polygon in addition to the individual polygon closures, then this would be nothing else but the sum of the remaining condition equations, and consequently, not independent.

Whether the indeterminacy (6) is always clearly expressed in the numerical computation is doubtful; because of the errors in rounding off, it may happen that, for instance, there does not result exactly $k = \frac{0.000}{0.000}$

but, say, $k = \frac{0.002}{0.001} = 2$, and if the computation were carried further with this, then something absurd would have to result.

In addition to the case of nonindependent condition equations, which we have finished herewith, there is also to be considered the case in which a condition equation is *wrongly* set up or entirely omitted. This does not result in anything conspicuous in the formation and solution of the normal equations, but it manifests itself at the end of the whole adjustment, after the calculation of the corrections v, very disagreeably in the fact that the condition equation, which has been wrongly set up or forgotten, is not satisfied.

After this, we can examine further the *agreement of various function forms and their weights after the adjustment*:

We will use here, as an illustrative example, the adjustment of a triangulation net with angle measurements. Let function F, of which we spoke in section 46, p. 143, be a triangle side, which can be computed from the base line in various ways.

Before the adjustment, these various methods of computation will also result in various values of F; *after* the adjustment, however, all ways must lead to the same value F. We have

Before the adjustment: $$(F) = f_0 + f_1 l_1 + f_2 l_2 + f_3 l_3 + f_4 l_4 , \tag{7}$$

After the adjustment: $$F = f_0 + f_1 (l_1 + v_1) + f_2 (l_2 + v_2) + f_3 (l_3 + v_3) + f_4 (l_4 + v_4) , \tag{8}$$

and for this, we can also find another, very illustrative, form, namely according to (18), section 46, p. 146

$$\left.\begin{aligned} F = f_0 + f_1 l_1 &+ f_2 l_2 + f_3 l_3 + f_4 l_4 \\ + \quad r_1 w_1 &+ r_2 w_2 + r_3 w_3 \end{aligned}\right\}, \tag{9}$$

i.e., we can compute the adjusted value of function F not only by means of the adjusted values $l + v$, but also by means of the observational values l themselves by adding further, in the latter case, the supplemental terms $r_1 w_1$, $r_2 w_2$, $r_3 w_3$. If all w's were equal to zero, i.e. if all condition equations were satisfied by the observations l themselves, then no adjustment would be necessary; the $l + v$'s would have to be taken equal to the l's, and the formulae (7) and (9) would become identical due to the cancellation of the w's.

We can compute function F in various ways from the adjusted values $l + v$. Assuming that we had chosen, instead of (8), the form

$$F = f_0' + f_1'(l_1 + v_1) + f_2'(l_2 + v_2) + f_3'(l_3 + v_3) + f_4'(l_4 + v_4) \tag{10}$$

where such terms, which are missing in (8), occur, and conversely, terms, which are found in (8), will not exist, then, formulae (8) and (10), however, would have to yield the *same* value F, for (8) and (10) can be related only by any condition equations between the $l + v$'s; such equations, however, are all rigorously satisfied after the adjustment.

We can ask further whether the two different forms (8) and (10) of the same function F after the adjustment also lead to the same weight P.

However, upon different assumptions of the f's also the coefficients $[af]$, $[bf]$ . . . and, with them, also the transformation coefficients r become different; on the other hand, in the formula (4), section 46, p. 143, namely

$$F = F_0 + F_1 l_1 + F_2 l_2 + F_3 l_3 + F_4 l_4 , \tag{11}$$

the coefficients F_1, F_2, F_3, F_4 become always the same again, for a function of *all* observations l, is involved here, and this function cannot be influenced by the *form* in which it originally occurred as a function of individual $l + v$'s any more than the adjustment results v are influenced by the *form* in which the condition equations enter the adjustment.

In the little example of section 48, the adjustment of the three angles of a plane triangle, we find this confirmed.

The first function form on p. 150 yields:

(9) $F = x_1$

$f_1 = 1\,, \quad f_2 = 0\,, \quad f_3 = 0;$

(11) $r = -\frac{1}{3};$

(12) $F_1 = +\frac{2}{3}, F_2 = -\frac{1}{3}, F_3 = -\frac{1}{3}.$

The second function form on p. 150 yields:

(9*) $F = 180° - x_2 - x_3$

$f_1 = 0\,, \quad f_2 = -1\,, \quad f_3 = -1\,;$

(11*) $r = +\frac{2}{3};$

(12*) $F_1 = +\frac{2}{3}, F_2 = -\frac{1}{3}, F_3 = -\frac{1}{3}.$

A *formal* proof that the coefficients F in (11) are independent of the first form of the function is not easy to conduct; but a formal proof is also hardly necessary if we understand that, objectively, the adjusted function F can have only *one* mean error, since according to (11) it presents itself as a function of all original measurements l.

We consider further the total differential of (11):

$$dF = F_1 dl_1 + F_2 dl_2 + F_3 dl_3 + F_4 dl_4 \,. \tag{12}$$

$F_1 dl_1$ represents the change of function F which is generated if we change only the observation l_1 by dl_1 and, retaining all other observations l, repeat the whole method of adjustment.

In general, the individual terms of (12) show what amount each observational error dl supplies to the function error dF. This amount is independent of the *form* of the calculation of the function.

Section 53. Most Favorable Distribution of Weight

Schreiber's Theorem

In concluding the adjustment of conditioned observations, we treat a theorem found by General Schreiber in regard to the distribution of weight in the case of conditioned observations. Since this theorem was found in the case of trigonometric adjustments (base net of Göttingen), and finds its main application to trigonometric nets, we express the theorem itself, in the interest of illustrative clearness, in trigonometric application:

> If in a triangulation net with condition equations, *one* side with a weight P as large as possible is to be determined, the sum $[p]$ of the weights of the measured angles $p_1, p_2 \ldots$ being constant, then there is, for all cases, among the possible distributions of the weights $p_1, p_2 \ldots$ one distribution, in which only as many weights p actually occur as the number of the angles (or directions, etc.) indispensably necessary for the determination of that one side, while the remaining weights p are all to be set equal to zero.

The general theorem treats, in place of a triangle side, any function of the adjusted observations.

For this, the simplest example of all consists in the arithmetic mean with unequal weights.

Let us have n independent measurements $l_1, l_2, \ldots, l_n$ with the weights $p_1, p_2, \ldots, p_n$. The measurements $l_1, l_2, \ldots, l_n$ are related to an unknown x as follows:

$$a_1 x - l_1 = 0\,, \qquad a_2 x - l_2 = 0 \ldots, \qquad a_n x - l_n = 0\,;$$

then we have, for the determination of x, the equation

$$x = \frac{[p\,a\,l]}{[p\,a\,a]}\,, \tag{1}$$

and the weight of x is

$$P = [p\,a\,a] = p_1 a_1^{\,2} + p_2 a_2^{\,2} + \ldots\, p_n a_n^{\,2}\,.$$

If we ask for the maximum of P with respect to the weights p and the *distribution* of weight, the sum $[p]$ being constant, then we understand offhand that P becomes largest, if we choose of all coefficients

$a_1, a_2, \ldots, a_n$ the *largest*, assign it the weight $[p]$ and set all other p's equal to *zero*, e.g., if the first value a_1 is the largest, then, according to this, we will have

$$P = [p]\, a_1^{\,2}\,. \tag{2}$$

Hence, we carry out only that measurement l, which leads, in the most favorable way, to the determination of x, and do not measure all other l's at all.

(This does not hold, of course, for the case in which we measure also the other l's in order to avoid one-sided errors. We shall not speak here and in the following problem of such practical secondary considerations.)

Before we pass over to the general theorem, we will take up further a simple example with *one* condition equation and three observations. According to the summary of formulae of section 47, p. 147, we have, for this, the following equations:

Condition equation:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + w = 0\,. \tag{3}$$

Function:

$$F = f_1 x_1 + f_2 x_2 + f_3 x_3\,. \tag{4}$$

Transformation equation:

$$\left[\frac{a\,a}{p}\right] r + \left[\frac{a\,f}{p}\right] = 0\,. \tag{5}$$

Weight coefficients after the adjustment:

$$F_1 = f_1 + a_1 r\,, \qquad F_2 = f_2 + a_2 r\,, \qquad F_3 = f_3 + a_3 r\,. \tag{6}$$

The weight of the function after the adjustment P is determined by

$$\frac{1}{P} = \frac{F_1^2}{p_1} + \frac{F_2^2}{p_2} + \frac{F_3^2}{p_3}\,. \tag{7}$$

We apply these formulae (3) to (7), which hold for an adjustment with *one* condition equation, to an example:

In Fig. 1 in the margin, we deal with the determination of the height h of a triangle, in which the base b and three angles (1), (2), (3) are measured. There exists a condition equation according to equation (3):

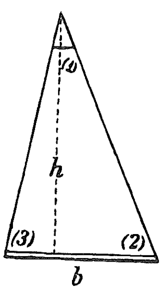


Fig. 1.

$$v_1 + v_2 + v_3 + w = 0\,,$$

hence,

$$a_1 = 1\,, \quad a_2 = 1\,, \quad a_3 = 1\,.$$

The function F is in this case

$$h = \frac{b}{\sin(1)} \sin(2) \sin(3)\,.$$

In order to make this function linear, we need

$$\frac{\partial h}{\partial (1)} = b \sin(2) \sin(3) \left(\frac{-\cos(1)}{\sin^2(1)}\right) = -h \cot\;(1)\,,$$

likewise,

$$\frac{\partial h}{\partial (2)} = +h \cot\;(2) \text{ and } \frac{\partial h}{\partial (3)} = +h \cot\;(3)\,.$$

For the maximum of the function, with which we are concerned here, we can leave aside everything constant, and as the linear function F in the sense of equation (4) we can therefore take

$$F = -\cot(1)\, x_1 + \cot(2)\, x_2 + \cot(3)\, x_3 \,.$$

Hence, if we write for abbreviation $\cot(1) = c_1$, etc., we have

$$f_1 = -c_1\,, \qquad f_2 = +c_2\,, \qquad f_3 = +c_3\,. \tag{8}$$

The general equation (5) yields herewith

$$\left[\frac{1}{p}\right] r + \left(-\frac{c_1}{p_1} + \frac{c_2}{p_2} + \frac{c_3}{p_3}\right) = 0\,.$$

Solution:

$$r = \frac{\dfrac{c_1}{p_1} - \dfrac{c_2}{p_2} - \dfrac{c_3}{p_3}}{\left[\dfrac{1}{p}\right]}\,. \tag{9}$$

Hence, according to (6): $$F_1 = -c_1 + r\,, \qquad F_2 = c_2 + r\,, \qquad F_3 = c_3 + r\,, \tag{10}$$

and according to (7): $$\frac{1}{P} = \frac{(-c_1 + r)^2}{p_1} + \frac{(c_2 + r)^2}{p_2} + \frac{(c_3 + r)^2}{p_3}\,. \tag{11}$$

This with (9) and (10) can be brought further into several other forms, which are communicated later in (15) and (16). Formulae (9), (10), (11), obtained directly, are sufficient for our immediate purpose.

The point in question is to make function (11) a minimum with the side condition

$$[p] = p_1 + p_2 + p_3 = \text{constant}\,. \tag{12}$$

We will obtain this by continued experiments with a numerical example, which corresponds approximately to Fig. 1, namely

$$\left.\begin{array}{ll} (1) = 34^\circ & \cot\ 34^\circ = c_1 = 1.483 \\ (2) = 68^\circ & \cot\ 68^\circ = c_2 = 0.404 \\ (3) = 78^\circ & \cot\ 78^\circ = c_3 = 0.213 \end{array}\right\}. \tag{13}$$

In regard to the weights we will assume first $p_1 = 1$, $p_2 = 1$, $p_3 = 1$, $[p] = 3$, and with this, the preceding formulae (9), (10), (11) yield the following:

$$r = 0.289\,, \qquad F_1 = -1.194\,, \qquad F_2 = +0.693\,, \qquad F_3 = +0.502\,,$$
$$\frac{1}{P} = 2.1578\,. \qquad \text{Absolute sum } [\pm F] = 2.389\,.$$

According to this, we have made a *new* assumption of weight, namely, adjusted the p_1's, p_2's, p_3's, proportionally to the F_1's, F_2's, F_3's, and again to the previous sum $[p] = 3$, i.e. the new assumption is:

$$p_1 = 1.50\,, \qquad p_2 = 0.87\,, \qquad p_3 = 0.63\,, \qquad [p] = 3.00\,.$$

In this manner, we have repeated the computation six times, as is seen from the following table with the numbers 1, 2, 3, . . . , 7; the numbers 0 and 0.5, coming first in the table, have been calculated in addition afterwards with new arbitrary assumptions of weight.

The series 1 to 7 shows quite a good convergence and leads finally to the most favorable distribution of weight:

$$p_1 = 2.696\,, \qquad p_2 = 0.304\,, \qquad p_3 = 0.000 \text{ with } [p] = 3.000\,. \tag{14}$$

Hence, we will have $p_3 = 0$, i.e., in this case, we shall not measure at all the third angle (3).

Number	p_1	p_2	p_3	$[p]$	r	F_1	F_2	F_3	$\left[\frac{FF}{p}\right]$
0	0.0	1.5	1.5	3.0	1.483	0.000	1.887	1.696	5.292
0.5	0.5	1.0	1.5	3.0	0.660	−0.823	1.064	0.873	2.995
1	1.00	1.00	1.00	3.00	0.289	−1.194	+0.693	+0.502	2.158
2	1.50	0.87	0.63	3.00	0.059	−1.424	+0.463	+0.272	1.715
3	1.979	0.643	0.378	3.000	−0.0940	−1.577	+0.310	+0.119	1.443
4	2.358	0.464	0.178	3.000	−0.1756	−1.659	+0.228	+0.037	1.287
5	2.586	0.356	0.058	3.000	−0.2072	−1.690	+0.197	+0.006	1.214
6	2.679	0.312	0.009	3.000	−0.2128	−1.696	+0.191	+0.000	1.191
7	2.696	0.304	0.000	3.000	−0.2130	−1.696	+0.191	+0.000	1.187

If we will substitute this $p_3 = 0$ in (9), (10), (11), then the formal difficulty arises that $r = \frac{0}{0}$ appears; therefore, it is necessary to bring (11) first into another form. We will be able to convince ourselves easily that the following new forms for (11) are algebraically correct:

$$\frac{1}{P} = \frac{c_1^2}{p_1} + \frac{c_2^2}{p_2} + \frac{c_3^2}{p_3} - \frac{1}{\left[\frac{1}{p}\right]}\left(\frac{c_1}{p_1} - \frac{c_2}{p_2} - \frac{c_3}{p_3}\right)^2 \tag{15}$$

$$\frac{1}{P} = \frac{p_1(c_3 - c_2)^2 + p_2(c_1 + c_3)^2 + p_3(c_1 + c_2)^2}{p_1 p_2 + p_1 p_3 + p_2 p_3}\,. \tag{16}$$

We can set here $p_3 = 0$; whereby we obtain

$$\left(\frac{1}{P}\right)_{min} = \frac{1}{P_{max}} = \frac{(c_2 - c_3)^2}{p_2} + \frac{(c_1 + c_3)^2}{p_1}\,. \tag{17}$$

With this, our purpose to premise a simple example to the general theory of Schreiber's theorem is reached; we have added, also, formulae (15) to (17) partly because we shall meet with these formulae also otherwise in geodesy. –

Now passing over to the general theory of the theorem of the most favorable distribution of weight, we use the general summary of formulae of section 47 and thereby consider n observations with n weights $p_1, p_2, p_3, \ldots, p_n$ and 3 condition equations.

If from the adjusted observations a function

$$F = f_1(l_1 + v_1) + f_2(l_2 + v_2) + f_3(l_3 + v_3) + f_4(l_4 + v_4) \tag{18}$$

is computed, then, in general, only individual observations, e.g., the quantities l_1, l_2, l_3, l_4 will occur herein. We are to bear in mind, however, that the corrections v_1, v_2, v_3, v_4 are computed by adjustment from all observations $l_1, l_2, \ldots, l_n$, hence, are linear functions of the l's. Consequently, we can write F also in the form

$$F = F_1 l_1 + F_2 l_2 + F_3 l_3 + \ldots + F_n l_n\,; \tag{19}$$

here the coefficients F have the following meaning according to (13), section 47, p. 148:

$$\left.\begin{aligned} F_1 &= f_1 + a_1 r_1 + b_1 r_2 + c_1 r_3 \\ F_2 &= f_2 + a_2 r_1 + b_2 r_2 + c_2 r_3 \\ &\ldots\ldots\ldots \\ F_n &= f_n + a_n r_1 + b_n r_2 + c_n r_3 \end{aligned}\right\}, \tag{20}$$

and the transformation coefficients r result from equations (12), section 47, p. 148

$$\left.\begin{aligned} \left[\frac{a\,a}{p}\right] r_1 + \left[\frac{a\,b}{p}\right] r_2 + \left[\frac{a\,c}{p}\right] r_3 + \left[\frac{a\,f}{p}\right] &= 0 \\ \left[\frac{a\,b}{p}\right] r_1 + \left[\frac{b\,b}{p}\right] r_2 + \left[\frac{b\,c}{p}\right] r_3 + \left[\frac{b\,f}{p}\right] &= 0 \\ \left[\frac{a\,c}{p}\right] r_1 + \left[\frac{b\,c}{p}\right] r_2 + \left[\frac{c\,c}{p}\right] r_3 + \left[\frac{c\,f}{p}\right] &= 0 \end{aligned}\right\}. \tag{21}$$

For our purposes, we will bring these transformation equations into another form and, for this, write the first equation as follows:

$$\begin{aligned} &\frac{f_1}{p_1} a_1 + \frac{a_1 a_1}{p_1} r_1 + \frac{a_1 b_1}{p_1} r_2 + \frac{a_1 c_1}{p_1} r_3 \\ +&\frac{f_2}{p_2} a_2 + \frac{a_2 a_2}{p_2} r_1 + \frac{a_2 b_2}{p_2} r_2 + \frac{a_2 c_2}{p_2} r_3 \\ &\ldots\ldots\ldots\ldots \\ +&\frac{f_n}{p_n} a_n + \frac{a_n a_n}{p_n} r_1 + \frac{a_n b_n}{p_n} r_2 + \frac{a_n c_n}{p_n} r_3 = 0\,. \end{aligned}$$

or

$$\begin{aligned} &\left(f_1 + a_1 r_1 + b_1 r_2 + c_1 r_3\right) \frac{a_1}{p_1} \\ +&\left(f_2 + a_2 r_1 + b_2 r_2 + c_2 r_3\right) \frac{a_2}{p_2} \\ &\ldots\ldots\ldots\ldots \\ +&\left(f_n + a_n r_1 + b_n r_2 + c_n r_3\right) \frac{a_n}{p_n} = 0\,. \end{aligned}$$

This is according to (20)

$$\frac{F_1}{p_1} a_1 + \frac{F_2}{p_2} a_2 + \ldots + \frac{F_n}{p_n} a_n = 0\,.$$

If the second and the third equations of (21) are transformed likewise, then we have in all

$$\left.\begin{aligned} \frac{F_1}{p_1} a_1 + \frac{F_2}{p_2} a_2 + \ldots + \frac{F_n}{p_n} a_n &= 0 \\ \frac{F_1}{p_1} b_1 + \frac{F_2}{p_2} b_2 + \ldots + \frac{F_n}{p_n} b_n &= 0 \\ \frac{F_1}{p_1} c_1 + \frac{F_2}{p_2} c_2 + \ldots + \frac{F_n}{p_n} c_n &= 0 \end{aligned}\right\}. \tag{22}$$

If we pass over now to the weight of function F in (19) and denote the reciprocal weight by y, then we have according to (14), section 47, p. 148

$$y = \frac{1}{P} = \frac{F_1^2}{p_1} + \frac{F_2^2}{p_2} + \ldots + \frac{F_n^2}{p_n} = \left[\frac{FF}{p}\right]. \tag{23}$$

This expression shall now be made a minimum, while, at the same time, we have the side condition

$$p_1 + p_2 + \ldots + p_n = [p] = S = \text{constant} . \tag{24}$$

Since the weights p must be positive, then we set for the time being

$$p_1 = x_1^2 , \; p_2 = x_2^2 , \; \ldots , \; p_n = x_n^2 , \tag{25}$$

and if we denote by λ a factor still undetermined at first, then we have

$$\left[\frac{FF}{x^2}\right] + \lambda \left([x^2] - S\right) = \text{minimum} . \tag{26}$$

If we differentiate this expression with respect to x_1, then we obtain as the first minimum condition

$$-\frac{F_1^2}{x_1^4} 2 x_1 + 2 \lambda x_1 = 0 \quad \text{or} \quad \left(\frac{F_1^2}{x_1^4} - \lambda\right) x_1 = 0 .$$

In all, there result the n equations

$$\left.\begin{aligned} \left(\frac{F_1^2}{x_1^4} - \lambda\right) x_1 &= 0 \\ \left(\frac{F_2^2}{x_2^4} - \lambda\right) x_2 &= 0 \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; &\cdot \; \cdot \\ \left(\frac{F_n^2}{x_n^4} - \lambda\right) x_n &= 0 \end{aligned}\right\} . \tag{27}$$

It follows hence that for the individual F's it must be true that either

$$\frac{F^2}{x^4} = \lambda \quad \text{or} \quad x = 0$$

for which we will write

$$\frac{F^2}{p^2} = \lambda \quad \text{or} \quad p = 0 . \tag{28}$$

But, at the same time, there also exist for the n values $\frac{F}{p}$ the three equations (22), which represent the conditions for function F being computed according to the method of least squares.

Therefore, if we assume of the n quantities $\frac{F}{p}$, the values $\frac{F}{p} = \sqrt{\lambda}$ for $n - 3$ of them, then the remaining three values $\frac{F}{p}$ are determined by equations (22). With this, it is excluded that these three remaining values satisfy the equations $\frac{F^2}{p^2} = \lambda$ and the equations $p = 0$ must hold for them. But since these three quantities contain values already determined by equation (22), then this discrepancy is solvable only by the fact that, for this, not only $p = 0$, but also $F = 0$, with which the quotients $\frac{F}{p}$ can very well assume the values determined by equations (22).

We thus have for the minimum of $\frac{1}{P}$, according to (23) and (28), the value

$$\left(\frac{1}{P}\right)_{min} = \left(p_1 + p_2 + \ldots + p_{n-3}\right)\lambda = S\lambda.$$

From equations $\frac{F^2}{p^2} = \lambda$ it follows, since the p's can only be positive, that the absolute amount of each F is equal to the corresponding $p\sqrt{\lambda}$.

Therefore, the result is that we reach the smallest value of y; hence, the largest weight P of the function F, if we set equal to zero as many weights p of the observations as there exist condition equations, i.e., only as many observations are measured as are required for the determination of the function, in the present case, therefore, $n - 3$ observations. For these $n - 3$ observations, we compute the values of the F's from (20). Of all possible combinations of the $n - 3$ measured quantities, that which yields the smallest absolute sum of the F's is to be chosen. The weights are then proportional to the absolute values of the F's, and the factor of proportionality λ results from the sum S of the weights fixed from the outset.

As far as the actual calculation of a weight distribution is concerned, we will always proceed best according to the instruction of our small example (8) to (14), pp. 169 and 170, in such a way that we assume tentatively a weight distribution, calculate the F's, and repeat the calculation with new weights in proportion to the F's obtained for the first time; if individual terms converge clearly with respect to zero, then we will set the p's in question at once entirely equal to zero, and so on.

If such a problem, objectively considered in itself, lets us assume a considerably unequally favorable distribution of weight, then the successive approximate computation with the F's and p's will converge quickly; also, *before* the beginning of the whole calculation there will then already be hints in what sense, for instance, weight intensifications or weight reductions down to zero may be appropriate.

If, on the other hand, the tentative computations do *not* converge, then we may have to do with a case in which unequal distribution of weight is out of place altogether.

"Schreiber's Theorem" about the most favorable weight distribution was first communicated by Schreiber himself in the treatise "Die Anordnung der Winkelbeobachtungen im Göttinger Basisnetz" in *Zeitschrift für Vermessungswesen*, 1882, pp. 129-161. Further literature on this subject is:

Jordan, "Über günstigste Gewichtsverteilung," *Zeitschrift für Vermessungswesen*, 1888, pp. 641-649.

Schreiber, "Über günstigste Gewichtsverteilung," *Zeitschrift für Vermessungswesen*, 1889, pp. 57-59.

Runge, "Der Schreibersche Satz," *Zeitschrift für Vermessungswesen*, 1890, pp. 21-24.

Helmert, *Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate*, 2nd Edition, Leipzig and Berlin, 1907, pp. 551-563.

Sommer, "Minimumsprobleme für Summen absoluter Beträge," *Zeitschrift für Vermessungswesen*, 1919, pp. 105-125.

Krüger, "Über die Bestimmung der Winkelgewichte in Basisnetzen," *Veröffentlichung des Preuss. Geod. Inst.*, N.F. Nr. 81, Berlin, 1920.

Jung, *Über die günstigste Gewichtsverteilung in Basisnetzen*. Akademische Abhandlung, Upsala, 1924.

The last-mentioned paper gives a very detailed treatment of the problem and its practical applications.

Earlier, another treatise on this subject also appeared: "Über eine Aufgabe der Ausgleichungsrechnung," by H. Bruns, Vol. XIII of *Abh. d. m. ph. Kl. der Kgl. Sächs. Gesellschaft der Wissenschaften*, Nr. VII, pp. 517-563, Leipzig, 1886. This treatise contains a generalization of Schreiber's theorem; it does not deal only with *one* function whose weight becomes a maximum with a constant sum of weights $[p]$, but with a totality of several functions.

The forms of adjustment treated by us until now - namely, first, direct observations with the arithmetic mean; second, indirect observations; and third, conditioned observations - were the only problems treated at the time of the origin of the method of least squares. Gauss did not go further in his main work of "Theoria combinationis," 1823-1826, and Gerling, in 1843, still did not know anything further in his textbook.

Meanwhile, however, the geodetic need had produced a still higher form of adjustment, "indirect observations with condition equations."

In his *Gradmessung in Ostpreussen, 1838*, Bessel first set up this problem, solved it essentially and applied it to the adjustment of triangulation nets with incomplete sets of directions, but did not add computations of weight and errors.

The whole problem, adjustment *and accuracy determinations* for indirect observations with condition equations, seems to have been solved for the first time by J. Zech ("Einladungsschrift zur Königs-Geburtstags-Feier 1857 [invited paper for the Royal Birthday celebration, 1857], Tübingen bei Fues"). This was a theoretical solution: The application of the problem to geodesy was soon undertaken by the first-class geodesists of that time, especially by Andrae in Copenhagen and Hansen in Gotha, and at the time of the origin of the Internationale Erdmessung, after 1862, this problem played an important role in the then-existing geodetic literature.

In Prussia, the problem in question was used to great advantage in the land survey from about 1870-1880.

Our treatment of the problem presented in the following sections 54 to 58 is developed independently of the cited authorities, according to the author's own course. We have explained Hansen's theories in Jordan-Steppe's *Deutsches Vermessungswesen*, 1882, I, pp. 53 to 59.

The first treatment of the adjustment of indirect observations with condition equations consists in the theory that we eliminate the condition equations, and since we obtain in this way, at the same time, the clearest insight into the determination of the mean error, we set this form of solution up first, although it is less important in practice:

We have direct observations l, which are connected with u unknowns $x, y, z, t \ldots$ by n error equations. Besides, between the unknowns there exist r condition equations which are to be satisfied rigorously.

Error equations:

$$\text{Number} = n \left\{ \begin{array}{l} v_1 = a_1 x + b_1 y + c_1 z + d_1 t \ldots - l_1 \\ v_2 = a_2 x + b_2 y + c_2 z + d_2 t \ldots - l_2 \\ v_3 = a_3 x + b_3 y + c_3 z + d_3 t \ldots - l_3 \\ \ldots\ldots\ldots\ldots\ldots \\ v_n = \underbrace{a_n x + b_n y + c_n z + d_n t \ldots}_{\text{Number} = u} - l_n \end{array} \right\}. \tag{1}$$

Condition equations:

$$\text{Number} = r \left\{ \begin{array}{l} A_0 + A_1 x + A_2 y + A_3 z + A_4 t + \ldots = 0 \\ B_0 + B_1 x + B_2 y + B_3 z + B_4 t + \ldots = 0 \\ \ldots \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{Number} = u} \ldots \end{array} \right\}. \tag{2}$$

We can reduce this adjustment to the adjustment of indirect observations without condition equations by eliminating a number r of the u unknowns by means of the condition equations (2). This is done as follows:

We can imagine the equations (2) solved as follows:

$$\text{Number} = r \left\{ \begin{array}{l} x = A_1' z + A_2' t + \ldots \\ y = B_1' z + B_2' t + \ldots \\ \ldots \underbrace{\ldots\ldots\ldots\ldots}_{\text{Number} = u - r} \end{array} \right\}. \tag{3}$$

If we substitute these expressions in (1), then there is obtained:

$$\text{Number} = n \left\{ \begin{array}{l} v_1 = c_1' z + d_1' t + \ldots - l_1' \\ v_2 = c_2' z + d_2' t + \ldots - l_2' \\ v_3 = c_3' z + d_3' t + \ldots - l_3' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n = \underbrace{c_n' z + d_n' t + \ldots}_{\text{Number} = u - r} - l_n' \end{array} \right\}. \tag{4}$$

Therefore, we have to deal now with an adjustment of indirect observations, which is to be treated according to the previous rules. After the adjustment we have the mean error

$$m = \sqrt{\frac{[v\,v]}{n - (u - r)}} \quad \text{or} \quad = \sqrt{\frac{[v\,v]}{(n - u) + r}}. \tag{5}$$

Between n, u, and r there exist the following relations:

Since the condition equations (2) must be satisfied rigorously but the x's, y's, z's, t's must thereby not be completely determined, then it must be true that

$$u > r\,, \tag{6}$$

and if the equations (4), which remain after the elimination of the r unknowns, are still to have the character of "error equations," then it must also be true that

$$n > u - r\,. \tag{7}$$

Because of (6) and (7), the denominator $n - (u - r)$ will therefore always be a positive number.

If there exists only a *small* number of condition equations (2), this method may be suitable, and the choice of the unknowns to be eliminated will then depend on the nature of the problem in each case.

In the following, we will however deal now also with the direct solution of the problem.

Direct adjustment of indirect observations with condition equations

We limit ourselves to four error equations with three unknowns and to two condition equations with the designations of equations (1) and (2), p. 174.

If we introduce two correlates k_1 and k_2, still undetermined at first, then the minimum condition of the sum of squares $[v\,v]$, taking into account the two condition equations, reads

$$\begin{aligned} &[v\,v] + 2\,k_1\,(A_0 + A_1\,x + A_2\,y + A_3\,z) \\ &\qquad + 2\,k_2\,(B_0 + B_1\,x + B_2\,y + B_3\,z) = \text{minimum}\,. \end{aligned} \tag{8}$$

The differentiation of this equation with respect to x with the use of the error equations (1) yields as the first condition

$$2\,v_1\,a_1 + 2\,v_2\,a_2 + \ldots + 2\,k_1\,A_1 + 2\,k_2\,B_1 = 0$$

or

$$[a\,v] + A_1\,k_1 + B_1\,k_2 = 0\,.$$

If we carry out, in the same way, the differentiation with respect to y and z, then we obtain in all

$$\left.\begin{array}{l} [a v]+A_1 k_1+B_1 k_2=0 \\ [b v]+A_2 k_1+B_2 k_2=0 \\ [c v]+A_3 k_1+B_3 k_2=0 \end{array}\right\}. \tag{9}$$

Herein we replace the corrections v by their values in the error equations (1), and if we add the two condition equations (2), then we have

$$\left.\begin{array}{l} [a a] x+[a b] y+[a c] z+A_1 k_1+B_1 k_2-[a l]=0 \\ [a b] x+[b b] y+[b c] z+A_2 k_1+B_2 k_2-[b l]=0 \\ [a c] x+[b c] y+[c c] z+A_3 k_1+B_3 k_2-[c l]=0 \\ A_1 x+A_2 y+A_3 z+A_0=0 \\ B_1 x+B_2 y+B_3 z+B_0=0 \end{array}\right\}. \tag{10}$$

From these equations, which have the character of normal equations, the unknowns x, y, z and the correlates k_1, k_2 can be computed. Likewise, there result the weight reciprocals $[\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$ of the three unknowns. If, in addition, the corrections v are computed from the error equations, then the mean error of an observation can be determined from (5).

We find another solution of the problem, in which equations (1) and (2) are treated separately, by the aid of the theory of equivalent observations of section 33, p. 106.

If we determine the unknowns x, y, z from the error equations alone, then we obtain preliminary values x_0, y_0, z_0, which still require further corrections in order to satisfy the condition equations. If we denote these corrections by δx, δy, δz, then the final values of the three unknowns are

$$x=x_0+\delta x, \qquad y=y_0+\delta y, \qquad z=z_0+\delta z. \tag{11}$$

For the preliminary values x_0, y_0, z_0 we have the normal equations

$$\left.\begin{array}{l} [a a] x_0+[a b] y_0+[a c] z_0-[a l]=0 \\ [a b] x_0+[b b] y_0+[b c] z_0-[b l]=0 \\ [a c] x_0+[b c] y_0+[c c] z_0-[c l]=0 \end{array}\right\}. \tag{12}$$

We obtain hence the following expressions for the unknowns according to (14), section 33, p. 106:

$$\left.\begin{array}{l} x_0=u_1-\dfrac{[a b]}{[a a]} u_2-\left(\dfrac{[a c]}{[a a]}-\dfrac{[a b][b c \cdot 1]}{[a a][b b \cdot 1]}\right) u_3 \\ y_0=u_2-\dfrac{[b c \cdot 1]}{[b b \cdot 1]} u_1 \\ z_0=u_3 \end{array}\right\}, \tag{13}$$

where

$$u_1=\frac{[a l]}{[a a]}, \qquad u_2=\frac{[b l \cdot 1]}{[b b \cdot 1]}, \qquad u_3=\frac{[c l \cdot 2]}{[c c \cdot 2]} \tag{14}$$

Here the quantities u_1, u_2, u_3 are to be treated as measured quantities to which the weights $[a a]$, $[b b \cdot 1]$, $[c c \cdot 2]$ are to be assigned.

By means of the condition equations these fictitious measured quantities receive corrections

$\lambda_1, \lambda_2, \lambda_3$, by which the unknowns change into their final values. We thus have

$$\left.\begin{aligned} x &= x_0 + \delta x = u_1 + \lambda_1 - \frac{[a\,b]}{[a\,a]}(u_2 + \lambda_2) - \left(\frac{[a\,c]}{[a\,a]} - \frac{[a\,b]\,[b\,c\cdot 1]}{[a\,a]\,[b\,b\cdot 1]}\right)(u_3 + \lambda_3) \\ y &= y_0 + \delta y = u_2 + \lambda_2 - \frac{[b\,c\cdot 1]}{[b\,b\cdot 1]}(u_1 + \lambda_1) \\ z &= z_0 + \delta z = u_3 + \lambda_3 \end{aligned}\right\}. \tag{15}$$

With the expressions (11) the two condition equations obtain the form

$$\left.\begin{aligned} A_0 + A_1(x_0 + \delta x) + A_2(y_0 + \delta y) + A_3(z_0 + \delta z) = 0 \\ B_0 + B_1(x_0 + \delta x) + B_2(y_0 + \delta y) + B_3(z_0 + \delta z) = 0 \end{aligned}\right\}. \tag{16}$$

If the values (15) are substituted herein, then we have in the two equations only the corrections $\lambda_1, \lambda_2, \lambda_3$, and the adjustment can then be carried out by taking into account the above indicated weights according to section 44.

Section 55. Separation of the Adjustment into Two Parts, and First Adjustment

In triangulation adjustments on a large scale, in the case of which this problem has formerly played an important role, there has been applied the following symmetrical method, in the case of which the error equations and the condition equations are likewise adjusted separately.

By limiting ourselves again to four error equations, three unknowns, and two condition equations, we summarize once more:

Error equations

$$\text{Number} = n \left\{\begin{aligned} v_1 &= a_1 x + b_1 y + c_1 z - l_1 \\ v_2 &= a_2 x + b_2 y + c_2 z - l_2 \\ v_3 &= a_3 x + b_3 y + c_3 z - l_3 \\ v_4 &= \underbrace{a_4 x + b_4 y + c_4 z}_{\text{Number } = u} - l_4 \end{aligned}\right\}. \tag{1}$$

Condition equations

$$\text{Number} = r \left\{\begin{aligned} \underbrace{\begin{aligned} A_0 + A_1 x + A_2 y + A_3 z = 0 \\ B_0 + B_1 x + B_2 y + B_3 z = 0 \end{aligned}}_{\text{Number } = u} \end{aligned}\right\}. \tag{2}$$

The adjustment of the error equations (1) without taking into account the condition equations yields the normal equations

$$\left.\begin{aligned} [a\,a]\,x_0 + [a\,b]\,y_0 + [a\,c]\,z_0 - [a\,l] = 0 \\ [a\,b]\,x_0 + [b\,b]\,y_0 + [b\,c]\,z_0 - [b\,l] = 0 \\ [a\,c]\,x_0 + [b\,c]\,y_0 + [c\,c]\,z_0 - [c\,l] = 0 \end{aligned}\right\}. \tag{3}$$

We solve these equations (3) for x_0, y_0, z_0, and at the same time we can determine all weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc., according to section 30 or 36; these weight coefficients will be needed in the second part of the adjustment.

After this, we compute further the corrections of the observations

$$\left.\begin{aligned} v_1' &= a_1 x_0 + b_1 y_0 + c_1 z_0 - l_1 \\ v_2' &= a_2 x_0 + b_2 y_0 + c_2 z_0 - l_2 \\ v_3' &= a_3 x_0 + b_3 y_0 + c_3 z_0 - l_3 \\ v_4' &= a_4 x_0 + b_4 y_0 + c_4 z_0 - l_4 \end{aligned}\right\}, \tag{4}$$

which we denote by v' to distinguish from the final v's.

The values x_0, y_0, z_0 will obtain corrections $\delta x, \delta y, \delta z$ through the second part of the adjustment so that

$$x = x_0 + \delta x, \qquad y = y_0 + \delta y, \qquad z = z_0 + \delta z. \tag{5}$$

The final corrections of the observations are then

$$\left.\begin{aligned} v_1 &= a_1 (x_0 + \delta x) + b_1 (y_0 + \delta y) + c_1 (z_0 + \delta z) - l_1 = v_1' + v_1'' \\ v_2 &= a_2 (x_0 + \delta x) + b_2 (y_0 + \delta y) + c_2 (z_0 + \delta z) - l_2 = v_2' + v_2'' \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\}. \tag{6}$$

To the v''s already computed there are thus added further the quantities

$$\left.\begin{aligned} v_1'' &= a_1 \delta x + b_1 \delta y + c_1 \delta z \\ v_2'' &= a_2 \delta x + b_2 \delta y + c_2 \delta z \\ &\cdots\cdots\cdots \end{aligned}\right\}. \tag{7}$$

The quantities x_0, y_0, z_0 and v_1', v_2', v_3', v_4' are to be conceived as approximate values, which are to be treated for the second part of the adjustment like constant numbers. The values of $\delta x, \delta y, \delta z$ and $v_1'', v_2'', v_3'', v_4''$ are now to be determined in such a way that $[v v]$ becomes a minimum and that, at the same time, the two condition equations (2) are satisfied.

Before we pass over to this, we have to consider once more the sum of squares $[v v]$. We have

$$v = v' + v'',$$

hence,

$$[v v] = [v' v'] + [v'' v''] + 2 [v' v'']. \tag{8}$$

Of this, however, $[v' v''] = 0$, as is easily proved by multiplying (7) by v':

$$[v' v''] = [a v'] \delta x + [b v'] \delta y + [c v'] \delta z. \tag{9}$$

But according to (4) and (3) we have:

$$[a v'] = [a a] x_0 + [a b] y_0 + [a c] z_0 - [a l] = 0,$$

and, likewise, $[b v'] = 0$ and $[c v'] = 0$; therefore in all $[v' v''] = 0$, and (8) changes into

$$[v v] = [v' v'] + [v'' v'']. \tag{10}$$

After the sum $[v v]$, whose minimum is to be obtained, is divided into two parts $[v' v']$ and $[v'' v'']$, of which the first part $[v' v']$ is already known, we have to make only just $[v'' v'']$ a minimum.

The separation of the x's, y's, z's into $x_0 + \delta x$, $y_0 + \delta y$, $z_0 + \delta z$ can be expressed also in the condition equations (2). For these yield

$$\left.\begin{aligned} A_0 + A_1 (x_0 + \delta x) + A_2 (y_0 + \delta y) + A_3 (z_0 + \delta z) = 0 \\ B_0 + B_1 (x_0 + \delta x) + B_2 (y_0 + \delta y) + B_3 (z_0 + \delta z) = 0 \end{aligned}\right\}. \tag{11}$$

Therefore, if we set

$$\left.\begin{aligned} A_0 + A_1 x_0 + A_2 y_0 + A_3 z_0 = w_1 \\ B_0 + B_1 x_0 + B_2 y_0 + B_3 z_0 = w_2 \end{aligned}\right\}, \tag{12}$$

then the comparison of (11) with (12) yields

$$\left.\begin{aligned} A_1 \delta x + A_2 \delta y + A_3 \delta z + w_1 = 0 \\ B_1 \delta x + B_2 \delta y + B_3 \delta z + w_2 = 0 \end{aligned}\right\}. \tag{13}$$

These are the condition equations for the second part of the adjustment, to which we pass over now.

Section 56. Correlation Adjustment of the Second Part

We have convinced ourselves that we are to make $[v'' v''] = \text{minimum}$ with the side conditions

$$\left.\begin{aligned} A_1 \delta x + A_2 \delta y + A_3 \delta z + w_1 = 0 \\ B_1 \delta x + B_2 \delta y + B_3 \delta z + w_2 = 0 \end{aligned}\right\}. \tag{1}$$

If these condition equations (1) referred, instead of to the corrections δx, δy, δz, directly to the v'''s, then we *would* proceed according to the correlation method. But this is not the case; therefore, we must replace the δx's, δy's, δz's in the condition equations (1) by the v'''s.

By multiplying equations (7), section 55, p. 178, in succession by $a_1, a_2 \ldots$ and forming the sum, we obtain $[a v'']$, and if we set up, accordingly, $[b v'']$ and $[c v'']$, then we obtain

$$\left.\begin{aligned} [a a] \delta x + [a b] \delta y + [a c] \delta z = [a v''] \\ [a b] \delta x + [b b] \delta y + [b c] \delta z = [b v''] \\ [a c] \delta x + [b c] \delta y + [c c] \delta z = [c v''] \end{aligned}\right\}. \tag{2}$$

These equations have exactly the form of the normal equations of indirect observations, in which $v''_1, v''_2 \ldots$ is set in place of the l_1's, l_2's . . . . Now we know according to (6), section 30, p. 95, from the theory of the adjustment of indirect observations that the general solution of the normal equations is contained in the following formulae:

$$\left.\begin{aligned} \delta x = \alpha_1 v_1'' + \alpha_2 v_2'' + \alpha_3 v_3'' + \alpha_4 v_4'' \\ \delta y = \beta_1 v_1'' + \beta_2 v_2'' + \beta_3 v_3'' + \beta_4 v_4'' \\ \delta z = \gamma_1 v_1'' + \gamma_2 v_2'' + \gamma_3 v_3'' + \gamma_4 v_4'' \end{aligned}\right\}. \tag{3}$$

We have to set these expressions (3) into the condition equations (1), whereby there result equations which refer to the corrections v'' and have the following form:

$$\left.\begin{aligned} \mathrm{I}_1 v_1'' + \mathrm{I}_2 v_2'' + \mathrm{I}_3 v_3'' + \mathrm{I}_4 v_4'' + w_1 = 0 \\ \mathrm{II}_1 v_1'' + \mathrm{II}_2 v_2'' + \mathrm{II}_3 v_3'' + \mathrm{II}_4 v_4'' + w_2 = 0 \end{aligned}\right\}. \tag{4}$$

The coefficients I and II have the following meanings according to their origin from (3) and (1):

$$\left.\begin{array}{ll} \mathrm{I}_1 = A_1\alpha_1 + A_2\beta_1 + A_3\gamma_1 & \mathrm{II}_1 = B_1\alpha_1 + B_2\beta_1 + B_3\gamma_1 \\ \mathrm{I}_2 = A_1\alpha_2 + A_2\beta_2 + A_3\gamma_2 & \mathrm{II}_2 = B_1\alpha_2 + B_2\beta_2 + B_3\gamma_2 \\ \mathrm{I}_3 = A_1\alpha_3 + A_2\beta_3 + A_3\gamma_3 & \mathrm{II}_3 = B_1\alpha_3 + B_2\beta_3 + B_3\gamma_3 \\ \mathrm{I}_4 = A_1\alpha_4 + A_2\beta_4 + A_3\gamma_4 & \mathrm{II}_4 = B_1\alpha_4 + B_2\beta_4 + B_3\gamma_4 \end{array}\right\}. \tag{5}$$

Now the problem is reduced to the adjustment of direct conditioned observations, i.e., with the introduction of the correlates k_1 and k_2 to the condition equations (4) we have to form the normal equations

$$\left.\begin{array}{l} [\mathrm{I\ I}]\,k_1 + [\mathrm{I\ II}]\,k_2 + w_1 = 0 \\ [\mathrm{I\ II}]\,k_1 + [\mathrm{II\ II}]\,k_2 + w_2 = 0 \end{array}\right\}, \tag{6}$$

whereupon the individual v'''s can then be found, namely, according to the columns in (4)

$$\left.\begin{array}{l} v_1'' = \mathrm{I}_1 k_1 + \mathrm{II}_1 k_2 \\ v_2'' = \mathrm{I}_2 k_1 + \mathrm{II}_2 k_2 \\ v_3'' = \mathrm{I}_3 k_1 + \mathrm{II}_3 k_2 \\ v_4'' = \mathrm{I}_4 k_1 + \mathrm{II}_4 k_2 \end{array}\right\}. \tag{7}$$

If we pursue this course of computation more closely, then, by squaring and multiplying the coefficients I and II, we find the sum coefficients of the normal equations (6) as follows:

$$\begin{array}{rr} [\mathrm{I\ I}] = & [\mathrm{I\ II}] = \\ A_1A_1[\alpha\alpha] + 2A_1A_2[\alpha\beta] + 2A_1A_3[\alpha\gamma] & +A_1B_1[\alpha\alpha] + A_1B_2[\alpha\beta] + A_1B_3[\alpha\gamma] \\ +\ A_2A_2[\beta\beta] + 2A_2A_3[\beta\gamma] & +A_2B_1[\alpha\beta] + A_2B_2[\beta\beta] + A_2B_3[\beta\gamma] \\ +\ A_3A_3[\gamma\gamma] & +A_3B_1[\alpha\gamma] + A_3B_2[\beta\gamma] + A_3B_3[\gamma\gamma] \end{array} \tag{8}$$

$$\begin{array}{r} [\mathrm{II\ II}] = \\ B_1B_1[\alpha\alpha] + 2B_1B_2[\alpha\beta] + 2B_1B_3[\alpha\gamma] \\ +2B_2B_2[\beta\beta] + 2B_2B_3[\beta\gamma] \\ +\ B_3B_3[\gamma\gamma]. \end{array} \tag{9}$$

If we introduce (5) also into (7), then we obtain first

$$\left.\begin{array}{l} v_1'' = (A_1\alpha_1 + A_2\beta_1 + A_3\gamma_1)\,k_1 + (B_1\alpha_1 + B_2\beta_1 + B_3\gamma_1)\,k_2 \\ v_2'' = (A_1\alpha_2 + A_2\beta_2 + A_3\gamma_2)\,k_1 + (B_1\alpha_2 + B_2\beta_2 + B_3\gamma_2)\,k_2 \\ v_3'' = (A_1\alpha_3 + A_2\beta_3 + A_3\gamma_3)\,k_1 + (B_1\alpha_3 + B_2\beta_3 + B_3\gamma_3)\,k_2 \\ v_4'' = (A_1\alpha_4 + A_2\beta_4 + A_3\gamma_4)\,k_1 + (B_1\alpha_4 + B_2\beta_4 + B_3\gamma_4)\,k_2 \end{array}\right\}, \tag{10}$$

and if we substitute these expressions again in (3), then we obtain:

$$\left.\begin{array}{rl} \delta x = x - x_0 = & [\alpha\alpha]\,A_1k_1 + [\alpha\alpha]\,B_1k_2 \\ & +[\alpha\beta]\,A_2k_1 + [\alpha\beta]\,B_2k_2 \\ & +[\alpha\gamma]\,A_3k_1 + [\alpha\gamma]\,B_3k_2 \\ \delta y = y - y_0 = & [\alpha\beta]\,A_1k_1 + [\alpha\beta]\,B_1k_2 \\ & +[\beta\beta]\,A_2k_1 + [\beta\beta]\,B_2k_2 \\ & +[\beta\gamma]\,A_3k_1 + [\beta\gamma]\,B_3k_2 \\ \delta z = z - z_0 = & [\alpha\gamma]\,A_1k_1 + [\alpha\gamma]\,B_1k_2 \\ & +[\beta\gamma]\,A_2k_1 + [\beta\gamma]\,B_2k_2 \\ & +[\gamma\gamma]\,A_3k_1 + [\gamma\gamma]\,B_3k_2 \end{array}\right\}. \tag{11}$$

With this, the adjustment is complete, and we have the following course of computation:

With the coefficients A, B of the condition equations (1) and with the weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$ . . . , which have already been determined on the occasion of the first part of the adjustment (in the previous section 55, p. 177), we calculate the coefficients [I I], [I II] . . . according to (8) and (9), set up, with this, the normal equations (6), and solve them for the correlates k_1 and k_2.

We substitute these correlates in equations (11) and have, with this, all the corrections $\delta x, \delta y, \delta z$.

We do not obtain in this way the individual corrections v'' according to (10); however, we do not usually need them, but only the sum of their squares $[v''v'']$, which we shall treat in section 57.

Although everything is thus taken care of in principle, we introduce further a few intermediate designations, by which the formulae (8), (9), and (11) shall be made more readily conceivable and more convenient for numerical calculation.

According as we collect the terms in (11), by rows or by columns, we obtain partial sums, for which we introduce the new symbols [1], [2], [3], or $\mathfrak{A}, \mathfrak{B}$, and which are named as "auxiliary quantities [1], [2], [3]" and "coefficients of transformation $\mathfrak{A}, \mathfrak{B}$."

Auxiliary quantities:

$$\left.\begin{aligned}[1] &= A_1 k_1 + B_1 k_2\\ [2] &= A_2 k_1 + B_2 k_2\\ [3] &= A_3 k_1 + B_3 k_2\end{aligned}\right\}. \qquad (12)$$

Coefficients of transformation:

$$\left.\begin{aligned}\mathfrak{A}_1 &= A_1[\alpha\alpha] + A_2[\alpha\beta] + A_3[\alpha\gamma] & \mathfrak{B}_1 &= B_1[\alpha\alpha] + B_2[\alpha\beta] + B_3[\alpha\gamma]\\ \mathfrak{A}_2 &= A_1[\alpha\beta] + A_2[\beta\beta] + A_3[\beta\gamma] & \mathfrak{B}_2 &= B_1[\alpha\beta] + B_2[\beta\beta] + B_3[\beta\gamma]\\ \mathfrak{A}_3 &= A_1[\alpha\gamma] + A_2[\beta\gamma] + A_3[\gamma\gamma] & \mathfrak{B}_3 &= B_1[\alpha\gamma] + B_2[\beta\gamma] + B_3[\gamma\gamma]\end{aligned}\right\}. \qquad (13)$$

With these, equations (11) can be newly written in two different ways:

$$\left.\begin{aligned}\delta x &= [\alpha\alpha][1] + [\alpha\beta][2] + [\alpha\gamma][3] & & & \delta x &= \mathfrak{A}_1 k_1 + \mathfrak{B}_1 k_2\\ \delta y &= [\alpha\beta][1] + [\beta\beta][2] + [\beta\gamma][3] & &\text{or} & \delta y &= \mathfrak{A}_2 k_1 + \mathfrak{B}_2 k_2\\ \delta z &= [\alpha\gamma][1] + [\beta\gamma][2] + [\gamma\gamma][3] & & & \delta z &= \mathfrak{A}_3 k_1 + \mathfrak{B}_3 k_2\end{aligned}\right\}. \qquad (14)\alpha \text{ and } (14)\mathfrak{A}$$

Also the coefficients of the normal equations in (6) can now, in addition, be represented differently:

$$\left.\begin{aligned}[\mathrm{I\,I}] = [A\mathfrak{A}], \qquad [\mathrm{I\,II}] &= [A\mathfrak{B}] \quad \text{or} \quad = [\mathfrak{A}B]\\ [\mathrm{II\,II}] &= [B\mathfrak{B}]\end{aligned}\right\}. \qquad (15)$$

The nonquadratic coefficients, e.g. [I II], can be obtained twice, which serves as a check. Individually, we have here

$$\left.\begin{aligned}[A\mathfrak{B}] &= A_1\mathfrak{B}_1 + A_2\mathfrak{B}_2 + \dots\\ [\mathfrak{A}B] &= \mathfrak{A}_1 B_1 + \mathfrak{A}_2 B_2 + \dots\end{aligned}\right\}. \qquad (15a)$$

With these auxiliary quantities [1], [2], [3] and the coefficients of transformation $\mathfrak{A}, \mathfrak{B}$, the adjustment assumes the following course:

1. Coefficients A, B . . . of the condition equations according to (1).
2. Weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$ . . . according to section 30 or 36.
3. Coefficients of transformation $\mathfrak{A}, \mathfrak{B}$ . . . according to (13).
4. Coefficients of the normal equations according to (15).

5. Solution of the normal equations (6).
6. Auxiliary quantities [1], [2], [3] according to (12) } , or $\delta x, \delta y, \delta z$ according to (14) 𝔄.
7. Corrections $\delta x, \delta y, \delta z$ according to (14) α.

Section 57. Sum of Squares of the Corrections and Mean Errors

According to (10), section 55, p. 178, we have

$$[v v] = [v' v'] + [v'' v''] . \tag{1}$$

We do not need to calculate the components $[v' v']$ and $[v'' v'']$, which originate from the first part and from the second part of the adjustment, from the individual v''s and v'''s, but we can derive them from the normal equations.

First, as far as the first part is concerned, we have all that is necessary for this according to (8), section 29, p. 91, namely for u unknowns

$$[v' v'] = [l l \cdot u] = [l l] - \frac{[a l]^2}{[a a]} - \frac{[b l \cdot 1]^2}{[b b \cdot 1]} - \frac{[c l \cdot 2]^2}{[c c \cdot 2]} - \dots , \tag{2}$$

and also for the second part of the adjustment we have already learned, in section 45, all forms into which the $[v'' v'']$'s can be brought, for these forms can all be transferred to the new case, if we only take into account the meaning of the new symbols in sections 55 and 56.

We have therefore according to (4) and (5), section 45, p. 142

$$[v'' v''] = - [w k] = - w_1 k_1 - w_2 k_2 - \dots \tag{3}$$

and

$$[v'' v''] = \frac{w_1{}^2}{[\mathrm{I\,I}]} + \frac{[w_2 \cdot 1]^2}{[\mathrm{II\,II} \cdot 1]} + \frac{[w_3 \cdot 2]^2}{[\mathrm{III\,III} \cdot 2]} + \dots . \tag{4}$$

Although these formulae can be put down at once corresponding to section 45, because the sum $[v'' v'']$, which was made a minimum by the correlation adjustment of section 56, is involved, it may also be desirable to prove these formulae directly, where, at the same time, a new form (5), usable in practice, will result.

The individual v'''s, which are given in (10), section 56, p. 180, can first be represented with the use of the auxiliary quantities [1], [2], [3] as follows:

$$\begin{aligned} v_1'' &= \alpha_1 [1] + \beta_1 [2] + \gamma_1 [3] \\ v_2'' &= \alpha_2 [1] + \beta_2 [2] + \gamma_2 [3] \\ &\dots\dots\dots\dots ; \end{aligned}$$

therefore:

$$\begin{aligned} [v'' v''] = [\alpha \alpha] [1] [1] &+ 2 [\alpha \beta] [1] [2] + 2 [\alpha \gamma] [1] [3] \\ &+ [\beta \beta] [2] [2] + 2 [\beta \gamma] [2] [3] \\ &+ [\gamma \gamma] [3] [3] . \end{aligned}$$

With the use of (14) α, section 56, p. 181, this becomes

$$[v'' v''] = [1]\, \delta x + [2]\, \delta y + [3]\, \delta z , \tag{5}$$

and with (12), section 56, p. 181, if we collect terms, at the same time, with respect to k,

$$[v'' v''] = (A_1 \delta x + A_2 \delta y + A_3 \delta z) k_1 + (B_1 \delta x + B_2 \delta y + B_3 \delta z) k_2 .$$

The coefficients of k_1 and k_2 are nothing else here than the left-hand parts of the condition equations (1), section 56, p. 179, i.e., we have now

$$[v'' v''] = - w_1 k_1 - w_2 k_2 = - [w k] ,$$

as was already written on p. 182 in (3); and there follows hence (4) in just the same way as the previous corresponding equation in section 45.

After we have thus learned the component parts of $[v v]$, the formula for the mean error m can also be brought into a more illustrative meaning. For, according to (5), section 54, p. 175, we have now

$$m^2 = \frac{[v\,v]}{n - u + r} = \frac{[v'\,v'] + [v''\,v'']}{(n - u) + r} . \tag{6}$$

Here the numerators and denominators fit together again in parts. For we have

$$\frac{[v'\,v']}{n - u} = m_1{}^2 \qquad \text{and} \qquad \frac{[v''\,v'']}{r} = m_2{}^2 ,$$

where m_1 represents the mean error only from the first part of the adjustment, and m_2 the mean error only from the second part of the adjustment.

In general, m_1 and m_2 will turn out neither equal to each other nor equal to the total mean error m. If m_1 and m_2 turn out very considerably different, then this points to error sources which have remained hidden in the first or in the second part.

Section 58. Function Weight After the Adjustment

After the correlation adjustment in the previous section 57 is reduced by means of the coefficients I, II . . . to the earlier case of section 43, we can also put down the function weight after the adjustment, without further developments, according to section 46, if only the meaning of the coefficients α, β . . . and I, II . . . is used correctly.

We assume that it is a question of the following function:

$$F = f_1 x + f_2 y + f_3 z , \tag{1}$$

whose weight P is to be determined. We split up as before:

$$F = f_1\, (x_0 + \delta x) + f_2\, (y_0 + \delta y) + f_3\, (z_0 + \delta z) . \tag{2}$$

In order to represent this as a function of $(l + v'')$, we have first according to (3), section 55, p. 177, and according to (6), section 30, p. 95,

$$\begin{aligned} x_0 &= \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 \\ y_0 &= \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \beta_4 l_4 \\ z_0 &= \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3 + \gamma_4 l_4 . \end{aligned}$$

This yields together with (3), section 56, p. 179,

$$\begin{aligned} (x_0 + \delta x) &= \alpha_1 (l_1 + v_1'') + \alpha_2 (l_2 + v_2'') + \alpha_3 (l_3 + v_3'') + \alpha_4 (l_4 + v_4'') \\ (y_0 + \delta y) &= \beta_1 (l_1 + v_1'') + \beta_2 (l_2 + v_2'') + \beta_3 (l_3 + v_3'') + \beta_4 (l_4 + v_4'') \\ (z_0 + \delta z) &= \gamma_1 (l_1 + v_1'') + \gamma_2 (l_2 + v_2'') + \gamma_3 (l_3 + v_3'') + \gamma_4 (l_4 + v_4'') . \end{aligned}$$

If we substitute these expressions in (2), then the following form will result:

$$F = \varphi_1 (l_1 + v_1'') + \varphi_2 (l_2 + v_2'') + \varphi_3 (l_3 + v_3\ ) + \varphi_4 (l_4 + v_4''), \tag{3}$$

where the coefficients φ have the following meanings:

$$\left.\begin{array}{l} \varphi_1 = f_1 \alpha_1 + f_2 \beta_1 + f_3 \gamma_1 \\ \varphi_2 = f_1 \alpha_2 + f_2 \beta_2 + f_3 \gamma_2 \\ \varphi_3 = f_1 \alpha_3 + f_2 \beta_3 + f_3 \gamma_3 \\ \varphi_4 = f_1 \alpha_4 + f_2 \beta_4 + f_3 \gamma_4 \end{array}\right\}. \tag{4}$$

Now the weight of function (3) is according to the rules for conditioned observations according to (14), section 46, p. 145,

$$\frac{1}{P} = [\varphi\varphi] - \left\{\frac{[\mathrm{I}\,\varphi]^2}{[\mathrm{I\,I}]} + \frac{[\mathrm{II}\,\varphi \cdot 1]^2}{[\mathrm{II\,II} \cdot 1]} + \frac{[\mathrm{III}\,\varphi \cdot 2]^2}{[\mathrm{III\,III} \cdot 2]} + \ldots\right\}. \tag{5}$$

Here the first part according to (4) is

$$\left.\begin{array}{rl} [\varphi\varphi] = [\alpha\alpha] f_1^2 & + 2[\alpha\beta] f_1 f_2 + 2[\alpha\gamma] f_1 f_3 \\ & + [\beta\beta] f_2^2 + 2[\beta\gamma] f_2 f_3 \\ & + [\gamma\gamma] f_3^2 \end{array}\right\}, \tag{6}$$

i.e., according to (3), section 31, p. 100, $1:[\varphi\varphi]$ is the weight of function F after the first adjustment and *before* the second adjustment.

Also all other forms of section 31 are valid for this, hence, after the introduction of the auxiliary quantities q we have the formula (5), p. 100,

$$[\varphi\varphi] = q_1 f_1 + q_2 f_2 + q_3 f_3 = [q f] \tag{7}$$

and the elegant final formula (13), p. 102,

$$[\varphi\varphi] = \frac{f_1^2}{[a a]} + \frac{[f_2 \cdot 1]^2}{[b b \cdot 1]} + \frac{[f_3 \cdot 2]^2}{[c c \cdot 2]} + \ldots . \tag{8}$$

As for the second part of (5), we only have to take, for this, the meanings of the φ's in (4) together with the coefficients I, II, III of (5), section 56, p. 180. And if we form, according to this, the product sums $[\mathrm{I}\,\varphi]$, $[\mathrm{II}\,\varphi]$, $[\mathrm{III}\,\varphi] \ldots$, then we see at once that we come again to the same forms as in the case of $[\mathrm{I\,II}]$, $[\mathrm{I\,III}] \ldots$. The terms $[\mathrm{I}\,\varphi]$, $[\mathrm{II}\,\varphi] \ldots$ are connected to this just as if a further equation with the coefficients $f_1, f_2, f_3 \ldots$ had been added to the condition equations with the coefficients $A, B \ldots$, and the q's are connected to the series of coefficients of transformation $\mathfrak{A}, \mathfrak{B} \ldots$ as follows:

$$\left.\begin{array}{ccccc} \mathfrak{A}_1 & \mathfrak{A}_2 & \mathfrak{A}_3 & \mathfrak{A}_4 & \ldots \\ \mathfrak{B}_1 & \mathfrak{B}_2 & \mathfrak{B}_3 & \mathfrak{B}_4 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_1 & q_2 & q_3 & q_4 & \ldots \end{array}\right\}. \tag{9}$$

For the $[I\varphi]$'s, $[II\varphi]$'s . . . we obtain here also *two* formulae each, just as the nonquadratic [I II]'s, [I III]'s . . . according to (15), section 56, p. 181, could also be expressed in a double manner. The results are

$$\left.\begin{array}{lcl} [\mathrm{I}\ \varphi] = [\mathfrak{A} f] & \text{or} & = [A\,q] \\ [\mathrm{II}\ \varphi] = [\mathfrak{B} f] & & = [B\,q] \\ [\mathrm{III}\ \varphi] = [\mathfrak{C} f] & & = [C\,q] \\ \cdots & & \cdots \end{array}\right\} . \tag{10}$$

We annex these terms to the elimination of the normal equations and compute then further in just the same manner as was already taught in section 46.

Section 59. Indirect Observations with Partial Condition Equations

If the error equations contain unknowns which do not occur in the condition equations, then the formulae developed above can still be applied, for we can then assume that the coefficients pertaining to the condition equations are equal to zero.

According to this, it would not be necessary at all to treat this case separately; but we can use to advantage the circumstance in question already in the first part of the adjustment by already eliminating in the normal equations of the first part the unknowns not occurring in the condition equations.

Since this case is of importance in triangulation adjustments, we enter here on the subject more closely and also take the occasion at the same time to prepare the application of the general theory to triangulation adjustment otherwise.

Error equations:

$$\text{Number} = n \left\{ \begin{array}{l} v_1' = h_1(x) + i_1(y) + a_1'x + b_1'y + c_1'z - l_1' \\ v_2' = h_2(x) + i_2(y) + a_2'x + b_2'y + c_2'z - l_2' \\ v_3' = h_3(x) + i_3(y) + a_3'x + b_3'y + c_3'z - l_3' \\ v_4' = h_4(x) + i_4(y) + a_4'x + b_4'y + c_4'z - l_4' \\ \cdots \underbrace{\cdots\cdots}_{\text{Number} = s} \cdots \underbrace{\cdots\cdots\cdots}_{\text{Number} = u} \cdots \end{array} \right\} \tag{1}$$

Total number $= s + u$.

Condition equations:

$$\text{Number} = r \left\{ \begin{array}{l} A_0 + A_1 x + A_2 y + A_3 z = 0 \\ B_0 + \underbrace{B_1 x + B_2 y + B_3 z}_{\text{Number} = u} = 0 \end{array} \right\} \tag{2}$$

The s unknowns (x), (y) (the zero-point corrections of the individual sets) occur only in the error equations (1), but not in the condition equations (2). This has no influence on the beginning of the first adjustment; we set up, regardless of this, the normal equations belonging to (1), and start with these to eliminate (x) and (y).

After (x) and (y) are eliminated, there remains a system which must be written in our hitherto used notation as follows:

$$\left.\begin{array}{r} \underline{[a'a'\cdot 2]}\,x + [a'b'\cdot 2]\,y + [a'c'\cdot 2]\,z - [a'l'\cdot 2] = 0 \\ \underline{[b'b'\cdot 2]\,y} + [b'c'\cdot 2]\,z - [b'l'\cdot 2] = 0 \\ \underline{[c'c'\cdot 2]\,z} - [c'l'\cdot 2] = 0 \\ \underline{[l'l'\cdot 2]} = [v_0 v_0] \end{array}\right\} . \tag{3}$$

According to the previous considerations about reduced error equations, section 28, the coefficients occurring here, $[a' a' \cdot 2]$ . . . , have again the meanings of sums and products; therefore, we set again $[a' a' \cdot 2] = [a a]$ . . . and have instead of (3)

$$\left.\begin{aligned} \underline{[a a]} x + [a b] y + [a c] z - [a l] &= 0 \\ \underline{[b b]} y + [b c] z - [b l] &= 0 \\ \underline{[c c]} z - [c l] &= 0 \\ [l l] &= [v_0 v_0] \end{aligned}\right\} . \tag{4}$$

With these coefficients $[a a]$, $[a b]$ . . . we compute now further just as with the former coefficients of the first part of the adjustment, likewise denoted by $[a a]$, $[a b]$ . . . .

The sum of squares $[v_0 v_0]$ in (3) and (4) is completely reduced stepwise to $[v' v']$ in the following manner:

$$[v' v'] = [v_0 v_0] - \frac{[a l]^2}{[a a]} - \frac{[b l \cdot 1]^2}{[b b \cdot 1]} - \frac{[c l \cdot 2]^2}{[c c \cdot 2]} - \dots . \tag{5}$$

The mean square of the error of the first adjustment becomes

$$m_1^2 = \frac{[v' v']}{n - (s + u)} . \tag{6}$$

We will now emphasize also another circumstance, which appears in triangulation adjustments by itself, without influencing the general validity of the developments hitherto carried out, namely the splitting of the error equations (1) and the normal equations (4) into different, quite independent groups. For these equations belong to the total of *all* station adjustments.

We will indicate this in general formulae, and assume that it is a question of three unknowns with the following error equations, divided into two systems, in which, for instance, certain other unknowns (x), (y) . . . are assumed to be already eliminated:

$$\begin{aligned} v_1 &= a_1 x + b_1 y \dots - l_1 \\ v_2 &= a_2 x + b_2 y \dots - l_2 \\ v_3 &= a_3 x + b_3 y \dots - l_3 \\ v_4 &= \dots\dots \ c_4 z - l_4 \\ v_5 &= \dots\dots \ c_5 z - l_5 . \end{aligned}$$

If we form the normal equations to these, then they become

$$\left.\begin{aligned} \underline{[a a]} x + [a b] y \dots - [a l] &= 0 \\ \underline{[b b]} y \dots - [b l] &= 0 \\ \underline{[c c]} z - [c l] &= 0 \\ \underline{[l l]} = (l_1^2 + l_2^2 + l_3^2) &+ (l_4^2 + l_5^2) \\ = [v_0 v_0]_1 \qquad &+ [v_0 v_0]_2 \end{aligned}\right\} . \tag{7}$$

This form (7) divided into two systems is included in the general form (4), and the circumstance that a part of the coefficients in (7) is equal to zero makes itself agreeably felt in the further computation.

Formula (5), also, is divided then by itself into individual separate parts:

$$\begin{aligned} [v' v']_1 &= [v_0 v_0]_1 - \frac{[a l]^2}{[a a]} - \frac{[b l \cdot 1]^2}{[b b \cdot 1]} \\ [v' v']_2 &= [v_0 v_0]_2 + \dots\dots\dots - \frac{[c l]^2}{[c c]} . \end{aligned}$$

For because of the falling out of $[a c]$ and $[b c]$ in (7) we have $[c l \cdot 2] = [c l]$ and $[c c \cdot 2] = [c c]$. Then we have $[v' v']_1 + [v' v']_2 = [v' v']$.

$[v' v']_1$ corresponds to a first station, $[v' v']_2$ to a second station, and so forth.

We can now put down formula (6) for each individual station as well as for the total of all stations.

The adjustment of indirect observations with condition equations has in earlier times played an important role in the operations of the Prussian Land Survey. In this connection, special "Rechnungsvorschriften" [computational instructions] are indicated in the work, *Die Königlich Preussische Landes-Triangulation,* Hauptdreiecke, Erster Teil, zweite verbesserte Auflage, herausgegeben vom Bureau der Landes-Triangulation, Berlin, 1870, pp. 1-24. The designations are there in part different ones than those used by us. Therefore, in the former editions of this volume, e.g. in the 7th Edition, 1920, pp. 179-183, a special comparison of our formulae with those of the Land Survey was communicated. We have however excluded taking up this comparison again, since the adjustment of indirect observations with condition equations has no longer been applied in newer triangulations.

Section 60. Error Equations with Several Quantities of Observation

Until now we have treated only such error equations, in which there occurred, in addition to the unknowns, one observational quantity each with its correction. In the most general form of the problem of adjustment, the error equations will contain an arbitrary number of observations so that we can speak of error equations with several observations as well as of condition equations with unknowns. Before we turn to this most general problem, we consider first the simpler case of indirect observations with equations of *differences* of errors of the following form:

$$\text{Number} \atop = n-1 \left\{ \begin{array}{l} v_2 - v_1 = a_1 x + b_1 y + c_1 z - l_1 \\ v_3 - v_1 = a_2 x + b_2 y + c_2 z - l_2 \quad (n-1 > u) \\ \cdots\cdots\cdots\cdots \\ v_n - v_1 = \underbrace{a_n x + b_n y + c_n z}_{\text{Number} = u} - l_n \end{array} \right\} . \tag{1}$$

Here only the *differences* $v_2 - v_1$, $v_3 - v_1 \ldots$ occur as the left-hand sides of the error equations, never the v_1's, v_2's ... individually; but we can produce the usual error equations by introducing v_1 itself as an unknown, hence, as follows:

$$\text{Number} = n \left\{ \begin{array}{l} v_1 = v_1 \quad \cdots\cdots\cdots \\ v_2 = v_1 + a_1 x + b_1 y + c_1 z - l_1 \\ v_3 = v_1 + a_2 x + b_2 y + c_2 z - l_2 \quad (n > u+1) \\ \cdots\cdots\cdots\cdots \\ v_n = \underbrace{v_1 + a_n x + b_n y + c_n z}_{\text{Number} = u+1.} - l_n \end{array} \right\} . \tag{2}$$

The normal equations belonging to these become

$$\left. \begin{array}{l} n v_1 + [a] x + [b] y + [c] z - [l] = 0 \\ [a] v_1 + [a a] x + [a b] y + [a c] z - [a l] = 0 \\ [b] v_1 + [a b] x + [b b] y + [b c] z - [b l] = 0 \\ [c] v_1 + [a c] x + [b c] y + [c c] z - [c l] = 0 \end{array} \right\} . \tag{3}$$

and the first reduction with the elimination of v_1 yields

$$\left. \begin{array}{c} \left([aa] - \frac{[a]}{n}[a]\right) x + \left([a b] - \frac{[a]}{n}[b]\right) y + \left([ac] - \frac{[a]}{n}[c]\right) z - \left([a l] - \frac{[a]}{n}[l]\right) = 0 \\ \left([b b] - \frac{[b]}{n}[b]\right) y + \ldots, \text{ etc.} \end{array} \right\} . \tag{4}$$

We shall find this case again in different practical-geodetic problems (especially, the elimination of the unknowns of orientation in direction measurements). Here the principal thing was to show that, by the trick of setting one of the v's itself as unknown, the problem (1) with $n - 1$ equations and u unknowns could be reduced to the adjustment with n usual error equations and $u + 1$ unknowns. The method is also applicable to more than *one* auxiliary unknown v, if each error equation, in addition to other corrections, contains one which does not occur in any other error equation.

Most general case

All problems of adjustment hitherto treated can be considered as individual cases of the following general problem:

$$\text{Number} = r \left\{ \begin{array}{l} a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots a_n v_n + A_1 x + B_1 y + C_1 z + w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + \ldots b_n v_n + A_2 x + B_2 y + C_2 z + w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots c_n v_n + A_3 x + B_3 y + C_3 z + w_3 = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \underbrace{r_1 v_1 + r_2 v_2 + r_3 v_3 + \ldots r_n v_n}_{\text{Number} = n} + \underbrace{A_r x + B_r y + C_r z}_{\text{Number} = u} + w_r = 0 \end{array} \right\} \tag{5}$$

$$r > u, \; n > r - u.$$

The v_1's, v_2's, . . . , v_n's are corrections of observations whose sum of squares $[v\,v]$ is to become a minimum; x, y, z are unknowns so that in the present case $u = 3$ is assumed.

If $r = u$, then we have, without an adjustment, a rigorous solution since all v's become 0 and the thus remaining n equations are solved for the u unknowns x, y, z.

An adjustment is thus possible only for $r > u$, and then we can reduce the problem to r condition equations between the v's without the x's, y's, z's by expressing, with the help of u chosen equations, the u unknowns x, y, z in terms of the v's pertaining to them and, with it, eliminating them in the $r - u$ remaining equations. There remain then $r - u$ condition equations between the n values v; therefore, for their adjustment it is still necessary that $r - u < n$, and the mean error of the unit of weight is then

$$m = \sqrt{\frac{[v\,v]}{r - u}} \, . \tag{6}$$

The problem contained in equations (5) is herewith solved in principle.

We arrive at a direct solution of the problem by finding the minimum of the function

$$\left. \begin{array}{l} [v\,v] - 2 k_1 (a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n + A_1 x + B_1 y + C_1 z + w_1) \\ \quad - 2 k_2 (b_1 v_1 + b_2 v_2 + b_3 v_3 + \ldots + b_n v_n + A_2 x + B_2 y + C_2 z + w_2) \\ \quad - 2 k_3 (c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n + A_3 x + B_3 y + C_3 z + w_3) \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad - 2 k_r (r_1 v_1 + r_2 v_2 + r_3 v_3 + \ldots + r_n v_n + A_r x + B_r y + C_r z + w_r) \end{array} \right\} . \tag{7}$$

If the differential quotients of this expression with respect to v_1, v_2, . . . , x, y, z are set equal to zero, then there result $n + u$ equations, which, together with the r equations (5), are sufficient for the determination of the n corrections v, the u unknowns, and the r correlates k.

The differentiation with respect to v_1, v_2, . . . , v_n yields

$$\left. \begin{array}{l} v_1 - a_1 k_1 - b_1 k_2 - c_1 k_3 - \ldots = 0 \\ v_2 - a_2 k_1 - b_2 k_2 - c_2 k_3 - \ldots = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right\} \text{Number} = n \tag{8}$$

and by differentiation with respect to x, y, z we find

$$\left.\begin{array}{l} A_1 k_1 + A_2 k_2 + A_3 k_3 \ldots = 0 \\ B_1 k_1 + B_2 k_2 + B_3 k_3 \ldots = 0 \\ C_1 k_1 + C_2 k_2 + C_3 k_3 \ldots = 0 \end{array}\right\} \text{Number} = u\,. \tag{9}$$

These equations (8) and (9) correspond to the correlate equations in the case of the adjustment of conditioned observations. If we substitute the expressions for $v_1, v_2, \ldots, v_n$ from (8) in (5) and add equations (9), then there follows

$$\left.\begin{array}{l} [aa]k_1 + [ab]k_2 + [ac]k_3 + \ldots + A_1 x + B_1 y + C_1 z + w_1 = 0 \\ [ab]k_1 + [bb]k_2 + [bc]k_3 + \ldots + A_2 x + B_2 y + C_2 z + w_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_1 k_1 + A_2 k_2 + A_3 k_3 \quad + \ldots\ldots\ldots\ldots = 0 \\ B_1 k_1 + B_2 k_2 + B_3 k_3 \quad + \ldots\ldots\ldots\ldots = 0 \\ C_1 k_1 + C_2 k_2 + C_3 k_3 \quad + \ldots\ldots\ldots\ldots = 0 \end{array}\right\}. \tag{10}$$

By applying the Gaussian method we can solve these normal equations, and if all k's are eliminated here, then there remain three equations of the form:

$$\left.\begin{array}{l} (AA)x + (AB)y + (AC)z - (Aw) = 0 \\ (AB)x + (BB)y + (BC)z - (Bw) = 0 \\ (AC)x + (BC)y + (CC)z - (Cw) = 0 \end{array}\right\} \tag{11}$$

which form a system of normal equations for the unknowns x, y, z. From these equations, as in the case of indirect observations, we can determine also the weights of the unknowns.

The problem of the "conditioned observations with unknowns" was for the first time worked up in the above form by Helmert in *Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate*, Leipzig, 1872, pp. 215-222 (2nd Edition, 1907, pp. 285-293), where, at the same time, the mean error for a function of the adjusted observations and of the final values of the unknowns is also developed.

The treatment of the above problem turns out considerably simpler if each correction occurs only in one equation of (5). If we assume for the sake of simplicity that each equation contains only two corrections, and that there are only two unknowns x and y, then we can write equations (5) in the following form:

$$\left.\begin{array}{l} a_1 v_1 + a_1' v_1' \ldots\ldots\ldots\ldots + A_1 x + B_1 y + w_1 = 0 \\ \ldots\ldots b_2 v_2 + b_2' v_2' \ldots\ldots + A_2 x + B_2 y + w_2 = 0 \\ \ldots\ldots\ldots\ldots c_3 v_3 + c_3' v_3' \ldots + A_3 x + B_3 y + w_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\}. \tag{12}$$

With this, the first group of the normal equations (10) becomes

$$\left.\begin{array}{l} (a_1^2 + a_1'^2)k_1 + A_1 x + B_1 y + w_1 = 0 \\ (b_2^2 + b_2'^2)k_2 + A_2 x + B_2 y + w_2 = 0 \\ (c_3^2 + c_3'^2)k_3 + A_3 x + B_3 y + w_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{13}$$

or

$$\left.\begin{array}{l} k_1 = -\dfrac{A_1}{a_1^2 + a_1'^2}x - \dfrac{B_1}{a_1^2 + a_1'^2}y - \dfrac{w_1}{a_1^2 + a_1'^2} \\ k_2 = -\dfrac{A_2}{b_2^2 + b_2'^2}x - \dfrac{B_2}{b_2^2 + b_2'^2}y - \dfrac{w_2}{b_2^2 + b_2'^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\}. \tag{14}$$

For simplification we set

$$\frac{1}{a_1^2+a_1'^2}=p_1\,, \qquad \frac{1}{b_2^2+b_2'^2}=p_2\,, \text{ and so forth}\,. \tag{15}$$

If we introduce (14) and (15) into the second group of (10), then we obtain

$$\left.\begin{array}{l}[AAp]\,x+[ABp]\,y+[Awp]=0\\ [ABp]\,x+[BBp]\,y+[Bwp]=0\end{array}\right\}. \tag{16}$$

These normal equations (16) found according to the above generally valid solution show now that we can arrive at the same result by a simpler method. For if we set

$$\left.\begin{array}{ll}a_1 v_1+a_1' v_1'=-\lambda_1 & \text{weight } p_1\\ b_2 v_2+b_2' v_2'=-\lambda_2 & \text{weight } p_2\\ \cdots\cdots & \cdots\end{array}\right\}, \tag{17}$$

then there follow the new error equations

$$\left.\begin{array}{ll}\lambda_1=A_1 x+B_1 y+w_1 & \text{weight } p_1\\ \lambda_2=A_2 x+B_2 y+w_2 & \text{weight } p_2\\ \cdots\cdots & \cdots\end{array}\right\}, \tag{18}$$

whose normal equations agree with the normal equations (16).

If, after the completion of the adjustment, the individual corrections v and v' are also indicated, then we have for this, according to (8), p. 188,

$$\left.\begin{array}{ll}v_1=a_1 k_1 & v_1'=a_1' k_1\\ v_2=b_2 k_2 & v_2'=b_2' k_2\\ v_3=c_3 k_3 & v_3'=c_3' k_3\\ \cdots & \cdots\end{array}\right\}, \tag{19}$$

and with this, we will have according to (17)

$$\left.\begin{array}{ll}a_1^2 k_1+a_1'^2 k_1=-\lambda_1 & k_1=-\lambda_1 p_1\\ b_2^2 k_2+b_2'^2 k_2=-\lambda_2 & k_2=-\lambda_2 p_2\\ c_3^2 k_3+c_3'^2 k_3=-\lambda_3 & k_3=-\lambda_3 p_3\\ \cdots\cdots & \cdots\end{array}\right\}. \tag{20}$$

By means of these expressions for the correlates, we can also compute the individual corrections (19). We have

$$\left.\begin{array}{ll}v_1=-a_1 p_1 \lambda_1 & v_1'=-a_1' p_1 \lambda_1\\ v_2=-b_2 p_2 \lambda_2 & v_2'=-b_2' p_2 \lambda_2\\ v_3=-c_3 p_3 \lambda_3 & v_3'=-c_3' p_3 \lambda_3\\ \cdots & \cdots\end{array}\right\}. \tag{21}$$

The computation of the mean error of a measurement can be carried out either from the v's according to equation (6), p. 188, or from the λ's. Hence, we have for m the two expressions

$$m = \sqrt{\frac{[v\,v] + [v'\,v']}{r-u}}, \qquad m = \sqrt{\frac{[\lambda\,\lambda\,p]}{r-u}}, \tag{22}$$

whose agreement is readily understood.

From the error equations (18) there can also be determined the weights of the unknowns x and y as well as their mean error.

Example

As a final example for the above developments we use the determination of the equation of a straight line from the measured coordinates of several points of the same. On a drawing sheet there are measured, by means of a coordinatograph, the coordinates of 10 points of a straight line, which, as measured quantities, may be denoted by l and l'. The measured coordinates are;

	l	l'
1	+ 55.58 mm	+ 40.50 mm
2	+ 68.50	+ 46.40
3	+ 78.00	+ 50.70
4	+ 82.15	+ 52.51
5	+ 95.44	+ 58.52
6	+ 115.80	+ 67.54
7	+ 120.05	+ 69.50
8	+ 130.18	+ 74.00
9	+ 144.40	+ 80.55
10	+ 154.50	+ 84.98

(23)

For the equation of the straight line we assume the following form:

$$x - (l + v) + (l' + v') \cot y = 0\,, \tag{24}$$

in which x is the length which the straight line cuts from the axis of abscissae, while y is the direction angle of the straight line with respect to the axis of abscissae; v and v' are the corrections of the measured coordinates.

There exists further the difficulty that the quantity of observation l' and its correction v' occur as factors of the expression $\cot y$, which itself is again a nonlinear function of the unknown y. This difficulty is removed, and at the same time a simplification of the numerical computation is also obtained if we introduce approximate values for the measured quantities as well as for the unknowns by setting

$$\left.\begin{array}{ll} x = (x) + \Delta x & y = (y) + \Delta y \\ l_1 = (l_1) + \Delta l_1 & l_1' = (l_1') + \Delta l_1' \\ l_2 = (l_2) + \Delta l_2 & l_2' = (l_2') + \Delta l_2' \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}\right\}, \tag{25}$$

so that

$$\cot\, y = \cot\, (y) - \frac{1}{\sin^2 (y)} \Delta y \tag{26}$$

can then be assumed. Equation (24) changes then into

$$(x) + \Delta x - \big((l) + \Delta l + v\big) + \big((l') + \Delta l' + v'\big)\Big(\cot\, (y) - \frac{1}{\sin^2 (y)} \Delta y\Big) = 0\,,$$

and by neglecting all terms of second order we will have

$$\left.\begin{aligned}(x) - (l) + (l')\cot(y) - \Delta l + \cot(y)\,\Delta l' + \Delta x - (l')\frac{1}{\sin^2(y)}\Delta y \\ - v + \cot(y)\,v' = 0\end{aligned}\right\}. \tag{27}$$

If we set herein

$$\left.\begin{aligned}-\Delta l + \cot(y)\,\Delta l' &= -w, \\ -v + \cot(y)\,v' = -\lambda \qquad \text{weight } p &= \frac{1}{1+\cot^2(y)} = \sin^2(y)\end{aligned}\right\} \tag{28}$$

and choose the approximate values so that we will have

$$(x) - (l) + (l')\cot(y) = 0 \tag{29}$$

then we have the error equation

$$\lambda = -w + \Delta x - (l')\frac{1}{\sin^2(y)}\Delta y \qquad p = \sin^2(y). \tag{30}$$

First it is a question of choosing the approximate values according to equation (29). For this, we write this equation in the form

$$(x)\tan(y) - (l)\tan(y) + (l') = 0$$

and compute with the approximate values

$$(y) = 24°10'$$

for all points the difference $l' - l\tan(y)$, whose mean value we set equal to $-(x)\tan(y)$. With this value of $(x)\tan(y)$ and the values $(l)\tan(y)$ already computed, we compute the approximate values (l') pertaining to them and the corrections $\Delta l'$, while we assume, for the sake of simplicity, $(l) = l$, hence, $\Delta l = 0$.

The numerical computation belonging to this is summarized in the following table:

$(y) = 24°\,10'$ (31)

	$l\tan(y)$	$l' - l\tan(y)$	(l')	$\Delta l'$
1	+ 24.94	+ 15.56	+ 40.59	— 0.09
2	+ 30.74	+ 15.66	+ 46.39	+ 0.01
3	+ 35.00	+ 15.70	+ 50.65	+ 0.05
4	+ 36.86	+ 15.65	+ 52.51	0.00
5	+ 42.82	+ 15.70	+ 58.47	+ 0.05
6	+ 51.96	+ 15.58	+ 67.61	— 0.07
7	+ 53.87	+ 15.63	+ 69.52	— 0.02
8	+ 58.41	+ 15.59	+ 74.06	— 0.06
9	+ 64.79	+ 15.76	+ 80.44	+ 0.11
10	+ 69.33	+ 15.65	+ 84.98	0.00

Mean: $+15.648 = -(x)\tan(y)$

$(x) = -34.873$ mm .

The error equations (30) become then:

$$
\left.\begin{aligned}
\lambda_1 &= -0.09\, cot\,(y) + \Delta x - \frac{1}{sin^2\,(y)}\, 40.59\,\Delta y \\
\lambda_2 &= +0.01 \quad '' \quad + \Delta x - \quad '' \quad 46.39\,\Delta y \\
\lambda_3 &= +0.05 \quad '' \quad + \Delta x - \quad '' \quad 50.65\,\Delta y \\
\lambda_4 &= 0.00 \quad '' \quad + \Delta x - \quad '' \quad 52.51\,\Delta y \\
\lambda_5 &= +0.05 \quad '' \quad + \Delta x - \quad '' \quad 58.47\,\Delta y \\
\lambda_6 &= -0.07 \quad '' \quad + \Delta x - \quad '' \quad 67.61\,\Delta y \\
\lambda_7 &= -0.02 \quad '' \quad + \Delta x - \quad '' \quad 69.52\,\Delta y \\
\lambda_8 &= -0.06 \quad '' \quad + \Delta x - \quad '' \quad 74.06\,\Delta y \\
\lambda_9 &= +0.11 \quad '' \quad + \Delta x - \quad '' \quad 80.44\,\Delta y \\
\lambda_{10} &= 0.00 \quad '' \quad + \Delta x - \quad '' \quad 84.98\,\Delta y
\end{aligned}\right\} \qquad (32)
$$

weight $p = sin^2\,(y)$

Now we do not pass over at once to the normal equations, but form from (32) the reduced error equations by elimination of Δx according to equation (5), section 33, p. 104. For this, we need the first normal equation of (32), for which we find

$$
0 = 0 + 10\,\Delta x - \frac{1}{sin^2\,(y)}\, 625.22\,\Delta y
$$

or

$$
0 = 0 - \Delta x + \frac{1}{sin^2\,(y)}\, 62.52\,\Delta y\,, \qquad (33)
$$

with which the reduced error equations become:

$$
\left.\begin{aligned}
\lambda_1 &= -0.09\, cot\,(y) + \frac{1}{sin^2\,(y)}\, 21.93\,\Delta y \\
\lambda_2 &= +0.01 \quad '' \quad + \quad '' \quad 16.13\,\Delta y \\
\lambda_3 &= +0.05 \quad '' \quad + \quad '' \quad 11.87\,\Delta y \\
\lambda_4 &= 0.00 \quad '' \quad + \quad '' \quad 10.01\,\Delta y \\
\lambda_5 &= +0.05 \quad '' \quad + \quad '' \quad 4.05\,\Delta y \\
\lambda_6 &= -0.07 \quad '' \quad + \quad '' \quad 5.09\,\Delta y \\
\lambda_7 &= -0.02 \quad '' \quad + \quad '' \quad 7.00\,\Delta y \\
\lambda_8 &= -0.06 \quad '' \quad + \quad '' \quad 11.54\,\Delta y \\
\lambda_9 &= +0.11 \quad '' \quad + \quad '' \quad 17.92\,\Delta y \\
\lambda_{10} &= 0.00 \quad '' \quad + \quad '' \quad 22.46\,\Delta y
\end{aligned}\right\}\,. \qquad (34)
$$

weight $p = sin^2\,(y)$.

We obtain therefrom the normal equation

$$
2032.26\,\frac{1}{sin^2\,(y)}\,\Delta y - 1.7989\, cot\,(y) = 0\,, \qquad (35)
$$

$$
\Delta y = 0.0003\,3063 \quad \text{or} \quad \Delta y = 68.2''
$$

and from (33)

$$
\Delta x = +0.123 \text{ mm}\,.
$$

The final values of the unknowns are therefore

$$
y = 24°11'\,08.2''\,, \qquad x = -34.750 \text{ mm}\,. \qquad (36)
$$

The computation of the λ's is carried out from the error equation (34), and for the individual corrections v and v' we have then according to (21)

$$
v = +\lambda \sin^2\,(y)\,, \qquad v' = -\lambda \sin\,(y) \cos\,(y)\,,
$$

or

$$v = +0.1676\,\lambda\,, \qquad v' = -0.3735\,\lambda\,.$$

With this, we obtain the following summary:

	λ	λ^2	v	v^2	v'	v'^2
	mm		mm		mm	
1	− 0.16	0.0256	− 0.03	0.0009	+ 0.06	0.0036
2	+ 0.05	0.0025	+ 0.01	0.0001	− 0.02	0.0004
3	+ 0.13	0.0169	+ 0.02	0.0004	− 0.05	0.0025
4	+ 0.02	0.0004	0.00	0.0000	− 0.01	0.0001
5	+ 0.12	0.0144	+ 0.02	0.0004	− 0.04	0.0016
6	− 0.17	0.0289	− 0.03	0.0009	+ 0.06	0.0036
7	− 0.05	0.0025	− 0.01	0.0001	+ 0.02	0.0004
8	− 0.15	0.0225	− 0.03	0.0009	+ 0.06	0.0036
9	+ 0.20	0.0400	+ 0.03	0.0009	− 0.07	0.0049
10	− 0.04	0.0016	− 0.01	0.0001	+ 0.01	0.0001
		0.1553		0.0047		0.0208

$$[\lambda\lambda p] = 0.0260 \qquad [v\,v] + [v'\,v'] = 0.0255$$

$$m = \sqrt{\frac{0.0260}{10-2}} = \pm\, 0.057 \text{ mm}\,. \tag{37}$$

The normal equation (35) yields at the same time the weight:

$$p_y = \frac{2032.26}{\sin^2 (y)}\,, \qquad m_y = \pm\, 0.000518 \quad \text{or} \quad m_y = \pm\, 106.8''\,,$$

and if we determine further, from the original error equations (32), the sums of the coefficients $[a\,a]$, $[a\,b]$, and $[b\,b]$, then we find easily the weight:

$$p_x = 0.0828\,, \qquad m_x = \pm\, 0.198 \text{ mm}\,.$$

As the result of the adjustment we obtain the following equation of the straight line:

$$\begin{aligned} -\,34.750 \;&-\; x \;+\; 2.2266\; y = 0 \\ \pm\; 0.198 \;& \qquad\quad \pm\, 0.0031\,, \end{aligned}$$

where, as usual, x and y again designate the abscissae and ordinates.

We have treated the problem of adjustment given by equations (12), p. 189, and the above example on pp. 190-195, in *Zeitschrift für Vermessungswesen*, 1918, pp. 1-16, in reference to a treatise by R. Schumann in *Sitzungsbericht der Kaiserlichen Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Klasse Abteilung II a*, 125 Band, 10 Heft, 1916, pp. 1429 to 1466, from which also the measured numbers (23), p. 191, on which the numerical example has been based, are taken.

In equations (5), p. 188, we can find again all problems of adjustment treated earlier, through simplifying assumptions concerning the coefficients; e.g. indirect observations result from (5) by the theory that in each equation there remains only a single v, which occurs then as a linear function of the x's, y's, z's; for this, it is necessary that $r > n$ or $r = n$. If $r < n$, then it is not possible to obtain for each v a special error equation, but we can then pick out individual v's as special unknowns, as has happened in (2) with a single v_1.

The case of the conditioned observations is contained in (5) very simply if x, y, z become equal to zero. The problem of the indirect observations with condition equations results from (5) by a splitting into two groups of equations, the first of which contains only a single v each and the x's, y's, z's, and the second of which contains no v's, but the x's, y's, z's alone.

By the above considerations it has also become clear that *all* problems of adjustment can be reduced to the case of indirect observations. For this reason, it was only necessary to prove the proposition for the mean error in regard to the division by $n-u$ in (19), section 29, p. 93, for indirect observations; this proposition could then be transferred to the remaining cases by simple considerations about the number of the observations and the unknowns, as, for instance, has been done for the most general case (5), p. 188, in the preceding equation (6), p. 188.

Section 61. Functions of the Adjusted Observations in the Case of Adjustment According to Conditioned Observations

We return once more to the problem already treated in section 46, for which we can find a new solution in connection with the theories developed in the preceding section 60.

We assume that there exists an adjustment according to conditioned observations, where, for simplicity, we will limit ourselves to three condition equations and four measured quantities l_1, l_2, l_3, l_4. Let the following condition equations exist for them:

$$\left.\begin{array}{l} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 + w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 + w_3 = 0 \end{array}\right\} . \tag{1}$$

From the adjusted observations the following function is to be computed

$$F = f_0 + f_1 (l_1 + v_1) + f_2 (l_2 + v_2) + f_3 (l_3 + v_3) + f_4 (l_4 + v_4) . \tag{2}$$

Instead of computing this function after the adjustment with the help of the corrections v, we will connect the computation of F with the adjustment by adding equation (2) as a fourth condition equation to equations (1). If we set

$$f_0 + f_1 l_1 + f_2 l_2 + f_3 l_3 + f_4 l_4 = W , \tag{3}$$

then we have now the four equations

$$\left.\begin{array}{l} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \quad . \quad + w_1 = 0 \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 \quad . \quad + w_2 = 0 \\ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 \quad . \quad + w_3 = 0 \\ f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 - F + W = 0 \end{array}\right\} . \tag{4}$$

Since the quantity F can be interpreted as unknown, then the four equations (4) correspond to the previous equations (5), section 60, p. 188, and we can therefore apply the method of adjustment of section 60. According to (10), section 60, p. 189, we obtain the normal equations

$$\left.\begin{array}{l} [a a] k_1 + [a b] k_2 + [a c] k_3 + [a f] k_4 \quad . \quad + w_1 = 0 \\ [a b] k_1 + [b b] k_2 + [b c] k_3 + [b f] k_4 \quad . \quad + w_2 = 0 \\ [a c] k_1 + [b c] k_2 + [c c] k_3 + [c f] k_4 \quad . \quad + w_3 = 0 \\ [a f] k_1 + [b f] k_2 + [c f] k_3 + [f f] k_4 - F + W = 0 \\ . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; - k_4 \; . \; . \; . \quad = 0 \end{array}\right\} . \tag{5}$$

It is true that it follows directly from this that the correlate $k_4 = 0$; however, we will for the present retain k_4 as an auxiliary quantity.

We obtain from (5), after reducing three times, the two equations

$$\left.\begin{array}{l}[ff\cdot 3]\,k_4 - F + [W\cdot 3] = 0 \\ -k_4 \quad . \quad . \quad . \quad . \quad = 0\end{array}\right\}, \tag{6}$$

where we will denote by $[W \cdot 3]$ the value of W three times reduced.

The repeated reduction of equations (6) yields

$$-\frac{1}{[ff\cdot 3]}F + \frac{[W\cdot 3]}{[ff\cdot 3]} = 0\,. \tag{7}$$

From this equation it follows first that

$$F = [W \cdot 3]\,. \tag{8}$$

But since further, according to (16), section 47, p. 148,

$$P = \frac{1}{[ff\cdot 3]}\,, \tag{9}$$

then we have in the negative coefficient of F in equation (7) the weight P of F.

We will apply this to the simple example of the adjustment of the three angles of a plane triangle of section 48, p. 149. According to (4) and (9), pp. 149-150, we have the condition equation

$$v_1 + v_2 + v_3 + w = 0$$

with

$$w = l_1 + l_2 + l_3 - 180^\circ$$

and the function to be computed

$$F = l_1 + v_1\,.$$

However, our equations (4), p. 195, assume the form

$$\left.\begin{array}{l}v_1 + v_2 + v_3 \quad . \quad + w = 0 \\ v_1 \quad . \quad . \quad . \quad - F + l_1 = 0\end{array}\right\} \tag{10}$$

and the normal equations (5) change into

$$\left.\begin{array}{l}3\,k_1 + k_2 \quad . \quad + w = 0 \\ k_1 + k_2 - F + l_1 = 0 \\ . \quad - k_2 \quad . \quad . \quad . \quad = 0\end{array}\right\}. \tag{11}$$

The first reduction of these equations yields

$$\left.\begin{array}{l}\frac{2}{3}k_2 - F + l_1 - \frac{1}{3}w = 0 \\ -k_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = 0\end{array}\right\}, \tag{12}$$

and by repeated reduction we obtain

$$-\frac{3}{2}F+\frac{3}{2}l_1-\frac{1}{2}w=0\,. \tag{13}$$

There follows hence at first

$$F=l_1-\frac{1}{3}w\,, \tag{14}$$

and with

$$w=l_1+l_2+l_3-180^\circ$$

we will have

$$F=\frac{1}{3}(2\,l_1-l_2-l_3+180^\circ).$$

Further, the weight of F becomes

$$P=\frac{3}{2}\,,$$

which agrees with section 48, p. 149.

We see at the same time that the substitution of the value F of (14) in the first equation (12) for k_2 yields the value $k_2=0$, which was known from the outset.

Simultaneous computation of several functions

We can also apply the theory developed in the above to the case in which several functions are to be computed from the adjusted observations. Let there be, e.g., the two functions

$$\left.\begin{aligned} F_1 &= f_0+f_1(l_1+v_1)+f_2(l_2+v_2)+f_3(l_3+v_3)+f_4(l_4+v_4)\\ F_2 &= g_0+g_1(l_1+v_1)+g_2(l_2+v_2)+g_3(l_3+v_3)+g_4(l_4+v_4)\end{aligned}\right\}, \tag{15}$$

while we will assume again the three condition equations (1). If we set further

$$\left.\begin{aligned} f_0+f_1\,l_1+f_2\,l_2+f_3\,l_3+f_4\,l_4 &= W_1\\ g_0+g_1\,l_1+g_2\,l_2+g_3\,l_3+g_4\,l_4 &= W_2\end{aligned}\right\}, \tag{16}$$

then we have now in all the 5 equations

$$\left.\begin{aligned} a_1v_1+a_2v_2+a_3v_3+a_4v_4 \quad\ldots\ldots\quad +w_1 &= 0\\ b_1v_1+b_2v_2+b_3v_3+b_4v_4 \quad\ldots\ldots\quad +w_2 &= 0\\ c_1v_1+c_2v_2+c_3v_3+c_4v_4 \quad\ldots\ldots\quad +w_3 &= 0\\ f_1v_1+f_2v_2+f_3v_3+f_4v_4-F_1 \quad\ldots\quad +W_1 &= 0\\ g_1v_1+g_2v_2+g_3v_3+g_4v_4 \quad\ldots\quad -F_2+W_2 &= 0\end{aligned}\right\}. \tag{17}$$

These equations have again the form of the general condition equations (5), section 60, p. 188; therefore, we can also set up the normal equations (10), section 60, p. 189. We obtain

$$\left.\begin{aligned} [aa]k_1+[ab]k_2+[ac]k_3+[af]k_4+[ag]k_5 \quad\ldots\ldots\quad +w_1 &= 0\\ [ab]k_1+[bb]k_2+[bc]k_3+[bf]k_4+[bg]k_5 \quad\ldots\ldots\quad +w_2 &= 0\\ [ac]k_1+[bc]k_2+[cc]k_3+[cf]k_4+[cg]k_5 \quad\ldots\ldots\quad +w_3 &= 0\\ [af]k_1+[bf]k_2+[cf]k_3+[ff]k_4+[fg]k_5-F_1 \quad\ldots\quad +W_1 &= 0\\ [ag]k_1+[bg]k_2+[cg]k_3+[fg]k_4+[gg]k_5 \quad\ldots\quad -F_2+W_2 &= 0\\ \ldots\ldots\ldots\ldots -k_4 \ldots\ldots\ldots\ldots &= 0\\ \ldots\ldots\ldots\ldots\ldots -k_5 \ldots\ldots\ldots &= 0\end{aligned}\right\}. \tag{18}$$

The correlates k can be eliminated therefrom in succession in the usual manner. After the third elimination, the following four equations remain:

$$\left.\begin{array}{l} [ff\cdot 3]\,k_4 + [fg\cdot 3]\,k_5 - F_1 \quad . \quad . + [W_1\cdot 3] = 0 \\ [fg\cdot 3]\,k_4 + [gg\cdot 3]\,k_5 \quad . \quad . - F_2 + [W_2\cdot 3] = 0 \\ . \quad . - k_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . = 0 \\ . \quad . \quad . \quad . \quad - k_5 \quad . \quad . \quad . \quad . \quad . \quad . \quad = 0 \end{array}\right\}. \qquad (19)$$

where by $[W_1 \cdot 3]$ and $[W_2 \cdot 3]$ the three-times-reduced values of W_1 and W_2 are again denoted.

Since $k_4 = 0$ and $k_5 = 0$ we obtain from (19) as the first result

$$\left.\begin{array}{l} F_1 = [W_1\cdot 3] \\ F_2 = [W_2\cdot 3] \end{array}\right\}, \qquad (20)$$

which agrees with our above found equation (8), p. 196.

We can now carry further the reduction of equations (19), and since we have eliminated further the last two correlates k_4 and k_5, there remain two equations with the two unknowns F_1 and F_2, which we will write with a symbolic notation for the coefficients in the form

$$\left.\begin{array}{l} [AA]\,F_1 + [AB]\,F_2 - [AL] = 0 \\ [AB]\,F_1 + [BB]\,F_2 - [BL] = 0 \end{array}\right\}. \qquad (21)$$

With this, we are in a position to determine the weights P_1 and P_2 of the two functions F_1 and F_2 according to (18), section 18, p. 58.

But equations (21) obtain further a special significance by the fact that we can use them for the setting up of equivalent observations according to section 33, p. 106. Hence, if to the original adjustment of the condition equations (1), p. 195, there shall be connected a further adjustment, in which the quantities F_1 and F_2 occur as elements of the adjustment, then we have to carry out the first adjustment in the form of the above equations (17) and (18) and set up from equations (21) the two equivalent observations, which can then be submitted to an arbitrary further adjustment.

Chapter II

TRIANGULATION NETS

The adjustment of triangulation nets is one of the most important problems of the method of least squares and has, through its urgency, contributed, most of all, to the development of the theories of adjustment. This adjustment is carried out in most cases according to conditioned observations and was taught for the first time by Gauss in 1826 in the treatise, "Supplementum theoriae combinationis" (cf. p. 2).

In this chapter II there shall first be treated the net adjustment according to the method of conditioned observations, and in the later chapter III, the adjustment of coordinates according to the method of indirect observations.

The first task for the adjustment of a triangulation net consists in the setting up of the condition equations, with which we shall therefore have to deal first in this chapter.

Before we pass over to this, however, we will for the present consider simple station adjustments, which are involved in the case of the simple nets of this chapter, as well as in the case of the point intercalations of chapter III.

In the case of triangulations, the measurements are in many cases adjusted independently at the stations according to simple rules before entering them into the actual net adjustment, and the result of these "station adjustments" is then introduced into the further adjustment like directly measured, uniform sets of directions. Without investigating now the theoretical justification of this method (see the later consideration in the following paragraphs), it shall only be noted here that it is a good approximation method, which is often used and is greatly recommended for all measurements of second and lower order.

If in the case of such measurements themselves no ray is improperly neglected, and the measurement is rather arranged so that each ray receives its right compared with the others, then it is justified to introduce all station directions as *having equal weights* into the net, as will be done in the first examples of this chapter. In such a manner, Gauss adjusted his degree-measurement of Hannover in the years 1820-1830; cf. *Zeitschrift für Vermessungswesen* 1879, p. 141; also Jordan-Steppe's *Deutsches Vermessungswesen* I, p. 11.

Section 62. Station Adjustment with Angle Measurements

If angle measurements are distributed in such a way that at one station with s rays more than $s-1$ angles are present, then we can insert a condition into the net adjustment for each excessive angle at the station, as, e.g., will be shown later in the example of S c h w e r d' s base net, section 75, by Fig. 1 with the station Mannheim.

But if the number of the excessive angles at the stations is large, then this method becomes very detailed and, therefore, is not used as a whole; the stations are rather to be adjusted first independently.

For the present, leaving aside the question as to how the station adjustments are to pass over into the net, we now treat the station adjustment with angles as an independent preliminary problem.

I. *Sum check at the horizon*

A simple case, which often occurred in earlier times when repetition angles were measured and even now is still considered for triangulations of lower order, is present when several angles, which fill the whole horizon of 360°, are measured individually and then must come out to the horizon check; e.g., in Fig. 2, section 67, we shall have two such horizon checks with (3) + (6) + (8) + (10) + (24) + (25) at the horizon of Oggersheim, and with (12) + (15) + (16) + (19) + (22) at the horizon of Speyer. [Also, e.g., the check on Mannheim in Fig. 1, section 75, appears to be of this kind, if we interpret (7), (8) and 360° – (9) as three individual angle measurements, which must fill the horizon 360°.]

The adjustment of such a horizon consists in the case of equal weights of the individual angles only in that we relay the discrepancy at the horizon in equal parts to all participating angles, and if the angles have unequal weights, then the distribution of the discrepancy is carried out in inverse proportion to the weights.

We can treat this according to the principle of the arithmetic mean of *two* measurements of unequal accuracy, similarly to the angle adjustment in a triangle treated already previously in section 10.

We assume for this the following:

Measured:	A_1	A_2	A_3	...	A_n
With the weights:	p_1	p_2	p_3	...	p_n
We should have:	$A_1 + A_2 + A_3 + \ldots + A_n - 360° = 0$				
We have:	$A_1 + A_2 + A_3 + \ldots + A_n - 360° = w$.				(1)

Let the adjusted value of the first angle be x; therefore, two observational results exist for the determination of x:

1. $x = A_1$ with the weight p_1,
2. $x = 360° - (A_2 + A_3 + \ldots + A_n) = A_1 - w$ with the weight p'.

The weight p' is determined by

$$\frac{1}{p'} = \frac{1}{p_2} + \frac{1}{p_3} \cdots \frac{1}{p_n} = \left[\frac{1}{p}\right] - \frac{1}{p_1} . \tag{2}$$

The adjusted value x is according to (4), section 8, p. 21.

$$x = \frac{A_1 p_1 + (A_1 - w)p'}{p_1 + p'} = A_1 - \frac{p'}{p_1 + p'} w$$

and with the introduction of the value of p' from (2)

$$x = A_1 - \frac{1}{p_1} \frac{w}{\left[\frac{1}{p}\right]} . \tag{3}$$

Since a similar formula holds for the remaining angles, then we have the result in words: The discrepancy w is distributed among the individual angles in inverse proportion to their weights.

The correction v_1 of the first measured angle amounts to

$$v_1 = - \frac{1}{p_1} \frac{w}{\left[\frac{1}{p}\right]} , \tag{4}$$

and since a similar formula holds also for the other angles, then we have

$$[p\,v\,v] = \frac{1}{p_1} \frac{w^2}{\left[\frac{1}{p}\right]^2} + \frac{1}{p_2} \frac{w^2}{\left[\frac{1}{p}\right]^2} + \ldots = \frac{w^2}{\left[\frac{1}{p}\right]} . \tag{5}$$

The mean error m of an observation of weight 1 is

$$m = \sqrt{\frac{[p\,v\,v]}{2-1}}, \text{ therefore } m = \frac{w}{\sqrt{\left[\frac{1}{p}\right]}} . \tag{6}$$

The mean error M_1 of the first adjusted angle x is obtained by dividing m by the square root of the sum of weights $p_1 + p'$; the execution yields

$$M_1 = w \frac{\sqrt{\frac{1}{p_1}\left(\left[\frac{1}{p}\right] - \frac{1}{p_1}\right)}}{\left[\frac{1}{p}\right]} . \tag{7}$$

If all weights become $p_1 = p_2 = \ldots p_n = 1$ then the formulae become simpler, namely

$$x = A_1 - \frac{w}{n}, \tag{8}$$

$$m = \frac{w}{\sqrt{n}}, \tag{9}$$

$$M = \frac{w}{n}\sqrt{n-1} = \frac{w}{\sqrt{n}}\sqrt{\frac{n-1}{n}}. \tag{10}$$

From (10) and (9) it follows that the accuracy of an angle *after* the adjustment in comparison to the accuracy *before* the adjustment has increased in the ratio $\sqrt{n} : \sqrt{n-1}$.

The gain of accuracy for an angle is therefore relatively small if many angles are measured at the horizon.

II. *Horizontal closure according to indirect observations*

In Fig. 1 we treat the measurement of six individual angles 1 to 6 between four rays and, in fact, angle observations in all combinations, i.e., between the rays AB, AC, AD, BC, BD, CD, and adjust according to indirect observations. The six measurements are:

$$\left.\begin{array}{llll} 1. = & 48^\circ 17' \ 1.4'' & 4. = & 48^\circ 35' \ 14{,}3'' \\ 2. = & 96 \ \ 52 \ 16.8 & 5. = & 104 \ \ 37 \ \ 7.8 \\ 3. = & 152 \ \ 54 \ \ 6.8 & 6. = & 56 \ \ 1 \ 48.9 \end{array}\right\} . \tag{11}$$

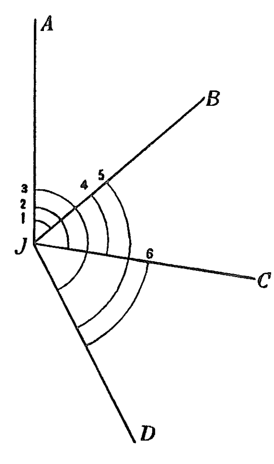


Fig. 1.

For the reciprocal determination of the four rays A, B, C, D, three angles are necessary; therefore, we introduce three angles as independent unknowns and have:

Unknowns: AJB, AJC, AJD .

As the first approximate values of the unknowns we take the first three measurements themselves.

$$\left.\begin{array}{lll} & \text{Approximate values:} & \text{Corrections:} \\ (AJB) = & 48^\circ 17' \ \ 1.4'' & x \\ (AJC) = & 96 \ \ 52 \ \ 16.8 & y \\ (AJD) = & 152 \ \ 54 \ \ \ 6.8 & z \end{array}\right\} . \tag{12}$$

We denote further, as usual, the six corrections of the observations by v_1, v_2, v_3, v_4, v_5, v_6, and now the first three error equations are obviously very simple:

$$v_1 = x, \qquad v_2 = y, \qquad v_3 = z, \tag{13}$$

because, in particular, the first three observations themselves are taken as approximations. The fourth error equation becomes

$$48^\circ\,35'\,14.3'' + v_4 = (96^\circ\,52'\,16.8'' + y) - (48^\circ\,17'\,1.4'' + x), \qquad v_4 = -x + y + 1.1''.$$

In a similar way, also the fifth and the sixth error equations are formed; we have, therefore, the summary of all six error equations:

(13)

$$\left.\begin{array}{l} v_1 = +x \;.\;.\;\;\;.\;.\;\;\;\;.\;. \\ v_2 = \;.\;.\; +y \;.\;.\;\;\;\;.\;. \\ v_3 = \;.\;.\;\;\;.\;.\; + \;\;\;\;\;.\;. \\ v_4 = -x + y \;.\;.\; + 1.1'' \\ v_5 = -x \;.\;.\; + z - 2.4'' \\ v_6 = \;.\;.\; - y + z + 1.1'' \end{array}\right\} . \tag{14}$$

With equal weights we obtain the normal equations

$$\begin{array}{l} 3x - y - z + (+2.4 - 1.1) = 0 \\ -x + 3y - z + (+1.1 - 1.1) = 0 \\ -x - y + 3z + (-2.4 + 1.1) = 0 \end{array} \quad \text{or} \quad \left.\begin{array}{r} \underline{3x} - y - z + 1.3 = 0 \\ \underline{3y} - z + 0.0 = 0 \\ \underline{3z} - 1.3 = 0 \end{array}\right\} . \tag{15}$$

The solution yields:

$$x = -0.3'', \qquad y = 0.0'', \qquad z = +0.3''. \tag{16}$$

If we add these corrections to the measured angles 1, 2, 3, or to the approximations (1), (2), (3), which are the same here, then we obtain the *station result*:

$$\left.\begin{array}{lrrr} \text{Angle } AB = & 48^\circ & 17' & 1.1'' \\ \text{Angle } AC = & 96 & 52 & 16.8 \\ \text{Angle } AD = & 152 & 54 & 7.1 \end{array}\right\} . \tag{17}$$

We can treat such an adjustment also more generally. With four rays and six angle measurements, the absolute terms of the error equations, which in (14) had the values $+1.1''$, $-2.4''$, $+1.1''$, are denoted generally by $-l_4$, $-l_5$, $-l_6$, and with this, we obtain

Error equations | Normal equations

$$\left.\begin{array}{l} v_1 = +x \;.\;.\;\;.\;.\;\;.\;. \\ v_2 = \;.\;.\; +y \;.\;.\;\;.\;. \\ v_3 = \;.\;.\;\;.\;.\; +z \;.\;. \\ v_4 = -x + y \;.\;.\; - l_4 \\ v_5 = -x \;.\;.\; + z - l_5 \\ v_6 = \;.\;.\; - y + z - l_6 \end{array} \qquad \begin{array}{l} 3x - y - z + l_4 + l_5 = 0 \\ -x + 3y - z - l_4 + l_6 = 0 \\ -x - y + 3z - l_5 - l_6 = 0 \end{array}\right\} . \tag{18}$$

The solution of the normal equations yields

$$x = \frac{-l_4 - l_5}{4}, \qquad y = \frac{+l_4 - l_6}{4}, \qquad z = \frac{+l_5 + l_6}{4}. \tag{19}$$

In the numerical example treated above with (14) we had

$$l_4 = -1.1'', \qquad l_5 = +2.4'', \qquad l_6 = -1.1'',$$

whence there follows

$$x = -0.3'', \qquad y = 0.0'', \qquad z = +0.3''; \tag{20}$$

this is in agreement with the previous (16).

The above treatment of the angle adjustment of Fig. 1, where all six angles between four rays are measured with equal weight, has presented itself at the first sight of the matter. The *more general* and much more elegant treatment of equal-weight angle measurements in all combinations will be given later in section 86.

If the weights are *not equal* a priori, then, even with symmetrical arrangement of the measurements, the adjustment no longer becomes symmetrical.

We will assume the following weights to the six measurements, which were treated above, or, as the case may be, to the six error equations (14):

$$p_1 = 30, \quad p_2 = 20, \quad p_3 = 26, \quad p_4 = 25, \quad p_5 = 28, \quad p_6 = 44. \tag{21}$$

With this, we will have

$$[p\,a\,a] = +30 + 25 + 28 = +83$$
$$[p\,a\,b] = -25 \quad \text{and so on}$$

and the normal equations in an abbreviated manner of writing are

$$\begin{array}{rrrr} \underline{83.0\,x} - 25.0\,y & - 28.0\,z & + 39.7 & = 0 \\ \underline{89.0\,y} & - 44.0\,z & - 20.9 & = 0 \\ & \underline{98.0\,z} & - 18.8 & = 0 \\ & & \underline{244.77}\,. & \end{array}$$

The solution yields

$$\left.\begin{array}{ll} x = -0.34'' & p_x = 54.6 \\ y = +0.24 & p_y = 50.4 \\ z = +0.20 & p_z = 54.8 \end{array}\right\} \tag{22}$$

$$[p\,l\,l\cdot 3] = [p\,v\,v] = 222.5\,. \tag{23}$$

With this, the mean error of an observation with the weight 1 becomes

$$m = \sqrt{\frac{222.5}{6-3}} = \pm 8.61''.$$

The mean errors of the adjusted x's, y's, z's become, therefore,

$$m_x = \frac{8.61}{\sqrt{54.6}} = \pm 1.17'', \quad m_y = \frac{8.61}{\sqrt{50.4}} = \pm 1.21'', \quad m_z = \frac{8.61}{\sqrt{54.8}} = \pm 1.16''. \tag{24}$$

We obtain all six adjusted angle values by adding the corrections x, y, z to the assumed approximate values (12); at the same time, we also form, by subtraction, the three remaining angles and find with these:

Angles	Measured	v	Adjusted	v^2	p	$p\,v^2$
1. = AB	48° 17′ 1.4″	− 0.34″	48° 17′ 1.06″	0.12	30	3.60
2. = AC	96 52 16.8	+ 0.24	96 52 17.04	0.06	20	1.20
3. = AD	152 54 6.8	+ 0.20	152 54 7.00	0.04	26	1.04
4. = BC	48 35 14.3	+ 1.68	48 35 15.98	2.82	25	70.50
5. = BD	104 37 7.8	− 1.86	104 37 5.94	3.46	28	96.89
6. = CD	56 1 48.9	+ 1.06	56 1 49.96	1.12	44	49.28
						222.51 .

(25)

The agreement of $[p\,v\,v]$ according to (23) and (25) confirms the correctness of the computation. We can set up this adjustment result also in the form of an adjusted set of directions:

$$\left.\begin{array}{ll}\text{Direction } JA = & 0°\ 0'\ 0.00''\\ \text{Direction } JB = & 48\ \ 17\ \ 1.06\\ \text{Direction } JC = & 96\ \ 52\ 17.04\\ \text{Direction } JD = & 152\ \ 54\ \ 7.00\end{array}\right\}. \quad (26)$$

And if we aim to exclude the different weight distinctions which are contained in such an adjusted set, then we introduce the set (26) simply like a directly fully measured set into a net adjustment, which is then carried further in the form of section 68 or 70.

In such a manner, Gerling, as a student of Gauss, treated the horizontal closures and, particularly, their introduction into the net adjustment, and the above example is borrowed from Gerling's *Adjustment Computations of Practical Geometry*, 1843, sections 56-57. The unequal weights p according to (21) (between 20 and 44) may represent numbers of repetition of the angle measurements; however, this is not indicated with certainty for the present case. (Cf. also *Zeitschrift für Vermessungswesen* 1901, p. 23.) The adjustment of equal-weight angle measurements in all combinations, to which we shall return in greater detail later in section 86, is much more important than an adjustment with unequal weights.

III. *Horizontal closure according to the correlation method*

We will now handle the hitherto treated case of Fig. 1 also according to conditioned observations, with correlates, and find that the following three condition equations exist between the six measured angles:

$$\left.\begin{array}{ll}\text{a)}\ -v_1+v_2\ \ .\ \ .-v_4\ \ .\ \ .\ \ .\ \ .\ \ +w_4=0\,, & w_4=+1.1''\\ \text{b)}\ -v_1\ \ .\ \ +v_3\ \ .\ \ .-v_5\ \ .\ \ .\ \ +w_5=0\,, & w_5=-2.4\\ \text{c)}\ \ .\ \ .\ \ -v_2+v_3\ \ .\ \ .\ \ .\ \ .\ -v_6\ \ +w_6=0\,, & w_6=+1.1\end{array}\right\}. \quad (27)$$

We can read these equations directly from the figure, with the substitution of the observations (11). But these equations (27) are also contained already in the second half of the previous error equations (18) if we set there

$$x=v_1,\quad y=v_2,\quad z=v_3,\quad \text{and}\quad l_4=-w_4,\quad l_5=-w_5,\quad l_6=-w_6. \quad (28)$$

The condition equations (27) yield the following normal equations:

$$\left.\begin{array}{l} +3k_1 + k_2 - k_3 + w_4 = 0 \\ + k_1 + 3k_2 + k_3 + w_5 = 0 \\ - k_1 + k_2 + 3k_3 + w_6 = 0 \end{array}\right\}. \tag{29}$$

The general solution of this is:

$$k_1 = \frac{-2w_4 + w_5 - w_6}{4}, \quad k_2 = \frac{+w_4 - 2w_5 + w_6}{4}, \quad k_3 = \frac{-w_4 + w_5 - 2w_6}{4}, \tag{30}$$

and the correlation equations, which follow the condition equations (27) according to columns, yield

$$\left.\begin{array}{l|l} 4v_1 = -4(k_1 + k_2) = +w_4 + w_5 & 4v_4 = -4k_1 = +2w_4 - w_5 + w_6 \\ 4v_2 = +4(k_1 - k_3) = -w_4 + w_6 & 4v_5 = -4k_2 = - w_4 + 2w_5 - w_6 \\ 4v_3 = +4(k_2 + k_3) = -w_5 - w_6 & 4v_6 = -4k_3 = + w_4 - w_5 + 2w_6 \end{array}\right\}. \tag{31}$$

With this, we have again the same as was contained previously in (19), according to the change of notation indicated in (28).

Therefore, if we have angle observations with equal weights in all combinations, as in this case, and if it is a question of general formulae, then the correlation method leads by a further way to the same result as the method of the indirect observations.

If the measurements are not equivalent or not carried out in all combinations, then the correlation method is, under certain circumstances, very useful for the numerical adjustment.

For the comparison of the adjustment according to indirect or according to conditioned observations we make the following consideration:

For fixing s rays, $s-1$ angles are necessary; therefore, if we have measured W angles, then there exist

for indirect observations: $s-1$ independent unknowns (32)

for conditioned observations: $W-(s-1)$ condition equations, (33)

and the number of the normal equations to be solved is in accord in both cases; e.g., with $s =$ four rays and $W =$ six angles, we have to solve, in both cases, three equations.

However, with many rays and relatively few condition equations, the correlation method is very suitable, as the following example shows:

According to Fig. 2, six independent angles (1), (2), (3), (4), (5), (6), are measured. In addition to these six, only two control angles are measured, namely the sum

$$(1) + (2) = (7)$$

and the total sum

$$(1) + (2) + (3) + (4) + (5) + (6) = (8);$$

therefore, there exist only two condition equations.

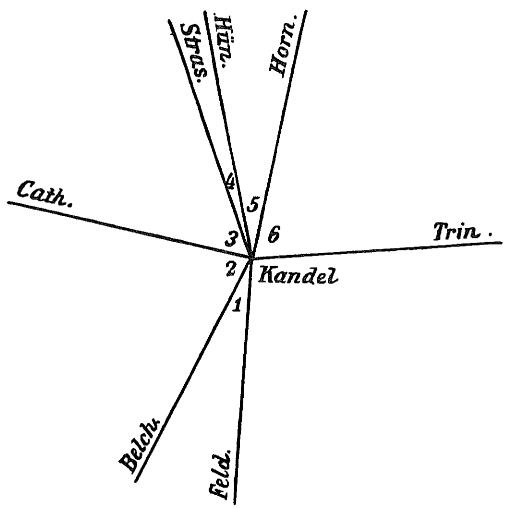


Fig. 2.

Following are the measurements in the new [centesimal] graduation and the weights:

No.	Target Point to the Left	Target Point to the Right	Measured Angles	Weight p
1	Feldberg	Belchen	(1) = 27.94758 g	5
2	Belchen	Catharina . .	(2) = 86.18972	12
3	Catharina . . .	Strassburg . . .	(3) = 63.70226	4
4	Strassburg . . .	Hünersedel . . .	(4) = 8.76737	3
5	Hünersedel . . .	Hornisgrinde . .	(5) = 26.27865	4
6	Hornisgrinde . .	Trinitatis	(6) = 82.73775	3
7	Feldberg	Catharina . . .	(7) = 114.13753	5
8	Feldberg	Trinitatis . . .	(8) = 295.62484	5
				41

(34)

The first condition equation reads

$$27.94758 + v_1 + 86.18972 + v_2 = 114.13753 + v_7$$

or, collecting terms, with the assumption of a hundredth of a second as a unit

$$v_1 + v_2 - v_7 - 2.3 = 0. \tag{35}$$

In the same manner, we set up the second condition equations:

$$v_1 + v_2 + v_3 + v_4 + v_5 + v_6 - v_8 - 15.1 = 0. \tag{36}$$

The coefficients of the two normal equations become

$$\left[\frac{a\,a}{p}\right] = +\frac{1}{5} + \frac{1}{12} + \frac{1}{7} = 0.200 + 0.083 + 0.143 = +0.426, \text{ etc.}$$

The two normal equations themselves are

$$+\underline{0.426k_1} + 0.283k_2 - 2.3 = 0$$

$$+0.283k_1 + \underline{1.649k_2} - 15.1 = 0.$$

Their solution yields

$$k_1 = -0.8, \qquad k_2 = +9.3.$$

With this, we determine the corrections v according to (3), section 44, p. 140:

$$v_1 = \frac{1}{5}(-0.8 + 9.3) = +1.7.$$

The remaining v's are computed likewise; altogether are

$$v_1 = +1.7 \quad v_2 = +0.7 \quad v_3 = +2.3 \quad v_4 = +3.1 \quad v_5 = +2.3 \quad v_6 = +3.1 \quad v_7 = +0.1 \quad v_8 = -1.9.$$

These v's in units of 1 cc = 0.0001 g added to the measured angles (34) yield the adjusted angles, which no longer show discrepancies.

The sum $[p\,v\,v]$ is obtained from the individual v's in agreement with the formula $-[w\,k]$:

$$[p\,v\,v] = 138.6,$$

hence, the mean error of an angle with weight 1 is:

$$m = \sqrt{\frac{138.6}{2}} = \pm 8.3\,cc \quad \text{or} \quad = \pm 2.7''.$$

(In our 3rd Edition, 1888, p. 226, this example was treated in more detail.)

Section 63. Computation of Complete Sets of Directions

If a complete set measurement is repeated several times in a homogeneous manner, with only an adjusted circle, then the computation of the final result consists only in forming the mean, as is already shown in detail in our Volume II, 1st half-volume, 9th Edition, 1931, section 88.

The circle adjustment between every two sets is done, as is known, because of the elimination of the division errors with equal intervals, e.g., from 45° to 45°, if we want to take four sets, from 30° to 30° with six sets and so forth. We make the different sets usually comparable by bringing any sight, say, as a start, to a common value, e.g. 0°0'0" (or, also, to an approximate direction angle) in all sets; we could, however, carry out the forming of the mean also *without* such previous moving closer together, if we so desired, since we can always move the set arbitrarily once more even *after* taking the mean.

We take the opportunity to recall here this known fact only because we want to show, in addition, the computation of the mean error from several repetitions of sets in (2), section 69; and indeed bearing in mind our later example of the triangulation of Hannover, Station Schanze.

There are measured twelve sets, of which we will, however, use here only six, i.e. half, first, because twelve sets are unusually many, and second, because we have in mind here a readily conceivable school example in which too many repetitions would disturb the over-all clarity.

Thus we have obtained the six sets of directions, which are inserted in the following table in section I,

Computation of the Mean Error of a Direction Measurement from Six Sets with Four Target Points (Table 1)

	Set No. / Position of the Circle	1. 0°	2. 30°	3. 60°	4. 90°	5. 120°	6. 150°	Transverse Sum	Means of Directions A
I	Target Pt. P° 359° 59'	60.0"	60.0"	60.0"	60.0"	60.0"	60.0"	360.0"	60.00" = A°
	Target Pt. P' 56 04	6.0	10.5	11.0	10.0	8.5	5.5	51.5	8.58 = A'
	Target Pt. P'' 307 54	63.0	61.0	60.5	62.0	56.0	58.0	360.5	60.08 = A''
	Target Pt. P''' 345 43	40.5	45.0	43.5	38.5	40.0	41.5	249.0	41.50 = A'''
	Sums	169.5	176.5	175.0	170.5	164.5	165.0	1021.0	170.16
	Mean of the Sets B	42.38	44.12	43.75	42.62	41.12	41.25	255.25	42.54
								255.24	$= A_0$
	Shifting of Sets $A_0 - B$	+0.16	−1.58	−1.21	−0.08	+1.42	+1.29	0.00	
II = I + $(A_0 - B)$	P°	60.16	58.42	58.79	59.92	61.42	61.29	360.00	60.00 = A°
	P'	6.16	8.92	9.79	9.92	9.92	6.79	51.50	8.58 = A'
	P''	63.16	59.42	59.29	61.92	57.42	59.29	360.50	60.08 = A''
	P'''	40.66	43.42	42.29	38.42	41.42	42.79	249.00	41.50 = A'''
	Sums	170.14	170.18	170.16	170.18	170.18	170.16	1021.00	
	Mean of the Sets B'	42.54	42.54	42.54	42.54	42.54	42.54	255.24	
III	$v^\circ = A^\circ - P^\circ$	−0.16	+1.58	+1.21	+0.08	−1.42	−1.29	0.00	
	$v' = A' - P'$	+2.42	−0.34	−1.21	−1.34	−1.34	+1.79	−0.02	
	$v'' = A'' - P''$	−3.08	+0.66	+0.79	−1.84	+2.66	+0.79	−0.02	
	$v''' = A''' - P'''$	+0.84	−1.92	−0.79	+3.08	+0.08	−1.29	0.00	
	Sums	+0.02	−0.02	0.00	−0.02	−0.02	0.00	−0.04	
IV	$v^{\circ 2}$	0.03	2.50	1.46	0.01	2.02	1.66	7.68	
	v'^2	5.86	0.12	1.46	1.80	1.80	3.20	14.24	
	v''^2	9.49	0.44	0.62	3.39	7.08	0.62	21.64	
	v'''^2	0.71	3.69	0.62	9.49	0.01	1.66	16.18	
		16.09	6.75	4.16	14.69	10.91	7.14	59.74 / 59.74	$= [v\,v]$

$$m = \sqrt{\frac{[v\,v]}{(n-1)(s-1)}} = \sqrt{\frac{59.74}{(6-1)(4-1)}} = \sqrt{\frac{59.74}{15}} = \pm 2.00\,. \qquad (1)$$

with 359°59'60.0" as the starting direction to P° instead of 0.0", in order that no negative values result in the shiftings of sets.

In section I we form now, as usual, the transverse sums and the means of the directions, and for the check also the sums of sets by columns. The transverse sum, 1021.0, must agree with the vertical addition of the transverse sums of the individual sets. The means of the directions A, which result from the transverse sums by dividing by 6, contain now already all that belongs to the computation of the direction sets in itself, for we have the following result by adding the degrees and minutes:

$P^\circ = 60.00''$	Ägidius	=	0° 00′ 00.00″
$P' = 8.58$	Burg	=	56 04 08.58
$P'' = 60.08$	Steuerndieb	=	307 55 0.08
$P''' = 41.50$	Dreifaltigkeit	=	345 43 41.50 .

In order to arrive further at the computation of a mean direction error, we form also the means of sets B, in the case of which we will look at once if they are nearly *equal*, and then we form again, from these means of sets B, a mean $A_0 = 42.54$, which, compared with the individual B's, yields the set shiftings $A_0 - B$. These $A_0 - B$'s added to the direction values in section I yield the shifted directions in section II (p. 207).

If we proceed in II just as previously in I, then we do not obtain anything new at all with regard to the means of directions A, but the means of sets B' become different than before, i.e., the B''s become all *equal*, as was intended, and if we now form the errors v by comparison of the means of directions A°, A', A'', A''' with the shifted directions of section II, as has been done in III, then we will have, adding by columns and rows, everywhere the sum $= 0$, or at the most 0.02 because of rounding off. Finally, these v's are squared, as is seen in section IV, and the sum of all v^2's, adding by columns and rows, becomes $[v^2] = 59.74$. The denominator for the computation of the mean error is $= (n-1)(s-1)$, where n means the number of the sets (turns) and s the number of the directions (rays) in each set. This denominator is to be explained in such a way that $n \cdot s$ is the number of all measured directions, to which correspond, however, as unknowns n, unknown set shiftings and $(s-1)$ independent angles between s sightings so that we will have $n \cdot s - (s-1) - n = (n-1)(s-1)$. Hence, in our case $n = 6$ and $s = 4$, or $(n-1)(s-1) = 15$, with which there follows

$$m = \sqrt{\frac{59.74}{15}} = \pm 2.00'' . \qquad (2)$$

This is the mean error of a direction in a set, and by repeating the set six times, the mean error of a direction, obtained by taking the mean six times (i.e. adjusted), will be

$$M = \frac{2.00}{\sqrt{6}} = \pm 0.82''. \qquad (3)$$

If we compute the actually measured twelve sets [of (2) section 69] together, just as the six sets treated here, then we obtain corresponding to Fig. 1, section 69,

$$m = \pm 2.31'', \qquad M = \pm 0.67''. \qquad (4)$$

Instead of forming the set displacements from $A_0 - B$ and then by adding the same, producing the newly oriented sets in section II, as has been done in the preceding Table 1 (p. 207), we can also form at once the deviations between the means of directions A and the observations P, namely $d = A - P$, where in the horizontal lines the transverse sum must then be equal to zero according to the arithmetic mean, as the following Table 2 on p. 209 shows.

In these deviations d, however, there is still contained the set displacement, which we will denote here, with reference to the later treatment (section 85), in sets, by $z_1, z_2, z_3 \ldots$. This follows for each set from $z = \frac{[d]}{s}$ in agreement with the $A_0 - B$'s computed previously in Table 1, as the comparison of Table 1 of

Computation of the Mean Error of a Direction Measurement from Six Sets with Four Target Points (Table 2)

Set No. / Position of the Circle	1. 0°	2. 30°	3. 60°	4. 90°	5. 120°	6. 150°	Transverse Sum	
Section I as in Table 1, p. 207								
Section II. $d = A - P$								
P°	0.00″	0.00″	0.00″	0.00″	0.00″	0.00″	0.00	
P'	+ 2.58	− 1.92	− 2.42	− 1.42	+ 0.08	+ 3.08	− 0.02	
P''	− 2.92	− 0.92	− 0.42	− 1.92	+ 4.08	+ 2.08	− 0.02	
P'''	+ 1.00	− 3.50	− 2.00	+ 3.00	+ 1.50	0.00	0.00	
Sums $[d]$	+ 0.66	− 6.34	− 4.84	− 0.34	+ 5.66	+ 5.16	− 0.04	
Shifting of Sets $z = \frac{[d]}{4}$	+ 0.16	− 1.58	− 1.21	− 0.08	+ 1.42	+ 1.29	0.00	
from this $v = d - z$ as in Section III, Table 1								
Section III. Squares $[d]$ $[d]$	0.44	40.20	23.43	0.12	32.04	26.63	122.86	$= [[d]^2]$
							30.72	$= \frac{[[d]^2]}{4}$
Squares: $d\ d$								
P°	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
P'	6.66	3.69	5.86	2.02	0.01	9.49	27.73	
P''	8.53	0.85	0.18	3.69	16.65	4.33	34.23	
P'''	1.00	12.25	4.00	9.00	2.25	0.00	28.50	
Sums	16.19	16.79	10.04	14.71	18.91	13.82	90.46	$= [d\ d]$
							30.72	$= \frac{[[d]^2]}{4}$
							59.74	$= [v\ v]$

p. 207 with the above Table 2 shows.

If we subtract now, in sets, these z's from the deviations d, then we obtain the corrections $v = d - z$ in agreement with section III of Table 1, as is seen from the comparison of II, Table 2, with III, Table 1. The sum of the squares $[v\,v]$ can then again be computed therefrom as in IV, Table 1.

We can however find also the sum of the squares $[v\,v]$ directly from the deviations d.

From $v = d - z$ there follows for each set the following summary:

Set 1

$$s \text{ Target Points} \begin{cases} v'_1\,v'_1 = d'_1\,d'_1 - 2\,d'_1\,z_1 + z_1^2 \\ v''_1 v''_1 = d''_1 d''_1 - 2\,d''_1 z_1 + z_1^2 \\ \cdots \end{cases}$$

$$\text{Sum: } [v_1\,v_1] = [d_1\,d_1] - 2\,[d_1]\,z_1 + s\,z_1^2 = [d_1\,d_1] - 2\,\frac{[d_1]^2}{s} + \frac{[d_1]^2}{s} = [d_1\,d_1] - \frac{[d_1]^2}{s}$$

Set 2

$$\begin{cases} v'_2\,v'_2 = d'_2\,d'_2 - 2\,d'_2\,z_2 + z_2^2 \\ v''_2 v''_2 = d''_2 d''_2 - 2\,d''_2 z_2 + z_2^2 \\ \cdots \end{cases}$$

$$[v_2\,v_2] = [d_2\,d_2] - 2\,[d_2]\,z_2 + s\,z_2^2 = [d_2\,d_2] - 2\,\frac{[d_2]^2}{s} + \frac{[d_2]^2}{s} = [d_2\,d_2] - \frac{[d_2]^2}{s}.$$

and therefrom the total of squares:

$$[v\,v] = [d\,d] - \frac{[[d]^2]}{s}. \tag{5}$$

According to this, $[v\,v] = 90.46 - \frac{122.86}{4} = 59.74$ is formed in III of Table 2 in agreement with Table 1.

This latter derivation is somewhat shorter and also more convenient than that explained first by Table 1.

As for the proof of the computational method with regard to the individual v's and $[v^2]$'s, then it may first be sufficient that the v's, by columns and rows, give the sums zero throughout, so that in every respect the principle of the arithmetic mean is retained. The more rigorous proof is to be taken from the later section 83.

It is advisable to derive the mean errors in this manner, at least for a part of the stations of a net, already during the observations, in order to obtain, according to this, a judgment about the performance of the instrument and observer as well as about the equality of weight of the station results aimed at.

Note about divisional errors

If the set displacements $A_0 - B$ or z in the computation of pp. 207 and 209 show a regular course (flowing curve by plotting the $A_0 - B$'s as ordinates to the positions on the circle as abscissae), then this contains a suggestion of systematic division errors but, at the same time, also the consolation that the actual mean direction error, as it enters later into the net, will be *smaller* than the error m or, as the case may be, M after the computations (2) and (3), for a part of the amounts of the values of the v's will have its cause in the systematic division errors and, therefore, be eliminated to a great extent through the symmetric positions on the circle.

In our case, the $A_0 - B$'s or, as the case may be, z's on pp. 207 and 209, show, in fact, a regular course, and the final error $M = \pm 0.67''$ according to (4) would therefore still be too great. The net adjustment, section 70, yields in (4) $m = \pm 1.04''$, which would be comparable with the previous $M = \pm 0.67''$, if there were not added a number of other circumstances, the many centerings on high (perhaps vibrating) towers, etc.

Section 64. Approximate Computation of Incomplete Sets of Directions

If we have observations of directions only in full sets, then the adjustment consists only in forming the mean for all readings of every target point, as is shown in the previous section 63, p. 207.

If the individual sets are not all complete, then the rigorous adjustment is to be made according to our later section 83, p. 308.

But there is a suitable approximate method for the adjustment of incomplete sets of directions, which is indicated first in the work, *Ordnance trigonometrical survey of Great Britain and Ireland*, London, 1858, pp. 62-66, and is treated in Helmert, *Ausgleichsrechnung nach der Methode der kleinsten Quadrate*, 2nd Edition, 1907, p. 199.

As a numerical example for this, on p. 211, we have made an arbitrary choice from the measurements at the station Trenk of the degree-measurement in East Prussia and assumed here that at least *one* target point (Mednicken) is intersected in all sets, so that all sets can be reduced to Mednicken = $0°0'0''$. This, however, is not essential; e.g., if in the 8th set only Wargelitten or Galtgarben were measured, but not Mednicken, then we would, say, bring this set to the mean of all previous readings of Wargelitten or Galtgarben. With a little skill, we will easily be able to make a shift in such cases, so that all sets are available in a common approximate orientation at the beginning of the adjustment. It is best, of course, to take into account, immediately during the observation, as good an orientation of the sets as possible.

If we have thus arranged the observational material in Table I a, then we form in all columns the means A, and if the sets showed only very few gaps, or if it were a question only of a cursory adjustment, then we would at once retain the mean values A as results.

The further adjustment is carried out as follows:

I b. We form the differences $A - l = v$ between the means A of the first step and their l's above them, e.g.:

Fuchsberg No.	1.	$35.8'' - 36.2'' = -0.4''$
	2.	$35.8 - 37.5 = -1.7$
	3.	. . . . . .
	4.	$35.8 - 33.7 = +2.1$
	5.	$35.8 - 36.1 = -0.3$
	6.	$35.8 - 34.7 = +1.1$
	7.	$35.8 - 36.5 = -0.7$
Total:		$+3.2 - 3.1 = +0.1$; should be $= 0.0$.

The algebraic sums of these $A - l = v$'s, as is known, are equal to zero, which serves as a

Set No.	I a. Observed Directions, First Step Mednicken l^0	Fuchsberg l'	Wargelitten l''	Galtgarben l'''	x	
1	0° 0′ 0.0″	83° 30′ 36.2″	287° 14′ 11.0″	346° 24′ 18.4″	+ 0.4	carried up from I b
2	0.0	37.5	14.5	. .	— 1.1	
3	0.0	. .	12.5	18.0	+ 0.3	
4	0.0	33.7	14.1	. .	+ 0.3	
5	0.0	36.1	13.4	. .	— 0.3	
6	0.0	34.7	. .	19.6	0.0	
7	0.0	36.5	. .	. .	— 0.3	
8	0.0	. .	13.7	20.5	— 0.9	
9	0.0	. .	11.2	. .	+ 0.8	
10	0.0	. .	. .	16.5	+ 1.0	
Sums	10. 0.0″	6. 214.7″	7. 90.4″	5. 93.0″	10 + 6 + 7 + 5 = 28	
Mean A	0° 0′ 0.0″	83° 30′ 35.8″	287° 14′ 12.9″	346° 24′ 18.6″		

	I b. Differences $A - l = v$				Transverse Sum	Number q	Mean x
1	0.0″	— 0.4″	+ 1.9″	+ 0.2″	+ 1.7	4	+ 0.4″
2	0.0	— 1.7	— 1.6	. .	— 3.3	3	— 1.1
3	0.0	. .	+ 0.4	+ 0.6	+ 1.0	3	+ 0.3
4	0.0	+ 2.1	— 1.2	. .	+ 0.9	3	+ 0.3
5	0.0	— 0.3	— 0.5	. .	— 0.8	3	— 0.3
6	0.0	+ 1.1	. .	— 1.0	+ 0.1	3	0.0
7	0.0	— 0.7	. .	. .	— 0.7	2	— 0.3
8	0.0	. .	— 0.8	— 1.9	— 2.7	3	— 0.9
9	0.0	. .	+ 1.7	. .	+ 1.7	2	+ 0.8
10	0.0	. .	. .	+ 2.1	+ 2.1	2	+ 1.0
Sums	0.0″	+ 3.2″ — 3.1	+ 4.0″ — 4.1	+ 2.9″ — 2.9		28	
Theoretical Check:	0.0″ 0.0	+ 0.1″ 0.0	— 0.1″ 0.0	0.0″ 0.0			

	II a. Corrected Directions $l + x$, Second Step				
1	359° 59′ 60.4″	83° 30′ 36.6″	287° 14′ 11.4″	346° 24′ 18.8″	These values are formed by additions $l + x$ of I.
2	58.9	36.4	13.4	. .	
3	60.3	. .	12.8	18.3	
4	60.3	34.0	14.4	. .	
5	59.7	35.8	13.1	. .	
6	60.0	34.7	. .	19.6	
7	59.7	36.2	. .	. .	
8	59.1	. .	12.8	19.6	
9	60.8	. .	12.0	. .	
10	61.0	. .	. .	17.5	
Sums	10. 600.2″	6. 213.7″	7. 89.9″	5. 93.8″	
Mean B	0° 0′ 0.0″	83° 30′ 35.6″	287° 14′ 12.8″	346° 24′ 18.8″	

computational check of section II and is here in sufficient agreement with $+0.1$ instead of 0.0.

We form further, for section I b, the transverse sums and the transverse means,

e.g.
$$\frac{0.0'' - 0.4'' + 1.9'' + 0.2''}{4} = \frac{+1.7''}{4} = +0.4'' = x\,.$$

These x's of the last column of I b, which represent, therefore, a set displacement, are set up in I a in an unchanged manner.

II. The just discussed x's are added to the l's in I a, and the sums set down in II a. Here we form again, in columns, the means B, which hold as the result.

The computation can of course be carried out also in an arrangement other than on p. 211 by entering the observations as in the example of the previous section 63, p. 207, i.e., therefore interchanging the rows and columns.

In order to prove this method, we recall that the main point is always to make the differences v between the results and the observations as small as possible in the sum of their squares $[v\,v]$. In this case, we can now imagine the total $[v\,v]$ split up in a twofold manner:

1. Splitting up by columns $\quad [v\,v] = [v^0 v^0] + [v'\,v'] + [v''v''] + \ldots \quad (1)$
2. Splitting up by rows $\quad [v\,v] = [v_1 v_1] + [v_2 v_2] + [v_3 v_3] + [v_4 v_4] + \ldots \quad (2)$

By forming the means of columns A, the sum $[v^0 v^0]$ or, as the case may be, $[v'v']$ or $[v''v'']$ is made as small as possible independently in each column, and in the individual rows, $[v_1 v_1]$ or, as the case may be, $[v_2 v_2]$, etc., is then further decreased by applying the set displacements x_1, x_2, etc., which are the arithmetic means of the v's of each line.

The latter shows most clearly in that column, which has served at the beginning for the main orientation, in our example in the column Mednicken; this receives all $v = 0$'s in Table I b, i.e., all discrepancies are thereby pushed to the other columns and not until after the introduction of the displacements x does a fairer distribution of the v's set in.

The *gradual* decrease of $[v\,v]$, first in the columns, then in the rows, corresponds approximately to the method of least squares, which, if rigorously applied, would require the *simultaneous* taking into account of all relations in the columns and in the rows.

The numerical values of Table II a still have the character of *original* observations, for only the readings of each set have been displaced by a *constant* quantity x, and in the case of sets of observations of directions, such a displacement is always arbitrarily admissible.

Therefore, beginning anew with Table II a, we can repeat the whole computation, form a third step III, and it is possible that we come unlimitedly close to the results of a rigorous adjustment according to the method of least squares by continued repetitions of this kind. In practical application, the amount of the x's in comparison to the measuring error to be expected indicates when the computation can be finished.

We can also calculate here, for each step of adjustment, the sum of the squares $[v^2]$ of the corrections and see by its gradual decrease if the course of computation converges. We have carried this through to the third step in our example on p. 211 and obtained the following:

Station Trenk

	Step I	Step II	Step III
Mednicken	0° 0′ 0.0″	0° 0′ 0.0″	0° 0′ 0.00″
Fuchsberg	83 30 35.8	83 30 35.6	83 30 35.55
Wargelitten	287 14 12.9	287 14 12.8	287 14 12.67
Galtgarben	346 24 18.6	346 24 18.8	346 24 18.81
	$[v^2] = 30.22$	$[v^2] = 18.42$	$[v^2] = 17.97$

From the IInd to the IIIrd step there is hardly still a noticeable change. From the sum $[v^2]$ we can also compute a mean error of a direction, namely

$$m = \sqrt{\frac{17.97}{28-13}} = \sqrt{\frac{17.97}{15}} = \pm 1.09' \,.$$

The denominator 28 – 13 has resulted here from the fact that there are 28 measured directions, to which, however, first correspond as unknowns three independent angles between four sightings and ten unknowns of orientation in ten measured sets, hence 13 unknowns.

The approximation adjustment treated in the foregoing is advisable for many reasons for triangulations of IInd to IVth order, especially if we bear in mind in the measurement that a sufficient connection between the sets (or groups of sets) is obtained. Even if at stations with several eccentric base points the centered directions are collected as a uniform set of directions, this method can be applied if it is not preferable to introduce the data observed at different base points separately into the adjustment (cf. section 69).

If the observations l in the individual sets (for instance, as means of groups of sets) are not equal weighted, then the same computational procedure can be maintained, except that the means of directions are to be formed according to the rules of the general arithmetic mean of section 8. The rigorous adjustment of incomplete sets of directions, which we shall teach in the later section 83, with the pertinent, formally rigorous net adjustment (section 84) is hardly still in practical use today. We shall treat newer methods of angle measurement for triangulations in the later sections 85-88.

The station results adjusted according to the preceding sections 62-64 are compiled in Abrisse [station data collected according to a systematic way] for further adjustment, as is already indicated in Volume II, 1st half volume, 9th Edition, 1931, section 91.

Section 65. Condition Equations in a Central System

Before we look for the conditions which exist between the measured angles or directions for nets of triangles in general, we will treat the especially important case of a *central system*, which occurs in all nets of triangles.

We mean by a central system an arbitrarily closed polygon, all corner points of which are connected with a common central point, so that a system of triangles results, which have a corner point, the *central point* of the system, in common. In Fig. 1 is represented the simplest case of such a central system, in which the corner points of a triangle ABC are connected with a central point D. Hence, we have to deal with three triangles, in which the nine angles 1, 2, 3, . . ., 9 are assumed to be measured independently of each other. We shall consider later the case of direction measurements, in which three directions would have to be measured at each of the four points in Fig. 1.

It follows from Fig. 1 that four angles would be sufficient for the construction of the three triangles, if one side is assumed as given; e.g., if the side AC is given, the whole figure could be constructed by means of the four angles 6, 1, 2, 3. Hence, five angles are excessive, and five condition equations independent of each other must correspond to these.

If we look for these conditions, then we see first that there exist three closed triangles, for which we obtain:

Triangle ABD (5) + (6) + (7) = 180° (1)

Triangle BCD (3) + (4) + (9) = 180° (2)

Triangle ACD (1) + (2) + (8) = 180° . (3)

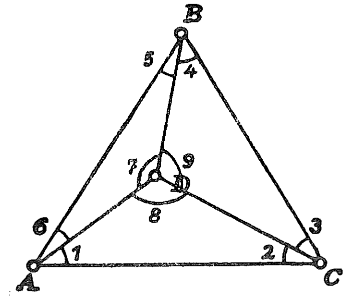


Fig. 1.

Further, there exists the condition that the three angles lying around the point D must yield the sum 360°, hence

$$(7) + (8) + (9) = 360^\circ . \tag{4}$$

We could think further of setting up the condition

$$(1) + (2) + (3) + (4) + (5) + (6) = 180^\circ \tag{5}$$

for the large triangle A, B, C. However, we see at once that this condition is not independent of equations (1) to (4); for if we form the sum of equations (1) to (3) and subtract equation (4) from this, then equation (5) results. This equation (5) is therefore no longer considered as an independent condition equation for the adjustment.

Hence, since we have found only four independent equations of sums of angles, but five condition equations must exist altogether, then we have to find, in addition, a fifth equation, which can be easily indicated by the construction according to Fig. 2.

For although the three triangles, in their sum of angles, conclude to 180° each, and the three angles around the point D yield the sum 360°, it is not yet insured with this that the three sides AB, BC, and CA yield a closed triangle. If we imagine, starting from AC, the triangle ADC constructed by means of two of its angles, then the triangle DCB adjoined to DC likewise with the use of two of its angles and, finally, the angle (5) laid off at B, then, as a rule, instead of the triangle ABD another triangle $A'BD$ will result. We see that in the triangles ADC, CDB, and $A'DB$ the sum of angles is equal to 180°, as well as at point D, the three angles (7), (8), and (9) yielding the sum 360°, but that these four conditions are not sufficient in order to avoid the deviation of point A' from A.

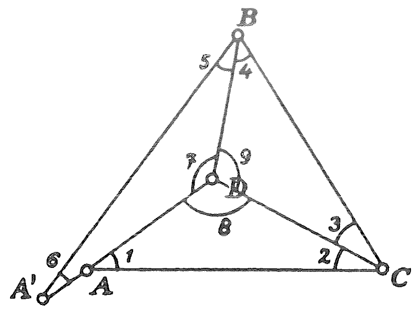


Fig. 2.

In order to express that the two points A and A' coincide, we can set up the requirement $DA = DA'$, and this can be brought into the form of a condition equation between the measured angles, if we compute $A'D$ from AD. We have

$$DC = DA\frac{\sin(1)}{\sin(2)}, \qquad DB = DC\frac{\sin(3)}{\sin(4)}, \qquad DA' = DB\frac{\sin(5)}{\sin(6)}$$

or

$$DA\frac{\sin(1)\,\sin(3)\,\sin(5)}{\sin(2)\,\sin(4)\,\sin(6)} = DA',$$

and since we are to have $DA' = DA$, then the condition equation reads

$$\frac{\sin(1)\,\sin(3)\,\sin(5)}{\sin(2)\,\sin(4)\,\sin(6)} = 1\,. \tag{6}$$

This is the fifth condition equation which was missing for Fig. 1, p. 213, which we call, due to its origin, a *side equation*.

We can express the requirement that $DA' = DA$ also by the identity

$$\frac{DA}{DC}\cdot\frac{DC}{DB}\cdot\frac{DB}{DA} = 1\,,$$

in which we replace each of the three quotients by the ratio of the sines of the corresponding angles. This yields

$$\frac{\sin(2)}{\sin(1)}\cdot\frac{\sin(4)}{\sin(3)}\cdot\frac{\sin(6)}{\sin(5)} = 1\,,$$

which agrees again with (6).

We have determined here the side equation for the simplest case of a central system. But we understand readily that such a side equation exists for any arbitrary central system; e.g., if in Fig. 3 there is a hexagon whose corner points are connected with the central point G, then the side equation can be set up in the same manner as above by starting from the identity

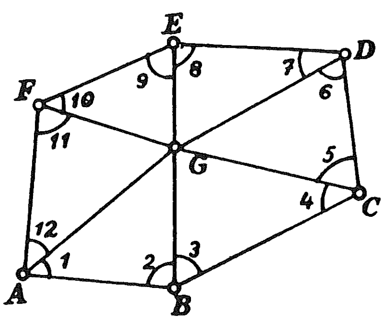


Fig. 3.

$$\frac{GA}{GB}\cdot\frac{GB}{GC}\cdot\frac{GC}{GD}\cdot\frac{GD}{GE}\cdot\frac{GE}{GF}\cdot\frac{GF}{GA} = 1\,.$$

We obtain therefrom at once

$$\frac{\sin(2)}{\sin(1)}\cdot\frac{\sin(4)}{\sin(3)}\cdot\frac{\sin(6)}{\sin(5)}\cdot\frac{\sin(8)}{\sin(7)}\cdot\frac{\sin(10)}{\sin(9)}\cdot\frac{\sin(12)}{\sin(11)} = 1\,. \tag{7}$$

If it is assumed in Fig. 3 that only the denoted 12 angles are measured, then neither an equation of sums of angles in the individual triangles nor one around the central point G exists. On the other hand, we are now to set up the sum of the angles in the hexagon, hence we have the condition

$$(1) + (2) + (3) + \dots + (12) \times 180^\circ . \qquad (8)$$

Hence, for the central system of Fig. 3 with the 12 measured angles there exist the two condition equations (7) and (8). That only two condition equations can exist follows also readily from the theory that for the construction of the whole point system 10 angles, e.g., the angles (1), (2), . . . , (10), are required so that two excessive measurements therefore exist.

If in Fig. 3 the angles of the triangles lying around the central point G were measured also, then eight excessive measurements would exist in all; hence, in addition to the side equation, seven equations of sums of angles would have to be set up. In the later section 67 we will occupy ourselves more thoroughly with all these questions which concern the number of the condition equations in a net of triangles.

But, in addition, we will now indicate the condition equations for the simple central system of Fig. 1, p. 213, completely by introducing the corrections of the observed angles.

Let the symbols (1), (2), etc., in the equations hitherto mentioned denote the angles in general, and, e.g., equation (1), p. 213, says that the sum (5) + (6) + (7) shall yield the amount 180°, which is not the case with observed angles because of the unavoidable observational errors. We shall therefore distinguish between observed angles (1), (2), (3) . . . , adjusted angles [1] · [2] · [3] . . . , and angle corrections $\delta_1 \cdot \delta_2 \cdot \delta_3 \dots$ in this manner:

$$(1) + \delta_1 = [1] , \qquad (2) + \delta_2 = [2] , \qquad (3) + \delta_3 = [3] \dots . \qquad (9)$$

The angular corrections are called δ here, because the symbol v otherwise used by us for observational corrections shall be reserved for the corrections to *directions* following afterwards.

With this, our equations (1) through (4) and (6), pp. 213 and 214 obtain then the form

$$\left.\begin{aligned}
&(5) + \delta_5 + (6) + \delta_6 + (7) + \delta_7 = 180^\circ \\
&(3) + \delta_3 + (4) + \delta_4 + (9) + \delta_9 = 180^\circ \\
&(1) + \delta_1 + (2) + \delta_2 + (8) + \delta_8 = 180^\circ \\
&(7) + \delta_7 + (8) + \delta_8 + (9) + \delta_9 = 360^\circ \\
&\frac{\sin\big((1)+\delta_1\big)\sin\big((3)+\delta_3\big)\sin\big((5)+\delta_5\big)}{\sin\big((2)+\delta_2\big)\sin\big((4)+\delta_4\big)\sin\big((6)+\delta_6\big)} = 1
\end{aligned}\right\} . \qquad (10)$$

While the first four equations have already a linear form, and hence, correspond directly to the general condition equations (4), section 47, p.147, the last condition equation (10), the side equation, is not linear with respect to the corrections δ and needs a special transformation, which we will show by an example in the following section 66, p. 216.

Observations of directions

Until now we have assumed that we have pure *angle* measurements, i.e., in each set of measurements only *two* sights so that in Fig. 1, p.213, the nine angles (1), (2), . . . , (9) are measured individually.

Instead of this, we will now assume in Fig. 4, which represents the same central system as Fig. 1, p.213, that at each of the four points a full set of directions is measured, hence, e.g., at A the three sights AB, AD, AC are combined into a set.

Then, instead of nine angles we have twelve directions, as is written in Fig. 4 at the arrow points.

We can now change easily from the angle corrections δ of the condition equations (10) to the corrections to the directions, which we will denote by v. If we introduce further the denotations l_1, l_2 . . . for the observed

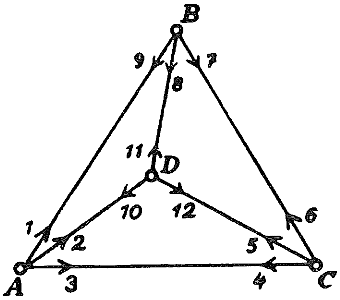


Fig. 4.

directions, then, by comparison of Fig. 1, p. 213, with Fig. 4 we have

$$(1) + \delta_1 = l_3 + v_3 - l_2 - v_2$$

$$(2) + \delta_2 = l_5 + v_5 - l_4 - v_4 \, .$$

$$\cdots\cdots\cdots\cdots$$

With this, we can easily reverse the above five condition equations (10) to the corrections to the directions. But it is obvious at once that the fourth equation (10), which comprises the angles lying around the central point D, vanishes, which is clear without further consideration, since no condition equation can exist between the three directions l_{10}, l_{11}, l_{12} of Fig. 4. The remaining four equations assume then the form

$$\left.\begin{aligned} & l_9 + v_9 - l_8 - v_8 + l_2 + v_2 - l_1 - v_1 + l_{11} + v_{11} - l_{10} - v_{10} = 180^\circ \\ & l_6 + v_6 - l_5 - v_5 + l_8 + v_8 - l_7 - v_7 + l_{12} + v_{12} - l_{11} - v_{11} = 180^\circ \\ & l_3 + v_3 - l_2 - v_2 + l_5 + v_5 - l_4 - v_4 + l_{10} + v_{10} - l_{12} - v_{12} = 180^\circ \\ & \frac{\sin(l_3 + v_3 - l_2 - v_2)\sin(l_6 + v_6 - l_5 - v_5)\sin(l_9 + v_9 - l_8 - v_8)}{\sin(l_5 + v_5 - l_4 - v_4)\sin(l_8 + v_8 - l_7 - v_7)\sin(l_2 + v_2 - l_1 - v_1)} = 1 \end{aligned}\right\} . \tag{11}$$

These are the four condition equations upon which the adjustment of the central system Fig. 4 with direction measurements is to be based.

Section 66. Condition Equations in the Quadrilateral

For the system of triangles represented in Fig. 1, p. 213, equation (6), p. 214, does not constitute the only form of side equation; e.g., the triangle ABD can also be regarded as the basic triangle and point C as the central point, where the side equation obtains the form

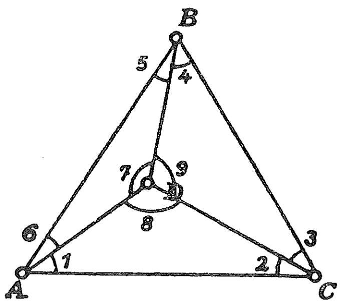


Fig. 1.

$$\frac{CA}{CB} \cdot \frac{CB}{CD} \cdot \frac{CD}{CA} = 1 \, .$$

If we replace, according to Fig. 1, the three quotients by the three corresponding sine conditions, then we will have

$$\frac{\sin(4 + 5)}{\sin(1 + 6)}, \quad \frac{\sin(9)}{\sin(4)}, \quad \frac{\sin(1)}{\sin(8)} = 1 \, . \tag{1}$$

In the same manner, we can regard A as the central point for the triangle BCD or B as the central point for the triangle ACD, with which we obtain two additional forms of the side equation for Fig. 1. The side equation can therefore be set up in four different forms for the system of triangles of Fig. 1. We will in this connection note at once that in general it is not immaterial which one of these four side equations is introduced into the adjustment; however, we will not until later, in section 77, come to this subject of discussion.

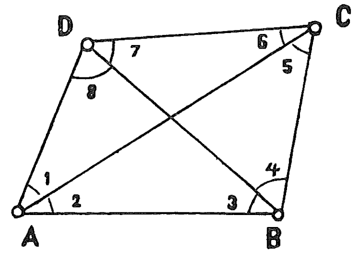


Fig. 2.

Now we consider, in Fig. 2, a quadrilateral $ABCD$ with the two diagonals, in which the eight individual angles denoted by 1, 2, 3, . . ., 8 are measured. Here, also, four angles are required for the unique construction of the point system; therefore, there exist four excessive measurements, which lead to four condition equations.

First we see that there are four closed triangles, for which the following equations result:

$$\text{Triangle } ABD \quad (1) + (2) + (3) + (8) = 180^\circ \tag{2}$$

$$\text{Triangle } BCD \quad (4) + (5) + (6) + (7) = 180^\circ \tag{3}$$

$$\text{Triangle } ADC \quad (1) + (6) + (7) + (8) = 180^\circ \tag{4}$$

$$\text{Triangle } ABC \quad (2) + (3) + (4) + (5) = 180^\circ . \tag{5}$$

These four equations, however, are not independent of one another; e.g., if we add (2) and (3) and subtract (4), then we obtain (5), or more generally, if three triangles close, then the fourth closes by itself.

Likewise, the equation

$$(1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) = 360^\circ , \tag{6}$$

which refers to the whole quadrilateral $ABCD$, can also no longer be regarded as an independent equation. Of the five equations (2) to (6) we can therefore use only three for the adjustment.

The fourth condition equation still missing is a side equation, since the quadrilateral, Fig. 2, is to be considered, likewise, as a central system. If we take the triangle ADC as the basic polygon, then we can regard point B as the central point, which, this time, lies outside the polygon. The side equation reads then

$$\frac{BD}{BA} \cdot \frac{BA}{BC} \cdot \frac{BC}{BD} = 1$$

or

$$\frac{\sin(1+2)\,\sin(5)\,\sin(7)}{\sin(8)\,\sin(2)\,\sin(5+6)} = 1 . \tag{7}$$

We have already found the side equation for Fig. 1 in four different forms. The same also holds for the quadrilateral of Fig. 2, in which each of the four corners can be used as the central point; e.g., for central point A we have the equation

$$\frac{AC}{AD} \cdot \frac{AD}{AB} \cdot \frac{AB}{AC} = 1$$

or

$$\frac{\sin(7+8)\,\sin(3)\,\sin(5)}{\sin(6)\,\sin(8)\,\sin(3+4)} = 1 . \tag{8}$$

Later we shall see that there are even seven different forms of side equations in a quadrilateral; meanwhile, the equations hitherto found are sufficient to pass over to a practical application.

As a numerical example we will assume, to Fig. 2, the following eight angles as observed, which we group according to the four triangles. We determine here, at the same time, the deviation of the sum of the angles from 180° or the discrepancy w for each triangle.

$$\left.\begin{array}{ll}
\begin{array}{rrrrl}
(1) = & 37^\circ & 26' & 41'' & \rbrace\ 72^\circ\,2'\ 6'' \\
(2) = & 34 & 35 & 25 & \\
(3) = & 41 & 17 & 34 & \\
(8) = & 66 & 40 & 24 & \\
\hline
 & 180^\circ & 00' & 04'' & \\
 & & w = & +4'' & \\
(4) = & 55^\circ & 58' & 2'' & \\
(5) = & 48 & 9 & 2 & \rbrace\ 78^\circ\,0'\,28'' \\
(6) = & 29 & 51 & 26 & \\
(7) = & 46 & 1 & 27 & \\
\hline
 & 179^\circ & 59' & 57'' & \\
 & & w = & -3'' &
\end{array}
&
\begin{array}{rrrrl}
(2) = & 34^\circ & 35' & 25'' & \\
(3) = & 41 & 17 & 34 & \rbrace\ 97^\circ\,15'\,36'' \\
(4) = & 55 & 58 & 2 & \\
(5) = & 48 & 9 & 2 & \\
\hline
 & 180^\circ & 00' & 03'' & \\
 & & w = & +3'' & \\
(1) = & 37^\circ & 26' & 41'' & \\
(8) = & 66 & 40 & 24 & \rbrace\ 112^\circ\,41'\,51'' \\
(7) = & 46 & 1 & 27 & \\
(6) = & 29 & 51 & 26 & \\
\hline
 & 179^\circ & 59' & 58'' & \\
 & & w = & -2'' &
\end{array}
\end{array}\right\rbrace . \tag{9}$$

The following four condition equations correspond to these four triangles:

$$\delta_1 + \delta_2 + \delta_3 + \delta_8 + 4'' = 0 \tag{10}$$

$$\delta_2 + \delta_3 + \delta_4 + \delta_5 + 3'' = 0 \tag{11}$$

$$\delta_4 + \delta_5 + \delta_6 + \delta_7 - 3'' = 0 \tag{12}$$

$$(\delta_1 + \delta_6 + \delta_7 + \delta_8 - 2'' = 0)\,. \tag{13}$$

Of these, as we have already seen at the beginning, however, only three equations are independent (which we have indicated by putting the last one (13) within parentheses).

According to this, we must also bring the side equation (7) into linear form and calculate, for this, first logarithmically:

	log sin	Diff. for 10″
(5) = 48° 09′ 02″	9.872 0983	189
(7) = 46 01 27	9.857 1109	203
(1 + 2) = 72 02 06	9.978 2924	69
	9.707 5016	
(8) = 66° 40′ 24″	9.962 9667	90
(2) = 34 35 25	9.754 1220	306
(5 + 6) = 78 00 28	9.990 4169	45
	9.707 5056	

(14)

Check: 9.707 5016 — 9.707 5056 = — 0.000 0040

or in units of the 7th place: $w = -40\,.$ (15)

In order to produce a relation between this logarithmic discrepancy and the angle corrections δ, in the logarithmic computation (14) we have noted the logarithmic differences for 10″ from the logarithmic-trigonometric table thereby used, e.g., 189 for the angle (5) = 48°9′. We thus know that a change of this angle by 1″ yields the amount 18.9 units of the last place, or a change of angle $= \delta_5$ seconds will bring 18.9 δ_5 last units. The same is true of the other angles, and therefore we soon come to the conclusion that the changes of angles δ_5, δ_7, $\delta_1 + \delta_2$ . . . must bring together the discrepancy $w = -40$ in the following manner:

$$+18.9\delta_5 + 20.3\delta_7 + 6.9(\delta_1 + \delta_2) - 9.0\delta_8 - 30.6\delta_2 - 4.5(\delta_5 + \delta_6) - 40 = 0\,. \tag{16}$$

We have computed here in units of the 7th place of logarithms, something which is not essential, however; if we compute in units of the 6th place, then we obtain

$$+1.89\delta_5 + 2.03\delta_7 + 0.69(\delta_1 + \delta_2) - 0.90\delta_8 - 3.06\delta_2 - 0.45(\delta_5 + \delta_6) - 4.0 = 0\,. \tag{17}$$

If we collect and arrange this, then we obtain

$$+0.69\delta_1 - 2.37\delta_2 + 1.44\delta_5 - 0.45\delta_6 + 2.03\delta_7 - 0.90\delta_8 - 4.0 = 0\,. \tag{18}$$

We can in general multiply or divide such an equation by any arbitrary number without changing its material significance for the adjustment, and for formal reasons it is very useful to make the coefficients of the side equations as nearly as possible = 1, because the coefficients of sum equations in (10) to (13) are all also = 1 .

Computational check for the absolute term of the side equation

We do not want to leave the logarithmic-trigonometric computation (14) without a check, because an error in the absolute term w would propagate itself through the whole computation and would appear unpleasantly only at the end. For *spherical* triangles, we have here, apart from the direct double computation according to the same method, the desirable check that the side equation holds for the direct angles of the triangle as well as for the angles of the triangle decreased each by one-third of the excess according to Legendre's theorem. Since only a *plane* quadrilateral is assumed in our case (9) to (14), we cannot show this check here, but we reserve the right of treating it later (section 68).

Theoretical computation of the coefficients of the side equation

Although the coefficients 18.9, 20.3, and so forth, of the linear side equation (16) have resulted directly, empirically, as logarithmic sine differences, it is not superfluous to compute these coefficients also theoretically.

For this, we will assume, somewhat more generally, a side equation thus:

$$\frac{\sin(1)\,\sin(3)\,\ldots}{\sin(2)\,\sin(4)\,\ldots} = 1\,, \tag{19}$$

or in logarithmic form:

$$\log\sin(1) - \log\sin(2) + \log\sin(3) - \log\sin(4)\,\ldots = 0\,.$$

This *should* be, but in general is not satisfied, but yields a logarithmic discrepancy w:

$$\log\sin(1) - \log\sin(2) + \log\sin(3) - \log\sin(4)\,\ldots = w\,. \tag{20}$$

The discrepancy w is to be eliminated by adding the corrections $\delta_1, \delta_2 \ldots$ to the observed angles (1), (2) . . ., i.e. we shall have

$$\log\sin\big((1)+\delta_1\big) - \log\sin\big((2)+\delta_2\big) + \ldots = 0\,.$$

For this, we find according to Taylor's theorem

$$\log\sin\big((1)+\delta_1\big) = \log\sin(1) + \mu\cot(1)\,\frac{\delta_1}{\varrho}\,. \tag{21}$$

Here $\mu = 0.43429\ldots$ means the logarithmic modulus, and the denominator $\rho = 206\,265''$ is inserted in order that δ_1 in the last term can be calculated in seconds. By calculation we find

$$\log\frac{\mu}{\varrho} = 4.32336 - 10\,, \qquad \frac{\mu}{\varrho} = 0.0000021{\cdot}055\,,$$

or

$$\left.\begin{aligned} \log\frac{\mu}{\varrho} &= 1.32336 & \frac{\mu}{\varrho} &= 21.055 \text{ for the 7th log place} \\ \log\frac{\mu}{\varrho} &= 0.32336 & \frac{\mu}{\varrho} &= 2.1055 \text{ for the 6th log place} \end{aligned}\right\}\,. \tag{22}$$

We will denote the coefficients of δ_1 in (21) briefly by a_1 and have then

$$\log\sin\big((1)+\delta_1\big) - \log\sin(1) = a_1\,\delta_1 \quad \text{with} \quad a_1 = \frac{\mu}{\varrho}\cot(1)\,. \tag{23}$$

If we take the calculation (14) as an example, then we have

$$\begin{aligned} (5) = 48^\circ\, 09'\, 02'' \qquad & \log \cot\ (5) = 9.95214 \\ & \log\,(\mu : \varrho) \ = 1.32336 \quad \text{for 7th place in the logarithm} \\ \hline & \log a_5 = 1.27550 \\ & a_5 = 18.858 \quad \text{for 7th place in the logarithm.} \end{aligned}$$

This is in sufficient agreement with 18.9 in (16).
If we calculate all coefficients in such a manner, then we obtain instead of (16)

$$+ 18.858\,\delta_5 + 20.222\,\delta_7 + 6.827(\delta_1 + \delta_2) - 9.079\,\delta_8 - 30.532\,\delta_2 - 4.472(\delta_5 + \delta_6) - 40.0 = 0\,. \tag{24}$$

Obviously, it is superfluous, strictly speaking, to compute the coefficients in such a manner more accurately than they result from that logarithmic table with which we compute the discrepancy w, or with which we intend to compute later the triangulation net itself; however, since we often write the coefficients at the beginning with more places than we finally need because of the many rounding errors accumulating in the course of the long computation, in the case of very extended nets we might calculate the coefficients according to (23) rather than fill the additional places in with zeros.

Observations of directions

In order to show further the case of observations of directions by our numerical example, we will assume that instead of the eight angles the twelve directions indicated in Fig. 3 are measured.

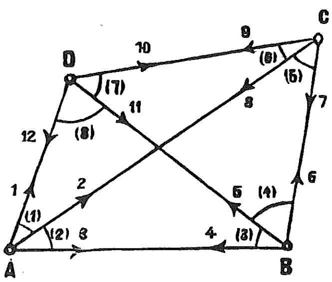


Fig. 3.

We have then to solve, according to section 65, p. 213, the angle corrections δ into two corrections to the directions v each, hence obtain

$$\delta_1 = v_2 - v_1, \qquad \delta_2 = v_3 - v_2, \qquad \delta_1 + \delta_2 = v_3 - v_1, \text{ and so on.}$$

In this manner, the previous equations (10) to (12) and (17) change to the following:

$$\begin{aligned} (v_3 - v_1) + (v_5 - v_4) + (v_{12} - v_{11}) + 4'' &= 0 \\ (v_3 - v_2) + (v_6 - v_4) + (v_8 - v_7) + 3 &= 0 \\ (v_6 - v_5) + (v_9 - v_7) + (v_{11} - v_{10}) - 3 &= 0 \end{aligned}$$

$$1.89\,(v_8 - v_7) + 2.03\,(v_{11} - v_{10}) + 0.69\,(v_3 - v_1) - 0.90\,(v_{12} - v_{11}) - 3.06\,(v_3 - v_2) - 0.45\,(v_9 - v_7) - 4.0 = 0\,.$$

Arranged according to the numbers and, as the case may be, collected, this yields:

$$- v_1 + v_3 - v_4 + v_5 - v_{11} + v_{12} + 4'' = 0 \tag{25}$$

$$- v_2 + v_3 - v_4 + v_6 - v_7 + v_8 + 3 = 0 \tag{26}$$

$$- v_5 + v_6 - v_7 + v_9 - v_{10} + v_{11} - 3 = 0 \tag{27}$$

$$- 0.69\,v_1 + 3.06\,v_2 - 2.37\,v_3 - 1.44\,v_7 + 1.89\,v_8 - 0.45\,v_9 - 2.03\,v_{10} + 2.93\,v_{11} - 0.90\,v_{12} - 4.0 = 0\,. \tag{28}$$

The adjustment of the direction of the quadrilateral (Fig. 3) is to be based on these four equations (25) to (28).

In each of these equations (25) to (28) the algebraic sum of all coefficients is equal to zero, namely in (25) to (27): $3 - 3 = 0$, and in (28): $+7.88 - 7.88 = 0$. This is not accidental, but always occurs in the case of condition equations for direction measurements, as is easily seen from the origin of the equations. The corrections to the directions v themselves, added by stations, also yield always, in the case of equal weights, the sums zero, as will be shown more thoroughly at the end of the next section 67.

Section 67. General Consideration of the Condition Equations

With what was taught in the preceding sections 65 and 66 about condition equations, we can already go to the adjustment of small nets, as in section 68 or 70; meanwhile, some further details are to be said about condition equations in nets of triangles. First we will consider the *number* of the independent condition equations.

We consider a net of triangles, in which *one* base line is measured, with the following geometric relations:

$$\left.\begin{array}{ll} \text{Number of the corner points} \dots\dots\dots\dots & p \\ \text{Number of all connecting lines} \dots\dots\dots\dots & l \\ \text{Number of the connecting lines observed only on } one \text{ side} \dots & l' \\ \text{Hence, number of the connecting lines observed on both sides} \dots & l - l' \end{array}\right\} . \qquad (1)$$

According to the angle measurements or direction measurements available let

$$\text{the number of the measurements be} \dots\dots\dots\dots W , \qquad (2)$$

or

$$\text{the number of the direction measurements be} \dots\dots R . \qquad (3)$$

The direction measurements shall be carried out here for each station in a full set.

First, we will now assume that we have angle measurements, and later treat the case of the direction measurements.

I. *Angle measurements*

The number of the independent condition equations is equal to the number of the excessive observations, for each excessive observation yields an independent measuring check. We denote:

$$\text{the number of the independent condition equations by} \dots\dots\dots\dots r . \qquad (4)$$

We find the number of the necessary angles thus: With the assumption of *one* base, the first two points, namely the two base end points, are fixed without angle measurements, and each additional point needs two angles for the attachment to the given points, hence:

	a base with two points requires	0	angles
$(p-2)$ points	the third point requires	2	angles
	the fourth point requires	2	angles
	. . . .		
	the pth point requires	2	angles
	Total: $0 + (p-2)\,2 = 2p - 4$		angles .

Therefore, if W angles are measured, then $W - (2p - 4)$ are excessive, or we have

$$r = W - 2p + 4 \text{ condition equations .} \qquad (5)$$

These condition equations are divided into two essentially different kinds, namely angle equations and side equations.

The angle equations are divided again into two different kinds, namely: (1) triangle equations or, more generally, polygon equations, and (2) horizontal equations or station equations.

The first kind of the angle equations is concerned, first, with the sums $(1) + (2) + (3) = 180^\circ$ or, as the case may be, $180^\circ +$ spherical excess, in the individual triangles, but then, also, with sums of quadrilaterals $(1) + (2) + (3) + (4) = 360^\circ$, and more generally, $(1) + (2) + \ldots (n) = (n - 2)\,180^\circ$, and so forth.

If we try to find the number of these equations, we notice first that lines observed on *one* side are not considered here; e.g., in a triangle, in which measurements are made only at two corners, we have only two angles, consequently no sum check, and so forth.

It follows hence further that such *points* which are connected with other points only by lines observed on one side, i.e., either only the intersected or only the resected points, also are not considered in counting the angle equations.

The number of the independent triangle equations is in general *not* simply equal to the number of the triangles themselves, even if the sides observed on one side are eliminated, for a triangle equation can already be contained in the sum or difference of other such equations, as we have already seen in the preceding section 66 in the case of (2) to (5), p. 217. It also has become evident there that equation (6) of a quadrilateral in connection with two triangle equations can express the same thing as three individual triangle equations.

If we have convinced ourselves according to this example that every polygon equation has the same use as a triangle equation inasmuch as it is possible to set up the polygon equations in question in various ways, then we also come to the determination of the number of independent polygon equations in the following manner: We imagine a closed path laid over all p points. This path also comprises p lines. If we have l lines, then $l - p$ lines are not yet hit by the path, and each of these lines yields a new equation. Hence, a polygon path with p lines yields one equation, the remaining $l - p$ lines yield $l - p$ equations; we thus have in all

$$1 + l - p \text{ triangle equations (in general, polygon equations)} . \qquad (6)$$

This holds if the l lines are all measured back and forth; if among them there are l' lines measured on one side, then we have instead of (6)

$$1 + (l - l') - p \text{ triangle equations or polygon equations} . \qquad (7)$$

But here, too, points which may be only intersected or only resected are not to be counted among the p points.

As examples of horizon equations, we have in Fig. 1, section 65, the horizon around D, as well as in Fig. 2, p. 214, following, the two horizons around Oggersheim and around Speyer; at Speyer there are measured the five individual angles (12), (15), (16), (19), (22), from which there results the check

$$(12) + (15) + (16) + (19) + (22) = 360^\circ ,$$

and similarly in the case of Oggersheim.

Or if, e.g., in the same net (Fig. 2, p. 214) at Donnersberg there were measured, in addition to the two individual angles (1) and (27), also the summation angle Klobberg-Calmit = (28), then we would further have the condition

$$(1) + (27) - (28) = 0 .$$

More generally, we can say about horizon equations: At a station with s rays, $s - 1$ angles are inevitably necessary for the fixing of these rays; if more angles, say n angles, are measured, then we have at this station

$$n - s + 1 \text{ station equations} . \qquad (8)$$

In the case of large adjustments, this kind of equations in most cases does not occur in the net, since we treat them independently at the individual stations. (In the case of direction observations in whole sets there are no station equations at all.)

Before we determine the number of station equations in the triangle net, we occupy ourselves with the side equations.

The number of the side equations becomes clear in the best way from the following consideration: We distinguish "triangle net" and "triangle chain" by calling a triangle chain such a joining of triangles, in the case of which we can arrive from any triangle at another one only by *one* definite way, whereas in the case of a triangle net different ways are possible. According to this interpretation, a quadrilateral with a diagonal is a chain, a quadrilateral with two diagonals is a net. Whether the sides intersect or not is not decisive; if the sides intersect, we have in any case a "net," but there are also "nets" without intersections of diagonals, e.g., with central systems; cf. the following Fig. 2, p. 225.

If we measure *one* additional angle in a chain, whereby we insert a sight which makes the chain into a net, then there results a new condition equation in so far as we can now express the length of the newly inserted side in a twofold way. Each side not necessary for the connection of the chain yields such a side equation. A triangle chain has no side equations; hence, there follows the rule:

The number of the side equations of a triangle net is equal to the number of sides which must be canceled in order that the net changes into a chain.

In order to form a formula for this, we consider that for the connection of the first three points, three lines and for each following point two further lines are needed, therefore:

	for the first three points	3 lines
$(p-3)$ points	for the fourth point.	2 lines
	for the fifth point.	2 lines
	. . . .	
	for the pth point	2 lines
	Sum:	$(3 + (p-3)2)$ lines $= (2p-3)$ lines .

If we have l lines, then of them $l - (2p - 3)$ are excessive; consequently, there exist

$$l - 2p + 3 \text{ side equations .} \tag{9}$$

Since in the preceding section 66 side equations in a quadrilateral have already been treated, we will take a further case for further explanation:

In Fig. 1 let the indicated five angles be measured. The above formula (9) yields here with $l = 6$ and $p = 4$:

$$6 - 8 + 3 = 1 \text{ side equation .}$$

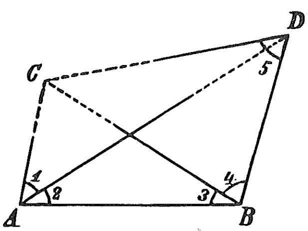


Fig. 1.

Although none of the four triangles of Fig. 1 is closed with respect to 180°, the quadrilateral has a check; for we can derive the side CD from AB in a twofold way:

$$1.\quad CD = AB \frac{sin\,(1+2)}{sin\,(1+2+3)} \frac{sin\,(4)}{sin\,(5)}$$

$$2.\quad CD = AB \frac{sin\,(2)}{sin\,(2+3+4)} \frac{sin\,(4)}{sin\,(4+5)} .$$

If we set these two expressions equal to one another, then we obtain the equation of a side condition:

$$\frac{sin\,(1+2)\,sin\,(2+3+4)\,sin\,(4+5)}{sin\,(1+2+3)\,sin\,(2)\,sin\,(5)} = 1 . \tag{10}$$

If this condition is not satisfied, then we cannot construct the quadrilateral of Fig. 1 without discrepancy with the five angles (1), (2), (3), (4), (5), but we will obtain, in the construction, at any point a triangle which shows error.

With this, we can also answer the question, left open above, for the number of the station equations by subtracting, from expression (5), the expressions (9) and (6) or, as the case may be, (7). We have then, if all lines are measured back and forth,

$$W - 2l + p \text{ station equations .} \tag{11}$$

If among the l lines there are l' lines measured on one side, and of the p points p' are only intersected or only resected, then we have

$$W + p - p' - 2l + l' \text{ station equations .} \tag{12}$$

In summary we have now the following:

Condition equations for angle observations

W angle measurements, p points, l lines, all observed back and forth:

$$\left.\begin{array}{rl} l - 2p + 3 & \text{side equations,} \\ l - \;\; p + 1 & \text{triangle or polygon equations,} \\ W - 2l + p & \text{station equations,} \\ r = W - 2p + 4 & \text{condition equations in general.} \end{array}\right\} \tag{13}$$

If among the l lines a number l' is observed only on one side, and among the p points a number p' is only intersected or only resected, then we have

$$\left.\begin{array}{rl} l - 2p + 3 & \text{side equations,} \\ l - l' - (p - p') + 1 & \text{triangle or polygon equations,} \\ W + p - p' - 2l + l' & \text{station equations,} \\ r = W - 2p + 4 & \text{condition equations in general.} \end{array}\right\} \tag{14}$$

Example of a triangle net with angle observations

In the following Fig. 2, which represents a triangle net of Baden, 27 individual angles are measured, and it is a question of finding the condition equations.

According to the rules (13), in this net we have two side equations and nine triangle equations, further at Oggersheim and Speyer one horizon equation each.

It is also not difficult to actually set up these $2 + 9 + 2$ equations:

The horizons at Oggersheim and at Speyer yield

$$\left.\begin{array}{r} (3) + (6) + (8) + (10) + (24) + (25) - 360^\circ = 0 \\ (12) + (15) + (16) + (19) + (22) - 360^\circ = 0 \end{array}\right\}. \tag{15}$$

The nine angle equations are according to the nine triangles I, II, III, . . . , IX

$$\left.\begin{array}{ll} (1) + (2) + (3) - (180^\circ + \text{spher. excess}) & = 0 \\ (4) + (5) + (6) - \ldots & = 0 \\ \ldots\ldots\ldots\ldots & \\ (25) + (26) + (27) - (180^\circ + \text{spher. excess}) & = 0 \end{array}\right\}. \tag{16}$$

The two side equations refer to the central points Oggersheim and Speyer; for instance, it is expressed that from the base Oggersheim-Speyer the base must come out again by sine computation through the triangles

VIII, IX, I, II, III, IV, hence:

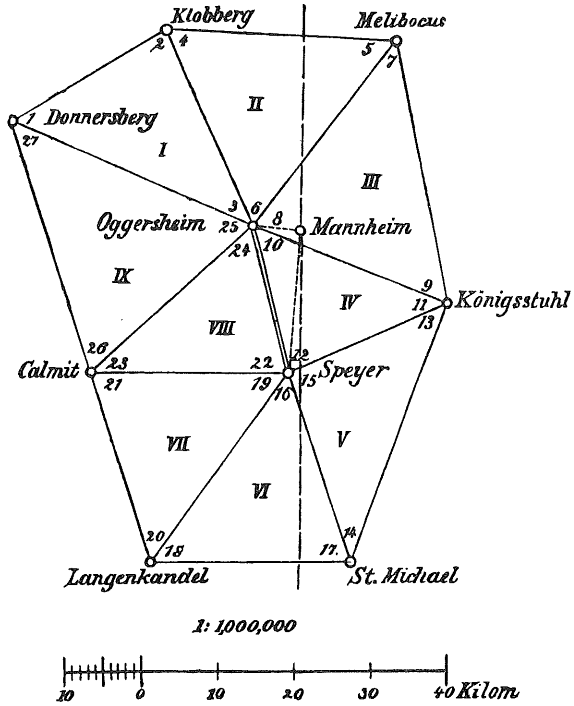
$$\frac{\text{Ogg.-Sp.}\; sin\,(22)\; sin\,(26)\; sin\,(1)\; sin\,(4)\; sin\,(7)\; sin\,(11)}{sin\,(23)\; sin\,(27)\; sin\,(2)\; sin\,(5)\; sin\,(9)\; sin\,(12)} = \text{Ogg. Sp.}$$

In the same manner we also treat the triangles around the central point Speyer, and the two side equations thus resulting are:

$$\left.\begin{array}{ll} (O) & \dfrac{sin\,(12)\; sin\,(9)\; sin\,(5)\; sin\,(2)\; sin\,(27)\; sin\,(23)}{sin\,(11)\; sin\,(7)\; sin\,(4)\; sin\,(1)\; sin\,(26)\; sin\,(22)} = 1 \\ (S) & \dfrac{sin\,(10)\; sin\,(13)\; sin\,(17)\; sin\,(20)\; sin\,(23)}{sin\,(11)\; sin\,(14)\; sin\,(18)\; sin\,(21)\; sin\,(24)} = 1 \end{array}\right\} . \quad (17)$$

With this, in (15), (16), (17) we have the $2 + 9 + 2 = 13$ equations, upon which the adjustment of the net with the 27 individual angle measurements is to be based.

(The adjustment of this example, completely carried out, was contained in the first three editions of this volume. In the later editions, the execution of the example is no longer included, because adjustments with individual *angle* measurements have moved more into the background. We shall treat the adjustment for a part of the triangle net of Fig. 2 in the later section 74.)



$W = 27$ angles, $\quad l - 2p + 3 = \;\; 2$ side equations,
$p = \;\; 9$ points, $\quad l - p \;\; + 1 = \;\; 9$ triangle equations,
$l = 17$ lines, $\quad W - 2l \; + p = \;\; 2$ station equations.
$W - 2p + 4 = 13$ equations.

Fig. 2.

(The point Mannheim does not belong to the net.)

II. *Direction measurements*

Since we have treated, in the foregoing, the setting up of the condition equations for *angle* measurements, it is not difficult to pass over from this also to direction measurements, which play the main role in present-day trigonometric practice. Nevertheless, in setting up the condition equations, the *angle* form is always (at least in thought) decisive, and only from there we pass over to directions.

Since we have already shown the basic idea of the change from angle corrections δ to corrections to directions v in section 65, p. 215, and in section 66, p. 220, i.e. splitting up $\delta = v_2 - v_1 \; . \; . \; . \; ,$ then we need not say much more in addition, but only interpret the subject still somewhat more generally.

We assume *full* sets of directions (or formation of the mean from repeated measurements of full sets of directions or results of station adjustments according to sections 62 to 64, which, obviously, is the same here), and indeed equally weighted at all stations. Unequal repetitions of the sets that may happen at various stations would introduce different weights into the adjustment, of which we shall not now speak, however.

In the case of this kind of measurement, the horizon equations with which we had to deal in the case of the adjustment of angle measurements are simply omitted.

We imagine a station, on one hand, with W angle measurements, without horizon checks, and the same sights with R direction measurements, on the other. Then we have obviously $R = W + 1$, i.e., at each station there is one direction more than the independent angles, and in a whole net with p points and W angles, without horizon checks, or R directions we have, obviously,

$$W = R - p\,; \qquad (18)$$

only *intersected* points that may occur among p are not taken into account here, however.

Observed: R directions, p points, l lines back and forth. With (18), (13) changes to:

$$\left.\begin{array}{ll} l-2p+3 & \text{side equations,} \\ l-\ \ p+1 & \text{triangle or polygon equations,} \\ \left.\begin{array}{l} r=2l-3p+4 \\ r=R-3p+4 \end{array}\right\} & \text{condition equations in general.} \end{array}\right\} \quad (19)$$

Here $2l = R$, for if all lines are measured back and forth, then we have twice as many directions as lines.

If among the l lines there is a number l' only observed on one side, and among the p points there is a number p' only intersected or only resected, then we have

$$\left.\begin{array}{rl} l-2p+3 & \text{side equations,} \\ (l-l')-(p-p')+1 & \text{triangle or polygon equations,} \\ r=2l-l'-3p+p'+4 & \text{condition equations in general.} \end{array}\right\} \quad (20)$$

The two formulae counting $l - p + 1$ triangle equations and $l - 2p + 3$ side equations are communicated as "Gaussian *theorems*" and proved in Gerling's *Ausgleichungsrechnungen*, Hamburg, 1843, pp. 273 and 277, with reference to *Beiträge zur Geographie Kurhessens, etc.*, by Gerling, Kassel, 1839, p. 166 (cf. also *Zeitschr. f. Verm., 1901*, p. 30).

Bessel gives another instruction for the counting of condition equations according to the geometric point of view, especially for direction measurements, on pp. 139 and 140 of the *Degree-measurement in East Prussia.*

We were led, by the need that had arisen, to the special treatment of lines and points, intersected only on *one* side, in (14) and (20).

If the triangle conditions refer to directions, then the algebraic sum of the coefficients in each condition equation is equal to zero, and the corrections to the directions v, added by stations, yield, likewise, in each case their sum equal to zero, assuming that all weights of measurements are *equal.*

These two theorems can be understood at once according to sections 65 and 67 and according to section 43. For instance, if a condition equation referred to angles reads:

$$a\delta + a'\delta' + a''\delta'' + \ldots = 0,$$

where δ, δ', δ'' ... are angle corrections, then the corresponding equation for corrections to directions v_l and v_r reads:

$$a\,(v_r - v_l) + a'\,(v_r' - v_l') + a''\,(v_r'' - v_l'') + \ldots = 0$$
$$a\,v_r - a\,v_l + a'\,v_r' - a'\,v_l' + a''v_r'' - a''v_l'' + \ldots = 0\,;$$

here we have:

$$a - a + a' - a' + a'' - a'' + \ldots \qquad = 0\,.$$

As for the v's, their formulae are given in (9), section 43, p. 139; if we add there, then we obtain

$$[v] = [a]k_1 + [b]k_2 + [c]k_3\,.$$

Now if we understand by $[v]$ the summation for *one* station in each case, then $[a]$, $[b]$, $[c]$ also hold only by stations, hence, are equal to zero; consequently, $[v]$ by stations is also equal to zero.

As for the actual setting up of the condition equations, the different equations of sums of angles will never offer any difficulties, and if the net can only be divided into individual triangles, then the side equations as sine products for central systems will also easily result according to the instructions of sections 65 and 66 and the above example (Fig. 2, p. 225). The matter may become somewhat more difficult in the case of points which are only intersected or only resected, yet nothing general can be said here on this subject; often it will also be advisable to exclude such points from the total adjustment and only to intercalate them afterwards.

We shall discuss further condition equations, as they occur in the attachment of new triangulation nets to already adjusted nets, in the later section 80.

There is one more question concerning the most favorable form of the side equation in a quadrilateral, which could be attached here; however, we will postpone this to the later sections 77 to 78 in order not to delay too long the transition to practical adjustment computations.

Section 68. Adjustment of a Quadrilateral with Four Full Sets of Directions

As the first example for computational execution we take a quadrilateral of the triangulation of Baden with four full, equally weighted, sets of directions.

The results of the measurements, referred in each case to a starting ray as the zero direction, are:

Station Catharina					Station Kandel				
Kandel . .	(1) =	0°	0′	0.00″	Feldberg .	(10) =	0°	0′	0.00″
Feldberg .	(2) =	34	52	27.44	Belchen .	(11) =	25	9	9.67
Belchen .	(3) =	57	49	20.90	Catharina.	(12) =	102	43	24.53
Station Belchen					Station Feldberg				
Catharina .	(4) =	0°	0′	0.00″	Belchen .	(7) =	0°	0′	0.00″
Kandel . .	(5) =	44	36	27.07	Catharina	(8) =	72	58	55.84
Feldberg .	(6) =	84	04	12.94	Kandel .	(9) =	115	23	06.40

(1)

An arbitrary sight is assumed here in each case as the *zero* ray with $0°0'\,0.00''$; this is unessential, however, and we should very much guard against attributing any special material significance to the formally preferred direction. Especially, we must not believe that, with this, we have before us *angles* in the sense as was set up in sections 65 and 66 in contrast to directions. The starting value $0°0'\,0.00''$ must rather receive an adjustment correction just as the second directional value $34°52'\,27.44''$, etc., and we do not change anything at all in the matter, if we move all directions of a set by an arbitrary measure.

For this reason, we frequently make the matter as convenient as possible for ourselves for *later* purposes by setting up, already from the outset, all sets of directions in the form of approximate trigonometric direction angles. We always have the orientation for such trigonometric direction angles from approximate attached computations.

In order to put the advantages of such an orientation immediately into the right light, we will make use of it at once in this our first example and write once more the sets of directions communicated under (1) in an approximate orientation (with $+x$ north and the counting of direction angles from north through east).

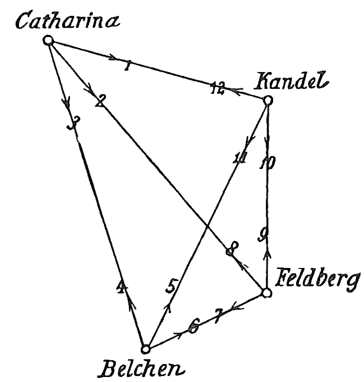


Fig. 1.
Quadrilateral with full sets of directions.
Scale 1:800,000.

Catharina				Kandel			
(1) =	104°	33′	24.00″	(10) =	181°	49′	59.47″
(2) =	139	25	51.44	(11) =	206	59	9.14
(3) =	162	22	44.90	(12) =	284	33	24.00
Belchen				Feldberg			
(4) =	342°	22′	40.13″	(7) =	246°	26′	53.07″
(5) =	26	59	7.20	(8) =	319	25	48.91
(6) =	66	26	53.07	(9) =	1	49	59.47

(2)

The direction (1) agrees with the opposite direction (12), likewise (10) agrees with (9), (7) with (6), because this has so been assumed in the arbitrary fixing of the starting directions; the fact that the remaining directions, e.g. (5) and (11), do not agree has its reason partly in the observational errors, but partly also in the circumstance that the triangles are not plane, but spherical.

For the net adjustment we need first the spherical excesses, and for this, *one* side of the net, e.g., Catharina – Belchen = 34433 m, log = 4.53697. The geographic latitude is approximately 48°, and the logarithm of the earth's radius, therefore, is $\log r = 6.80479$, with which we calculate, in the familiar manner, from the triangle area Δ:

$$\text{Spherical excess } \varepsilon = \frac{\Delta}{r^2}\rho, \qquad \text{with } \log\frac{\rho}{r^2} = 1.70485;$$

e.g., the first triangle Catharina – Belchen – Kandel has approximately

Catharina	(1,3) = 57°49'	log sin 9.92755
Belchen	(4,5) = 44 36	log sin 9.84643
Kandel	(11,12) = 77 34	log sin 9.98969,

with which, taking as a basis $\log CB = 4.53697$, we calculate $\log CK = 4.39371$ and then $\Delta = \frac{1}{2} CB \cdot CK \sin C$, $\log \Delta = 8.55720$ and $\varepsilon = 1.828''$. Or briefly, with provisional sides and angles we compute the excesses, as they are indicated in the following summary which contains all four sum checks:

C. (1,3) =	57° 49′ 20.90″	C. (2,3) =	22° 56′ 53.46″
B. (4,5) =	44 36 27.07	F. (7,8) =	72 58 55.84
K. (11,12) =	77 34 14.86	B. (4,6) =	84 4 12.94
Sum:	180° 0′ 2,83″	Sum:	180° 0′ 2.24″
Should be:	180 0 1,83	Should be:	180 0 1.22
	$w = +1{,}00''$		$w = +1.02''$
B. (5,6) =	39° 27′ 45.87″	F. (8,9) =	42° 24′ 10.56″
F. (7,9) =	115 23 6.40	K. (10,12) =	102 43 24.53
K. (10,11) =	25 9 9.67	C. (1,2) =	34 52 27.44
Sum:	180° 0′ 1.94″	Sum:	180° 0′ 2.53″
Should be:	180 0 0.67	Should be:	180 0 1.28
	$w = +1.27''$		$w = +1.25''$

(3)

If we denote the corrections to the directions by $v_1, v_2, \ldots, v_{12}$, then the following four condition equations follow from these check sums:

$$\left.\begin{aligned}
v_3 - v_1 + v_5 - v_4 + v_{12} - v_{11} + 1.00'' &= 0\\
v_3 - v_2 + v_8 - v_7 + v_6 - v_4 + 1.02 &= 0\\
v_6 - v_5 + v_9 - v_7 + v_{11} - v_{10} + 1.27 &= 0\\
v_9 - v_8 + v_{12} - v_{10} + v_2 - v_1 + 1.25 &= 0
\end{aligned}\right\}. \tag{4}$$

In each of these equations, the sum of the coefficients = zero, as results directly from the origin of these equations (cf. the note at the end of section 67 and on p. 227).

If any three of these four condition equations (4) are satisfied, then the fourth is satisfied by itself; therefore, we have to introduce into the adjustment only three of equations (4).

Further there exists an equation of a side condition (central point Feldberg):

$$\text{it should be} \qquad \frac{\sin(1,2)\,\sin(4,6)\,\sin(10,11)}{\sin(2,3)\,\sin(5,6)\,\sin(10,12)} = 1. \tag{5}$$

This equation also holds for a *spherical* quadrilateral, something which we see at once if, in Fig. 2, section 65, p. 214, we set now in place of the requirement $DA = DA'$ the requirement $\sin\frac{DA}{r} = \sin\frac{DA'}{r}$ and also introduce $\sin\frac{DC}{r}$ and $\sin\frac{DB}{r}$ for DC and DB. With the help of the law of sines of spherical trigonometry we arrive then, likewise, at the side equation (6), section 65, p. 214.

To the side equation (5) there corresponds the following logarithmic-trigonometric calculation with seven-place logarithms:

				Diff. for 10″
(1,2) =	34° 52′ 27.44″	log sin (1,2)	9.757 2273	+ 302
(4,6) =	84 4 12.94	log sin (4,6)	9.997 6699	+ 22
(10,11) =	25 9 9.67	log sin (10,11)	9.628 4216	+ 449
			9.383 3188	
(2,3) =	22° 56′ 53.46″	log sin (2,3)	9.590 9515	+ 498
(5,6) =	39 27 45.87	log sin (5,6)	9.803 1677	+ 256
(10,12) =	102 43 24.53	log sin (10,12)	9.989 2025	— 48
			9.383 3217	

(6)

Discrepancy: 9.383 3188 — 9.383 3217 = — 0.000 0029 = w.

In order to obtain a check for this very important logarithmic-trigonometric calculation of the error term w, we can make use of the spherical excesses, which are already indicated in (3), by decreasing each angle of the triangle by one-third of the spherical excess, calculating according to Legendre's theorem. Thus we obtain from (3):

C.	(1,3) =	57° 49′ 20.29″	C.	(2,3) =	22° 56′ 53.05″
B.	(4,5) =	44 36 26.46	F.	(7,8) =	72 58 55.43
K.	(11,12) =	77 34 14.25	B.	(4,6) =	84 4 12.53
		+ 1.00″			+ 1.01″
B.	(5,6) =	39° 27′ 45.65″	F.	(8,9) =	42° 24′ 10.13″
F.	(7,9) =	115 23 6.18	K.	(10,12) =	102 43 24.10
K.	(10,11) =	25 9 9.45	C.	(1,2) =	34 52 27.01
		+ 1.28″			+ 1.24″

(7)

The sums + 1.00″, etc., are to be equal to the corresponding w's in (3), and if this is not completely the case, then this is due only to rounding off in the formation of the thirds of the excesses. With the angles (7) we now carry out again a logarithmic-trigonometric calculation of the form (6):

				Diff. for 10″
(1,2) =	34° 52′ 27.01″	log sin (1,2)	9.757 2260	+ 302
(4,6) =	84 4 12.53	log sin (4,6)	9.997 6699	+ 22
(10,11) =	25 9 9.45	log sin (10,11)	9.628 4206	+ 449
			9.383 3165	
(2,3) =	22° 56′ 53.05″	log sin (2,3)	9.590 9495	+ 498
(5,6) =	39 27 45.65	log sin (5,6)	9.803 1672	+ 256
(10,12) =	102 43 24.10	log sin (10,12)	9.989 2027	— 48
			9.383 3194	

(8)

Discrepancy: 9.383 3165 — 9.383 3194 = — 0.000 0029 = w.

Now we obtain from (6) and (8) in agreement the linear side equation in units of the seventh decimal of logarithms:

$$\left.\begin{aligned} &+30.2\,(v_2-v_1)+\ \ 2.2\,(v_6-v_4)+44.9\,(v_{11}-v_{10}) \\ &-49.8\,(v_3-v_2)-25.6\,(v_6-v_5)+4.8\,\ (v_{12}-v_{10})-29=0 \end{aligned}\right\}. \qquad (9)$$

We can so use this equation further, but in comparison to the equations (4) of sums of angles, equation (9) has too large coefficients, which is formally disturbing for the further computation, and therefore, we will rather divide equation (9) throughout by 10, i.e., calculate in units of the sixth decimal of logarithms:

$$\left.\begin{aligned} &+3.02\,(v_2 - v_1) + 0.22\,(v_6 - v_4) + 4.49\,(v_{11} - v_{10}) \\ &-4.98\,(v_3 - v_2) - 2.56\,(v_6 - v_5) + 0.48\,(v_{12} - v_{10}) - 2.9 = 0 \end{aligned}\right\}. \tag{10}$$

We can multiply or divide such a condition equation (9) or (10) by any arbitrary number without changing its significance for the adjustment (while we cannot multiply *error* equations without changing the proportions of weight; cf. (6), section 22, p. 70). It is best if the coefficients of the linear side equations have in the mean approximately the value 1, because the equations (4) of sums of angles, also, have all the coefficients 1; therefore, we could think of dividing equation (10) once more by 2 or 3; however, with the convenient computation in units of the sixth place according to (10) we can usually manage, and therefore, we stop at (10) and find therefrom by collecting terms with respect to $v_1, v_2, v_3 \ldots$:

$$-3.02v_1 + 8.00v_2 - 4.98v_3 - 0.22v_4 + 2.56v_5 - 2.34v_6 - 4.97v_{10} + 4.49v_{11} + 0.48v_{12} - 2.9 = 0. \tag{11}$$

The sum of coefficients $-3.02 + 8.00 \ldots + 0.48$ also is again $= 0$, as in the case of equations (4).

The linear side equation (11) beside *three* of the four angle equations of group (4) form the complete system of the condition equations. With the elimination of the third equation of group (4) we form the table of the coefficients of the condition equations:

Coefficients of the condition equations

		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	w
k_1	a	-3.02	$+8.00$	-4.98	-0.22	$+2.56$	-2.34	. .	. .	. .	-4.97	$+4.49$	$+0.48$	-2.90
k_2	b	-1	. . .	$+1$	-1	$+1$	. . .	. .	. .	. .	. . .	-1	$+1$	$+1.00$
k_3	c	. . .	-1	$+1$	-1	. . .	$+1$	-1	$+1$	. .	. . .	. . .	. . .	$+1.02$
k_4	d	-1	$+1$	. . .	. . .	. . .	. . .	. .	-1	$+1$	-1	. . .	$+1$	$+1.25$

(12)

The formation of the sum coefficients is very simple; we have, e.g.:

$$\begin{aligned} [aa] &= 3.02^2 + 8.00^2 + 4.98^2 + \ldots\ldots + 0.48^2 = 155.09\,, \\ [ab] &= +3.02 - 4.98 + 0.22 + 2.56 - 4.49 + 0.48 = -3.19\,, \\ [ac] &= -8.00 - 4.98 + 0.22 + 2.34 = -15.10\,, \\ [ad] &= +3.02 + 8.00 + 4.97 + 0.48 = +16.47\,; \\ [bb] &= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 6\,, \\ [bc] &= +1 + 1 = +2\,, \\ [bd] &= +1 + 1 = +2\,; \\ [cc] &= 1^2 + \ldots \qquad = 6\,, \\ [cd] &= -1 - 1 \qquad = -2\,, \\ [dd] &= 1^2 + \ldots \qquad = 6\,. \end{aligned}$$

Hence, the normal equations are in abbreviated manner of writing

$$\left.\begin{array}{lllll} a] & b] & c] & d] & w] \\ [a\ \ \underline{155.09\,k_1} & -\,3.19\,k_2 & -\,15.10\,k_3 & +\,16.47\,k_4 & -\,2.90 = 0 \\ & [b\ \ \underline{6.00\,k_2} & +\,2.00\,k_3 & +\,2.00\,k_4 & +\,1.00 = 0 \\ & & [c\ \ \underline{6.00\,k_3} & -\,2.00\,k_4 & +\,1.02 = 0 \\ & & & [d\ \ \underline{6.00\,k_4} & +\,1.25 = 0 \end{array}\right\}. \tag{13}$$

The elimination and solution yield

$$k_1 = +0.044, \qquad k_2 = +0.078, \qquad k_3 = -0.232, \qquad k_4 = -0.431. \tag{14}$$

According to the instruction of the table of the condition equations (12) we make the following tabular computation of the corrections v:

k	1	2	3	4	5	6	7	8	9	10	11	12
+0.044	−0.132	+0.352	−0.218	−0.010	+0.112	−0.103	. . .	. . .	. . .	−0.217	+0.197	+0.021
+0.078	−0.078	. . .	+0.078	−0.078	+0.078	. . .	. . .	. . .	. . .	. . .	−0.078	+0.078
−0.232	. . .	+0.232	−0.232	+0.232	. . .	−0.232	+0.232	−0.232	. . .	. . .	. . .	. . .
−0.431	+0.431	−0.431	. . .	. . .	. . .	. . .	. . .	+0.431	−0.431	+0.431	. . .	−0.431
v	+0.221	−0.153	−0.372	+0.144	+0.190	−0.335	+0.232	+0.199	−0.431	+0.214	+0.119	−0.332
Sum		− 0.002			− 0.001			0.000			+ 0.001	

(15)

These values v yield, by stations, the sum 0.000" in each case, disregarding small uncertainties of rounding off 0.002", etc.

If we add the thus obtained v's to the observed directions (2), where we round off to 0.01", then we obtain the following:

Observed	v	Adjusted	v^2
(1) = 104° 33′ 24.00″	+ 0.22″	[1] = 104° 33′ 24.22″	0.0484
(2) = 139 25 51.44	+ 0.15	[2] = 139 25 51.59	0.0225
(3) = 162 22 44.90	− 0.37	[3] = 162 22 44.53	0.1369
(4) = 342° 22′ 40.13″	+ 0.14″	[4] = 342° 22′ 40.27″	0.0196
(5) = 26 59 7.20	+ 0.19	[5] = 26 59 7.39	0.0361
(6) = 66 26 53.07	− 0.33	[6] = 66 26 52.74	0.1089
(7) = 246° 26′ 53.07″	+ 0.23″	[7] = 246° 26′ 53.30″	0.0529
(8) = 319 25 48.91	+ 0.20	[8] = 319 25 49.11	0.0400
(9) = 1 49 59.47	− 0.43	[9] = 1 49 59.04	0.1849
(10) = 181° 49′ 59.47″	+ 0.21″	[10] = 181° 49′ 59.68″	0.0441
(11) = 206 59 9.14	+ 0.12	[11] = 206 59 9.26	0.0144
(12) = 284 33 24.00	− 0.33	[12] = 284 33 23.67	0.1089
		$[v\,v]$ =	0.8176

(16)

By subtractions of these adjusted directions we form the adjusted angles and we group these again by triangles as previously in the case of (3).

In distinction from the measured angles, which were denoted by (1,3), (4,5), . . . , we denote the adjusted angles by [1,3], [4,5], . . . :

C. [1,3] =	57° 49′ 20.31″	C. [2,3] =	22° 56′ 52.94″
B. [4,5] =	44 36 27.12	F. [7,8] =	72 58 55.81
K. [11,12] =	77 34 14.41	B. [4,6] =	84 4 12.47
Sum:	180° 0′ 1.84″	Sum:	180° 0′ 1.22″
Should be:	1.83	Should be:	1.22
B. [5,6] =	39° 27′ 45.35″	F. [8,9] =	42° 24′ 9.93″
F. [7,9] =	115 23 5.74	K. [10,12] =	102 43 23.99
K. [10,11] =	25 9 9.58	C. [1,2] =	34 52 27.37
Sum:	180° 0′ 0.67″	Sum:	180° 0′ 1.29″
Should be:	0.67	Should be:	1.28

(17)

To examine if the side equation comes out right, we carry out, instead of the previous logarithmic computation (6), the following new logarithmic computation:

$$\left.\begin{array}{lll|l} [1,2] = 34°\,52'\,27.37'' & & \log\sin\ [1,2] & 9.757\,2271 \\ [4,6] = 84\ \ \ 4\ \ 12.47'' & & \log\sin\ [4,6] & 9.997\,6699 \\ [10,11] = 25\ \ \ 9\ \ \ 9.58 & & \log\sin\ [10,11] & 9.628\,4212 \\ \hline & & & 9.383\,3182 \\ [2,3] = 22°\,56'\,52.94'' & & \log\sin\ [2,3] & 9.590\,9490 \\ [5,6] = 39\ \ 27\ \ 45.35'' & & \log\sin\ [5,6] & 9.803\,1664 \\ [10,12] = 102\ \ 43\ \ 23.99 & & \log\sin\ [10,12] & 9.989\,2028 \\ \hline & & & 9.383\,3182 \end{array}\right\} . \qquad (18)$$

With the adjusted directions we compute all triangle sides in agreement by all methods. As the base we take for this the side Catharina – Belchen as indicated in the land triangulation of Baden:

$$\text{Catharina – Belchen} = 34{,}432.57 \text{ m.} \qquad (19)$$

By assuming this length, the following distances were computed free from discrepancy, with the adjusted angles and sine logarithms (18) according to the method of additaments:

$$\left.\begin{array}{lll} \textbf{Catharina—Belchen} \ \ldots & = 34{,}432.57 \text{ m} & \log = 4.536\,9695 \\ \textbf{Catharina—Feldberg} \ \ldots & = 35{,}816.62 & 4.554\,0846 \\ \textbf{Catharina—Kandel} \ \ldots & = 24{,}760.43 & 4.393\,7582 \\ \textbf{Belchen—Feldberg} \ \ldots & = 14{,}039.83 & 4.147\,3617 \\ \textbf{Belchen—Kandel} \ \ldots & = 29{,}843.17 & 4.474\,8450 \\ \textbf{Feldberg—Kandel} \ \ldots & = 20{,}994.59 & 4.322\,1074 \end{array}\right\} . \qquad (20)$$

We can also compute in correspondence with (7) and (8) according to Legendre's theorem, which must yield the same values as (20).

As far as determination of accuracy is concerned, we have to keep in sight first the sum $[v\,v] = 0.8176$ according to (16). We obtain a check for this by the formula $[v\,v] = -[w\,k]$, according to (4), section 45, p. 142. The calculation for this is:

$$\begin{array}{lll} w_1 = -2.90 & k_1 = +0.044 & -w_1 k_1 = +0.1276 \\ w_2 = +1.00 & k_2 = +0.078 & -w_2 k_2 \qquad\qquad\quad -0.0780 \\ w_3 = +1.02 & k_3 = -0.232 & -w_3 k_3 = +0.2366 \\ w_4 = +1.25 & k_4 = -0.431 & -w_4 k_4 = +0.5388 \\ \hline & & \quad +0.9030 - 0.0780 \\ & & -[w\,k] = +0.8250\,. \end{array} \qquad (21)$$

We can also include further formula (5), section 45, p. 142, which we omit here, however, since we also have not presented in detail the elimination (13) to (14), and because the check (21), beside the direct calculation of the individual v^2's in (16), is completely sufficient.

We take in the mean $[v\,v] = 0.82$ and have, with this, the mean error of an observed direction:

$$m = \sqrt{\frac{0.82}{4}} = \pm\, 0.45'' \qquad (22)$$

and the mean error of an angle before the adjustment:

$$= m\sqrt{2} = \pm\, 0.64''. \qquad (23)$$

With this, the adjustment is completed, and with the triangle sides (20) beside the angles (18) we can now compute the coordinates of the points of the quadrilateral in any system which is considered.

The coordinates of the four points in the system of Baden are given in *Zeitschrift für Vermessungswesen*, 1878, pp. 33 and 34. There, on pp. 20 to 25, are also given the original measurements which have led to this net adjustment. They are many individual angles measured according to the repetition method, which we have first adjusted by stations and then treated further like full sets of directions.

The measurement of full equally weighted sets of directions at all four points, which we have assumed at the beginning in (1) or (2), p. 227, has thus actually *not* taken place, and therefore, we have to conceive our preceding adjustment of a quadrilateral either as a school example with fictitious direction measurements, or we have before us a practical example which, it is true, is not formally rigorous but very useful, and had already been applied by Gauss, for the first time in 1826 in "Supplementum theoriae combinationis"; and which consists in the theory that we adjust angle measurements or also incomplete measurements of directions in sets first by stations in any way (perhaps only approximately) and then introduce the station results into the net adjustment like full sets of directions.

Section 69. Triangulation Net of the City of Hannover

In 1891, the triangulation net of the city of Hannover, drawn in Fig. 1, p. 234, was surveyed by the author, who was active at that time, temporarily, as "trigonometer" of the trigonometric division of the Land Survey.

We have already reported about the technical part of these measurements in our Volume II, 1st half-volume (9th Edition, 1931, pp. 377 to 421) with centerings, etc., and let us repeat here only that the angle measurements have been carried out with a small microscope theodolite with a 14-cm circle, and indeed eight points of Fig. 1 in 12 sets in each case.

Following are the mean values of these measurements in the form of sets of directions, where the direction to Ägidius or Linden, water tower, is introduced as $0°0'0''$ in each case.

1. *Ägidius.*

Linden, Wasserturm	1.	0°	0′	0.00″
Hochschule		63	38	53.31
Burg	2.	70	56	34.82
Schanze	3.	110	42	21.36
Dreifaltigkeit		143	40	8.40
Steuerndieb	4.	163	44	49.52
Willmer	5.	259	4	4.67

2. *Linden, Wasserturm*

Burg	6.	284°	21′	15.98″
Hochschule		309	13	10.25
Ägidius	7.	0	0	0.00
Willmer	8.	45	5	26.24

3. *Willmer*

Linden, Wasserturm	9.	326°	1′	19.33″
Ägidius	10.	0	0	0.00
Dreifaltigkeit		22	9	36.10
Steuerndieb	11.	50	15	28.80

4. *Steuerndieb*

Willmer	12.	325°	34′	46.28″
Ägidius	13.	0	0	0.00
Dreifaltigkeit		12	56	15.08
Hochschule		24	5	46.54
Burg	14.	44	9	14.00
Schanze	15.	74	52	31.12

5. *Schanze*

Steuerndieb	16.	307°	55′	0.00″
Dreifaltigkeit		345	43	43.08
Ägidius	17.	0	0	0.00
Burg	18.	56	4	7.29

6. *Burg*

Schanze	19.	275°	49′	51.50″
Steuerndieb	20.	316	57	24.36
Dreifaltigkeit		336	24	35.87
Ägidius	21.	0	0	0.00
Hochschule		6	43	55.93
Linden, Wasserturm	22.	33	24	40.16

7. *Hochschule*

Schanze	249°	12′	49.37″
Steuerndieb	304	11	45.10
Dreifaltigkeit	316	13	35.82
Ägidius	0	0	0.00
Linden, Wasserturm	65	34	18.81
Burg	194	1	35.18

8. *Dreifaltigkeit*

Willmer	317°	33′	31.20″
Ägidius	0	0	0.00
Hochschule	56	12	25.93
Burg	83	41	5.14
Schanze	132	45	54.74

(1)

Instead of assuming here an arbitrary direction $= 0°0'0.00''$ in each set, we could also list at once all sets in the form of approximate direction angles, as in the example of the previous section 68 in (2), p. 227; we did not do this this time, first, because it was directed by the trigonometric division in 1891 to begin the station data of the individual sets with $0°0'0''$ in each case, and second, because it is useful for school examples, like this and the previous in section 68, to illustrate *both* forms. In practice, however, we always calculated with station measuring data oriented approximately according to trigonometric direction angles.

The details of the measurements of sets shall be presented for *one* case; for this, we choose the example *Schanze*, where the measurements could be carried out on the ground and centrally, and indeed in sets of four sights. This was repeated twelve times, as is seen from the above table in which, omitting the degrees and minutes, only the seconds are indicated.

Position on circle	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	Mean
	"	"	"	"	"	"	"	"	"	"	"	"	"
Ägidius . . .	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00
Burg	6.0	4.5	10.5	6.0	11.0	8.5	10.0	6.0	8.5	3.5	5.5	7.5	7.29
Steuerndieb . .	63.0	59.5	61.0	61.0	60.5	61.5	62.0	59.0	56.0	56.0	58.0	62.5	60.00
Dreifaltigkeit .	40.5	38.0	45.0	46.5	43.5	47.5	38.5	48.5	40.0	40.0	41.5	47.5	43.08
Set mean . .	27.38	25.50	29.12	28.38	28.75	29.37	27.62	28.37	26.12	24.88	26.25	29.38	27.59

(2)

The total result of these 12 sets is, therefore, if the degrees and minutes are now also inserted:

$$\left.\begin{array}{lrrr} \text{Ägidius} \ldots\ldots & 0° & 0' & 0.00'' \\ \text{Burg} \ldots\ldots & 56 & 4 & 7.29 \\ \text{Steuerndieb} \ldots\ldots & 307 & 55 & 0.00 \\ \text{Dreifaltigkeit} \ldots\ldots & 345 & 43 & 43.08 \end{array}\right\} . \qquad (2a)$$

This is the same as is indicated above on p. 233, under 5, Schanze.

Now it should be said that such *full* sets, as at Schanze, are measured only at the fewest possible of the eight stations of our net, for different reasons, partly because no more than four sights in one set could be taken on towers, but especially because of the numerous excentric settings, since at Ägidius, Hochschule, Dreifaltigkeit, and Burg a *single* base point could not be found in general, from which all sights would have been possible. In these cases, the partial sets were simply moved to one another after the centering.

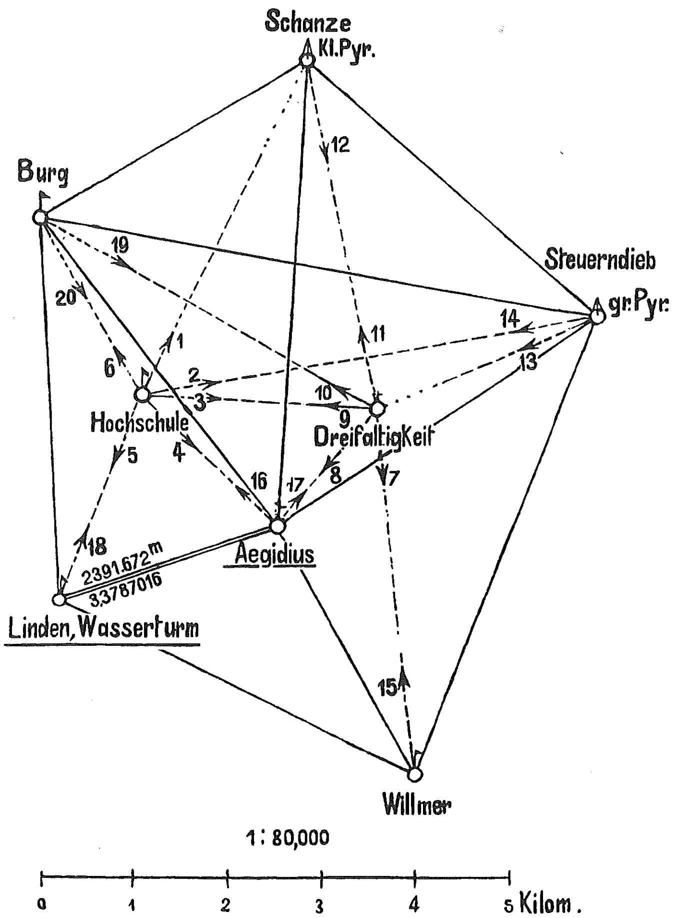


Fig. 1.

Now, strictly speaking, we have not the right to introduce such sets moved together into the adjustment as *full* sets; but we do this nevertheless in order to have a smooth computation and note to this in general that for a triangulation of third to fourth order, perhaps even in second order, such an approximate method is customary and advisable, for keeping all partial sets apart would be much too bothersome and would disturb exceedingly the total view (cf. also section 64, conclusion).

We have already indicated by the different drawing of the sights in Fig. 1, i.e., some solid lines, some broken lines, and further by numbering some of the sights, that the 42 directions afterwards are to be treated differently, since the unnumbered 22 directions enter together in a correlate adjustment and the remaining 20 directions are then to be used for the intercalation of the points Hochschule and Dreifaltigkeit.

If we disregard this first, then we can use our net as an example for the counting of the condition equations according to the rules of section 67. If all directions are used, then we have:

$R = 42$ directions,
$p = \ 8$ points,
$l = 22$ lines,
$l' = \ 2$ on one side (Hochschule-Schanze and Steuerndieb-Dreifaltigkeit);

this yields according to (20), section 67, p. 226

$$\begin{aligned} l - 2p + 3 &= 9 \text{ side equations,} \\ (l - l') - p + 1 &= 13 \text{ triangle equations } (180°), \\ 2l - l' - 3p + 4 &= 22 \text{ condition equations in all.} \end{aligned}$$

If, on the other hand, the dotted lines and, with this, the two points Hochschule and Dreifaltigkeit are omitted from the main net, then there remains an adjustment with only eight condition equations, which will be taken up in detail in the next section 70.

Before that, we still have to indicate the two base points of the net, namely Ägidius and Linden, water tower, with the base:

$$\text{Ägidius – Linden, Wasserturm} \quad \log S = 3.3787016$$
$$S = 2391.672 \text{ m}.$$

To this, the following further details shall be given with reference to section 71.

These two points were already adjusted to the first-order net by the trigonometric division of the Land Survey in 1887, as has been reported by us in *Zeitschrift für Vermessungswesen*, 1889, pp. 8 to 14. Besides, the point Celle is taken into account as the origin of our cadastral system of coordinates hitherto existing.

Following are the afore-mentioned geographic coordinates of the Land Survey:

	Latitude	Longitude		
Ägidius, Helmstange	52° 22′ 14.9611″	27° 24′ 24.6290″	(1888).	(3)
Wasserturm, Fahnenstange	52 21 49.9080	27 22 25.0168		
Differences: **Äg.-Wassert.**	0′ 25.0531″	1′ 59.6122″		

The official communication of 1891 yields:

Celle Koord.-Nullpunkt	52° 37′ 32.6709″	27° 44′ 54.8477″	(1891).	(4)
Ägidius, Helmstange	52 22 14.9611	27 24 24.6289		
Wasserturm, Fahnenstange	52 21 49.9080	27 22 25.0167		
Differences:	0′ 25.0531″	1′ 59.6122″		

The longitude data of 1891 are 0.0001″ smaller than those of 1888, which correspond only to a displacement of about 2 mm, hence, is *materially indifferent;* however, this small matter had to be mentioned here, because we have already given repeatedly the data of 1888.

In the rectangular system of the conformal double projection of the Trigonometric Division of the Reichsamt für Landesaufnahme, our two base points have the following coordinates:

Ägidius	$y = -244656.090\ m$	$x = -30624.971\ m$	(5)
Linden, Wasserturm	-246956.479	-31285.875	
Differences:	$+\ 2300.389\ m$	$+\ 660.904\ m$	

From the data (4) we have computed the following rectangular coordinates in the cadastral system:

Ägidius	$y = -23271.813\ m$	$x = -28308.395\ m$	(6)
Linden, Wasserturm	-25538.488	-29071.474	
Differences:	$+\ 2266.675\ m$	$+\ 763.079\ m$	

In order to derive now our base Ägidius-Wasserturm, we can adopt three ways, namely, compute this base from data (4) or (5) or (6). We will find, with sufficient agreement, the value already indicated above:

$$\left.\begin{aligned}\text{Ägidius-Linden, Wasserturm} \quad \log S &= 3.3787016 \\ S &= 2391.672 \text{ m}\end{aligned}\right\}. \tag{7}$$

Incidentally, let us note here: If we wish to compute the distance S directly from the geographic coordinates (4), we refer to our Volume III, 7th Edition, p. 456. To (5), there belong the formulae of the same volume, 7th Edition, p. 292 (cf. also *Zeitschrift für Vermessungskunde*, 1894, pp. 167-175), and to (6), there belong the usual Soldner formulae in that IIIrd Volume, 7th Edition, pp. 277 to 281.

Later, in section 71, we shall have to deal further with these different coordinates; for the present we only need the base side Ägidius-Wasserturm, according to (7).

We could question why our whole net (Fig. 1) was attached to the relatively *short* side Ägidius–Wasserturm and why, say, Steuerndieb was not taken as a second base point together with Linden, Wasserturm. The point Ägidius (geodetically sacred, so to speak, because of measurements by Gauss himself in 1823) has so many advantages that it will always be retained as first or second order; in 1887 Linden, Wasserturm, was taken here as the point which follows, at a time when Steuerndieb, 1891, was not yet constructed. The base Ägidius–Wasserturm was used in 1888 first for Linden; if the whole layout had not been decided until 1891 together with Steuerndieb, then the latter point would probably have been taken in addition as a base point.

Section 70. Adjustment of the Hannover Pentagon

Of all the direction measurements of Hannover given in the previous section 69, p. 233, we use first only those which refer to the main pentagon with the central point Ägidius, for which the side Ägidius-Wasserturm = 2391.672 m is used as the base. The measuring sights pertaining to this are provided with arrows and numbers in Fig. 1.

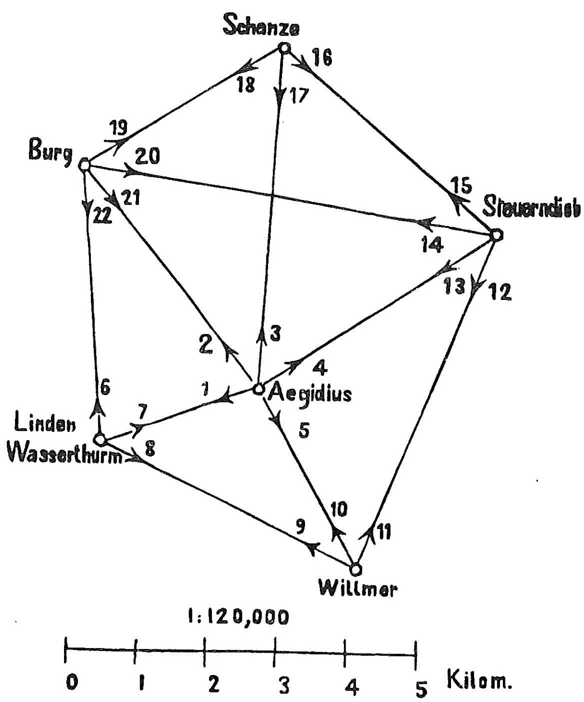


Fig. 1.

From section 69, p. 233, we have extracted especially once more the direction measurements pertaining to the main pentagon and summarized them in the following:

1. *Ägidius*

Wasserturm	(1) =	0°	0′	0.00″
Burg	(2) =	70	56	34.82
Schanze	(3) =	110	42	21.36
Steuerndieb	(4) =	163	44	49.52
Willmer	(5) =	259	4	4.67

2. *Wasserturm*

Burg	(6) =	284°	21′	15.98″
Ägidius	(7) =	0	0	0.00
Willmer	(8) =	45	5	26.24

3. *Willmer*

Wasserturm	(9) =	326°	1′	19.33″
Ägidius	(10) =	0	0	0.00
Steuerndieb	(11) =	50	15	28.80

4. *Steuerndieb*

Willmer	(12) =	325°	34′	46.28″
Ägidius	(13) =	0	0	0.00
Burg	(14) =	44	9	14.00
Schanze	(15) =	74	52	31.12

5. *Schanze*

Steuerndieb	(16) =	307°	55′	0.00″
Ägidius	(17) =	0	0	0.00
Burg	(18) =	56	4	7.29

6. *Burg*

Schanze	(19) =	275°	49′	51.50″
Steuerndieb	(20) =	316	57	24.36
Ägidius	(21) =	0	0	0.00
Wasserturm	(22) =	33	24	40.16

(1)

We consider first the number and type of the condition equations. We have $p = 6$ points, $l = 11$ lines observed on both sides, $R = 2l = 22$ observed directions. With this, we obtain according to (19), section 67, p. 226:

$$\begin{aligned} l - 2p + 3 &= 2 \quad \text{side equations,} \\ l - p + 1 &= 6 \quad \text{triangle equations,} \\ R - 3p + 4 &= 8 \quad \text{equations in all.} \end{aligned}$$

These equations are also easily proved in detail. We will first set up the six triangle equations and consider to this if the triangles can be treated directly as *plane*, or if the spherical excesses still make themselves felt. We have for our latitude of approximately 52°:

$$\varepsilon = \frac{\Delta}{r^2}\rho, \quad \text{where} \quad \log \frac{\rho}{r^2} = 1.7044,$$

and a small additional computation yields here the excesses, as indicated in the following, namely, between 0.02" and 0.04" for a triangle, and for the whole pentagon the amount of 0.15".

Although a quantity 0.02" has now no longer a real value in our measurements, since the second decimals 0.01" are in general carried along only as purely *computational* places for protection against a rounded-off error of 0.1 seconds, in order to have a complete example, we have not neglected the excesses in the amount of 0.02" to 0.04", since in all they accumulate to 0.15", especially because it does not cause any trouble at all to carry them along.

Now we put together the first triangle Ägidius-Wasserturm-Burg from the observations of directions (1):

$$\begin{array}{lrrr} (2) & = 70° & 56' & 34.82'' \\ (1) & = 0 & 0 & 0.00 \\ \hline (2)-(1) & = 70° & 56' & 34.82'' \end{array} \qquad \begin{array}{lrrr} (7) & = 0° & 0' & 0.00'' \\ (6) & = 284 & 21 & 15.98 \\ \hline (7)-(6) & = 75° & 38' & 44.02'' \end{array} \qquad \begin{array}{lrrr} (22) & = 33° & 24' & 40.16'' \\ (21) & = 0 & 0 & 0.00 \\ \hline (22)-(21) & = 33° & 24' & 40.16'' \end{array}$$

$$\begin{array}{llrrr} \text{Ägidius} & (2)-(1) = & 70° & 56' & 34.82'' \\ \text{Wasserturm} & (7)-(6) = & 75 & 38 & 44.02 \\ \text{Burg} & (22)-(21) = & 33 & 24 & 40.16 \\ \hline \text{Sum:} & & 179° & 59' & 59.00'' \\ \text{Should be } 180° + \varepsilon = & & 180 & 0 & 0.02 \\ & w = & & & -1.02''. \end{array} \qquad (c')$$

Now if we denote by (1), (2), (3) . . . the observed directions, by [1], [2], [3] . . . the adjusted directions, and by [1] − (1) = v_1, [2] − (2) = v_2, etc., the corrections to the directions, then we have

$$
\begin{aligned}
(2) - (1) + (7) - (6) + (22) - (21) &= 179^\circ\ 59'\ 59.00''\\
[2] - [1] + [7] - [6] + [22] - [21] &= 180\quad 0\quad 0.02
\end{aligned}
$$

hence,

$$v_2 - v_1 + v_7 - v_6 + v_{22} - v_{21} - 1.02'' = 0. \tag{c}$$

We have denoted this equation by (c), because in the later enumeration of all condition equations it will be the third.

We also form the remaining triangle equations just like this equation (c):

Ägidius	(1) − (5) =	100° 55′ 55.33″
Willmer	(10) − (9) =	33 58 40.67
Wasserturm	(8) − (7) =	45 5 26.24
		180° 0′ 2.24″
	Should be	180 0 0.02
		$w = +2.22''$

$$v_1 - v_5 + v_{10} - v_9 + v_8 - v_7 + 2.22'' = 0 \tag{d}$$

Ägidius	(5) − (4) =	95° 19′ 15.15″
Steuerndieb	(13) − (12) =	34 25 13.72
Willmer	(11) − (10) =	50 15 28.80
		179° 59′ 57.67″
	Should be	180 0 0.03
		$w = -2.36''$

$$v_5 - v_4 + v_{13} - v_{12} + v_{11} - v_{10} - 2.36'' = 0 \tag{e}$$

Ägidius	(4) − (3) =	53° 2′ 28.16″
Steuerndieb	(15) − (13) =	74 52 31.12
Schanze	(17) − (16) =	52 5 0.00
		179° 59′ 59.28″
	Should be	180 9 0.04
		$w = -0.76''$

$$v_4 - v_3 + v_{15} - v_{13} + v_{17} - v_{16} - 0.76'' = 0 \tag{f}$$

Ägidius	(3) − (2) =	39° 45′ 46.54″
Schanze	(18) − (17) =	56 4 7.29
Burg	(21) − (19) =	84 10 8.50
		180° 0′ 2.33″
	Should be	180 0 0.03
		$w = +2.30''$

$$v_3 - v_2 + v_{18} - v_{17} + v_{21} - v_{19} + 2.30'' = 0 \tag{g}$$

Ägidius	(4) − (2) =	92° 48′ 14.70″
Steuerndieb	(14) − (13) =	44 9 14.00
Burg	(21) − (20) =	43 2 35.64
		180° 0′ 4.34″
	Should be	180 0 0.04
		$w = +4.30''$

$$v_4 - v_2 + v_{14} - v_{13} + v_{21} - v_{20} + 4.30'' = 0 \tag{h}$$

Burg	(20) − (19) =	41° 7′ 32.86″
Steuerndieb	(15) − (14) =	30 43 17.12
Schanze	(18) − (16) =	108 9 7.29
		179° 59′ 57.27″
	Should be	180 0 0.03
		$w = -2.76''$ (i)

This last triangle Burg-Steuerndieb-Schanze no longer belongs to the independent condition equations, because it is already contained in (*f*) + (*g*) − (*h*).

Passing over to the side equations we note at once that the central system around Ägidius offers the first side equation, and the northern quadrilateral the second side equation. If we imagine the base Ägidius-Wasserturm computed back to itself through all five triangles of the central system, then we obtain

$$\frac{\sin(7-6)\,\sin(10-9)\ \ \sin(13-12)\,\sin(17-16)\,\sin(21-19)}{\sin(8-7)\,\sin(11-10)\,\sin(15-13)\,\sin(18-17)\,\sin(22-21)} = 1. \tag{A}$$

If, secondly, we express that any side of the northern diagonal quadrilateral starting from Schanze, computed back to itself, must come out right, then we obtain

$$\frac{\sin(3-2)\,\sin(15-13)\,\sin(20-19)}{\sin(4-3)\,\sin(15-14)\,\sin(21-19)} = 1. \tag{B}$$

Here we need not take into account the small spherical excesses, because the law of sines for the angles of a triangle holds also on the sphere, and we can imagine the side computation carried out according to the so-called method of additaments. The side equation (A) thus yields the following logarithmic calculation:

			Diff. for 10″
(7)—(6)	$= 75^\circ\ 38'\ 44.02''$	$\log\sin = 9.986\ 2255$	54
(10)—(9)	$= 33\ \ 58\ \ 40.67$	$9.747\ 3139$	313
(13)—(12)	$= 34\ \ 25\ \ 13.72$	$9.752\ 2497$	307
(17)—(16)	$= 52\ \ \ 5\ \ \ 0.00$	$9.897\ 0249$	164
(21)—(19)	$= 84\ \ 10\ \ \ 8.50$	$9.997\ 7471$	21
		$Z = 9.380\ 5611$	
(8)—(7)	$= 45^\circ\ \ 5'\ 26.24''$	$\log\sin = 9.850\ 1708$	210
(11)—(10)	$= 50\ \ 15\ \ 28.80$	$9.885\ 8874$	175
(15)—(13)	$= 74\ \ 52\ \ 31.12$	$9.984\ 6894$	57
(18)—(17)	$= 56\ \ \ 4\ \ \ 7.29$	$9.918\ 9250$	142
(22)—(21)	$= 33\ \ 24\ \ 40.16$	$9.740\ 8703$	319
		$N = 9.380\ 5429$	

$$Z - N = +0.000\ 0182\,.$$

The linear side equation belonging to it is for units of the sixth place of logarithms:

$$\begin{aligned}&+0.54\,(v_7-v_6)+3.13\,(v_{10}-v_9)+3.07\,(v_{13}-v_{12})+1.64\,(v_{17}-v_{16})+0.21\,(v_{21}-v_{19})\\&-2.10\,(v_8-v_7)-1.75\,(v_{11}-v_{10})-0.57\,(v_{15}-v_{13})-1.42\,(v_{18}-v_{17})-3.19\,(v_{22}-v_{21})+18.2=0\end{aligned}$$

or, collecting terms:

$$\left.\begin{aligned}&+2.64\,v_7 \qquad\qquad +4.88\,v_{10} \qquad\qquad +3.64\,v_{13} \qquad\qquad +3.06\,v_{17}\\&-0.54\,v_6-2.10\,v_8-3.13\,v_9-1.75\,v_{11}-3.07\,v_{12}-0.57\,v_{15}-1.64\,v_{16}-1.42\,v_{18}\\&\qquad\qquad\qquad\qquad\qquad\qquad +3.40\,v_{21}\\&\qquad\qquad\qquad\qquad\qquad\qquad -0.21\,v_{19}-3.19\,v_{22}+18.2=0\end{aligned}\right\}. \tag{a}$$

Likewise, the second side equation (B) yields the following calculation:

			Diff. for 10″
(3)—(2)	$= 39^\circ\ 45'\ 46.54''$	$\log\sin = 9.805\ 9169$	253
(15)—(13)	$= 74\ \ 52\ \ 31.12$	$9.984\ 6894$	57
(20)—(19)	$= 41\ \ \ 7\ \ 32.86$	$9.818\ 0374$	241
		$Z = 9.608\ 6437$	
(4)—(3)	$= 53^\circ\ \ 2'\ 28.16''$	$\log\sin = 9.902\ 5835$	158
(15)—(14)	$= 30\ \ 43\ \ 17.12$	$9.708\ 3056$	354
(21)—(19)	$= 84\ \ 10\ \ \ 8.50$	$9.997\ 7471$	21
		$N = 9.608\ 6362$	

$$Z - N = +0.000\ 0075$$

$$\begin{aligned}&+2.53\,(v_3-v_2)+0.57\,(v_{15}-v_{13})+2.41\,(v_{20}-v_{19})\\&-1.58\,(v_4-v_3)+3.54\,(v_{15}-v_{14})-0.21\,(v_{21}-v_{19})+7.5=0\,,\end{aligned}$$

collecting terms:

$$\left.\begin{aligned}&+4.11\,v_3 \qquad\qquad -2.97\,v_{15} \qquad\qquad -2.20\,v_{19}\\&-2.53\,v_2-1.58\,v_4-0.57\,v_{13}+3.54\,v_{14}+2.41\,v_{20}-0.21\,v_{21}+7.5=0\end{aligned}\right\}. \tag{b}$$

Now we have scattered, on pp. 238 to 239, the eight linear condition equations (c), (d), (e), (f), (g), (h), (a), (b), which we write now in an arranged form and in the succession (a), (b), . . . , (h) in Table I on the following page.

From this Table I we compute easily all sum coefficients $[a\,a]$, $[a\,b]$, etc; e.g.:

a	a^2	a	b	ab	$a\,c$
0.54	0.2916	+3.64	−0.57	2.0748	+0.54
2.64	6.9696	−0.57	−2.97	+1.6929	+2.64
. .	. . .	−0.21	−2.20	+0.4620	−3.40
3.19	10.1942	+3.40	−0.21	−0.7140	−3.19
$[a\,a] = 107.1942$		$[a\,b] = -0.6339$			$[a\,c] = -3.41 \ldots$.

With this, we form the normal equations, which, in an abbreviated manner of writing and at the same time with rounded-off coefficients (e.g. 107.19 instead of 107.1942), are represented thus [abbreviated manner of writing according to (2) and (3) on p. 86]:

$$
\begin{aligned}
&\underline{+107.19k_1} - 0.63k_2 - 3.41k_3 + 3.27k_4 + 0.08k_5 + 0.49k_6 - 0.87k_7 - 0.24k_8 + 18.20 = 0\\
&\underline{+58.16k_2} - 2.32k_3 \;\ldots\; + 1.01k_5 - 8.09k_6 + 8.63k_7 + 2.44k_8 + 7.50 = 0\\
&\underline{+6.00k_3} - 2.00k_4 \;\ldots\;\ldots\; - 2.00k_7 - 2.00k_8 - 1.02 = 0\\
&\underline{+6.00k_4} - 2.00k_5 \;\ldots\;\ldots\;\ldots\; + 2.22 = 0\\
&\underline{+6.00k_5} - 2.00k_6 \;\ldots\; - 2.00k_8 - 2.36 = 0\\
&\underline{+6.00k_6} - 2.00k_7 + 2.00k_8 - 0.76 = 0\\
&\underline{+6.00k_7} + 2.00k_8 + 2.30 = 0\\
&\underline{+6.00k_8} + 4.30 = 0\,.
\end{aligned}
$$

The solution of these 8 equations yielded

$$
\begin{array}{llll}
k_1 = -0.177 & k_2 = -0.048 & k_3 = -0.387 & k_4 = -0.367\\
k_5 = +0.112 & k_6 = +0.422 & k_7 = -0.014 & k_8 = -0.931\,;
\end{array}
$$

with this, we can calculate the corrections v according to equations (9), section 43, p. 139, namely

$$
\begin{aligned}
v_1 &= a_1\,k_1 + b_1\,k_2 + c_1\,k_3 + \ldots\\
v_2 &= a_2\,k_1 + b_2\,k_2 + c_2\,k_3 + \ldots\,.
\end{aligned}
$$

We follow here the condition equations by columns and obtain a calculation, which we have presented on the following page in tabular form in II immediately below the table of the condition equations I.

We have the first computational check here in the fact that the v's grouped by stations must yield in each case the sum zero. This is indicated on p. 241 by heavier vertical lines, e.g.:

$$+\,0.020 + 0.679 - 0.633 - 0.545 + 0.479 = 0.$$

However, this check is not conclusive, for even if the k's were completely incorrect, the sums of the v's by stations would be $= 0$ because of the grouping of the coefficients of the condition equations. We will hurry as quickly as possible to the decisive check if the corrections v lead to full agreement of the condition equations; we will not treat this however until later on p. 243 and first (for some formal reasons) take up the computation of $[v^2] = -\,[w\,k]$, which contains a computational check and also leads immediately to the mean error.

I. *Condition Equations*, *e.g.* $-0.54\,v_6 + 2.64\,v_7 - 2.10\,v_8 \ldots -3.19\,v_{22} + 18.20 = 0$

		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}	v_{21}	v_{22}	w
k_1	*a*	. .	. .	. .	. .	. .	− 0.54	+ 2.64	− 2.10	− 3.13	+ 4.88	− 1.75	− 3.07	+ 3.64	. .	− 0.57	− 1.64	+ 3.06	− 1.42	− 0.21	. .	+ 3.40	− 3.19	+ 18.20
k_2	*b*	. .	− 2.53	+ 4.11	− 1.58	. .	. .	. .	. .	. .	. .	. .	. .	− 0.57	+ 3.54	− 2.97	. .	. .	. .	− 2.20	+ 2.41	− 0.21	. .	+ 7.50
k_3	*c*	− 1	+ 1	. .	. .	. .	− 1	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 1	+ 1	− 1.02
k_4	*d*	+ 1	. .	. .	. .	− 1	. .	− 1	+ 1	− 1	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	+ 2.22
k_5	*e*	. .	. .	. .	− 1	+ 1	. .	. .	. .	. .	− 1	+ 1	− 1	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 2.36
k_6	*f*	. .	. .	− 1	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	− 1	. .	+ 1	− 1	+ 1	. .	. .	. .	. .	. .	− 0.76
k_7	*g*	. .	− 1	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 1	+ 1	− 1	. .	+ 1	. .	+ 2.30
k_8	*h*	. .	− 1	. .	+ 1	. .	. .	. .	. .	. .	. .	. .	. .	− 1	+ 1	. .	. .	. .	. .	. .	− 1	+ 1	. .	+ 4.30

II. *Calculation of Corrections* v *from the Correlates* k, *e.g.* $v_1 = a_1k_1 + b_1k_2 + \ldots = +0.387 - 0.367 = +0.020$

$k_1 = -0.177$	. .	. .	. .	. .	. .	+ 0.096	− 0.467	+ 0.372	+ 0.554	− 0.864	+ 0.310	+ 0.543	− 0.644	. .	+ 0.101	+ 0.290	− 0.541	+ 0.251	+ 0.037	. .	− 0.602	+ 0.565
$k_2 = -0.048$	. .	+ 0.121	− 0.197	+ 0.076	. .	. .	. .	. .	. .	. .	. .	. .	+ 0.027	− 0.169	+ 0.142	. .	. .	. .	+ 0.106	− 0.116	+ 0.010	. .
$k_3 = -0.387$	+ 0.387	− 0.387	. .	. .	. .	+ 0.387	− 0.387	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	+ 0.387	− 0.387
$k_4 = -0.367$	− 0.367	. .	. .	. .	+ 0.367	. .	+ 0.367	− 0.367	+ 0.367	− 0.367	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .
$k_5 = +0.112$	. .	. .	. .	− 0.112	+ 0.112	. .	. .	. .	. .	− 0.112	+ 0.112	− 0.112	+ 0.112	. .	. .	. .	. .	. .	. .	. .	. .	. .
$k_6 = +0.422$	. .	. .	− 0.422	+ 0.422	. .	. .	. .	. .	. .	. .	. .	. .	− 0.422	. .	+ 0.422	− 0.422	+ 0.422	. .	. .	. .	. .	. .
$k_7 = -0.014$	. .	+ 0.014	− 0.014	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	+ 0.014	− 0.014	+ 0.014	. .	− 0.014	. .
$k_8 = -0.931$	. .	+ 0.931	. .	− 0.931	. .	. .	. .	. .	. .	. .	. .	. .	+ 0.931	− 0.931	. .	. .	. .	. .	. .	+ 0.931	− 0.931	. .
	+ 0.020	+ 0.679	− 0.633	− 0.545	+ 0.479	+ 0.483	− 0.487	+ 0.005	+ 0.921	− 1.343	+ 0.422	+ 0.431	+ 0.004	− 1.100	+ 0.665	− 0.132	− 0.105	+ 0.237	+ 0.157	+ 0.815	− 1.150	+ 0.178
	$= v_1$	$= v_2$	$= v_3$	$= v_4$	$= v_5$	$= v_6$	$= v_7$	$= v_8$	$= v_9$	$= v_{10}$	$= v_{11}$	$= v_{12}$	$= v_{13}$	$= v_{14}$	$= v_{15}$	$= v_{16}$	$= v_{17}$	$= v_{18}$	$= v_{19}$	$= v_{20}$	$= v_{21}$	$= v_{22}$

By rounding off the v's to 0.01", we have:

	v	v^2		v	v^2		v	v^2
1.	+0.02"	0.0004	9.	+0.92"	0.8464	17.	−0.11"	0.0121
2.	+0.68	0.4624	10.	−1.34	1.7956	18.	+0.24	0.0576
3.	−0.63	0.3969	11.	+0.42	0.1764	19.	+0.16	0.0256
4.	−0.55	0.3025	12.	+0.43	0.1849	20.	+0.81	0.6561
5.	+0.48	0.2304	13.	+0.00	0.0000	21.	−1.15	1.3225
6.	+0.48	0.2304	14.	−1.10	1.2100	22.	+0.18	0.0324
7.	−0.49	0.2401	15.	+0.67	0.4489			2.1063
8.	+0.01	0.0001	16.	−0.13	0.0169			4.6791
		1.8632			4.6791			1.8632
								8.6486

(2)

Check computation:

	w	k	$-wk$	
1.	+18.20	−0.177	3.2214	
2.	+ 7.50	−0.048	0.3600	
3.	− 1.02	−0.387		−0.3947
4.	+ 2.22	−0.367	0.8147	
5.	− 2.36	+0.112	0.2643	
6.	− 0.76	+0.422	0.3207	
7.	+ 2.30	−0.014	0.0322	
8.	+ 4.30	−0.931	4.0033	
			8.6219	

(3)

The agreement between 8.65 and 8.62 is sufficient; we compute therefrom the mean error of a direction, for which we obtain, on account of the 8 condition equations,

$$m = \sqrt{\frac{8.63}{8}} = \pm\, 1.04''. \tag{4}$$

But now we will pass over to the last decisive check, namely, satisfying the condition equations and, for this, must add the just computed v's to the observed directions given earlier under (1) on p. 233.

Here we have to explain a few formal deviations, since individual v's of the following table on p. 243 deviate by up to 0.01' from the v's just summarized in (2), e.g. $v_9 = +0.92''$ appears later $= +0.91''$. Since the 0.01" have in our case, in general, no longer any real value, but run along only as a last computational check place, we hardly need to give an account about a variation of 0.01"; it often occurs in the case of such adjustments that we eliminate small disagreements still remaining after the adjustment in the condition equations entirely, offhand, at our own discretion. In our case, however, we have carried out, in addition to the adjustment given above, a second adjustment with a more rigorous form of the second side equation and used this second adjustment also in writing down the final v's, from which the few formal deviations of 0.01" mentioned have resulted.

We shall come back to this in section 77 and also mention further another formal circumstance concerning the *sequence* of the condition equations in Table I, on p. 241.

	Observed (cf. p. 233)	Correction v	Adjusted
1. *Ägidius*			
Wasserturm	(1) = 0° 0′ 0.00″	+ 0.02″	0° 0′ 0.02″
Burg	(2) = 70 56 34.82	+ 0.68	70 56 35.50
Schanze	(3) = 110 42 21.36	− 0.63	110 42 20.73
Steuerndieb	(4) = 163 44 49.52	− 0.55	163 44 48.97
Willmer	(5) = 259 4 4.67	+ 0.48	259 4 5.15
	50.37″	0.00″	50.37″
2. *Wasserturm*			
Burg	(6) = 284° 21′ 15.98″	+ 0.48″	284° 21′ 16.46″
Ägidius	(7) = 0 0 0.00	− 0.48	359 59 59.52
Willmer	(8) = 45 5 26.24	− 0.00	45 5 26.24
	42.22″	0.00″	42.22″
3. *Willmer*			
Wasserturm	(9) = 326° 1′ 19.33″	+ 0.91″	326° 1′ 20.24″
Ägidius	(10) = 0 0 0.00	− 1.33	359 59 58.67
Steuerndieb	(11) = 50 15 28.80	+ 0.42	50 15 29.22
	48.13″	0.00″	48.13″
4. *Steuerndieb*			
Willmer	(12) = 325° 34′ 46.28″	+ 0.43″	325° 34′ 46.71″
Ägidius	(13) = 0 0 0.00	+ 0.01	0 0 0.01
Burg	(14) = 44 9 14.00	− 1.09	44 9 12.91
Schanze	(15) = 74 52 31.12	+ 0.66	74 52 31.78
	31.40″	+ 0.01″	31.41″
5. *Schanze*			
Steuerndieb	(16) = 307° 55′ 0.00″	− 0.13″	307° 54′ 59.87″
Ägidius	(17) = 0 0 0.00	− 0.10	359 59 59.90
Burg	(18) = 56 4 7.29	+ 0.23	56 4 7.52
	7.29″	0.00″	7.29″
6. *Burg*			
Schanze	(19) = 275° 49′ 51.50″	+ 0.16″	275° 49′ 51.66″
Steuerndieb	(20) = 316 57 24.36	+ 0.81	316 57 25.17
Ägidius	(21) = 0 0 0.00	− 1.15	359 59 58.85
Wasserturm	(22) = 33 24 40.16	+ 0.17	33 24 40.33
	56.02″	− 0.01″	56.01″.

With these adjusted directions we form again the angles of the triangles as differences, just as previously in (c) to (h) on p. 238 with the observed directions. We write this in detail only for the first triangle:

$$\begin{array}{llll}
[2] = 70^\circ\,56'\,35.50'' & [7] = 359^\circ\,59'\,59.52'' & [22] = 33^\circ\,24'\,40.33'' \\
[1] = 0\;\;0\;\;0.02 & [6] = 284\;\;21\;\;16.46 & [21] = 359\;\;59\;\;58.85 \\
\hline
[2]-[1] = 70^\circ\,56'\,35.48'' & [7]-[6] = 75^\circ\,38'\,43.06'' & [22]-[21] = 33^\circ\,24'\,41.48''
\end{array}$$

$$\begin{array}{lll}
\text{Ägidius} & [2]-[1] & = 70^\circ\,56'\,35.48'' \\
\text{Wasserturm} & [7]-[6] & = 75\;\;38\;\;43.06 \\
\text{Burg} & [22]-[21] & = 33\;\;24\;\;41.48 \\
\hline
 & & 180^\circ\;\;0'\;\;0.02''. \qquad [c]
\end{array}$$

In the same manner we form also the remaining adjusted triangles:

$$\begin{array}{lll}
\text{Ägidius} & [1]-[5] & = 100^\circ\,55'\,54.87'' \\
\text{Willmer} & [10]-[9] & = 33\;\;58\;\;38.43 \\
\text{Wasserturm} & [8]-[7] & = 45\;\;5\;\;26.72 \\
\hline
 & & 180^\circ\;\;0'\;\;0.02'' \quad [d]
\end{array}
\qquad
\begin{array}{lll}
\text{Ägidius} & [3]-[2] & = 39^\circ\,45'\,45.23'' \\
\text{Schanze} & [18]-[17] & = 56\;\;4\;\;7.62 \\
\text{Burg} & [21]-[19] & = 84\;\;10\;\;7.19 \\
\hline
 & & 180^\circ\;\;0'\;\;0.04'' \quad [g]
\end{array}$$

$$\begin{array}{lll}
\text{Ägidius} & [5]-[4] & = 95^\circ\,19'\,16.18'' \\
\text{Steuerndieb} & [13]-[12] & = 34\;\;25\;\;13.30 \\
\text{Willmer} & [11]-[10] & = 50\;\;15\;\;30.55 \\
\hline
 & & 180^\circ\;\;0'\;\;0.03'' \quad [e]
\end{array}
\qquad
\begin{array}{lll}
\text{Ägidius} & [4]-[2] & = 92^\circ\,48'\,13.47'' \\
\text{Steuerndieb} & [14]-[13] & = 44\;\;9\;\;12.89 \\
\text{Burg} & [21]-[20] & = 43\;\;2\;\;33.68 \\
\hline
 & & 180^\circ\;\;0'\;\;0.04'' \quad [h]
\end{array}$$

$$\begin{array}{lll}
\text{Ägidius} & [4]-[3] & = 53^\circ\,2'\,28.24'' \\
\text{Steuerndieb} & [15]-[13] & = 74\;\;52\;\;31.77 \\
\text{Schanze} & [17]-[16] & = 52\;\;5\;\;0.03 \\
\hline
 & & 180^\circ\;\;0'\;\;0.04'' \quad [f]
\end{array}
\qquad
\begin{array}{lll}
\text{Burg} & [20]-[19] & = 41^\circ\,7'\,33.51'' \\
\text{Steuerndieb} & [15]-[14] & = 30\;\;43\;\;18.88 \\
\text{Schanze} & [18]-[16] & = 108\;\;9\;\;7.65 \\
\hline
 & & 180^\circ\;\;0'\;\;0.04''. \quad [i]
\end{array}$$

For the computation of the triangles, we still distribute the small excesses, which amount to between 0.02" and 0.04", to the angles of the triangles, and then we only need, in addition, a base length in order to be able to compute all triangle sides according to the sine law. As was mentioned at the end of the previous section 69, p. 235, the base side

$$\text{Ägidius-Wasserturm} \qquad S = 2391.672 \text{ m}$$
$$\log S = 3.3787016.$$

With this, all remaining triangle sides were also computed free of discrepancy in the manner indicated with the following results:

Ägidius—Wasserturm	$\log S = 3.878\,7016$	(5)
Ägidius—Willmer	$\log S = 3.481\,3665$	
Ägidius—Steuerndieb	$\log S = 3.615\,2086$	
Ägidius—Schanze	$\log S = 3.702\,8735$	
Ägidius—Burg	$\log S = 3.624\,0521$	
Wasserturm—Willmer	$\log S = 3.623\,4413$	
Willmer—Steuerndieb	$\log S = 3.727\,4425$	
Steuerndieb—Schanze	$\log S = 3.620\,7673$	
Schanze—Burg	$\log S = 3.511\,0402$	
Burg—Wasserturm	$\log S = 3.618\,3487$	
Burg—Steuerndieb	$\log S = 3.780\,5588$	

Section 71. Station Data [Abrisse] and Coordinates of the Triangulation of Hannover

After conclusion of the net adjustment, it is a question of summarizing in an easily visible manner the results of the triangulation and computing the coordinates of the triangle points. We are concerned here with the kind of coordinate system to be taken as a basis.

As already stated at the end of section 69, p. 236, we have used for our two base points Ägidius and Linden, Wasserturm the connecting coordinates in the rectangular plane system of the conformal double projection of the Reichsamt für Landesaufnahme. Further details about this coordinate system, which for decades has played an important role in Prussian geodesy, are given in our Vol. III, 7th Ed., 1923, section 109.

For cadastral and city surveys there were used hitherto rectangular spheroidal coordinate systems, more narrowly bounded, about which we have written in Vol. II, 1st half-volume, 1931, section 34, as well as in Vol. III, 7th Ed., 1923, section 57. At the same places, the plane conformal Gauss-Krüger coordinates, to be used in the future, have also been treated.

In order to have side by side, in an easily visible example, the adjustment results for the different kinds of coordinates, the results of the triangulation of Hannover shall in the following be brought to a conclusion separately in the different systems.

I. Station data and coordinates in the system of the conformal double projection of the Land Survey

The theory of conformal rectangular coordinates is treated in Volume III of this handbook, 7th Edition, 1923, section 48, pp. 288 to 297 (also *Zeitschrift für Vermessungswesen*, 1894, pp. 167-171). Let us insert here the practical formulae only with reference to Fig. 1.

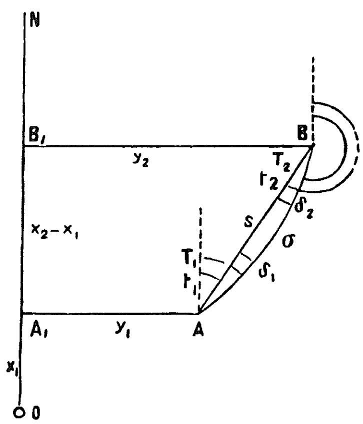


Fig. 1.

Letting a point A have the coordinates of projection x_1, y_1, and a point B, accordingly, x_2, y_2, then we have for the rectilinear distance s and the direction angle t_1 in the plane rectangular system as always:

$$\left.\begin{aligned} \tan t_1 &= \frac{y_2-y_1}{x_2-x_1} \\ s = \frac{y_2-y_1}{\sin t_1} &= \frac{x_2-x_1}{\cos t_1} = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2} \end{aligned}\right\}. \qquad (1)$$

In addition, let S be the spherical distance of the points A and B; for this, we use the ratio of magnification

$$\left.\begin{aligned} \frac{ds}{dS} = m = 1+\frac{y^2}{2r^2} \quad \text{or} \quad \log m = \frac{\mu}{2r^2}y^2, \\ \log\frac{\mu}{2r^2} = 2.72670. \end{aligned}\right\} \qquad (2)$$

where for 7-place logarithms

For a distance S and its projection s we have in logarithmic form

$$\left.\begin{aligned} \log s - \log S &= \frac{\mu}{12\,r^2}(y_1{}^2 + 4\,y_0{}^2 + y_2{}^2)\,, \\ \text{where } y_0 = \frac{y_1 + y_2}{2} &\text{ and } \log\frac{\mu}{12\,r^2} = 1.94855\,. \end{aligned}\right\} \tag{3}$$

If y_1 and y_2 are nearly equal, i.e. if the difference $y_2 - y_1$ is relatively small in comparison to y itself, we can compute approximately

$$\left.\begin{aligned} \text{Approximately } \log s - \log S &= \frac{\mu}{8\,r^2}(y_1 + y_2)^2\,, \\ \text{where } \log\frac{\mu}{8\,r^2} &= 2.12464\,. \end{aligned}\right\} \tag{4}$$

Furthermore, let T_1 be the spherical (also spheroidal) direction angle of the ray AB at A. For the reduction from t_1 to T_1 we have

$$T_1 - t_1 = (x_2 - x_1)\,\frac{(2\,y_1 + y_2)}{3}\,\frac{\varrho}{2\,r^2}\,,$$

and, accordingly,

$$\left.\begin{aligned} T_2 - t_2 &= (x_1 - x_2)\,\frac{(2\,y_2 + y_1)}{3}\,\frac{\varrho}{2\,r^2}\,, \end{aligned}\right\} \tag{5}$$

$$\text{where } \log\frac{\varrho}{2\,r^2} = 1.40334\,.$$

If $y_2 - y_1$ is small in comparison to y_1 and y_2 itself, we can also take these two reductions immediately, namely approximately

$$\left.\begin{aligned} T_1 - t_1 = t_2 - T_2 &= (x_2 - x_1)\,(y_2 + y_1)\,\frac{\varrho}{4\,r^2}\,, \\ \text{where } \log\frac{\varrho}{4\,r^2} &= 1.10231\,. \end{aligned}\right\} \tag{6}$$

The origin of the system is at the meridian for 31° longitude at latitude 52°42' 2.53251" (corresponding to 52°40' on the conformal sphere of Gauss). The $+x$'s are counted to the north and the $+y$'s are counted to the east; in the region of Hannover we have, therefore, $-y$ and $-x$, and our two base points are

Ägidius	$y_1 = -\,244\,656.090$ m	$x_1 = -\,30\,624.971$ m
Wasserturm	$y_2 = -\,246\,956.479$	$x_2 = -\,31\,285.875$
	$y_1 - y_2 = +\quad 2\,300.389$ m	$x_1 - x_2 = +\quad 660.904$ m.

If we compute therefrom the distance and the two direction angles according to the formulae (3) to (6), then we obtain

$$\begin{aligned} \log\,(y_1 - y_2) &= 3.361\,8013 \\ \log\,(x_1 - x_2) &= 2.820\,1384 \\ \hline \log\tan\,t_2 &= 0.541\,6629 \qquad t_2 = 73^\circ\,58'\,14.12'' \end{aligned} \tag{7}$$

$$s = \frac{y_1 - y_2}{\sin t_2} = \frac{x_1 - x_2}{\cos t_2}\,..\,3.379\,0236$$

$\log s - \log S = 3220$ according to formula (3) or (4);

therefore

$$\log S = 3.378\,7016\,. \tag{8}$$

This is log S, as is already indicated in the adjustment of triangles in section 70, p. 243. We compute further $t - T$ according to (5) or (6) and have then

$$\begin{array}{rr} t_1 = 253^\circ\, 58'\, 14.12'' & t_2 = 73^\circ\, 58'\, 14.12'' \\ T_1 - t_1 = \quad +0.41 & T_2 - t_2 = \quad -0.41 \\ \hline T_1 = 253^\circ\, 58'\, 14.53'' & T_2 = 73^\circ\, 58'\, 13.71'' \end{array} \qquad (9)$$

In order to compute $\log s - \log S$ and $t - T$ also for the remaining triangle sides, we must first determine preliminary approximate values by rounded-off assumptions, something which, moreover, is useful also for other reasons. Such approximate values are

	y	x	
1. Ägidius	$y = -$ 244 656.1 m	$x = -$ 30 625.0 m	
2. Wasserturm	− 246 956.5	− 31 285.9	
3. Willmer	− 243 280.9	− 33 328.4	
4. Steuerndieb	− 241 167.9	− 28 421.4	(10)
5. Burg	− 247 076.5	− 27 179.2	
6. Schanze	− 244 244.4	− 25 592.9	

In the case of the computation of the $\log s - \log S$'s it is very convenient that these reductions depend only on the ordinates y and not at all on the abscissae x; therefore, we can take care of the computation of the $\log m$'s once and for all by means of a table. We have computed, for our measured area of Hannover, such a table, of which we indicate here, however, for the sake of saving space, only the main values for y for increasing intervals of 1 km, and, in fact, $\log m$ in units of the seventh place of logarithms

y	$\log m$	y	$\log m$	y	$\log m$	y	$\log m$
230 km	2819·4	240 km	3069·9	250 km	3331·0	260 km	3602·8
231	2844·0	241	3095·5	251	3357·7	261	3630·6
232	2868·6	242	3121·3	252	3384·6	262	3658·5
233	2893·4	243	3147·1	253	3411·5	263	3686·5
234	2918·3	244	3173·1	254	3438·5	264	3714·6
235	2943·3	245	3199·1	255	3465·6	265	3742·8
236	2968·4	246	3225·3	256	3492·8	266	3771·1
237	2993·6	247	3251·6	257	3520·2	267	3799·5
238	3018·9	248	3278·0	258	3547·6	268	3828·0
239	3044·4	249	3304·4	259	3575·2	269	3856·6

(11)

We can treat this matter also graphically: We cover the picture of the net of triangulation (Fig. 1, section 69, p. 234) with a series of ordinate lines of constant $\log m$'s and can then scale off the $\log m$ belonging to each point by interpolation. In this manner, we still obtain the seventh place of logarithms nearly correct even at the small scale 1:80,000 of Fig. 1, p. 234, and in any case, we have, with this, a suitable check of the computation by tables or the direct computation according to formula (3) or (4), p. 245 (cf. also Volume III, 7th Ed., section 49).

The Land Survey used a detailed table of $\log m$'s according to the principle of our preceding example (11) for the whole of Prussia, besides a small indication for showing how much more convenient the computation in conformal coordinates is in comparison to the Soldner coordinates. –

In this manner (cf. (5), p. 245), the following values have resulted:

Triangle Side	$\log S$ (spherical)	$\log s - \log S$	$\log s$ (plane)
Ägidius—Wasserturm	3.378 7016	+ 3220	3.379 0236
Ägidius—Willmer	3.481 5665	+ 3172	3.481 8837
Ägidius—Steuerndieb	3.615 2086	+ 3145	3.615 5231
Ägidius—Schanze	3.702 8735	+ 3185	3.703 1920
Ägidius—Burg	3.624 0521	+ 3222	3.624 3743
Wasserturm—Willmer	3.623 4413	+ 3202	3.623 7615
Willmer—Steuerndieb	3.727 4425	+ 3127	3.727 7552
Steuerndieb—Schanze	3.620 7673	+ 3140	3.621 0813
Schanze—Burg	3.511 0402	+ 3216	3.511 3618
Burg—Wasserturm	3.613 3487	+ 3252	3.613 6739
Burg—Steuerndieb	3.780 5583	+ 3177	3.780 8760

(12)

After this computation of the sides there comes the setting up of the direction data in the following table in which there occur, however, not only the directions of the pentagon of section 70, p. 236, but also those directions which refer to the directions to the two points Hochschule and Dreifaltigkeit in the tables of measurement of section 69, p. 233.

In these station data there are, in the first column, the observed directions T, i.e., such values which, with respect to the directly observed directions of section 69, p. 233, are shifted by a constant amount at a station.

Station data in the system of the conformal double projection of the Land Survey (13)

	Observed T (spherical)	v	Adjusted T (spherical)	$t-T$	Adjusted t (plane)
1. *Ägidius*			(γ=2° 50′ 49.56″)		
Wasserturm . . 1.	253° 58′ 14.51″	+ 0.02″	253 58 14.53	− 0.41″	253° 58′ 14.12″
Hochschule . .	317 37 7.82			+ 0.93	(317 37 8.75)
Burg . . . . . 2.	324 54 49.33	+ 0.68	324 54 50.01	+ 2.14	324 54 52.15
Schanze . . . . 3.	4 40 35.87	− 0.63	4 40 35.24	+ 3.11	4 40 38.35
Dreifaltigkeit . .	37 38 22.91			+ 0.83	(37 38 23.74)
Steuerndieb . . 4.	57 43 4.03	− 0.55	57 43 3.48	+ 1.36	57 43 4.84
Willmer . . . . 5.	153 2 19.18	+ 0.48	153 2 19.66	− 1.67	153 2 17.99
2. *Wasserturm*			(γ=2° 52′ 23.46″)		
Burg . . . . . 6.	358° 19′ 30.17″	+ 0.48″	358 19 30.65	+ 2.57″	358° 19′ 33.22″
Hochschule . .	23 11 24.44			+ 1.35	(23 11 25.79)
Ägidius . . . 7.	73 58 14.19	− 0.48	73 58 13.71	+ 0.41	73 58 14.12
Willmer . . . . 8.	119 3 40.43	0.00	119 3 40.43	− 1.27	119 3 39.16
3. *Willmer*			(γ=2° 49′ 43.18″)		
Wasserturm . . 9.	299° 3′ 36.98″	+ 0.91″	299 3 37.89	+ 1.26″	299° 3′ 39.15″
Ägidius . . . . 10.	333 2 17.65	− 1.33	333 2 16.32	+ 1.67	333 2 17.99
Dreifaltigkeit . .	355 11 53.75			+ 2.49	(355 11 56.24)
Steuerndieb . . 11.	23 17 46.45	+ 0.42	23 17 46.87	+ 3.01	23 17 49.88
4. *Steuerndieb*			(γ=2° 48′ 31.02″)		
Willmer . . . 12.	203° 17′ 52.46″	+ 0.43″	203 17 52.89	− 3.00″	203° 17′ 49.89″
Ägidius . . . . 13.	237 43 6.18	+ 0.01	237 43 6.19	− 1.35	237 43 4.84
Dreifaltigkeit . .	250 39 21.26			− 0.53	(250 39 20.73)
Hochschule . .	261 48 52.72			− 0.43	(261 48 52.29)
Burg . . . . . 14.	281 52 20.18	− 1.09	281 52 19.09	+ 0.76	281 52 19.85
Schanze . . . . 15.	312 35 37.30	+ 0.66	312 35 37.96	+ 1.73	312 35 39.69
5. *Schanze*			(γ=2° 50′ 49.06″)		
Steuerndieb . . 16.	132° 35′ 41.56″	− 0.13″	132 35 41.43	− 1.74″	132° 35′ 39.69″
Dreifaltigkeit . .	170 24 24.64			− 2.28	(170 24 22.36)
Ägidius . . . . 17.	184 40 41.56	− 0.10	184 40 41.46	− 3.11	184 40 38.35
Burg . . . . . 18.	240 44 48.85	+ 0.23	240 44 49.08	− 0.98	240 44 48.10
6. *Burg*			(γ=2° 52′ 42.26″)		
Schanze . . . . 19.	60° 44′ 46.95″	+ 0.16″	60 44 47.11	+ 0.99″	60° 44′ 48.10″
Steuerndieb . . 20.	101 52 19.81	+ 0.81	101 52 20.62	− 0.77	101 52 19.85
Dreifaltigkeit . .	121 19 31.32			− 1.31	(121 19 30.01)
Ägidius . . . . 21.	144 54 55.45	− 1.15	144 54 54.30	− 2.15	144 54 52.15
Hochschule . .	151 38 51.38			− 1.21	(151 38 50.17)
Wasserturm . . 22.	178 19 35.61	+ 0.17	179 19 35.78	− 2.57	179 19 33.21

Inasmuch as we can always rotate observed sets by an arbitrary angle at one time (just as the assumption 0°0′0″ for each ray in the measured sets of section 69, p. 233, already rests on an arbitrary twisting), we can still designate the numbers of the first column as "observed," if their shifting is constant, and it is only just a question of the amount of this shifting, which we will determine in the first example of Station Ägidius by starting with the introduction of the *adjusted* T_1 in the third column. The value $T = 253°58'14.53''$ computed from the base coordinates in (9), p. 246, is to be substituted here, and since the net corrections v of the second column are known from section 70, p. 242, we also have the first observed $T = 253°58'14.53''$ $- 0.02'' = 253°58'14.51''$, and the values indicated previously as observed at Station Ägidius of p. 243 appear all increased by 253°58' 14.51" in the first column of p. 247, and the adjusted directions of p. 243 also appear now in the third column of p. 247 with an increase of the same amount. If we have set the $t - T$'s into the fourth column of p. 247, then we also obtain all values t of the last column of p. 247 and, with this, also the passage to the *other* stations. For since the t's correspond to *plane* rectilinear coordinates, they change from one point to the point on the other side by adding ±180°.

The values γ added next to the adjusted T's are the meridian convergences.

Let us make only an additional remark in regard to those rays which, until now, have not yet occurred at all in the net adjustment, namely, concerning the points Hochschule and Dreifaltigkeit; these rays have simply been taken over from section 69, p. 233, and at the already adjusted stations Ägidius, Wasserturm, etc., these new rays are oriented quite simply and automatically *with* the old rays.

At the stations Hochschule and Dreifaltigkeit themselves, this is not yet possible, and could only be carried out perhaps approximately, of which we shall however speak only in the next chapter in the case of the double intercalation of points Hochschule-Dreifaltigkeit.

Now the adjusted sides and directions of the tables of p. 246 and p. 247 contain everything which is required for the computation of coordinates; we compute with the *plane* sides s and the *plane* direction angles t as always in the plane:

$$y_2 = y_1 + s_1 \sin t_1, \qquad x_2 = x_1 + s_1 \cos t_1, \text{ etc.}$$

Following are the results of this computation of coordinates:

Coordinates in the system of the conformal double projection of the Land Survey

1. Ägidius	$y = -244\,656.090$ m	$x = -30\,624.971$ m	(14)
2. Wasserturm	$-246\,956.479$	$-31\,285.875$	
3. Willmer	$-243\,280.909$	$-33\,328.385$	
4. Steuerndieb	$-241\,167.896$	$-28\,421.362$	
5. Burg	$-247\,076.504$	$-27\,179.218$	
6. Schanze	$-244\,244.387$	$-25\,592.941$	

II. Station data [Abrisse] and coordinates in the cadastral system with the zero point Celle

Quite independently of the foregoing, we will further utilize the adjusted directions, angles, and sides of section 70 for station data [Abrisse] and coordinates in the hitherto existing cadastral system Celle. This system is a rectangular-spheroidal system of coordinates, which can however be treated with great approximation as a spherical system, for which we have developed the required formulae in our Volume III, 7th Ed., 1923, sections 44 to 47.

The coordinates of our two base points already indicated in (6), section 69, p. 235, are:

Ägidius	$y_1 = -23271.813$	$x_1 = -28308.395$	(15)
Wasserturm	$y_2 = -25538.488$	$x_2 = -29071.474$	
	$y_1 - y_2 = +\ 2266.675$	$x_1 - x_2 = +\ 763.079$	

In purely *plane* computation [corresponding to t and s of (1), p. 244] we obtain

$$\tan\ \alpha_0 = \frac{y_1 - y_2}{x_1 - x_2}, \qquad s_0 = \frac{y_1 - y_2}{\sin\ \alpha_0} = \frac{x_1 - x_2}{\cos\ \alpha_0},$$

$$\alpha_0 = 71°\,23'\,38.85'', \qquad \log s_0 = 3.378\,7019\,. \tag{16}$$

Considering the matter from a practical viewpoint, we could now, perhaps, compute the whole net (Fig. 1, section 69, p. 234) in this manner simply as *plane* according to coordinates, but the usual spherical reductions for Soldner coordinates (which are indicated in our Volume III, *Handbuch der Vermessungskunde*, 7th Ed., sections 44 to 46) still bring, however, in our case about 0.5" in the directions, and since we have adjusted everything to 0.01", we will not now neglect already again 0.5", especially because the mean error of a measured direction is only ±1.0" after the net adjustment.

Otherwise in the coordinates we would already obtain discrepancies in the centimeters in purely plane computation, while it is convenient in the case of a city survey to calculate everything basic to 1 mm formally, but in any case to 1 cm rigorously. In short, for these reasons we have not been satisfied with the results (15) and (16), but have taken into account, also, the small correction terms touched, without proving them in detail, since they can be assumed as known according to Volume III, sections 44 to 46, and have found:

$$\alpha_1 - \alpha_0 = +0.41'', \quad \alpha_2 - \alpha_0 \pm 180° = +0.40'', \quad \log s - \log s_0 = -0.000\,0003, \tag{17}$$

therefore, added to (16):

$$\alpha_1 = 71°23'39.26'', \quad \alpha_2 = 251°23'39.35'', \quad \log s = 3.378\,7016. \tag{18}$$

The last $\log s$ agrees with $\log S$ of (8), p. 245. The α_1's and α_2's correspond to the previous T_1's and T_2's of p. 245; they are about 2°35' smaller than those T's, because the α's contain only the meridian convergence with respect to Celle, the T's, on the other hand, the meridian convergence with respect to the 31st degree of longitude (at Berlin).

We find the just computed direction angle (cadastral designation "inclination") $\alpha_2 = 251°23'39.35''$ again in the station data [Abriss] of p. 250 as the first value of adjusted α, and with $v_1 = +0.02''$ from the net adjustment, section 70, p. 242, we obtain as the first observed $A = 251°23'39.33''$, from which it follows that all directions observed at Ägidius of section 69, p. 233, must now be shifted by 251°23'39.33", because Ägidius-Wasserturm had previously the value 0°0'0.00".

We proceed likewise at station Wasserturm with the direction angle Ägidius = 71°23'39.26", according to computation (18) just carried out.

In order to orient then also the remaining sets completely, we have first Willmer-Ägidius as the inversion of Ägidius-Willmer = 330°27'44.48", which changes, however, to 330°27'44.19", because of the spherical corrections.

We have no longer shown these small spherical corrections in detail; they correspond to the $t - T$'s in the station data of the Land Survey of p. 247. In that conformal system, the $t - T$'s as well as the $\log s - \log S$'s were to be treated much more smoothly and clearly than in the Soldner cadastral system, in which the small direction and distance corrections, in the case of the usual manner of computation, are hidden in the coordinate corrections, and also otherwise are not computed as conveniently as in a conformal system.

The $\log S$'s indicated in the table on p. 250 are the same as in (12), p. 246.

Noting this only incidentally, we give in the following the final coordinates:

Coordinates in the cadastral system Celle

	y	x	
1. Ägidius	$y = -$ 23 271.813 *m*	$x = -$ 28 308.395 *m*	
2. Wasserturm	− 25 538.488	− 29 071.474	
3. Willmer	− 21 777.609	− 30 945.359	(19)
4. Steuerndieb	− 19 888.668	− 25 951.884	
5. Schanze	− 23 086.933	− 23 266.607	
6. Burg	− 25 842.799	− 24 977.399	

Station data [Abriss] in the cadastral system Celle

	Observed A	v	Adjusted $\alpha = A + v$	$\log S$
1. *Ägidius*				
Wasserturm . . 1.	251° 23′ 39.33″	+ 0.02″	251° 23′ 39.35″	3.378 7016
Hochschule . .	315 2 32.64			
Burg . . . . . 2.	322 20 14.15	+ 0.68	322 20 14.83	3.624 0521
Schanze . . . . 3.	2 6 0.69	— 0.63	2 6 0.07	3.702 8735
Dreifaltigkeit . .	35 3 47.73			
Steuerndieb . . 4.	55 8 28.85	— 0.55	55 8 28.30	3.615 2086
Willmer . . . . 5.	150 27 44.00	+ 0.48	150 27 44.48	3.481 5665
2. *Wasserturm*				
Burg . . . . . 6.	355° 44′ 55.72″	+ 0.48″	355° 44′ 56.20″	3.613 3487
Hochschule . .	20 36 49.99			
Ägidius . . . . 7.	71 23 39.74	— 0.48	71 23 39.26	3.378 7016
Willmer . . . . 8.	116 29 5.98	0.00	116 29 5.98	3.623 4413
3. *Willmer*				
Wasserturm . . 9.	296° 29′ 4.85″	+ 0.91″	296° 29′ 5.76″	3.623 4413
Ägidius . . . . 10.	330 27 45.52	— 1.33	330 27 44.19	3.481 5665
Dreifaltigkeit . .	352 37 21.62			
Steuerndieb . 11.	20 43 14.32	+ 0.42	20 43 14.74	3.727 4425
4. *Steuerndieb*				
Willmer . . . . 12.	200° 43′ 14.83″	+ 0.43″	200° 43′ 15.26″	3.727 4425
Ägidius . . . . 13.	235 8 28.55	+ 0.01	235 8 28.56	3.615 2086
Dreifaltigkeit . .	248 4 43.63			
Hochschule . .	259 14 15.09			
Burg . . . . . 14.	279 17 42.55	— 1.09	279 17 41.46	3.780 5583
Schanze . . . . 15.	310 0 59.67	+ 0.66	310 1 0.31	3.620 7673
5. *Schanze*				
Steuerndieb . . 16.	130° 1′ 0.73″	— 0.13″	130° 1′ 0.60″	3.620 7673
Dreifaltigkeit . .	167 49 43.84			
Ägidius . . . . 17.	182 6 0.73	— 0.10	182 6 0.66	3.702 8735
Burg . . . . . 18.	238 10 8.02	+ 0.23	238 10 8.24	3.511 0402
6. *Burg*				
Schanze . . . . 19.	58° 10′ 7.89″	+ 0.16″	58° 10′ 8.03″	3.511 0402
Steuerndieb . . 20.	99 17 40.75	+ 0.81	99 17 41.58	3.780 5583
Dreifaltigkeit . .	118 44 52.26			
Ägidius . . . 21.	142 20 16.39	— 1.15	142 20 15.24	3.624 0521
Hochschule . .	149 4 12.32			
Wasserturm . . 22.	175 44 56.55	+ 0.17	175 44 56.73	3.613 3487

III. Geographic coordinates and azimuths

Not belonging to the adjustment itself, the geographic coordinates and the azimuths of the net of Fig. 1, p. 236, are however to be connected here because of completeness.

The basic values of the geographic longitudes and latitudes in the system of the Land Survey for the two base points Ägidius and Wasserturm are already given under (4) p. 235, where the geographic longitudes refer to the meridian of Ferro. Since at the Reichsamt für Landesaufnahme the meridian of Greenwich has been introduced as zero meridian, we will now likewise change to Greenwich and, to do so, have to subtract from the geographic longitudes on p. 235 the difference of longitude Ferro-Greenwich = 17°40′. With the thus obtained connecting values and with the adjusted angles and sides of section 70 and section 71 we have computed the following, partly according to the computational instructions of the Land Survey, partly according to the formulae of Gauss for the mean latitude in our Volume III, 7th Ed., 1923, p. 456:

Point	*Latitude*	*Longitude*	
Ägidius	52° 22′ 14.9611″	9° 44′ 24.6289″	(20)
Linden Wasserturm	52 21 49.9080	9 42 25.0167	
Willmer	52 20 49.8613	9 45 44.2318	
Steuerndieb	52 23 31.6871	9 47 22.9670	
Schanze	52 24 58.1187	9 44 33.1515	
Burg	52 24 2.3220	9 42 7.7967	

Furthermore, all azimuths (counted from north through east) and, to this, once again the logarithms of the triangle sides of (5), p. 244, or (12), p. 246, are:

Station Direction	Azimuth α	*log S*
1. *Ägidius*		
Wasserturm . . .	$\alpha = 251°\ 7'\ 24.97''$	3.378 7016
Burg	.. 322 4 0.45	3.624 0521
Schanze	.. 1 49 45.68	3.702 8735
Steuerndieb . . .	.. 54 52 13.92	3.615 2086
Willmer	.. 150 11 30.10	3.481 5665
2. *Wasserturm*		
Burg	$\alpha = 355°\ 27'\ 7.19''$	3.613 3487
Ägidius	.. 71 5 50.25	3.378 7016
Willmer	.. 116 11 16.97	3.623 4413
3. *Willmer*		
Wasserturm . . .	$\alpha = 296°\ 13'\ 54.71''$	3.623 4413
Ägidius	.. 330 12 33.14	3.481 5665
Steuerndieb . . .	.. 20 28 3.69	3.727 4425
4. *Steuerndieb*		
Willmer	$\alpha = 200°\ 29'\ 21.87''$	3.727 4425
Agidius	.. 234 54 35.17	3.615 2086
Burg	.. 279 3 48.07	3.780 5583
Schanze	.. 309 47 6.94	3.620 7673
5. *Schanze*		
Steuerndieb . . .	$\alpha = 129°\ 44'\ 52.37''$	3.620 7673
Ägidius	.. 181 49 52.40	3.702 8735
Burg	.. 237 54 0.02	3.511 0402
6. *Burg*		
Schanze	$\alpha = 57°\ 52'\ 4.85''$	3.511 0402
Steuerndieb . . .	.. 98 59 38.36	3.780 5583
Ägidius	.. 142 2 12.04	3.624 0521
Wasserturm . . .	.. 175 26 53.52	3.613 3487

If we compute the azimuths and the log S's again backward from the previously given latitudes and longitudes, then we will not find the last places completely in agreement, first, for conceivable reasons of rounding off, second however, also because in the computations of the Land Survey the azimuths were transformed not only from geographic coordinates, but also by the *more rigorous* way of rectangular coordinates (14), p. 248, which has been done in the above α's.

Our values summarized here as well as previously (section 70) are those which have been published in "Abrisse und Koordinaten" of the trigonometric division of the Land Survey, Volume XVII, 1907. These publications, however, still give the geographic longitudes with reference to Ferro.

IV. Gauss-Krüger coordinates

Since Gauss-Krüger systems of coordinates have been introduced for the whole territory of the Reich (cf. Vol. II, 1, 9th Edition, 1931, p. 145), we will also indicate, in addition, the Gauss-Krüger conformal plane coordinates for the net of Hannover, where the meridian 9° east of Greenwich holds as zero meridian.

The computation of these coordinates was carried out in two different ways. First, the plane coordinates were computed for the two connecting points Ägidius and Wasserturm from the geographic coordinates of p. 235 with the use of the formulae of Volume III, 7th Edition, 1923, section 92, p. 512. Then there followed a preliminary computation of the plane coordinates of all remaining points by a purely trigonometric method, whereupon the reductions of the measured directions could be computed according to Volume III, 7th Edition, 1923, section 94, p. 527. This formed the basis for a new net adjustment with a following computation of coordinates on the plane.

In order to obtain an unobjectionable check, there was carried out a second direct computation of the conformal plane coordinates from the geographic coordinates of p. 250 according to Volume III, 7th Edition, 1923, section 92, p. 512.

In the following we only give the result of these computations and note in addition that instead of the ordinates and abscissae the values of the eastings [*Rechts*] and northings [*Hoch*] corresponding to Vol. II, 1, 9th Edition, 1931, p. 146, are communicated.

Gauss-Krüger coordinates in the system 9° east of Greenwich

Point	*Easting*	*Northing*
Ägidius	3,550,406.109	5,804,265.553
Linden Wasserturm	3,548,151.032	5,803,468.646
Willmer	3,551,939.624	5,801,651.158
Steuerndieb	3,553,753.742	5,806,672.467
Schanze	3,550,515.554	5,809,309.666
Burg	3,547,785.512	5,807,557.793

Finally, in the following table we summarize further the station data [Abrisse] of the individual stations, and add also the meridian convergence γ and the scale factor m.

Station data [Abriss] in the Gauss-Krüger system 9° east of Greenwich

	Observed T (spheroid)	$t-T$	v	Adjusted t (plane)	*log s* (plane)
1. *Ägidius*				(γ = 35′ 10.37″)	(m = 1.000 031)
Wasserturm . . 1.	250° 32′ 14.64″	+0.10″	+0.02″	250° 32′ 14.76″	3.378 7144
Burg . . . . . 2.	321 28 49.46	−0.41	+0.68	321 28 49.73	3.624 0648
Schanze . . . . 3.	1 14 36.00	−0.64	−0.64	1 14 34.72	3.702 8868
Steuerndieb . . 4.	54 17 04.15	−0.31	−0.54	54 17 03.30	3.615 2229
Willmer . . . 5.	149 36 19.31	+0.34	+0.48	149 36 20.13	3.481 5804
2. *Wasserturm*				(γ = 33′ 35.45″)	(m = 1.000 028)
Burg . . . . . 6.	354° 53′ 31.32″	−0.50″	+0.48″	354° 53′ 31.30″	3.613 3609
Ägidius . . . . 7.	70 32 15.34	−0.10	−0.48	70 32 14.76	3.378 7144
Willmer . . . . 8.	115 37 41.57	+0.23	0.00	115 37 41.80	3.623 4546
3. *Willmer*				(γ = 36′ 12.73″)	(m = 1.000 033)
Wasserturm . . 9.	295° 37′ 41.12″	−0.23″	+0.91″	295° 37′ 41.80″	3.623 4546
Ägidius . . . . 10.	329 36 21.80	−0.34	−1.33	329 36 20.13	3.481 5804
Steuerndieb . . 11.	19 51 50.59	−0.67	+0.42	19 51 50.34	3.727 4572
4. *Steuerndieb*				(γ = 37′ 32.27″)	(m = 1.000 035)
Willmer . . . . 12.	199° 51′ 49.24″	+0.68″	+0.42″	199° 51′ 50.34″	3.727 4572
Ägidius . . . . 13.	234 17 02.96	+0.32	+0.02	234 17 03.30	3.615 2229
Burg . . . . . 14.	278 26 16.97	−0.12	−1.10	278 26 15.75	3.780 5720
Schanze . . . . 15.	309 09 34.08	−0.35	+0.66	309 09 34.39	3.620 7817
5. *Schanze*				(γ = 35′ 18.42″)	(m = 1.000 032)
Steuerndieb . . 16.	129° 09′ 34.18″	+0.34″	−0.13″	129° 09′ 34.39″	3.620 7817
Ägidius . . . 17.	181 14 34.18	+0.64	−0.10	181 14 34.72	3.702 8868
Burg . . . . . 18.	237 18 41.47	+0.22	+0.23	237 18 41.92	3.511 0530
6. *Burg*				(γ = 33′ 22.80″)	(m = 1.000 028)
Schanze . . . . 19.	57° 18′ 41.97″	−0.21″	+0.16″	57° 18′ 41.92″	3.511 0530
Steuerndieb . . 20.	98 26 14.83	+0.11	+0.81	98 26 15.75	3.780 5720
Ägidius . . . . 21.	141 28 50.46	+0.41	−1.14	141 28 49.73	3.624 0648
Wasserturm . . 22.	174 53 30.63	+0.50	+0.17	174 53 31.30	3.613 3609

Since with the adjusted directions, angles, and triangle sides of section 70 everything required for further geodetic use is completely computed in section 71, we have satisfied the common need of practice.

However, we will now make a further examination concerning the accuracy of any triangle side derived from the base Ägidius-Wasserturm.

Since the base Ägidius-Wasserturm is rather small in comparison to the extent of the whole net, it is not without interest to compute the transfer of error from this base to one of the longer sides, e.g., to the diagonal Burg-Steuerndieb, nearly 2-1/2 times as long. It is a question here only of those errors which are produced by the angle measurements and carried forward, while the base Ägidius-Wasserturm itself is considered here as free from error.

With this, we will make a computation of the weight of a function according to the theory of section 46 or, as the case may be, section 61.

If we set the base Ägidius-Wasserturm $= b$ and the diagonal Burg-Steuerndieb $= s$, then according to Fig. 1, section 70, p. 236, there exists the relation:

$$s = b\,\frac{\sin\,[7-6]\,\sin\,[4-2]}{\sin\,[22-21]\,\sin\,[14-13]}\,. \tag{1}$$

The two triangles required for this are specified with their adjusted angles under [c] and [h] on p. 238, namely with the addition of the plane Legendre angles:

			Legendre
Ägidius	[2 — 1]	= 70° 56′ 35.48″	70° 56′ 35.47″
Wasserturm	[7 — 6]	= 75 38 43.06	75 38 43.06
Burg	[22 — 21]	= 33 24 41.48	33 24 41.47
		180° 0′ 0.02″	180° 0′ 0.00″
Ägidius	[4 — 2]	= 92° 48′ 13.47″	92° 48′ 13.46″
Steuerndieb	[14 — 13]	= 44 9 12.89	44 9 12.88
Burg	[21 — 20]	= 43 2 33.68	43 2 33.66
		180° 0′ 0.04″	180° 0′ 0.00″.

With the Legendre angles we compute:

		Diff. for 10″			Diff. for 10″
$\log\sin\,[7-6]$	= 9.986 2250	54	$\log\sin\,[22-21]$	= 9.740 8745	319
$\log\sin\,[4-2]$	= 9.999 4798	—11	$\log\sin\,[14-13]$	= 9.842 9735	216
	9.985 7048			— 9.583 8480	
				= + 0.416 1520	

$$\begin{array}{lll} \text{Ägidius—Wasserturm} & \log b = & 3.378\,7016 \\ & & +\,9.985\,7048 \\ & & +\,0.416\,1520 \\ \text{Burg—Steuerndieb} & \log s = & 3.780\,5584\,, \qquad s = 6033.349\ \text{m}\,. \end{array} \tag{2}$$

This agrees sufficiently with 3.780 5583, as is indicated on p. 246, and, incidentally, the small deviation of one unit of the seventh place in $\log s$ results only from the fact that the values (5) at the end of section 70, p. 244, were obtained by taking the mean in the last place, if needed, in the case of the computations from the various triangles.

From the above logarithmic computation with the inserted differences for log sin we can write the error function:

$$\log s - \log b = +\,0.54\,(d_7 - d_6) - 0.11\,(d_4 - d_2) - 3.19\,(d_{22} - d_{21}) - 2.16\,(d_{14} - d_{13}), \tag{3}$$

here d_7, d_6, etc., mean the changes of the measured directions (7), (6) . . . , and, in fact, d_7, d_6 . . . are in seconds if $\log s - \log b$ is understood in units of the *sixth* decimal of logarithms. This method, obviously, is in complete correspondence with the setting up of the linear side equations, which we have taught in section 66, p. 218; and we could also prove the coefficients 0.54, etc., as there, as differential quotients of the log sin-function, which we need not carry out once more.

Now we consider the above equation (3) as a function F in the sense of section 46 and section 61 (while our d's have the same character as the $l + v$'s of (3), section 46, p. 143, and with this, the coefficients f become

$$f_7 = +0.54 \quad f_6 = -0.54 \quad f_4 = -0.11 \quad f_2 = +0.11$$
$$f_{22} = -3.19 \quad f_{21} = +3.19 \quad f_{14} = -2.16 \quad f_{13} = +2.16;$$

all other f's are equal to zero. We form a table of all f's and of all those a's, b's, c's . . . , which occur together with the existing f's. The a's, b's, c's . . . are carried over here from the earlier large table of coefficients of section 70, p. 241.

	2	4	6	7	13	14	21	22
a	. .	. .	−0.54	+2.64	+3.64	. .	+3.40	−3.19
b	−2.53	−1.58	. .	. .	−0.57	+3.54	−0.21	. .
c	+1	. .	−1	+1	. .	. .	−1	+1
d	. .	. .	. .	−1	. .	. .	. .	. .
e	. .	−1	. .	. .	+1	. .	. .	. .
f	. .	+1	. .	. .	−1	. .	. .	. .
g	−1	. .	. .	. .	. .	. .	+1	. .
h	−1	+1	. .	. .	−1	+1	+1	. .
f	+0.11	−0.11	−0.54	+0.54	+2.16	−2.16	+3.19	−3.19

Now we compute all af's, bf's, cf's, etc., e.g.

a	*f*	*af*	*cf*
−0.54	−0.54	+ 0.2916	+0.11
+2.64	+0.54	+ 1.4256	+0.54
+3.64	+2.16	+ 7.8624	+0.54
+3.40	+3.19	+10.8460	−3.19
−3.19	−3.19	+10.1761	−3.19
		$[af] = 30.6017$	$[cf] = -5.19$ and so on.

These are the substitute terms for the absolute terms w of the normal equations, and according to (17), p. 146, we have the following system of coefficients:

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	f
a	+107.19	−0.63	−3.41	+3.27	+0.08	+0.49	−0.87	−0.24	+30.60
b		+58.16	−2.32	−0.00	+1.01	−8.09	+8.63	+2.44	−9.65
c			+6.00	−2.00	. . .	. . .	−2.00	−2.00	−5.19
d				+6.00	−2.00	. . .	. . .	. . .	−0.54
e					+6.00	−2.00	. . .	−2.00	+2.27
f						+6.00	−2.00	+2.00	−2.27
g							+6.00	+2.00	+3.08
h								+6.00	−1.35
f									+30.29 = [ff].

This system is reduced eight times, i.e., we eliminate k_1, k_2 as far as k_8 and carry here always the final term. By means of these eight eliminations we will obtain [according to equation (14), p. 145]

$$30.29 - (8.74 + 1.54 + 3.63 + 1.68 + 0.33 + 3.01 + 1.39 + 0.87),$$

$$\text{i. e. } \frac{1}{P} = 30.29 - 21.19 = 9.10.$$

This whole computation is only done with the slide rule, therefore, at best, accurate to 0.1; hence, we set approximately

$$\frac{1}{P} = 9.1.$$

Since the mean error of the unit of weight was found previously in (4), section 70, p. 242, to be approximately $m = 1.0''$, we have now the mean error of a function

$$M = m\sqrt{\frac{1}{P}} = 1.0\sqrt{9.1} = \pm 3.0\,. \tag{4}$$

This is a mean error in units of the sixth place of logarithms; we have therefore according to (2) and (3)

$$\log s = 3.780\,5584 \pm 30,$$
$$\text{therefore } s = 6033.349 \text{ m} \pm 0.042 \text{ m}.$$

This is obtained directly from the table of logarithms, since ± 30 at the place in question in the case of the tabular difference 72 amounts to ± 0.042 m. More theoretically, we have

$$d \log s = \frac{\mu}{s}\, ds = \pm\, 0.000\,0030, \qquad \frac{ds}{s} = \frac{0.000\,0030}{0.434\ldots} = 0.000\,007,$$

i.e., seven millionths or 7 mm per 1 km, which, computed for 6 km, yields again ± 42 mm as previously.

This is the mean error to be anticipated of the diagonal Burg-Steuerndieb, if this side is derived trigonometrically from the side Ägidius-Wasserturm assumed free from error. The accuracy obtained, obviously, is very satisfactory.

In the same manner, we could also determine a *twisting error* of the net, which is not insignificant in view of the shortness of the base.

It is easy to understand that in a net of triangles in which no side equations occur, the corrections of two reciprocal directions are equal to each other with opposite sign. If side equations exist besides, then the corrections of the reciprocal directions, apart from the signs, differ from each other only by the terms depending on the side equations. These properties of the corrections to the directions are the basis of an adjustment method, which Major Anér in Stockholm has published in *Zeitschrift für Vermessungswesen*, 1926, pp. 69 to 73.

If we consider, e.g., the condition equations of the pentagon of Hannover of section 70, p. 241, then the correlated equations for the corrections v_4 and v_{13} are

$$\begin{aligned} v_4 &= \quad \cdot \qquad - 1.58 k_2 - k_5 + k_6 + k_8 \\ v_{13} &= + 3.64 k_1 - 0.57 k_2 + k_5 - k_6 - k_8 . \end{aligned}$$

Therefore, we have

$$v_4 + v_{13} = + 3.64 k_1 - 2.15 k_2 .$$

If, generally, v_{ik} and v_{ki} are the corrections of the two directions between the points P_i and P_k, and if the side equations of the net are

$$\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + w_a = 0 \quad &\text{Correlate } k_a \\ b_1 v_1 + b_2 v_2 + b_3 v_3 + \ldots + w_b = 0 \quad &\text{Correlate } k_b \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &, \end{aligned}$$

then we will have

$$v_{ik} + v_{ki} = (a_{ik} + a_{ki}) k_a + (b_{ik} + b_{ki}) k_b + \ldots . \tag{1}$$

If we denote further the quantities of orientation of the two points by z_i and z_k, and the two measured directions by r_{ik} and r_{ki}, then we have

$$\begin{aligned} r_{ik} + v_{ik} + z_i &= r_{ki} + v_{ki} + z_k \pm 180^\circ \\ v_{ik} - v_{ki} &= - r_{ik} + r_{ki} - z_i + z_k \pm 180^\circ . \end{aligned} \tag{2}$$

We will assume here that the directions are already oriented approximately so that z_i and z_k mean only just corrections of the preliminary orientation.

From (1) and (2) we obtain

$$2 v_{ik} = - r_{ik} + r_{ki} - z_i + z_k \pm 180^\circ + (a_{ik} + a_{ki}) k_a + (b_{ik} + b_{ki}) k_b + \ldots . \tag{3}$$

At the end of section 67, p. 226, we have established further that the corrections to the directions, by stations, must yield the sum zero. Therefore, if we set up equation (3) for all directions of station P_i and form the sum of all equations, then we obtain an equation free from the corrections v between the quantities of orientation of the stations and the correlates of the side equations.

We will show this by our pentagon of Hannover of section 71, where we use the preliminarily oriented directions T after the addition of the reductions $t - T$ of p. 247 in the system of the conformal double projection. These directions are summarized once more in the following.

1. *Ägidius*

Wasserturm	(1)	= 253° 58′ 14.10″
Burg	(2)	= 324 54 51.47
Schanze	(3)	= 4 40 38.98
Steuerndieb	(4)	= 57 43 5.39
Willmer	(5)	= 153 2 17.51

2. *Wasserturm*

Burg	(6)	= 358° 19′ 32.74″
Ägidius	(7)	= 73 58 14.60
Willmer	(8)	= 119 3 39.16

3. *Willmer*

Wasserturm	(9)	= 299° 3′ 38.24″
Ägidius	(10)	= 333 2 19.32
Steuerndieb	(11)	= 23 17 49.46

4. *Steuerndieb*

Willmer	(12)	= 203° 17′ 49.46″
Ägidius	(13)	= 237 43 4.83
Burg	(14)	= 281 52 20.94
Schanze	(15)	= 312 35 39.03

5. *Schanze*

Steuerndieb	(16)	= 132° 35′ 39.82″
Ägidius	(17)	= 184 40 38.45
Burg	(18)	= 240 44 47.87

6. *Burg*

Schanze	(19)	= 60° 44′ 47.94″
Steuerndieb	(20)	= 101 52 19.04
Ägidius	(21)	= 144 54 53.30
Wasserturm	(22)	= 178 19 33.04 .

For the triangle side Ägidius-Wasserturm we obtain then according to equation (2)

$$\begin{aligned} v_1 - v_7 = & - 253^\circ\, 58'\, 14.10'' - z_1 \\ & + \;\; 73 \;\; 58 \;\; 14.60 + z_2 \\ & \pm 180 \end{aligned}$$

or

$$v_1 - v_7 = + 0.50'' - z_1 + z_2 .$$

On the other hand, we obtain from the condition equations, section 70, p. 241

$$\begin{aligned} v_1 &= \quad \cdot \qquad - k_3 + k_4 \\ v_7 &= + 2.64\, k_1 + k_3 - k_4 \end{aligned}$$

or

$$v_1 + v_7 = + 2.64\, k_1 .$$

Thus there follows for equation (3)

$$\begin{aligned} 2\, v_1 &= + 0.50'' - z_1 + z_2 + 2.64\, k_1 \\ 2\, v_7 &= - 0.50 \;\; - z_2 + z_1 + 2.64\, k_1 . \end{aligned}$$

In the same manner, we have set up these equations for all triangle sides so that we obtain the following summary:

Ägidius (1)

$$\begin{aligned} 2\, v_1 &= - z_1 + z_2 + 2.64\, k_1 \quad . \quad . \quad . \quad + 0.50 \\ 2\, v_2 &= - z_1 + z_6 + 3.40\, k_1 - 2.74\, k_2 + 1.83 \\ 2\, v_3 &= - z_1 + z_5 + 3.06\, k_1 + 4.11\, k_2 - 0.53 \\ 2\, v_4 &= - z_1 + z_4 + 3.64\, k_1 - 2.15\, k_2 - 0.56 \\ 2\, v_5 &= - z_1 + z_3 + 4.88\, k_1 \quad . \quad . \quad . \quad + 1.81 \\ \hline 0 &= - 5\, z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + 17.62\, k_1 - 0.78\, k_2 + 3.05 \end{aligned}$$

Wasserturm (2)

$$\begin{aligned} 2\, v_6 &= - z_2 + z_6 - 3.73\, k_1 \quad . \quad . \quad . \quad . \quad + 0.30 \\ 2\, v_7 &= - z_2 + z_1 + 2.64\, k_1 \quad . \quad . \quad . \quad . \quad - 0.50 \\ 2\, v_8 &= - z_2 + z_3 - 5.23\, k_1 \quad . \quad . \quad . \quad . \quad - 0.92 \\ \hline 0 &= - 3\, z_2 + z_1 + z_3 + z_6 - 6.32\, k_1 - 1.12 \end{aligned}$$

Willmer (3)

$$
\begin{aligned}
2\,v_{9} &= -z_3 + z_2 - 5.23\,k_1 \;.\;.\;.\;.\; + 0.92 \\
2\,v_{10} &= -z_3 + z_1 + 4.88\,k_1 \;.\;.\;.\;.\; - 1.81 \\
2\,v_{11} &= -z_3 + z_4 - 4.82\,k_1 \;.\;.\;.\;.\; + 0.00 \\
\hline
0 &= -3\,z_3 + z_1 + z_2 + z_4 - 5.17\,k_1 - 0.89
\end{aligned}
$$

Steuerndieb (4)

$$
\begin{aligned}
2\,v_{12} &= -z_4 + z_3 - 4.82\,k_1 \;.\;.\;.\;.\; + 0.00 \\
2\,v_{13} &= -z_4 + z_1 + 3.64\,k_1 - 2.15\,k_2 + 0.56 \\
2\,v_{14} &= -z_4 + z_6 \;.\;.\;.\;.\; + 5.95\,k_2 - 1.90 \\
2\,v_{15} &= -z_4 + z_5 - 2.21\,k_1 - 2.97\,k_2 + 0.79 \\
\hline
0 &= -4\,z_4 + z_1 + z_3 + z_5 + z_6 - 3.39\,k_1 + 0.83\,k_2 - 0.55
\end{aligned}
$$

Schanze (5)

$$
\begin{aligned}
2\,v_{16} &= -z_5 + z_4 - 2.21\,k_1 - 2.97\,k_2 - 0.79 \\
2\,v_{17} &= -z_5 + z_1 + 3.06\,k_1 + 4.11\,k_2 + 0.53 \\
2\,v_{18} &= -z_5 + z_6 - 1.63\,k_1 - 2.20\,k_2 + 0.07 \\
\hline
0 &= -3\,z_5 + z_1 + z_4 + z_6 - 0.78\,k_1 - 1.06\,k_2 - 0.19
\end{aligned}
$$

Burg (6)

$$
\begin{aligned}
2\,v_{19} &= -z_6 + z_5 - 1.63\,k_1 - 2.20\,k_2 - 0.07 \\
2\,v_{20} &= -z_6 + z_4 \;.\;.\;.\;.\; + 5.95\,k_2 + 1.90 \\
2\,v_{21} &= -z_6 + z_1 + 3.40\,k_1 - 2.74\,k_2 - 1.83 \\
2\,v_{22} &= -z_6 + z_2 - 3.73\,k_1 \;.\;.\;.\;.\; - 0.30 \\
\hline
0 &= -4\,z_6 + z_1 + z_2 + z_4 + z_5 - 1.96\,k_1 + 1.01\,k_2 - 0.30\,.
\end{aligned}
$$

To the thus found 6 sum equations we take further the two side equations of section 70, p. 241, from which we eliminate, however, the corrections v by means of the above 22 equations. We obtain then two additional equations, in which the quantities of orientation z and the two correlates k_1 and k_2 occur.

We are to bear in mind, however, that the 6 sum equations are not independent of one another, since one of the 6 unknowns of orientation is assumed arbitrarily. We set arbitrarily $z_1 = 0$, omit the first sum equation, and have then the 7 following equations for the determination of $z_2, z_3, \ldots, z_6$ and k_1, k_2.

$$
\begin{array}{rrrrrrrrl}
-\;3\,z_2 + & 1\,z_3 & .\,. & .\,.\;+ & 1\,z_6 - & 6.32\,k_1 & .\,. & -\;1.12 & = 0 \\
+\;1\,z_2 - & 3\,z_3 + & 1\,z_4 & .\,. & .\,.\;- & 5.17\,k_1 & .\,. & -\;0.89 & = 0 \\
.\,.\;+ & 1\,z_3 - & 4\,z_4 + & 1\,z_5 + & 1\,z_6 - & 3.39\,k_1 + & 0.83\,k_2 - & 0.55 & = 0 \\
.\,. & .\,.\;+ & 1\,z_4 - & 3\,z_5 + & 1\,z_6 - & 0.78\,k_1 - & 1.06\,k_2 - & 0.19 & = 0 \\
+\;1\,z_2 & .\,.\;+ & 1\,z_4 + & 1\,z_5 - & 4\,z_6 - & 1.96\,k_1 + & 1.01\,k_2 - & 0.30 & = 0 \\
-\;6.32\,z_2 - & 5.17\,z_3 - & 3.39\,z_4 - & 0.78\,z_5 - & 1.96\,z_6 - & 137.00\,k_1 - & 5.58\,k_2 - & 24.30 & = 0 \\
.\,. & .\,. & +\;0.83\,z_4 - & 1.06\,z_5 + & 1.01\,z_6 - & 5.58\,k_1 - & 78.08\,k_2 - & 4.80 & = 0\,.
\end{array}
$$

The solution of these 7 equations is carried out appropriately according to Gauss' method of elimination. We omit, however, further details and indicate at once the result:

$$
\begin{array}{lll}
z_1 = \, 0.000'' & z_4 = +\,0.001'' & k_1 = -\,0.1753 \\
z_2 = -\,0.003 & z_5 = -\,0.001 & k_2 = -\,0.0483. \\
z_3 = +\,0.005 & z_6 = -\,0.002 &
\end{array}
$$

With these, we compute the individual corrections for the 6 stations according to p. 257 and above. We will have

$$2\,v_1 = -\,0.003 - 0.463 + 0.50 = +\,0.034''$$

$$2\,v_2 = -\,0.002 - 0.595 + 0.132 + 1.83 = +\,1.365'', \text{ etc.}$$

Altogether we obtain the following values:

$v_1 = +0.02''$	$v_6 = +0.48''$	$v_9 = +0.91''$	$v_{12} = +0.42''$	$v_{16} = -0.13''$	$v_{19} = +0.16''$
$v_2 = +0.68$	$v_7 = -0.48$	$v_{10} = -1.33$	$v_{13} = +0.01$	$v_{17} = -0.10$	$v_{20} = +0.81$
$v_3 = -0.63$	$v_8 = 0.00$	$v_{11} = +0.42$	$v_{14} = -1.10$	$v_{18} = +0.23$	$v_{21} = -1.15$
$v_4 = -0.55$	0.00	0.00	$v_{15} = +0.66$	0.00	$v_{22} = +0.18$
$v_5 = +0.48$			-0.01		0.00 .
0.00					

The station sums of the corrections yield zero almost everywhere; the individual values, also, show nearly complete agreement with the earlier results of section 70, p. 242.

Approximate adjustment of a triangle net with observations of directions

In section 50 we have already learned methods according to which the adjustment of a net of triangles can be divided into several parts. The method of adjustment in groups is not advisable for nets of triangles, since sufficient approximation is obtained only after a repetition of several times. However, the groupwise adjustment given on p. 159 with transformed side equations can be applied to advantage in those cases in which the normal equations resulting from the angle equations can be easily solved.

We will now in addition treat an approximation method which leads to a usable result for triangulation nets with observations of directions.

Let the measured direction for the ray $P_i P_k$ obtain in the adjustment the correction $v_{i\,.\,k}$, and the opposite direction $P_k P_i$ the correction $v_{k\,.\,i}$. Instead of these corrections we introduce two new values $\varepsilon_{i\,.\,k}$ and $\delta_{i\,.\,k}$ by setting:

$$v_{i\,.\,k} = \varepsilon_{i\,.\,k} + \delta_{i\,.\,k} \qquad \text{and} \qquad v_{k\,.\,i} = \varepsilon_{i\,.\,k} - \delta_{i\,.\,k}\,. \tag{1}$$

In the angle condition equations, the corrections v occur only in the form $v_{i\,.\,k} - v_{k\,.\,i} = 2\delta_{i\,.\,k}$; hence, the ε's will vanish in them entirely and only the δ's will occur. In the side equations, however, the ε's and the δ's remain.

The angle equations are now adjusted first independently, and with this, the values of the δ's are determined. We substitute these values in the side equations and now adjust these independently, whereby we arrive at the values of the ε's. Since the corrections δ are not changed hereby, then the angle equations also remain untouched by this second adjustment; the angle equations are satisfied by arbitrary values of the ε's.

With this, the theory of the method is sufficiently explained. For practical application, we can however disregard substitution (1) altogether. Since in the angle equations there occur only the differences $v_{i\,.\,k} - v_{k\,.\,i}$, it is indifferent if the individual v's or only their differences δ are determined by the first part of the adjustment; in any case there follows $v_{i\,.\,k} = -v_{k\,.\,i} = \delta_{i\,.\,k}$. With the thus adjusted directions we set up the side equations and adjust them independently after having set the corrections of the reciprocal directions equal to each other. These corrections which are to be newly determined are then the values ε from above. From the second adjustment there result the same directions which we have obtained above with the corrections $v_{i\,.\,k}$ and $v_{k\,.\,i}$.

We apply this approximation method to the pentagon of Hannover of section 70. In the summary I, p. 241, we neglect first the side equations and set up, from the angle equations alone, the normal equations, which read in the abbreviated form thus:

$$\begin{array}{l}
\underline{+6k_1} - 2k_2 \;\ldots\ldots\; - 2k_5 - 2k_6 - 1.02 = 0 \\
\qquad \underline{+6k_2} - 2k_3 \;\ldots\ldots\ldots\; + 2.22 = 0 \\
\qquad\qquad \underline{+6k_3} - 2k_4 \;\ldots\; - 2k_6 - 2.36 = 0 \\
\qquad\qquad\qquad \underline{+6k_4} - 2k_5 + 2k_6 + 0.76 = 0 \\
\qquad\qquad\qquad\qquad \underline{+6k_5} + 2k_6 + 2.30 = 0 \\
\qquad\qquad\qquad\qquad\qquad \underline{+6k_6} + 4.30 = 0\,.
\end{array}$$

The solution of these equations yields

$$k_1 = -0.293 \quad k_2 = -0.437 \quad k_3 = +0.091$$
$$k_4 = +0.460 \quad k_5 = -0.022 \quad k_6 = -0.930$$

and with the coefficients of the angle condition equations of p. 241 we find the preliminary corrections of the measured directions

$$\begin{aligned} v_1 &= -v_7 = -0.14'' & v_8 &= -v_9 = -0.44'' \\ v_2 &= -v_{21} = +0.66 & v_{11} &= -v_{12} = +0.09 \\ v_3 &= -v_{17} = -0.48 & v_{14} &= -v_{20} = -0.93 \\ v_4 &= -v_{13} = -0.56 & v_{15} &= -v_{16} = +0.46 \\ v_5 &= -v_{10} = +0.53 & v_{18} &= -v_{19} = -0.02\,. \\ v_6 &= -v_{22} = +0.29 & & \end{aligned}$$

In order to set up the two side equations we have to collect the directions, preliminarily corrected, and to repeat the computational procedure of p. 238 and p. 239. We will however omit these details and give at once the side equations resulting therefrom. We have:

$$\left.\begin{aligned} -0.54\,v_6 + 2.64\,v_7 - 2.10\,v_8 - 3.13\,v_9 + 4.88\,v_{10} - 1.75\,v_{11} - 3.07\,v_{12} \\ +3.64\,v_{13} - 0.57\,v_{15} - 1.64\,v_{16} + 3.06\,v_{17} - 1.42\,v_{18} - 0.21\,v_{19} + 3.40\,v_{21} \\ -3.19\,v_{22} + 18.2 = 0 \end{aligned}\right\}, \qquad \text{(a)}$$

$$\left.\begin{aligned} -2.53\,v_2 + 4.11\,v_3 - 1.58\,v_4 - 0.57\,v_{13} + 3.54\,v_{14} - 2.97\,v_{15} - 2.20\,v_{19} \\ +2.41\,v_{20} - 0.21\,v_{21} + 2.1 \doteq 0 \end{aligned}\right\}. \qquad \text{(b)}$$

Herein the corrections of the reciprocal directions are to be set equal to one another according to the above computational rule, therefore, e.g.

$$v_1 = v_7, \qquad v_2 = v_{21}, \qquad v_3 = v_{17} \ldots .$$

With this, the two equations (a) and (b) change to

$$\left.\begin{aligned} -3.73\,v_6 + 2.64\,v_7 - 5.23\,v_8 + 4.88\,v_{10} - 4.82\,v_{11} + 3.64\,v_{13} \\ -2.21\,v_{15} + 3.06\,v_{17} - 1.63\,v_{18} + 3.40\,v_{21} + 18.2 = 0 \end{aligned}\right\}. \qquad \text{(a*)}$$

$$\left.\begin{aligned} -2.74\,v_2 + 4.11\,v_3 - 2.15\,v_4 + 5.95\,v_{14} - 2.97\,v_{15} - 2.20\,v_{19} \\ +2.1 = 0 \end{aligned}\right\}. \qquad \text{(b*)}$$

The normal equations for these two equations are

$$\begin{aligned} \underline{137.06\,k_1} + 5.59\,k_2 + 18.2 &= 0 \\ \underline{+78.08\,k_2} + 2.1 &= 0\,, \end{aligned}$$

from which there follows

$$k_1 = -0.132\,, \qquad k_2 = -0.017\,,$$

and

$$\begin{aligned} v_1 &= v_7 = -0.35'' & v_5 &= v_{10} = -0.64'' & v_{14} &= v_{20} = -0.10'' \\ v_2 &= v_{21} = -0.40 & v_6 &= v_{22} = +0.49 & v_{15} &= v_{16} = +0.34 \\ v_3 &= v_{17} = -0.48 & v_8 &= v_9 = +0.69 & v_{18} &= v_{19} = +0.25\,. \\ v_4 &= v_{13} = -0.44 & v_{11} &= v_{12} = +0.64 & & \end{aligned}$$

These corrections would have to be added to the preliminarily corrected directions and would then yield the final values of the directions. But we can also collect the corrections of the two adjustments and obtain the final corrections. We have

$v_1 = -0.49''$	$v_7 = -0.21''$	$v_{13} = +0.12''$	$v_{19} = +0.27''$
$v_2 = +0.26$	$v_8 = +0.25$	$v_{14} = -1.03$	$v_{20} = +0.83$
$v_3 = -0.96$	$v_9 = +1.13$	$v_{15} = +0.80$	$v_{21} = -1.06$
$v_4 = -1.00$	$v_{10} = -1.17$	$v_{16} = -0.12$	$v_{22} = +0.20$.
$v_5 = -0.11$	$v_{11} = +0.73$	$v_{17} = 0.00$	
$v_6 = +0.78$	$v_{12} = +0.55$	$v_{18} = +0.23$	

For comparison with the result of the rigorous adjustment in section 70 we must further reduce to zero the sum of the corrections for each station by the addition of a constant. Then we obtain

$v_1 = -0.03''$	$v_7 = -0.48''$	$v_{13} = +0.01''$	$v_{19} = +0.21''$
$v_2 = +0.72$	$v_8 = -0.02$	$v_{14} = -1.14$	$v_{20} = +0.77$
$v_3 = -0.50$	$v_9 = +0.90$	$v_{15} = +0.69$	$v_{21} = -1.12$
$v_4 = -0.54$	$v_{10} = -1.40$	$v_{16} = -0.16$	$v_{22} = +0.14$.
$v_5 = +0.35$	$v_{11} = +0.50$	$v_{17} = -0.04$	
$v_6 = +0.51$	$v_{12} = +0.44$	$v_{18} = +0.19$	

The above values deviate from the corrections found according to the method of least squares on an average by 0.04'', at the most by 0.13''.

Section 74. Net Adjustment with Angle Measurement

After having learned the adjustment of a triangle net with direction measurements by the two examples of sections 68 and 70, we will now also explain the net adjustment with angle measurements by a few examples. In the last decades, such adjustments have moved into the background for main triangulations, since, for these, the angle measurements are always arranged now so that they can be introduced into the net adjustment in the form of complete sets of directions. However, for some purposes, e.g. colonial triangulations of first order, we will have to return to the accurate measurement of as few angles as possible because of the difficulty of the signaling by heliotropes.

A complete example of such an adjustment is contained in the 3rd Edition of this volume 1888 in sections 70 and 71, pp. 194 to 207. We have already used this example in section 67, pp. 224 and 225, without specifying the measurement numbers, in order to explain the setting up of the condition equations. The latter is now already sufficiently discussed; therefore, we take at once the condition equations from the 3rd Edition, p. 198, and show the further numerical treatment. Since it is a question only of a computational example, then we will cut out the upper part from Fig. 2, section 67, p. 225, with the triangles I-IV, VIII, and IX and adjust it independently. According to p. 224, to this there belong six triangle equations, one horizon equation, and one side equation, hence, together eight condition equations, which we have collected in the table on p. 263.

About the setting up of the normal equations, also, we no longer have to add anything essential. We obtain

$$
\begin{array}{cccccccccc}
3k_1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & +k_7 & -1.020\,k_8 & -0.792 & =0 \\
\cdot\cdot & +3k_2 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & +k_7 & +0.307\,k_8 & -2.826 & =0 \\
\cdot\cdot & \cdot\cdot & +3k_3 & \cdot\cdot & \cdot\cdot & \cdot\cdot & +k_7 & -0.719\,k_8 & -1.821 & =0 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & +3k_4 & \cdot\cdot & \cdot\cdot & +k_7 & -1.670\,k_8 & -4.073 & =0 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & +3k_5 & \cdot\cdot & +k_7 & +1.784\,k_8 & -0.560 & =0 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & +3k_6 & +k_7 & +0.993\,k_8 & -1.994 & =0 \\
k_1 & +k_2 & +k_3 & +k_4 & +k_5 & +k_6 & +6k_7 & \cdot\;\cdot & -3.338 & =0 \\
-1.020\,k_1 & +0.307\,k_2 & -0.719\,k_3 & -1.670\,k_4 & +1.784\,k_5 & +0.993\,k_6 & \cdot\cdot & +25.308\,k_8 & -11.635 & =0\,.
\end{array}
$$

In view of the special form of the above normal equations, it is advisable to apply Schleiermacher's method on p. 157, instead of the usual elimination method. For from the first six equations we obtain

$$\begin{aligned}
k_1 &= -0.333\,k_7 + 0.340\,k_8 + 0.264\\
k_2 &= -0.333\,k_7 - 0.102\,k_8 + 0.942\\
k_3 &= -0.333\,k_7 + 0.240\,k_8 + 0.607\\
k_4 &= -0.333\,k_7 + 0.557\,k_8 + 1.358\\
k_5 &= -0.333\,k_7 - 0.595\,k_8 + 0.187\\
k_6 &= -0.333\,k_7 - 0.331\,k_8 + 0.665\,.
\end{aligned}$$

With these, the last two normal equations change to

$$\begin{aligned}
4.000\,k_7 &+ 0.109\,k_8 + 0.685 = 0\\
0.109\,k_7 &+ 22.437\,k_8 - 13.326 = 0\,.
\end{aligned}$$

The two equations must again show the symmetric form of the normal equations, which can serve as a check. The solution yields

$$k_7 = -0.188\,, \qquad k_8 = +0.595\,,$$

and by substitution in the above equations we also obtain the values of the remaining k's. Together we have then

$$\begin{aligned}
k_1 &= +0.529 & k_5 &= -0.105\\
k_2 &= +0.944 & k_6 &= +0.530\\
k_3 &= +0.813 & k_7 &= -0.188\\
k_4 &= +1.753 & k_8 &= +0.595\,.
\end{aligned}$$

With the coefficients of the condition equations and the k's we form now the corrections v. We will have

$$\begin{aligned}
v_1 &= -0.27'' & v_7 &= -0.38'' & v_{22} &= -0.43''\\
v_2 &= +0.72 & v_8 &= +0.62 & v_{23} &= +1.28\\
v_3 &= +0.34 & v_9 &= +1.58 & v_{24} &= -0.29\\
v_4 &= +0.32 & v_{10} &= +1.56 & v_{25} &= +0.34\\
v_5 &= +1.75 & v_{11} &= +0.52 & v_{26} &= -0.02\\
v_6 &= +0.76 & v_{12} &= +1.99 & v_{27} &= +1.68\,.
\end{aligned}$$

After the adjustment we convince ourselves that all condition equations are satisfied by the corrections. We will however no longer carry this out and rather turn once more to the treatment of the condition equations according to the theory developed in section 50, p. 159, which is applied to advantage just in the case of triangle nets of the present form.

Groupwise adjustment of the triangle net

We divide the above condition equations into two groups, one of which contains the first six equations, the other, the last two equations. These two groups correspond to the two systems (27) and (28) of section 50, p. 159. Before the transformation of the second group, we adjust the first six equations independently, which amounts to a division into three parts of the absolute term w of each equation. We obtain hereby the preliminary corrections:

$$v_1 = v_2 = v_3 = +0.264''$$
$$v_4 = v_5 = v_6 = +0.942$$
$$v_7 = v_8 = v_9 = +0.607$$
$$v_{10} = v_{11} = v_{12} = +1.358$$
$$v_{22} = v_{23} = v_{24} = +0.187$$
$$v_{25} = v_{26} = v_{27} = +0.665\,,$$

with which the preliminarily corrected angles can be computed.

According to p. 157, for the final adjustment of these angles, all condition equations would have to be adjusted once more together. With the already corrected angles, the absolute terms of the first six equations become all equal to zero, and those of the two last equations $= +0.685$ or, as the case may be, $= -13.325$.

Condition Equations

		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	w
k_1	a	+1	+1	+1	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 0.792
k_2	b	. .	. .	. .	+1	+1	+1	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 2.826
k_3	c	. .	. .	. .	. .	. .	. .	+1	+1	+1	. .	. .	. .	. .	. .	. .	. .	. .	. .	− 1.821
k_4	d	. .	. .	. .	. .	. .	. .	. .	. .	. .	+1	+1	+1	. .	. .	. .	. .	. .	. .	− 4.073
k_5	e	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	+1	+1	+1	. .	. .	. .	− 0.560
k_6	f	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	. .	+1	+1	+1	− 1.994
k_7	g	. .	. .	+1	. .	. .	+1	. .	+1	. .	+1	. .	. .	. .	. .	+1	+1	. .	. .	− 3.338
k_8	h	−1.349	+0.329	. .	−1.049	+1.356	. .	−2.006	. .	+1.285	. .	−2.072	+0.402	−0.548	+2.332	. .	. .	−0.933	+1.926	−11.635

Now we turn to the transformation of the last two equations, for which we have to compute the coefficients ξ_1, η_1, ζ_1 and so forth according to equations (33), p. 160. We will have

$$\xi_1 = -\frac{1}{3}, \qquad \eta_1 = -\frac{1}{3}, \qquad \zeta_1 = -\frac{1}{3}\ldots,$$

consequently, according to (31), section 50, p. 160.

$$A_1 = -\frac{1}{3}, \quad A_2 = -\frac{1}{3}, \quad A_3 = +\frac{2}{3}, \quad A_4 = -\frac{1}{3}, \quad A_5 = -\frac{1}{3}, \quad A_6 = +\frac{2}{3}, \quad \text{and so on,}$$

$$W_1 = +0.685 + 0 + 0\ldots;$$

therefore, the next-to-the-last condition equation changes to

$$-\frac{1}{3}v_1 - \frac{1}{3}v_2 + \frac{2}{3}v_3 - \frac{1}{3}v_4 - \frac{1}{3}v_5 + \frac{2}{3}v_6 - \frac{1}{3}v_7 + \frac{2}{3}v_8 - \frac{1}{3}v_9 + \frac{2}{3}v_{10}$$
$$-\frac{1}{3}v_{11} - \frac{1}{3}v_{12} - \frac{1}{3}v_{22} - \frac{1}{3}v_{23} + \frac{2}{3}v_{24} + \frac{2}{3}v_{25} - \frac{1}{3}v_{26} - \frac{1}{3}v_{27} + 0.685 = 0\,.$$

Since condition equations can be multiplied by arbitrary numbers, then we write in a simplified manner

$$-v_1 - v_2 + 2v_3 - v_4 - v_5 + 2v_6 - v_7 + 2v_8 - v_9 + 2v_{10} - v_{11} - v_{12} - v_{22} - v_{23}$$
$$+2v_{24} + 2v_{25} - v_{26} - v_{27} + 2.055 = 0\,.$$

For the transformation of the last equation we have

$$\xi_2 = \frac{+1.349 - 0.329}{3}, \qquad \eta_2 = \frac{+1.049 - 1.356}{3}, \qquad \zeta_2 = \frac{+2.006 - 1.285}{3}, \quad \text{etc.,}$$

or $\xi_2 = +0.340, \quad \eta_2 = -0.102, \quad \zeta_2 = +0.239,$ etc.

Therefore:

$$B_1 = -1.349 + 0.340 = -1.009, \qquad B_2 = +0.329 + 0.340 = +0.669$$
$$B_3 = \quad 0 \quad + 0.340 = +0.340, \qquad B_4 = -1.049 - 0.102 = -1.151$$
$$B_5 = +1.356 - 0.102 = +1.254, \qquad B_6 = \quad 0 \quad - 0.102 = -0.102, \text{ etc.},$$

and the last condition equation thus reads after having been transformed:

$$\begin{aligned}&-1.009\,v_1 + 0.669\,v_2 + 0.340\,v_3 - 1.151\,v_4 + 1.254\,v_5 - 0.102\,v_6\\&-1.764\,v_7 + 0.240\,v_8 + 1.524\,v_9 + 0.557\,v_{10} - 1.516\,v_{11} + 0.959\,v_{12}\\&-1.142\,v_{22} + 1.737\,v_{23} - 0.595\,v_{24} - 0.331\,v_{25} - 1.264\,v_{26} + 1.595\,v_{27} - 13.325 = 0\ .\end{aligned}$$

From these two transformed condition equations we have now to form further the normal equations:

$$\begin{aligned}36\,k_7 + \ 0.327\,k_8 + \ 2.055 &= 0,\\ 0.327\,k_7 + 22.439\,k_8 - 13.325 &= 0,\end{aligned}$$

from which we obtain

$$k_7 = -0.062, \qquad k_8 = +0.594.$$

It is however noted that the two normal equations would agree with the normal equations found on p. 261, if we had omitted the multiplication by 3 in the transformed, next-to-the-last, condition equation.

With the two correlates k_7 and k_8 and the coefficients of the two transformed condition equations we can now compute the second corrections of the angles. In order to find their final values, we will add at the same time the preliminary corrections found on p. 262. We thus will have

$$\begin{aligned}v_1 &= +0.264 + \quad 0.062 - 1.349 \cdot 0.594 = -0.27''\\ v_2 &= +0.264 + \quad 0.062 + 0.669 \cdot 0.594 = +0.72\\ v_3 &= +0.264 - 2 \cdot 0.062 + 0.340 \cdot 0.594 = +0.34\\ v_4 &= +0.942 + \quad 0.062 - 1.152 \cdot 0.594 = +0.32, \quad \text{etc.}\end{aligned}$$

According to this, we compute all corrections and obtain altogether

$v_1 = -0.27''$	$v_7 = -0.38''$	$v_{22} = -0.43''$
$v_2 = +0.72$	$v_8 = +0.63$	$v_{23} = +1.28$
$v_3 = +0.34$	$v_9 = +1.57$	$v_{24} = -0.29$
$v_4 = +0.32$	$v_{10} = +1.56$	$v_{25} = +0.34$
$v_5 = +1.75$	$v_{11} = +0.52$	$v_{26} = -0.02$
$v_6 = +0.76$	$v_{12} = +1.99$	$v_{27} = +1.67$,

which is in good agreement with the result of the previous adjustment of p. 262.

The method of the groupwise adjustment with transformed condition equations, for nets of triangles of the type of the above, can be summarized briefly as follows. First we distribute the closure errors of the triangles equally over the three angles. With the thus corrected angles we set up the horizon and side equations of the net. In these equations we then collect the terms with the corrections of each triangle in groups and subtract, in each group, the arithmetic mean of the three coefficients from the individual coefficients. (Check: The sum of the reduced coefficients of each group is equal to zero.) We adjust the thus transformed condition equations independently and add the found corrections to the preliminarily corrected angles.

This adjustment method, which has great advantages over the method of Schleiermacher, can be applied in the case of all nets of triangles with angle measurement. The preliminary distribution of the closure errors, however, is applicable only to the triangles independent of one another. All remaining triangle equations must be treated like the horizon and side equations; e.g., in the quadrilateral treated in section 66, p. 216, with Fig. 2, the two condition equations

$$(1) + (2) + (3) + (8) = 180°$$
$$(4) + (5) + (6) + (7) = 180$$

can be satisfied by equal distribution of the closure errors over the four angles. For the second part of the adjustment there remains then further the third angle condition, for instance

$$(1) + (6) + (7) + (8) = 180°,$$

as well as the side equation.

The groupwise adjustment remains most valuable for central systems as for Fig. 2 in section 67, p. 225, in which the 13 condition equations can be replaced, with little labor, by four equations, namely the two horizon equations and the two side equations.

Section 75. Schwerd's Base Net with Angle Measurements

In order to be able to treat also the net adjustment with angle measurement of unequal accuracy, we use the base net of Schwerd as a further example.

Professor Schwerd, at the Lyceum in Speyer, with his Lyceum students in 1820 measured (as a work done competitively and in opposition to the official base measurement Speyer-Oggersheim) a short base, 860 m long, and connected it by a trigonometric net with the line Speyer-Oggersheim.

Schwerd published his measurements in the paper, *Die kleine Speyrer Basis, oder Beweis, dass man mit einem geringen Aufwand an Zeit, Mühe und Kosten durch eine kleine, genau gemessene Linie die Grundlage einer grossen Triangulation bestimmen kann* [*The small base of Speyer, or a proof that with a small expenditure of time, labor and expenses one can determine the foundation of a large triangulation by means of a small, accurately measured line*], Speyer, printed by Jakob Christian Kolb, 1822. But since the author only adjusted his trigonometric measurements as he thought proper ("in the most natural manner," p. 64), a new adjustment of Schwerd's measurements seemed to us all the more appropriate because the excellent weight distinction, which Schwerd made in correct recognition of the different significance of the greater or smaller acute angles, can thereby be fully utilized.

The part of Schwerd's net which we will treat is drawn in the quadrilateral of Fig. 1. The dotted triangle Mannheim-Speyer-Oggersheim does not belong to our adjustment, but serves as the connection with the triangulation of Baden.

As the base we take the line derived trigonometrically from the actual, measured base, only 860 m long (Schwerd, p. 69):

$$D\ Sp - H = 4962.8282 \text{ m}. \qquad (1)$$

($D\ Sp$ = Dom in Speyer, H = Heiligenstein.)

The original angle measurements which are considered for our net of Fig. 1 are given by Schwerd on pp. 48 to 50 of his book, and the centering reductions belonging to them as well as the thirds of spherical excesses are specified on pp. 56 and 57 there. By applying the centering reductions to the measured angles, but leaving out of account here the spherical excesses included by Schwerd, we obtain the following table of the measured angles:

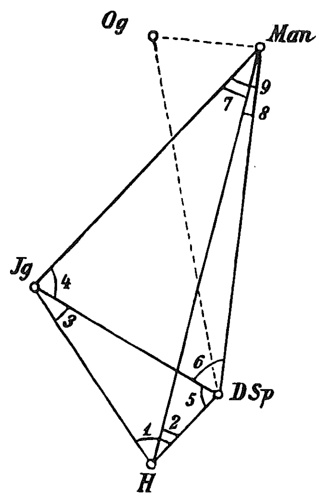


Fig. 1.
Schwerd's base net.
Scale 1:400,000.

Schwerd Side	No.	Angle	Weight p	$\frac{1}{p}$
48 and 56	58	(1) = 81° 21′ 43.36″	70	0.0143
48 and 56	60	(2) = 31 37 39.73	7	0.1429
49 and 56	64	(3) = 25 16 28.85	101	0.0099
49 and 56	65	(4) = 76 33 44.65	47	0.0213
48 and 55	61	(5) = 73 21 46.35	85	0.0118
49 and 55	62	(6) = 67 4 27.96	57	0.0175
50 and 57	68	(7) = 28 25 42.53	10	0.1000
50 and 57	67	(8) = 7 56 6.92	28	0.0357
50 and 57	66	(9) = 36 21 49.55	30	0.0333
		Sum $[p]$ =	435	0.3867

(2)

The repetition numbers given by Schwerd are taken here as the weights p.

As we see, the weights p are very unequal. Schwerd has measured rightly the acute angles, which lie opposite the base lines, more carefully.

We find the number of condition equations according to instructions from section 67. The number of the measured angles $W = 9$, the number of the points $p = 4$, and the number of the connecting lines $l = 6$; therefore, we have according to (13), section 67, p. 224.

$$W - 2p + 4 = 5 \text{ condition equations},$$

and in particular

$$\left.\begin{array}{l} l - 2p + 3 = 1 \text{ side equation} \\ l - \ \ p + 1 = 3 \text{ triangle equations} \\ \text{and for point } M \ 1 \text{ horizon equation} \end{array}\right\}, \tag{3}$$

hence, $1 + 3 + 1 = 5$ equations in all as above.

We form the side equation for the central point $D\,Sp$, i.e., we express that the side $D\,Sp - Man$ is obtained equally from the base $D\,Sp - H$ in both possible ways. This yields:

$$\text{We should have } \frac{\sin(1)\sin(4)\sin(8)}{\sin(3)\sin(9)\sin(2)} = 1. \tag{4a}$$

The three triangle closures are:

$$\text{We should have } (1) + (3) + (5) \qquad - (180° + \varepsilon_b) = 0 \tag{4b}$$

$$\text{We should have } (4) + (6) + (9) \qquad - (180 \ + \varepsilon_c) = 0 \tag{4c}$$

$$\text{We should have } (2) + (5) + (6) + (8) - (180 \ + \varepsilon_d) = 0 \tag{4d}$$

and the horizon equation at Mannheim:

$$\text{We should have } (7) + (8) - (9) = 0. \tag{4e}$$

For the spherical excesses we compute all triangles preliminarily on the strength of the base $DH = 4962.8282$ m according to (1) with the measured angles; the mean geographic latitude is about 49°30', hence, $\log r = 6.80487$; with this, we find according to the formula $\varepsilon = \frac{\rho}{r^2}\Delta$:

$$\left.\begin{array}{l} \text{For the triangle } DHJ \ \ \varepsilon_b = 0.138'' \\ \text{For the triangle } DJM \ \ \varepsilon_c = 0.505 \\ \text{For the triangle } DHM \ \ \varepsilon_d = 0.151 \end{array}\right\}. \tag{5}$$

We need these excesses in any case for the setting up of the triangle equations, but we can also use them for the formation of the side equation.

We form first the checks of the sums of triangles:

$$\begin{array}{rl} (1) = & 81^\circ\,21'\,43.36'' \\ (3) = & 25^\circ\,16\;\;28.85 \\ (5) = & 73^\circ\,21\;\;46.35 \\ \hline & 179^\circ\,59'\,58.560'' \\ \text{Should be} & 180\;\;\;0\;\;\;0.138 \\ \hline & w_b = -1.578 \end{array} \qquad \begin{array}{rl} (4) = & 76^\circ\,33'\,44.65'' \\ (6) = & 67\;\;\;4\;\;27.96 \\ (9) = & 36\;\;21\;\;49.55 \\ \hline & 180^\circ\;\;0'\;\;2.160'' \\ \text{Should be} & 180\;\;\;0\;\;\;0.505 \\ \hline & w_c = +1.655 \end{array} \qquad \left.\begin{array}{rl} (2) = & 31^\circ\,37'\,39.73'' \\ (5) = & 73\;\;21\;\;46.35 \\ (6) = & 67\;\;\;4\;\;27.96 \\ (8) = & 7\;\;56\;\;\;6.92 \\ \hline & 180^\circ\;\;0'\;\;0.960'' \\ \text{Should be} & 180\;\;\;0\;\;\;0.151 \\ \hline & w_d = +0.809 \end{array}\right\}. \qquad (6)$$

The condition equations are formed according to the instruction of sections 65 to 67; they are:

a) side equation:

$$+0.320\,v_1 + 0.503\,v_4 + 15.105\,v_8 - 4.459\,v_3 - 2.860\,v_9 - 3.419\,v_2 + 4.715 = 0. \qquad (7)$$

We have computed here in units of the sixth decimal of logarithms.
The three triangle equations become according to (6):

$$\left.\begin{array}{ll} \text{b)} & v_2 + v_5 + v_6 + v_8 + 0.809'' = 0 \\ \text{c)} & v_1 + v_3 + v_5 - 1.578 = 0 \\ \text{d)} & v_4 + v_6 + v_9 + 1.655 = 0 \end{array}\right\}. \qquad (8)$$

Finally, the horizon equation is:

$$\text{e)}\quad v_7 + v_8 - v_9 - 0.100'' = 0. \qquad (9)$$

The angle corrections are denoted here by v, as the corrections in general. In section 65 we had distinguished between corrections to the *angles* δ and corrections to the *directions* v; it will hardly be necessary to explain that this distinction is no longer made here, because in this section 75 it is a question of only one kind of correction, for which the general symbol v is taken.

We write the coefficients of the condition equations (7), (8), (9) in addition to the weights in a table:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	w
p	70	7	101	47	85	57	10	28	30	
$\frac{1}{p}$	0.0143	0.1429	0.0099	0.0213	0.0118	0.0175	0.1000	0.0357	0.0333	
1. a	+0.320	−3.419	−4.459	+0.503	. .	. .	. .	+15.105	−2.860	+4.715
2. b	. .	+1	. .	. .	+1	+1	. .	+1	. .	+0.809
3. c	+1	. .	+1	. .	+1	. .	. .	. .	. .	−1.578
4. d	. .	. .	. .	+1	. .	+1	. .	. .	+1	+1.655
5. e	. .	. .	. .	. .	. .	. .	+1	+1	−1	−0.100

(10)

Now if we form the sums $\left[\frac{a\,a}{p}\right]\left[\frac{a\,b}{p}\right]$, etc., then we note that it was so arranged by Schwerd with good judgment that to the large coefficients there also come the larger weights. The calculation yields the normal equations in the abbreviated manner of writing [according to (2) and (3) on p. 86].

$$\left.\begin{array}{rl} +\underline{10.2941\,k_1} + 0.0511\,k_2 - 0.0396\,k_3 - 0.0846\,k_4 + 0.6348\,k_5 + 4.7150 = 0 \\ +\underline{0.2079\,k_2} + 0.0118\,k_3 + 0.0175\,k_4 + 0.0357\,k_5 + 0.8090 = 0 \\ +\underline{0.0360\,k_3} \quad \cdot\,\cdot \quad\quad \cdot\,\cdot \quad - 1.5780 = 0 \\ +\underline{0.0721\,k_4} - 0.0333\,k_5 + 1.6550 = 0 \\ +\underline{0.1690\,k_5} - 0.1000 = 0 \end{array}\right\} . \tag{11}$$

The solution of these five equations yields

$$k_1 = -0.3436, \quad k_2 = -4.0814, \quad k_3 = +44.7898, \quad k_4 = -23.2072, \quad k_5 = -1.8279. \tag{12}$$

In the computation of the corrections v we follow, as always, Table (10) of the condition equations according to columns, e.g.

$$v_1 = +\frac{0.320}{70}k_1 + \frac{1}{70}k_3 = +0.638'',$$

$$v_2 = -\frac{3.419}{7}k_1 + \frac{1}{7}k_2 = -0.415\ ,$$

$$\cdots\cdots\cdots\cdots$$

The table of all v's in addition to its use for the correction of the measured angles shows the following

Measured	v	Corrected
(1) = 81° 21′ 43.360″	+ 0.638″	[1] = 81° 21′ 43.998″
(2) = 31 37 39.730	− 0.415	[2] = 31 37 39.315
(3) = 25 16 28.850	+ 0.459	[3] = 25 16 29.309
(4) = 76 33 44.650	− 0.497	[4] = 76 33 44.153
(5) = 73 21 46.350	+ 0.479	[5] = 73 21 46.829
(6) = 67 4 27.960	− 0.479	[6] = 67 4 27.481
(7) = 28 25 42.530	− 0.183	[7] = 28 25 42.347
(8) = 7 56 6.920	− 0.396	[8] = 7 56 6.524
(9) = 36 21 49.550	− 0.680	[9] = 36 21 48.870

(13)

After completion of the adjustment we first convince ourselves as to whether the condition equations are satisfied. This is the case, and we can now compute all triangle sides uniquely. The results for four sides are:

$DH =$ 4,962.8282 m (Base)		$\log DH =$ 3.695 7292·3	
$DM =$ 18,851.510	± 0.12 m	$\log DM =$ 4.275 3461·4	
$HM =$ 22,896.729		$\log HM =$ 4.359 7734·4	
$JM =$ 17,851.153	± 0.11 m	$\log JM =$ 4.251 6662·4	

(14)

(The mean errors ±0.12 m and ±0.11 m specified in the case of two sides are inserted from later computations.)

Turning to computations of accuracy, we form first the sum $[p\,v\,v]$ directly from the individual v's and p's, and then for the check according to the formula $[p\,v\,v] = -[w\,k]$, as follows from the calculation in detail:

	v	v^2	p	$p\,v^2$
1)	+ 0.638	0.4070	70	28.49
2)	− 0.415	0.1722	7	1.21
3)	+ 0.459	0.2107	101	21.28
4)	− 0.497	0.2470	47	11.61
5)	+ 0.479	0.2294	85	19.50
6)	− 0.479	0.2294	57	13.08
7)	− 0.183	0.0335	10	0.34
8)	− 0.396	0.1568	28	4.39
9)	− 0.680	0.4624	30	13.87
			435	113.77
				$=[p\,v\,v]$.

	w	k	$-wk$
1)	+ 4.715	− 0.3436	+ 1.62
2)	+ 0.809	− 4.0814	+ 3.30
3)	− 1.578	+ 44.7898	+ 70.68
4)	+ 1.655	− 23.2072	+ 38.41
5)	− 0.100	− 1.8279	− 0.18
			113.83
			$=-[w\,k]$.

We have in sufficient agreement $[p v v] = -[w k] = 113.8$.
The mean error of the unit of weight becomes therefrom:

$$m = \sqrt{\frac{113.8}{5}} = \pm 4.77''. \tag{15}$$

There shall further be determined the weight and the mean error of the side DM after the adjustment. The side DM is derived from DH in the shortest way by the equation

$$DM = \frac{DH}{\sin[8]} \sin[2]. \tag{16}$$

As always in such cases, it is more convenient to convert the function first to logarithmic form, i.e., to form from (16) with the omission of the constant DH

$$F = \log \sin[2] - \log \sin[8]. \tag{17}$$

The computation takes now the same course as in section 72, p. 254. We will have

$$f_2 = +3.419, \quad f_8 = -15.105; \tag{18}$$

all other f's are equal to zero. Then we compute

$$\left[\frac{af}{p}\right] = -9.82, \quad \left[\frac{bf}{p}\right] = -0.05, \quad \left[\frac{cf}{p}\right] = 0.00, \quad \left[\frac{df}{p}\right] = 0.00, \quad \left[\frac{ef}{p}\right] = -0.54,$$
$$\left[\frac{ff}{p}\right] = +9.82 .$$

We set these terms in place of the end terms of (11), eliminate them throughout, and find

$$\left[\frac{ff}{p} \cdot 5\right] = 0.31 = \frac{1}{P}; \tag{19}$$

therefrom the mean error of a function is:

$$M = m\sqrt{\frac{1}{P}} = 4.77\sqrt{0.31} = \pm 2.66 \quad \text{units of the sixth place of logarithms,}$$

i.e.

$$d \log s = \pm 0.000\,0026{\cdot}6 = \frac{\mu}{s}\, ds, \qquad s = 18852 \text{ m}$$

$$ds = \pm 0.115 \text{ m}.$$

Therefore, in all we have the distance

$$DM = 18{,}851.510 \text{ m} \pm 0.115 \text{ m} \quad \text{(error of triangulation)}, \tag{20}$$

as is already indicated in the case of (14).

The base of computation $DH = 4962.8282$ m is assumed here as free from error. But now we will also take into account the mean error of the base itself, and compute its propagation to the line DM.

The mean error of the base DH was estimated by Schwerd at ± 9.7 mm on the strength of different data.

This error increases by trigonometric transfer to the line DM, 18852 m long, so that it yields

$$\frac{18852}{4963}\, 9.7 = \pm\, 36.8 \text{ mm}, \tag{21}$$

and to this, there is added the error of triangulation of the line DM, which has the value ± 115 mm according to (20); hence, the mean total error of the line DM is

$$= \sqrt{36.8^2 + 115^2} = \pm\, 124 \text{ mm}.$$

Instead of (20) we shall therefore write now:

$$DM = 18{,}851.510 \text{ m} \pm 0.124 \text{ m} \quad \text{(total error)}. \tag{22}$$

Mean angle error for mean weight

It is true that in (15) we have determined the mean error of the unit of weight $m = \pm\, 4.77''$, and this value forms the foundation of all further rigorous computations of accuracy, as is seen from the example of the computation of error (16) to (20). But that error of the unit of weight m does not satisfy the desire which often arises of possessing an illustrative measure for the accuracy of angles obtained in the mean at the station. For this, the mean error for the mean weight is used, whereby the mean weight must however not simply be taken as the average value $\frac{[p]}{n}$ of the nine individual weights, but is to be computed corresponding to the definition of the weights as reciprocals of the mean squares of errors and the principle of the mean error itself, as follows:

$$\frac{1}{g} = \frac{1}{n}\left[\frac{1}{p}\right], \tag{23}$$

where n is the number of the individual weights. In our case, the $\frac{1}{p}$'s are given in Table (2) with the sum 0.3867; hence, we have

$$\frac{1}{g} = \frac{0.3867}{9} = 0.04297, \quad g = 23.37, \tag{24}$$

and the mean error for this mean weight is

$$\mu = m\sqrt{\frac{1}{g}} = 4.77\sqrt{0.04297} = \pm\, 0.989''. \tag{25}$$

If at each station not more than the number of angles required for the determination of the station itself were measured, then this μ *would* also represent the mean error of the mean weight of an angle adjusted at the station. But in our case, one angle is excess at one station, namely Mannheim [cf. horizon equation (9)], and therefore, our μ in (25) has a somewhat different meaning, which shall not now be further discussed. On the whole, we have presented (25) only as a formal computational example here, for if the angles, intentionally, are very unequally weighted, the computation (25) does not have the inner sense, which belongs to it in the case of smooth continuous chains or uniformly extended nets.

Schwerd's Base Net

If the angle adjustment of the previous section 75 was not already an example for the current practice, then this is still less true in the case of an adjustment of a net of triangles with angles according to *indirect* observations, which we will add further. But it is useful to understand that we must not necessarily make such an adjustment according to correlates, but that other forms are also available.

In the general theory of section 42 it was already stated that we can reduce measurements, which first present themselves in the form of conditioned observations, also to indirect observations. Whether or not this is useful depends on the individual case; with our base net by Schwerd we can proceed in this way, and in fact, we only have to solve then four normal equations, compared with five in the case of the correlate method. But the reduction to indirect observations will also offer further advantages in examinations of accuracy.

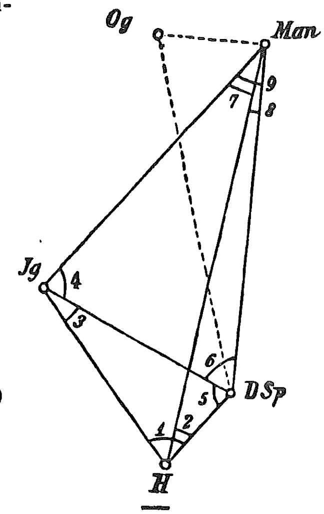


Fig. 1.
Schwerd's base net.
Scale 1:400,000.

According to the previous section 75 (7) to (10), p. 267, the following five independent condition equations exist for Schwerd's net, Fig. 1:

$$\left.\begin{aligned} &+0.320\,v_1 - 3.419\,v_2 - 4.459\,v_3 + 0.503\,v_4 \\ &\qquad + 15.105\,v_8 - 2.860\,v_9 + 4.715 = 0 \\ &v_2 + v_5 + v_6 + v_8 + 0.809 = 0 \\ &v_1 + v_3 + v_5 - 1.578 = 0 \\ &v_4 + v_6 + v_9 + 1.655 = 0 \\ &v_7 + v_8 - v_9 - 0.100 = 0 \end{aligned}\right\} . \qquad (1)$$

Of the nine measured angles we choose the four angles (1), (2), (3), and (8) as independent unknowns, or, turning at once to the corrections, we take the corrections v_1, v_2, v_3, v_8 as independent unknowns of the adjustment, and there arises the problem to express all other v's in terms of these v_1's, v_2's, v_3's, v_8's, with the help of the condition equations (1).

According to the usual algebraic methods we obtain for this:

$$\left.\begin{aligned} v_5 &= -v_1 - v_3 + 1.578 \\ v_6 &= +v_1 - v_2 + v_3 - v_8 - 2.387 \\ v_9 &= -0.054\,v_1 - 0.867\,v_2 - 1.475\,v_3 + 4.641\,v_8 + 1.511 \\ v_4 &= -0.946\,v_1 + 1.867\,v_2 + 0.475\,v_3 - 3.641\,v_8 - 0.779 \\ v_7 &= -0.054\,v_1 - 0.867\,v_2 - 1.475\,v_3 + 3.641\,v_8 + 1.611 \end{aligned}\right\} . \qquad (2)$$

To these there are added further the error equations for the independent v's themselves, namely

$$v_1 = v_1, \qquad v_2 = v_2, \qquad v_3 = v_3, \qquad v_8 = v_8. \qquad (3)$$

The weights p are already indicated in (2), p. 266, but this time it is more convenient for us to take the unit of weight 100 times smaller, and with this, we obtain the following table of the coefficients of the error equations (2) and (3) and the weights p, in addition to $\sqrt{p}$:

Error equations: (4)

						p	$\sqrt{p}$
$v_1 =$	$+1.000\,v_1$	..	..	..	..	0.70	0.837
$v_2 =$	..	$+1.000\,v_2$	..	..	..	0.07	0.265
$v_3 =$	..	..	$+1.000\,v_3$	..	..	1.01	1.005
$v_8 =$	..	..	..	$+1.000\,v_8$	..	0.28	0.529
$v_5 =$	$-1.000\,u_1$	..	$-1.000\,v_3$	..	$+1.578$	0.85	0.922
$v_6 =$	$+1.000\,v_1$	$-1.000\,v_2$	$+1.000\,v_3$	$-1.000\,v_8$	-2.387	0.57	0.755
$v_4 =$	$-0.946\,v_1$	$+1.867\,v_2$	$+0.475\,v_3$	$-3.641\,v_8$	-0.779	0.47	0.686
$v_7 =$	$-0.054\,v_1$	$-0.867\,v_2$	$-1.475\,v_3$	$+3.641\,v_8$	$+1.611$	0.10	0.316
$v_9 =$	$-0.054\,v_1$	$-0.867\,v_2$	$-1.475\,v_3$	$+4.641\,v_8$	$+1.511$	0.30	0.548

In order to take into account the weights once for all, we multiply the coefficients and absolute terms of the error equations, respectively, by $\sqrt{p}$ and obtain, with this, the following new table:

Equally weighted error equations

$$\left.\begin{array}{rcrrrrr}
\sqrt{p_1}\,v_1 & = & +0.84\,v_1 & .. & .. & .. & .. \\
\sqrt{p_2}\,v_2 & = & .. & +0.26\,v_2 & .. & .. & .. \\
\sqrt{p_3}\,v_3 & = & .. & .. & +1.00\,v_3 & .. & .. \\
\sqrt{p_8}\,v_8 & = & .. & .. & .. & +0.53\,v_8 & .. \\
\sqrt{p_5}\,v_5 & = & -0.92\,v_1 & .. & -0.92\,v_3 & .. & +1.46 \\
\sqrt{p_6}\,v_6 & = & +0.75\,v_1 & -0.75\,v_2 & +0.75\,v_3 & -0.75\,v_8 & -1.80 \\
\sqrt{p_4}\,v_4 & = & -0.65\,v_1 & +1.28\,v_2 & +0.33\,v_3 & -2.49\,v_8 & -0.53 \\
\sqrt{p_7}\,v_7 & = & -0.02\,v_1 & -0.27\,v_2 & +0.46\,v_3 & +1.15\,v_8 & +0.51 \\
\sqrt{p_9}\,v_9 & = & -0.03\,v_1 & -0.48\,v_2 & -0.81\,v_3 & +2.55\,v_8 & +0.83
\end{array}\right\}. \tag{5}$$

Rounding off to 0.01 is sufficient in order to obtain an accuracy of angles of 0.01".

The normal equations belonging to this are in the abbreviated manner of writing

$$\left.\begin{array}{rrrrr}
+\underline{2.54\,v_1} & -1.37\,v_2 & +1.23\,v_3 & +0.96\,v_8 & -2.38 = 0 \\
 & +\underline{2.57\,v_2} & +0.37\,v_3 & -4.16\,v_8 & +0.14 = 0 \\
 & & +\underline{3.39\,v_3} & -3{,}98\,v_8 & -3.77 = 0 \\
 & & & +\underline{14.87\,v_8} & +5.37 = 0 \\
 & & & & +6.60
\end{array}\right\}. \tag{6}$$

The solution of this group of four equations yielded:

$$v_1 = +0.64'', \qquad v_2 = -0.41'', \qquad v_3 = +0.46'', \qquad v_4 = -0.40''. \tag{7}$$

By setting these first four v's into the error equations (4) we also obtain all other v's, and in fact, everywhere in agreement to about 0.01" with the previous results v of (13), section 75, p. 268; therefore, we do not insert these numbers once again.

The elimination of the normal equations (6) yields also an end term:

$$[ll \cdot 4] = 1.14, \tag{8}$$

which likewise agrees sufficiently with 113.8 according to (15), section 75, p. 269, taking into account the unit of weight now 100 times smaller.

The mean angle error for the unit of weight is now

$$m = \sqrt{\frac{[p\,v\,v]}{9-4}} = \sqrt{\frac{1.14}{5}} = \pm 0.477''. \tag{9}$$

This corresponds to the previous 4.77" according to (15), section 75, p. 269, in the case of a 100 times larger unit of weight.

We will determine further the weight of a function of the adjusted angles, and in fact, this function shall be the side JM. The side DH holds as an error-free base; the function is therefore

$$JM = DH \frac{\sin[1]\,\sin[6]}{\sin[3]\,\sin[9]}, \tag{10}$$

or logarithmically

$$\log JM = \log DH + \log\sin[1] + \log\sin[6] - \log\sin[3] - \log\sin[9]. \tag{11}$$

The differential is

$$d \log J M = \frac{\mu}{\rho} \cot [1] \, d[1] + \frac{\mu}{\rho} \cot [6] \, d[6] - \frac{\mu}{\rho} \cot [3] \, d[3] - \frac{\mu}{\rho} \cot [9] \, d[9]$$

$$d \log J M = 0.32 \, d[1] + 0.90 \, d[6] - 4.46 \, d[3] - 2.86 \, d[9]. \tag{12}$$

But only [1] and [3] are independent unknowns of the adjustment here, and therefore, only $d[1]$ and $d[3]$, or the differential $d[2]$ and $d[8]$ of the likewise independent adjusted unknowns [2] and [8], can occur in (12). Therefore, in order to eliminate $d[6]$ and $d[9]$, we consider the error equations (2) and form therefrom

$$\left.\begin{aligned} d[9] &= -0.054 \, d[1] - 0.867 \, d[2] - 1.475 \, d[3] + 4.641 \, d[8], \\ d[6] &= d[1] - d[2] + d[3] - d[8] \end{aligned}\right\}. \tag{13}$$

These equations are correct because the absolute terms l vanish in the error equations, if we start from the adjusted angles [1], [2] . . . instead from the observed angles (1), (2) . . . and, correspondingly, set $d[1]$, $d[2]$ . . . instead of v_1, v_2 . . . . We can read the second equation (13) directly from the figure.
Therefore, if we substitute (13) in (12), then we obtain

$$d \log J M = +1.37 \, d[1] + 1.58 \, d[2] + 0.66 \, d[3] - 14.17 \, d[8], \tag{14}$$

hence,

$$f_8 = -14.17, \quad f_3 = +0.66, \quad f_1 = +1.37, \quad f_2 = +1.58. \tag{15}$$

We have assumed here the order f_8, f_3, f_1, f_2, because it is of interest to us for later purposes to have v_2 as the last unknown, in order to obtain the weight of v_2 at the same time. If, in order to conform with the instruction of section 31, p. 102, we set now the values f computed in (15) in place of the absolute terms of the normal equations (6) and collect terms according to the sequence f_8, f_3, f_1, f_2, then we find

$$\left.\begin{aligned} +\underline{14.87 \, q_8} - 3.98 \, q_3 + 0.96 \, q_1 - 4.16 \, q_2 + 14.17 &= 0 \\ +\underline{3.39 \, q_3} + 1.23 \, q_1 + 0.37 \, q_2 - 0.66 &= 0 \\ +\underline{2.54 \, q_1} - 1.37 \, q_2 - 1.37 &= 0 \\ +\underline{2.57 \, q_2} - 1.58 &= 0 \end{aligned}\right\}. \tag{16}$$

The further elimination yields

$$\left.\begin{aligned} +\underline{2.32 \, q_3} + 1.49 \, q_1 - 0.74 \, q_2 + 3.12 &= 0 \\ +\underline{2.48 \, q_1} - 1.10 \, q_2 - 2.28 &= 0 \\ +1.41 \, q_2 + 2.39 &= 0 \end{aligned}\right\}. \tag{17}$$

$$\left.\begin{aligned} +\underline{1.52 \, q_1} - 0.62 \, q_2 - 4.28 &= 0 \\ +\underline{1.17 \, q_2} + 3.38 &= 0 \end{aligned}\right\}. \tag{18}$$

$$+\underline{0.92 \, q_2} - 1.61 = 0. \tag{19}$$

The q's written here are only symbols for the order of the previous v's.
According to the instruction of (13), section 31, p. 102, we have now

$$\frac{1}{P} = \frac{14.17^2}{14.87} + \frac{3.12^2}{2.32} + \frac{4.28^2}{1.52} + \frac{1.61^2}{0.92} = 13.50 + 4.20 + 12.05 + 2.81 = 32.56, \tag{20}$$

$$M = m \sqrt{\frac{1}{P}} = 0.477 \sqrt{32.56} = \pm 2.72. \tag{21}$$

This is the mean error of function (11) in units of the sixth decimal, hence:

$$\log J M = 4.251\ 6662{\cdot}4 \pm 27{\cdot}2, \tag{22}$$
$$\log J M = 17{,}851.15 \text{ m} \pm 0.11 \text{ m}. \tag{23}$$

This has already been added in (14), section 75.

At the same time we have also obtained the weight of the angle correction v_2 in the last coefficient of q_2 in (19), namely

$$P_2 = 0.92 \tag{24}$$

(the same coefficient 0.92 would also be obtained if we solved equations (6), already previously existing, so that v_2 would be the last unknown).

We also find therefrom the mean error of the adjusted angle $(2) + v_2 = [2]$, namely:

$$M_2 = \frac{m}{\sqrt{0.92}} = \pm 0.50''$$

(the eliminations of this section are only done with the slide rule, hence are accurate everywhere only to about 2-3 places).

Section 77. Most Favorable Choice of the Side Equation in the Quadrilateral

At the end of section 67, p. 227, we have briefly mentioned a question concerning the most favorable form of the side equation in a quadrilateral and postponed it to the end of this discussion, so that we now have to treat this matter.

Just as the conditions concerning sums of angles could be expressed in various forms [see section 66, equations (10) to (13), p. 218; cf. also the following section 79], so can also the condition of coinciding side computation in a quadrilateral be determined in different forms.

The nature of the side equation consists in the fact that for a central point all quotients of the lengths of the consecutive rays are to be expressed by the sines of the angles of triangles belonging to them. The

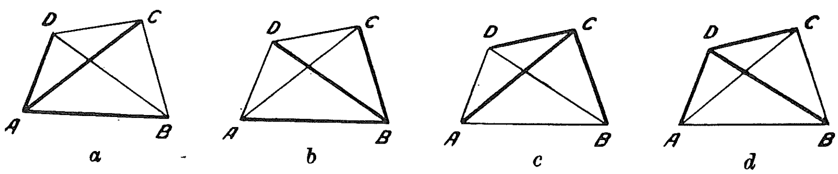


Fig. 1. Side equations.

product of all of these quotients, which must be equal to unity, yields the side equation.

For the four corner points of a quadrilateral as central points, the three rays starting from each central point are especially marked in Fig. 1 *a*, *b*, *c*, *d*.

According to this, we have, e.g., for the case a the three quotients

$$\frac{AD}{AC}, \quad \frac{AC}{AB}, \quad \frac{AB}{AD},$$

whose product must be equal to one. If we express, according to Fig. 2, the quotients by the sine conditions, then we have

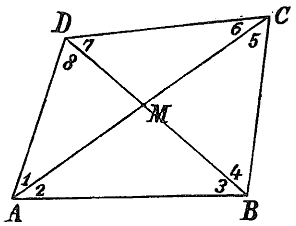


Fig. 2.
Complete quadrilateral.

$$\frac{AD}{AC} = \frac{\sin(6)}{\sin(7+8)}, \quad \frac{AC}{AB} = \frac{\sin(3+4)}{\sin(5)}, \quad \frac{AB}{AD} = \frac{\sin(8)}{\sin(3)} \tag{1}$$

and this yields the side equation

$$(A)\quad \frac{\sin(6)\sin(3+4)\sin(8)}{\sin(7+8)\sin(5)\sin(3)} = 1\,. \tag{2}$$

Likewise, we obtain for the central points B, C, D the equations:

$$\left.\begin{array}{ll} (B) & \dfrac{\sin(8)\sin(5+6)\sin(2)}{\sin(1+2)\sin(7)\sin(5)} = 1 \\ (C) & \dfrac{\sin(2)\sin(4)\sin(7+8)}{\sin(1)\sin(7)\sin(3+4)} = 1 \\ (D) & \dfrac{\sin(4)\sin(1+2)\sin(6)}{\sin(5+6)\sin(3)\sin(1)} = 1 \end{array}\right\}. \tag{3}$$

In addition to the four corner points there are three further central points for the quadrilateral, namely the points of intersection of each two opposite sides and the point of intersection of the diagonals. In Fig. 3 we obtain hereby the three new central points M, S, S'.

For the central point M we set up the product of the quotients:

$$\frac{MA}{MB}\cdot\frac{MB}{MC}\cdot\frac{MC}{MD}\cdot\frac{MD}{MA} = 1$$

which yields the side equation

$$(M)\quad \frac{\sin(3)\sin(5)\sin(7)\sin(1)}{\sin(2)\sin(4)\sin(6)\sin(8)} = 1\,. \tag{4}$$

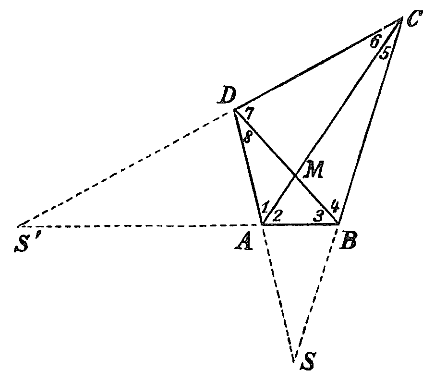


Fig. 3.
Central points M, S, and S'.

Likewise, we have for the central point S:

$$\frac{SA}{SC}\cdot\frac{SC}{SD}\cdot\frac{SD}{SB}\cdot\frac{SB}{SA} = 1\,,$$

$$(S)\quad \frac{\sin(5)\sin(7+8)\sin(4)\sin(1+2)}{\sin(1)\sin(5+6)\sin(8)\sin(3+4)} = 1 \tag{5}$$

and, finally, for S':

$$\frac{S'A}{S'C}\cdot\frac{S'C}{S'B}\cdot\frac{S'B}{S'D}\cdot\frac{S'D}{S'A} = 1\,,$$

$$(S')\quad \frac{\sin(6)\sin(3+4)\sin(7)\sin(1+2)}{\sin(2)\sin(5+6)\sin(3)\sin(7+8)} = 1\,. \tag{6}$$

We also see that equation (S) is equal to the quotient of equations (A) and (D) or (B) and (C). By analogy, (S') is equal to the quotient of equations (A) and (B) or (D) and (C), and M is equal to the product of equations (A) and (C) or (B) and (D).

The side equations (A) to (D) have six terms, while (M), (S), and (S') have eight terms.

Insofar as it is a question of the side equations in the above closed form, all seven forms are of course equally weighted. The matter is different, however, if we pass over to the linear forms now, and it is not immaterial for the rigorousness of numerical computation which one of these formulae is used.

We will first consider the six-term side equations on the basis of a numerical example according to Fig. 4. For the sake of simplicity, the values of the angles are rounded off here to 1', and, in fact, in such a way that the sums of the angles in the triangles yield exactly 180°.

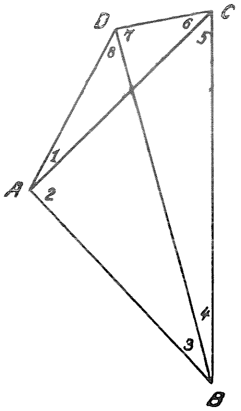


Fig. 4.

(1) = 16° 40′	(1) = 16° 40′	(2) = 91° 11′	(7) = 84° 38′	
(2) = 91 11	(8) = 44 33	(3) = 27 36	(4) = 14 14	
(3) = 27 36	(7) = 84 38	(4) = 14 14	(6) = 34 09	(7)
(8) = 44 33	(6) = 34 09	(5) = 46 59	(5) = 46 59	
180° 00′	180° 00′	180° 00′	180° 00′	

We have already set up the four side equations (A) to (D) belonging to this on p. 275. These equations, however, cannot directly be compared with one another, because they refer to different angles. We will therefore reduce all four equations to the 5 angles (1), (2), (5), (6), and (8), for which we have:

$$\left.\begin{aligned} (3+4) &= 180° - (2+5) \\ (7+8) &= 180 \;\; - (1+6) \\ (3) \quad &= 180 \;\; - (1+2+8) \\ (7) \quad &= 180 \;\; - (1+6+8) \\ (4) \quad &= (1) \;\; - (5) + (8) \end{aligned}\right\} . \tag{8}$$

With this, equations (A) to (D) of p. 275 pass over to:

$$(A') \quad \frac{\sin(1+2+8)\sin(5)\sin(1+6)}{\sin(6)\sin(8)\sin(2+5)} = 1 \tag{9}$$

$$(B') \quad \frac{\sin(8)\sin(5+6)\sin(2)}{\sin(1+2)\sin(1+6+8)\sin(5)} = 1 \tag{10}$$

$$(C') \quad \frac{\sin(2)\sin(1-5+8)\sin(1+6)}{\sin(1)\sin(1+6+8)\sin(2+5)} = 1 \tag{11}$$

$$(D') \quad \frac{\sin(1-5+8)\sin(1+2)\sin(6)}{\sin(5+6)\sin(1+2+8)\sin(1)} = 1 . \tag{12}$$

These four equations, also, are completely equivalent.

We add now corrections v to the individual angle values and set up the linear equations. Equation (A') leads to the following computational procedure:

			log Diff. for 1′
(1 + 2 + 8) = 152° 24′	*log sin* (1 + 2 + 8)	= 9.66586	− 24
(5) = 46 59	*log sin* (5)	= 9.86401	+ 12
(1 + 6) = 50 49	*log sin* (1 + 6)	= 9.88937	+ 11
		9.41924	

			log Diff. for 1′
(6) = 34° 09′	*log sin* (6)	= 9.74924	+ 19
(8) = 44 33	*log sin* (8)	= 9.84605	+ 13
(2 + 5) = 138 10	*log sin* (2 + 5)	= 9.82410	− 14
		9.41939	

$$9.41924 - 9.41939 = -0.00015 ,$$

hence, in units of the fifth decimal we have the side equation:

$$-24(v_1+v_2+v_8)+12v_5+11(v_1+v_6)-19v_6-13v_8+14(v_2+v_5)-15=0$$

or

$$-13v_1-10v_2+26v_5-8v_6-37v_8-15=0 . \tag{13}$$

In entirely the same way we also find the three linear side equations for the central points B, C, and D, so that we have altogether:

$$\left.\begin{array}{ll} (A'') & -13\,v_1 - 10\,v_2 + 26\,v_5 - 8\,v_6 - 37\,v_8 - 15 = 0 \\ (B'') & -5\,v_1 - 3\,v_2 + 10\,v_5 - 3\,v_6 - 14\,v_8 - 7 = 0 \\ (C'') & -19\,v_1 - 13\,v_2 + 36\,v_5 - 12\,v_6 - 51\,v_8 - 22 = 0 \\ (D'') & -27\,v_1 - 20\,v_2 + 52\,v_5 - 17\,v_6 - 74\,v_8 - 30 = 0 \end{array}\right\}. \tag{14}$$

Apart from minor differences, these four equations agree perfectly from the algebraic point of view, as it must. From the practical point of view, however, the fourth (D'') is to be preferred to all others, since it has the largest coefficients; in (D''), e.g., these coefficients are about five to six times as accurate as the coefficients of equation (B'').

For the comparison of the four forms (14) it is sufficient to compare the four absolute terms with each other, and that equation which has the largest absolute term will be the most favorable one.

Now we will pursue this further geometrically on the basis of Fig. 5. The occurrence of an absolute term in the side equation expresses the fact that in combining the individual triangles into a quadrilateral, or in constructing the quadrilateral from the angles of the individual triangles, at any point there is formed a triangle which shows error, as, e.g. $D_1 D_2 D_3$ in Fig. 5.

We consider the two side equations (9) and (11) for the central points A and C:

$$(A')\quad \frac{\sin(1+2+8)\sin(5)\sin(1+6)}{\sin(6)\sin(8)\sin(2+5)} = 1 \tag{15}$$

$$(C')\quad \frac{\sin(2)\sin(1-5+8)\sin(1+6)}{\sin(1)\sin(1+6+8)\sin(2+5)} = 1\,. \tag{16}$$

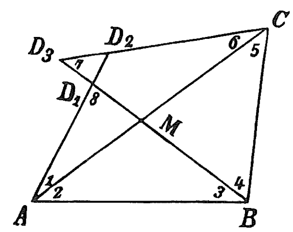


Fig. 5.

Because of the observational errors, these equations are not satisfied, but there appear discrepancies, which occur in the computation in logarithmic form. If we compare equation (15), according to its origin from (1), with Fig. 5, then we find

$$\frac{\sin(1+2+8)\sin(5)\sin(1+6)}{\sin(6)\sin(8)\sin(2+5)} = \frac{A\,D_1}{A\,D_2} = 1 - \frac{D_1\,D_2}{A\,D_2}\,,$$

or if we carry equation (15) out logarithmically, then we obtain on the right-hand side, instead of $\log 1 = 0$, the discrepancy

$$\log\left(1 - \frac{D_1\,D_2}{A\,D_2}\right) = -\,\mu\,\frac{D_1\,D_2}{A\,D_2}\,, \tag{17}$$

where $\mu = 0.43429$ is the logarithmic modulus. This value (17) is the absolute term of the linear side equation, which results by differentiation of (15); therefore, without taking into account the sign, we have for the central point A the relative measure for the absolute term of the side equation

$$(a) = \frac{D_1 D_2}{A D}\,, \tag{18}$$

where we write in the denominator simply AD instead of AD_2 with sufficient accuracy.

If we do everything by analogy for the central point C, then we obtain

$$(c) = \frac{D_2 D_3}{C D}\,. \tag{19}$$

From (18) and (19) we find the comparison

$$\frac{(a)}{(c)} = \frac{D_1 D_2}{D_2 D_3} \frac{C D}{A D}. \tag{20}$$

But we have

$$\frac{D_1 D_2}{D_2 D_3} = \frac{\sin(7)}{\sin(8)}$$

and according to Fig. 2

$$\frac{C D \sin(7)}{A D \sin(8)} = \frac{C M}{A M}; \tag{21}$$

consequently, the ratio of the absolute terms in the side equations (15) and (16) is

$$\frac{(a)}{(c)} = \frac{C M}{A M}. \tag{22}$$

We would obtain the same ratio if we did not present for inspection the absolute terms of the side equations for A and C by the triangle which shows error at D, but by the triangle which shows error at B, as we would find by repetition of the examination in regard to B, which already appears from the fact that in the ratio $(a) : (c)$ there does not occur a quantity of the quadrilateral referring to D or B.

The fundamental theorem expressed in the comparison (22) was found for the first time by General Zachariae in *Den danske Gradmaaling*, Vol. II, 1872, pp. 483 to 487, and later represented once again in the work, *Die geodätischen Hauptpunkte und ihre Koordinaten*, translated from Danish into German by L a m p, Berlin, 1878, pp. 152 and following.

We make the comparison referring to Fig. 6 and obtain the following:

$$\left.\begin{array}{llll} A & \text{is more favorable than } C, & \text{because} & AM < CM \\ B & \text{is more favorable than } D, & \text{because} & BM < DM \\ A & \text{is more favorable than } B, & \text{because} & AS' < BS' \\ A & \text{is more favorable than } D, & \text{because} & AS < DS \\ B & \text{is more favorable than } C, & \text{because} & BS < CS \\ D & \text{is more favorable than } C, & \text{because} & DS' < CS' \end{array}\right\}. \tag{23}$$

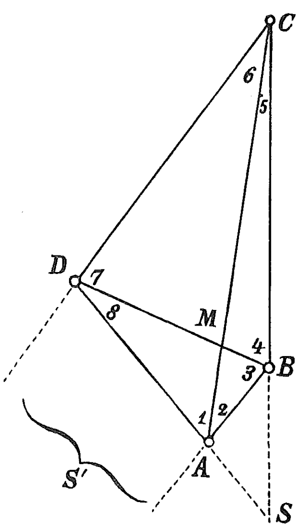


Fig. 6.

From all six comparisons it follows that A is the most favorable central point for the setup of a side equation. Only the six-term side equations (corresponding to Fig. 1*a* to *d*) are compared here.

All of this refers to the case in which the sums of the angles in the individual triangles yield exactly 180°, or in which only the five angles (1), (2), (5), (6), (8) are measured, which means the same thing.

If more than five angles are measured, then discrepancies will exist in the equations of the sums of angles, and the question is to what extent the above theory changes in this case.

If the side equation is set up in the four different forms (A) to (D), according to equations (2) and (3), p. 275, then, with the help of the equations of the sums of angles, we can again convert the four equations to five determined angles, and there exists the same case as if only these five angles were measured. It is immaterial which five angles are chosen here; the ratio of the coefficients and absolute terms of the converted side equations remains always the same, since it depends only on the shape of the quadrilateral.

It is immaterial here with regard to the accuracy of the adjustment if we introduce in the computation the transformed or the original side equations, since the condition equations can be transformed arbitrarily, as we have already shown in section 52, p. 164.

Eight-term side equations. The material hitherto discussed has for the most part been given by Zachariae in the above-mentioned works; only the four forms a, b, c, d, Fig. 1, p. 274, have been taken into account here.

A completion of this theory is now, first of all, possible in the sense that the eight-term side equations of Fig. 3 also are included. We have already specified the three eight-term side equations (M), (S), and (S') in (4), (5), and (6), p. 275, and have also shown that

$$\left.\begin{array}{l} (M) = (A) \times (C) = (B) \times (D) \\ (S) = \dfrac{(A)}{(D)} = \dfrac{(B)}{(C)}, \qquad (S') = \dfrac{(A)}{(B)} = \dfrac{(D)}{(C)} \end{array}\right\}. \tag{24}$$

If we thus denote the logarithmic absolute terms of the three side equations (M), (S), and (S') by (m), (s), and (s'), as hitherto, then we have

$$\left.\begin{array}{l} (m) = (a) + (c) = (b) + (d) \\ (s) = (a) - (d) = (b) - (c) \\ (s') = (a) - (b) = (d) - (c) \end{array}\right\}. \tag{25}$$

Triangle areas as a measure of suitability

Zachariae's theory requires, for finding the most favorable central point according to (23), a *six*fold comparison of diagonal lengths in order to find, finally, the most favorable case from all six comparisons.

This can be reduced to a much more convenient *area* comparison, to be accomplished at *one* glance, as we shall now show:

We consider the formula (22) in connection with Fig. 7 and find

$$\frac{(a)}{(c)} = \frac{CM}{AM} = \frac{\triangle BDC}{\triangle ABD},$$

i.e., the absolute terms (a) and (c) are proportional to the *areas* of the triangles BDC and ABD. If we include now further equations (25), then we obtain the following summary, and, with this, the total result of our investigation

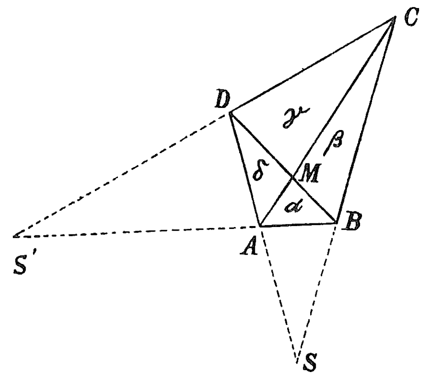


Fig. 7.
Area proportions α, β, γ, δ to equations (26) and (27).

Central point	Measure of suitability (Fig. 7)	
A with six-term side equation	area $\beta + \gamma$	(26)
B with six-term side equation	area $\gamma + \delta$	
C with six-term side equation	area $\alpha + \delta$	
D with six-term side equation	area $\alpha + \beta$	
S with eight-term side equation	area $\gamma - \alpha$	(27)
S' with eight-term side equation	area $\beta - \delta$	
M with eight-term side equation	area $\alpha + \beta + \gamma + \delta$	

The side equation (4), p. 275, for the central point M is the most favorable; however, it has eight terms.

If two sides of a quadrilateral become *parallel,* then the eight-term equation corresponding to their intersection fails completely; if, e.g., AB is parallel to DC, then we will have $\beta = \delta$, and the equation for S' is reduced to $0 \ldots + 0 \ldots \ldots = 0$, which we can also prove directly.

If the four points $ABCD$ do not form a real quadrilateral as Fig. 7, but a triangle ABC with an inner point D, as, e.g. Fig. 8, then we obtain the following comparison concerning the suitability of the seven possible side equations:

Central point	Measure of suitability (Fig. 8)	
A with six-term side equation	area α	
B with six-term side equation	area β	(28)
C with six-term side equation	area γ	
D with six-term side equation	area $\alpha+\beta+\gamma$	
S with eight-term side equation	area $\beta+\gamma$	
S' with eight-term side equation	area $\alpha+\beta$	(29)
M with eight-term side equation	area $\alpha+\gamma$	

In this case, the side equation for the central point D is absolutely the most favorable, for it has only six terms and has the largest number of coefficients.

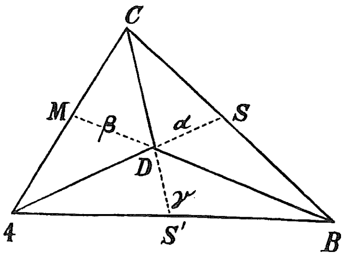


Fig. 8.
Area proportions α, β, γ for equations (28) and (29).

If, in the above, also the eight-term forms were drawn in for comparison, this happened because of theoretical completeness, for in the case of real adjustment computations the consideration of a small number of terms often prevails, and we will then have to choose only among the four cases $A\,B\,C\,D$ and, in this case, make the selection according to (26) or (28).

The elimination of the eight-term side equations, however, is not at all always justified; on the contrary, we regard the absolutely most favorable side equation (4) for the fictitious central point M of Fig. 3, which, in addition, has so symmetrical a construction that it can be written down from memory for every quadrilateral, as the only one recommendable in cases in which rigorous computation is required.

As an example, we have once again carried out the adjustment of the Hannover pentagon of Fig. 1, p. 236, with the introduction of the *most rigorous* eight-term form for the quadrilateral Ägidius-Burg-Schanze-Steuerndieb, namely according to Fig. 1, p. 236:

$$\frac{\sin(17-16)\sin(14-13)\sin(3-2)\sin(20-19)}{\sin(18-17)\sin(15-14)\sin(4-3)\sin(21-20)}=1.$$

The calculation with the angles of p. 237 yielded in units of the sixth decimal of logarithms:

$$\begin{aligned}&+1.64\,(v_{17}-v_{16})+2.16\,(v_{14}-v_{13})+2.53\,(v_3-v_2)+2.41\,(v_{20}-v_{19})\\&+1.42\,(v_{17}-v_{18})+3.54\,(v_{14}-v_{15})+1.58\,(v_3-v_4)+2.25\,(v_{20}-v_{21})+6.5=0,\end{aligned}$$

or collecting terms:

$$\begin{aligned}&-2.53\,v_2+4.11\,v_3-1.58\,v_4-2.16\,v_{13}+5.70\,v_{14}-3.54\,v_{15}\\&-1.64\,v_{16}+3.06\,v_{17}-1.42\,v_{18}-2.41\,v_{19}+4.66\,v_{20}-2.25\,v_{21}+6.5=0.\end{aligned}$$

This equation was set in place of the second equation on p. 241, and all the rest was left; the normal equations, in place of p. 240, become the following (in the abbreviated manner of writing according to p. 86):

$$\begin{array}{l}
\underline{+107.19k_1}+1.08k_2-3.41k_3+3.27k_4+0.08k_5+0.49k_6-0.87k_7-0.24k_8+18.20=0\\
\underline{+122.13k_2}-0.28k_3\ \ldots\ -0.58k_5-2.37k_6+2.32k_7+1.90k_8+6.50=0\\
\underline{+6.00k_3}-2.00k_4\ \ldots\ \ldots\ -2.00k_7-2.00k_8-1.02=0\\
\underline{+6.00k_4}-2.00k_5\ \ldots\ \ldots\ \ldots\ +2.22=0\\
\underline{+6.00k_5}-2.00k_6\ \ldots\ -2.00k_8-2.36=0\\
\underline{+6.00k_6}-2.00k_7+2.00k_8-0.76=0\\
\underline{+6.00k_7}+2.00k_8+2.30=0\\
\underline{+6.00k_8}+4.30=0.
\end{array}$$

The solution yielded:

$$k_1 = -0.175 \quad k_2 = -0.027 \quad k_3 = -0.386 \quad k_4 = -0.366$$
$$k_5 = +0.111 \quad k_6 = +0.462 \quad k_7 = -0.059 \quad k_8 = -0.941\,.$$

The further computation yielded v's somewhat different than were obtained in Table II of p. 241, in favor of the *new* adjustment; therefore, on the bottom of p. 241 we have set these *new* v's, not the old ones before, as is already explained on p. 242 in small print. The new computation with the more rigorous side equation has thus brought corrections up to the amount of about 0.01" without causing here more labor than a less favorable side equation.

Division and multiplication of the side equations

Referring to the above investigation about the most favorable form of side equation, let us further discuss here another question of form.

The point in question is that a side equation may arbitrarily be divided (or also multiplied) by 10, 100, 1000, in short, by *any* arbitrary number, if its coefficients become inconveniently large, without changing its mathematical sense or its rigorousness, provided that no casting off of decimals occurs. This is already specified in section 52, p. 164.

The nomenclature "*large*" coefficients can mean here two things; e.g., in equation (14), p. 277, $74\,v_8$ has a *large* coefficient 74 in comparison to $14\,v_8$, which has only the *small* coefficient 14. But if we divide the fourth equation of (14) by 10 and straightway take to this the second equation, then we have:

$$-2.7\,v_1 - 2.0\,v_2 + 5.2\,v_5 - 1.7\,v_6 - 7.4\,v_8 - 3.0 = 0$$
$$-\;\;5\,v_1 - \;\;3\,v_2 + 10\,v_5 - \;\;3\,v_6 - 14\,v_8 - \;\;7 = 0\,.$$

Now we could say that 7.4 is a *small* coefficient in comparison to 14; but in another sense 7.4 is a larger coefficient than 14, because 7.4 formed from $\frac{74}{10}$ is accurate with ±1 unit of its last place, that is, to $\frac{1}{74}$ or 1.3% of its value, but 14 with ±1 of its last place is accurate only to $\frac{1}{14}$ or 7.1% of its value, or if we take the coefficients of v_5, namely 5.2 and 10, then it is true that 5.2 in this form is smaller than 10, yet 5.2 in the sense of the theory treated here is the larger coefficient, because the position of the decimal point is immaterial.

Or, in short, the *large* coefficients are the better ones because, even divided by 10, 100 and arbitrarily, they are *relatively* still more accurate than the originally smaller coefficients.

In the next section 78, we shall give a further explanation to this matter by a numerical example.

Section 78. Favorable and Unfavorable Side Equation in the Quadrilateral

For further clarification of the theory treated in the previous section 77 for a more or less rigorous setting up of the side equation in the quadrilateral, we will in the following calculate a numerical example with a group of triangles composed of two quadrilaterals (Fig. 1), for which two side equations exist. For the sake of simplicity, we will assume here that only the eight angles (1), (2), (5), (6), (7), (9), (10), (11) are measured so that condition equations for the sums of angles are not involved.

Since only 6 angles are required for the determination of the whole group of triangles, then we have two excessive angles, therefore also two condition equations, namely the two side equations already mentioned. The same also results according to the first equation (13), section 67, p. 224.

The measured angles are the following:

$$\left.\begin{array}{ll} (1) = 50^\circ\,20'\,07'' & (7) = 40^\circ\,45'\,28'' \\ (2) = \;\;1\;\;44\;\;56 & (9) = 49\;\;18\;\;25 \\ (5) = \;\;1\;\;40\;\;56 & (10) = 48\;\;39\;\;26 \\ (6) = 49\;\;15\;\;56 & (11) = 40\;\;17\;\;00 \end{array}\right\}. \qquad (1)$$

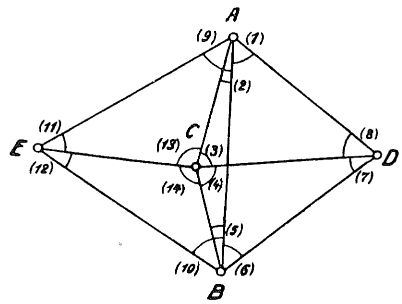


Fig. 1.

The application of the theorem of areas (26), section 77, p. 279, shows that for the two quadrilaterals $ADBC$

and $ACBE$ the point C is the most favorable central point, while the two central points D and E yield the most unfavorable side equations.

We begin with setting up the two latter equations and have for this:

$$\text{Central point } D \left\{ \begin{array}{l} \dfrac{DA \cdot DC \cdot DB}{DC \cdot DB \cdot DA} = 1 \\ \dfrac{\sin(3)\sin(5+6)\sin(1)}{\sin(1+2)\sin(4)\sin(6)} = 1 \end{array} \right. \tag{2}$$

$$\text{Central point } E \left\{ \begin{array}{l} \dfrac{EA \cdot EC \cdot EB}{EC \cdot EB \cdot EA} = 1 \\ \dfrac{\sin(13)\sin(10-5)\sin(9)}{\sin(9-2)\sin(14)\sin(10)} = 1\,. \end{array} \right. \tag{3}$$

Herein:

$$\begin{aligned} (3) &= (6+7-2) &&= 88^\circ\,16'\,28'' \\ (4) &= 180^\circ - (5+6+7) &&= 180^\circ - 91^\circ\,42'\,20'' \\ (13) &= 180^\circ - (9+11-2) &&= 180^\circ - 87^\circ\,50'\,29'' \\ (14) &= (5+9+11) &&= 91^\circ\,16'\,21''\,. \end{aligned}$$

This yields for the central point D the following trigonometric calculation with seven-place logarithms:

		Diff. for 10″			Diff. for 10″
$\sin(6+7-2)$	9.999 8030	+ 6	$\sin(1+2)$	9.897 0298	+ 164
$\sin(5+6)$	9.890 1817	+ 171	$\sin(5+6+7)$	9.999 8076	− 7
$\sin(1)$	9.886 3738	+ 174	$\sin(6)$	9.879 5215	+ 181
	9.776 3585			9.776 3589.	

The linear side equation belonging to this is for units of the seventh place of logarithms:

$$+1.0\,v_1 - 17.0\,v_2 + 17.8\,v_5 + 0.3\,v_6 + 1.3\,v_7 - 4 = 0. \tag{4}$$

The second central system E yields according to equation (3):

		Diff. for 10″			Diff. for 10″
$\sin(9+11-2)$	9.999 6917	+ 8	$\sin(9-2)$	9.868 0337	+ 192
$\sin(10-5)$	9.863 9507	+ 196	$\sin(5+9+11)$	9.999 8929	− 5
$\sin(9)$	9.879 7914	+ 181	$\sin(10)$	9.875 5076	+ 185.

Linear side equation for units of the seventh decimal:

$$+18.4\,v_2 - 19.1\,v_5 + 0.2\,v_9 + 1.1\,v_{10} + 1.3\,v_{11} - 4 = 0. \tag{5}$$

For the group of triangles in Fig. 1 we have thus found the two side equations (4) and (5), which we collect now once again:

$$\left. \begin{array}{l} +1.0\,v_1 - 17.0\,v_2 + 17.8\,v_5 + 0.3\,v_6 + 1.3\,v_7 \quad . \quad . \quad . \quad - 4 = 0 \\ \quad . \quad + 18.4\,v_2 - 19.1\,v_5 \quad . \quad . \quad + 0.2\,v_9 + 1.1\,v_{10} + 1.3\,v_{11} - 4 = 0 \end{array} \right\}. \tag{6}$$

The normal equations are:

$$\left.\begin{array}{r} \underline{+608.62\,k_1} - 652.78\,k_2 - 4 = 0 \\ \underline{+706.31\,k_2} - 4 = 0 \end{array}\right\} \tag{7}$$

$$k_1 = +1.461 \qquad k_2 = +1.356\,. \tag{8}$$

If we calculate further with these correlates (8), then we obtain the corrections to the angles:

$$\left.\begin{array}{llll} v_1 = +1.461'' & v_2 = +0.113'' & v_5 = +0.107'' & v_6 = +0.438'' \\ v_7 = +1.899 & v_9 = +0.271 & v_{10} = +1.492 & v_{11} = +1.763 \end{array}\right\}. \tag{9}$$

The corrected angles are then:

$$\left.\begin{array}{ll} [1] = 50^\circ\ 20'\ 08.461'' & [7] = 40^\circ\ 45'\ 29.899'' \\ [2] = \ \ 1\ \ 44\ \ 56.113 & [9] = 49\ \ 18\ \ 25.271 \\ [5] = \ \ 1\ \ 40\ \ 56.107 & [10] = 48\ \ 39\ \ 27.492 \\ [6] = 49\ \ 15\ \ 56.438 & [11] = 40\ \ 17\ \ 01.763 \end{array}\right\}. \tag{10}$$

For the examination of the adjustment, we now set up once again the two side equations (2) and (3) with the adjusted angles (10).

Central system D		Central system E	
sin [3] . .	9.999 8031	*sin* [13] . .	9.999 6919
sin [5 + 6] . .	9.890 1827	*sin* [10 — 5] . .	9.863 9534
sin [1] . .	9.886 3763	*sin* [9] . .	9.879 7919
	9.776 3621		9.743 4372
sin [1 + 2] . .	9.897 0324	*sin* [9 — 2] . .	9.868 0340
sin [4] . .	9.999 8074	*sin* [14] . .	9.999 8928
sin [6] . .	9.879 5223	*sin* [10] . .	9.875 5103
	9.776 3621		9.743 4371 .

With this, the adjustment has been brought to a satisfactory conclusion. However, there arises now the further question as to whether the above adjusted angles (10) also satisfy the side equations of Fig. 1 for other central points. According to the theorem of areas (26), section 77, p. 279, we obtain the most favorable side equations for the two quadrilaterals $ADBC$ and $ACBE$ if we use the central point C for both quadrilaterals. These two side equations are:

$$\frac{CA \cdot CD \cdot CB}{CD \cdot CB \cdot CA} = 1 \qquad \text{or} \qquad \frac{\sin(1+2)\sin(7)\sin(5)}{\sin(8)\sin(5+6)\sin(2)} = 1 \tag{11}$$

$$\frac{CA \cdot CE \cdot CB}{CE \cdot CB \cdot CA} = 1 \qquad \text{or} \qquad \frac{\sin(9-2)\sin(12)\sin(5)}{\sin(11)\sin(10-5)\sin(2)} = 1 \tag{12}$$

$$\begin{array}{l} (8) = 180^\circ - (1+6+7) \\ (12) = 180 \quad - (9+10+11)\,. \end{array}$$

If we substitute in (11) and (12) the adjusted angles (10), then we obtain the following computation:

sin [1 + 2]	9.897 0324	sin [9 — 2]	9.868 0340	
sin [7]	9.814 8264	sin [12]	9.823 4100	
sin [5]	8.467 7060	sin [5]	8.467 7060	
	8.179 5648		8.159 1500	
sin [1 + 6 + 7]	9.804 7977	sin [11]	9.810 6185	(13)
sin [5 + 6]	9.890 1828	sin [10 — 5]	9.863 9534	
sin [2]	8.484 5799	sin [2]	8.484 5799	
	8.179 5604		8.159 1518	
	$w = +44$		$w = -18$	

The large discrepancies +44 and –18 lead to the realization that the adjustment hitherto carried out with the two unfavorable side equations is not sufficient. Therefore, we repeat the adjustment with the two side equations (11) and (12).

Equation (11) leads with the measured angles (1) to the following trigonometric computation:

		Diff. for 10″			Diff. for 10″
sin (1 + 2)	9.897 0298	+ 164	sin (8)	9.804 8074	+ 254
sin (7)	9.814 8218	+ 245	sin (5 + 6)	9.890 1817	+ 171
sin (5)	8.467 6983	+ 7170	sin (2)	8.484 5721	+ 6895
	8.179 5499			8.179 5612 .	

There follows hence the side equation:

$$+41.8\,v_1 - 673.1\,v_2 + 699.9\,v_5 + 8.3\,v_6 + 49.9\,v_7 - 113 = 0. \qquad (14)$$

Correspondingly, we obtain from equation (12):

		Diff. for 10″			Diff. for 10″
sin (9 — 2)	9.868 0337	+ 192	sin (11)	9.810 6141	+ 249
sin (12)	9.823 4182	+ 235	sin (10 — 5)	9.863 9507	+ 196
sin (5)	8.467 6983	+ 7170	sin (2)	8.484 5721	+ 6845
	8.159 1502			8.159 1369	

and the side equation:

$$-708.7\,v_2 + 736.6\,v_5 - 4.3\,v_9 - 43.1\,v_{10} - 48.4\,v_{11} + 133 = 0. \qquad (14a)$$

For the new adjustment we have therefore the two condition equations:

$$\left.\begin{array}{l} +41.8\,v_1 - 673.1\,v_2 + 699.9\,v_5 + 8.3\,v_6 + 49.9\,v_7 \quad . \quad . \quad . \quad -113 = 0 \\ \qquad -708.7\,v_2 + 736.6\,v_5 \quad . \quad . \quad -4.3\,v_9 - 43.1\,v_{10} - 48.4\,v_{11} + 133 = 0 \end{array}\right\}. \qquad (15)$$

Strictly speaking, these equations (15) would have to agree with the previous condition equations (6). In order to be able to compare the large coefficients of equations (15) with those of equations (6), we divide the former by 40 and have then:

$$\left.\begin{array}{l} +1.04\,v_1 - 16.83\,v_2 + 17.50\,v_5 + 0.21\,v_6 + 1.25\,v_7 \quad . \quad . \quad . \quad . \quad -2.82 = 0 \\ \qquad -17.72\,v_2 + 18.42\,v_5 \quad . \quad . \quad -0.11\,v_9 - 1.08\,v_{10} - 1.21\,v_{11} + 3.32 = 0 \end{array}\right\}. \qquad (16)$$

We see hence that the agreement with the previous equations (6) in fact exists. But at the same time we also realize that the two equations (16) and, likewise, the original equations (15), with the same computational rigorousness, are considerably more accurate than equations (6). This must also confirm itself in the further course of the computation.

The normal equations belonging to (15) are:

$$\left.\begin{aligned} +\underline{947\,230\,k_1} + \;\;992\,572\,k_2 - 113 &= 0 \\ +\underline{1049\,055\,k_2} + 133 &= 0 \end{aligned}\right\} \tag{17}$$

$$k_1 = +0.029\,4844 \qquad k_2 = -0.028\,0236\,. \tag{18}$$

Corrections:

$$\left.\begin{matrix} v_1 = +1.2324'' & v_2 = +0.0144'' & v_5 = -0.0061'' & v_6 = +0.2447'' \\ v_7 = +1.4713 & v_9 = +0.1205 & v_{10} = +1.2078 & v_{11} = +1.3563 \end{matrix}\right\}. \tag{19}$$

Adjusted angles:

$$\left.\begin{matrix} [1] = 50^\circ\,20'\,08.2324'' & [7] = 40^\circ\,45'\,29.4713'' \\ [2] = \;\;1 \;\; 44 \;\; 56.0144 & [9] = 49 \;\; 18 \;\; 25.1205 \\ [5] = \;\;1 \;\; 40 \;\; 55.9939 & [10] = 48 \;\; 39 \;\; 27.2078 \\ [6] = 49 \;\; 15 \;\; 56.2447 & [11] = 40 \;\; 17 \;\; 01.3563 \end{matrix}\right\}. \tag{20}$$

Final check:

$$\left.\begin{matrix} \sin[1+2] \;.\,. & 9.897\,0319 & \sin[9-2] \;.\,. & 9.868\,0339 \\ \sin[7] \;.\,. & 9.814\,8254 & \sin[12] \;.\,. & 9.823\,4120 \\ \sin[5] \;.\,. & \underline{8.467\,6979} & \sin[5] \;.\,. & \underline{8.467\,6978} \\ & 8.179\,5552 & & 8.159\,1437 \\ \sin[8] \;.\,. & 9.804\,7998 & \sin[11] \;.\,. & 9.810\,6175 \\ \sin[5+6] \;.\,. & 9.890\,1822 & \sin[10-5] \;.\,. & 9.863\,9530 \\ \sin[2] \;.\,. & \underline{8.484\,5731} & \sin[2] \;.\,. & \underline{8.484\,5731} \\ & 8.179\,5551 & & 8.159\,1436 \\ & w = +1 & & w = +1 \end{matrix}\right\}. \tag{21}$$

Both final checks thus agree sufficiently.

The comparison of the corrections (9) and (19) shows quite considerable differences in the values of the v's. Therefore, it remains to be examined further if the corrections (19) and the corrected angles (20) of the second adjustment also satisfy the two side equations of the first adjustment. For this, we have to substitute the angle values (20) in the two equations (2) and (3) and obtain the following:

$$\left.\begin{matrix} \sin[6+7-2] \;.\,. & 9.999\,8031 & \sin[9+11-2] \;.\,. & 9.999\,6919 \\ \sin[5+6] \;.\,. & 9.890\,1827 & \sin[10-5] \;.\,. & 9.863\,9534 \\ \sin[1] \;.\,. & \underline{9.886\,3763} & \sin[9] \;.\,. & \underline{9.879\,7919} \\ & 9.776\,3621 & & 9.743\,4372 \\ \sin[1+2] \;.\,. & 9.897\,0324 & \sin[9-2] \;.\,. & 9.868\,0340 \\ \sin[5+6+7] \;.\,. & 9.999\,8074 & \sin[5+9+11] \;.\,. & 9.999\,8928 \\ \sin[6] \;.\,. & \underline{9.879\,5223} & \sin[10] \;.\,. & \underline{9.875\,5103} \\ & 9.776\,3621 & & 9.743\,4371 \end{matrix}\right\}. \tag{22}$$

With this, the comparison of the two forms of the side equations is decided absolutely in favor of the second adjustment, which also corresponds to the theorem of areas (26), section 77, p. 279.

However, with respect to the two condition equations (15) there remains a further remark. On account of the two small angles (2) and (5), the coefficients of v_2 and v_5 in the two equations are very large in proportion to the remaining coefficients, which has the further consequence that the proportion of the coefficients of k_1 and k_2 in the two normal equations (17) becomes nearly equal. In the present case, this has not disturbed the solution of the normal equations and the computation of the corrections; however, in the case of very acute angles (2) and (5) the coefficients of v_2 and v_5 in the condition equations would crush all remaining coefficients and make the adjustment impossible. In such cases, it is therefore appropriate to carry out a transformation of the side equations before setting up the normal equations. If we subtract, for instance, the first equation of (15) from the second, then we obtain:

$$-41.8\,v_1 - 35.6\,v_2 + 36.7\,v_5 - 8.3\,v_6 - 49.9\,v_7 - 4.3\,v_9 - 43.1\,v_{10} - 48.4\,v_{11} + 246 = 0. \qquad (23)$$

This equation also results directly as a side equation for the quadrilateral $ADBE$ with the central point C in Fig. 1, p. 281, for we have:

$$\frac{CA\cdot CD\cdot CB\cdot CE}{CD\cdot CB\cdot CE\cdot CA} = 1$$

or

$$\frac{\sin(8)\sin(5+6)\sin(12)\sin(9-2)}{\sin(1+2)\sin(7)\sin(10-5)\sin(11)} = 1,$$

which leads to equation (23), according to logarithmic-trigonometric calculation.

If we combine now the first equation of (15) with equation (23), then there follow the two normal equations:

$$\left.\begin{array}{r} +\underline{947{,}230\,k_1} + 45{,}342\,k_2 - 113 = 0 \\ +\underline{11{,}139\,k_2} + 246 = 0 \end{array}\right\}, \qquad (24)$$

in the case of which the inconvenient uniformity of the coefficients no longer exists. The solution of the normal equations (24), which we will no longer discuss, leads to the same corrections (19) which we have found in the case of the second adjustment.

Sequence of the condition equations

If the condition equations are set up individually, then so much is decided otherwise about the adjustment that a definite correlate k belongs to every equation, independently of the sequence in which the condition equations are treated further; however, for the question of the convenience of the elimination, this sequence is not in every case immaterial. We must always aim at having the quotients of the first reductions, namely $\frac{[ab]}{[aa]}$, $\frac{[ac]}{[aa]}$, etc., as *small* as possible, and in particular it should be avoided that such a quotient becomes larger than 1, because the uncertainties of rounding off would also thereby be carried forward in an increased manner. For the comparison of different forms of elimination of this kind there can be used the pentagon of Hannover with normal equations on p. 240 and the earlier hexagon of Linden in *Handbuch*, Volume II, 3rd Edition, 1893, p. 291.

In *Zeitschrift für Vermessungswesen*, 1875, p. 411, for a solution of 34 normal equations, Koppe has appended to his triangulation of St. Gothard two tables, Enclosure A and Enclosure B, the first of which has an inconvenient order and the second a more favorable one and, in fact, such a sequence that the complete terms are together as much as possible, crowded together in the neighborhood of the diagonals of the quadratic terms.

In the adjustment of the Saxon triangulation net by Nagel with a solution of 159 equations, the normal equations, pp. 579 to 604, are so arranged that in the neighborhood of the diagonals of the quadratic terms the terms are complete and at a distance from it, the terms are empty.

Section 79. Most Favorable Angle Equations in the Quadrilateral

The theory of the most favorable side equation in the quadrilateral, which has been treated in sections 77 and 78, with the area measures as decisive quantities for the suitability (Fig. 7, p. 279), also has an analogy for the choice of the *angle* equations in the quadrilateral, with which we will deal further.

We have already seen in section 66, p. 217, that the four triangle closures, which we can form into a quadrilateral with two diagonals, are not independent of each other, and that the quadrilateral sum of 360° can also be introduced as a condition, for instance, with two further triangle closures of 180°, which themselves can again be chosen differently. In the case of such arbitrariness in mathematical questions, it is now very often best not to take *any* of the different ways, which offer themselves for selection with equal justification, but to adopt a new method which lies symmetrical with respect to the previous methods.

In this sense we will try to find *three* quadrilateral equations, namely, first, the angle equation for the usual total quadrilateral with sum of angles 360°, and then, further, two equations for two so-called "cross" quadrilaterals with sum of angles 0°, i.e., for the quadrilateral of Fig. 3, section 66, p. 220, we will form:

$$\begin{array}{ll} \text{Full quadrilateral} & A\,B\,C\,D\,A \\ \text{Cross quadrilateral} & A\,C\,B\,D\,A \\ \text{Cross quadrilateral} & A\,B\,D\,C\,A\,. \end{array} \qquad \begin{array}{r}(1)\\(2)\\(3)\end{array}$$

Or if we express everything in terms of direction corrections to the 12 directions of Fig. 3, p. 220, then we have:

$$\text{I. } ABCDA \qquad -v_1+v_3-v_4+v_6-v_7+v_9-v_{10}+v_{12}+w' = 0 \qquad (4)$$

$$\text{II. } ACBDA \qquad -v_1+v_2+v_5-v_6+v_7-v_8-v_{11}+v_{12}+w'' = 0 \qquad (5)$$

$$\text{III. } ABDCA \qquad -v_2+v_3-v_4+v_5+v_8-v_9+v_{10}-v_{11}+w''' = 0\,. \qquad (6)$$

We have written these three equations from the geometrical point of view of Fig. 3, p. 220, and since they are independent, there is no doubt that they can be taken as a basis for an adjustment as the three angle equations, which we must have.

If we collect the three equations (4), (5), (6) according to the numbers $v_1, v_2, v_3, \ldots, v_{12}$, especially if we write, for this, a table as in (12), section 68, p. 230, for b, c, d, then we will see at once that the normal equations belonging to this become:

$$\left.\begin{array}{llll} 8\,k_1 & \ldots & \ldots & +w' = 0 \\ \ldots & 8\,k_2 & \ldots & +w'' = 0 \\ \ldots & \ldots & 8\,k_3 & +w''' = 0 \end{array}\right\}. \qquad (7)$$

The nonquadratic terms in these normal equations become equal to zero, and the correlates become:

$$k_1 = -\frac{w'}{8} \qquad k_2 = -\frac{w''}{8} \qquad k_3 = -\frac{w'''}{8}\,, \qquad (8)$$

we can also specify at once the 12 corrections to the directions v by forming a table of the form (15), p. 231:

$8\,v_1 = -w' - w'''$	$8\,v_4 = -w' - w''$	$8\,v_7 = -w' + w'''$	$8\,v_{10} = -w' + w''$
$8\,v_2 = -w'' + w'''$	$8\,v_5 = +w'' + w'''$	$8\,v_8 = +w'' - w'''$	$8\,v_{11} = -w'' - w'''$
$8\,v_3 = +w' + w''$	$8\,v_6 = +w' - w'''$	$8\,v_9 = +w' - w''$	$8\,v_{12} = +w' + w'''$
Sum = 0	Sum = 0	Sum = 0	Sum = 0

(9)

Equations (7), (8), (9) show already by their simplicity and symmetry the advantages of the new method, since, e.g., in the case of the normal equations (7) the operation of elimination is omitted, and in view of the rigorousness of the computation, the closure terms of the quadrilateral w', w'', w''' have the advantage that in general they are larger than the closure terms of the triangle, and with this, they yield a more rigorous computation.

For the total adjustment there is added further the side equation, which, consequently, we shall also form as rigorously as possible, i.e. according to the form M in (27), section 77, p. 279; in the normal equations (7) there are then added further four terms for k_4, but the terms indicated in (7) by . . remain equal to zero.

As a numerical example we will take the angle values of section 66, p. 217, but bring them in the form of directions according to Fig. 3, p. 220.

For the triangle of Fig. 3, p. 220, we will assume the following 12 directions as measured:

$$\left.\begin{array}{llll} D & \begin{array}{l} 10. = 0^\circ\ \ 0'\ \ 0'' \\ 11. = 46\ \ \ 1\ 27 \\ 12. = 112\ 41\ 51 \end{array} & (66^\circ\,40'\,24'') & C \quad \begin{array}{l} 7. = 0^\circ\ \ 0'\ \ 0'' \\ 8. = 48\ \ \ 9\ \ \ 2 \\ 9. = 78\ \ \ 0\ 28 \end{array} \ (29^\circ\,51'\,26'') \\ A & \begin{array}{l} 1. = 0^\circ\ \ 0'\ \ 0'' \\ 2. = 37\ 26\ 41 \\ 3. = 72\ \ \ 2\ \ \ 6 \end{array} & (34^\circ\,35'\,25'') & B \quad \begin{array}{l} 4. = 0^\circ\ \ 0'\ \ 0'' \\ 5. = 41\ 17\ 34 \\ 6. = 97\ 15\ 36 \end{array} \ (55^\circ\,58'\ \ 2'') \end{array}\right\} . \qquad (10)$$

If we collect therefrom the four triangles, then we obtain:

$$\left.\begin{array}{ll} \begin{array}{l} A = (1,3) = 72^\circ\ \ 2'\ \ 6'' \\ B = (4,5) = 41\ \ 17\ \ 34 \\ D = (11,12) = 66\ \ 40\ \ 24 \\ \hline \ 180^\circ\ \ 0'\,04'' \\ \ w_1 = +\,4'' \end{array} & \begin{array}{l} A = (2,3) = 34^\circ\,35'\,25'' \\ B = (4,6) = 97\ \ 15\ \ 36 \\ C = (7,8) = 48\ \ \ 9\ \ \ 2 \\ \hline \ 180^\circ\ \ 0'\,03'' \\ \ w_3 = +\,3'' \end{array} \\ \begin{array}{l} B = (5,6) = 55^\circ\,58'\ \ 2'' \\ C = (7,9) = 78\ \ \ 0\ \ 28 \\ D = (10,11) = 46\ \ \ 1\ \ 27 \\ \hline \ 179^\circ\,59'\,57'' \\ \ w_2 = -\,3'' \end{array} & \begin{array}{l} A = (1,2) = 37^\circ\,26'\,41'' \\ D = (10,12) = 112\ \ 41\ \ 51 \\ C = (8,9) = 29\ \ 51\ \ 26 \\ \hline \ 179^\circ\,59'\,58'' \\ \ w_4 = -\,2'' \end{array} \end{array}\right\} . \qquad (11)$$

These are the same four triangle closures for directions which are given on p. 217 for angles.

The following four condition equations for directions belong to the above four triangle closures:

$$\left.\begin{array}{l|l} w_1 & -v_1 \cdot\cdot + v_3 - v_4 + v_5 \cdot\cdot \qquad\qquad - v_{11} + v_{12} + 4'' = 0 \\ w_2 & \cdot\cdot \qquad - v_5 + v_6 - v_7 \cdot\cdot + v_9 - v_{10} + v_{11} \cdot\cdot - 3'' = 0 \\ w_3 & \cdot\cdot - v_2 + v_3 - v_4 \cdot\cdot + v_6 - v_7 + v_8 \cdot\cdot \qquad + 3'' = 0 \\ w_4 & -v_1 + v_2 \cdot\cdot \qquad\qquad - v_8 + v_9 - v_{10} \cdot\cdot + v_{12} - 2'' = 0 \end{array}\right\} . \qquad (12)$$

We can form therefrom:

$$+w_1 + w_2 \quad \text{or} \quad +w_3 + w_4 \ \Big|\ -v_1 + v_3 - v_4 + v_6 - v_7 + v_9 - v_{10} + v_{12} + 1'' = 0 \qquad (13)$$

$$+w_1 - w_3 \quad \text{or} \quad -w_2 + w_4 \ \Big|\ -v_1 + v_2 + v_5 - v_6 + v_7 - v_8 - v_{11} + v_{12} + 1 \ = 0 \qquad (14)$$

$$+w_1 - w_4 \quad \text{or} \quad -w_2 + w_3 \ \Big|\ -v_2 + v_3 - v_4 + v_5 + v_8 - v_9 + v_{10} - v_{11} + 6 \ = 0. \qquad (15)$$

These three equations agree again with the original equations (4), (5), (6), with which it is shown that those equations (4), (5), (6) written briefly from the geometrical viewpoint also result as simple algebraic consequences of the four usual triangle closures (12).

Now we will also form the most favorable side equation, in the form M in (27), p. 279, quite similarly as we have already formed such an eight-term side equation on p. 280. This equation for the angles given in group (10), belonging to the quadrilateral on p. 220, yields according to logarithmic calculation in units of the sixth place of logarithms:

$$\left.\begin{array}{l} +\,3.06\,(v_3 - v_2) + 1.42\,(v_6 - v_5) + 3.67\,(v_9 - v_8) + 0.90\,(v_{12} - v_{11}) \\ +\,2.75\,(v_1 - v_2) + 2.40\,(v_4 - v_5) + 1.89\,(v_7 - v_8) + 2.03\,(v_{10} - v_{11}) - 7.2 = 0 \end{array}\right\}. \tag{16}$$

If we arrange this equation according to $v_1, v_2 \ldots,$ then we obtain the fourth line in the following table, while the first three lines correspond to equations (13), (14), (15).

Table of Condition Equations (13), (14), (15), (16)

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	w
−1	+1	. .	. .	+1	−1	+1	−1	. .	. .	−1	+1	+1″
. .	−1	+1	−1	+1	. .	. .	+1	−1	+1	−1	. .	+6″
−1	. .	+1	−1	. .	+1	−1	. .	+1	−1	. .	+1	+1″
+2.75	−5.81	+3.06	+2.40	−3.82	+1.42	+1.89	−5.56	+3.67	+2.03	−2.93	+0.90	−7.2″

(17)

The normal equations belonging to this become:

$$\left.\begin{array}{l} +\,\underline{8\,k_1}\ \ldots\ \ldots\ -0.0025\,(8\,k_4) + 1.00 = 0 \\ \quad +\,\underline{8\,k_2}\ \ldots\ -0.3150\,(8\,k_4) + 1.00 = 0 \\ \qquad +\,\underline{8\,k_3} - 0.2025\,(8\,k_4) + 6.00 = 0 \\ \qquad\quad +\,\underline{16.8152}\,(8\,k_4) - 7.20 = 0 \end{array}\right\}, \tag{18}$$

$$\begin{array}{llll} 8\,k_1 = -0.99916 & 8\,k_2 = -0.89294 & 8\,k_3 = -5.93117 & 8\,k_4 = +0.33988 \\ k_1 = -0.125 & k_2 = -0.112 & k_3 = -0.741 & k_4 = +0.04248\,. \end{array}$$

We will not carry this numerical example further; the further computation is the usual one. It yields, with a table in the arrangement as, e.g. (15), p. 231, and II, p. 241, the following 12 corrections to the directions v [in the same order as in the case of (17) above]:

$$+0.35'' + 0.38'' - 0.74''\ |\ +0.97'' - 1.01'' + 0.05''\ |\ +0.09'' - 0.87'' + 0.77''\ |\ -0.53'' + 0.73'' - 0.20''.$$

If we add these v's to the 12 directions, which were given above under (10), and form the sum checks and so on, then we will find everything in agreement.

The principal thing only was for us here to show, with table (17) and the normal equations (18), that the condition equations (17) become more complete than in the usual treatment (12), p. 230, but on the other hand the normal equations (18) become more convenient than (13), p. 230, since several terms are omitted in the new treatment; and the new treatment has the advantage of absolutely greater rigorousness.

It is true that we cannot express, by so small an example of *one* quadrilateral, the advantage of the computational rigorousness; but if such a quadrilateral is the component part of a rather large net adjustment with 30 to 40 condition equations, then it is very important to introduce the conditions as rigorously as possible from the outset in order to counteract the accumulation of errors of the long elimination. The table (17) has already the formal advantage that the grouping of its terms yields easily recognizable checks (e.g. 2.75 + 3.06 = 5.81, etc.), which is not the case in the usual treatment on p. 230.

The net adjustments hitherto treated referred to completely independent nets of triangles, i.e., such which either are in no connection with another net of triangles, or only use as base line a side of an already completed, existing net, as is the case with the net of Hannover, Fig. 1, section 70, p. 236. For the number of the condition equations, the rules summarized in section 67 are then decisive.

As an example of a net of triangles, attached several times, we have represented, in the central part of Fig. 1, the triangulation net of the Palatinate of the Prussian Land Survey, which was measured during the years 1896-97, and which is published in the work, *Die Königlich Preussische Landestriangulation*, Hauptdreiecke, 11 Teil, Berlin, 1901. The net rests on the line path Loeberg, Erbeskopf, Donnersberg, Melibocus, on the Rhinish-Hessian triangulation chain already published in "Hauptdreiecke, 9. Teil, 1897." The adjustment of the new net had therefore to take place in such a manner that, in the latter, the lengths of the connecting sides as well as the angles between them agree with those of the given triangulation chain.

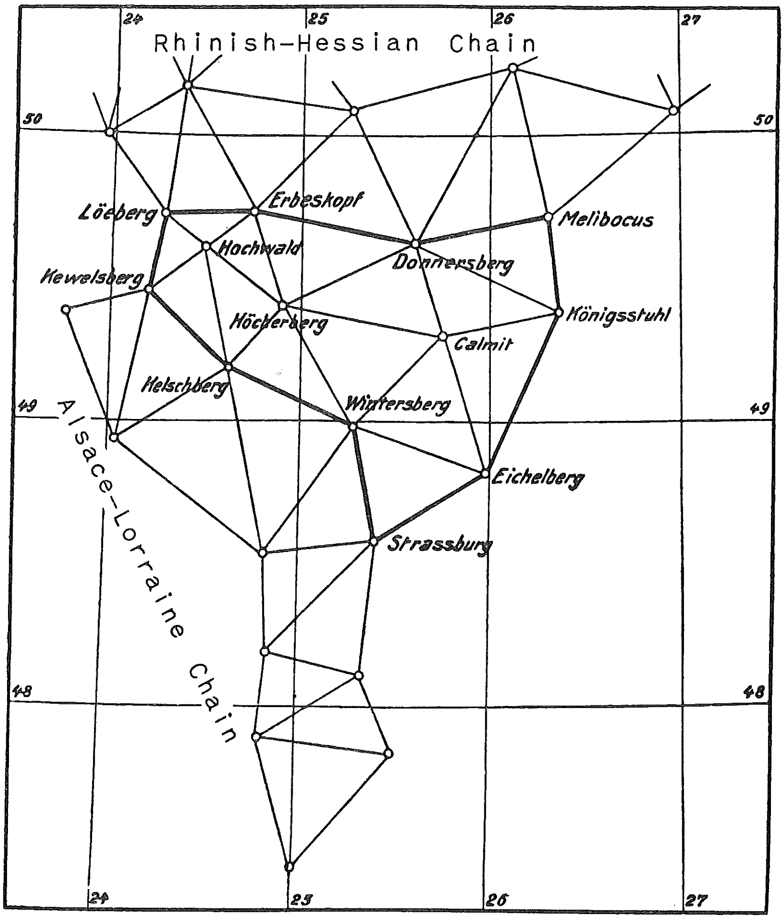


Fig. 1. Triangulation net of the Palatinate.

Further requirements, however, resulted for the adjustment of the net of the Palatinate. For the triangulation chain of Alsace-Lorraine, likewise represented in Fig. 1, had already been measured in 1876, and since at that time the connection of this chain with the older nets of the Land Survey was not yet possible, it was adjusted completely independently and oriented on the basis of astronomical determinations in Strassburg. The final orientation of the chain could take place only by the adjustment of the net of the Palatinate, where nothing was to be changed any more in the lengths and angles of the chain of Alsace-Lorraine.

The net of the Palatinate taken alone by itself contains, since the angle measurement exists in the form of complete sets of directions, a total of 17 condition equations, namely 14 triangle equations and 3 side equations corresponding to the 3 central systems with the points Hochwald, Höcherberg, Calmit as central points. We will explain the setting up of the further condition equations, which are required for the forcing of attachment, on the basis of Fig. 2, which represents the northwestern part of the net by itself with the notation of the directions according to the publication of the Land Survey mentioned on p. 290.

First of all, we have the angle between the measured directions (13) and (16), which must become equal to the difference of the two given direction angles (Loeberg, Muxerath) and (Loeberg, Erbeskopf).

There thus exists the condition equation

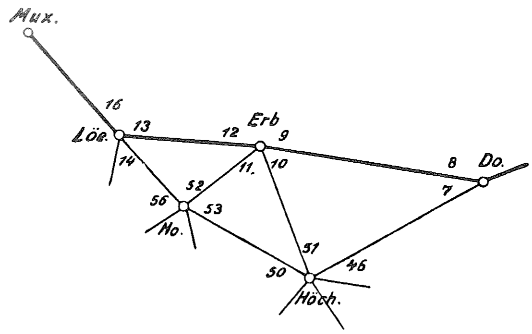


Fig. 2.

$$(16) + v_{16} - (13) - v_{13} = (\text{Loeberg} - \text{Muxerath}) - (\text{Loeberg} - \text{Erbeskopf}). \tag{1}$$

Correspondingly, we obtain:

$$(12) + v_{12} - (9) - v_9 = (\text{Erbeskopf, Loeberg}) - (\text{Erbeskopf} - \text{Donnersberg}), \text{ etc.} \tag{2}$$

As Fig. 1, p. 290, shows, of such equations there exist four for the attachment to the Rhinish-Hessian chain and, likewise, four for the attachment for the chain of Alsace-Lorraine.

For the connecting sides the requirement is to be set up that their ratio in the net of the Palatinate is the same as in the connecting chains. Since the net has three triangle sides in common with each of the two connecting chains, then on each side there will have to be set up two condition equations each. For instance, the following expression is thus found for the sine condition of the two sides Loeberg-Erbeskopf and Erbeskopf-Donnersberg in the net of the Palatinate with the help of the three spherical triangles Loeberg-Erbeskopf-Hochwald, Erbeskopf-Höcherberg-Hochwald and Erbeskopf-Donnersberg-Höcherberg:

$$\frac{\sin\left((14)-(13)+v_{14}-v_{13}\right)\sin\left((53)-(52)+v_{53}-v_{52}\right)\sin\left((46)-(51)+v_{46}-v_{51}\right)}{\sin\left((52)-(56)+v_{52}-v_{56}\right)\sin\left((51)-(50)+v_{51}-v_{50}\right)\sin\left((8)-(7)+v_8-v_7\right)}. \tag{3}$$

On the other hand, we can take the lengths of the two sides Loeberg-Erbeskopf and Erbeskopf-Donnersberg from the Rhinish-Hessian chain and compute therefrom the sine condition

$$\frac{\sin\dfrac{L-E}{r}}{\sin\dfrac{E-B}{r}}, \tag{4}$$

where r denotes the earth's radius. By setting the two expressions (3) and (4) equal to each other we obtain the required condition equation.

If we have set up in this manner the four condition equations for the forcing of attachment, then we are to express further that the ratio of the northern to the southern connecting sides in the net of the Palatinate, also, becomes the same as in the two given chains. In connection with the preceding four equations it is sufficient for this to express the sine condition of one side each of the northern and of the southern attachment by the sines of the angles of the triangles lying between, hence, e.g., to connect the two sides Donnersberg-Melibocus and Wintersberg-Strassburg with each other.

The conversion of expression (3) to the linear form with respect to the corrections v is carried out most appropriately with the help of the logarithmic differences, as in the previous examples in sections 66, 68, and 70.

Besides, expression (4) holds, first, for the spherical-trigonometric computation. But we can at once turn to plane computation if we use the logarithmic additaments of section 40, p. 248, from Vol. III, 7th Edition, 1923.

Therefore, 8 angle equations and 5 side equations, in all, are added further to the actual net condition equations as connecting conditions.

Now we consider further the case in which upon attachment on both sides of a new net the two connecting nets are already fixed so far as their position is concerned, so that not only the connecting angles and sides, but also the coordinates of all connecting points must not be changed by the adjustment.

Of the triangulation net of Baden already used several times, Fig. 2, p. 225, we have cut out, in Fig. 3, the eastern chain and will now assume that the four points Klobberg, Melibocus, Langenkandel, and St. Michael belong to older nets and are given by their coordinates.

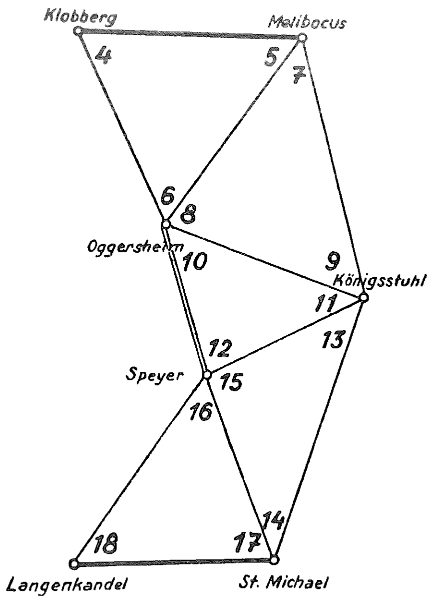


Fig. 3. Triangulation chain of Baden.

At the same time we will make the further assumption that the length of the side Speyer-Oggersheim is determined by a base measurement and hereby is to be considered, likewise, as unchanged.

The triangle net in itself would require only 5 equations of sums of angles for the 5 triangles, since it is a question of a simple chain without connection of diagonals. First of all, two side equations are required by the attachment in order to connect, as in Fig. 1, the three given lengths with each other. But even if these 7 condition equations are satisfied by the adjustment, there still exist further discrepancies. For if we compute the coordinates of the points Klobberg and Melibocus, starting from the points Langenkandel and St. Michael by means of the adjusted angles, then the computed coordinates will not agree with the given coordinates. This discrepancy is eliminated if the differences of coordinates of two points, e.g., the points St. Michael and Melibocus, and the difference of the two direction angles (Langenkandel, St. Michael) and (Klobberg, Melibocus) computed from the adjusted angles agree with the values computed from the given coordinates.

Hence, in addition to the 7 condition equations already mentioned, three further equations are to be set up.

In the net of Baden, Fig. 3, there are measured the individual angles of triangles, whose measured values are given in the earlier 3rd Edition of this volume, p. 194. Furthermore, we can take from Volume III, 7th Edition, 1923, p. 298, the Soldner spherical as well as the Gauss plane coordinates of the four points considered as given.

Although it is immaterial from the theoretical viewpoint in which form the coordinates are used, the computation on the plane is considerably more convenient for the setting up of the two coordinate conditions; therefore, we have first of all to convert everything to the plane.

The plane conformal coordinates of the four given points are according to Volume III, 7th Edition, p. 298:

Point	y	x	
Klobberg	— 18 104.652 m	+ 28 049.296 m	(5)
Melibocus	+ 12 727.478	+ 26 509.100	
Langenkandel . . .	— 19 467.751	— 44 893.918	
St. Michael . . .	+ 7 407.500	— 44 332.386	

According to this, there are computed the following logarithms of the distances to which we attach further the logarithm of the given base Speyer-Oggersheim, likewise transformed to the plane.

Klobberg—Melibocus . . . $\log s = 4.489\,54473$

Langenkandel—St. Michael . $\log s = 4.429\,44731$ (6)

Speyer—Oggersheim . . . $\log s = 4.296\,54773$.

Likewise, there are computed the two connecting direction angles from the coordinates (5), namely:

$$\begin{aligned}(\text{La., St. M.}) &= 88^\circ\, 48'\, 10.926'' \\ (\text{Mel., Klobb.}) &= 272\;\; 51\;\; 35.246\;.\end{aligned} \tag{7}$$

The transformation of the measured angles to the plane could likewise be taken from Volume III, 7th Edition, p. 300. However, since the conversion quantities are given there only to 2 decimals, and for the calculation of the example a greater accuracy appeared desirable for formal reasons, then the reductions of the directions were computed once more according to Volume III, 7th Edition, section 48, p. 295. The following values result hereby:

Angle	Spherical	Red.	Plane
(4)	63° 30′ 31.054″	— 0.955″	63° 30′ 30.099″
(5)	57 13 02.870	— 0.439	57 13 02.431
(6)	59 16 25.363	— 0.719	59 16 24.644
	179 59 59.287	— 2.113	179 59 57.174
(7)	46 24 50.850	— 0.928	46 24 49.922
(8)	74 59 26.909	+ 0.077	74 59 26.986
(9)	58 35 42.565	— 1.294	58 35 41.271
	180 00 00.324	— 2.145	179 59 58.179
(10)	55 21 08.132	— 0.275	55 21 07.857
(11)	45 27 47.758	— 0.575	45 27 47.183
(12)	79 11 01.162	— 0.276	79 11 00.886
	179 59 57.052	— 1.126	179 59 55.926
(13)	46 07 40.660	— 1.070	46 07 39.590
(14)	37 41 59.793	— 0.725	37 41 59.068
(15)	96 10 20.212	+ 0.247	96 10 20.459
	180 00 00.665	— 1.548	179 59 59.117
(16)	53 39 29.318	— 0.590	53 39 28.728
(17)	72 32 22.104	— 0.291	72 32 21.813
(18)	53 48 11.347	— 0.869	53 48 10.478
	180 00 02.769	— 1.750	180 00 01.019

(8)

In the present case, also, there is advisable Krüger's method of the groupwise adjustment of section 50, p. 159, of which we have already made use in the example of section 74, p. 263. According to the practical rule specified on p. 292 for this method, the closure errors of the triangles are first to be distributed equally among the three angles. This yields the following angles, preliminarily corrected:

(4) = 63° 30′ 31.041″	(7) = 46° 24′ 50.529″	(10) = 55° 21′ 09.215″
(5) = 57 13 03.373	(8) = 74 59 27.593	(11) = 45 27 48.541
(6) = 59 16 25.586	(9) = 58 35 41.878	(12) = 79 11 02.244
180° 00′ 00.000″	180° 00′ 00.000″	180° 00′ 00.000″

(13) = 46° 07′ 39.884″	(16) = 53° 39′ 28.389″
(14) = 37 41 59.362	(17) = 72 32 21.473
(15) = 96 10 20.754	(18) = 53 48 10.138
180° 00′ 00.000″	180° 00′ 00.000″.

(9)

The five condition equations becoming necessary by the attachment are now to be set up with these angles. We begin with the two side equations, which connect the base Speyer-Oggersheim with the two connecting sides Klobberg-Melibocus and Langenkandel-St. Michael. These two equations read according to Fig. 3:

$$\left.\begin{aligned} \frac{(\text{Sp.—Ogg.})\,\sin(6)\,\sin(9)\,\sin(12)}{(\text{Klobb.—Mel.})\,\sin(4)\,\sin(7)\,\sin(11)} &= 1 \\ \frac{(\text{Lang.—St. Mich.})\,\sin(11)\,\sin(14)\,\sin(18)}{(\text{Sp.—Ogg.})\,\sin(10)\,\sin(13)\,\sin(16)} &= 1 \end{aligned}\right\}. \tag{10}$$

The above angles yield, for this, the following logarithmic-trigonometric computation by means of the eight-place table of logarithms by Bauschinger-Peters:

		Diff. for 1″
log (Sp.-Ogg.)	4.296 54773	
log sin (6)	9.934 30575	+ 126
log sin (9)	9.931 20608	+ 129
log sin (12)	9.992 21529	+ 41
Sum Z	4.154 27485	

		Diff. for 1″
log (Lang.-St. Mich.)	4.429 44731	
log sin (11)	9.852 96988	+ 207
log sin (14)	9.786 41393	+ 273
log sin (18)	9.906 86780	+ 154
Sum Z	3.975 69892	

		Diff. for 1″
log (Klobb.-Mel.)	4.489 54473	
log sin (4)	9.951 82376	+ 105
log sin (7)	9.859 94293	+ 201
log sin (11)	9.852 96988	+ 207
Sum N	4.154 28130	

Sum Z – Sum N = — 0.000 00645
Should be 0.000 00000
Discrepancy $w = -645$

		Diff. for 1″
log (Sp.-Ogg.)	4.296 54773	
log sin (10)	9.915 22350	+ 146
log sin (13)	9.857 86708	+ 202
log sin (16)	9.906 06172	+ 155
Sum N	3.975 70003	

Sum Z – Sum N = — 0.000 00111
Should be 0.000 00000
Discrepancy $w = -111$.

Here we have indicated the discrepancies, as well as the logarithmic differences for 1″, in units of the eighth decimal, with which the two condition equations become then:

$$\left.\begin{aligned} -105\,v_4 + 126\,v_6 - 201\,v_7 + 129\,v_9 - 207\,v_{11} + 41\,v_{12} - 645 &= 0 \\ -146\,v_{10} + 207\,v_{11} - 202\,v_{13} + 273\,v_{14} - 155\,v_{16} + 154\,v_{18} - 111 &= 0 \end{aligned}\right\}. \tag{11}$$

The condition between the two direction angles given as fixed (St. Michael, Langenkandel) and (Melibocus, Klobberg) can be represented in the following form:

$$(\text{St. Mich., Lang.}) + (17) + (14) + (13) + (11) + (9) + (7) + (5) - (\text{Mel., Klobb.}) \pm 2 \times 180^\circ = 0. \tag{12}$$

If we substitute herein the angle values from (7) and (9), and add the corresponding correction to each of the angles, then we obtain:

$$v_5 + v_7 + v_9 + v_{11} + v_{13} + v_{14} + v_{17} + 0.720'' = 0. \tag{13}$$

Now there remain further the two coordinate attachment equations, by which we will connect the points St. Michael and Melibocus with each other. The two equations read according to Fig. 3, p. 292:

$$\left.\begin{aligned} y_{St.\,M.} &+ (\text{St. Mich.—Kön.})\,\sin\,(\text{St. Mich., Kön.}) \\ &+ (\text{Kön.—Mel.})\,\sin\,(\text{Kön., Mel.}) - y_{Mel.} = 0 \\ x_{St.\,M.} &+ (\text{St. Mich.—Kön.})\,\cos\,(\text{St. Mich., Kön.}) \\ &+ (\text{Kön.—Mel.})\,\cos\,(\text{Kön., Mel.}) - x_{Mel.} = 0 \end{aligned}\right\}. \tag{14}$$

In the case of these equations, the advantage of the computation in the plane makes itself felt especially, since the three equations (13) and (14) assume exactly the same form as the equations for the computation of a polygon line connected on both sides.

We now have to express first the direction angles and distances occurring in (14) by the angle values (9), where the correction v is to be added to each angle. For the two direction angles we have the following computation:

$$\left.\begin{array}{lcl}
\text{(St. Mich., Lang.)} & = & 268^\circ\ 48'\ 10.926'' \\
(17) & = & 72\ \ 32\ \ 21.473\ \ + v_{17} \\
(14) & = & 37\ \ 41\ \ 59.362\ \ + v_{14} \\
\hline
\text{(St. Mich., Kön.)} & = & 19^\circ\ 02'\ 31.761'' + v_{14} + v_{17} \\
(13) & = & 46^\circ\ 07'\ 39.884'' + v_{13} \\
(11) & = & 45\ \ 27\ \ 48.541\ \ + v_{11} \\
(9) & = & 58\ \ 35\ \ 41.878\ \ + v_9 \\
\hline
\text{(Kön., Mel.)} & = & 349^\circ\ 13'\ 42.064'' + v_9 + v_{11} + v_{13} + v_{14} + v_{17}
\end{array}\right\} . \qquad (15)$$

On the other hand, the following expressions result for the two triangle sides St. Michael-Königstuhl and Königstuhl-Melibocus according to Fig. 3, p. 292:

$$\left.\begin{array}{l}
\text{St. Mich.—Kön.} = \text{(St. Mich.—Lang.)}\,\dfrac{\sin(15)\,\sin(18)}{\sin(13)\,\sin(16)} \\
\text{Kön.—Mel.} \quad = \text{(Klobb.—Mel.)}\,\dfrac{\sin(4)\,\sin(8)}{\sin(6)\,\sin(9)}
\end{array}\right\} , \qquad (16)$$

and with the numerical values (6) and (9):

	Diff. for 1″		Diff. for 1″
St. Mich.—Lang.	4.429 44731	Klobb.—Mel.	4.489 54473
sin (15)	9.997 47499 — 23	*sin* (4)	9.951 82376 + 105
sin (18)	9.906 86780 + 154	*sin* (8)	9.984 92549 + 57
— *sin* (13)	0.142 13292 — 202	— *sin* (6)	0.065 69425 — 126
— *sin* (16)	0.093 93828 — 155	— *sin* (9)	0.068 79392 — 129
St. Mich.—Kön.	4.569 86130	Kön.—Mel.	4.560 78215

$$\left.\begin{array}{l}
\log\,\text{(St. Mich.—Kön.} = 4.569\,86130 - 202\,v_{13} - 23\,v_{15} - 155\,v_{16} + 154\,v_{18} \\
\log\,\text{(Kön.—Mel.)} = 4.560\,78215 + 105\,v_4 - 126\,v_6 - 57\,v_8 - 129\,v_9
\end{array}\right\} . \qquad (17)$$

To this, we take further from (15):

$$\left.\begin{array}{l}
\log\sin\text{(St.Mich.,Kön.)} = 9.513\,56881 + 610v_{14} + 610v_{17} \\
\log\cos\text{(St.Mich.,Kön.)} = 9.975\,55991 - 72v_{14} - 72v_{17} \\
\log\sin\text{(Kön., Mel.)} = 9.271\,59823_n - 1107v_9 - 1107v_{11} - 1107v_{13} - 1107v_{14} - 1107v_{17} \\
\log\cos\text{(Kön., Mel.)} = 9.992\,27945 + 40v_9 + 40v_{11} + 40v_{13} + 40v_{14} + 40v_{17}
\end{array}\right\} . \qquad (18)$$

With this, we have collected all numerical values for the computation of the differences of coordinates. For the differences of ordinates we obtain:

$$\left.\begin{array}{l}
\log(y_{Kö.} - y_{St.M.}) = 4.083\,43011 - 202v_{13} + 610v_{14} - 23v_{15} - 155v_{16} + 610v_{17} + 154v_{18} \\
\log(y_{M.} - y_{Kö.}) = 3.832\,38038_n + 105v_4 - 126v_6 + 57v_8 - 1236v_9 - 1107v_{11} - 1107v_{13} \\
\qquad\qquad - 1107v_{14} - 1107v_{17}
\end{array}\right\} . \qquad (19)$$

Condition Equations

v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	w
. .	+ 1.000	. .	+ 1.000	. .	+ 1.000	. .	+ 1.000	. .	+ 1.000	+ 1.000	. .	. .	+ 1.000	. .	+ 0.720
− 0.165	. .	+ 0.197	. .	− 0.089	+ 1.934	. .	+ 1.732	. .	+ 1.169	+ 3.431	− 0.064	− 0.432	+ 3.431	+ 0.429	+ 0.110
+ 0.864	. .	− 1.037	. .	+ 0.469	− 0.733	. .	+ 0.329	. .	− 1.304	− 0.253	− 0.186	− 1.253	− 0.253	+ 1.246	+ 0.870
− 1.050	. .	+ 1.260	− 2.010	. .	+ 1.290	. .	− 2.070	+ 0.410	. .	. .	. .	. .	. .	. .	− 6.450
. .	. .	. .	. .	. .	. .	− 1.460	+ 2.070	. .	− 2.020	+ 2.730	. .	− 1.550	. .	+ 1.540	− 1.110

Transformed Condition Equations

v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	w
− 0.333	+ 0.666	− 0.333	+ 0.333	− 0.666	+ 0.333	− 0.333	+ 0.666	− 0.333	+ 0.333	+ 0.333	− 0.666	− 0.333	+ 0.666	− 0.333	+ 0.720
− 0.176	− 0.011	+ 0.187	− 0.615	− 0.704	+ 1.319	− 0.577	+ 1.154	− 0.577	− 0.343	+ 1.919	− 1.576	− 1.575	+ 2.288	− 0.713	+ 0.110
+ 0.922	+ 0.058	− 0.980	+ 0.088	+ 0.557	− 0.645	− 0.110	+ 0.220	− 0.110	− 0.723	+ 0.328	+ 0.395	− 1.166	− 0.166	+ 1.332	+ 0.870
− 1.120	− 0.070	+ 1.190	− 1.770	+ 0.240	+ 1.530	+ 0.553	− 1.517	+ 0.964	. .	. .	. .	. .	. .	. .	− 6.450
. .	. .	. .	. .	. .	. .	− 1.663	+ 1.867	− 0.204	− 2.257	+ 2.493	− 0.236	− 1.547	+ 0.003	+ 1.544	− 1.110

For the transition to the antilogarithm we have to use again the logarithmic differences. We find, e.g.:

$$\text{Antilogarithm of } 4.083\,43011 = 12{,}117.977 \text{ m} + \frac{1}{3590} \text{ m},$$

where $\frac{1}{3590}$ m denotes the change of the antilogarithm for 1 unit of the eighth decimal place of the logarithm. Therefore, $-202v_{13}$ in the first equation of (19) yields a change of the antilogarithm of $-0.0563v_{13}$ in meters, where v_{13} is to be computed in seconds. In this manner we obtain:

$$\left.\begin{aligned} y_{Kö.} - y_{St.M.} &= +12117.977 - 0.0563\,v_{13} + 0.1699\,v_{14} - 0.0064\,v_{15} - 0.0432\,v_{16} \\ &\qquad + 0.1699\,v_{17} + 0.0429\,v_{18} \\ y_{M.} - y_{Kö.} &= -\;6\,797.988 - 0.0165\,v_4 + 0.0197\,v_6 - 0.0089\,v_8 + 0.1934\,v_9 \\ &\qquad + 0.1732\,v_{11} + 0.1732\,v_{13} + 0.1732\,v_{14} + 0.1732\,v_{17} \end{aligned}\right\}. \tag{20}$$

In the same manner we treat the differences of abscissae by collecting the logarithms from (17) and (18) and then pass over to the numerical values. We obtain:

$$\begin{aligned} \log(x_{Kö.} - x_{St.M.}) &= 4.545\,42121 - 202\,v_{13} - 72\,v_{14} - 23\,v_{15} - 155\,v_{16} \\ &\qquad - 72\,v_{17} + 154\,v_{18} \\ \log(x_{M.} - x_{Kö.}) &= 4.553\,06160 + 105\,v_4 - 126\,v_6 + 57\,v_8 - 89\,v_9 \\ &\qquad + 40\,v_{11} + 40\,v_{13} + 40\,v_{14} + 40\,v_{17} \end{aligned}$$

$$\left.\begin{aligned} x_{Kö.} - x_{St.M.} &= +35\,109.222 - 0.1633\,v_{13} - 0.0582\,v_{14} - 0.0186\,v_{15} - 0.1253\,v_{16} \\ &\qquad - 0.0582\,v_{17} + 0.1246\,v_{18} \\ x_{M.} - x_{Kö.} &= +35\,732.351 + 0.0864\,v_4 - 0.1037\,v_6 + 0.0469\,v_8 - 0.0733\,v_9 \\ &\qquad + 0.0329\,v_{11} + 0.0329\,v_{13} + 0.0329\,v_{14} + 0.0329\,v_{17} \end{aligned}\right\}. \tag{21}$$

If the differences of coordinates (20) and (21) as well as the given coordinates (5) are substituted in the two equations (14), then we obtain the following condition equations:

$$\left.\begin{aligned} &-0.0165\,v_4 + 0.0197\,v_6 - 0.0089\,v_8 + 0.1934\,v_9 + 0.1732\,v_{11} + 0.1169\,v_{13} + 0.3431\,v_{14} \\ &\qquad - 0.0064\,v_{15} - 0.0432\,v_{16} + 0.3431\,v_{17} + 0.0429\,v_{18} + 0.011 = 0 \\ &+0.0864\,v_4 - 0.1037\,v_6 + 0.0469\,v_8 - 0.0733\,v_9 + 0.0329\,v_{11} - 0.1304\,v_{13} - 0.0253\,v_{14} \\ &\qquad - 0.0186\,v_{15} - 0.1253\,v_{16} - 0.0253\,v_{17} + 0.1246\,v_{18} + 0.087 = 0 \end{aligned}\right\}. \tag{22}$$

The 5 condition equations found, (11), (13), and (22), are collected once again in the above summary, where the side equations (11) are divided by 100 and the equations of coordinates (22) are multiplied by 10, in order to obtain nearly equivalent coefficients.

The condition equations are to be submitted to a further transformation according to the rule specified in section 74, p. 264, so that in each equation the coefficients belonging to a triangle yield the sum zero. The condition equations transformed in this manner are likewise summarized above, with which the method of adjustment can now be carried out. We obtain the following normal equations in the abbreviated manner of writing:

$$\begin{aligned} \underline{3.3267\,k_1} + 5.7053\,k_2 - 0.8392\,k_3 - 1.8252\,k_4 + 2.1039\,k_5 + 0.720 &= 0 \\ \underline{19.1852\,k_2} - 0.5005\,k_3 + 0.7322\,k_4 + 10.5044\,k_5 + 0.110 &= 0 \\ \underline{6.5681\,k_3} - 3.7124\,k_4 + 6.8323\,k_5 + 0.870 &= 0 \\ \underline{11.7432\,k_4} - 3.9485\,k_5 - 6.450 &= 0 \\ \underline{22.4348\,k_5} - 1.110 &= 0 \end{aligned}$$

and therefrom the correlates:

$$\begin{aligned} k_1 &= +0.88\,149 \\ k_2 &= -0.45\,885 \\ k_3 &= +0.11\,974 \\ k_4 &= +0.85\,196 \\ k_5 &= +0.29\,513\,. \end{aligned}$$

The corrections v and the adjusted angles yield the following summary:

Preliminarily Corrected:		Adjusted:
(4) = 63° 30′ 31.041″	— 1.056″	[4] = 63° 30′ 29.985″
(5) = 57 13 03.373	+ 0.539	[5] = 57 13 03.912
(6) = 59 16 25.586	+ 0.517	[6] = 59 16 26.103
(7) = 46 24 50.529	— 0.922	[7] = 46 24 49.607
(8) = 74 59 27.593	+ 0.007	[8] = 74 59 27.600
(9) = 58 35 41.878	+ 0.915	[9] = 58 35 42.793
(10) = 55 21 09.215	— 0.062	[10] = 55 21 09.153
(11) = 45 27 48.541	— 0.657	[11] = 45 27 47.884
(12) = 79 11 02.244	+ 0.719	[12] = 79 11 02.963
(13) = 46 07 39.884	— 0.302	[13] = 46 07 39.582
(14) = 37 41 59.362	+ 0.188	[14] = 37 41 59.550
(15) = 96 10 20.754	+ 0.114	[15] = 96 10 20.868
(16) = 53 39 28.389	— 0.167	[16] = 53 39 28.222
(17) = 72 32 21.473	— 0.482	[17] = 72 32 20.991
(18) = 53 48 10.138	+ 0.649	[18] = 53 48 10.787.

We will not present with all details the examination of the adjustment by the substitution of the final angles in the condition equations; it is sufficient to give the results.

The connection of directions is represented as follows:

(St. Mich., Lang.)	= 268° 48′ 10.926″
[17] + [14]	= 110 14 20.541
(St. Mich., Kön.)	= 19 02 31.467
[13] + [11] + [9] ± 180°	= 330 11 10.259
(Kön., Mel.)	= 349 13 41.726
[7] + [5] ± 180°	= 283 37 53.519
(Mel., Klobb.)	= 272 51 35.245
	Should be 35.246.

For this, we find the two side logarithms:

$$\begin{aligned} \log\,(\text{St. Mich.—Kön.}) &= 4.569\,86314 \\ \log\,(\text{Kön.—Mel.}) &= 4.560\,77921 \end{aligned}$$

and obtain then the following computation of coordinates:

St. Michael	+ 7 407.500	— 44 332.386
	+ 12 117.978	+ 35 109.388
Königsstuhl	+ 19 525.478	— 9 222.998
	— 6 798.000	+ 35 732.098
Melibocus	+ 12 727.478	+ 26 509.100
	Should be 7.478	9.100.

The two side equations (10) are satisfied just as satisfactorily so that everything is now sufficiently examined.

The adjusted angles would now have to be retransformed to the sphere by means of the reductions from (8), which has nothing to do, however, with the adjustment in itself, and therefore we shall omit it here.

Wreath systems

If a chain of triangles encloses an area wreathlike, then there results the same problem of adjustment as in the case of the connection of the chain in Fig. 3 to the four points Klobberg, Melibocus, Langenkandel, and St. Michael. If we consider, e.g., the chain represented in Fig. 4, then we have herein 12 equations of sums of angles for the 12 triangles as well as one side equation, by which the equality of the two triangle sides AB and $A'B'$ is expressed. But even when all of these 13 equations are satisfied, there result still further discrepancies, since the wreath does not come to a closure in the case of the computation of coordinates. If the coordinates of the triangle points, starting, say, from A and B, are computed one after the other, then, at the end, we obtain two points A' and B', which do not coincide with the starting points.

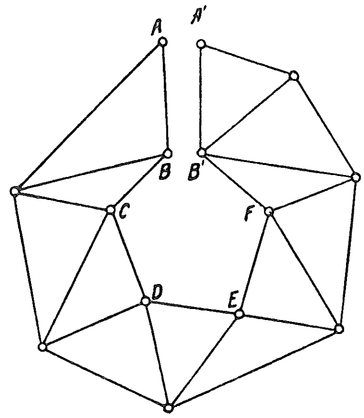


Fig. 4. Wreath system.

If this gap of the chain is to be eliminated, then three connecting conditions are to be introduced again in the adjustment, namely a condition that the direction angles (A,B) and (A',B') are equal to each other, and the two further conditions that the abscissae and ordinates of B and B' (or of A and A') coincide.

We thus have again one direction equation and two coordinate equations, three equations which we also designate as *wreath or polygon condition equations*.

The setting up of these three equations is carried out in complete agreement with the example just treated, where either spherical or, as the case may be, spheroidal or, better, conformal plane coordinates are taken as a basis. The three wreath condition equations can be treated here either together with the remaining condition equations of the net, which cannot be avoided in the case of direction measurements, or if only the angles in the individual triangles are measured, we can apply Krüger's method of the groupwise adjustment, as we have shown in the previous example.

Wreath systems have played an important role in the adjustment operations of the Prussian Land Survey, and we will therefore present the historical development, which the theories pertaining to this have taken, according to a communication by Major Haupt in *Astronomische Nachrichten*, vol. 107, No. 2549-2550 (Sept. 1883):

When in the sixties of the past century at the Prussian Land Survey, chains, which surrounded a part of the country free of triangles, joined up with themselves again, there turned out the inconvenience that, in spite of setting up all existing and necessary angle equations and side equations, identical points computed from different sides did *not* receive the same geographic coordinates, and the inner space free of triangles, the polygon, did *not* receive the sum of angles corresponding to its area, for the three so-called polygon equations were missing.

The first one who gave a method for setting up these polygon equations was Premierleutnant v. Prondzynski (*Astronomische Nachrichten*, No. 1690, vol. 71, 1868, pp. 145-154 and No. 1782, vol. 75, 1869, pp. 81-90). He drew corrections to the directions of the most recent chain through the whole longitude, latitude, and azimuth computations and thus formed three new equations.

Thereafter, Professor Börsch indicated another, but in general not simpler, method (*Astronomische Nachrichten*, No. 1697 and No. 1704, vol. 71, 1868, pp. 265-268 and 379-380), which is based on the theory of covering the inner free space between the different chains with triangles to be computed and so to set up the missing equations.

There is also to be mentioned here Zachariae, *Die geodätischen Hauptpunkte und ihre Koordinaten*, German by Lamp, Berlin, 1878, p. 156, where for "wreath systems" the necessary equations are obtained by the introduction of fictitious measurements and re-elimination of the same.

These methods were not used in the Prussian land triangulations; the method applied in reality by this authority was indicated by the then Captain Schreiber (later General and chief of the Prussian Land Survey). It yields two equations by the projection of the inner polygon wreath boundary on an arbitrarily assumed rectangular spheroidal coordinate system.

Schreiber's method of the polygon wreath connection by rectangular coordinates is given in the work, *Die Königlich Preussische Landestriangulation*, Hauptdreiecke, Erster Teil, Berlin, 1870, p. 421, and Hauptdreiecke, II Teil, Berlin, 1874, p. 605.

We have given a report about it with an example in Jordan-Steppe's *Deutsches Vermessungswesen*, 1882, pp. 81-85 and 103, and note to this that the first Schreiber formulae of 1870 ("Hauptdreiecke I," p. 421) can be developed purely spherically with a mean radius of curvature, and thus coincide with the known Soldner formulae of coordinates, while the second Schreiber formulae of 1874 ("Hauptdreiecke II," p. 605) add still higher terms, corresponding to Gauss' treatise, "Disquisitiones generales circa superficies curvas."

The setting up of the condition equations for the polygon closure in the case of a wreath system is treated further by Krüger in, "Beiträge zur Berechnung von Lotabweichungssystemen," *Veröff. d. Geod. Inst.*, Potsdam, 1898, pp. 29-30.

The groupwise adjustment, according to Krüger's method, which we have applied in the previous example on pp. 293-298, is developed for a wreath system in the form of a simple closed chain of triangles by P. H a t t in *Compt. rend. hebd. des séances de l'acad. des sciences*, 1908, pp. 887-895. The same method is represented once again in *Verhandlungen der 17. Allgemeinen Konferenz der Internationalen Erdmessung im September 1912 in Hamburg*, Part II, 1914, Annex B, XVIII, last treatise.

Of further literature we mention:

W e i g e l, "Compensation des chaînes fermées," *Bull. géod.*, 1925, pp. 74-82.

Weigel, "Nouvelle méthode de compensation des chaînes fermées," *Bull. géod.*, 1928, pp. 431-443.

J e n n e, "Uber die Bildung von Polygonbedingungsgleichungen mit Hilfe fingierter Beobachtungen," *Zeitschrift für Vermessungswesen*, 1932, pp. 273-283.

Section 81. The B o l t z Development

The development method by Boltz is aimed at the gradual adjustment of a triangulation net with the goal that the final result is to equal the result of a uniform total adjustment of the net. It is assumed here that direction measurements with the same accuracy exist.

The *Veröffentlichung des Preuss. Geod. Inst.*, N.F. Nr. 90, "Entwicklungsverfahren zum Ausgleichen geodätischer Netze nach der Methode der kleinsten Quadrate," by H. Boltz, Berlin, 1923, gives a thorough representation of this method.

As preparation we consider, in Fig. 1, a simple chain of triangles, which is composed of an arbitrary number of triangles, and in which we assume the discrepancies of the individual triangles to be w_1, w_2, w_3 . . . . With the directions denoted in Fig. 1 we have the condition equations

$$\left.\begin{array}{l} -v_1+v_2-v_3+v_4 \quad\cdot\quad\cdot\quad\cdot\; -v_8+v_9 \;\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\; +w_1=0 \\ \cdot\quad\cdot\quad\cdot\; -v_4+v_5 \;\cdot\; -v_7+v_8 \;\cdot\quad\cdot\quad\cdot\; -v_{12}+v_{13} \;\cdot\quad\cdot\quad\cdot\; +w_2=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\; -v_5+v_6 \;\cdot\quad\cdot\quad\cdot\quad\cdot\; -v_{11}+v_{12} \;\cdot\; -v_{14}+v_{15} \;\cdot\quad\cdot\; +w_3=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} . \qquad (1)$$

To these there belong the normal equations:

$$\left.\begin{array}{l} +6k_1-2k_2 \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\; +w_1=0 \\ -2k_1+6k_2-2k_3 \quad\cdot\quad\cdot\quad\cdot\quad\cdot\; +w_2=0 \\ \cdot\quad -2k_2+6k_3-2k_4 \quad\cdot\quad\cdot\quad\cdot\; +w_3=0 \\ \cdot\quad\cdot\quad -2k_3+6k_4-2k_5 \quad\cdot\quad\cdot\; +w_4=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} . \qquad (2)$$

These normal equations can be easily solved in the general form due to their symmetrical construction, so that we obtain the correlates k_1, k_2, k_3 . . . as linear functions of the discrepancies w_1, w_2, w_3 . . . , e.g., for a chain of five triangles we have

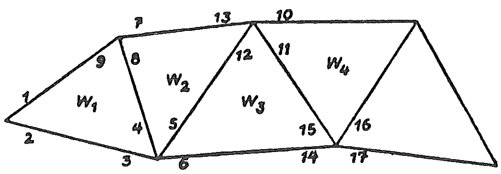


Fig. 1.

$$\left.\begin{array}{l} k_1=-0{,}191\,w_1-0.073\,w_2-0.028\,w_3-0.010\,w_4-0.003\,w_5 \\ k_2=-0.073\,w_1-0.219\,w_2-0{,}083\,w_3-0.031\,w_4-0.010\,w_5 \\ k_3=-0.028\,w_1-0{,}083\,w_2-0.222\,w_3-0.083\,w_4-0.028\,w_5 \\ k_4=-0.010\,w_1-0{,}031\,w_2-0{,}083\,w_3-0.219\,w_4-0{,}073\,w_5 \\ k_5=-0.003\,w_1-0.010\,w_2-0{,}028\,w_3-0.073\,w_4-0{,}191\,w_5 \end{array}\right\} . \qquad (3)$$

Quite generally, we can write

$$\left.\begin{array}{l} k_1=f_{11}\,w_1+f_{12}\,w_2+f_{13}\,w_3+\quad\cdot\quad\cdot\quad\cdot \\ k_2=f_{21}\,w_1+f_{22}\,w_2+f_{23}\,w_3+\quad\cdot\quad\cdot\quad\cdot \\ k_3=f_{31}\,w_1+f_{32}\,w_2+f_{33}\,w_3+\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \qquad (4)$$

where the f's are to be computed from the coefficients of the normal equations.

After these preparations we will assume that two groups of linear condition equations are given in the following form:

$$\text{1st group:} \quad \left.\begin{aligned} a_1 v_1 + a_2 v_2 + \quad \cdots \quad + a_n v_n + w_1 &= 0 \\ b_1 v_1 + b_2 v_2 + \quad \cdots \quad + b_n v_n + w_2 &= 0 \\ c_1 v_1 + c_2 v_2 + \quad \cdots \quad + c_n v_n + w_3 &= 0 \end{aligned}\right\} \tag{5}$$

$$\text{2nd group:} \quad \left.\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \quad \cdots \quad + \alpha_n v_n + w_4 &= 0 \\ \beta_1 v_1 + \beta_2 v_2 + \quad \cdots \quad + \beta_n v_n + w_5 &= 0 \end{aligned}\right\}. \tag{6}$$

The total normal equations for this are:

$$\left.\begin{aligned} [a\,a]\,k_1 + [a\,b]\,k_2 + [a\,c]\,k_3 \;&|\; + [a\,\alpha]\,k_4 + [a\,\beta]\,k_5 + w_1 = 0 \\ [a\,b]\,k_1 + [b\,b]\,k_2 + [b\,c]\,k_3 \;&|\; + [b\,\alpha]\,k_4 + [b\,\beta]\,k_5 + w_2 = 0 \\ [a\,c]\,k_1 + [b\,c]\,k_2 + [c\,c]\,k_3 \;&|\; + [c\,\alpha]\,k_4 + [c\,\beta]\,k_5 + w_3 = 0 \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} [a\,\alpha]\,k_1 + [b\,\alpha]\,k_2 + [c\,\alpha]\,k_3 \; + [\alpha\alpha]\,k_4 + [\alpha\,\beta]\,k_5 + w_4 = 0 \\ [a\,\beta]\,k_1 + [b\,\beta]\,k_2 + [c\beta]\,k_3 \; + [\alpha\beta]\,k_4 + [\beta\,\beta]\,k_5 + w_5 = 0 \end{aligned}\right\}. \tag{8}$$

We denote in (7) the last three terms by w_1', w_2', w_3', and therefore set

$$\left.\begin{aligned} w_1' &= w_1 + [a\,\alpha]\,k_4 + [a\,\beta]\,k_5 \\ w_2' &= w_2 + [b\,\alpha]\,k_4 + [b\,\beta]\,k_5 \\ w_3' &= w_3 + [c\,\alpha]\,k_4 + [c\,\beta]\,k_5 \end{aligned}\right\}. \tag{9}$$

From equations (7), the correlates k_1, k_2, k_3 can then be expressed by the quantities w_1', w_2', w_3', and we obtain in brief form:

$$\left.\begin{aligned} k_1 &= f_{11}\,w_1' + f_{12}\,w_2' + f_{13}\,w_3' \\ k_2 &= f_{21}\,w_1' + f_{22}\,w_2' + f_{23}\,w_3' \\ k_3 &= f_{31}\,w_1' + f_{32}\,w_2' + f_{33}\,w_3' \end{aligned}\right\}. \tag{10}$$

If we compare this with the indeterminate solution of the normal equations (19), section 30, p. 97, then we see that for the coefficients of equations (10) the relations

$$f_{12} = f_{21}, \qquad f_{13} = f_{31}, \qquad f_{23} = f_{32} \tag{11}$$

exist.

If we use in (10) only the original w's instead of the w''s, then we obtain only preliminary values k° instead of the final correlate values k, i.e.,

$$\left.\begin{aligned} k_1^\circ &= f_{11}\,w_1 + f_{12}\,w_2 + f_{13}\,w_3 \\ k_2^\circ &= f_{21}\,w_1 + f_{22}\,w_2 + f_{23}\,w_3 \\ k_3^\circ &= f_{31}\,w_1 + f_{32}\,w_2 + f_{33}\,w_3 \end{aligned}\right\}. \tag{12}$$

If we substitute in (10) the values of the w''s from (9), then we obtain for k_1, taking into account (12),

$$\begin{aligned} k_1 = k_1^\circ &+ (f_{11}\,[a\,\alpha] + f_{12}\,[b\,\alpha] + f_{13}\,[c\,\alpha])\,k_4 \\ &+ (f_{11}\,[a\,\beta] + f_{12}\,[b\,\beta] + f_{13}\,[c\,\beta])\,k_5 \end{aligned},$$

and corresponding expressions result for k_2 and k_3.

We set for abbreviation

$$\left.\begin{aligned} f_{11}[a\alpha]+f_{12}[b\alpha]+f_{13}[c\alpha]&=\xi_1\\ f_{11}[a\beta]+f_{12}[b\beta]+f_{13}[c\beta]&=\eta_1\\ f_{21}[a\alpha]+f_{22}[b\alpha]+f_{23}[c\alpha]&=\xi_2\\ f_{21}[a\beta]+f_{22}[b\beta]+f_{23}[c\beta]&=\eta_2\\ f_{31}[a\alpha]+f_{32}[b\alpha]+f_{33}[c\alpha]&=\xi_3\\ f_{31}[a\beta]+f_{32}[b\beta]+f_{33}[c\beta]&=\eta_3 \end{aligned}\right\} . \tag{13}$$

Then

$$\left.\begin{aligned} k_1&=k_1{}^\circ+\xi_1 k_4+\eta_1 k_5\\ k_2&=k_2{}^\circ+\xi_2 k_4+\eta_2 k_5\\ k_3&=k_3{}^\circ+\xi_3 k_4+\eta_3 k_5 \end{aligned}\right\} . \tag{14}$$

The six auxiliary quantities ξ and η can be computed here from equations (12), if we replace herein the discrepancies w_1, w_2, w_3 by the quantities $[a\alpha]$, $[b\alpha]$, $[c\alpha]$ or, as the case may be, $[a\beta]$, $[b\beta]$, $[c\beta]$.

We turn now first to the computation of the correlates k_4 and k_5 of the second system. If we substitute the values (14) in the two normal equations (8), then we obtain

$$\left.\begin{aligned} &[a\alpha]k_1{}^\circ+[a\alpha]\xi_1 k_4+[a\alpha]\eta_1 k_5\\ &+[b\alpha]k_2{}^\circ+[b\alpha]\xi_2 k_4+[b\alpha]\eta_2 k_5\\ &+[c\alpha]k_3{}^\circ+[c\alpha]\xi_3 k_4+[c\alpha]\eta_3 k_5\\ &\qquad+[\alpha\alpha]k_4\quad+[\alpha\beta]k_5\quad+w_4=0\\ &[a\beta]k_1{}^\circ+[a\beta]\xi_1 k_4+[a\beta]\eta_1 k_5\\ &+[b\beta]k_2{}^\circ+[b\beta]\xi_2 k_4+[b\beta]\eta_2 k_5\\ &+[c\beta]k_3{}^\circ+[c\beta]\xi_3 k_4+[c\beta]\eta_3 k_5\\ &\qquad+[\alpha\beta]k_4\quad+[\beta\beta]k_5\quad+w_5=0 \end{aligned}\right\} . \tag{15}$$

For simplification we introduce the following designations:

$$\left.\begin{aligned} [\alpha\alpha]+[a\alpha]\xi_1+[b\alpha]\xi_2+[c\alpha]\xi_3&=[AA]\\ [\alpha\beta]+[a\alpha]\eta_1+[b\alpha]\eta_2+[c\alpha]\eta_3&=[AB]\\ w_4+[a\alpha]k_1{}^\circ+[b\alpha]k_2{}^\circ+[c\alpha]k_3{}^\circ&=W_4\\ [\alpha\beta]+[a\beta]\xi_1+[b\beta]\xi_2+[c\beta]\xi_3&=[BA]\\ [\beta\beta]+[a\beta]\eta_1+[b\beta]\eta_2+[c\beta]\eta_3&=[BB]\\ w_5+[a\beta]k_1{}^\circ+[b\beta]k_2{}^\circ+[c\beta]k_3{}^\circ&=W_5 \end{aligned}\right\} . \tag{16}$$

Taking into account (11) we can easily prove that $[BA]=[AB]$. We thus obtain for equations (15) the simple form

$$\left.\begin{aligned} [AA]k_4+[AB]k_5+W_4&=0\\ [AB]k_4+[BB]k_5+W_5&=0 \end{aligned}\right\} . \tag{17}$$

The two correlates k_4 and k_5 can be computed therefrom, and equations (14) yield then the values of k_1, k_2, k_3.

The practical significance of the above method lies in its application to chains of triangles. As we have already shown in (3), p. 300, for a simple chain of triangles with direction measurements, the correlates can be developed as linear functions of the discrepancies for which the coefficients can be collected in tables. With this, we obtain, without further consideration, equations (10), and in the case of connecting a second chain the correlates of equations (17) also can be expressed in the same way, so that in this case the numerical computation turns out very simple.

For this, we will present, with Fig. 2, a simple example in which, first, the triangles 4 and 5 are to be connected to the chain 1, 2, 3, and after this, the triangles 6 and 7 are to be combined with the previous ones into a uniform adjustment.

In Fig. 2 we consider first the triangles 1 to 5, for which, without specifying the condition equations, we write down at once the normal equations. Corresponding to equations (7) and (8), we will separate here the correlates of the first three triangles by a special framing.

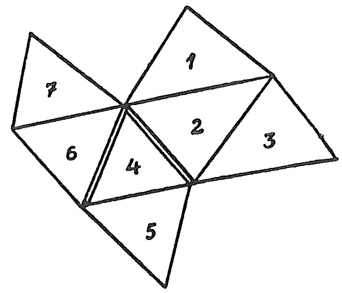


Fig. 2.

We obtain then:

$$\left.\begin{array}{ccc|cc} 6k_1 - 2k_2 & \cdot & & \cdot \quad \cdot & + w_1 = 0 \\ -2k_1 + 6k_2 - 2k_3 & & & -2k_4 \quad \cdot & + w_2 = 0 \\ \cdot \quad -2k_2 + 6k_3 & & & \cdot \quad \cdot & + w_3 = 0 \end{array}\right\} \qquad (7^*)$$

$$\left.\begin{array}{l} \cdot \quad -2k_2 \quad \cdot \quad + 6k_4 - 2k_5 + w_4 = 0 \\ \cdot \quad \cdot \quad \cdot \quad -2k_4 + 6k_5 + w_5 = 0 \end{array}\right\}. \qquad (8^*)$$

From the tables contained in the above-mentioned work by Boltz we take for the preliminary correlates of the first three triangles the values

$$\left.\begin{array}{l} 42\,k_1{}^\circ = -8\,w_1 - 3\,w_2 - 1\,w_3 \\ 42\,k_2{}^\circ = -3\,w_1 - 9\,w_2 - 3\,w_3 \\ 42\,k_3{}^\circ = -1\,w_1 - 3\,w_2 - 8\,w_3 \end{array}\right\}. \qquad (12^*)$$

For the determination of the auxiliary quantities ξ and η according to (13) we take from (7*) $[b\alpha] = -2$, while all other coefficients are equal to zero. With this, equations (12*) yield

$$\left.\begin{array}{ll} 42\,\xi_1 = -3 \times -2 = +\ 6 & \eta_1 = 0 \\ 42\,\xi_2 = -9 \times -2 = +18 & \eta_2 = 0 \\ 42\,\xi_3 = -3 \times -2 = +\ 6 & \eta_3 = 0 \end{array}\right\}. \qquad (13^*)$$

Equations (14) yield, with this,

$$\left.\begin{array}{l} 42\,k_1 = 42\,k_1{}^\circ + \ 6\,k_4 \quad . \\ 42\,k_2 = 42\,k_2{}^\circ + 18\,k_4 \quad . \\ 42\,k_3 = 42\,k_3{}^\circ + \ 6\,k_4 \quad . \end{array}\right\}. \qquad (14^*)$$

To this, we take the two normal equations (8*) after multiplication by 21

$$\begin{array}{l} -42\,k_2 + 126\,k_4 - \ 42\,k_5 + 21\,w_4 = 0 \\ \quad . \quad - \ 42\,k_4 + 126\,k_5 + 21\,w_5 = 0\,. \end{array}$$

The substitution of the values (14*) herein is limited to k_2, with which we obtain:

$$\begin{array}{l} +108\,k_4 - \ 42\,k_5 - 42\,k_2{}^\circ + 21\,w_4 = 0 \\ - \ 42\,k_4 + 126\,k_5 \quad . \quad + 21\,w_5 = 0 \end{array}$$

or

$$\left.\begin{array}{l} + \frac{36}{7}k_4 - \ 2\,k_5 - \ 2\,k_2{}^\circ + w_4 \ = 0 \\ - \ 2\,k_4 + \ 6\,k_5 \quad . \quad + w_5 \ = 0 \end{array}\right\}. \qquad (17^*)$$

We could substitute the value of k_2° from (12*) in (17*); we will postpone this, however, and preliminarily combine the term $-2k_2^\circ$ with the term w_4.

Equations (17*) are the two normal equations for the triangles 4 and 5. These form in themselves a *simple* chain; but since in the term with k_4 the influence of the first chain, i.e., the triangles 1, 2, 3, is already contained, then this term does no longer read $+6k_4$, but has received the value $+\frac{36}{7}k_4$. Therefore, we call the chain of triangles 4 and 5 a *disturbed simple* chain.

In the work by Boltz mentioned on p. 300 there are also given tables for the correlates of a disturbed simple chain, which we will likewise use here. We set for this

$$\frac{36}{7} = A \qquad -2k_0 + w_4 = W_4$$

and find then the following values in Boltz's tables for the two correlates k_4 and k_5

$$2(3A-2)k_4 = -6W_4 - 2w_5$$
$$2(3A-2)k_5 = -2W_4 - Aw_5 .$$

If we substitute here the values of A and W_4, then we obtain

$$\frac{94}{7}k_4 = +6k_2^\circ - 3w_4 - w_5$$
$$\frac{94}{7}k_5 = +2k_2^\circ - w_4 - \frac{18}{7}w_5$$

and with the value of k_2° from (12*)

$$\left.\begin{aligned} k_4 &= -\frac{3}{94}w_1 - \frac{9}{94}w_2 - \frac{3}{94}w_3 - \frac{21}{94}w_4 - \frac{7}{94}w_5 \\ k_5 &= -\frac{1}{94}w_1 - \frac{3}{94}w_2 - \frac{1}{94}w_3 - \frac{7}{94}w_4 - \frac{18}{94}w_5 \end{aligned}\right\} . \qquad (18)$$

With this, we can now also compute from (14*) and (12*) the correlates k_1, k_2, k_3. There follows

$$\left.\begin{aligned} 3k_1 &= -\frac{55}{94}w_1 - \frac{24}{94}w_2 - \frac{8}{94}w_3 - \frac{9}{94}w_4 - \frac{3}{94}w_5 \\ 3k_2 &= -\frac{24}{94}w_1 - \frac{72}{94}w_2 - \frac{24}{94}w_3 - \frac{27}{94}w_4 - \frac{9}{94}w_5 \\ 3k_3 &= -\frac{8}{94}w_1 - \frac{24}{94}w_2 - \frac{55}{94}w_3 - \frac{9}{94}w_4 - \frac{3}{94}w_5 \end{aligned}\right\} . \qquad (19)$$

With (18) and (19) there are found the final values of the correlates, which correspond to a uniform adjustment of all 5 triangles.

A further simple chain of triangles, consisting of the two triangles 6 and 7, is now to be connected. We omit here also the condition equations and give at once the seven normal equations of the whole net:

$$\left.\begin{array}{ccccc|cc} 6k_1 - 2k_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot + w_1 = 0 \\ -2k_1 + 6k_2 - 2k_3 - 2k_4 & & & & \cdot & \cdot & \cdot + w_2 = 0 \\ \cdot & -2k_2 + 6k_3 & \cdot & & \cdot & \cdot & \cdot + w_3 = 0 \\ \cdot & -2k_2 & \cdot & +6k_4 - 2k_5 & & -2k_6 & \cdot + w_4 = 0 \\ \cdot & \cdot & \cdot & -2k_4 + 6k_5 & & \cdot & \cdot + w_5 = 0 \end{array}\right\} \qquad (7^{**})$$

$$\left.\begin{array}{cccccc} \cdot & \cdot & \cdot & -2k_4 & \cdot & +6k_6 - 2k_7 + w_6 = 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2k_6 + 6k_7 + w_7 = 0 \end{array}\right\} . \qquad (8^{**})$$

In (18) and (19) we have already found the correlates for the first five triangles, which we now use as preliminary ones. We collect these values once more:

$$\left.\begin{aligned} 282\,k_1{}^\circ &= -55\,w_1 - 24\,w_2 - 8\,w_3 - 9\,w_4 - 3\,w_5 \\ 282\,k_2{}^\circ &= -24\,w_1 - 72\,w_2 - 24\,w_3 - 27\,w_4 - 9\,w_5 \\ 282\,k_3{}^\circ &= -8\,w_1 - 24\,w_2 - 55\,w_3 - 9\,w_4 - 3\,w_5 \\ 282\,k_4{}^\circ &= -9\,w_1 - 27\,w_2 - 9\,w_3 - 63\,w_4 - 21\,w_5 \\ 282\,k_5{}^\circ &= -3\,w_1 - 9\,w_2 - 3\,w_3 - 21\,w_4 - 54\,w_5 \end{aligned}\right\}. \tag{12**}$$

If the coefficients of the fourth and fifth condition equations are denoted by d and by e, then we have in equations (7**) only the coefficient $[d\alpha] = -2$ for the determination of the new auxiliary quantities ξ and η, while all other coefficients are equal to zero. In equations (12**) w_4, therefore, is to be replaced everywhere by -2; whereby we obtain

$$\left.\begin{aligned} 282\,\xi_1 &= +18 & \eta_1 &= 0 \\ 282\,\xi_2 &= +54 & \eta_2 &= 0 \\ 282\,\xi_3 &= +18 & \eta_3 &= 0 \\ 282\,\xi_4 &= +126 & \eta_4 &= 0 \\ 282\,\xi_5 &= +42 & \eta_5 &= 0 \end{aligned}\right\}. \tag{13**}$$

Then, according to (14),

$$\left.\begin{aligned} 47\,k_1 &= 47\,k_1{}^\circ + 3\,k_6 \\ 47\,k_2 &= 47\,k_2{}^\circ + 9\,k_6 \\ 47\,k_3 &= 47\,k_3{}^\circ + 3\,k_6 \\ 47\,k_4 &= 47\,k_4{}^\circ + 21\,k_6 \\ 47\,k_5 &= 47\,k_5{}^\circ + 7\,k_6 \end{aligned}\right\}. \tag{14**}$$

If the value of k_4 is substituted in equations (8**), then we obtain

$$\left.\begin{aligned} +\frac{240}{47}\,k_6 - 2\,k_7 - 2\,k_4{}^\circ + w_6 &= 0 \\ -2\,k_6 + 6\,k_7 \quad . \quad + w_7 &= 0 \end{aligned}\right\}. \tag{17**}$$

If we now set again

$$\frac{240}{47} = A \qquad -2k_4{}^\circ + w_6 = W_6,$$

then we have according to Boltz's tables for a disturbed chain with two triangles

$$2\,(3A - 2)\,k_6 = -6W_6 - 2w_7$$
$$2\,(3A - 2)\,k_7 = -2W_6 - Aw_7$$

and with the values of A and W_6

$$626\,k_6 = 282\,k_4{}^\circ - 141\,w_6 - 47\,w_7$$
$$626\,k_7 = 94\,k_4{}^\circ - 47\,w_6 - 120\,w_7.$$

If we take to this, in addition, k_4° from (12**), then we will have

$$\left.\begin{aligned} 626\,k_6 &= -9\,w_1 - 27\,w_2 - 9\,w_3 - 63\,w_4 - 21\,w_5 - 141\,w_6 - 47\,w_7 \\ 626\,k_7 &= -3\,w_1 - 9\,w_2 - 3\,w_3 - 21\,w_4 - 7\,w_5 - 47\,w_6 - 120\,w_7 \end{aligned}\right\}. \qquad (18^*)$$

Finally, we find now from (14**) and (12**)

$$\left.\begin{aligned} 1878\,k_1 &= -368\,w_1 - 165\,w_2 - 55\,w_3 - 72\,w_4 - 24\,w_5 - 27\,w_6 - 9\,w_7 \\ 1878\,k_2 &= -165\,w_1 - 495\,w_2 - 165\,w_3 - 216\,w_4 - 72\,w_5 - 81\,w_6 - 27\,w_7 \\ 1878\,k_3 &= -55\,w_1 - 165\,w_2 - 368\,w_3 - 72\,w_4 - 24\,w_5 - 27\,w_6 - 9\,w_7 \\ 1878\,k_4 &= -72\,w_1 - 216\,w_2 - 72\,w_3 - 504\,w_4 - 168\,w_5 - 189\,w_6 - 63\,w_7 \\ 1878\,k_5 &= -24\,w_1 - 72\,w_2 - 24\,w_3 - 168\,w_4 - 369\,w_5 - 63\,w_6 - 21\,w_7 \end{aligned}\right\}. \qquad (19^*)$$

With this, the third chain also is attached, and the correlates (18*) and (19*) correspond to the uniform adjustment of all three chains.

The development method turns out very simple in the case of attachment of triangle chains, as is seen from the above example. If, in addition, a side equation belongs to the triangulation net, then a further normal equation would be added to the normal equations (7**) and (8**), while equations (7**) and (8**) are extended by one correlate k_8. In this case, equations (18*) and (19*) would yield the preliminary values $k_1^\circ, k_2^\circ, \ldots k_7^\circ$. The computation of the auxiliary quantities ξ, however, does then not become as simple as in equations (13*) and (13**).

H. Boltz gives a rather large example for the adjustment of a triangulation net according to the development method in the work mentioned on p. 300. There is to be mentioned further here the adjustment of the connecting chain Berlin-Schubin, for which the development method was likewise used. This adjustment is published in *Die Preussische Landesvermessung*, Hauptdreiecke, Neue Folge, Dritter Teil, Berlin, 1932, pp. 74-154.

Section 82. Distinction Between Angle Measurement and Direction Measurement

After having dealt in sections 62 to 64 with the simplest forms of station adjustment, we will now return once again to this problem, in order to treat it in a more general form. Before we enter on the subject, let us premise, however, a few further remarks on angle measurements and direction measurements.

Hitherto we have already made several times the distinction between angle measurements and direction measurements without giving thereby clear explanations, since these nomenclatures are used today so generally, already in the case of the measurements, that their meaning could be assumed as known, namely that an *angle measurement* (in the narrower sense) always refers to *two* sight rays, left and right, while a set of *direction* measurements can contain an arbitrary number of sight rays.

The angle measurement, therefore, appears as a special case of the direction measurement, and if we put several angle measurements together, then we obtain for the trigonometric computations the same which a set of direction measurements offers, and only if we deal with adjustment and accuracy questions is the distinction between angles and directions in the specified sense of importance.

The present-day nomenclatures of angle measurement and direction measurement, in this sense, did not establish themselves until the middle of the past century. Gauss distinguished in 1821 (*Astronomische Nachrichten*, Vol. 1, p. 81) "angle measurements made at one time" (directions) and "angle measurements made independently" (angle measurements in the narrower sense); in 1837 Bessel and Hagen used the words angle and direction without a rigorous distinction in the present-day sense, which, as it seems, is derived from Hansen, who gave an adjustment according to "directions" in 1841 in *Astronomische Nachrichten*, Vol. 18, p. 173. But Gerling in his *Ausgleichungsrechnungen*, 1843, and in his work, *Beiträge zur Geographie Kurhessens*, Kassel, 1839, already uses "angles" and "directions" as is customary today.

After these preliminary remarks we will establish the most important concepts about mean errors and weights of angles and directions.

An angle can always be considered as the difference of two directions; e.g., in Fig. 1 we have:

$$\text{Angle } AJB = \text{direction } JB - \text{direction } JA. \tag{1}$$

Therefore, if r is a mean direction error and m is a mean angle error, then, according to (1):

$$\begin{aligned} m^2 &= r^2 + r^2 = 2r^2 \\ m &= r\sqrt{2}, \end{aligned} \tag{2}$$

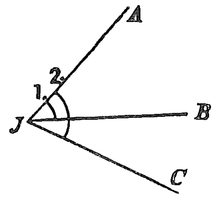


Fig. 1.
3 directions ABC
2 angles 1, 2.

or in proportions of weight, if p is an angle weight and q is a direction weight:

$$\left. \begin{aligned} p:q &= \frac{1}{2}:1 \quad \text{or} = 1:2 \\ \text{or } p &= \frac{q}{2}, \quad q = 2p \end{aligned} \right\}, \tag{3}$$

i.e., an angle measurement has only half the weight p in comparison to the weight q of a direction measurement.

Since by repetition of the measurement the weight also is doubled, it also follows that the double measurement of an angle has the same weight as the simple measurement of a direction.

Let us remember that a single direction measurement is worthless for practical purposes. *One* such measurement only determines the angle which a theodolite sight makes with the sight for the zero reading, e.g., if, upon setting up a theodolite, we aim at a point A and read 26° 17' 20" then this means that the sight A makes the angle 26° 17' 20" with that sight line which we have when reading 0° 0' 0"; if we took the theodolite away now, then we would have obtained absolutely nothing for trigonometric computations. For geodetic purposes we must always measure at least *two* directions together. The directions which are measured at one and the same position on the circle form together a *set* (or gyrus). (Cf. Vol. II, 1st half-volume, 9th Edition, 1931, section 88.)

If, according to Fig. 1, the three directions ABC are intersected in one set, then each direction is to be provided with a correction vA, vB, vC, and if the three direction measurements enter in an adjustment, then we must have:

$$(vA)^2 + (vB)^2 + (vC)^2 + \ldots = \text{minimum}, \tag{4}$$

since the squares of all corrections are now to be collected.

If, on the other hand, the three rays ABC are fixed by two angle measurements AB and AC with respect to each other, then two corrections δ_1 and δ_2 are to be added to the angles, and we must then have:

$$\delta_1{}^2 + \delta_2{}^2 + \ldots = \text{minimum}. \tag{5}$$

We can also adjust the angles in the form of directions by regarding each angle as a set with two directions, and instead of the corrections to the angles δ_1 and δ_2 we have then to introduce in the adjustment the four corrections to the directions vA_1, vB_1, vA_2, vC_2, namely:

$$\left. \begin{aligned} \delta_1 &= vB_1 - vA_1 \\ \delta_2 &= vC_2 - vA_2 \end{aligned} \right\}. \tag{6}$$

Then there exists the condition:

$$(vA_1)^2 + (vB_1)^2 + (vA_2)^2 + (vC_2)^2 + \ldots = \text{minimun}. \tag{7}$$

Such an adjustment of the angle measurements according to directions, however, has no practical value, since we can determine the corrections to the directions only in the form $vB_1 - vA_1$ and $vC_2 - vA_2$, hence only as corrections to the angles, while the individual corrections to the directions vA_1, vB_1, vA_2, vC_2 remain unknown.

Direction measurements have further the characteristic that all readings of a set may be varied by an arbitrary quantity, as the following example shows:

Target Point	Measured Directions	Reduction to $A = 0$	Reduction to $A = 57°\,37'\,3''$
A	165° 46′ 12″	0° 0′ 0″	57° 37′ 3″
B	205 45 37	39 59 25	97 36 28
C	286 54 20	121 8 8	178 45 11

(8)

The *differences* of the directions, i.e. the angles, which alone are involved in the case of trigonometric computations, remain always the same, whether we use the original readings or reduce them to a ray = 0° 0' 0", or assign to a ray its trigonometric direction angle as its direction, as is assumed in the last column of the summary (8).

Section 83. Rigorous Adjustment of Incomplete Sets of Directions (Bessel's Method)

If several incomplete sets are measured at a station, as, for instance, in the example of section 64, p. 211, then, first of all, it is always convenient if at least *one* sight occurs in all sets, in order that we can annex all sets to this one sight, for instance as the zero ray and preliminarily collect all the rest. However, this is not important; the strictly theoretical adjustment of such incomplete sets of directions is independent of such an arrangement to a common starting ray, and only makes use of it, say, for obtaining first approximations.

At the first point of observation J_1 (Fig. 1) there is set up a theodolite whose sighting axis has the direction J_1O_1 when reading 0°; and when setting the targets $P°$, P', P'', P''', the readings $l_1°$, l_1', l_1'', l_1''' are made on the circle. If these measured directions of the set in connection with other sets are submitted to an adjustment, then they obtain corrections $v_1°$, v_1', v_1'', v_1''' . . . , so that the adjusted direction measurements are $l_1° + v_1°$, $l_1' + v_1'$, $l_1'' + v_1''$ . . . .

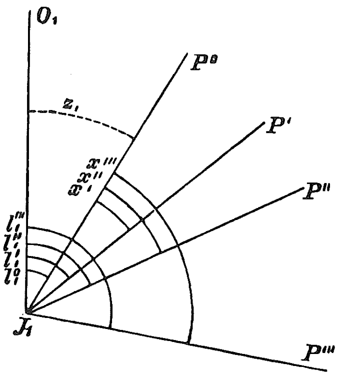


Fig. 1.
Direction measurements, set 1.

The observational values l are *directions*. The observations l have been made for the determination of the angles which the rays $P°$, P', P'', P''', form with each other, and since these four rays are fixed reciprocally by three angles, then we now introduce here as *independent unknowns* the following *angles*:

$$\left.\begin{array}{l} \text{Angle } P°P' = x' \\ \text{Angle } P°P'' = x'' \\ \text{Angle } P°P''' = x''' \end{array}\right\} . \quad (1)$$

It is unimportant here that these angles have a ray, namely $P°$, in common; we *could* also introduce, e.g., the three angles $P°P'$, $P'P''$, $P''P'''$ as independent unknowns.

Now we can set up the following equations:

$$\left.\begin{array}{l} l_1' + v_1' - l_1° - v_1° = x' \\ l_1'' + v_1'' - l_1° - v_1° = x'' \\ l_1''' + v_1''' - l_1° - v_1° = x''' \\ . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \end{array}\right\} . \quad (2)$$

We replace the direction l_1° with its correction v_1° occurring in all equations by a further unknown z_1, as we have already done in section 60, p. 187, hence set

$$l_1^\circ + v_1^\circ = z_1 . \tag{3}$$

With this, we can write equations (2) in the form

$$\left.\begin{aligned} v_1^\circ &= z_1 \quad\;\; . \quad\; - l_1^\circ \\ v_1' &= z_1 + x' \;\; - l_1' \\ v_1'' &= z_1 + x'' \; - l_1'' \\ v_1''' &= z_1 + x''' - l_1''' \end{aligned}\right\} . \tag{4}$$

These are the error equations for the first set.

From equation (3) it follows at the same time that the unknown z_1 is the angle which the adjusted direction to point P° forms with the zero direction of the theodolite. This is also indicated in Fig. 1 which, however, is not quite correct insofar as the illustration is supposed to represent the rays after the adjustment; instead of the measured directions $l_1^\circ, l_1', l_1'' \ldots$ the adjusted directions $l_1^\circ + v_1^\circ, l_1' + v_1', l_1'' + v_1'' \ldots$ would thus have to be indicated.

All this is repeated with the circle shifted, as is indicated in Fig. 2; and by numbering now the sets with $_1, _2, _3 \ldots$, but distinguishing the target points and all which refers to them by $^\circ, ', '', ''' \ldots$, we have the following summary:

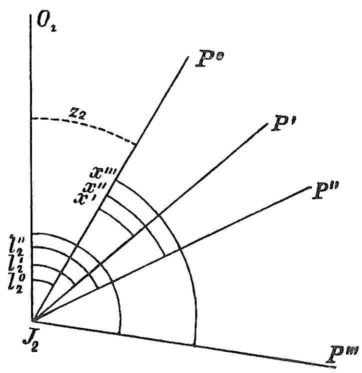


Fig. 2.
Direction measurements, set 2.

$$\left.\begin{array}{l} \text{Set}_1 \; l_1^\circ \, l_1' \, l_1'' \, l_1''' \ldots z_1 \\ \text{Set}_2 \; l_2^\circ \, l_2' \, l_2'' \, l_2''' \ldots z_2 \\ \text{Set}_3 \; l_3^\circ \, l_3' \, l_3'' \, l_3''' \ldots z_3 \\ \ldots\ldots\ldots\ldots\ldots \\ \text{Set}_n \; l_n^\circ \, l_n' \, l_n'' \, l_n''' \ldots z_n \end{array}\right\} . \tag{5}$$

Common to all sets:

$$x', \; x'', \; x''' \ldots .$$

The weights of the directions will in general be equal to unity. But if the same directions occur in several sets, then we will combine them into mean values and assume the number of these sets as the weight of the directions. Likewise, the individual sets will then have to receive different weights if they are measured with different theodolites.

In order to treat this more general case, we assume the weights $p_1, p_2, p_3 \ldots$ for the individual sets.

The whole system of the error equations is then:

$$\left.\begin{array}{l|l|l|l|l} \text{Set}_1 & v_1^\circ = z_1 - l_1^\circ & v_1' = z_1 + x' - l_1' & v_1'' = z_1 + x'' - l_1'' & v_1''' = z_1 + x''' - l_1''' \\ \text{Set}_2 & v_2^\circ = z_2 - l_2^\circ & v_2' = z_2 + x' - l_2' & v_2'' = z_2 + x'' - l_2'' & v_2''' = z_2 + x''' - l_2''' \\ \text{Set}_3 & v_3^\circ = z_3 - l_3^\circ & v_3' = z_3 + x' - l_3' & v_3'' = z_3 + x'' - l_3'' & v_3''' = z_3 + x''' - l_3''' \\ \ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}\right\} . \tag{6}$$

If we stop at three sets and four target points, for the sake of over-all clarity, then the coefficients of the error equations form the table on p. 310:

	Set	Target Point	z_1	z_2	z_3	x'	x''	x'''	$-l$	p
Generally:										
Number = s	1	P°	+1						$-l_1^{\circ}$	p_1
	1	P'	+1			+1			$-l_1'$	
	1	P''	+1				+1		$-l_1''$	
	1	P'''	+1					+1	$-l_1'''$	
Number = s	2	P°		+1					$-l_2^{\circ}$	p_2
	2	P'		+1		+1			$-l_2'$	
	2	P''		+1			+1		$-l_2''$	
	2	P'''		+1				+1	$-l_2'''$	
Number = s	3	P°			+1				$-l_3^{\circ}$	p_3
	3	P'			+1	+1			$-l_3'$	
	3	P''			+1		+1		$-l_3''$	
	3	P'''			+1			+1	$-l_3'''$	
			Number = n			Number = $s-1$				

(7)

According to the composition of the sets, individual error equations will fall out here.

To set up the normal equations we set further the sums of the weights of a set equal to $[p_1]$ or, as the case may be, $[p_2]$, and so on, and the sums of the weights of the directions to a target point $= [p']$ or, as the case may be, $[p'']$, and so on.

Correspondingly, we denote further by $-[l_1]$, $-[l_2]$, and so on, the sum of the absolute terms of the individual sets and by $-[p\,l']$, $-[p\,l'']$, and so on, the sum of the absolute terms for the individual target points multiplied by their weights.

We thus obtain the following normal equations:

$$\left.\begin{array}{llllllll}
[p_1]\,z_1 & \cdot & \cdot & +p_1\,x' & +p_1\,x'' & +p_1\,x''' & -p_1\,[l_1] & =0 \\
\cdot & +[p_2]\,z_2 & \cdot & +p_2\,x' & +p_2\,x'' & +p_2\,x''' & -p_2\,[l_2] & =0 \\
\cdot & \cdot & +[p_3]\,z_3 & +p_3\,x' & +p_3\,x'' & +p_3\,x''' & -p_3\,[l_3] & =0 \\
p_1\,z_1 & +p_2\,z_2 & +p_3\,z_3 & +[p']\,x' & \cdot & \cdot & -[p\,l'] & =0 \\
p_1\,z_1 & +p_2\,z_2 & +p_3\,z_3 & \cdot & +[p'']\,x'' & \cdot & -[p\,l''] & =0 \\
p_1\,z_1 & +p_2\,z_2 & +p_3\,z_3 & \cdot & \cdot & +[p''']\,x''' & -[p\,l'''] & =0
\end{array}\right\}, \qquad (8)$$

where, according to the composition of the error equations, individual terms will fall out, however.

We now obtain from the first three equations

$$\left.\begin{array}{l}
-z_1 = (x' + x'' + x''' - [l_1])\,\dfrac{p_1}{[p_1]} \\[2mm]
-z_2 = (x' + x'' + x''' - [l_2])\,\dfrac{p_2}{[p_2]} \\[2mm]
-z_3 = (x' + x'' + x''' - [l_3])\,\dfrac{p_3}{[p_3]}
\end{array}\right\} \qquad (9)$$

and, with this, can eliminate the quantities of orientation z from the further normal equations. We obtain with symbolic notation of the coefficients:

$$\left.\begin{array}{r}
\underline{(a\,a)\,x'} + (a\,b)\,x'' + (a\,c)\,x''' - (a\,l) = 0 \\
\underline{(b\,b)\,x''} + (b\,c)\,x''' - (b\,l) = 0 \\
\underline{(c\,c)\,x'''} - (c\,l) = 0
\end{array}\right\}. \qquad (10)$$

This system is treated further as usual (cf. section 27) and yields then the unknowns x', x'', x'''. From equations (9) we find the quantities z_1, z_2, z_3, and the error equations (7) yield the corrections v.

For the mean square of error we have the expression:

$$m^2 = \frac{[v\,v\,p]}{R - n - (s - 1)} \tag{11}$$

where R is the number of all measured directions. The further terms in the denominator are to be understood in such a way that n means the number of the unknowns of orientation and $s - 1$ the number of the unknown angles.

Turning to an example, we take from *Gradmessung in Ostpreussen,* pp. 101 and 102, the direction measurements carried out at the Station Nidden. Three sets, of which only the third was complete, were measured here to the three targets Kalleninken (P°), Gilge (P'), and Lattenwalde (P''). After collecting and taking the mean of homogeneous sets we have the following:

	1st Set $p_1 = 12$	
Kalleninken	0° 00′ 00″	$l_1{}^\circ = 0''$
Lattenwalde	87 04 53.104	$l_1{}' = 3.104''$
	2nd Set $p_2 = 19$	
Kalleninken	0° 00′ 00″	$l_2{}^\circ = 0''$
Gilge	26 14 52.553	$l_2{}' = 2.553''$
	3rd Set $p_3 = 12$	
Kalleninken	0° 00′ 00″	$l_3{}^\circ = 0''$
Gilge	26 14 51.646	$l_3{}' = 1.646''$
Lattenwalde	87 04 52.792	$l_3{}'' = 2.792''$.

As unknowns we introduce, in addition to the three quantities of orientation z_1, z_2, z_3, the two angles:

$$\left.\begin{array}{l} \text{Kalleninken—Gilge} = 26^\circ\, 14'\, 50'' + x' \\ \text{Kalleninken—Lattenwalde} = 87\;\; 04\;\; 50\;\; + x'' \end{array}\right\}. \tag{12}$$

We have decreased here the measured quantities l, l', l'' as well as the unknowns x', x'' by rounded-off values in order to be able to compute with small numbers.

With this, we obtain the error equations

$$\begin{array}{llll} v_1{}^\circ = z_1 \;.\;.\;.\;\;. & & \left.\right\} p_1 = 12 \\ v_1{}'' = z_1 \;.\;.\;.\; + x'' - 3.104 & & \\ v_2{}^\circ = \;.\; z_2 \;.\;.\;. & & \left.\right\} p_2 = 19 \\ v_2{}' = \;.\; z_2 \;.\; + x' \;.\; - 2.553 & & \\ v_3{}^\circ = \;.\;.\; z_3 \;.\;. & & \\ v_3{}' = \;.\;.\; z_3 + x' \;.\; - 1.646 & & \left.\right\} p_3 = 12 \\ v_3{}'' = \;.\;.\; z_3 \;.\; + x'' - 2.792 & & \end{array} \tag{13}$$

and the normal equations

$$\left.\begin{array}{rrrrrl} 24\,z_1 & . & . & . & +\,12\,x'' & -\,37.25 = 0 \\ . & +\,38\,z_2 & . & +\,19\,x' & . & -\,48.50 = 0 \\ . & . & +\,36\,z_3 & +\,12\,x' & +\,12\,x'' & -\,53.25 = 0 \\ . & +\,19\,z_2 & +\,12\,z_3 & +\,31\,x' & . & -\,68.25 = 0 \\ 12\,z_1 & . & +\,12\,z_3 & . & +\,24\,x'' & -\,70.75 = 0 \end{array}\right\}. \tag{14}$$

For the elimination of z_1, z_2, z_3 we obtain from the first three normal equations

$$\left.\begin{aligned} -12\,z_1 &= \quad . \qquad\quad + 6\,x'' - 18.625 \\ -19\,z_2 &= +9.5\,x' \qquad\qquad - 24.250 \\ -12\,z_3 &= +4.0\,x' + 4\,x'' - 17.750 \end{aligned}\right\}, \tag{15}$$

and with this, the last two normal equations change to

$$\left.\begin{aligned} +17.50\,x' - 4.00\,x'' - 26.25 &= 0 \\ -4.00\,x' + 14.00\,x'' - 34.37 &= 0 \end{aligned}\right\}. \tag{16}$$

If we reduce here once, then we obtain

$$+13.086\,x'' - 40.370 = 0\,, \tag{17}$$

and with this there follow the unknowns

$$x' = +2.205'' \qquad x'' = +3.085''. \tag{18}$$

If we add these x''s and x'''s to the approximate values for Gilge and Lattenwalde, then we have the angles adjusted at the Station Nidden

$$\left.\begin{aligned} &\text{Kalleninken—Gilge} &&= 26^\circ\,14'\,50'' + x' &&= 26^\circ\,14'\,52.205'' \\ &\text{Kalleninken—Lattenwalde} &&= 87 \quad\; 4 \;\; 50 \;\; + x'' &&= 87 \quad\; 4 \;\; 53.085 \end{aligned}\right\}. \tag{19}$$

For the computation of the mean error, in addition we have to compute first the quantities of orientation z_1, z_2, z_3 from (15), for which we obtain

$$z_1 = +0.010'', \qquad z_2 = +0.174'', \qquad z_3 = -0.284'', \tag{20}$$

and with this there follow from the error equations (13) the corrections

$$\begin{aligned} &v_1^\circ = +0.010'' &\qquad &v_2^\circ = +0.174'' &\qquad &v_3^\circ = -0.284'' \\ &v_1'' = -0.009 & &v_2' = -0.174 & &v_3' = +0.275 \\ & & & & &v_3'' = +0.009\;. \end{aligned}$$

As a computational check we have $[v_1] = 0,\ [v_2] = 0,\ [v_3] = 0$.
We obtain further

$$[v\,v\,p] = 3.03\,,$$

and since $R = 7,\ n = 3,\ s = 3,$ then, according to (11),

$$m^2 = \frac{3.03}{2} = 1.52, \qquad m = \pm\,1.23''. \tag{21}$$

This is the mean error of the unit of weight, hence, the mean error of a measured direction.

The station adjustment treated in the foregoing was first taught by Bessel in *Gradmessung in Ostpreussen*, 1838, pp. 69-71, however without the computation of the mean error according to the formulae, which were not added until later.
On p. 70 of *Gradmessung in Ostpreussen* Bessel has the designations $m,\ m_1',\ m_2',$ etc., instead of our l's and

the symbols x, x_1, x_2, etc., instead of our z_1's, z_2's, z_3's; furthermore, Bessel's A's, B's, C's correspond to our x''s, x'''s, x''''s. Our z_1's, z_2's, z_3's bear a certain relation to the quantity which is denoted by z by Bessel on p. 134 of *Gradmessung in Ostpreussen*.

Section 84. Adjustment of a Net of Triangles with Incomplete Sets of Directions

From the net of the degree measurement in East Prussia we excerpt the quadrilateral with two diagonals drawn in Fig. 1 and choose from the original measurements of the degree measurement in East Prussia those which refer to this quadrilateral. (The measurements at Kalleninken are carried out with a less reliable instrument than the measurements at Nidden, Lattenwalde, and Gilge; but since only a simple computational example is involved here, we shall disregard this distinction and introduce all original measurements in the adjustment as equally justified.)

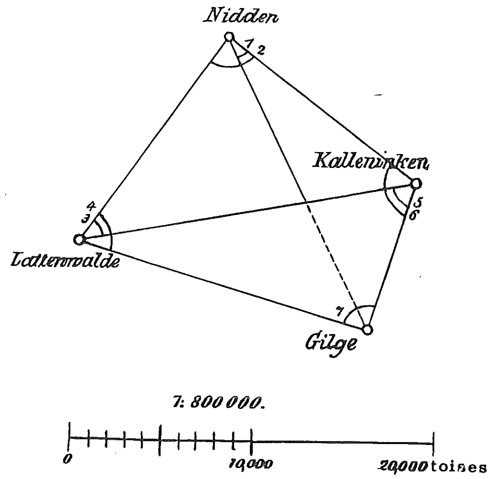


Fig. 1.
Quadrilateral of the degree measurement in East Prussia.

We have already treated the observations and the adjustment for the Station Nidden in the previous section 83; for the remaining stations we no longer give the original measurements but only the results of the station adjustments.

We will carry out the net adjustment according to the method specified at the end of section 54, p. 176, and must therefore collect the results of the station adjustments in the form of equations (13) there on p. 176.

For the Station Nidden we have, for this, according to (16) and (17), section 83, p. 312

$$\begin{aligned} x_0' + \delta x' - 0.229\,(x_0'' + \delta x'') &= 15.0 \quad + \lambda_1 \\ x_0'' + \delta x'' &= 3.085 + \lambda_2 \end{aligned}$$

or

$$\begin{aligned} x_0' + \delta x' &= 2.205 + \lambda_1 + 0.229\,\lambda_2 \\ x_0'' + \delta x'' &= 3.085 \qquad\qquad + \lambda_2\,. \end{aligned}$$

According to (12), section 83, p. 311, the final two angles of Station Nidden are then

$$\begin{aligned} \text{Kalleninken—Gilge} &= 26^\circ\,14'\,52.205'' + \lambda_1 + 0.229\,\lambda_2 \\ \text{Kalleninken—Lattenwalde} &= 87\;\;04\;\;53.085 \qquad\qquad + \lambda_2\,. \end{aligned}$$

In the same manner we have carried out the station adjustments for the other three points of our quadrilateral of Fig. 1, and we summarize the results of all four stations in the following:

Station Nidden

$$\left.\begin{aligned} \text{Kalleninken—Gilge}\;.\;.\;\; A^1 &= 26^\circ\,14'\,52.205'' + \lambda_1 + 0.229\,\lambda_2 \\ \text{Kalleninken—Lattenwalde}\;\; A^2 &= 87\;\;04\;\;53.085 \qquad\qquad + \lambda_2 \end{aligned}\right\} \qquad (1)$$

Weights: 17.5 13.0

Station Lattenwalde

$$\left.\begin{aligned} \text{Nidden—Kalleninken}\;.\;.\;\; A^3 &= 45^\circ\,25'\,23.827'' + \lambda_3 + 0.923\,\lambda_4 \\ \text{Nidden—Gilge}\;.\;.\;.\;.\;\; A^4 &= 72\;\;48\;\;58.486 \qquad\qquad + \lambda_4 \end{aligned}\right\} \qquad (2)$$

Weights: 13.0 12.0

Station Kalleninken

$$\left.\begin{array}{lll} \text{Gilge—Lattenwalde} \;\ldots & A^5 = & 62^\circ\ 58'\ 36.167'' + \lambda_5 + 0.500\,\lambda_6 \\ \text{Gilge—Nidden} \;\ldots\ldots & A^6 = & 110\ \ 28\ \ 23.667 \qquad\qquad\quad + \lambda_6 \\ & \text{Weights:} & \quad 8.0 \qquad\qquad 6.0 \end{array}\right\} \tag{3}$$

Station Gilge

$$\begin{array}{lll} \text{Lattenwalde—Kalleninken} & A^7 = & 89^\circ\ 37'\ 54.583'' + \lambda_7 \\ & \text{Weight:} & \quad 3.0\,. \end{array} \tag{4}$$

The number of condition equations is easily given according to the general rules of section 67; we have:

$W = 7$ angles,
$p = 4$ points,
$l = 6$ lines, among them $l' = 1$ line measured on one side, hence, according to (14), section 67, p. 224,
$l - 2p + 3 = 1$ side equation,
$l - l' - p + 1 = 2$ triangle equations,
$W - 2p + 4 = 3$ equations in all.

According to p. 168 of *Gradmessung in Ostpreussen* we choose as base line the side:

$$\left.\begin{array}{l} \text{Nidden—Lattenwalde} = 14\,047.7228 \text{ toises} = 27\,379.522 \text{ m} \\ \qquad\qquad (log = 4.147\,6059{\cdot}3) \quad (log = 4.437\,4258{\cdot}6) \end{array}\right\}. \tag{5}$$

As radius of curvature for the computation of the excess, on p. 253 of *Gradmessung in Ostpreussen*, Bessel takes the equatorial radius of the Earth, on which the metric system is based, namely r = 3,271,628.89 toises = 6,376,522 m (log = 6.80458 for meters), and with this:

$$\left.\begin{array}{ll} \text{For toises:} & \text{For meters:} \\ log\,\dfrac{\varrho''}{2\,r^2} = 1.98387 - 10 & log\,\dfrac{\varrho''}{2\,r^2} = 1.40423 - 10 \end{array}\right\}. \tag{6}$$

Without discussing this assumption of the radius of curvature here further, we retain the constant (6), compute the net preliminarily with the base (5) and the angles given in (1) to (4), and find the spherical excesses:

For the triangle LGK: $\varepsilon = 1.430''$
For the triangle LNK: $\varepsilon = 1.835''$.

Now we collect the angles according to the data (1) to (4) in triangles:

$$\left.\begin{array}{lrl|lrl} \text{Lattenwalde} & A^4 - A^3 = & 27^\circ\ 23'\ 34.659'' & \text{Nidden}\ \ldots & A^2 = & 87^\circ\ \ 4'\ 53.085'' \\ \text{Gilge}\ \ldots & A^7 = & 89\ \ 37\ \ 54.583 & \text{Lattenwalde} & A^3 = & 45\ \ 25\ \ 23.827 \\ \text{Kalleninken} & A^5 = & 62\ \ 58\ \ 36.167 & \text{Kalleninken} & A^6 - A^5 = & 47\ \ 29\ \ 47.500 \\ \hline & & 180^\circ\ \ 0'\ \ 5.409'' & & & 180^\circ\ \ 0'\ \ 4.412'' \\ & \text{Should be} & 180\ \ \ 0\ \ \ 1.430 & & \text{Should be} & 180\ \ \ 0\ \ \ 1.835 \\ \hline & & w = +3.979'' & & & w = +2.577'' \end{array}\right\}. \tag{7}$$

By denoting the corrections of the angles (1) to (4) for the time being, collectively, by (1), (2), . . . , (7), we obtain the condition equations corresponding to summary (7):

$$\left.\begin{array}{l} (4) - (3) + (7) + (5) + 3.979'' = 0 \\ (2) + (3) + (6) - (5) + 2.577 \ \ = 0 \end{array}\right\}. \tag{8}$$

For the formation of the side equation we can derive, for G as the central point, the side GK from GL in a twofold manner; this yields:

$$\text{We should have}\quad \frac{\sin A^4 \sin A^5 \sin A^1}{\sin (A^2 - A^1) \sin A^6 \sin (A^4 - A^3)} = 1\,,$$

which leads to the following logarithmic computation:

				Diff. for 10″
$A^4 =$	72° 48′ 58.486″	$\log \sin A^4$	9.980 1680·4	+ 65
$A^5 =$	62 58 36.167	$\log \sin A^5$	9.949 7909·0	+ 107
$A^1 =$	26 14 52.205	$\log \sin A^1$	9.645 6725·3	+ 427
		Sum Z	9.575 6314·7	
$A^2 - A^1 =$	60° 50′ 0.880″	$\log \sin (A^2 - A^1)$	9.941 1176·2	+ 117
$A^6 =$	110 28 23.667	$\log \sin A^6$	9.971 6634·1	− 78
$A^4 - A^3 =$	27 23 34.659	$\log \sin (A^4 - A^3)$	9.662 8434·6	+ 406
		Sum N	9.575 6244·9	

$$\text{Sum } Z - \text{Sum } N = w = +69{\cdot}8\,.$$

With this, the side equation will be for units of the sixth decimal of logarithms:

$$+0.65\,(4) + 1.07\,(5) + 4.27\,(1) - 1.17\,(2-1) + 0.78\,(6) - 4.06\,(4-3) + 6.98 = 0\,.$$

If we compute the coefficients with the cotangents more accurately according to (23), section 66, p. 219, and also check the absolute term according to Legendre's theorem (cf. section 68, p. 229, then the above equation with more decimals will be:

$$\left.\begin{array}{c} +0.6511\,(4) + 1.0739\,(5) + 4.2698\,(1) - 1.1751\,(2-1) + 0.7861\,(6) \\ -4.0630\,(4-3) + 7.0100 = 0 \end{array}\right\}. \tag{9}$$

If we collect by numbers, and drop a decimal here again, we obtain:

$$+5.445\,(1) - 1.175\,(2) + 4.063\,(3) - 3.412\,(4) + 1.074\,(5) + 0.786\,(6) + 7.010 = 0\,. \tag{10}$$

Now everything is prepared so far that the adjustment can be started according to section 54, p. 177. First we have to replace the corrections (1), (2) . . . by the corrections λ_1, λ_2 . . . . According to the values (1) to (4), pp. 313-314, we have for this

$$\left.\begin{array}{l} (1) = \lambda_1 + 0.229\,\lambda_2 \\ (2) = \lambda_2 \\ (3) = \lambda_3 + 0.923\,\lambda_4 \\ \;\vdots \quad\;\; \vdots \qquad\quad \vdots \end{array}\right\}, \tag{11}$$

and if we substitute this in the condition equations (8) and (10), then we obtain:

Condition equations

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	w
	.	.	−1	+0.077	+1	+0.500	+1	+3.979
	.	+1	+1	+0.923	−1	+0.500	.	+2.577
	+5.445	+0.072	+4.063	+0.338	+1.074	+1.323	.	+7.010
p	17.5	13.0	13.0	12.0	8.0	6.0	3.0	

(12)

To set up the normal equations, in order to arrive at a more convenient numerical computation, we do not use the weight reciprocals $\frac{1}{p}$, but the values $\frac{100}{p}$, which is to be taken into account in the computation of the corrections.

Normal equations

$$\left.\begin{array}{r} +57.75\,k_1 - 15.44\,k_2 - 6.59\,k_3 + 3.979 = 0 \\ -15.44\,k_1 + 39.14\,k_2 + 32.00\,k_3 + 2.577 = 0 \\ -6.59\,k_1 + 32.00\,k_2 + 340.98\,k_3 + 7.010 = 0 \end{array}\right\} . \tag{13}$$

The solution of these equations yields

$$\left.\begin{array}{l} k_1 = -\,0.09510 \\ k_2 = -\,0.09212 \\ k_3 = -\,0.01376 \end{array}\right\} , \tag{14}$$

and with the above weight reciprocals there follow therefrom the corrections

$$\left.\begin{array}{ll} \lambda_1 = -\,0.4281'' & \lambda_5 = -\,0.2220'' \\ \lambda_2 = -\,0.7162 & \lambda_6 = -\,1.8635 \\ \lambda_3 = -\,0.4071 & \lambda_7 = -\,3.1700 \\ \lambda_4 = -\,0.8083 & \end{array}\right\} . \tag{15}$$

With this, we can also give the corrections of the angles summarized in (1) to (4), pp. 313-314, which we have denoted in (11) by (1), (2) . . . . We will have

$$\left.\begin{array}{ll} (1) = -\,0.594'' & (5) = -\,1.154'' \\ (2) = -\,0.716 & (6) = -\,1.864 \\ (3) = -\,1.153 & (7) = -\,3.170 \\ (4) = -\,0.808 & \end{array}\right\} . \tag{16}$$

If we add this to the angles resulting from the station adjustment, then we have the following finally adjusted angles:

$$\left.\begin{array}{lll} \text{Nidden} & A^1 + (1) = & 26^\circ\ 14'\ 51.611'' \\ & A^2 + (2) = & 87\ \ \ 4\ \ 52.369 \\ \text{Kalleninken} & A^3 + (3) = & 45^\circ\ 25'\ 22.674'' \\ & A^4 + (4) = & 72\ \ 48\ \ 57.678 \\ \text{Lattenwalde} & A^5 + (5) = & 62^\circ\ 58'\ 35.013'' \\ & A_6 + (6) = & 110\ \ 28\ \ 21.803 \\ \text{Gilge} & A_7 + (7) = & 89^\circ\ 37'\ 51.413'' \end{array}\right\} . \tag{17}$$

With this, the net adjustment is completed. Summarizing the angles in the triangles according to (7) does not show closure errors; likewise, the logarithmic calculation of the side equation does not yield a discrepancy. In concluding the adjustment, we give in the following, further data [Abriss] for the four stations.

Data [Abriss] of the Net Adjustment to Fig. 1, p. 313

Target Point	Station Adjustment A	Net Correction (a)	Net Adjustment $A+$(a)
	1. Station Nidden		
Kalleninken	0° 0′ 0.000″		0° 0′ 0.000″
Gilge . . . . . A^1	26 14 52.205	— 0.594″	26 14 51.611
Lattenwalde . A^2	87 4 53.085	— 0.716	87 4 52.369
	2. Station Lattenwalde		
Nidden	0° 0′ 0.000″		0° 0′ 0.000″
Kalleninken . . A^3	45 25 23.827	— 1.153″	45 25 22.674
Gilge . . . . . A^4	72 48 58.486	— 0.808	72 48 57.678
	3. Station Kalleninken		
Gilge	0° 0′ 0.000″		0° 0′ 0.000″
Lattenwalde . . A^5	62 58 36.167	— 1.154″	62 58 35.013
Nidden . . . . A^6	110 28 23.667	— 1.864	110 28 21.803
	4. Station Gilge		
Lattenwalde	0° 0′ 0.000″		0° 0′ 0.000″
Kalleninken . . A^7	89 37 54.583	— 3.170″	89 37 51.413

In the earlier editions of this volume, the last being in the 7th Edition, 1920, the adjustment of the quadrilateral of Fig. 1, p. 313, has been carried out according to the method developed in sections 55-56, a method which Bessel also applied in the adjustment of the degree measurement in East Prussia. Our adjustment result shows small differences with respect to the earlier editions, which are attributed to the fact that the weights introduced by us do not completely correspond to the earlier adjustment.

We have foregone the reproduction of Bessel's adjustment as represented very completely in the earlier editions, since a net adjustment with incomplete sets of directions will hardly still be used in the practice of triangulation.

Section 85. Theory of Full Sets of Directions

We have already treated the computation of full sets of directions in section 63; it appeared that this computation in itself consists only in forming the mean from the individual readings, and the computation of the mean direction error also has easily followed. We could however not give as yet the proof for that error computation rigorously in section 63, and since otherwise, a few more things are to be said, we treat the full sets of directions here once more by regarding them as a special case of the general adjustment of directions of section 83.

If, in the case of s sights or target points, we have a number of n full sets in each set, and put all weights equal to one, then we have now

$$[p_1] = [p_2] = [p_3] = s, \qquad [p'] = [p''] = [p'''] = n. \tag{1}$$

The normal equations (8), section 83, p. 310, will therefore change to

$$\left.\begin{array}{llllllll}
s z_1 & . & . & + x' + & x'' + & x''' & - [l_1] & = 0 \\
. & s z_2 & . & + x' + & x'' + & x''' & - [l_2] & = 0 \\
. & . & s z_3 + & x' + & x'' + & x''' & - [l_3] & = 0 \\
z_1 + & z_2 + & z_3 + & n x' & . & . & - [l'] & = 0 \\
z_1 + & z_2 + & z_3 & . & + n x'' & . & - [l''] & = 0 \\
z_1 + & z_2 + & z_3 & . & . & + n x''' & - [l'''] & = 0
\end{array}\right\} . \tag{2}$$

From the first three equations there follows

$$\left.\begin{aligned} -z_1 &= \frac{x'+x''+x'''-[l_1]}{s} \\ -z_2 &= \frac{x'+x''+x'''-[l_2]}{s} \\ -z_3 &= \frac{x'+x''+x'''-[l_3]}{s} \end{aligned}\right\} ; \tag{3}$$

therefore,

$$z_1+z_2+z_3 = -\frac{n}{s}\left(x'+x''+x'''\right)+\frac{1}{s}[l] \cdot \tag{4}$$

This substituted in the last three equations (2) yields

$$\left.\begin{aligned} \left(n-\frac{n}{s}\right)x' \quad -\frac{n}{s}x'' \quad -\frac{n}{s}x''' - \left([l'] - \frac{[l]}{s}\right) &= 0 \\ -\frac{n}{s}x' + \left(n-\frac{n}{s}\right)x'' \quad -\frac{n}{s}x''' - \left([l''] - \frac{[l]}{s}\right) &= 0 \\ -\frac{n}{s}x' \quad -\frac{n}{s}x'' + \left(n-\frac{n}{s}\right)x''' - \left([l'''] - \frac{[l]}{s}\right) &= 0 \end{aligned}\right\} . \tag{5}$$

The number of these equations is equal to $s - 1$, and if we form their sum, then we obtain

$$n\,x' - (s-1)\frac{n}{s}x' + n\,x'' - (s-1)\frac{n}{s}x'' + n\,x''' - (s-1)\frac{n}{s}x'''$$
$$- [l] + \frac{s-1}{s}[l] = 0$$

or

$$\frac{n}{s}x' + \frac{n}{s}x'' + \frac{n}{s}x''' - \frac{[l]}{s} = 0 \; . \tag{6}$$

If we add this equation (6) to equations (5), then we find the values of the unknowns

$$\left.\begin{aligned} n\,x' - [l'] &= 0 \qquad & x' &= \frac{[l']}{n} \\ n\,x'' - [l''] &= 0 \qquad & x'' &= \frac{[l'']}{n} \\ n\,x''' - [l'''] &= 0\,, \qquad & x''' &= \frac{[l''']}{n} \end{aligned}\right\} , \tag{7}$$

i.e., a simple result, which agrees with the formation of the mean in section 63.

The mean square of error of a direction becomes, according to (11), section 83, p. 311, because the number of all measured directions $R = n\,s$:

$$m^2 = \frac{[v\,v]}{R-n-(s-1)} = \frac{[v\,v]}{n\,s-n-(s-1)} = \frac{[v\,v]}{(n-1)(s-1)} \cdot \tag{8}$$

We will now show that the method already found in section 63 almost without a theory corresponds to the formulae of section 83.

To this, we will follow the numerical computation of section 63 with our formulae and have then the following summary:

Target Point	Set_1	Set_2	Set_3	Mean of Directions
P^o	l_1^o	l_2^o	l_3^o	$\frac{[l^o]}{n} = 0$
P'	l_1'	l_2'	l_3'	$\frac{[l']}{n} = x'$
P''	l_1''	l_2''	l_3''	$\frac{[l'']}{n} = x''$
. .	. .	. .	. .	. . . . . .
Mean of the set	$\frac{[l_1]}{s}$	$\frac{[l_2]}{s}$	$\frac{[l_3]}{s}$	
Set shifting	$\frac{[l_1]}{s} - \frac{x'+x''\ldots}{s} = z_1$	$\frac{[l_2]}{s} - \frac{x'+x''\ldots}{s} = z_2$	$\frac{[l_3]}{s} - \frac{x'+x''\ldots}{s} = z_3$	$z_1 + z_2 + z_3 + \ldots = 0$
Shifted sets	$l_1^o - z_1$ $l_1' - z_1$ $l_1'' - z_1$	$l_2^o - z_2$ $l_2' - z_2$ $l_2'' - z_2$	$l_3^o - z_3$ $l_3' - z_3$ $l_3'' - z_3$	0 x' x'' . .
Mean of the set	$\frac{[l_1]}{s} - z_1$	$\frac{[l_2]}{s} - z_2$	$\frac{[l_3]}{s} - z_3$	
Corrections v	$v_1^o = 0 - (l_1^o - z_1)$ $v_1' = x' - (l_1' - z_1)$ $v_1'' = x'' - (l_1'' - z_1)$	$v_2^o = 0 - (l_2^o - z_2)$ $v_2' = x' - (l_2' - z_2)$ $v_2'' = x'' - (l_2'' - z_2)$	$v_3^o = 0 - (l_3^o - z_3)$ $v_3' = x' - (l_3' - z_3)$ $v_3'' = x'' - (l_3'' - z_3)$	

These v's are in agreement with the v's in (6), section 83, p. 309, only those in section 83 are inversely arranged by columns and rows in comparison to the first. If we introduce $d_1' = x' - l_1'$, etc., as in section 63, p. 208, then we will have the corrections $v_1' = d_1' - z_1$, etc., with which there follows the derivation of the squares of errors $[v\,v]$ given in (5), section 63, p. 209.

With this, we have proved theoretically the computation of the v's and $[v\,v]$'s shown in section 63 and then the computation of the mean error.

Weight computation for full sets

We obtain the weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$ . . . to the normal equations (5) according to (20), section 30, p. 97, by replacing the absolute terms of the normal equations by 1, 0, 0 . . . , i.e., we obtain from (5) the following weight equations:

$$\text{Number} = s-1 \left\{ \begin{array}{l} \left(n - \frac{n}{s}\right)[\alpha\alpha] - \frac{n}{s}[\alpha\beta] - \frac{n}{s}[\alpha\gamma] - \ldots = 1 \\ -\frac{n}{s}[\alpha\alpha] + \left(n - \frac{n}{s}\right)[\alpha\beta] - \frac{n}{s}[\alpha\gamma] - \ldots = 0 \\ -\frac{n}{s}[\alpha\alpha] - \frac{n}{s}[\alpha\beta] + \left(n - \frac{n}{s}\right)[\alpha\gamma] - \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots - \ldots = 0 \end{array} \right\}. \qquad (9)$$

In order to solve these equations, we form first their sum:

$$\frac{n}{s}[\alpha\,\alpha] + \frac{n}{s}[\alpha\,\beta] + \frac{n}{s}[\alpha\,\gamma] + \ldots = 1\,.$$

This sum equation added to each individual equation of group (9) yields:

$$\begin{array}{llll} n\,[\alpha\,\alpha] & \dots & & = 2 \\ & n\,[\alpha\,\beta] & \dots & = 1 \\ & & n\,[\alpha\,\gamma] & = 1 \end{array}$$

and since similar relations also hold for $[\beta\,\beta]$, and so on, we have now:

$$[\alpha\,\alpha] = [\beta\,\beta] = \quad [\gamma\,\gamma] = \dots \qquad = \frac{2}{n} \tag{10}$$

$$[\alpha\,\beta] = [\alpha\,\gamma] = \dots [\beta\,\gamma] = \dots \qquad = \frac{1}{n}. \tag{11}$$

The $[\alpha\alpha]$'s, $[\beta\beta]$'s, $[\gamma\gamma]$'s . . . are the weight reciprocals of the angles x', x'', x'''. . . , which the second, third . . . ray P', P'', P''' . . . form with the first ray P° (cf. Fig. 1, section 83, p. 308), and for the angle $P'\,P''$, which $= x'' - x'$, we obtain the weight reciprocal according to the theory of section 31

(with $f_1 = -1$, $f_2 = +1$, $f_3 = 0$ in (3), p. 100):

$$\frac{1}{P(x'\,x'')} = [\alpha\,\alpha] - 2\,[\alpha\,\beta] + [\beta\,\beta] = \frac{2}{n} - \frac{2}{n} + \frac{2}{n} = \frac{2}{n}. \tag{12}$$

The same holds for every angle between any two rays of a full set of directions, and, therefore, we can now express generally:

If a group of n homogeneous sets of directions with an arbitrary number of rays is contracted into *one* set by forming the mean, then every adjusted mean of directions has a weight $= n$, and every angle between two such directions has a weight $= \frac{n}{2}$, where a weight $= 1$ belongs to an original direction, or in summary:

$$\left.\begin{array}{ll} \text{Weight of an observed direction} & = 1 \\ \text{Weight of an adjusted direction} & = n \\ \text{Weight of an adjusted angle} & = \dfrac{n}{2} \end{array}\right\}. \tag{13}$$

Combining of angle measurements in directions

Angle measurements without excessive observations also can be combined, in the simplest manner, in sets of directions, which are equivalent to the measured full sets of directions in every direction.

For this, we choose, in addition to the given target points, a further easily visible target, not belonging to the net of triangles, and measure the individual angles between this auxiliary point and the target points.

If each of these measured angles has the weight p, then every angle between two arbitrary target points receives the weight $\frac{p}{2}$. Or we can now also combine all measured angles, by means of the common auxiliary target, in a set of directions, where the weight p is assigned to each direction.

Since the measurement of individual angles compared with observations in sets has great advantages, the above method offers the possibility of combining the advantages of full sets of directions with the advantages of the measurement of individual angles.

After the full sets (section 85) and the method given at the end of section 85, angle measurement in all combinations is a further important arrangement of angle measurements, because, as with the two previously mentioned, it will offer the adjustment with the form of the weights of the directions.

Already praised as ideal by Gauss and Gerling and treated theoretically by Hansen in 1871, angle measurement in all combinations was brought to new life by General Schreiber at the Land Survey, and since about 1880 forms the fundamental method of the Prussian triangulation of first order.

In section 62 already, we have treated the present problem with an example according to Gerling, and that with equal and with unequal weights. The unequal and irregularly scattered weights, however, are not desirable in this case; it is just the *equal* weights in connection with the regular distribution which give the method the flexibility and clearness, and which constitute its advantages.

With this, we now assume in Fig. 1 that we have measured homogeneously, between four rays, all six possible angles, namely:

Measured angles

$$(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4). \tag{1}$$

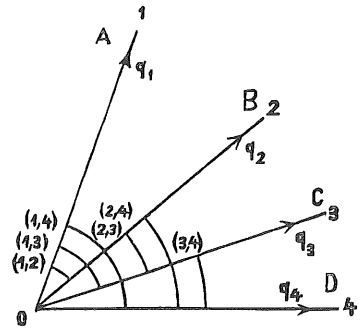


Fig. 1.

The three angles between the first ray and the three following rays, which shall now be denoted by [1,2], [1,3], [1,4] to distinguish them from the measured angles (1,2), (1,3), (1,4), shall hold as independent unknowns, and with this, we obtain the following six error equations for the six measured angles:

Error equations

	[1,2]	[1,3]	[1,4]	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
$v_{12} =$	[1,2]	. . .	. . .	— (1,2)	. . .	. . .	. . .	. . .	. . .
$v_{13} =$	. . .	[1,3]	. . .	. . .	— (1,3)	. . .	. . .	. . .	. . .
$v_{14} =$	. . .	. . .	[1,4]	. . .	. . .	— (1,4)	. . .	. . .	. . .
$v_{23} =$	— [1,2]	+ [1,3]	. . .	. . .	. . .	. . .	— (2,3)	. . .	. . .
$v_{24} =$	— [1,2]	. . .	+ [1,4]	. . .	. . .	. . .	. . .	— (2,4)	. . .
$v_{34} =$	. . .	— [1,3]	+ [1,4]	. . .	. . .	. . .	. . .	. . .	— (3,4)

(2)

Therefrom we form the

Normal equations

$$\left.\begin{array}{l} \underline{3\,[1,2]} - [1,3] - [1,4] - (1,2) \quad .. \quad .. + (2,3) + (2,4) \quad .. = 0 \\ -[1,2] + \underline{3\,[1,3]} - [1,4] \quad .. - (1,3) \quad .. - (2,3) \quad .. + (3,4) = 0 \\ -[1,2] - [1,3] + \underline{3\,[1,4]} \quad .. \quad .. - (1,4) \quad .. - (2,4) - (3,4) = 0 \end{array}\right\} . \tag{3}$$

In order to solve these normal equations, we form first their sum:

$$[1,2] + [1,3] + [1,4] - (1,2) - (1,3) - (1,4) = 0 . \tag{4}$$

This sum equation (4) added to each single one of equations (3) yields at once the solutions for the three unknowns without further elimination:

$$\text{Angle } [1,2] = \frac{2\,(1,2) + (1,3) - (2,3) + (1,4) - (2,4)}{4} \tag{5}$$

$$\text{Angle } [1,3] = \frac{2\,(1,3) + (1,4) - (3,4) + (1,2) + (2,3)}{4} \tag{6}$$

$$\text{Angle } [1,4] = \frac{2\,(1,4) + (1,2) + (2,4) + (1,3) + (3,4)}{4}\,. \tag{7}$$

We can describe these solutions in words thus: In order to compute finally any angle $[i, k]$, we take first its own direct measurement (i, k) with double the weight, and with this, with a *single* weight all those values of the angle which we can put together from the remaining angle measurements by subtraction or addition from every two other angles, and from all this we form the mean with respect to the weights.

As a numerical example for this, we will take once again the example of Gerling of section 62, p. 201, namely:

$$\left.\begin{array}{lll} (1,2) = 48^\circ\,17'\,1.4'' & (1,3) = 96^\circ\,52'\,16.8'' & (1,4) = 152^\circ\,54'\;\;6.8'' \\ & (2,3) = 48\;\;35\;\;14.3 & (2,4) = 104\;\;37\;\;\;7.8'' \\ & & (3,4) = \;\;56\;\;\;1\;\;48.9 \end{array}\right\}. \tag{8}$$

The following calculation corresponds to equations (5) to (7):

(1,2)	48° 17′ 1.4″	(1,3)	96° 52′ 16.8″	(1,4)	152° 54′ 6.8″
	17 1.4		16.8		6.8
	17 2.5		17.9		9.2
	16 59.0		15.7		5.7
	4.3		27.2		28.5 .

$$\left.\begin{array}{lll} [1,2] = 48^\circ\,17'\,1.075'' & [1,3] = 96^\circ\,52'\,16.800'' & [1,4] = 152^\circ\,54'\;\;7.125'' \\ \text{Likewise,} & [2,3] = 48\;\;35\;\;15.725 & [2,4] = 104\;\;37\;\;\;6.050 \\ & & [3,4] = \;\;56\;\;\;1\;\;50.325 \end{array}\right\}. \tag{9}$$

These adjusted [1,2]'s . . . compared with the measured (1,2)'s . . . yield the corrections v and their squares:

$$\left.\begin{array}{lll} v_{12} = -0.325'' & v_{13} = \;\;\;0.000'' & v_{14} = +0.325'' \\ & v_{23} = +1.425 & v_{24} = -1.750 \\ & & v_{34} = +1.425 \\ v^2 \ldots\; 0.0506 & 0.0000 & 0.1056 \\ & 2.0306 & 3.0625 \\ & & 2.0306 \\ \multicolumn{3}{c}{[v\,v] = 7.2799\,,\quad m = \sqrt{\dfrac{7.2799}{6-3}} = \pm 1.56''} \end{array}\right\}. \tag{10}$$

This m is the mean error of a measured angle (before the adjustment).

We will go one step further, and bring the calculation in the form of *directions* by introducing, in addition, (2,1) = – (1,2), (3,1) = – (1,3), etc., so that equation (5) assumes the following form:

$$\text{Angle } [1,2] = \frac{(1,2) + (1,3) + (1,4)}{4} - \frac{(2,1) + (2,3) + (2,4)}{4}\,.$$

The adjusted directions (or, as the case may be, their corrections) are denoted, according to Fig. 1, by A, B, C . . . ; then we have for three rays:

$$\left.\begin{aligned}\text{Direction } A &= -\frac{(1,2)+(1,3)+(1,4)+\ldots}{s}\\ \text{Direction } B &= -\frac{(2,1)+(2,3)+(2,4)+\ldots}{s}\\ \text{Direction } C &= -\frac{(3,1)+(3,2)+(3,4)+\ldots}{s}\\ \ldots\ldots \text{etc.},& \ldots\ldots\end{aligned}\right\};\qquad(11)$$

and we only need to subtract $A, B, C \ldots$, computed according to this, from one another, like directions measured independently in *one* set, in order to obtain all adjusted angles; or we can also compute further with the directions computed according to formulae (11) as if they were measured directions.

Adjustment according to directions

The conversion from the angle measurement in all combinations to a full set of directions can also be accomplished directly by the adjustment. We have in view here, from the outset, directions whose adjusted values shall be denoted by A, B, C, D according to Fig. 1, p. 321. Let the measured angles be (1,2), (1,3), etc., as before.

The error equations assume then the following form instead of (1):

$$\left.\begin{aligned}v_{12} &= -A + B \ldots - (1,2) \ldots\\ v_{13} &= -A \ldots + C \ldots - (1,3) \ldots\\ v_{14} &= -A \ldots + D \ldots - (1,4) \ldots\\ v_{23} &= \ldots - B + C \ldots - (2,3) \ldots\\ v_{24} &= \ldots - B \ldots + D \ldots - (2,4) \ldots\\ v_{34} &= \ldots - C + D \ldots - (3,4) \ldots\end{aligned}\right\}.\qquad(12)$$

The normal equations to this read:

$$\left.\begin{aligned}3A - B - C - D + (1,2) + (1,3) + (1,4) &= 0\\ -A + 3B - C - D + (2,1) + (2,3) + (2,4) &= 0\\ -A - B + 3C - D + (3,1) + (3,2) + (3,4) &= 0\\ -A - B - C + 3D + (4,1) + (4,2) + (4,3) &= 0\end{aligned}\right\}.\qquad(13)$$

These normal equations, however, are not independent of one another, for the sum of every three equations yields always the fourth equation. Equations (13) are therefore not sufficient for the determination of all four unknowns; and we must either assume an arbitrary value for one unknown or introduce an arbitrary condition between the four unknowns. We will use for this the condition

$$A + B + C + D = 0\qquad(14)$$

with which the normal equations (13) change then to

$$\left.\begin{aligned}4A + (1,2) + (1,3) + (1,4) &= 0\\ 4B + (2,1) + (2,3) + (2,4) &= 0\\ 4C + (3,1) + (3,2) + (3,4) &= 0\\ 4D + (4,1) + (4,2) + (4,3) &= 0\end{aligned}\right\},\qquad(15)$$

which agrees again with equations (11). We will show the application of formulae (11) or, as the case may be, (15) by a rather large example with five sighting rays, and, in fact, with Station Keulenberg of the Markish-Silesian chain in the work, *Die Kgl. Preuss. Landestriangulation*, Hauptdreiecke, Part II, Berlin, 1874, pp. 357-361.

If we contract all values given there by forming the mean, then we obtain:

I. Station Keulenberg. Ten Angle Measurements

	1. Collm	2. Strauch	3. Brautberg	4. Brandberg	5. Hochstein
1.	0° 0′ 0.000″	25° 54′ 20.367″	76° 40′ 49.008″	127° 17′ 42.100″	175° 20′ 21.075″
2.		25 54 20.000	76 40 47.658	127 17 42.958	175 20 20.392
3.			76 40 49.000	127 17 43.100	175 20 21.725
4.				127 17 42.000	175 20 20.667
5.					

(16)

These are ten *angles*, which are put together, however, in the *form* of 14 directions, since the five values of the first line were obtained from four angles, all of which have the left leg Collm = 0° 0′ 0″ in common, and likewise, in the second line three angles, all of which have the left leg Strauch = 25° 54′ 20″ in common.

From the above table we form the following second table with the separation of approximate values:

II. Station Keulenberg. Adjustment of Ten Angles in the Form of Directions

	1. 0° 0′0″	2. 25° 54′ 20″	3. 76° 40′ 49″	4. 127° 17′ 42″	5. 175° 20′ 21″	Transverse Sum
1.	0.000″	+0.367″	+0.008″	+0.100″	+0.075″	+0.550″
2.	−0.367	0.000	−1.342	+0.958	−0.608	−1.359
3.	−0.008	+1.342	0.000	+1.100	+0.725	+3.159
4.	−0.100	−0.958	−1.100	0.000	−0.333	−2.491
5.	−0.075	+0.608	−0.725	+0.333	0.000	+0.141
Sums S	−0.550″	+1.359″	−3.159″	+2.491″	−0.141″	
Mean	$A = -0.110$	$B = +0.272$	$C = -0.632$	$D = +0.498$	$E = -0.028$	
Adjusted direction	359° 59′ 59.890″	25° 54′ 20.272″	76° 40′ 48.368″	127° 17′ 42.498″	175° 20′ 20.972″	
or	0 0 0.000	25 54 20.382	76 40 48.478	127 17 42.608	175 20 21.082	

(17)

The last line has resulted from the preceding one by adding 0.110″, in order to bring the starting direction again to 0° 0′ 0″.

This adjustment (16) to (17) in a somewhat different form offers the same as first given in the treatise by General Schreiber in *Zeitschrift für Vermessungswesen*, 1878, pp. 220 and 232. Our sums S equal $s - \sigma$ of that treatise, p. 220.

In order to determine also the mean error, we have, in any case, the sure way of returning to the actual ten angle measurements, and comparing them with the adjusted angles (which we obtain by subtractions from (17)). We will do this in the following by denoting again by (1,2) a measured angle and by [1,2] an adjusted angle:

(1,2) = 25° 54′ 20.367″
[1,2] = 25 54 20.382
v_{12} = +0.015″

(1,3) = 76° 40′ 49.008″
[1,3] = 76 40 48.478
v_{13} = −0.530″

(1,4) = 127° 17′ 42.100″
[1,4] = 127 17 42.608
v_{14} = +0.508″

(1,5) = 175° 20′ 21.075″
[1,5] = 175 20 21.082
v_{15} = +0.007″

(2,3) = 50° 46′ 27.658″
[2,3] = 50 46 28.096
v_{23} = +0.438″

(2,4) = 101° 22′ 22.958″
[2,4] = 101 22 22.226
v_{24} = −0.732″

(2,5) = 49° 26′ 0.392″
[2,5] = 49 26 0.700
v_{25} = +0.308″

(3,4) = 50° 36′ 54.100″
[3,4] = 50 36 54.130
v_{34} = +0.030″

(3,5) = 98° 39′ 32.725″
[3,5] = 98 39 32.604
v_{35} = −0.121″

(4,5) = 48° 2′ 38.667″
[4,5] = 48 2 38.474
v_{45} = −0.193″

v	v^2
+0.015	0.0002
−0.530	0.2809
+0.508	0.2581
+0.007	0.0000
+0.438	0.1918
−0.732	0.5358
+0.308	0.0949
+0.030	0.0009
−0.121	0.0146
−0.193	0.0372
$[vv] =$	1.4144

$$m = \sqrt{\frac{1.4144}{10-4}} = \pm 0.486''. \tag{18}$$

This m is the mean error of a measured angle (mean angle error before the adjustment).

Besides this computation (18) which presents itself directly, we can also carry out a computation of $[v\,v]$ from the reduced normal equations (15) by following formula (8), p. 91. Since all terms $[a\,b]$ $[a\,c]$, etc., are equal to zero, then this formula reads in the case of four directions:

$$[v\,v] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \frac{[b\,l]^2}{[b\,b]} - \frac{[c\,l]^2}{[c\,c]} - \frac{[d\,l]^2}{[d\,d]}. \tag{19}$$

However,

$$[a\,l]^2 = 16\,A^2 \quad [b\,l]^2 = 16\,B^2 \quad \ldots \quad [d\,l]^2 = 16\,D^2$$
$$[a\,a] = [b\,b] \quad \ldots \quad [d\,d] = 4\,,$$

hence,

$$[v\,v] = (1,2)^2 + (1,3)^2 + \quad \ldots \quad + (3,4)^2 - 4\,(A^2 + B^2 + C^2 + D^2)\,. \tag{20}$$

The calculation yields for the above example with five directions:

l	l^2		
$+0.367$	0.1347	$A = -0.110$	$A^2 = 0.01210$
$+0.008$	0.0001	$B = +0.272$	$B^2 = 0.07398$
$+0.100$	0.0100	$C = -0.632$	$C^2 = 0.39942$
$+0.075$	0.0056	$D = +0.498$	$D^2 = 0.24800$
-1.342	1.8010	$E = -0.028$	$E^2 = 0.00078$
$+0.958$	0.9178		0.73428
-0.608	0.3697		
$+1.100$	1.2100		
$+0.725$	0.5256		
$+0.333$	0.1109		
	5.0854		

$$[v\,v] = 5.0854 - 5 \cdot 0.73428 = 5.0854 - 3.6714 = 1.4140\,. \tag{21}$$

This agrees sufficiently with 1.4144, hence, the mean error of a measured angle is also again

$$m = \sqrt{\frac{1.4140}{10-4}} = \pm\,0.486''\,. \tag{22}$$

The denominator $10 - 4 = 6$ holds here for five rays; in the general case with s rays, the number of the angles in all combinations $= \frac{s\,(s-1)}{2}$ and the number of the independent angles $= s - 1$, hence, according to (19), section 29, p. 93:

$$n - u = \frac{s\,(s-1)}{2} - (s-1) = \frac{(s-1)\,(s-2)}{2}\,,$$

therefore, the mean error of a measured angle is:

$$m = \sqrt{\frac{2\,[v\,v]}{(s-1)\,(s-2)}}\,, \tag{23}$$

and the mean error of a measured direction is:

$$\mu = \frac{m}{\sqrt{2}} \sqrt{\frac{[v\,v]}{(s-1)(s-2)}}. \tag{24}$$

The calculation (21) corresponds to *Kgl. Preuss. Landestriangulation*, Part II, 1874, p. 361, from which we have borrowed our example; for we read there:

$$(V\,V) = (V_0\,V_0) - \frac{(a\,n)^2}{(a\,a)} - \frac{(b\,n)^2}{(b\,b)} - \ldots = 82.05 - 22.01 = 60.04 \text{ with divisor} = 116.$$

This is not to be compared directly with our equations (21) and (22), because in *Landestriangulation*, pp. 357-360, 120 individual angles (each composed of l and r), which we contracted in our table (16) to ten *mean* values and also used further only as ten angles, were utilized. That "Divisor = 116" resulted from 120-4, corresponding to our 10-4. The result -22.01 is further also in close relation to our 18.3537 in (21), for in *Landestriangulation*, p. 360, $(a\,l) = -3.30$, $(a\,a) = 30$, etc., from which $\frac{3.30^2 + 8.15^2 + \ldots}{30} = \frac{660.63}{30} = 22.021$ or six times our result 3.6707.

The example used in the foregoing, Station Keulenberg, observer Schreiber, May to June 1872 (*Preussische Landestriangulation*, Part II, 2. Abt., 1874, pp. 357-361) originates from the time of the first introduction of angle measurements in all combinations and, in fact, with 12 repetitions of each individual angle in telescope position I or II. This number of repetitions, 12 in the case of $s = 5$ directions, however, does not correspond as yet to Schreiber's method, which was further developed later, and which takes only ten repetitions in the case of five directions, as will be shown in our later section 88.

Weight determination

From the normal equations (3), p. 321, which refer to angles, we obtain the following weight equations according to (20), section 30, p. 97:

$$\left.\begin{aligned} 3[\alpha\,\alpha] - [\alpha\,\beta] - [\alpha\,\gamma] &= 1 \\ -[\alpha\,\alpha] + 3[\alpha\,\beta] - [\alpha\,\gamma] &= 0 \\ -[\alpha\,\alpha] - [\alpha\,\beta] + 3[\alpha\,\gamma] &= 0 \end{aligned}\right\}. \tag{25}$$

In order to solve these equations, we form again their sum:

$$[\alpha\alpha] + [\alpha\beta] + [\alpha\gamma] = 1.$$

If we add this to each one of the three equations (25), then we obtain the values of $[\alpha\alpha]$, $[\alpha\beta]$, and $[\alpha\gamma]$, and if we carry out the same for the other two systems of weight equations, then we have

$$\left.\begin{aligned} [\alpha\,\alpha] = \frac{2}{4}\,, \quad [\alpha\,\beta] = \frac{1}{4}\,, \quad [\alpha\,\gamma] = \frac{1}{4} \\ [\beta\,\beta] = \frac{2}{4}\,, \quad [\beta\,\gamma] = \frac{1}{4} \\ , \quad [\gamma\,\gamma] = \frac{2}{4} \end{aligned}\right\}, \tag{26}$$

but this holds first only for the special case of four sighting rays; generally, for s rays we obtain:

$$\text{and} \quad \left.\begin{aligned} [\alpha\,\alpha] = \frac{2}{s}\,, \quad [\alpha\,\beta] = \frac{1}{s}\,, \quad [\alpha\,\gamma] = \frac{1}{s}\,, \quad [\alpha\,\delta] = \frac{1}{s} \\ [\alpha\,\beta] = \frac{1}{s}\,, \quad [\beta\,\beta] = \frac{2}{s}\,, \quad [\beta\,\gamma] = \frac{1}{s}\,, \quad [\beta\,\delta] = \frac{1}{s} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,, \text{ etc.} \end{aligned}\right\}. \tag{27}$$

The $[\alpha\alpha]$'s, $[\beta\beta]$'s . . . are the reciprocal weights of the unknowns [1,2], [1,3], [1,4] . . . after the adjustment. Therefore, the weight of these angles $P = \frac{s}{2}$, hence, in the small example of Gerling (8) to (10) with $s = 4$ rays $P = 2$, consequently, the mean error of an angle after the adjustment, M, is $m : \sqrt{2} = \pm 1.10''$.

A further question is now what weights are assigned to the angles [2,3], [2,4] . . . , hence, to the differences of every two unknowns after the adjustment. For this, we have to use the formulae (5) and (6), section 31, pp. 100-101, according to which we have for the case of four rays in accordance with (26):

$$\frac{1}{P} = \frac{2}{4}(f_1^2 + f_2^2 + f_3^2 + f_1 f_2 + f_1 f_3 + f_2 f_3),$$

from which we find by simple transformations:

$$\frac{1}{P} = \frac{1}{4}(f_1^2 + f_2^2 + f_3^2) + \frac{1}{4}(f_1 + f_2 + f_3)^2. \tag{28}$$

Now since

$$[2,3] = -[1,2] + [1,3]$$
$$[2,4] = -[1,2] + [1,4] \text{ , etc.,}$$

then, in any case, the first expression in parentheses in (28) is equal to 2, while the second expression in parentheses is equal to zero. We have, therefore, for any arbitrary angle between the four rays:

$$\frac{1}{P} = \frac{2}{4} \qquad P = 2$$

and, generally, in the case of s rays

$$P = \frac{s}{2}.$$

All angles between the s rays have thus the same weight $P = \frac{s}{2}$, as if the angles have resulted from the directions of a complete set of directions. Consequently, we can also split up the weights of angles into equal weights of directions; for, if q_1, q_2, q_3 . . . are the weights of directions after the adjustment, then it must be true that

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q_3} = \ldots = \frac{1}{P} = \frac{2}{s},$$

therefore,

$$\frac{1}{q_1} = \frac{1}{q_2} = \ldots = \frac{1}{s}$$
$$q_1 = q_2 = \ldots = s. \tag{29}$$

The weight of a measured angle is assumed here as unity; if we take as unity a direction weight which is equal to double the angle weight, then the P's and the q's become after the adjustment half the previous ones. We will summarize all this in an easily seen manner:

Between s rays there are measured the angles in all $\frac{s(s-1)}{2}$ combinations, where $(s-1)$ independent angles or s directions are determined.

If a measured *angle* has the weight $=1$, then every angle after the adjustment receives the weight $P = \frac{s}{2}$ or every adjusted direction the weight $q = s$ (e.g. $s = 2$ yields $P = 1$ and $q = 2$).

If a measured *direction* has the weight $=1$, hence, a measured angle the weight $= \frac{1}{2}$, then every angle after the adjustment receives the weight $P = \frac{s}{4}$ and every adjusted direction the weight $q = \frac{s}{2}$ (e.g., $s = 2$ yields $P = \frac{1}{2}$ and $q = 1$, as must be).

The mean error of a unit of weight is always computed according to the formulae (23) or (24) already given earlier, for which $[v\,v]$ is to be determined according to example (18) or (21).

Angle measurement in all combinations at several eccentric base points

If, due to unfavorable position of the target points, several eccentric base points must be used at a station, then the angle measurement in all combinations is not practicable without further consideration; e.g., if we assume that in Fig. 2 there exist the two base points I and II, then the angles (1,3) and (1,4) cannot be measured directly, because the directions 1, 3, and 4 are not visible from a single base point. In this case, we can however take recourse to target 2 by measuring the two angles (1,2) and (2,3) instead of angle (1,3). In order to obtain the weight 1 for all angles, the measurements are to be distributed in the following manner:

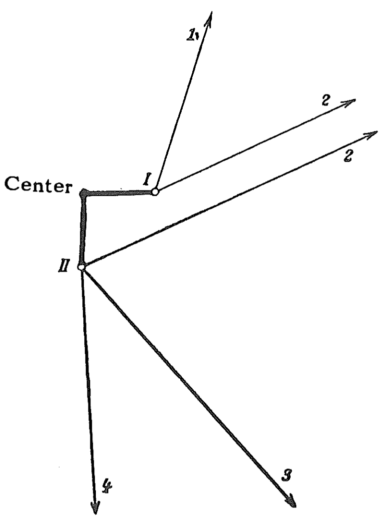


Fig. 2.

		Number of Measurements
(1,2)	directly	1
(1,3) =	(1,2)	2
	+ (2,3)	2
(1,4) =	(1,2)	2
	+ (2,4)	2
(2,3)	directly	1
(2,4)	directly	1
(3,4)	directly	1

In the case of the measurement of the sums of angles (1,2) + (2,3) and (1,2) + (2,4), target 2 only plays the role of an auxiliary target, which can also be replaced by any other target visible from I and II; in some cases, the inclusion of such a special auxiliary target may not be avoidable at all.

In unfavorable conditions, the combination of an angle of 3 or 4 partial angles can also become necessary, whereby the angle measurement in all combinations, however, becomes then very clumsy.

Angle measurement in symmetrical arrangement

Not only in the case of angle measurements in all combinations, i.e. combinations of the rays by *twos*, can we represent the adjustment results of the station in the form of directions with weights of directions, but also in the more general case of the station observations in some sort of symmetrical arrangement.

This case has been treated by Vogler and Helmert in *Zeitschrift für Vermessungswesen*, 1885, pp. 49-59 and 263-266. Here, also, as in angle measurements with two rays each, we find a splitting up of the weight equations into individual weights. If we observe p rays in symmetrical sets of i directions each, hence in $\binom{p}{i}$ combinations, then the weight of an adjusted angle $P = \frac{p}{2i}\binom{p-2}{i-2}$, or if the number of the rays is set $= s$, then $P = \frac{s}{2i}\binom{s-2}{i-2}$, therefore, with $i = 2$, we will have $\binom{s-2}{i-2} = 1$ and $P = \frac{s}{4}$. The weight of a measured direction is set $= 1$ here.

If we would set the weight of a measured *angle* $=1$, then the weight of an adjusted angle would become $P = \frac{s}{i}\left(\frac{s-2}{i-2}\right)$, hence, with $i=2$ we would have $P = \frac{s}{2}$, which agrees with our result of p. 328.

Section 87. Angle Measurement in All Combinations with Forcing Connections

In the adjustment of free triangulation nets, with which we mainly deal in this chapter, forcing connections occur only exceptionally. (The chain of the Elbe River to be treated in the later section 89 offers a practical example for this.)

However, in order to have everything together which refers to angle measurements in all combinations, we will also treat here the forcing connections with two fixed rays, according to Fig. 1, while we assume here that OA and OD are two *fixed* rays, invariably determined from older measurements and adjustments, to which two new rays OB and OC are to be connected by angle measurements.

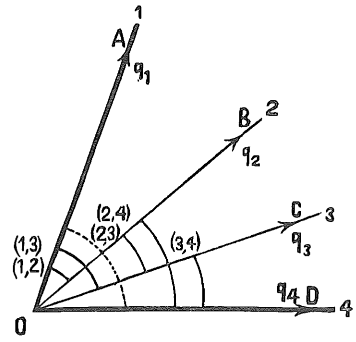


Fig. 1.

According to the rules of General Schreiber (*Zeitschrift für Vermessungswesen*, 1878, p. 217) two such fixed connecting rays are to be regarded, with respect to the arrangement of the observations, as *one* single direction, and every new direction to be determined is to be connected just as often with one as with the other connecting direction.

Let there be measured: Angle (1,2) with the weight 1
Angle (1,3) with the weight 1
Angle (1,4) with the weight 0, i.e *not* measured
Angle (2,3) with the weight 2
Angle (2,4) with the weight 1
Angle (3,4) with the weight 1.

In the case of this arrangement, the whole treatment can be reduced to the case of *one* fixed ray A, i.e., to the case of completely free angle measurement between three rays A, B, C by reducing the angle (2,4) to $W-(2,4)=(1,2)$ and, likewise, $W-(3,4)=(1,3)$ if an angle $AOD = W$ is given.

We will test this at once by an example, and, in fact, from *Vermessung der freien Hansestadt Bremen*, by G e i s s l e r, 1890, pp. 18-19, Station Weyerberg, where two fixed rays and two new rays exist, namely:

1. Bremen	$A = 207°\ 40'\ 32.91''$	fixed
2. Oberblockland	B	
3. Scharmbeck	C	
4. Garlstedt	$D = 291°\ \ 6'\ \ 9.47''$	fixed

$$[1,4] = W = D - A = 83°\,25'\,36.56''.$$

Now we summarize the five measured angles in addition to the fixed angle [1,4] in the same form as if [1,4] were also measured:

$$(1,2) = 4°\,53'\,20.18'' \qquad (1,3) = 71°\,38'\,27.73'' \qquad [1,4] = 83°\,25'\,36.56'' = W$$

$$\begin{cases}(2,3) = 66°\,45'\,\ 8.99''\\ (2,3) = 66\ \ 45\ \ \ 8.99\end{cases} \qquad (2,4) = 78°\,32'\,15.69''$$

$$(3,4) = 11\ \ 47\ \ \ 9.22\ .$$

According to the given fundamental theorem of the reduction from 4 to 1 we obtain:

$$\begin{array}{rl} (1,2) = & 4°\,53'\,20.18'' \\ W-(2,4) = & 4\ \ 53\ \ 20.87 \\ \hline (1,2)' = & 4°\,53'\,20.525'' \end{array} \qquad \begin{array}{rl} (1,3) = & 71°\,38'\,27.73'' \\ W-(3,4) = & 71\ \ 38\ \ 27.34 \\ \hline (1,3)' = & 71°\,38'\,27.535'' \\ (2,3)' = & 66\ \ 45\ \ \ 8.990\,. \end{array}$$

These three angles (1,2)', (1,3)', and (2,3)' are now adjusted among themselves as usual:

$$\begin{array}{rl} (1,2)' = & 4^\circ\, 53'\, 20.525'' \\ & 20.525 \\ (1,3)' - (2,3)' = & 18.545 \\ \hline \text{Mean } [1,2] = & 4^\circ\, 53'\, 19.865'' \end{array} \qquad \begin{array}{rl} (1,3)' = & 71^\circ\, 38'\, 27.535'' \\ & 27.535 \\ (1,2)' + (2,3)' = & 29.515 \\ \hline [1,3] = & 71^\circ\, 38'\, 28.195'' \end{array}$$

$$\begin{array}{rl} (2,3)' = & 66^\circ\, 45'\, 8.990'' \\ & 8.990 \\ (1,3)' - (1,2)' = & 7.010 \\ \hline \text{Mean } [2,3] = & 66^\circ\, 45'\, 8.330''\,. \end{array}$$

These adjusted angles agree with the values obtained by Geisler on p. 19 of the *Triangulation of Bremen* by another method, and by connecting these angles to the fixed directions we have the final result:

Station Weyerberg

1.	Bremen	$A =$ 207°	40'	32.91"	fixed
2.	Oberblockland	$B =$ 212	33	52.77	intercalated
3.	Scharmbeck	$C =$ 279	19	1.10	intercalated
4.	Garlstedt	$D =$ 291	6	9.47	fixed.

These are measuring data in the form of oriented direction angles; the two new directions B and C enter in a net adjustment or an adjustment of coordinates as observed.

In the case of an *angle measurement* at a station with connections to an arbitrary number of fixed rays, we can always reduce the angle adjustment and the error computation, in short everything which refers to such a station, by means of the given formal device, to the case of *one* fixed ray, or to the case of completely free rays, which is the same here, so that it is not necessary to set up special formulae for the adjustment of station angles with forcing connections to fixed rays.

Section 88. Triangulation According to Schreiber's Method

After we have recognized in section 86, p. 328, that angle measurement in all combinations can be represented after the station adjustment as *one* set of direction measurements with a certain weight for every adjusted direction, we can also understand that a whole triangulation net with all stations so treated can simply be adjusted according to directions, similarly as has been shown previously by the two small examples of sections 68 and 70.

The only difference consists in the fact that it is true that all adjusted directions of the *same* station are equally weighted, but that *different* stations with a different number of sights yield unequally weighted directions, because the weight after the adjustment depends on the number of sights themselves. General Schreiber took this circumstance into account by repeating the combination measurements several times, and the repetition numbers n of the stations were so apportioned that the direction weights, finally appearing, become as equal as possible, as the following summary table shows:

Number of Rays of a Station s	Without Repetition			Repetition No. $2n$	With $2n$ Times Repetition	
	No. of Angles $s\frac{s-1}{2}$	Dir. Weight for Measurement	Dir. Weight for Adjustment		Angle Weight $p=\frac{n\,s}{2}$	Direction Weight $q=2p=n\,s$
2	1	1	1.0	24	12	24
3	3	1	1.5	16	12	24
4	6	1	2.0	12	12	24
5	10	1	2.5	10	12.5	25
6	15	1	3.0	8	12	24
7	21	1	3.5	8	14	28
8	28	1	4.0	6	12	24

As the measurement for the unit of weight $p = 1$ we count here the measurement of a direction in *one* telescope position I or II, or the measurement of an angle in two telescope positions I and II, which means the same. The specified repetition number $2n$ is to be understood in such a way that n measurements of the angle in telescope position I and n measurements in telescope position II must be carried out. The angles are measured here in each telescope position just as often from left to right as from right to left.

The measurement of an angle back and forth is carried out at the same point on the circle; moreover, the repetitions for the elimination of the systematic division errors are made at different positions on the circle, and, in fact, for the n repetitions, with two microscopes, the circle is turned always by $\frac{360^\circ}{2n}$ or $\frac{180^\circ}{n} = d$ (with 4 microscopes by $\frac{360^\circ}{4n}$), so that the positions on the circle $0, d, 2d, 3d \ldots$ occur.

A further question is whether we will assume the same starting position of the circle for each of the $s\frac{s-1}{2}$ angles. In order to decrease the influence of the irregular division errors it is advisable to choose the starting positions of the circle for the individual angles in such a way that every direction is read only once at one and the same position of the circle.

For instance, if we have $s = 4$, therefore, according to the above table $n = 6$, and then the first angle 1,2 could be measured in the following positions of the circle.

Position of the telescope:	I	I	I	II	II	II
Position of the circle:	0°	30°	60°	90°	120°	150°

always once back and forth. In order to avoid a repeated reading of one of the four directions in these positions of the circle upon measuring of the further angles, we could divide the interval 30° once again by the number of the angles, therefore, form in the present case $\frac{30^\circ}{6} = 5^\circ$, and now assume, for every further angle, the starting position of the circle as 5° larger.

The same result, however, is also obtained if we group the 6 angles as follows:

1,2	1,3	1,4
3,4	2,4	2,3

and choose a new starting position of the circle for each of these three groups. We will thus begin for the angles 1,2 and 3,4 with the position of the circle at 0°, for the angles 1,3 and 2,4 with 10°, and for 1,4 and 2,3 with 20°

In the general case with s directions, we will so group the $s\frac{s-1}{2}$ angles that in each group a direction does not occur more than once and that the number of the groups is as small as possible. We find easily that with an even number of s the number of groups is equal to $s - 1$ and with uneven s the number of groups is equal to s. Therefore, the starting positions of the circle must be twisted for each angle in the case of even s by $\delta = \frac{d}{s-1} = \frac{180^\circ}{n(s-1)}$ and in the case of uneven s by $\delta = \frac{d}{s} = \frac{180^\circ}{ns}$.

According to these principles there have been set up the following tables (from a communication by General Schreiber in *Zeitschrift für Vermessungswesen*, 1878, pp. 215-217, and *Kgl. Preussische Landestriangulation*, Part IV, Die Elbkette, 1891, pp. 54-55). I and II mean here the positions of the telescope and 0°, 15°, 30° . . . the positions of the circle.

$s = 2$ directions, $s\frac{s-1}{2} = 1$ angle, $n(s-1) = 12$, $d = \frac{180^\circ}{12} = 15^\circ$, $\delta = \frac{180^\circ}{12} = 15^\circ$
$n = 12$

Angle	I	I	I	I	I	I	II	II	II	II	II	II
1,2	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°

$s=3$ directions, $s\frac{s-1}{2}=3$ angles, $ns=24$, $d=\frac{180°}{8}=22.5°$, $\delta=\frac{180°}{24}=7,5°$
$n=8$

Angle	I	I	I	I	II	II	II	II
1,2	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
1,3	7.5	30	52.5	75	97.5	120	142.5	165
2,3	15	37.5	60	82.5	105	127.5	150	172.5

$s=4$ directions, $s\frac{s-1}{2}=6$ angles, $n(s-1)=18$
$n=6$

Angle	I	I	I	II	II	II
1,2	0°	30°	60°	90°	120°	150°
1,3	10	40	70	100	130	160
1,4	20	50	80	110	140	170
2,3	20	50	80	110	140	170
2,4	10	40	70	100	130	160
3,4	0	30	60	90	120	150

$d=\frac{180°}{6}=30°$

$\delta=\frac{180°}{18}=10°$

$s=5$ directions, $s\frac{s-1}{2}=10$ angles, $ns=25$
$n=5$

Angle	I	I	I	II	II	II
1,2	0°	36°	72°		108°	144°
1,3	7.2	43.2	79.2		115.2	151.2
1,4	14.4	50.4	86.4		122.4	158.4
1,5	21.6	57.6	93.6		129.6	165.6
2,3	14.4	50.4	86.4		122.4	158.4
2,4	21.6	57.6		93.6°	129.6	165.6
2,5	28.8	64.8		100.8	136.8	172.8
3,4	28.8	64.8		100.8	136.8	172.8
3,5	0	36		72	108	144
4,5	7.2	43.2		79.2	115.2	151.2

$d=\frac{180°}{5}=36°$

$\delta=\frac{180°}{25}=7.2°$

To the table for $s = 5$ we will note that the earlier example of section 86, p. 324, Station Keulenberg, has also $s = 5$ rays, but $n = 6$, not $n = 5$, repetitions; i.e., the former example Keulenberg of 1872 was, to be sure, arranged according to the new Schreiber method with respect to the angle measurements in all combinations, but not yet with respect to the repetition numbers.

$s=6$ directions, $s\frac{s-1}{2}=15$ angles
$n=4$ " $d=\frac{180°}{4}=45°$
$n(s-1)=20$ $\delta=\frac{180°}{20}=9°$

Angle	I	I	II	II
1,2	0°	45°	90°	135°
1,3	9	54	99	144
1,4	18	63	108	153
1,5	27	72	117	162
1,6	36	81	126	171
2,3	36	81	126	171
2,4	27	72	117	162
2,5	9	54	99	144
2,6	18	63	108	153
3,4	0	45	90	135
3,5	18	63	108	153
3,6	27	72	117	162
4,5	36	81	126	171
4,6	9	54	99	144
5,6	0	45	90	135

$s=7$ directions, $s\frac{s-1}{2}=21$ angles
$n=4$ " $d=\frac{180°}{4}=45°$
$ns=28$ $\delta=\frac{180°}{28}=6.4286°$

Angle	I	I	II	II
1,2	0.0°	45.0°	90.0°	135.0°
1,3	6.4	51.4	96.4	141.4
1,4	12.9	57.9	102.9	147.9
1,5	19.3	64.3	109.3	154.3
1,6	25.7	70.7	115.7	160.7
1,7	32.1	77.1	122.1	167.1
2,3	12.9	57.9	102.9	147.9
2,4	19.3	64.3	109.3	154.3
2,5	25.7	70.7	115.7	160.7
2,6	32.1	77.1	122.1	167.1
2,7	38.6	83.6	128.6	173.6
3,4	25.7	70.7	115.7	160.7
3,5	32.1	77.1	122.1	167.1
3,6	38.6	83.6	128.6	173.6
3,7	0.0	45.0	90.0	135.0
4,5	38.6	83.6	128.6	173.6
4,6	0.0	45.0	90.0	135.0
4,7	6.4	51.4	96.4	141.4
5,6	6.4	51.4	96.4	141.4
5,7	12.9	57.9	102.9	147.9
6,7	19.3	64.3	109.3	154.3

These tables on the preceding pages hold for free stations; for stations with forced connections, however, we proceed according to section 87, as the following example shows, with two fixed directions 1 and 2 and three new directions 3, 4, and 5.

This small table is to be read as follows:

Angle 1,3 is measured:
in the circle position 0°, forward and backward in telescope position I,
in the circle position 60°, forward in I, backward in II,
in the circle position 120°, forward and backward in II.

Angle 2,3 is measured:
in the circle position 30°, forward and backward in telescope position I,
in the circle position 90°, forward in I, backward in II,
in the circle position 150°, forward and backward in II, etc.

Angle	I	I	I. II	I. II	II	II
1,3	0°	. .	60°	. .	120°	. .
2,3	. .	30°	. .	90°	. .	150°
1,4	10	. .	70	. .	130	. .
2,4	. .	40	. .	100	. .	160
1,5	20	. .	80	. .	140	. .
2,5	. .	50	. .	110	. .	170
	I	I	I	II	II	II
3,4	20°	50°	80°	110°	140	170°
3,5	10	40	70	100	130	160
4,5	0	30	60	90	120	150

Since every angle must altogether be measured as often in telescope position I as in telescope position II, then, with an odd number of the circle positions it cannot be done otherwise than to measure at least in *one* position of the circle forward and backward in various positions of the telescope.

If the angle measurements are arranged and adjusted according to the above rules at all stations, then the net adjustment assumes, essentially, the form of the direction adjustments of sections 68 and 70, only with the small difference that the individual directions do not all have exactly the same weight (weight = 1), but weights which vary between 24 and 28 according to the table on p. 330. We could perhaps also neglect the weight distinctions between 24, 25, and at the most 28; however, carrying them along does not offer any difficulty at all.

Section 89. The Chain of the Elbe River

As an example of the treatment of triangulation according to Schreiber's angle method we take the "chain of the Elbe River," whose triangulation net picture is given on p. 334, according to the official work, *Die Kgl. Preuss. Landestriangulation*, Hauptdreiecke, IV. Teil, Die Elbkette, Zweite Abteilung, Die Beobachtungen und deren Ausgleichung, gemessen und bearbeitet von der trigonometrischen Abteilung der Landesaufnahme, Berlin, 1891 (cf. *Zeitschrift für Vermessungswesen*, 1888, p. 399, and 1891, pp. 455-459). On the layout of this chain, in general we note:

The Elbe chain has fixed connections in the east and in the west, which are indicated in the drawing on p. 334 by *heavy lines*, namely:

In the east, Küstenvermessung [coast survey]:

Eichstädt-Eichberg $\log S = 4.619\ 7943{\cdot}5$

Azimuth $(Et - Eg) = 177°\ 19'\ 53.221''$

Azimuth $(Eg - Et) = 357°\ 21'\ 14.480''$

In the west: **Baursberg-Havighorst** $\log S = 4.430\ 7451{\cdot}8$

Baursberg-Kaiserberg $\log S = 4.638\ 2214{\cdot}6$

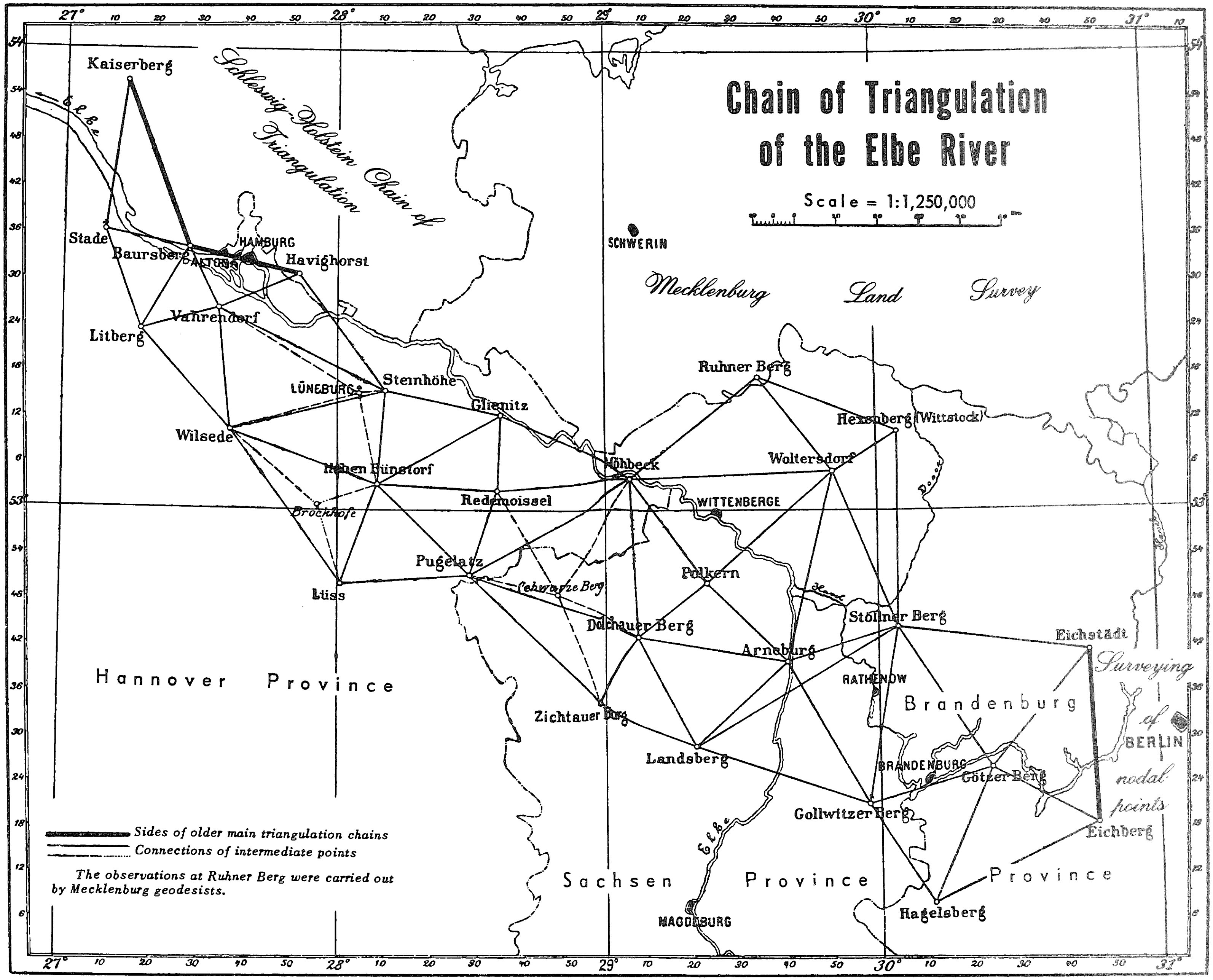
Angle: **Havighorst-Baursberg-Kaiserberg** $= 236°\ 39'\ 50.203''$.

The connection in the east is a fixed one in all respects, while in the west, with two sides and one angle, there occurs only a connection with respect to the *relative* position of the three points Kaiserberg, Baursberg, Havighorst.

Apart from the secondary points Schwarzenberg, Lüneburg, and Brockhöfe, the Elbe chain has:

$p = 28$ points

$l = 62$ two-sided sighting lines.



(Cf. section 89, pp. 333-337)

There follow therefrom according to the rules (13), p. 224:

$$l - 2p + 3 = 9 \text{ side equations}$$
$$\text{and } l - p + 1 = \underline{35} \text{ triangle equations}$$
$$\text{In all } 44 \text{ condition equations.}$$

There are added to this, however, two side equations (V and XLVI) for linear forced connections, first in the west with the two sides Kaiserberg-Baursberg and Baursberg-Havighorst, and second a side equation running through from west to east near Eichstädt-Eichberg. As for the *angle* forcing in Baursberg, it does not produce a new equation, but is expressed by giving the directions from Baursberg to Kaiserberg and to Havighorst both the *same* correction [correction (6) in the total numbering].

As for the arrangement of the measurements, there exists the peculiarity that the measurements were carried out partly according to the old Bessel method of sets, and partly according to Schreiber's method, in the following distribution (p. 334):

Stations with angle measurement in all combinations according to Schreiber's method

1. Kaiserberg . . .	2 sights
2. Stade	3 sights
3. Baursberg . . .	5 sights
4. Litberg	4 sights
5. Havighorst . . .	3 sights
6. Vahrendorf . . .	5 sights
7. Wilsede	5 sights
8. Steinhöhe . . .	5 sights
9. Hohen-Bünstorf .	6 sights
10. Lüss	3 sights
11. Glienitz	4 sights
12. Redemoissel . . .	4 sights
13. Pugelatz	6 sights
14. Höhbeck	7 sights
16. Zichtauer Berg . .	3 sights
Total	65 sights

Stations with direction measurements according to the old Bessel method

15. Dolchauer Berg .	6 sights
17. Polkern	4 sights
18. Woltersdorf . . .	6 sights
19. Ruhnerberg . . .	3 sights
20. Hexenberg . . .	3 sights
21. Landsberg . . .	5 sights
22. Arneburg	6 sights
23. Stöllner Berg . .	7 sights
24. Gollwitzer Berg .	5 sights
25. Götzer Berg . .	5 sights
26. Hagelsberg . . .	3 sights
27. Eichstädt . . .	3 sights
28. Eichberg . . .	3 sights
Total	59 sights

In all there are $65 + 59 = 124$ sights, corresponding to the two-sided measured $l = 62$ connecting lines, already previously mentioned.

The station Baursberg has *two* fixed connecting directions; the angle measurement and adjustment therefore took place according to the method described in section 87.

The station Baursberg gives once more an opportunity for the adjustment of angles with the connection to two fixed directions. For we have from *Kgl. Preuss. Landestriangulation*, Hauptdreiecke, IV. Teil, Die Elbkette, 1891, pp. 69-70, by forming the mean, the following nine measured angles and, in addition, a fixed given angle [1, 5].

1. Havighorst	2. Vahrendorf	3. Litberg	4. Stade	5. Kaiserberg
	$(1,2) = 50°\ 46'\ 35.17''$ $p = 1$	$(1,3) = 107°\ 55'\ 45.37''$ $p = 1$	$(1,4) = 178°\ 56'\ 37.13''$ $p = 1$	$[1,5] = \underline{236°\ 41'\ 37.75''}$ fixed
		$(2,3) = 57\ \ 9\ 10.24$ $p = 2$	$(2,4) = 128\ \ 10\ \ 3.29$ $p = 2$	$(2,5) = 186°\ 55'\ 0.50''$ $p = 1$
			$(3,4) = 71\ \ 0\ 52.95$ $p = 2$	$(3,5) = 128\ \ 45\ 51.98$ $p = 1$
				$(4,5) = 57\ \ 44\ 58.60$ $p = 1$.

First we eliminate the forcing connection, whereby we only write just to seconds:

$$\begin{array}{rl} (1,2) &= 35.17'' \\ [1,5]-(2,5) &= 37.25 \\ \hline \text{Mean}\quad (1,2) &= 36.21'' \\ \text{To this, from above:} & \end{array} \qquad \begin{array}{rl} (1,3) &= 45.37'' \\ [1,5]-(3,5) &= 45.77 \\ \hline \text{Mean}\quad (1,3) &= 45.57'' \\ (2,3) &= 10.24 \end{array} \qquad \begin{array}{rl} (1,4) &= 37.13'' \\ [1,5]-(4,5) &= 39.15 \\ \hline \text{Mean}\quad (1,4) &= 38.14'' \\ (2,4) &= 3.29 \\ (3,4) &= 52.95\ . \end{array}$$

These six angles have now all $p = 2$, hence are equally weighted. We adjust them thus:

$$\begin{array}{rl} (1,2) &= \begin{cases} 36.21'' \\ 36.21 \end{cases} \\ (1,3)-(2,3) &= 35.33 \\ (1,4)-(2,4) &= 34.85 \\ \hline \text{Mean } [1,2] &= 35.65'' \end{array} \quad \begin{array}{rl} (1,3) &= \begin{cases} 45.57'' \\ 45.57 \end{cases} \\ (1,4)-(3,4) &= 45.19 \\ (1,2)+(2,3) &= 46.45 \\ \hline [1,3] &= 45.70'' \end{array} \quad \begin{array}{rl} (1,4) &= \begin{cases} 38.14'' \\ 38.14 \end{cases} \\ (1,2)+(2,4) &= 39.50 \\ (1,3)+(3,4) &= 38.52 \\ \hline [1,4] &= 38.58'' \end{array} \quad \begin{array}{rl} (2,3) &= \begin{cases} 10.24'' \\ 10.24 \end{cases} \\ (1,3)-(1,2) &= 9.36 \\ (3,4)-(2,4) &= 10.34 \\ \hline [2,3] &= 10.04''. \end{array}$$

Likewise also $[2, 4] = 2.92''$ and $[3, 4] = 52.88''$.

If we take everything together, add again the degrees and minutes and write the result of the adjustment in the form of a set of directions with Havighorst as the beginning $= 0°0'0''$, then we have:

1. **Havighorst**	=	0°	0′	0.00″
2. **Vahrendorf**	=	50	46	35.65
3. **Litberg**	=	107	55	45.70
4. **Stade**	=	178	56	38.58
5. **Kaiserberg**	=	236	41	37.75 .

If we consider further, by way of example, the Station Litberg more closely, then we see first that it has four directions: Baursberg, Vahrendorf, Wilsede, Stade; therefore, it has been measured according to the table of section 88, p. 332, with $s = 4$, i.e. six angles with $n = 6$ times repetition in positions I and II or $2n = 12$ times repetition of the individual positions. Therefore, on pp. 72-73 of the official work, "Elbkette," there occur in all 72 angle measurements with the normal equations:

$$\left.\begin{array}{ll} 24A = +1.35'' & A = +0.056'' \\ 24B = +1.20 & B = +0.050 \\ 24C = -3.80 & C = -0.158 \\ 24D = +1.25 & D = +0.052 \end{array}\right\}. \tag{a}$$

The weight equations belonging to this become just as simple, namely:

$$\left.\begin{array}{l} (10) = 0.0417\ [10] \\ (11) = 0.0417\ [11] \\ (12) = 0.0417\ [12] \\ (13) = 0.0417\ [13] \end{array}\right\}. \tag{b}$$

For further details we refer to the official original work, "Die Elbkette," mentioned on p. 333.

Reference to literature to sections 85-89.

Hansen made the first general theoretical investigations about station and net adjustments with angles and directions in his treatises, *Von der Methode der kleinsten Quadrate*, 1868-1871 (cf. the note on p. 174). Hansen already recognized the three cases in which station adjustments can be presented in the form of independent directions, namely: 1. only full sets, 2. not more than three rays, 3. angle measurement in all combinations. Especially, Hansen treated this third case in "Fortgesetzte geodätische Untersuchungen, bestehend in zehn Supplementen," etc., *Abhandlungen der mathematisch-physikalischen Klasse der Königl. Sächsischen Gesellschaft der Wissenschaften*, Volume IX, 1871, pp. 169-184, "das Beobachtungsverfahren betreffend, welches Gauss in der hannoverschen Gradmessung angewendet hat." [Concerning the observation method, which Gauss applied in the degree-measurement of Hannover.]

General Schreiber treated the method of angle measurement in all combinations theoretically as well as utilizing it practically in the official work, *Die Kgl.Preuss. Landestriangulation*, Hauptdreiecke, II. Teil, 2. Abt., Berlin, 1874. To this, there also belongs: "Über die Anordnung von Horizontalwinkelbeobachtungen auf der Station," by Schreiber, *Zeitschrift für Vermessungswesen*, 1878, pp. 209-237, and "Richtungsbeobachtungen und Winkelbeobachtungen," Schreiber, *Zeitschrift für Vermessungswesen*, 1879, pp. 97-149.

The characterization of the method of the angle measurement in all combinations has been given by General Schreiber in *Zeitschrift für Vermessungswesen*, 1878, pp. 209-211, where the symbols p, n, q have *other* meanings than in our foregoing, as follows:

"[Translated] If n directions exist at a station and p full observation series or $\frac{p}{2}$ full sets are measured (two observation series forward and backward form a set), then with np settings the weight p is obtained for every direction, where the weight of a single direction observation is set equal to 1. If, on the other hand, we measure every angle q times, then the number of settings is equal to $n(n-1)q$ [since n directions form $1/2\,n(n-1)$ angles]. Therefore, in order to obtain equal weights of the results in both cases, we must have:

$$\frac{nq}{2} = p, \quad \text{from which} \quad q = \frac{2p}{n}.$$

"The weight p is thus obtained for every direction:

1. with np settings upon measuring full sets,
2. with $2(n-1)p$ settings upon measuring angles.

"Therefore, angle observations require more, but never twice as many settings as direction observations, not even in their most favorable case (with full sets). The less it has been possible at a station to observe only in full sets (which is already seldom possible at stations of four directions), the more is the superiority of the direction observations with respect to the number of settings lost; at any rate, the latter remain in the advantage with respect to angle observations. Or with other words: the longer the observation series at a station the less settings are necessary in order to obtain a definite weight of the results.

"In spite of this result and in contradiction to the geodetic practice generally followed since Bessel, I have become more and more convinced in the case of the Prussian land triangulation (1868-1874) that the superiority of long observation series compared to the short ones, in the sense of the weights computed as above, is not only a nominal one, but an illusory one as for actual accuracy; that short observation series are rather to be preferred to the long ones, but in particular the shortest to all others, i.e. *pure angle observations are to be preferred to the direction observations, because they give more accurate results with equal expenditure of time.*

"The reasons, which have led me to the above expressed opinion, are the following:

1. A short observation series yields the direction differences in general more accurately than a long one, and this is the case all the more, the less firm the setting up of the instrument and the instrument itself is. The above comparison of the numbers of settings, with equal weights of the results, is thereby modified considerably in favor of the angle observations. This is most conspicuous in the case of a turn of the pillar, which cannot be avoided at all in the case of triangulations in flat, and even in mountainous, but richly wooded, regions.

2. In the case of short observation series, we can make more settings in the same time than in the case of long ones, especially because, in the case of the latter, the loss of time due to the absence of lights and other interruptions is much greater than in the case of the former.

3. When angle observations are used, observations can be carried out at any station according to a definite observation plan designed in advance, which exactly prescribes the number of the measurements of each angle, the positions of telescope and circle, etc., while such an arrangement is impracticable for direction observations. In the case of the first, a far more complete elimination of constant errors and division errors becomes thereby possible.

4. At most stations there are one or more directions, whose observation is successful with far greater difficulty and more rarely than that of the remaining ones and only by quickly profiting by individual occasions of a short duration. In order to take advantage of such opportunities the angle method is far more suitable than the direction method.

"It is these considerations and this experience which have already led me, since 1871, to the exclusive use of angle observations (cf. Hauptdreiecke, 2. Band, 2. Abt.: the stations Marienberg, Brautberg, Keulenberg, Brandberg and Schneekoppe, of the Markish-Silesian chain, as well as in the third volume all stations of the Markish net); but not until 1875 have I given this method that improvement in which, ever since, it is consequently applied by the trigonometric section for all measurements of first order, and, in fact, with great gain not only in accuracy but also in time."

After having already considered hitherto some relations between angles and directions, e.g., the station adjustment of angle measurements in all combinations with angles as well as with directions as unknowns (section 86), we will in addition present a general theory, for further clarification and for the comprehension of the next section 91, from the work, *Die Kgl. Preussische Landestriangulation*, Hauptdreiecke, 2. Teil, 2. Abt., Berlin, 1874. On pp. 303-313, we find there a treatise by Schreiber: "Vereinfachte Form der Stations-ausgleichungsresultate," whose basic idea consists in that the results of station adjustments are given the form of *directions*, not the form of *angles* . . . (as in the case of Bessel).

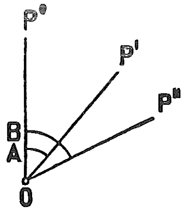


Fig. 1.

If three rays $OP^\circ\,OP'\,OP''$ are intersected several times, then we can consider, as unknowns of the adjustment, first the two *angles* $A, B,$ which the rays $OP'\,OP''$ form with the ray OP° (Fig. 1). In this manner, we proceed in the case of Bessel's adjustment, which was taught in our section 83. But now we pass over to another idea, and consider, according to Fig. 2, the three *directions* A, B, C as unknowns; then it is very clear first that $B - A$ from Fig. 2 $= A$ from Fig. 1, or more generally:

Assumption Fig. 1		Assumption Fig. 2	
Angle A	= difference of direction	$B - A$	
Angle B	= difference of direction	$C - A$	(1)
. . . .			

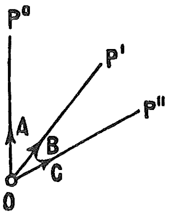


Fig. 2.

In order to make the second assumption result from the first, let us recall that "directions" are nothing else but angles which the geodetic rays $OP^\circ\,OP'\,OP''$ form with any *one* starting ray ON, being immaterial as far as its geodetic position is concerned, but common for all directions; therefore, the directions A, B, C of Fig. 2 can be represented more completely by Fig. 3.

We will now assume first that in Fig. 3 ON is likewise a simultaneously intersected ray so that we can then carry out a Bessel adjustment according to section 83 with the three angles A, B, C as unknowns.

In section 83 we had assumed that in the different sets the directions had different weights $p_1, p_2, p_3 \ldots,$ but that within the individual sets all directions had the same weight. Now we go one step further by assuming also different weights within the individual sets. We thus have the following collection of weights:

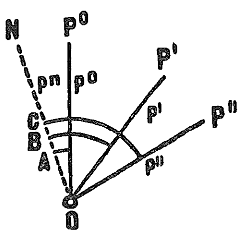


Fig. 3.

$$\left.\begin{array}{l} \text{Set } 1: \; p_1^n \;\; p_1^\circ \;\; p_1' \;\; p_1'' \\ \text{Set } 2: \; p_2^n \;\; p_2^\circ \;\; p_2' \;\; p_2'' \\ \ldots\ldots\ldots\ldots \end{array}\right\}. \qquad (1)$$

If individual directions fall out, then we have to set the corresponding p's of the set equal to zero.

With this, we come to the following normal equations for the three unknown angles A, B, C in Fig. 3 corresponding to equations (8), of section 83, p. 310.

$$\left.\begin{array}{llllllll}
[p_1]z_1 & . & . & +p_1^\circ A & +p_1' B & +p_1'' C & -[p_1 l_1] & =0 \\
. & [p_2]z_2 & . & +p_2^\circ A & +p_2' B & +p_2'' C & -[p_2 l_2] & =0 \\
. & . & [p_3]z_3 & +p_3^\circ A & +p_3' B & +p_3'' C & -[p_3 l_3] & =0 \\
p_1^\circ z_1 & +p_2^\circ z_2 & +p_3^\circ z_3 & +[p^\circ]A & . & . & -[p^\circ l^\circ] & =0 \\
p_1' z_1 & +p_2' z_2 & +p_3' z_3 & + \;\; . & +[p']B & . & -[p' l'] & =0 \\
p_1'' z_1 & +p_2'' z_2 & +p_3'' z_3 & + \;\; . & . & +[p'']C & -[p'' l''] & =0
\end{array}\right\}. \qquad (2)$$

The following equations correspond to equations (9), of section 83, p. 310:

$$\left.\begin{aligned}
-z_1 &= \frac{1}{[p_1]}(p_1{}^\circ A + p_1{}' B + p_1{}'' C - [p_1 l_1]) \\
-z_2 &= \frac{1}{[p_2]}(p_2{}^\circ A + p_2{}' B + p_2{}'' C - [p_2 l_2]) \\
-z_3 &= \frac{1}{[p_3]}(p_3{}^\circ A + p_3{}' B + p_3{}'' C - [p_3 l_3])
\end{aligned}\right\}. \tag{3}$$

If we substitute these values in the last three equations (2), then we obtain three equations of the form

$$\left.\begin{aligned}
(a\,a)\,A + (a\,b)\,B + (a\,c)\,C - (a\,l) &= 0 \\
(a\,b)\,A + (b\,b)\,B + (b\,c)\,C - (b\,l) &= 0 \\
(a\,c)\,A + (b\,c)\,B + (c\,c)\,C - (c\,l) &= 0
\end{aligned}\right\}, \tag{4}$$

where the coefficients have the following easily given values:

$$\left.\begin{aligned}
(a\,a) &= -\frac{p_1{}^\circ p_1{}^\circ}{[p_1]} - \frac{p_2{}^\circ p_2{}^\circ}{[p_2]} - \frac{p_3{}^\circ p_3{}^\circ}{[p_3]} + [p^\circ] \\
(a\,b) &= -\frac{p_1{}^\circ p_1{}'}{[p_1]} - \frac{p_2{}^\circ p_2{}'}{[p_2]} - \frac{p_3{}^\circ p_3{}'}{[p_3]} \\
(a\,c) &= -\frac{p_1{}^\circ p_1{}''}{[p_1]} - \frac{p_2{}^\circ p_2{}''}{[p_2]} - \frac{p_3{}^\circ p_3{}''}{[p_3]} \\
(b\,b) &= -\frac{p_1{}' p_1{}'}{[p_1]} - \frac{p_2{}' p_2{}'}{[p_2]} - \frac{p_3{}' p_3{}'}{[p_3]} + [p'] \\
(b\,c) &= -\frac{p_1{}' p_1{}''}{[p_1]} - \frac{p_2{}' p_2{}''}{[p_2]} - \frac{p_3{}' p_3{}''}{[p_3]} \\
-(a\,l) &= \frac{p_1{}^\circ [p_1 l_1]}{[p_1]} + \frac{p_2{}^\circ [p_2 l_2]}{[p_2]} + \frac{p_3{}^\circ [p_3 l_3]}{[p_3]} - [p^\circ l^\circ] \\
-(b\,l) &= \frac{p_1{}' [p_1 l_1]}{[p_1]} + \frac{p_2{}' [p_2 l_2]}{[p_2]} + \frac{p_3{}' [p_3 l_3]}{[p_3]} - [p' l'] \\
-(c\,l) &= \frac{p_1{}'' [p_1 l_1]}{[p_1]} + \frac{p_2{}'' [p_2 l_2]}{[p_2]} + \frac{p_3{}'' [p_3 l_3]}{[p_3]} - [p'' l'']
\end{aligned}\right\}. \tag{5}$$

All this holds good under the assumption that the direction ON is likewise *measured*. If this assumption does not prove true, then we have to set equal to zero only all p^n's, i.e. $p_1{}^n$, $p_2{}^n$, $p_3{}^n$. If we do this in equations (5), then the sums $[p_1]$, $[p_2]$, $[p_3]$, in contrast to section 83, p. 310, contain only just the weights of the directions to P°, P', P'', hence

$$\left.\begin{aligned}
[p_1] &= p_1{}^\circ + p_1{}' + p_1{}'' \\
[p_2] &= p_2{}^\circ + p_2{}' + p_2{}'' \\
&\ldots\ldots\ldots\ldots
\end{aligned}\right\}. \tag{6}$$

With this, we obtain

$$\begin{aligned}
(a\,a) + (a\,b) + (a\,c) &= [p^\circ] - \frac{p_1{}^\circ}{[p_1]}(p_1{}^\circ + p_1{}' + p_1{}'') \\
&\quad - \frac{p_2{}^\circ}{[p_2]}(p_2{}^\circ + p_2{}' + p_2{}'') \\
&\quad - \frac{p_3{}^\circ}{[p_3]}(p_3{}^\circ + p_3{}' + p_3{}'') \\
&= [p^\circ] - (p_1{}^\circ + p_2{}^\circ + p_3{}^\circ) = [p^\circ] - [p^\circ] = 0\,.
\end{aligned}$$

By the same way we find that $(a\,b)+(b\,b)+(b\,c)=0$ and $(a\,c)+(b\,c)+(c\,c)=0$. From the last three equations of (5) we obtain further

$$\begin{aligned}-(a\,l)-(b\,l)-(c\,l) &= \frac{[p_1\,l_1]}{[p_1]}(p_1^\circ+p_1'+p_1'')-[p^\circ\,l^\circ]\\ &+\frac{[p_2\,l_2]}{[p_2]}(p_2^\circ+p_2'+p_2'')-[p'\,l']\\ &+\frac{[p_3\,l_3]}{[p_3]}(p_3^\circ+p_3'+p_3'')-[p''\,l'']\end{aligned}$$

or with (6)

$$\begin{aligned}-(a\,l)-(b\,l)-(c\,l) &= [p_1\,l_1]+[p_2\,l_2]+[p_3\,l_3]\\ &-[p^\circ\,l^\circ]-[p'\,l']-[p''\,l''].\end{aligned}$$

Herein, the two rows on the right-hand side are equivalent, so that they yield together zero. In all we thus have

$$\left.\begin{aligned}(a\,a)+(a\,b)+(a\,c)&=0\\ (a\,b)+(b\,b)+(b\,c)&=0\\ (a\,c)+(b\,c)+(c\,c)&=0\end{aligned}\right\} \qquad (7)$$

$$(a\,l)+(b\,l)+(c\,l)=0\,. \qquad (8)$$

With this, we have found that the normal equations (4), which refer to *directions* A, B, C do not admit a single-valued solution, for these equations (4) are not independent of one another according to (7) and (8).

Although elementary algebra teaches the impossibility of a single-valued solution of equations (4), we will directly pursue this further: If A, B, C is a system, which satisfies equations (4), and we set in their places $A+z$, $B+z$, $C+z$, respectively, then we have from (4)

$$(a\,a)(A+z)+(a\,b)(B+z)+(a\,c)(C+z)-(a\,l)=0\,,$$

i.e.

$$(a\,a)\,A+(a\,b)\,B+(a\,c)\,C-(a\,l)+\{(a\,a)+(a\,b)+(a\,c)\}\,z=0\,.$$

The first part vanishes because of (4) and the second part because of (7); hence, the equation which was satisfied for A, B, C is also generally correct for $A+z$, $B+z$, $C+z$.

Since a direction remains undetermined, then we can set such a one $=0$, e.g. setting $A=0$ yields from (4):

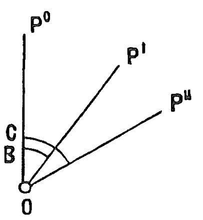


Fig. 4.

$$\left.\begin{aligned}(a\,b)\,B+(a\,c)\,C-(a\,l)&=0\\ (b\,b)\,B+(b\,c)\,C-(b\,l)&=0\\ (b\,c)\,B+(c\,c)\,C-(c\,l)&=0\end{aligned}\right\}. \qquad (9)$$

Because of (7) and (8) we can arbitrarily omit here an equation, e.g. the first, and then we obtain the angles B and C (Fig. 4) unequivocally determined by the equations:

$$\left.\begin{aligned}(b\,b)\,B+(b\,c)\,C-(b\,l)&=0\\ (b\,c)\,B+(c\,c)\,C-(c\,l)&=0\end{aligned}\right\}. \qquad (10)$$

The reverse procedure, namely to convert angle equations into direction equations, results likewise by means of equations (7) and (8); we will not however work this out further, since it is not necessary for the further development.

As we have seen in the two preceding sections, 89 and 90, the result of the adjustment of angle measurements at a station can in certain cases be summarized in the form of directions.

In these cases, the totality of the angle measurements made at a station for the purposes of the net adjustment is completely replaced by *one* set of directions with equal or different weights, so that the adjustment can be carried out according to section 43 or section 44.

In all other cases, the result of the station adjustment can be introduced only with the help of *weight equations* according to sections 54 to 56 into the net adjustment, whereby the latter becomes exceedingly clumsy and troublesome.

Consequently, it has already long been tried to derive, also in these cases, a set of directions from the station adjustment and to give, for the individual directions, *approximate* weights, with which we can approach as much as possible the rigorous adjustment with weight equations. The great British triangulation (Ordnance trigonometrical survey) was thus adjusted, already about 1850, according to individual weights, where partly the deviations of the individual direction measurements from their mean, and partly the number of the settings were used as a measure of accuracy. At that time, before 1858, the theory of net adjustments did not fully clearly exist at all, and at any rate, it was not available as yet to British geodesists; that method was the result of a sound empiricism.

Later, about 1870, we have a similar method in the new adjustment of the old Bavarian triangulation by v. Orff, who simply took the "number of cuts" as direction weights.

The determination of approximate direction weights made considerable progress in a theory by Helmert, which is contained in *Veröffentlichung des Kgl. Preuss. Geodätischen Instituts und Zentralbureaus der internationalen Erdmessung*: Die europäische Längengradmessung in 52° Breite von Greenwich bis Warschau I. Heft, Hauptdreiecke und Grundlinienanschlüsse von England bis Polen, Berlin, 1893, pp. 37-42, and in a treatise in *Astronomische Nachrichten*, Volume 134, 1893, pp. 281-296: "Über eine Vereinfachung bei der Einführung von Stationsergebnissen in der Ausgleichung eines Dreiecksnetzes." (Report about it in *Zeitschrift für Vermessungswesen*, 1894, pp. 212-222.)

While we give this theory here, we will also retain here Helmert's designations because of the connection to the original publications mentioned.

The weight coefficients, which were denoted in our earlier section 30 partly by Q_1, Q_2, Q_3 and partly by $[\alpha\alpha], [\alpha\beta], [\alpha\gamma]$, etc., are now denoted in the first place by Q_{22}, Q_{23}, Q_{24}, etc.

Further let q with the corresponding index represent a weight reciprocal.

If a station adjustment according to section 83 is present with two angles x' and x'' as independent unknowns, then we find the weights of x' and x'' in the case of the adjustment or $p' = \dfrac{1}{[\alpha\alpha]}$ and $p'' = \dfrac{1}{[\beta\beta]}$, where $[\alpha\alpha]$ and $[\beta\beta]$ are the weight coefficients according to sections 30 and 31. The weight P of the difference $x'' - x'$, i.e. the angle between the two rays P' and P'' of Fig. 1, section 83, p. 308, can also be given, for $x'' - x'$ is a linear function of the independent x''s and x'''s, namely, according to (1), p. 100:

$$x'' - x' = F = f_1 x' + f_2 x'' + f_3 x''' ,$$

where

$$f_1 = -1 , \qquad f_2 = +1 , \qquad f_3 = 0 ,$$

consequently, according to (3), p. 100: $\dfrac{1}{P} = [\alpha\alpha] - 2\,[\alpha\beta] + [\beta\beta]$.

Such a formula holds for every angle obtained after the adjustment, and if we designate the weight coefficients differently according to the above reference, namely, $[\alpha\alpha] = Q_{22}$, $[\alpha\beta] = Q_{23}$, $[\beta\beta] = Q_{33}$, and the angles themselves $x' = (1,2)$, $x'' = (1,3)$, hence, $x'' - x' = (2,3)$, then the weight reciprocal of the angle $(2,3)$ is expressed by

$$q_{23} = Q_{22} - 2\,Q_{23} + Q_{33} .$$

After these statements about the choice of the designations we consider first the case of a station adjustment with four rays.

Between these four rays 1, 2, 3, 4 there exist six angles, which shall have the following weights after the adjustment:

$$\left.\begin{array}{ll}
\text{Angle (1,2) with weight} = 1: Q_{22} & = 1: q_{12} \\
\text{Angle (1,3) with weight} = 1: Q_{33} & = 1: q_{13} \\
\text{Angle (1,4) with weight} = 1: Q_{44} & = 1: q_{14} \\
\text{Angle (2,3) with weight} = 1: (Q_{22}+Q_{33}-2\,Q_{23}) & = 1: q_{23} \\
\text{Angle (2,4) with weight} = 1: (Q_{22}+Q_{44}-2\,Q_{24}) & = 1: q_{24} \\
\text{Angle (3,4) with weight} = 1: (Q_{33}+Q_{44}-2\,Q_{34}) & = 1: q_{34}
\end{array}\right\} . \tag{1}$$

Now we shall set *one* set of direction measurements with individual weights of the individual rays in the place of the adjustment result; if the reciprocals of the individual weights of the four rays are q_1, q_2, q_3, q_4, respectively, then the following equations must exist:

$$\left.\begin{array}{lll}
q_{12} = q_1 + q_2 & q_{13} = q_1 + q_3 & q_{14} = q_1 + q_4 \\
 & q_{23} = q_2 + q_3 & q_{24} = q_2 + q_4 \\
 & & q_{34} = q_3 + q_4
\end{array}\right\} . \tag{2}$$

If only *three* rays existed, then only three of these equations would exist, namely:

$$\left.\begin{array}{ll}
q_{12} = q_1 + q_2 & q_{13} = q_1 + q_3 \\
 & q_{23} = q_2 + q_3
\end{array}\right\} \tag{3}$$

and the three unknowns q_1, q_2, q_3 can be determined directly therefrom, namely:

$$\left.\begin{array}{l}
q_1 = \dfrac{q_{12}+q_{13}-q_{23}}{2} \\
q_2 = \dfrac{q_{12}-q_{13}+q_{23}}{2} \\
q_3 = \dfrac{-q_{12}+q_{13}+q_{23}}{2}
\end{array}\right\} . \tag{4}$$

If we substitute here according to (1):

$$q_{12} = Q_{22}, \quad q_{13} = Q_{33}, \quad q_{23} = Q_{22} + Q_{33} - 2\,Q_{23},$$

then we obtain from (4) the weight reciprocals:

$$q_1 = Q_{23}, \quad q_2 = Q_{22} - Q_{23}, \quad q_3 = Q_{33} - Q_{23} . \tag{5}$$

Through these equations (3) to (5), we have found that in the special case of *three* rays the station adjustment can always be taken care of in the form of directions with individual weights $\frac{1}{q}$ of the adjusted directions.

Now we return to the case of four rays with equations (1) and (2).

Since we have here six equations in group (2), and only four unknowns q_1, q_2, q_3, q_4, a complete solution is not possible; therefore, the four values q shall so be determined that they satisfy as much as possible the equations (2) (according to the method of least squares). If we consider the above six equations (2) as error equations, then they yield the following four normal equations:

$$\left.\begin{array}{lll} \underline{3q_1} + q_2 + q_3 + q_4 - s_1 = 0, & \text{where} & s_1 = q_{12} + q_{13} + q_{14} \\ q_1 + \underline{3q_2} + q_3 + q_4 - s_2 = 0 & & s_2 = q_{12} + q_{23} + q_{24} \\ q_1 + q_2 + \underline{3q_3} + q_4 - s_3 = 0 & & s_3 = q_{13} + q_{23} + q_{34} \\ q_1 + q_2 + q_3 + \underline{3q_4} - s_4 = 0 & & \underline{s_4 = q_{14} + q_{24} + q_{34}} \\ & & S = s_1 + s_2 + s_3 + s_4 \end{array}\right\}. \tag{6}$$

The solution of these four equations yields:

$$\left.\begin{array}{ll} \text{Weight reciprocal } q_1 = \frac{s_1}{2} - \frac{S}{12} & \\ \text{Weight reciprocal } q_2 = \frac{s_2}{2} - \frac{S}{12} & \text{with the sum} \\ \text{Weight reciprocal } q_3 = \frac{s_3}{2} - \frac{S}{12} & q_1 + q_2 + q_3 + q_4 = \frac{S}{6} \\ \text{Weight reciprocal } q_4 = \frac{s_4}{2} - \frac{S}{12} & \end{array}\right\}. \tag{7}$$

These equations (6) and (7) hold for the special case with four rays; in the general case with n rays, an entirely corresponding, more general treatment is to be made, where the normal equations instead of (6) become the following:

$$\left.\begin{array}{lllll} \underline{(n-1)\,q_1} + & q_2 + & q_3 + \dots & + q_n - s_1 = 0 \\ q_1 + & \underline{(n-1)\,q_2} + & q_3 + \dots & + q_n - s_2 = 0 \\ q_1 + & q_2 + & \underline{(n-1)\,q_3} + \dots & + q_n - s_3 = 0 \\ \dots & \dots & \dots & \dots \\ q_1 + & q_2 + & q_3 + \dots & \underline{(n-1)\,q_n} - s_n = 0 \end{array}\right\}. \tag{8}$$

The absolute terms s have here the following meanings:

$$\left(\begin{array}{c} \text{here } q_{12} = q_{21} \\ q_{13} = q_{31}, \text{ etc.} \end{array}\right) \quad \left.\begin{array}{ll} q_{12} + q_{13} + q_{14} + \dots q_{1n} & = s_1 \\ q_{21} + q_{23} + q_{24} + \dots q_{2n} & = s_2 \\ q_{31} + q_{32} + q_{34} + \dots q_{3n} & = s_3 \\ \dots & \dots \\ \underline{q_{n1} + q_{n2} + q_{n3} + \dots q_{(n-1)n}} & \underline{= s_n} \\ s_1 + s_2 + s_3 + \dots s_n & = S \end{array}\right\}. \tag{9}$$

For a check we have directly from the Q's:

$$S = 2\,(n-1)\,(Q_{22} + Q_{33} + Q_{44} + \dots Q_{nn}) - 4\,(Q_{23} + Q_{24} + \dots + Q_{34} + \dots Q_{(n-1)n}). \tag{10}$$

The general formula for the computation of a q is:

$$\left\{\begin{array}{l} q_1 = \frac{s_1}{n-2} - \frac{S}{2\,(n-1)\,(n-2)} \\ q_2 = \frac{s_2}{n-2} - \frac{S}{2\,(n-1)\,(n-2)}, \text{ etc.,} \\ \dots \end{array}\right. \tag{11}$$

where, in addition, we have the sum check:

$$q_1 + q_2 + q_3 + \dots q_n = \frac{S}{2\,(n-1)}. \tag{12}$$

If we return to the meaning of s and S as functions of the Q's, then we find the formulae:

$$\left.\begin{aligned} q_1 &= \frac{2\,(Q_{23}+Q_{24}+\ldots)}{(n-1)\,(n-2)} \\ q_2 &= q_1 + Q_{22} \frac{2\,(Q_{23}+Q_{24}+Q_{25}+\ldots Q_{2n})}{n-2} \\ q_3 &= q_1 + Q_{33} - \frac{2\,(Q_{32}+Q_{34}+Q_{35}+\ldots Q_{3n})}{n-2} \end{aligned}\right\}, \tag{13}$$

etc., where only nonquadratic indices are to be considered in the sums, hence, only e.g. $Q_{32}, Q_{34} \ldots$, but not Q_{33}, etc.

We consider, in the case of n rays, the $(n-1)$ angles which any ray forms with the remaining rays, e.g. the ray 1 in connection with the rays 2, 3, 4 . . . ; then we can express the sum of the squares of errors for such $n-1$ angles in a twofold way, in terms of $q_{12}, q_{13} \ldots$, as well as in terms of $q_1, q_2, q_3 \ldots$, namely, because of (9):

$$q_{12} + q_{13} + q_{14} + \ldots = s_1 ,$$

or, second, because of the normal equations (8):

$$(q_1+q_2) + (q_1+q_3) + (q_1+q_4) + \ldots = (n-1)\,q_1 + q_2 + q_3 + \ldots = s_1 . \tag{14}$$

These two sums are thus *equal*, which speaks very much in favor of the approximations q_1, q_2, q_3, q_4, while it is not favorable that the deviations between q_{12} and q_1+q_2, etc., were treated in the adjustment as *equally* admissible. We have already treated the special case of only *three* rays as an intermediate remark in (3) to (5), and we have seen in (5) that we can simply express the reciprocals q of the direction weights in terms of the weight coefficients. In this simple case, however, we can also express everything by the coefficients of the normal equations $[a\,a]$, $[a\,b]$, $[b\,b]$ themselves, for according to (18) and (19), p. 58, we have:

$$[\alpha\,\alpha] = Q_{22} = \frac{[b\,b]}{D}, \quad [\alpha\,\beta] = Q_{23} = \frac{-[a\,b]}{D}, \quad [\beta\,\beta] = Q_{33} = \frac{[a\,a]}{D}, \tag{15}$$

where
$$D = [a\,a]\,[b\,b] - [a\,b]\,[a\,b] ,$$

we find therefrom, in connection with (5):

$$q_1 = \frac{-[a\,b]}{D}, \qquad q_2 = \frac{[b\,b]+[a\,b]}{D}, \qquad q_3 = \frac{[a\,a]+[a\,b]}{D}; \tag{16}$$

this equation (15) holds when the station adjustment is made with *two angles* as unknowns. If, on the other hand, *three directions* are introduced as unknowns, then the normal equations will assume the following forms:

$$\left.\begin{aligned} (a\,a)\,A + (a\,b)\,B + (a\,c)\,C - (a\,l) &= 0 \\ (a\,b)\,A + (b\,b)\,B + (b\,c)\,C - (b\,l) &= 0 \\ (a\,c)\,A + (b\,c)\,B + (c\,c)\,C - (c\,l) &= 0 \end{aligned}\right\} . \tag{17}$$

Since A, B, C are *directions* (not two angles), these three equations cannot be independent, but according to (7), section 90, p. 340, there exist the relations:

$$(a\,a) + (a\,b) + (a\,c) = 0 , \quad (a\,b) + (b\,b) + (b\,c) = 0 , \quad (a\,c) + (b\,c) + (c\,c) = 0 \tag{18}$$

and *one* of the three unknowns is arbitrary; therefore, we can set, for instance, $A = 0$ and then from (17) there remains only:

$$\left.\begin{aligned} (b\,b)\,B + (b\,c)\,C - (b\,l) = 0 \\ (b\,c)\,B + (c\,c)\,C - (c\,l) = 0 \end{aligned}\right\} . \tag{19}$$

In the case of *two* equations, however, we can express again the weight coefficients Q directly by the coefficients $(b\,b)\,(b\,c)$, namely, just as in (15):

$$\left.\begin{aligned} Q_{22} = \frac{(c\,c)}{(b\,b)\,(c\,c) - (b\,c)\,(b\,c)}, \quad Q_{33} = \frac{(b\,b)}{(b\,b)\,(c\,c) - (b\,c)\,(b\,c)} \\ Q_{22} = \frac{-(b\,c)}{(b\,b)\,(c\,c) - (b\,c)\,(b\,c)} \end{aligned}\right\} . \tag{20}$$

Because of (18), (20) with (5) can be brought into the following form:

$$\left.\begin{aligned} \frac{1}{q_1} &= (a\,a) - \frac{(a\,b)\,(a\,c)}{(b\,c)} \\ \frac{1}{q_2} &= (b\,b) - \frac{(a\,b)\,(b\,c)}{(a\,c)} \\ \frac{1}{q_3} &= (c\,c) - \frac{(a\,c)\,(b\,c)}{(a\,b)} \end{aligned}\right\} . \tag{21}$$

In the general formulae there must also be included the case that between n rays all $n\,\frac{n-1}{2}$ angles are measured equally weighted. If we set here the weight of a measured angle $= 1$, then, according to section 86, p. 327, all weights of angles $= \frac{n}{2}$, hence, the reciprocals of the weights after the adjustment will become:

$$q_{12} = q_{13} = q_{14} = \ldots q_{1}\,(n-1) = \frac{2}{n},$$

then according to (9):

$$q_{12} + q_{13} + q_{14} + \ldots + q_{1}\,(n-1) = \frac{2\,(n-1)}{n} = s_1 .$$

Likewise, $s_2, s_3, \ldots, s_n$, and therefore $S = n\,s_1 = 2\,(n-1)$.

Hence, now according to (11):

$$q_1 = \frac{2\,(n-1)}{n\,(n-2)} - \frac{2\,(n-1)}{2\,(n-1)\,(n-2)} = \frac{1}{n},$$

i.e. the weight of an adjusted direction $= n$, as a result of the approximate method, which agrees with the rigorous method.

If we have, finally, m full sets in the case of n rays, then after the formation of the mean, all direction weights will equal m or the reciprocals of the weights of angles are:

$$q_{12} = q_{13} = \ldots = \frac{2}{m},$$

therefore

$$s_1 = s_2 = \ldots = \frac{2\,(n-1)}{m}, \quad S = \frac{2\,n\,(n-1)}{m},$$

$$q_1 = \frac{2\,(n-1)}{(n-2)\,m} - \frac{2\,n\,(n-1)}{2\,m\,(n-1)\,(n-2)} = \frac{1}{m},$$

hence, are again in agreement with the rigorous theory.

In the three cases considered, in which rigorous direction weights are possible, Helmert's approximation method thus follows the rigorous theory and, moreover, conforms to it approximately.

In order to have a complete example, we take a station adjustment, which was calculated entirely elsewhere (Jordan-Steppe's *Deutsches Vermessungswesen*, I., p. 66), according to *Kgl. Preuss. Landestriangulation*, Hauptdreiecke, I. Teil, 2. Auflage, Berlin, 1870, p. 57.

Station Lautern:

Sternberg	Paulinen	Schippenbeil	Rössel	
$p_1^\circ = 18$	$p_1' = 18$	$p_1'' = 18$	$p_1''' = 18$	$[p_1] = 72$
$p_2^\circ = 6$	. .	. .	$p_2''' = 6$	$[p_2] = 12$
. .	$p_3' = 6$	$p_3'' = 6$	$p_3''' = 6$	$[p_3] = 18$
$[p^\circ] = 24$	$[p'] = 24$	$[p''] = 24$	$[p'''] = 30$	$[p] = 102.$

According to (10), section 83, p. 310, we compute:

$$(a\,a) = 24 - \frac{18}{72}18 - \ldots - \frac{6}{18}6 = \quad 17.50$$

$$(a\,b) = \quad -\frac{18}{72}18 \quad \ldots - \frac{6}{18}6 = - \quad 6.50\,, \text{ etc.}$$

The station adjustment itself is not of interest to us here, but only the weight determinations; therefore, we form at once the weight equations according to (20), p. 97:

Solution:

$$\begin{array}{llll} +\underline{17.50\,[\alpha\,\alpha]} - 6.50\,[\alpha\,\beta] - 6.50\,[\alpha\,\gamma] - 1 = 0 & [\alpha\,\alpha] = 0.094 \\ -\ 6.50\,[\alpha\,\alpha] + \underline{17.50\,[\alpha\,\beta]} - 6.50\,[\alpha\,\gamma] = 0 & [\alpha\,\beta] = 0.052 \\ -\ 6.50\,[\alpha\,\alpha] - 6.50\,[\alpha\,\beta] + \underline{20.50\,[\alpha\,\gamma]} = 0 & [\alpha\,\gamma] = 0.046\,. \end{array}$$

The other two groups of weight equations yield further:

$$[\beta\,\beta] = 0.093\,, \qquad [\beta\,\gamma] = 0.046\,, \qquad [\gamma\,\gamma] = 0.078\,.$$

Now if we use the designations Q according to (1), we have:

$$\begin{array}{ll} Q_{22} = [\alpha\,\alpha] = +0.094 = q_{12} & Q_{23} = 0.052 \\ Q_{33} = [\beta\,\beta] = +0.093 = q_{13} & Q_{24} = 0.046 \\ Q_{44} = [\gamma\,\gamma] = +0.078 = q_{14} & Q_{34} = 0.046 \end{array} \Bigg\}\ 0.144$$

$$\begin{array}{l} Q_{22} + Q_{33} - 2\,Q_{23} = 0.094 + 0.093 - 0.104 = +0.083 = q_{23} \\ Q_{22} + Q_{44} - 2\,Q_{24} = 0.094 + 0.078 - 0.092 = +0.080 = q_{24} \\ Q_{33} + Q_{44} - 2\,Q_{34} = 0.093 + 0.078 - 0.092 = +0.079 = q_{34} \end{array}$$

$$\left.\begin{array}{l} q_{12} + q_{13} + q_{14} = s_1 = 0.265 \\ q_{12} + q_{23} + q_{24} = s_2 = 0.257 \\ q_{13} + q_{23} + q_{34} = s_3 = 0.255 \\ q_{14} + q_{24} + q_{34} = s_4 = 0.237 \end{array}\right\} \quad S = 1.014\,, \quad \frac{S}{12} = 0.0845\,.$$

Then according to (7):

$$\left.\begin{array}{l} q_1 = 0.1325 - 0.0845 = 0.0480 \\ q_2 = 0.1285 - 0.0845 = 0.0440 \\ q_3 = 0.1275 - 0.0845 = 0.0430 \\ q_4 = 0.1185 - 0.0845 = 0.0340 \end{array}\right\} \quad 0.1690 = \frac{S}{6}\ \text{(Check).}$$

In order to apply, in addition, the formulae (13), we have with $n = 4$:

$$q_1 = \frac{0.144}{3} = 0.048$$

$$q_2 = 0.048 + 0.094 - 0.098 = 0.044$$
$$q_3 = 0.048 + 0.093 - 0.098 = 0.043$$
$$q_4 = 0.048 + 0.078 - 0.092 = 0.034\ .$$

With this, we have computed all q's and their reciprocals, i.e. the direction weights $\frac{1}{q}$, and even with checks.

Now it is a question to see further how accurately the approximate weights agree with the rigorous weights or the reciprocals of the approximate weights with the rigorous reciprocals of the weights of all angles; e.g., the first angle Sternberg-Paulinen has the rigorous weight reciprocal $q_{12} = 0.094$, and the approximate one $q_1 + q_2 = 0.092$, which agrees very closely.

For all six angles there follows the following comparison:

Rigorous	Approximate	Deviation	
$q_{12} = 0.094$	$q_1 + q_2 = 0.092$	-0.002	
$q_{13} = 0.093$	$q_1 + q_3 = 0.091$	-0.001	Average
$q_{14} = 0.078$	$q_1 + q_4 = 0.082$	$+0.004$	deviation
$q_{23} = 0.083$	$q_2 + q_3 = 0.087$	$+0.004$	= 0.0025
$q_{24} = 0.080$	$q_2 + q_4 = 0.078$	-0.002	or 3%.
$q_{34} = 0.079$	$q_3 + q_4 = 0.077$	-0.002	

These are the weight reciprocals; we will calculate further the weights themselves and compare them with the number of pointings:

	Weight	Number of Pointings	Deviation
$q_1 = 0.048$	$1 : q_1 = 20.8$	$[p^\circ] = 24$	$+3.2$
$q_2 = 0.044$	$1 : q_2 = 22.7$	$[p'] = 24$	$+1.3$
$q_3 = 0.043$	$1 : q_3 = 23.3$	$[p''] = 24$	$+0.7$
$q_4 = 0.034$	$1 : q_4 = 29.4$	$[p'''] = 30$	$+0.6$
		Average	1.45 or 6%.

The number of pointings deviates from the approximate weights in the mean only by 6%, therefore, the number of pointings could in this case also be used further as approximate weights, however, our case is a very simple one; in general, the number of pointings will yield larger deviations from the theoretical weights.

Helmert's treatise already mentioned in *Astronomische Nachrichten,* Volume 134, 1893, p. 287, gives a rather large example for the comparison between the rigorous and the approximate weights of angles, as well as between the approximate direction weights and the number of pointings; it is a Spanish station with 12 directions; the rigorous q_{12}'s, q_{13}'s . . . deviate from the approximate $q_1 + q_2$, $q_1 + q_3$ . . . in the mean by 11%, in the extreme by 30%. Between the q's and the reciprocals of the number of pointings, there exists an average deviation of 31%.

In the same place, on p. 289, a report is also given about the application of the approximate method at 12 stations with altogether 71 directions. Between the rigorous q_{12}'s, q_{13}'s as reciprocals of the weights of the angles and the approximate $q_1 + q_2$, $q_1 + q_3$ . . . there resulted in the mean a deviation of 9%.

For a further characterization of the method, there is given on pp. 290-291 of the treatise mentioned in *Astronomische Nachrichten* the adjustment carried out by Helmert according to his approximate method for the small triangulation net of Württemberg consisting of six triangles for the measurement of the earth (by E. Hammer, Stuttgart, 1892), in comparison with the rigorous adjustment. The largest difference of the results on both sides for the 33 possible angles amounts to 0.038''; the mean variation is ±0.014'' in the case of a mean error of an angle of ±0.47'' adjusted at the station and a mean error of an angle of ±0.4'' adjusted in the net. The latter manifests a mean variation of only ±3.5% and a maximum change of 9.5%.

The theory treated on the preceding pages concerns only the pure measuring errors in the narrower sense, namely, setting errors (with the telescope) and reading errors (at the microscope or nonius), or the plain ["nackt"] observational errors. The division errors of the circle on the theodolite can be partially eliminated by symmetrical circle adjustments in the different measuring sets; but they still play a role in the total error of a direction.

In general, we cannot simply assume *one* observational error for a sight, but a combined effect of different causes of errors, which are partly irregular (plain observational errors), partly regular or even constant.

As constant errors there occur: the instrumental errors, especially the regular division errors and the accidental division errors, the personal errors of the observer in fixing the target points, centering errors, time changes in the location of the stations - "for every possible reason" - and lateral refractions of the light rays in the air. If these errors act under certain circumstances in the case of the different groups of angle measurements of a station from group to group in a changing manner, then their influence is a symmetrical one.

Starting from this idea, Helmert on p. 36 of the publication mentioned above, on p. 341, proceeds further thus:

Assuming that we have a set of independent unequally weighted directions, one of which has the weight n, or with the setting of an ideal unit sighting n times is equally weighted, and that μ is the mean plain observational error for the unit of weight, then $\frac{\mu^2}{n}$ is the mean plain contribution of the observational error of the direction in question. Let further $\frac{\tau^2}{r}$ be the influence of the accidental division error in the case of r equally distributed settings. Then we have that the mean square of error of the direction according to the developments hitherto carried out $= \frac{\mu^2}{n} + \frac{\tau^2}{r}$.

But now we take to this further a direction error, ν, of the *net*, which is composed of the constant errors, which, as a rule, remain hidden at the stations and can only be recognized in the discrepancies of the *net*. To this there also belong residues of regular division errors and other systematic influences, which actually should be eliminated by the measuring procedure.

In this way, the complete mean square of the error of a direction is represented by:

$$M^2 = \frac{\mu^2}{n} + \frac{\tau^2}{r} + \nu^2 . \tag{22}$$

In order to learn ν^2, an average value of M^2 was taken from the discrepancies of the triangle closures or from older net adjustments and in part from the comparison of station results of different epochs, but μ^2 and τ^2 were estimated as far as possible. According to pp. 55, 56, 89, 109, 110, 125 of the publication mentioned, such determinations are $\nu^2 = 0.19, 0.43, 0.09, 0.18, 0.09, 0.15, 0.09$.

$$\text{In the mean} \qquad \nu^2 = 0.17 \quad \text{or} \quad \nu = \pm 0.4'' . \tag{23}$$

This error of the net $\nu = \pm 0.4''$ is a relatively *high* value!

A further peculiarity is added on p. 37: If at the same station, measurements from different years were to be connected with each other, then M^2 was computed first by itself for each epoch. The occurrence of *large differences* even in the best measurements of *different years* spoke in favor of this practice, where it remained unexplained, however, what may be the reasons of such changes with *time*.

From the publication of the Geodetic Institute of 1893 (see above p. 341), we take from pp. 87-95 the "Belgisch-Deutsche Verbindungsnetz," which is illustrated in Fig. 1 following, and we present the principal moments of the direction adjustment, which is carried out according to Helmert's weight theory (in the previous section 91).

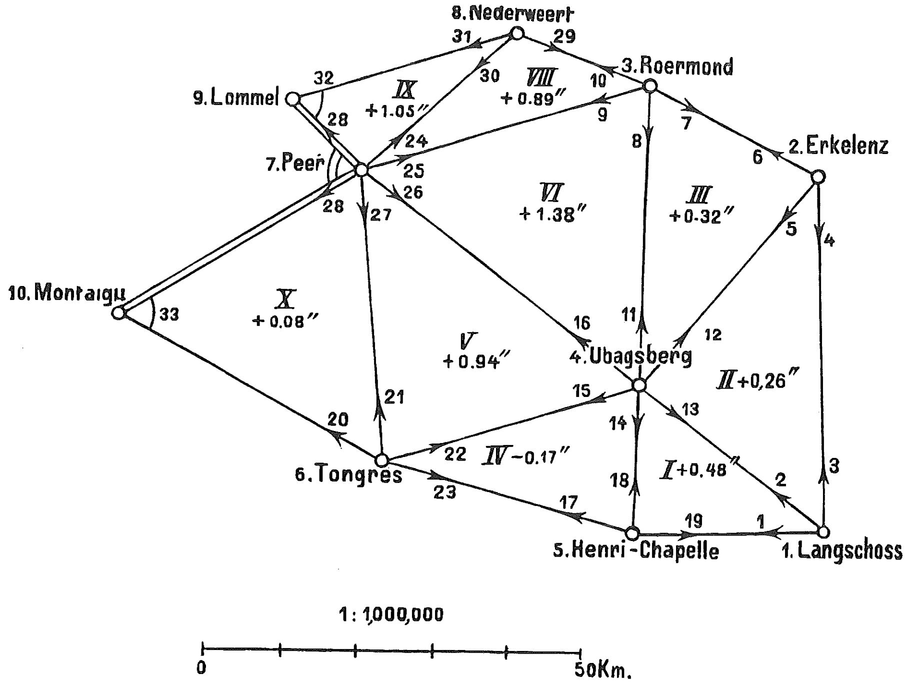


Fig. 1.
The Belgian-German connecting net.

The net has ten stations of the following origins:

Belgian Measurements	Belgian and Prussian Measurements	Prussian Measurements
Nederweert Lommel Peer Montaigu Tongres	Roermond Ubagsberg Henri-Chapelle	Erkelenz Langschoss
$M_B = \frac{0.89''}{\sqrt{2}} = \pm 0.63''$	$M_B = \pm 0.73''$, $M_P = \begin{cases} \pm 0.79'' \\ \pm 0.66'' \end{cases}$	$M_P = \frac{0.73''}{\sqrt{2}} = \pm 0.52''$.

The mean angle errors inserted here at once, $\pm 0.89''$ and $\pm 0.73''$, are in general determined from triangle closures of the Belgian triangulation and the newer Rhenish triangulation net of the Geodetic Institute, and yield the mean error of a Belgian direction adjusted in the net or Prussian direction equal to $\pm 0.63''$ and $\pm 0.52''$, respectively, while the comparisons at the connected stations yield larger amounts, namely, $\pm 0.73''$ for Belgium, and $\pm 0.79''$ for Prussia, 1861, $\pm 0.66''$ for Prussia, 1869.

From such considerations there was in general assumed for Belgium $M = \pm 0.64''$, $M^2 = 0.41$, and for Prussia the M^2's were determined individually according to the theory of equation (22) in the previous section 91, p. 348, which is to be looked up on pp. 89-90 of the publication itself.

Adjustment Station Data [Ausgleichungsabriss] of the Belgian-German Connecting Net

No.	Target Point	Before the Adjustment: Mean Error M	$q=\frac{1}{p}=\left(\frac{M}{0.412}\right)^2$	Observed Direction A	After the Adjustment: Net Corr. v	Adjusted Direction $A+v-v_1$, etc.	$v\sqrt{p}=\frac{0.412\,v}{M}$	$v^2 p=\frac{v^2}{q}$
				1. Langschloss				
1	H. Ch.	± 0.79″	3.7	0° 0′ 0.00″	+1.30″	0° 0′ 0.00″	0.67″	0.45
2	Ub.	0.34	0.7	37 39 39.16	−0.29	37 39 37.57	0.35	0.12
3	Er.	0.34	0.7	89 38 19.68	+0.04	89 38 18.42	0.05	0.00
				2. Erkelenz				
4	La.	± 0.32″	0.6	0° 0′ 0.00″	+0.27″	0° 0′ 0.00″	0.35″	0.12
5	Ub.	0.41	1.0	42 11 47.80	−0.29	42 11 47.24	0.29	0.08
6	Ro.	0.41	1.0	117 3 49.47	−0.16	117 3 49.04	0.16	0.03
				3. Roermond				
7	Er.	± 0.34″	0.7	0° 0′ 0.00″	+0.26″	0° 0′ 0.00″	0.31″	0.10
8	Ub.	0.34	0.7	64 6 33.96	−0.06	64 6 33.64	0.07	0.00
9	Pe.	0.37	0.8	140 18 33.79	−0.15	140 18 33.38	0.17	0.03
10	Ne.	0.37	2.9	182 4 19.06	−0.24	182 4 18.56	0.14	0.02
				4. Ubagsberg				
11	Ro.	± 0.34″	0.7	0° 0′ 0.00″	+0.07″	0° 0′ 0.00″	0.08″	0.01
12	Er.	0.34	0.7	41 1 27.05	−0.06	41 1 26.92	0.07	0.00
13	La.	0.34	0.7	126 51 1.84	−0.10	126 51 1.67	0.12	0.01
14	H. Ch.	0.49	1.4	183 39 50.22	+0.45	183 39 50.60	0.37	0.14
15	To.	0.37	0.8	255 2 35.84	+0.16	255 2 35.93	0.17	0.03
16	Pe.	0.37	0.8	309 6 7.89	−0.30	309 6 7.52	0.33	0.11
				5. Henri-Chapelle				
17	To.	± 0.47″	1.3	0° 0′ 0.00″	+0.50″	0° 0′ 0.00″	0.44″	0.19
18	Ub.	0.47	1.3	76 53 13.63	−0.53	76 53 12.60	0.46	0.21
19	La.	0.47	1.3	162 24 47.82	+0.03	162 24 47.35	0.03	0.00
				6. Tongres				
20	Mo.	± 0.64″	2.4	0° 0′ 0.00″	+0.58″	0° 0′ 0.00″	0.37″	0.14
21	Pe.	0.64	2.4	55 26 16.96	+0.35	55 26 16.73	0.22	0.05
22	Ub.	0.64	2.4	134 41 43.47	−1.22	134 41 41.67	0.77	0.59
23	H. Ch.	0.64	2.4	166 25 45.66	+0.28	166 25 45.36	0.18	0.03
				7. Peer				
24	Ne.	± 0.73″	3.1	0° 0′ 0.00″	+0.04″	0° 0′ 0.00″	0.02″	0.00
25	Ro.	0.70	2.9	28 32 11.13	+0.56	28 32 11.65	0.33	0.11
26	Ub.	0.70	2.9	81 26 24.22	−1.10	81 26 23.08	0.64	0.41
27	To.	0.43	1.1	128 7 30.03	−0.01	128 7 29.98	0.97	0.01
28	Mo.	0.34	0.7	192 34 27.58 }	+0.13	192 34 27.67	0.16	0.03
28	Lo.	0.34	0.7	267 43 23.04 }	+0.13	267 43 23.13	0.01	0.03
				75° 8′ 55.46″				
				8. Nederweert				
29	Ro.	± 0.64″	2.4	0° 0′ 0.00″	+0.56″	0° 0′ 0.00″	0.36″	0.13
30	Pe.	0.64	2.4	109 42 5.72	−0.76	109 42 4.40	0.48	0.23
31	Lo.	0.64	2.4	137 11 35.79	+0.21	137 11 35.44	0.13	0.02
				9. Lommel				
. .	Ne.	± 0.00″	. .	0° 0′ 0.00″		0° 0′ 0.00″		
32	Pe.	0.90	4.8	60 13 54.99	−1.93″	60 13 53.06	0.87″	0.76
				10. Montaigu				
. .	Pe.	± 0.00″	. .	0° 0′ 0.00″		0° 0′ 0.00″		
33	To.	0.71	3.0	60 6 48.90	+0.01″	60 6 48.91	0.01″	0.00
								4.19

$$m=\sqrt{\frac{4.19}{11}}=\pm 0.62''$$

As the unit of weight there is taken that value $p=\frac{1}{q}$, which belongs to the mean square of error $M^2 = 0.17$ of a direction entering in the net, hence:

$$\left.\begin{aligned} q=\frac{1}{p}=\frac{M^2}{0.17}=\left(\frac{M}{0.412}\right)^2 &= \text{reciprocal of the weight} \\ p=\frac{0.17}{M^2}=\left(\frac{0.412}{M}\right)^2 &= \text{weight of a direction} \end{aligned}\right\}. \quad (1)$$

Thus are apportioned the weights p, which are specified in the complete station data of the net adjustment above.

The net is not a completely free one, but a net tied firmly at the three stations Lommel, Peer, and

Montaigu, a fact by which the condition equations in that northwestern region are influenced.

Before we enter on the subject, it is noted to the "observed directions," in the case of the Stations Lommel and Montaigu, that in the station data on p. 350 for 32 and 33 there are not inserted true directions but angles, since, e.g., at Montaigu the direction from Peer is considered as fixed and error-free and only the direction to Tongres is introduced as faulty with the weight reciprocals $q = 3.0$. The reason why this is so arranged is probably related to the examinations of weight; considered purely formally, the matter could also have been treated in such a way that two measured directions Montaigu-Peer and Montaigu-Tongres with weight reciprocals $= 1.5$ each would be put down, and similarly at Lommel.

At the Station Peer the angle between Montaigu and Lommel is given invariably fixed $= 75°8'55.46''$, which is already expressed in the observed directions A on p. 350 by the fact that the directions Peer-Montaigu and Peer-Lommel are put down with this difference. (We can imagine, for instance, that the station adjustment at Peer is carried out according to section 87.) Now in order that this connecting angle is not changed once more by the net adjustment, the corrections to the directions Peer-Montaigu and Peer-Lommel are both to be assumed *equal* and, therefore, designated with the same number, 28 and 28.

Turning, after this, to the condition equations, we first consider the net as completely free, and then, with $p = 10$ points and $l = 18$ lines forward and backward, there would have to be taken into account $l - 2p + 3 = 1$ side equation and $l - p + 1 = 9$ triangle equations (with $180° +$ excess), according to the rules of (19), p. 226. These ten equations are present in any case, but because of the connection in the northwest there is added a forcing side equation, so that altogether 11 condition equations will occur.

If we consider first the nine triangle closures, they are carried out in quite the same way as previously in the small examples on p. 226 and p. 237; e.g., the first triangle I yields:

$$\begin{array}{rl} (1,2) = & 37°\,39'\,39.16'' \\ (13,14) = & 56\;\;48\;\;48.38 \\ (18,19) = & 85\;\;31\;\;34.19 \\ \hline & 180°\,00'\;\;0.73'' \\ \text{Should be } 180° + & \text{Excess}\;\;1.25 \\ \hline \text{Discrepancy} & W = +0.48''. \end{array}$$

Hence, the condition equation belonging to this is:

$$\text{I.}\; -v_1 + v_2 - v_{13} + v_{14} - v_{18} + v_{19} + 0.48'' = 0. \qquad \text{(a)}$$

The remaining triangle closure equations are formed likewise:

$$\begin{array}{rlr} \text{II.} & -v_2 + v_3 - v_4 + v_5 - v_{12} + v_{13} + 0.26'' = 0 & \text{(b)} \\ \text{III.} & -v_5 + v_6 - v_7 + v_8 - v_{11} + v_{12} + 0.32 = 0 & \text{(c)} \\ \text{IV.} & -v_{14} + v_{15} - v_{17} + v_{18} - v_{22} + v_{23} - 0.17 = 0 & \text{(d)} \\ \text{V.} & -v_{15} + v_{16} - v_{21} + v_{22} - v_{26} + v_{27} + 0.94 = 0 & \text{(e)} \\ \text{VI.} & -v_8 + v_9 + v_{11} - v_{16} - v_{25} + v_{26} + 1.38 = 0 & \text{(f)} \\ \text{VIII.} & -v_9 + v_{10} - v_{24} + v_{25} - v_{29} + v_{30} + 0.89 = 0 & \text{(h)} \\ \text{IX.} & +v_{24} - v_{28} - v_{30} + v_{31} + v_{32} + 1.05'' = 0 & \text{(i)} \\ \text{X.} & -v_{20} + v_{21} - v_{27} + v_{28} + v_{33} + 0.08 = 0. & \text{(k)} \end{array}$$

The discrepancies $+0.48''$ to I, $+0.26''$ to II, etc., are also entered in the triangulation net picture of Fig. 1, p. 349, in the triangles concerned.

Passing over to the side equations, we have the following condition for the central system Ubagsberg as the sine computation through the triangles I, II, III, IV, V, VI:

$$\frac{\sin(2,3)\sin(5,6)\sin(8,9)\sin(26,27)\sin(22,23)\sin(18,19)}{\sin(1,2)\sin(4,5)\sin(7,8)\sin(25,26)\sin(21,22)\sin(17,18)} = 1.$$

The calculation is just as previously already on pp. 229 and 230; therefore, we write only a part of the calculation here:

		Diff. for 10″
$(2,3) = 51° 58' 40.52''$	$\log \sin (2,3) = 9.896\,4014$	165
$(5,6) = 74 \;\; 52 \;\; 1.67$	$\log \sin (5,6) = 9.984\,6724$	57
. . . .	. . . .	
	$\log$ numerator $= 9.449\,8818$.	

Likewise log denominator = 9.449 8995

$w = \quad -177$.

Hence, first the linear side equation in units of the sixth decimal of logarithms is:

$$+1.65\,(v_3 - v_2) + 0.57\,(v_6 - v_5) + \ldots \; -2.73\,(v_2 - v_1) - 17.7 = 0$$

$$+1.65\,v_3 - 4.38\,v_2 \ldots \qquad -17.7 = 0\,. \qquad (g')$$

We will further divide this condition equation, which is to be rigorously satisfied, by 2, whereby, fully written out, it becomes now:

$$\left.\begin{array}{l} +1.37\,v_1 - 2.19\,v_2 + 0.82\,v_3 + 1.16\,v_4 - 1.45\,v_5 + 0.29\,v_6 + 0.51\,v_7 \\ -0.77\,v_8 + 0.26\,v_9 + 0.25\,v_{17} - 0.33\,v_{18} + 0.08\,v_{19} + 0.20\,v_{21} \\ -1.91\,v_{22} + 1.71\,v_{23} + 0.79\,v_{25} - 1.79\,v_{26} + 1.00\,v_{27} - 8.85 = 0 \end{array}\right\} . \qquad (g)$$

The circumstance that we have divided the original equation (g') by 2, in order to obtain the final equation (g), rests on the desire to make the coefficients of all condition equations *equal* to each other as far as possible, and since all previous equations (a) to (k) had the coefficients 1 (namely, +1 or −1), it was proper to decrease also the coefficients in (g') somewhat; this however is only a matter of computational convenience, and depending on the option of computing in units of the sixth decimal of logarithms, or in half of such units, as here.

We could also make a control computation according to Legendre's theorem for the side equation (g'), i.e. for its most important absolute term −17.7, as in (8), p. 229; however we do not discuss this now.

On the other hand, we will now take up the other one, namely the *forcing* side computation, for the connection to the *two* base lines Lommel-Peer and Peer-Montaigu.

As invariants there are given:

Lommel—Peer	Peer—Montaigu	
$\log s_1 = 4.155\,0338{\cdot}4$	$\log s_2 = 4.570\,5942{\cdot}6$.	(2)

We compute according to the so-called method of additaments (in our Volume III of *Handbuch der Vermessungskunde*, 7th Edition, 1923, pp. 246-250) and compute the logarithmic additament of a side s according to the formula: $\frac{\mu}{6r^2}s^2$, where $\mu = 0.43429$ and r is the earth's radius in our case for 51°10' latitude $\log r = 6.80495$; this yields:

$$\frac{\mu}{6r^2}s_1^2 = \ldots \;\; 3{\cdot}6 \qquad \frac{\mu}{6r^2}s_2^2 = \ldots \;\; 24{\cdot}6\,.$$

These values subtracted from the above $\log s_1$'s and $\log s_2$'s yield:

$$\log s_1' = 4.155\,0334{\cdot}8 \qquad \log s_2' = 4.570\,5918{\cdot}0 \tag{3}$$

$$\log \frac{s_1'}{s_2'} = 9.584\,4416{\cdot}8\,,$$

and now, according to the view of the triangulation net picture of Fig. 1, p. 349, there exists the following equation:

$$\frac{s_1' \sin(32) \sin(29{,}30) \sin(8{,}9) \sin(15{,}16) \sin(20{,}21)}{s_2' \sin(30{,}31) \sin(9{,}10) \sin(16{,}11) \sin(21{,}22) \sin(33)} = 1\,.$$

The calculation yields first in units of the sixth decimal of logarithms:

$$1.21\, v_{32} - 0.75\,(v_{30} - v_{29}) + 0.52\,(v_9 - v_8) + \ldots - 2.36\,(v_{10} - v_9) \ldots - 6.2 = 0\,.$$

If we divide again into halves and arrange and collect according to the numbers of the v's, we obtain the complete side equation of the forcing connection:

$$\left.\begin{array}{l} +0.26\, v_8 - 1.44\, v_9 + 1.18\, v_{10} + 0.86\, v_{11} + 0.76\, v_{15} - 1.62\, v_{16} \\ +0.73\, v_{20} - 0.93\, v_{21} + 0.20\, v_{22} - 0.38\, v_{29} - 1.65\, v_{30} + 2.03\, v_{31} \\ -0.61\, v_{32} + 0.61\, v_{33} + 3.10 = 0 \end{array}\right\}. \tag{1}$$

Now we possess in (a) (b) . . . (1) the correct number of eleven linear condition equations, which must first be arranged into a table, similarly as previously in (12), p. 230 or 241. In order to save space we can only indicate this table here:

		1	2	3	4	5	6	7	8	. .
$\frac{1}{p} = q =$		3.7	0.7	0.7	0.6	1.0	1.0	0.7	0.7	. .
1	a	−1	+1	.	.	.	.	.	.	. .
2	b	.	−1	+1	−1	+1	.	.	.	. .
3	c	.	.	.	.	−1	+1	−1	+1	. .
4	d	.	.	.	.	.	.	.	.	. .
5	e	.	.	.	.	.	.	.	.	. .
6	f	.	.	.	.	.	.	.	−1	. .
7	g	+1.37	−2.19	+0.82	+1.16	−1.45	+0.29	+0.51	−0.77	. .
. . .	. .	.	.	.	.	.	.	.	.	. .

(4)

There follows the computation of the sums $\left[\frac{aa}{p}\right]$, etc., as coefficients of the normal equations (5), p. 147, e.g.:

$$\left[\frac{aa}{p}\right] = [qaa] = 3.7 + 0.7 \ldots = +9.1$$

$$\left[\frac{ab}{p}\right] = [qab] = -0.7 \ldots = -1.4$$

$$\left[\frac{ac}{p}\right] = [qac] = \quad = 0$$

. . . .

$$\left[\frac{ag}{p}\right] = [qag] = -1.37 \times 3.7 - 2.19 \times 0.7 \ldots = -6.07$$

$$\left[\frac{gg}{p}\right] = [qgg] = 1.37^2 \times 3.7 + 2.19^2 \times 0.7 + 0.82^2 \times 0.7 \ldots$$

$$= 7.00 + 3.36 + 0.47 \ldots = 42.70$$

. . . .

After all coefficients are thus computed without difficulty, we form the normal equations (5), p. 147, which, in the abbreviated manner of writing (according to p. 86), are represented thus:

k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}	k_{11}	w
+ 9.1	− 1.4	.	− 2.7	.	.	− 6.07	.	.	.	.	+ 0.48
	+ 4.4	− 1.7	.	.	.	− 0.05	.	.	.	.	+ 0.26
		+ 4.8	.	.	− 1.4	+ 0.84	.	.	.	− 0.42	+ 0.32
			+ 9.6	− 3.2	.	+ 7.92	.	.	.	+ 0.13	− 0.17
				+ 10.4	− 3.7	+ 1.23	.	.	− 3.5	+ 0.80	+ 0.94
					+ 8.8	− 6.73	− 3.7	.	.	+ 0.57	+ 1.38
						+ 42.70	+ 2.08	.	− 0.62	− 1.81	− 8.85
							+ 14.5	− 5.5	.	+ 1.52	+ 0.89
								+ 13.4	− 0.7	+ 5.90	+ 1.05
									+ 9.6	− 2.15	+ 0.08
										+ 31.97	− 3.10

The solution of these normal equations yields:

$$k_1 = +0.0010 \qquad k_5 = -0.3748 \qquad k_9 = -0.2897$$
$$k_2 = -0.1459 \qquad k_6 = -0.2942 \qquad k_{10} = -0.1082$$
$$k_3 = -0.2329 \qquad k_7 = +0.2569 \qquad k_{11} = +0.1849\,.$$
$$k_4 = -0.3213 \qquad k_8 = -0.3025$$

With these values k we compute the corrections v according to the formulae (3), p. 140, where we follow table (4) by columns, e.g.:

$$p_1 v_1 = -k_1 + 1.37\,k_7 = -0.001 + 0.352 = +0.351\ ,$$

here

$$\frac{1}{p_1} = q_1 = 3.7\ ,$$

hence, $v_1 = +1.30''$, as is already entered on p. 350.

Moreover, we carry out this calculation of all v's itself by means of a table, as has been shown previously on p. 241; therefore, nothing is mentioned further to this now. The calculation of the corrected directions by adding the v's to the measured directions is already carried out in the table of station data on p. 350, and in addition, it should be noted only that the adjusted directions $A + v$ were shifted once more in each set in such a way that the set begins again with $0°0'0''$, which is a purely formal change.

The next thing is now to extract also the adjusted *angles* as differences of the directions, to check the nine triangles as to $180°0'0''$ + excess, and to calculate all triangle sides. Instruction for this is not necessary, since such calculations have already been taken up on pp. 231 and 243; the triangle sides can in our case be computed in connection with the base sides (2) or, as the case may be, (3) according to the "method of additaments" or also according to Legendre's theorem.

Following are the results for this, denoted only by numbers according to Fig. 1, p. 349:

1. log s = 4.415 6477 19.
2. log s = 4.491 6525 13.
3. log s = 4.663 3423 4.
5. log s = 4.650 8960 12.
6. log s = 4.423 9849 7.
8. log s = 4.591 5057 11.
9. log s = 4.579 5859 25.
10. log s = 4.284 9536 29.
14. log s = 4.279 0039 18.

15. log s = 4.546 5598 22.
16. log s = 4.676 9917 26.
17. log s = 4.534 6840 23.
20. log s = 4.610 2294 33.
21. log s = 4.592 9508 27.
24. log s = 4.429 2849 30.
28. log s = 4.570 5943 (33.)
28. log s = 4.155 0338 (32.)
31. log s = 4.490 4037 32.

The distribution of the v's and the calculations of accuracy to be drawn from the v's offer more interest. In the total station data on p. 350 we have computed, in addition to the v's themselves, also the $v\sqrt{p}$'s and the v^2p's besides the sum $[v^2 p]$, which yields the mean error of the unit of weight:

$$m = \sqrt{\frac{[v^2 p]}{11}} = \pm 0.62''. \tag{5}$$

(To this, we also have the check $-[wk]$, which, calculated, yields likewise very closely $= 4.2$, in agreement with $[v^2 p] = 4.19$.)

This $m = \pm 0.62''$ is to be compared with the original determination $M = \pm 0.412''$ in (1) as the error of the unit of weight *before* the adjustment, and the ratio $0.62:0.41 = 1.5$ is in such cases usually taken as the scale for the goodness of the estimate of accuracy before the adjustment. Hence, in our case with 1.5, the errors are in general after the adjustment 50% larger than was anticipated before the adjustment, and a similar ratio usually appears in such cases as a rule and is still regarded as satisfactory, while a considerably higher value of the ratio $m:M$ than 1.5 could perhaps give rise to repeating the adjustment with new assumptions of weight.—

Even apart from this ratio $m:M$, which in general holds good, the table of station data of p. 350 is further a matter for considerations as to whether the estimates were good in the *individual* cases of directions. For the $v\sqrt{p}$'s are the errors reduced to the unit of weight $p = 1$, and therefore, we will now examine which $v\sqrt{p}$'s are considerably larger than $0.41''$. Only the two cases v_{27} and v_{32} are twice as large as $0.41''$ or still larger; they now could perhaps give rise to a closer examination, which however we cannot discuss here, since we have taken up the whole only as a formal school example.

In conclusion, if we survey once again Helmert's theory of the net adjustment as a whole, then we find herein progress with respect to earlier theories in more ways than one. First, the theory with the weights of *directions* $1:q$ is a completion of the previous rougher assumptions of "number of pointings" as weights, and in such cases, which were the rule in the measurement of degrees of longitude, in the case of which rigorous adjustments according to Bessel's theory with the coefficients $[\alpha\alpha]$, $[\alpha\beta]$ . . . were already available in the most convenient way, the new theory of the q's was excellently suited. However, if the $[\alpha\alpha]$'s, $[\alpha\beta]$'s, etc., are to be calculated especially for the purpose of the q's, the convenience of the new method is decreased while the other advantages remain.

The second distinctive characteristic of Helmert's theory with respect to earlier methods is the introduction of the *net direction error* ν in equation (22), p. 348, which was found in the mean to be $\nu = \pm 0.4''$; and it is only to be asked with what degree of reliability such a ν can in the individual case be introduced?—

Chapter III

POINT DETERMINATION BY ADJUSTMENT OF COORDINATES

Section 93. Change of Direction and Change of Coordinates

In the case of all adjustments of coordinates, we shall see recur a fundamental problem, which we therefore premise once for all.

In Fig. 1 we have a fixed point P_1 with the rectangular coordinates x_1, y_1, and a second variable point P with the coordinates $x\,y$; let the distance from P_1 to P be $= s$, and the direction angle of the ray from P_1 to P be (P_1P) or designated more briefly $= \varphi$. Then there exist, as is known, the equations:

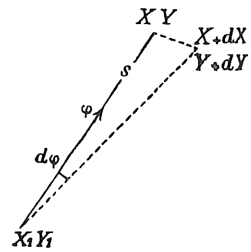


Fig. 1.

$$y - y_1 = s \sin \varphi \qquad x - x_1 = s \cos \varphi \tag{1}$$

$$\tan \varphi = \frac{y - y_1}{x - x_1} \tag{2}$$

or

$$\varphi = \arctan \frac{y - y_1}{x - x_1}\,. \tag{3}$$

(In Figs. 1 and 2 X and Y correspond to the symbols x and y of the text.)

Now we will let the point P move to P', whereby its coordinates x, y change to the closely adjacent $x + dx$, $y + dy$; and we ask what change does the direction angle φ thereby undergo? Insofar as this change $d\varphi$ is supposed to be only small, we shall determine it by differentiation, namely according to (2):

$$d\varphi = \frac{\partial \arctan \frac{y - y_1}{x - x_1}}{\partial x}\, dx + \frac{\partial \arctan \frac{y - y_1}{x - x_1}}{\partial y}\, dy\,; \tag{4}$$

the two parts, derived independently, yield:

$$\frac{\partial \arctan \frac{y - y_1}{x - x_1}}{\partial x} = \frac{1}{1 + \left(\frac{y - y_1}{x - x_1}\right)^2}\left(-\frac{y - y_1}{(x - x_1)^2}\right)$$

and

$$\frac{\partial \arctan \frac{y - y_1}{x - x_1}}{\partial y} = \frac{1}{1 + \left(\frac{y - y_1}{x - x_1}\right)^2}\left(\frac{1}{x - x_1}\right).$$

If we will collect hence the function (4), we note that in both parts there will arise in the denominator the expression $(y - y_1)^2 + (x - x_1)^2$ which is $= s^2$ and thereby we will have:

$$d\varphi = -\frac{y - y_1}{s^2} dx + \frac{x - x_1}{s^2} dy. \tag{5}$$

But it is to be noted here that $d\varphi$ is obtained from equation (5) in radian measure: and if we aim to calculate $d\varphi$ in degree measure, then we must multiply the right-hand side by ρ (= 206,265 for seconds of old [sexagesimal] division), i.e.

in seconds: $$d\varphi = -\frac{y - y_1}{s^2} \rho\, dx + \frac{x - x_1}{s^2} \rho\, dy. \tag{6}$$

Because of the original equations (1) we can write this (5) also thus:

$$d\varphi = -\frac{\rho \sin\varphi}{s} dx + \frac{\rho \cos\varphi}{s} dy; \tag{7}$$

either (7) or (6) is the basic formula mentioned at the beginning, necessary for *all* adjustments of coordinates (which the computer therefore will know even by memory soon).

Logarithmic derivation of the basic formula

However simple the direct differentiation of the equation (3) was, yet it leaves us the opportunity of doing the derivation also somewhat differently, namely in logarithmic form, in order to consider the matter from all sides. For if we write the equation (2) logarithmically, then we have:

$$\log \tan\varphi = \log(y - y_1) - \log(x - x_1)$$

and this differentiated on both sides yields:

$$d \log \tan\varphi = \frac{1}{\tan\varphi} \frac{d\varphi}{\cos^2\varphi} = \frac{dy}{y - y_1} - \frac{dx}{x - x_1}$$

$$d\varphi = \frac{\sin\varphi \cos\varphi}{y - y_1} dy - \frac{\sin\varphi \cos\varphi}{x - x_1} dx, \tag{8}$$

and because of the equations (1), this changes immediately to (7) (with the addition of ρ).

Geometrical interpretation of the basic formula

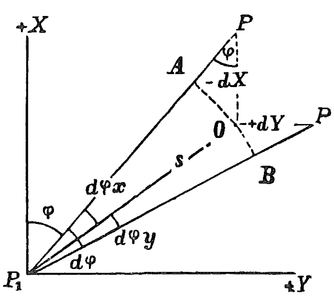


Fig. 2.

One can establish our basic formula also geometrically, according to Fig. 2 (in which dX and dY are supposed to mean the same as dx and dy in the text). We imagine $d\varphi$ broken into two parts $d\varphi_x$ and $d\varphi_y$, and do not let the point P move at *once* to P' but by the roundabout way of the two legs [Katheten] of a rectangular triangle $PO = dx$ and $OP' = dy$. But dx must be *negative* here in the sense of Fig. 2, as it is thus entered. If we further imagine the two segments dx and dy projected rectangularly to the ray direction φ, then we obtain:

$$O A = - d x \sin \varphi , \qquad O B = + d y \cos \varphi , \qquad d \varphi = d \varphi_x + d \varphi_y = \frac{O A}{s} + \frac{O B}{s} ,$$

hence:

$$d \varphi = - \frac{\sin \varphi}{s} d x + \frac{\cos \varphi}{s} d y . \tag{9}$$

Here we are only to add ρ again as factor, in order to obtain agreement with the previous formula (7).

This geometric foundation of the equation (7) is, it is true, more illustrative than the analytical differentiation in the case of (4) and (8); but the differentiation has the advantage that it is valid more generally, and renders superfluous the discussion of various *sign* questions which would be tied to the equations (9).

Various cases of change of direction and coordinates

In the original assumption of Fig. 1, P_1 was a fixed base point and P a variable target point, and for this, the equation (6) or (7) is valid; but now, conversely, the target point P can also be fixed and the base point P_1 variable, and then a quite similar formula will be valid as in the previous case, only with the difference that $\varphi \pm 180°$ takes the place of φ and that, therefore, $\sin\varphi$ and $\cos\varphi$ change their signs. This is made still clearer by Figs. 3 and 4.

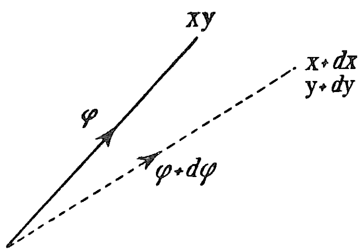


Fig. 3.
Direction angle at the fixed point.

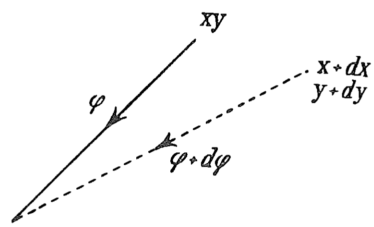


Fig. 4.
Direction angle at the variable point.

In order to be able to compare the two different cases, we will now drop the designations base point and target point and only distinguish a fixed point and a variable point. Then we have:

Case I (Fig. 3). The direction angle is measured from the fixed point.

Change of direction:

$$d \varphi = - \frac{\sin \varphi}{s} \varrho \, d x + \frac{\cos \varphi}{s} \varrho \, d y . \tag{10}$$

Case II (Fig. 4). The direction angle is measured from the variable point.

Change of direction:

$$d \varphi = + \frac{\sin \varphi}{s} \varrho \, d x - \frac{\cos \varphi}{s} \varrho \, d y . \tag{11}$$

Also both points can be variable (Figs. 5 and 6). In this case, we imagine at first P_1 fixed and P_2

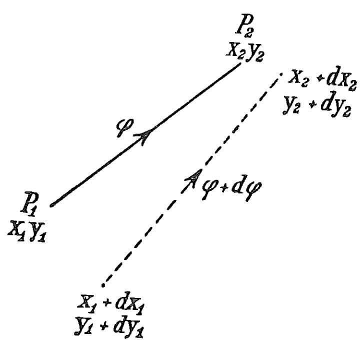


Fig. 5.
Direction angle at P_1.

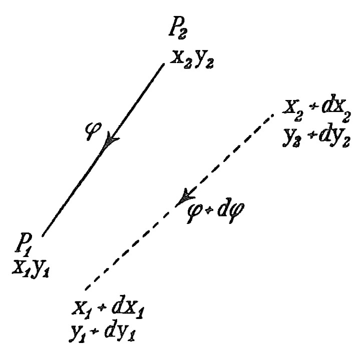


Fig. 6.
Direction angle at P_2.

variable, then P_1 variable and P_2 fixed. The total change of the direction angle is then equal to the sum of the two amounts (10) and (11). Here also we should note from which of the two points the direction angle is measured. After this, there is to be distinguished:

Case III (Fig. 5). The direction angle is measured from P_1. If we assume at first P_1 as fixed, then case I is present with equation (10); if P_2 is then regarded as fixed, then case II with equation (11) is present. The total change of direction is:

$$d\varphi = -\frac{\sin\varphi}{s}\varrho\, d x_2 + \frac{\cos\varphi}{s}\varrho\, d y_2 + \frac{\sin\varphi}{s}\varrho\, d x_1 - \frac{\cos\varphi}{s}\varrho\, d y_1 . \tag{12}$$

Case IV (Fig. 6). The direction angle is measured from point P_2. If we assume again at first P_1 as fixed, then equation (11) is valid, in the other case equation (10).

The total change of direction is:

$$d\varphi = +\frac{\sin\varphi}{s}\varrho\, d x_2 - \frac{\cos\varphi}{s}\varrho\, d y_2 - \frac{\sin\varphi}{s}\varrho\, d x_1 + \frac{\cos\varphi}{s}\varrho\, d y_1 . \tag{13}$$

Direction coefficients for meters or for decimeters

Let us especially designate the coefficients of the basic formula (6) or (7), p. 358, in which, according to case I, Fig. 3, the direction angle φ is thus measured from the fixed point, by a and b, namely:

$$a = -\frac{\varrho}{s}\sin\varphi \qquad b = +\frac{\varrho}{s}\cos\varphi \tag{14}$$

or

$$a = -\frac{\varrho}{s^2}\Delta y \qquad b = +\frac{\varrho}{s^2}\Delta x . \tag{15}$$

In this case, Δy means a change of ordinates and Δx a change of abscissae in the sense of direction of φ, which is always especially to be noted in (15), whereas according to (14) the a's and b's are entirely single-valued functions of φ and s.

The coefficients can also be written in the form

$$a = -\frac{\varrho}{\Delta y}\sin^2\varphi \qquad b = +\frac{\varrho}{\Delta x}\cos^2\varphi \tag{16}$$

in which case the computation of s is spared.

Now we can, of course, calculate for every given φ and s the functions a and b, e.g.: for $s = 2700$ m and $\varphi = 202°17'$:

$\log \varrho$	5.3144	$\log \varrho$	5.3144
$\log s$	3.4314	$\log s$	3.4314
$\log\left(\frac{-\varrho}{s}\right)$	1.8830_n	$\log\left(\frac{\varrho}{s}\right)$	1.8830
$\log \sin\varphi$	9.5789_n	$\log \cos\varphi$	9.9663_n
$\log a$	1.4619	$\log b$	1.8493_n
$a = +28.97$		$b = -70.68$.	

(17)

This computation is carried only with four-place logarithms, which is sufficient in most cases. Now the following formula corresponds to this example:

$$d\varphi = +28.97\, dx - 70.68\, dy \text{ for meters,} \tag{18}$$

where dx and dy are changes of coordinates in meter measure because s was introduced in meters.

Or the equation (18) says: If a length 2700 m long is displaced at the direction angle 202°17' at the opposite end point in x and in y by 1 m each, then the ray direction is changed by +28.97"−70.68" = −41.71".

In a great many cases, the displacements of coordinates dx and dy are considerably smaller than 1 m, for which reason the computation proves more convenient with dx and dy in *decimeters*; in this case, the a's and b's become likewise smaller, namely $\frac{1}{10}$ of the previous ones, or we have for decimeters:

$$\left.\begin{aligned} a &= -\frac{1}{10}\frac{\varrho}{s}\sin\varphi & b &= +\frac{1}{10}\frac{\varrho}{s}\cos\varphi \\ &= -\frac{1}{10}\frac{\varrho}{s^2}\Delta y & &= +\frac{1}{10}\frac{\varrho}{s^2}\Delta x \\ &= -\frac{1}{10}\frac{\varrho}{\Delta y}\sin^2\varphi & &= +\frac{1}{10}\frac{\varrho}{\Delta x}\cos^2\varphi \end{aligned}\right\}, \tag{19}$$

and the previous numerical example yields:

$$d\varphi = +\,2.897\,dx - 7.068\,dy \text{ for decimeters,}$$

i.e., if the displacements dx and dy amount to 1 decimeter or 0.1 m each, then the change of direction is $d\varphi = +\,2.897'' - 7.068'' = -\,4.171''$ or the same which we obtain in the first case with $dx = dy = 0.1$ m.

Since we shall compute later mostly with dx and dy in decimeters, and since the distances s are counted most conveniently in kilometers, we will introduce, for this, special symbols, namely for $s = 1000$ m = 1 km:

$$-\frac{\varrho}{10}\frac{\sin\varphi}{1000} = \mathfrak{x} \qquad +\frac{\varrho}{10}\frac{\cos\varphi}{1000} = \mathfrak{y}, \tag{20}$$

or because $\rho = 206{,}265$ for seconds of old [sexagesimal] division:

$$-\,20.6265\sin\varphi = \mathfrak{x} \qquad +\,20.6265\cos\varphi = \mathfrak{y}. \tag{21}$$

Then if we understand by S the distance in kilometers, hence $S = s{:}1000$, then we will have:

$$\left.\begin{aligned} a = \frac{\mathfrak{x}}{S} \qquad b = \frac{\mathfrak{y}}{S} \\ \text{for } dx \text{ and } dy \text{ in decimeters, } S \text{ in kilometers} \end{aligned}\right\}. \tag{22}$$

Concerning the direction coefficients a and b in the case of computation in new [centesimal] division, cf. section 111.

Section 94. Various Auxiliary Means for the Determination of the Direction Coefficients

The direction coefficients which we always need in the case of extended adjustment computations are according to (19) of the previous section 93 for dx and dy in *decimeters* and in the case of the measuring of the direction angle φ at the *fixed* point:

$$a = -\frac{1}{10}\frac{\varrho}{s}\sin\varphi \qquad b = +\frac{1}{10}\frac{\varrho}{s}\cos\varphi \tag{1}$$

or

$$a = -\frac{1}{10}\frac{\varrho}{s^2}\Delta y \qquad b = +\frac{1}{10}\frac{\varrho}{s^2}\Delta x \tag{2}$$

or

$$a = -\frac{1}{10}\frac{\varrho}{\Delta y}\sin^2\varphi \qquad b = +\frac{1}{10}\frac{\varrho}{\Delta x}\cos^2\varphi \tag{3}$$

and with arrangement for the distance S in kilometers:

$$\text{For old [sexagesimal] division: } \mathfrak{x} = -20.6265\sin\varphi \qquad \mathfrak{y} = +20.6265\cos\varphi \tag{4}$$

$$\text{For new [centesimal] division: } \mathfrak{x} = -63.6620\sin\varphi \qquad \mathfrak{y} = +63.6620\cos\varphi \tag{5}$$

$$a = \frac{\mathfrak{x}}{S} \qquad b = \frac{\mathfrak{y}}{S} \qquad (S \text{ in kilometers}). \tag{6}$$

We can determine these a's and b's in different ways, as we will present now:

1. Logarithmic computation

The simplest and most readily available means to determine these coefficients is the calculation with four- to five-place logarithms in addition to the logarithmic calculation of φ and s itself:

$$\left.\begin{array}{llllll}
P & y = -2784.96\text{ m} & & x = +8326.92\text{ m} & & \\
P_1 & y_1 = -1761.11 & & x_1 = +10825.29 & & \\
\hline
 & y - y_1 = -1023.85 & & x - x_1 = -2498.37 & & \\
 & = \Delta y & & = \Delta x & & \\
\log\Delta y & 3.010\,236_n & 3.010\,236 & & 3.397\,657 & \\
\log\Delta x & 3.397\,657_n & 9.578\,869 & & 9.966\,289 & \\
\hline
\log\tan\varphi & 9.612\,579 & 3.431\,367 & \log s & 3.431\,368 & \\
 & & & \log s^2 & 6.862\,735 & \\
 & & & s = 2700.12\text{ m} & & \\
 & \varphi = 202^\circ\,17'\,3.0'' & & S = 2.70\text{ km} & &
\end{array}\right\}. \tag{7}$$

This is the usual computation of φ and s, to which the computation of a and b with four-place logarithms is added now, written out fully

$$\left.\begin{array}{l|l}
\log -\frac{\varrho}{10} & 4.3144_n \\
\log(1:s) & 6.5686 \\
\log\sin\varphi & 9.5789_n \\
\hline
\log a & 0.4619 \\
a = +2.897 &
\end{array}\qquad
\begin{array}{l|l}
\log +\frac{\varrho}{10} & 4.3144 \\
\log(1:s) & 6.5686 \\
\log\cos\varphi & 9.9663_n \\
\hline
\log b & 0.8493_n \\
b = -7.068 &
\end{array}\right\}, \tag{8}$$

or according to the formulae (2):

$$\left.\begin{array}{r|l} \log -\frac{\varrho}{10} & 4.3144_n \\ \log(1:s^2) & 3.1373 \\ \log \Delta y & 3.0102_n \\ \hline \log a & 0.4619 \\ \multicolumn{2}{c}{a = +2.897} \end{array} \qquad \begin{array}{r|l} \log +\frac{\varrho}{10} & 4.3144 \\ \log(1:s^2) & 3.1373 \\ \log \Delta x & 3.3977_n \\ \hline \log b & 0.8494_n \\ \multicolumn{2}{c}{b = -7.068} \end{array}\right\}, \tag{9}$$

or finally according to the formulae (3):

$$\left.\begin{array}{r|l} \log -\frac{\varrho}{10} & 4.3144_n \\ \log(1:\Delta y) & 6.9898_n \\ \log \sin^2\varphi & 9.1578 \\ \hline & 0.4620 \\ \multicolumn{2}{c}{a = +2.897} \end{array} \qquad \begin{array}{r|l} \log +\frac{\varrho}{10} & 4.3144 \\ \log(1:\Delta x) & 6.6023_n \\ \log \cos^2\varphi & 9.9326 \\ \hline & 0.8493_n \\ \multicolumn{2}{c}{b = -7.068} \end{array}\right\}. \tag{10}$$

In regard to the signs we remember once for all that a has the opposite sign of Δy, and b the same sign as Δx. (For *intersecting* Fig. 3, section 93, p. 359.)

For the first introduction into the trigonometric adjustments of intersection, one will always take this logarithmic computation of the coefficients a and b, and only when, moreover, a certain computing skill is achieved, is it worth learning and comparing critically, also the other methods, which we describe afterwards under II-VIII.

II. As second auxiliary means we have computed auxiliary tables of the functions (4) and (5) and compiled in our Appendix, pages [8] to [17]; e.g., for $\varphi = 202°17'$ we have on page [9] of the Appendix at first quite roughly (only 202°):

$$\mathfrak{x} = +7.7 \qquad \mathfrak{y} = -19.1\,,$$

or somewhat more rigorously on page [10] for about 202°20' instead of 202°17':

$$\mathfrak{x} = +7.8 \qquad \mathfrak{y} = -19.1\,,$$

or with interpolation for 202°17':

$$\mathfrak{x} = +7.82 \qquad \mathfrak{y} = -19.09\,,$$

then for $S = 2.70$ km divided with the slide rule:

$$a = +\frac{7.82}{2.70} = +2.90 \qquad b = -\frac{19.09}{2.70} = -7.07\,. \tag{11}$$

Similarly also for the *new* division:

$$202°\,17' = 224\,\text{g}\,76\,\text{c} \qquad S = 2{,}70 \text{ km}$$

according to p. [16]:

$$\mathfrak{x} = +24.14 \qquad \mathfrak{y} = -58.91$$

$$a = +\frac{24.14}{2.70} = +8.9 \qquad b = -\frac{58.91}{2.70} = -21.8\,. \tag{12}$$

These latter a's and b's result also from (11) by multiplication by 3.09 or divisions by 0.321, because 1'' old division = 3.09 cc new division or 1 cc = 0.324''. The divisions with $S = 2.70$ in the case of (11) and (12) are done with the slide rule.

As for the distances s (or S in kilometers) needed here, we take them usually offhand from the net picture of the triangulation which, plotted at 1:10,000 for city triangulation, is fully sufficient for this, but is still enough also at smaller scale. We also compute, besides, the s's to four places logarithmically, which always costs still considerably less effort than to calculate the a's and b's themselves logarithmically.

There exists still another reason for preferring this method II, namely the introduction of the *distances* s into the adjustment form (sections 96 and 99). For, apart from their utilization for the determination of the coefficients a and b, these distances s are also desirable in the adjustment form because they serve for the explanation and judgment of the distribution of errors. If, for instance, one or the other correction v becomes especially large, then one need not worry if it belongs to a small s. A single glance at the column of the s's is often sufficient for the characterization of the whole case; e.g., all distances s smaller than 1 km is a case of lowest order in which considerable v's are to be tolerated much rather than in an adjustment with s between 1 km and 5 km, and so forth.

Briefly, for all these reasons we prefer a method of computation of the a's and b's which brings by itself into the adjustment form also the distances s.

Corresponding tables for dx and dy in meters are given by Lips in *Allgem. Verm. Nachr.*, 1928, pp. 329-335.

III. Logarithmic differences also lead to the coefficients a and b, as the following example, in addition to (7), may show:

			Logarithmic table differences
$\Delta y = -1023.85$	$\log \Delta y$	$3.010\,226_n$	$b)$ 43· for 0.1 m
$\Delta x = -2498.37$	$\log \Delta x$	$3.397\,657_n$	$a)$ 17· for 0.1 m
	$\log \tan \varphi$	$9.612\,569$	6·0 for 1''
	$\varphi = 202° 17' 1''$.		

Now we see at once, without any theory, that a change dy of 0.1 m expresses itself logarithmically by 0.000 043 = 43· and a change $d\varphi = 1''$ by 6·0, that, therefore, $dy = 0.1$ m must bring the value $d\varphi = \frac{43}{6.0} = 7.1''$ and a change $dx = 0.1$ m, the value $\frac{17}{6.0} = 2.8$. But these are just the required b's and a's, at first without taking into account the sign, and then according to the sign rule, which was indicated in the case of (9):

$$a = +\frac{17}{6.0} = +2.8 \qquad b = -\frac{43}{6.0} = -7.2\,. \tag{13}$$

These divisions are done with the slide rule.

These values (13) agree with (11) as closely as is possible when the main computation for φ and s is carried with six-place logarithms. If we made the main computation to 7 places, then also the coefficients a and b would become more accurate by one place, in the case of a five-place computation accordingly more inaccurate.

In formulas, this method expresses itself thus, in agreement with the logarithmic derivation in (8), section 93, p. 358:

$$a = -\frac{d \log \Delta x}{d \log \tan \varphi} \qquad b = +\frac{d \log \Delta y}{d \log \tan \varphi}, \tag{14}$$

where the tabular differences are denoted by $d \log \ldots$, where the signs are to be noticed, however; e.g. above $\Delta y = -1023.85$ yields $d \log \Delta y = -43$·, and so forth. By the way, one computes best, in the case of this method, at first without taking into account the signs and puts these in only afterwards according to the rule which was indicated on the preceding page in the case of (9).

IV. Auxiliary tables for $\rho \sin\varphi \cos\varphi$. This function is the subject of "Logarithmische Hilfstafel zur Berechnung der Fehlergleichungskoeffizienten beim Einschneiden nach der M. d. kl. Q." by O. Seiffert, Halle, 1892 (*Zeitschr. f. Verm.*, 1893, p. 221).

There is set here:

$$a = -\frac{\sin\varphi\cos\varphi}{\Delta x}\varrho \qquad b = +\frac{\sin\varphi\cos\varphi}{\Delta y}\varrho\,,$$

which, obviously, has the same significance as the previous (1) to (3) (the factor 0.1 excepted), and now the function is brought out:

$$\varrho \sin\varphi\cos\varphi = Z\,, \tag{15}$$

hence

$$a = -\frac{Z}{\Delta x} \qquad b = +\frac{Z}{\Delta y}\,. \tag{16}$$

From Seiffert's table, which goes from 1' to 1' and partially still narrower, we have formed a brief excerpt and compiled on page [19] of the Appendix, with the addition of a further function B, namely:

$$A = \frac{\varrho}{10}\sin\varphi\cos\varphi \text{ and } B = \frac{\varrho^2}{100}\sin\varphi\cos\varphi\,, \tag{17}$$

$\log A$ and $\log B$ being represented from 1° to 1° running through to 360°. We thus have then:

$$a = -\frac{A}{\Delta x}, \qquad b = +\frac{A}{\Delta y}, \qquad ab = -\frac{B}{s^2};$$

e. g., for our above-mentioned example with $\varphi = 202°$ approximately we have (at first without taking into account the sign):

Table, p. [19]

$\log A$	3.8552	$\log A$	3.8552	$\log B$	8.1696
$\log \Delta x$	3.3977	$\log \Delta y$	3.0102	$\log s^2$	6.8627
$\log a$	0.4575	$\log b$	0.8450	$\log ab$	1.3069
$a = 2.9$		$b = 7.0$		$ab = 20.3$.	

The sign of a and b is determined according to the rule indicated in the case of (9), and for ab we remember that it is negative for φ in the first and third quadrant, positive in the second and fourth quadrant. Accordingly, we will have:

$$a = +2.9\,, \qquad b = -7.0\,, \qquad ab = -20.3\,.$$

We have applied this method sometimes as a check *besides* other methods, especially, e.g., we have computed the ab's especially according to it, after the a's and the b's were already determined independently elsewhere and ab was already available as product from a and b, so that by ab, a and b themselves are also examined once again.

V. Scales for the determination of the factors $a - d$ for the normal equations in the case of trigonometric adjustment computations, etc., designed by Mr. Seyfert, surveyor, division head in the geodetic-technical office of the Royal General Commission for Silesia at Breslau. These scales consist of various pencils of rays with cross lines from which, according to the method of the "Isoplethes," the direction coefficients a and b are taken. A report in the *Zeitschr. f. Verm.*, 1893, pp. 219-221, gives information on the theory.

VI. Graphical table by F r a n k e, communicated in his paper, *Koordinatenausgleichung nach Näherungsmethoden* [Adjustment of coordinates according to methods of approximation], Munich, 1884, pp. 133-135, with lithographed table in the Appendix. This table gives in the form of isopleths the functions $\frac{\rho \sin \varphi}{10\, S}$ and $-\frac{\rho \cos \varphi}{10\, S}$, where one enters with φ in the direction angle and with S in the ray length of a radial system and the functions are taken off as rectangular coordinates.

VII. A further graphical table is: Hilfstafel zur Berechnung der Richtungskoeffizienten [Auxiliary table for the computation of the direction coefficients] by Dr. O. E g g e r t, designed by Fr. K r e i s e l, Berlin, 1903 (described in *Zeitschr. f. Verm.*, 1903, p. 666). This table printed on tracing paper presupposes likewise a triangulation net map (1:10,000) and is laid with the zero point on the map point, oriented to the positive direction of abscissae; then one reads at the positions of points the a's and after turning by 90° the b's with their signs directly in the drawn series of circles. If the scale of the triangulation net map deviates from 1:10,000, then the read-off a's and b's are to be increased or decreased accordingly. The basic thought of the construction of these series of circles can be expressed briefly thus: If we erect a right angle at the center of a length s given in the system of coordinates with the direction angle φ, e.g. of the length AP_1 in Fig. 2, p. 358, then this perpendicular intersects the ordinate $(+Y$ in Fig. 2, p. 358) subtended by the angle φ, whereby the center point of a circle is determined by means of the chord $s = AP_1$. As we know, for the radius of this circle there is valid $\Delta x^2 + \Delta y^2 = s^2 = 2\, \Delta y\, r$, hence $r = \frac{s^2}{2\, \Delta y}$, or with the introduction of (15), p. 360, $r = \frac{\rho}{2\, a}$. The corresponding relation follows for the coefficients b.

VIII. Slide rule by V o i g t. At the suggestion and invention of W. Voigt in Flatow, a special graduation has been placed, by D e n n e r t and P a p e, on the back of the "tongue" of the common slide rule stick, with the help of which the functions a and b can be read off according to the equations (15) and (16). Further details, directions for use and accuracy, are given by Voigt in *Zeitschrift für Vermessungswesen*, 1894, pp. 183-188.

If we attempt to compare all these methods of procedure for the determination of the direction coefficients a and b, the direct computation with four- to five-place logarithms deserves the preference in all important cases, e.g., of first to second order and also in the first instruction nothing else is to be advised.

In the case of smooth working of numerous adjustments of third to fourth order, however, the logarithmic calculation of the many a's and b's is too painful; we have had the best experiences here with the auxiliary tables, pp. [8] to [17], where the s's are taken from the representation of the net of the triangulation and the $\mathfrak{x}:s$'s and $\mathfrak{y}:s$'s are read off with the slide rule.

The method of the logarithmic differences (III, p. 364) is much praised, but it has the disadvantage that the logarithmic differences for Δx and Δy cannot be copied directly from the table of logarithms, but each time the *position of the decimal point* has to be considered especially. One is also unpleasantly hindered in the case of the rigorous calculation of the approximations (φ) by the secondary consideration of the a's and b's. In comparison with the smooth calculation with the differences for 10'', which we have learned on p. 229 in the case of triangulation nets, the calculation with the logarithmic differences is much less useful in the case of intersection adjustments.

We have found the Seiffert and similar auxiliary tables (procedure IV, p. 365) useful for check computations.

The different graphical auxiliary means, and the Voigt slide rule, are recommended for the sake of change in the deadening monotony of the computational work.

The fact that the need of such auxiliary means exists is proved by the numerous inventions and suggestions in this field; to make the right choice among them will be left to the experiences and the judgment of the individual computer.

For the coefficients of direction in the case of calculation in new [centesimal] division, compare section 111.

Computational check for the coefficients of direction. Apart from the check $\frac{a}{b} = -\frac{\Delta y}{\Delta x}$, resulting offhand from (15), p. 360, and easily evaluated with the slide rule, there follows from (1), p. 362:

$$a^2 + b^2 = \frac{\varrho^2}{100\, s^2}. \tag{18}$$

Hence, $a^2 + b^2$ becomes independent of the direction angle φ and remains only a function of the ray length s, for which a small auxiliary table, page [18], of the Appendix can easily be set up, the use of which is shown as follows, in connection with the numerical example (7) to (11), pp. 362 and 363:

$2.897^2 + 7.068^2 = 58.35$ or $2.90^2 + 7.07^2 = 58.39$ auxiliary table, p. [18], i.e. sufficiently correct within the accuracy aimed at.

A further check, which was indicated by Pinkwart in *Allgemeine Vermessungsnachrichten*, 1930, pp. 529-530, consists in computing, in addition to the a's and b's, for the given direction angle φ and the given distance s, also the values a_{45} and b_{45}, for the direction angle $\varphi + 45°$, and for the distance $s\sqrt{2}$. Thus we have

$$\left.\begin{aligned} a_{45} &= -\frac{\sin(\varphi + 45°)}{s\sqrt{2}}\varrho \\ b_{45} &= +\frac{\cos(\varphi + 45°)}{s\sqrt{2}}\varrho \end{aligned}\right\} . \tag{19}$$

Then we have

$$a_{45} + b_{45} = \frac{\varrho}{s\sqrt{2}}\Big(-\sin(\varphi + 45°) + \cos(\varphi + 45°)\Big)$$

$$a_{45} - b_{45} = \frac{\varrho}{s\sqrt{2}}\Big(-\sin(\varphi + 45°) - \cos(\varphi + 45°)\Big) .$$

Since $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$, there follows

$$a_{45} + b_{45} = -\frac{\sin\varphi}{s}\varrho = +a$$

$$a_{45} - b_{45} = -\frac{\cos\varphi}{s}\varrho = -b$$

or

$$\left.\begin{aligned} b_{45} + a_{45} &= a \\ b_{45} - a_{45} &= b \end{aligned}\right\} . \tag{20}$$

The computation of the two coefficients a_{45} and b_{45} means additional work, it is true; the above check, however, is more thorough than those mentioned previously.

Section 95. Intersection with Three Angles

As simplest case of intersection with adjustment we study the following in connection with Fig. 1, p. 369:

There are three points P_1, P_2, P_3, fixed by their rectangular coordinates and a new point P is to be determined by angle measurements on P_1, P_2 and P_3, e.g. by the measurement of the angle $A P_1 P_2 = \beta_1$, then $P_1 P_2 B = \beta_2$ and $P_2 P_3 C = \beta_3$.

If only *two* of these angles β were measured, the new point P would be determined unambiguously, and its coordinates could at once be completely determined according to familiar simple rules.

On the other hand, if there is measured also a third angle β, then we have before us a problem of adjustment, which we shall carry out now.

In connection with the fixed lines $P_1 P_2$ and $P_2 P_3$ there are determined, by means of the three measured angles β, to the new point P three rays $P_1 A$, $P_2 B$, and $P_3 C$, which do not intersect, however, at a *single* point P because of the observational errors, but will form a triangle $A B C$, which shows the error.

At first we shall convince ourselves that instead of the observed angles β we can introduce at once also the *direction* angles α as observed which lead from the fixed points $P_1 P_2 P_3$ to the new point P. In any case, we can fix by computation the direction angles $[P_1 P_2]$, in addition to $[P_2 P_3]$ and $[P_3 P_2]$ from the coordinates of the given points, namely in the well-known manner:

$$\left.\begin{aligned} \tan\ [P_1\, P_2] &= \frac{y_2 - y_1}{x_2 - x_1} \text{ and } [P_2\, P_1] = [P_1\, P_2] \pm 180^\circ \\ \tan\ [P_2\, P_3] &= \frac{y_3 - y_2}{x_3 - x_2} \end{aligned}\right\} . \tag{1}$$

Now if the angles β_1, β_2, β_3 are measured to these, then we have immediately also the new direction angles:

$$\left.\begin{aligned} (P_1\, A) &= [P_1\, P_2] - \beta_1 = \text{direction angle } \alpha_1 \\ (P_2\, B) &= [P_2\, P_1] + \beta_2 = \text{direction angle } \alpha_2 \\ (P_3\, C) &= [P_3\, P_2] + \beta_3 = \text{direction angle } \alpha_3 \end{aligned}\right\} . \tag{2}$$

Now we may be able to treat these direction angles α_1, α_2, α_3 further like directly *measured* quantities, for the established rays $[P_1 P_2]$, etc., hold as unchangeably fixed and without errors; therefore, the error of an angle β is carried over directly to α.

Now we have to correct the three direction angles in such a manner that the three corrected rays pass through a *single* point and that at the same time the sum of the squares of the corrections v becomes a minimum.

We denote the coordinates of the final point P by x, y and the final direction angles of the rays by φ. Thus we will have:

$$\left.\begin{aligned} \alpha_1 + v_1 &= \varphi_1 \\ \alpha_2 + v_2 &= \varphi_2 \\ \alpha_3 + v_3 &= \varphi_3 \end{aligned}\right\} . \tag{3}$$

The next requirement is to obtain approximate coordinates of the new point P, which are easily procured by determining the point P from *two* of the angles β, i.e. by computing, say, the intersection $P_1 P P_2$ with the base $P_1 P_2$ in the familiar manner, or else $P_2 P P_3$, or otherwise. For instance, if the calculation were carried with the base $P_1 P_2$ and with the two direction angles α_1 and α_2, then A would be obtained as intersection point for the approximation or B would be obtained as position of approximation from $P_2 P_3$; but it does not matter at all in which way we determine the approximate coordinates of P, and therefore we shall assume in Fig. 1 an arbitrary point (P) as approximate point, of which nothing else is required but that it does not lie far away from the adjusted point P. Let the coordinates of the approximate point be (x), (y).

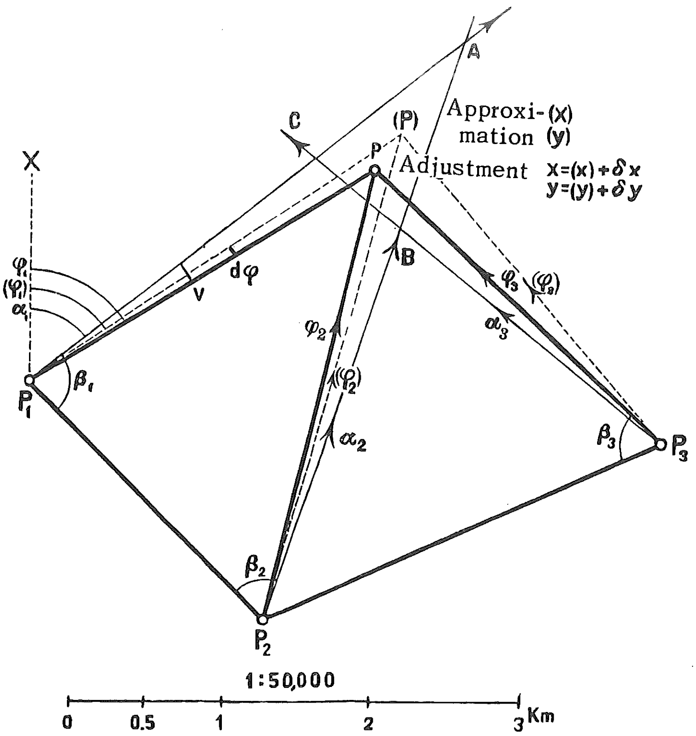


Fig. 1.
Intersection with three angles.

For the approximate point (P) we can compute the direction angles of the rays starting from the fixed points; let them be denoted by (φ_1), (φ_2), (φ_3). These approximate direction angles will change to the final ones if the coordinates (x), (y) change to x, y. If we set

$$\left.\begin{aligned} x &= (x) + \delta x \\ y &= (y) + \delta y \end{aligned}\right\}, \tag{4}$$

then we have according to (10), section 93, p. 359,

$$\varphi = (\varphi) - \frac{\sin(\varphi)}{s}\varrho\,\delta x + \frac{\cos(\varphi)}{s}\varrho\,\delta y,$$

where s is the length of the aiming ray. With the simple denotations on p. 360, we have

$$\varphi = (\varphi) + a\,\delta x + b\,\delta y. \tag{5}$$

With this we obtain according to (3)

$$v = -\alpha + (\varphi) + a\,\delta x + b\,\delta y. \tag{6}$$

If we set again

$$-\alpha + (\varphi) = -l, \tag{7}$$

then we have the error equations, which we write down now for all three rays:

$$\left.\begin{aligned} v_1 &= a_1\,\delta x + b_1\,\delta y - l_1 \\ v_2 &= a_2\,\delta x + b_2\,\delta y - l_2 \\ v_3 &= a_3\,\delta x + b_3\,\delta y - l_3 \end{aligned}\right\} . \tag{8}$$

With this, we already have now the whole theory of the adjustment for intersection, for as soon as we have set up the error equations (8) we can form the normal equations as always:

$$\left.\begin{aligned} &[a\,a]\,\delta x + [a\,b]\,\delta y - [a\,l] = 0 \\ &[a\,b]\,\delta x + [b\,b]\,\delta y - [b\,l] = 0 \text{ with } [l\,l] \end{aligned}\right\} , \tag{9}$$

and with these, we compute further in the familiar way according to section 14 and following.

We will test this immediately on a simple example which corresponds approximately to the conditions of Fig. 1, p. 369.

The following are the coordinates of the three fixed points:

$$\left.\begin{aligned} & \qquad\quad y && \qquad\quad x \\ P_1,\ y_1 &= -25014.26 \text{ m} \qquad & x_1 &= +42133.28 \text{ m} \\ P_2,\ y_2 &= -23406.93 & x_2 &= +40493.76 \\ P_3,\ y_3 &= -20728.34 & x_3 &= +41632.97 \end{aligned}\right\} . \tag{10}$$

At the same time the approximate coordinates of the point to be determined:

$$(P),\ (y) = -22501.20 \qquad (x) = +43512.40 . \tag{11}$$

From (10) we compute the fixed direction angles in the familiar way:

$$\left.\begin{aligned} [P_1\,P_2] &= 135^\circ\,34'\ \ 5'' \quad \text{or} \quad [P_2\,P_1] = 315^\circ\,34'\ \ 5'' \\ [P_2\,P_3] &= \ \ 66^\circ\,57'\,35'' \quad \text{or} \quad [P_3\,P_2] = 246^\circ\,57'\,35'' \end{aligned}\right\} . \tag{12}$$

and

Now the three angles are supposed to be measured:

$$\beta_1 = 74^\circ 19'\,41'' , \qquad \beta_2 = 61^\circ 8'\,10'' , \qquad \beta_3 = 69^\circ 42'\,28'' . \tag{13}$$

These measured angles attached to the fixed direction angles (12) yield the new direction angles α, i.e.:

$$\left.\begin{array}{rlrlrl} [P_1\,P_2] = & 135^\circ\,34'\ \ 5'' & [P_2\,P_1] = & 315^\circ\,34'\ \ 5'' & [P_3\,P_2] = & 246^\circ\,57'\,35'' \\ -\beta_1 = - & \ \ 74\ \ \ 19\ \ \ 41 & +\beta_2 = & \ \ 61\ \ \ \ 8\ \ 10 & +\beta_3 = & \ \ 69\ \ \ 42\ \ \ 28 \\ \hline \alpha_1 = & \ \ 61^\circ\,14'\,24'' & \alpha_2 = & \ \ 16^\circ\,42'\,15'' & \alpha_3 = & 316^\circ\,40'\,03'' \end{array}\right\} . \tag{14}$$

Now there follows the computation of the approximate direction angles (φ), etc., and with this we combine at once also the computation of the coefficients a and b according to section 93. This is contained in the following table:

Computation of (φ) *in addition to a and b* (15)

Points B A	$y_b - y_a = \Delta y$, $x_b - x_a = \Delta x$, $(AB) = (\varphi)$ $\tan\ (\varphi) = \frac{\Delta y}{\Delta x}$ y	x	(φ)	Distance s with Easting Column	$a = -\frac{0.1\,\varrho}{s^2}\Delta y$	$b = +\frac{0.1\,\varrho}{s^2}\Delta x$
(P)	− 22501.20	+ 43512.40	3.400203	3.4002	0,1 ϱ ..	4.3144
P_1	− 25014.26	+ 42133.28	3.139602	9.9428	s^2 ..	6.9148
	+ 2513.06	+ 1379.12	0.260601	3.4574	7.3996	7.3996
					3.4002	3.1396
			$(\varphi_1) =$	$s =$	0.7998	0.5392
			61° 14′ 34″	2867 m	$a = -6.31$,	$b = +3.46$
(P)	− 22501.20	+ 43512.40	2.956999	3.4798		4.3144
P_2	− 23406.93	+ 40493.76	3.479812	9.9813		6.9970
	+ 905.73	+ 3018.64	9.477187	3.4985	7.3174	7.3174
					2.9570	3.4798
			$(\varphi_2) =$	$s =$	0.2744	0.7972
			16° 42′ 6″	3152 m	$a = -1.88$,	$b = +6.29$
(P)	− 22501.20	+ 43512.40	3.248674	3.2740		4.3144
P_3	− 20728.34	+ 41632.97	3.274026	9.8618		6.8244
	− 1772.86	+ 1879.43	9.974648	3.4122	7.4900	7.4900
					3.2487	3.2740
			$(\varphi_3) =$	$s =$	0.7387	0.7640
			316° 40′ 17″	2584 m	$a = +5.48$,	$b = +5.81$

The comparison of the thus computed (φ)'s with the observed α's given under (14) yields the absolute terms $-l = (\varphi) - \alpha$ according to (7), and to this we also set at once the coefficients a and b computed in the above table:

Observed α	Approximation (φ)	$(\varphi) - \alpha = -l$	a	b
61° 14′ 24″	61° 14′ 34″	+ 10″	− 6.3	+ 3.5
16 42 15	16 42 6	− 9″	− 1.9	+ 6.3
316 40 3	316 40 17	+ 14″	+ 5.5	+ 5.8

(16)

Therefrom also immediately the sum-coefficients for the normal equations, all rounded off as much as possible:

a^2	b^2	ab	$-al$	$-bl$	l^2
40	12	− 22	− 63	+ 35	100
4	40	− 12	+ 17	− 57	81
30	34	+ 32	+ 77	+ 81	196
+ 74	+ 86	− 2	+ 31	+ 59	377

(17)

(In this connection, according to table [18] of the Appendix the check $a^2 + b^2$:51.8, 42.9, 63.9.)

The normal equations thus will be:

$$\begin{aligned} \underline{+74\,\delta x} - 2\,\delta y + 31 &= 0 \\ -\ 2\,\delta x \underline{+ 86\,\delta y} + 59 &= 0 \text{ with } [l\,l] = 377\,. \end{aligned}$$

The elimination in addition to the weight and error computation is done twice and, in fact, in the case of so small numbers simply with the slide rule, according to the instruction of (2) to (3), pp. 63 and 64:

$$\left.\begin{array}{llll|llll}
 & +74 & -2 & +31 & & +86 & -2 & +59 \\
 & & +86 & +59 & & & +74 & +31 \\
 & & -0 & +1 & & & -0 & +1 \\
 & & & +377 & & & & +377 \\
 & & & -13 & & & & -40 \\
\hline
[bb\cdot 1]=86 & & +86 & +60 & [aa\cdot 1]=74 & & +74 & +32 \\
 & & & +364 & & & & +337 \\
 & & & -42 & & & & -14 \\
\hline
 & & [ll\cdot 2] = & 322 & & & [ll\cdot 2] = & 323 \\
\end{array}\right\}$$

$$\delta y = -\frac{60}{86} = -0.70\ \partial\mathrm{m} = -0.070\ \mathrm{m} \qquad \delta x = -\frac{32}{74} = -0.43\ \partial\mathrm{m} = -0.043\ \mathrm{m} \tag{18}$$

to this

$$\left.\begin{array}{lcl}
(y) = -22501.20 & \text{approximation (11), p. 370,} & (x) = +43512.40 \\
\hline
y = -22501.27\ \mathrm{m} & & x = +43512.36\ \mathrm{m}
\end{array}\right\}. \tag{19}$$

With these adjusted coordinates y, x we compute the adjusted direction angles φ just as previously the approximations (φ) in the Table (15), p. 371. We again compare the thus obtained φ's with the measured α's, as the following table shows:

Observed α	Adjusted φ	$\varphi - \alpha = v$		v^2
61° 14′ 24″	61° 14′ 34″	+10″		100
16 42 15	16 42 2		−13	169
316 40 3	316 40 10	+ 7		49
				318

(20)

The thus obtained sum $[v^2] = 318$ must agree with $[ll \cdot 2] = 322$ or 323 of the elimination in the case of (18), which is sufficiently the case here; and now we have the mean error of an observation with approximately

$$m = \sqrt{\frac{[v^2]}{n-2}} = \sqrt{\frac{320}{1}} = \pm 18'' \tag{21}$$

and the mean coordinate errors according to (10) and (11) on p. 57:

$$m_y = \frac{18}{\sqrt{86}} = \pm 1.9\ \partial\mathrm{m} \qquad m_x = \frac{18}{\sqrt{74}} = \pm 2.1\ \partial\mathrm{m},$$

hence together with (19) we have the end result:

$$y = -22501.27\ \mathrm{m} \pm 0.19\ \mathrm{m} \qquad x = +43512.36\ \mathrm{m} \pm 0.21\ \mathrm{m}. \tag{22}$$

In order to show the gradual formation, all this which we have demonstrated here in individual equations and tabular groups will be demonstrated in the following section 96 in *one* tabular pattern on a more practical example, but meanwhile we shall make an additional remark about the check for $[v^2] = [ll \cdot 2]$, which we have indicated in section 29.

In the case of two unknowns we have the following check according to (11), p. 52, or else according to (8), p. 91:

$$[v\,v] = [l\,l \cdot 2] = [l\,l] - \frac{[a\,l]^2}{[a\,a]} - \frac{[b\,l \cdot 1]^2}{[b\,b \cdot 1]} . \tag{23}$$

This yields in our case directly from (20) $[v^2] = 318$ and then from (18):

$$[l\,l \cdot 2] = 377 - 13 - 42 = 322 \text{ and } 377 - 40 - 14 = 323 ,$$

and all three values 318, 322, 323 agree sufficiently according to the computational precision adhered to (rounding off to 1").

If we use, in addition, the computational check given in equation (1a) in section 29, p. 90, then we have in the case of only two unknowns δx and δy:

$$[v\,v] = [l\,l] - [a\,l]\,\delta x - [b\,l]\,\delta y \tag{24}$$

and with application to the above numerical example (18):

$$[v\,v] = 377 - 31 \times 0.43 - 59 \times 0.70$$
$$[v\,v] = 377 - 13 - 41 = 323.$$

It needs hardly to be mentioned that the connecting directions P_1P_2 and P_2P_3 of the station points did not necessarily have to be used as connecting directions in Fig. 1, p. 369; arbitrary directions going from P_1, P_2, P_3 to other fixed points could be used to this.

Now in Fig. 2 we study, in addition, the case that at the point B, the new point P is connected to the fixed directions BA and BC not only by *one* angle but by two angles. Let there be measured in Fig. 2 the angles

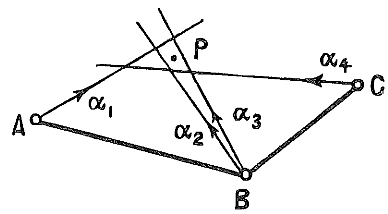


Fig. 2.

$$\left.\begin{array}{ll} PAB = \beta_1 & PBC = \beta_3 \\ ABP = \beta_2 & BCP = \beta_4 \end{array}\right\}, \tag{25}$$

which yield with the direction angles of the connecting directions:

$$\left.\begin{array}{l} \alpha_1 = (AB) - \beta_1 \\ \alpha_2 = (BA) + \beta_2 \\ \alpha_3 = (BC) - \beta_3 \\ \alpha_4 = (CB) + \beta_4 \end{array}\right\}. \tag{26}$$

Hence, *two* values α_2 and α_3 are found for the direction angle of the ray BP. The four direction angles α can again be treated like measured quantities.

We can carry out the further adjustment just like the pattern of the previous section.

Another consideration must however be attached to this. The second and third error equations will vary from one another only by the absolute terms l_2 and l_3 while the coefficients a and b are the same in both equations. We already have treated this case at the end of section 22, p. 72, and now we shall apply directly the result found there. According to this, we can replace the two error equations by a single one with the absolute term $\frac{l_2 + l_3}{2}$ and the weight 2 and then set up the normal equations. Now there is to be noted that this procedure yields the correct values of the unknowns, but not the correct sum of squares of the corrections v, and hence also not the correct value of the mean error m. In order to find the latter, we must compute the corrections v from the original four error equations.

Section 96. Intersection with Directions

After we have learned in section 95 the intersection with three rays fixed by simple or multiple angle measurement, we can easily change to the ray connection by measurements of directions.

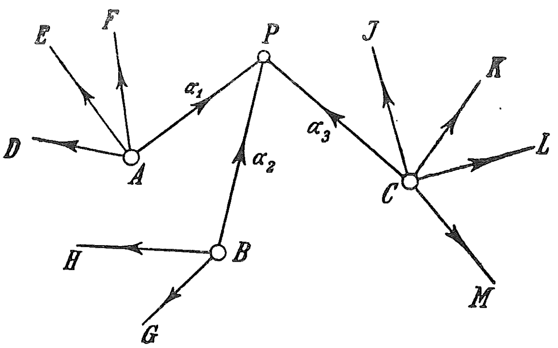


Fig. 1.

Let in Fig. 1 $A\,B\,C$ be three fixed points given by their coordinates, from which the point P is to be determined. For this purpose there are measured on the three fixed points sets of directions to the point P and to further points likewise given by their coordinates. The directions at the point A to the three fixed points D, E, F, at B to G and H, and at C to J, K, L, M are assumed to be observed.

Just as in section 95 we shall compute at first, from the coordinates, the direction angles from the three station points to the designated points, hence $[A\,D]$, $[A\,E]$, $[A\,F]$, $[B\,G]$, etc.

After this, the problem is to determine the direction angles $\alpha_1, \alpha_2, \alpha_3$ to the point P from the measured directions, or to orient the sets of directions measured at A, B, C. This orientation is carried out with the help of an Abriss [station data] according to Vol. II, 1st half-volume, 9th edition, 1931, section 91, pp. 414 and 417, which we must set up for every station.

If the direction angles $\alpha_1, \alpha_2, \alpha_3$ are found in the Abrisse [station data], then the further adjustment can be carried out according to the pattern of section 95.

In regard to the accuracy of the three direction angles there are differences, since, according to Fig. 1, the direction to the new point P is connected at A to three fixed points, at B to two and at C to four fixed points. For the time being, however, we will leave this question open and regard the three direction angles $\alpha_1, \alpha_2, \alpha_3$ as equally accurate. At the end of the section we shall investigate these relations more closely.

In order to be able to show all details of the adjustment with directions, we pass now to a numerical example in which we will intersect the point Hochschule according to Fig. 1, section 69, p. 234, from the points Steuerndieb, Ägidius, Wasserturm, Burg of the pentagon of Hannover.

We could take directly from the station data [Abrisse] of the triangulation of Hannover of section 71, p. 250, in which the points Hochschule and Dreifaltigkeit are already listed, the direction angles α; however, we will present here once again the station data for one of the points, e.g. Wasserturm.

Abriss [station data] for Wasserturm

Target Point	Observed r	Connection φ	Shifting $\varphi - r$	Oriented $\alpha = r + z$
Ägidius . .	0° 00′ 00″	71° 23′ 39.26″	71° 23′ 39.26″	(71° 23′ 39.74″)
Willmer . .	45 5 26.24	116 29 5.98	71 23 39.74	(116 29 05.98)
Burg . . .	284 21 15.98	355 44 56.20	71 23 40.22	(355 44 55.72)
Hochschule .	309 13 10.25			20 36 49.99

Mean: 71° 23′ 39.74″ = z.

We have taken the observed directions r from section 69, p. 233, while the connecting directions φ are the final direction angles resulting from the adjustment, which are contained in the Abriss, section 71, p. 250, in the fourth column.

In the same way we find the direction angles α for the other three fixed points and then we have in this connection by rounding off the seconds to one decimal place:

Steuerndieb . . .	$\alpha_1 =$	259° 14′ 15.1″
Ägidius	$\alpha_2 =$	315 2 32.6
Wasserturm . . .	$\alpha_3 =$	20 36 50.0
Burg	$\alpha_4 =$	149 4 12.3 .

In contrast to the first numerical example of section 95, now we will put the further computation on a *computation form* set up according to a regular plan, p. 376.

We begin with the filling-in of the first columns above on p. 376.

Computation of the Direction Angles for the Adjustment, p. 376

Point P Point A (p. 250)	y_p y_a $y_p - y_a = \Delta y$	x_p x_a $x_p - x_a = \Delta x$	$\log \Delta y$ $\log \Delta x$ $\log \tan\ (A\,P)$	Distance
		I. *Before the Adjustment*		
Hochschule . .	— 24709.800	— 26868.300	3.683149	3.6831
Steuerndieb . .	— 19888.668	— 25951.884	2.962093	9.9923
	— 4821.132	— 916.416	0.721056	3.6908
			$(\varphi_1) =$ 259° 14′ 14.7″	$s_1 =$ 4907 m
Hochschule . .	— 24709.800	— 26868.300	3.157755	3.1584
Ägidius	— 23271.813	— 28308.395	3.158391	9.8498
	— 1437.987	+ 1440.095	9.999364	3.3086
			$(\varphi_2) =$ 315° 2′ 31.0″	$s_2 =$ 2035 m
Hochschule . .	— 24709.800	— 26868.300	2.918391	3.3430
Wasserturm . .	— 25538.488	— 29071.474	3.343049	9.9713
	+ 828.688	+ 2203.174	9.575342	3.3717
			$(\varphi_3) =$ 20° 36′ 46.7″	$s_3 =$ 2353 m
Hochschule . .	— 24709.800	— 26868.300	3.054230	3.2767
Burg	— 25842.799	— 24977.399	3.276669	9.9334
	+ 1132.999	— 1890.901	9.777561	3.3433
			$(\varphi_4) =$ 149° 4′ 14.2″	$s_4 =$ 2204 m
		II. *After the Adjustment*		
Hochschule . .	— 24709.769	— 26868.306	3.683146	
Steuerndieb . .	— 19888.668	— 25951.884	3.962096	
	— 4821.101	— 916.422	0.721050	
			$\varphi_1 =$ 259° 14′ 14.2″	
Hochschule . .	— 24709.769	— 26868.306	3.157746	
Ägidius	— 23271.813	— 28308.395	3.158390	
	— 1437.956	+ 1440.089	9.999356	
			$\varphi_2 =$ 315° 2′ 32.9″	
Hochschule . .	— 24709.769	— 26868.306	2.918408	
Wasserturm . .	— 25538.488	— 29071.474	3.343048	
	+ 828.719	+ 2203.168	9.575360	
			$\varphi_3 =$ 20° 36′ 49.5″	
Hochschule . .	— 24709.769	— 26868.306	3.054241	
Burg	— 25842.799	— 24977.399	3.276671	
	+ 1133.030	— 1890.907	9.777570	
			$\varphi_4 =$ 149° 4′ 12.3″	

After this, we need for the intersected point Hochschule approximate coordinates which can be computed from any arbitrary two of the measured α's, perhaps according to the pattern of p. 423 of our Volume II, 2nd half-volume, 9th edition, 1931. In cases like ours on p. 234, we have such approximate coordinates a long time already from the centerings, etc. Where the approximations come from is of no importance; we have in our case:

$$\text{Hochschule approximation } (y) = -24709.8 \text{ m}, \quad (x) = -26868.3 \text{ m}.$$

For the whole trigonometric computation which now follows, direction angle (φ) and φ, etc., we note beforehand that a rectangular *plane* system of coordinates is assumed valid, hence that all the small corrections of the order $1/r^2$, which have been taken into account in section 71, p. 249, and in the station data [Abriss], p. 250, for the net on p. 236, are simply neglected now when passing to the intersection adjustments, so that

Intersection According to the Method of Least Squares

Intersected Point Hochschule (Fig. 1, p. 234)

No.	Base Point	Target Point	Measured Direction α (*A* p. 250)	Approximate Direction Angle (φ) (p. 375)	$(\varphi) - \alpha = -l$	l^2
1	**Steuerndieb**	Hochschule	259° 14′ 15.1″	259° 14′ 14.7″	− 0.4″	0.16
2	**Ägidius . .**	Hochschule	315 2 32.6	315 2 31.0	− 1.6	2.56
3	**Wasserturm**	Hochschule	20 36 50.0	20 36 46.7	− 3.3	10.89
4	**Burg. . .**	Hochschule	149 4 12.3	149 4 14.2	+ 1.9	3.61
			57′ 50.0″	57′ 46.6″	− 3.4″	17.22

No.	φ	$\mathfrak{x}$ (Auxiliary Table Appendix Pages [10] - [15])	$\mathfrak{y}$	s km	$\frac{\mathfrak{x}}{s} = a$	$\frac{\mathfrak{y}}{s} = b$	$-l$	a^2	b^2	ab (+ −)	$-al$ (+ −)	$-bl$ (+ −)
1	259° 10′	+ 20.3	− 3.9	4.91	+ 4.2	− 0.8	− 0.4	18	1	− 3	− 2	+ 0
2	315 0	+ 14.6	+ 14.6	2.04	+ 7.2	+ 7.2	− 1.6	52	52	+ 52	− 12	− 12
3	20 40	− 7.3	+ 19.3	2.35	− 3.1	+ 8.2	− 3.3	10	67	− 25	+ 10	− 27
4	149 0	− 10.6	− 17.7	2.20	− 4.8	− 8.0	+ 1.9	23	64	+ 38	− 9	− 15
								+ 103	+ 184	+ 62	− 13	− 54

	a]	b]	− l]	If Necessary, 3rd Check: $[vv] = [ll] - [al]\,\delta x - [bl]\,\delta y$	[b	[a	[− l	
[a	+ 103	+ 62	− 13		+ 184	+ 62	− 54	b]
	[b	+ 184	− 54			+ 103	− 13	a]
		− 37	+ 8			− 21	+ 18	
		[− l	+ 17				+ 17	− l]
			− 2				− 16	
$\delta y =$ + 0.31 dm	[b b · 1] = + 147		− 46		[a a · 1] =	+ 82	+ 5	$\delta x =$ − 0.06 dm
$\delta y =$ + 0.031 m			+ 15				+ 1	$\delta x =$ − 0.006 m
(y) = − 24709.800			− 14				− 0	(x) = − 26868.300
y = − 24709.769 m		[l l · 2] =	+ 1			[l l · 2] =	+ 1	x = − 26868.306 m

No.	Computation of the v's: $a\,\delta x + b\,\delta y - l$			v	Measured Direction α (see above)	Corrected Direction $\alpha + v$	Final Direction Angle φ (p. 375)	v^2
1	− 0.3	− 0.2	− 0.4	− 0.9″	259° 14′ 15.1″	259° 14′ 14.2″	259° 14′ 14.2″	0.81
2	− 0.4	+ 2.2	− 1.6	+ 0.2	315 2 32.6	315 2 32.8	315 2 32.9	0.04
3	+ 0.2	+ 2.5	− 3.3	− 0.6	20 36 50.0	20 36 49.4	20 36 49.5	0.36
4	+ 0.3	− 2.5	+ 1.9	− 0.3	149 4 12.3	149 4 12.0	149 4 12.3	0.09
				− 1.6″	57′ 50.0″	57′ 48.4″	57′ 48.9″	1.30

$[vv] = [ll \cdot 2] = 1.3$ $\qquad m^2 = \frac{1.3}{2} = 0.65 \qquad m_y{}^2 = \frac{m^2}{[bb \cdot 1]} = \frac{0.65}{147} \qquad m_x{}^2 = \frac{m^2}{[aa \cdot 1]} = \frac{0.65}{82}$

$n - 2 = 2$ $\qquad m = \pm\,0.8'' \qquad m_y = \pm\,0.07$ dm $\qquad m_x = \pm\,0.09$ dm

Final result:

Hochschule $y = -24709.769$ m $\qquad x = -26868.306$ m

$\pm\,0.007$ m $\qquad \pm\,0.009$ m

the tenths of a second (0.1'') can no longer be correct. If we compute formally nevertheless to 0.1'', this only has the meaning that the seconds themselves are supposed to be protected from being rounded off, and that we are still correct, at any rate, to approximately 0.3''.

The approximate coordinates (x), (y) obtained in any way are put on p. 375, in addition to the coordinates of the fixed points obtained from p. 249, and after this the computation of the (φ)'s and besides also the computation of the approximate distances s is carried out. We put the thus obtained (φ)'s on p. 376 above, and compute from these $(\varphi) - \alpha = -l$ and the l^2's.

There follows the second group of p. 376, in which case the coefficients a and b are computed with the auxiliary table, pp. [10] to [15], of the Appendix. We have computed the necessary s's on p. 375 with four-place logarithms on the side; but we can also very often take the s's well enough from a net representation, p. 234, if we adapt the computational rigorousness to the sense of the whole in the whole computation. The divisions $\frac{\mathfrak{x}}{s} = a$ and $\frac{\mathfrak{y}}{s} = b$ are done with the slide rule, and the computation of the a^2's, b^2's, ab's, etc., in the second group of p. 376 goes then also smoothly and rapidly.- We make here the check (18), indicated at the end of section 94, p. 366, according to $a^2 + b^2$ with the auxiliary table [18] of the Appendix (for which a special column in the computation form can be spared). In our example, p. 376, table [18] yields: 18, 102, 77, 88 in sufficient agreement with the sums $a^2 + b^2$ of p. 376.

Likewise also the elimination according to the instruction of (18) and (19), section 95, p. 372, with the slide rule and with the check $[ll \cdot 2] = +1$ on the left, and $[ll \cdot 2] = +1$ on the right. The third check $[ll \cdot 2] = [ll] - [al]\, x - [bl]\, y$ is written there only for the emergency case; it agrees also sufficiently in our case.

After the δy's and δx's are added to the approximations (y) and (x), and by so doing, the final coordinates y and x are obtained, the lower second half of p. 375 is filled in and the final φ's are computed, which are then to be put again, according to p. 376, on the right below.

Then we compute further the expressions $ax + by - l$ for the corrections v, and after the measured directions α are also taken from above, we obtain the corrected directions $\alpha + v$, which are to be compared with the final direction angles φ. This check agrees to 0.3'', which is sufficient in the case of the computational accuracy applied. At the same time we compute in the same section the v^2's whose sum 1.3 agrees sufficiently with $[ll \cdot 2] = 1$.

Now if in the case of $[vv] = [ll \cdot 2]$ disagreeing, the checks $\alpha + v = \varphi$ should partially not agree also then the error is to be sought for in the trigonometric computation of the (φ)'s or φ's; but if the $\alpha + v$'s are satisfactory and yet $[ll \cdot 2]$ deviates greatly from $[vv]$, then we would review the a's, b's, etc. In short, the various checks yield so many hints as to where the error is possibly to be looked for, that we will soon clear the whole thing up even though an error should creep in.

Regarding the computation in new [centesimal] division we refer to the later section 111.-

Taking into account the weights of the direction angles

Now we will still investigate the question of the weights of the direction angles α introduced into the adjustment, which was left open on p. 374.

If we start from the station data [Abriss] listed on p. 374 for the station Wasserturm, we have for the target point Hochschule

$$\alpha = r_4 + z = r_4 + \frac{[\varphi - r]}{3},$$

or

$$\alpha = \frac{\varphi_1 + \varphi_2 + \varphi_3}{3} + r_4 - \frac{r_1 + r_2 + r_3}{3}. \tag{1}$$

At first we are to investigate here whether it is admissible to introduce the direction angle α, which is computed according to (1) from the measured directions r_1, r_2, r_3, r_4, as a *measured quantity* into the adjustment, since the method of least squares requires that the sum of squares of the corrections of the *observations*, and hence here of the measured directions r, becomes a minimum.

We refer here to the previous section 60, p. 190, in which we have found that we can collect several observations in an error equation if these observations do not occur in another error equation. This holds true in the present case. As we have seen further in section 60, p. 190, we are to determine the weight for the quantity computed from the observations, hence here for α, which results according to the law of the propagation of errors. If we assume the mean error $\pm m$ for the measured directions, then we have for the mean error of α according to (1) the expression

$$m_\alpha{}^2 = m^2 + \frac{3\,m^2}{9} = \frac{4}{3}\,m^2 \tag{2}$$

and if the weight of a measured direction is set equal to one, then we have

$$p_\alpha = \frac{3}{4}\,. \tag{3}$$

We can easily generalize this by assuming that in addition to the new point fixed points are aimed at on the station. Then we find

$$m_\alpha{}^2 = m^2 + \frac{s\,m^2}{s^2} = \frac{s+1}{s}\,m^2$$

and

$$p_\alpha = \frac{s}{s+1}\,. \tag{4}$$

Hence, if we aim to take into account the differing accuracy of the direction angles determined on the individual stations to the new point, we are to attribute the weight p_α to each direction angle according to (4), and the further adjustment is then to be carried out according to section 22.

Remark on the slide rule

Our pattern, p. 376, is set up in such a way that no number need be computed or even only written any more on the side, other than that which actually is on p. 376, provided that we use the slide rule, with which we manipulate in fact everything suitable for it without guarantee for the accuracy of the last place. The use of the slide rule, however, is not a necessary presupposition for the applicability of the form, p. 376, itself. E.g., in the elimination we are to compute: $\frac{62}{103}\,62 = 37$, and $\frac{62}{103}\,13 = 8$, which is done by a *single* setting 62 over 103 on the slide rule; but that does not preclude dividing on another sheet $62:103 = 0.60$, and then multiplying $0.60 \times 62 = 37$ and $0.60 \times 13 = 8$. We can also use the table of products by Crelle and similar auxiliary means; but we found that everything else is much too troublesome in comparison to the ordinary slide rule.

The representation of the adjustment of the intersection with direction measurements, given in the preceding section, will in most cases be sufficient for the current practice. If *several* new points are aimed at from a fixed point, the method of the intercalation of individual points is likewise applied as a rule by taking at first the direction angle only for *one* point from the station data [Abriss] and computing this point completely. The final direction angles found hereby are entered into the station data [Abrisse] and treated further like the direction angles to the given fixed points. Then there follows the determination of the direction angles α to the second new point, etc.

Nevertheless, we will now treat also the simultaneous adjustment of several new points determined by intersection.

We assume that in Fig. 1 there are measured at the fixed point J the directions to three further fixed points A, B, C and to two points P and P' to be newly determined. Let the measured directions be $r_1, r_2, \ldots, r_5$, and then we have after the adjustment the corrected directions $r_1 + v_1, r_2 + v_2, \ldots, r_5 + v_5$.

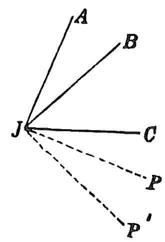


Fig. 1.

We begin again with the computation of the direction angles $\varphi_1, \varphi_2, \varphi_3$ for the rays to the fixed points A, B, C from the fixed coordinates.

Then we are to determine, in any arbitrary way, approximate coordinates (x), (y) and (x'), (y') for the new points P and P' and with these, we are to compute the preliminary direction angles (φ_4) and (φ_5).

If the approximate coordinates receive the corrections δx, δy and $\delta x'$, $\delta y'$ by the adjustment, then (φ_4) and (φ_5) change to the final values φ_4 and φ_5 and according to equation (10), section 93, p. 359, we have:

$$\varphi_4 = (\varphi_4) - \frac{\sin(\varphi_4)}{s}\,\varrho\,\delta x + \frac{\cos(\varphi_4)}{s}\,\varrho\,\delta y$$

$$\varphi_5 = (\varphi_5) - \frac{\sin(\varphi_5)}{s'}\,\varrho\,\delta x' + \frac{\cos(\varphi_5)}{s'}\,\varrho\,\delta y'$$

or with simplified notation:

$$\left.\begin{aligned} \varphi_4 &= (\varphi_4) + a\,\delta x + b\,\delta y \\ \varphi_5 &= (\varphi_5) + a'\,\delta x' + b'\,\delta y' \end{aligned}\right\} . \qquad (1)$$

If we denote by z the amount of orientation for the directions measured at the station J after the adjustment, then we have

$$\left.\begin{aligned} r_1 + v_1 + z &= \varphi_1 \\ r_2 + v_2 + z &= \varphi_2 \\ \cdots & \\ r_5 + v_5 + z &= \varphi_5 \end{aligned}\right\} . \qquad (2)$$

The amount of orientation z which could be determined finally in the Abriss [station data] in the case of the intercalation of individual points, is introduced now into the adjustment as a further unknown and then we have, with the help of (1), the five error equations:

$$\left.\begin{aligned} v_1 &= -z + \varphi_1 - r_1 \\ v_2 &= -z + \varphi_2 - r_2 \\ v_3 &= -z + \varphi_3 - r_3 \\ v_4 &= -z + (\varphi_4) - r_4 + a\,\delta x + b\,\delta y \\ v_5 &= -z + (\varphi_5) - r_5 \qquad\qquad + a'\,\delta x' + b'\,\delta y' \end{aligned}\right\} . \qquad (3)$$

In order to simplify the numerical computation, we introduce an approximate value (z) for z and set

$$z = (z) + \delta z. \tag{4}$$

With (z) we can orient provisionally the measured directions and obtain as previously

$$\left.\begin{array}{l} \alpha_1 = r_1 + (z) \\ \alpha_2 = r_2 + (z) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_5 = r_5 + (z) \end{array}\right\}. \tag{5}$$

If we introduce then, in addition,

$$\left.\begin{array}{l} \varphi_1 - \alpha_1 = -l_1 \\ \varphi_2 - \alpha_2 = -l_2 \\ \varphi_3 - \alpha_3 = -l_3 \\ (\varphi_4) - \alpha_4 = -l_4 \\ (\varphi_5) - \alpha_5 = -l_5 \end{array}\right\} \tag{6}$$

then the error equations will be:

$$\left.\begin{array}{llllll} v_1 = -\delta z & \cdot & \cdot & \cdot & \cdot & -l_1 \\ v_2 = -\delta z & \cdot & \cdot & \cdot & \cdot & -l_2 \\ v_3 = -\delta z & \cdot & \cdot & \cdot & \cdot & -l_3 \\ v_4 = -\delta z + a\,\delta x + b\,\delta y & & & \cdot & \cdot & -l_4 \\ v_5 = -\delta z & \cdot & \cdot & & + a'\,\delta x' + b'\,\delta y' & -l_5 \end{array}\right\}. \tag{7}$$

A suitable value can be found for the quantity (z) in any arbitrary way, e.g., we could set $(z) = \varphi_1 - r_1$. But if we take for (z) the mean value

$$(z) = \frac{\varphi_1 - r_1 + \varphi_2 - r_2 + \varphi_3 - r_3}{3} \tag{8}$$

then (z) becomes identical with the shifting of the set z, which we have found in the Abriss [station data], section 96, p. 374. Besides, we have then

$$l_1 + l_2 + l_3 = 0. \tag{9}$$

If we imagine the above error equations (7) set up for every station point, then a new unknown δz will occur each time. It is easy, however, to eliminate this unknown from each individual system. For this, we have as the most readily available means the sum-equation resulting from the error equations (7). For if in the case of the adjustment of indirect observations, one of the unknowns in the error equations has only the coefficients $+1$ or, as the case may be, -1, then we have according to (6), section 14, p. 43, $[v] = 0$. According to this, the above equations (7) yield

$$\left.\begin{array}{l} 0 = -5\,\delta z + a\,\delta x + b\,\delta y + a'\,\delta x' + b'\,\delta y' - [l] \\ \text{or} \quad 0 = +\,\delta z - \frac{a}{5}\,\delta x - \frac{b}{5}\,\delta y - \frac{a'}{5}\,\delta x' - \frac{b'}{5}\,\delta y' + \frac{[l]}{5} \end{array}\right\}. \tag{10}$$

By adding the second equations (10) to the error equations (7) we obtain

$$\left.\begin{aligned}
v_1 &= -\frac{1}{5}\,a\,\delta x - \frac{1}{5}\,b\,\delta y - \frac{1}{5}\,a'\,\delta x' - \frac{1}{5}\,b'\,\delta y' - \left(l_1 - \frac{[l]}{5}\right)\\
v_2 &= -\frac{1}{5}\,a\,\delta x - \frac{1}{5}\,b\,\delta y - \frac{1}{5}\,a'\,\delta x' - \frac{1}{5}\,b'\,\delta y' - \left(l_2 - \frac{[l]}{5}\right)\\
v_3 &= -\frac{1}{5}\,a\,\delta x - \frac{1}{5}\,b\,\delta y - \frac{1}{5}\,a'\,\delta x' - \frac{1}{5}\,b'\,\delta y' - \left(l_3 - \frac{[l]}{5}\right)\\
v_4 &= +\frac{4}{5}\,a\,\delta x + \frac{4}{5}\,b\,\delta y - \frac{1}{5}\,a'\,\delta x' - \frac{1}{5}\,b'\,\delta y' - \left(l_4 - \frac{[l]}{5}\right)\\
v_5 &= -\frac{1}{5}\,a\,\delta x - \frac{1}{5}\,b\,\delta y + \frac{4}{5}\,a'\,\delta x' + \frac{4}{5}\,b'\,\delta y' - \left(l_5 + \frac{[l]}{5}\right)
\end{aligned}\right\} . \qquad (11)$$

Such a system of error equations is to be set up for each station, where the corrections of coordinates of the two points P and P' as well as those of other new points can occur in each system. From all systems together, normal equations are to be set up, whose number is equal to double the number of the new points.

The sum-equation (10) is not the only means for the elimination of the unknown of orientation. However, we will discuss the further methods only in the later section 100 after having learned also to resect with directions.

For the computation of the mean error of a measured direction we have to group together the sum of squares of all corrections v whose number we assume to be equal to n. In addition to the corrections of coordinates, the corrections δz of the preliminary quantities of orientation (z) are also among the unknowns. Hence, the number of unknowns is equal to double the number of new points and the number of stations.

Example

For a numerical example we choose the intercalation of the two points Hochschule and Dreifaltigkeit into the pentagon of Hannover and for this, we insert here once again Fig. 1 originally from section 69, p. 234.

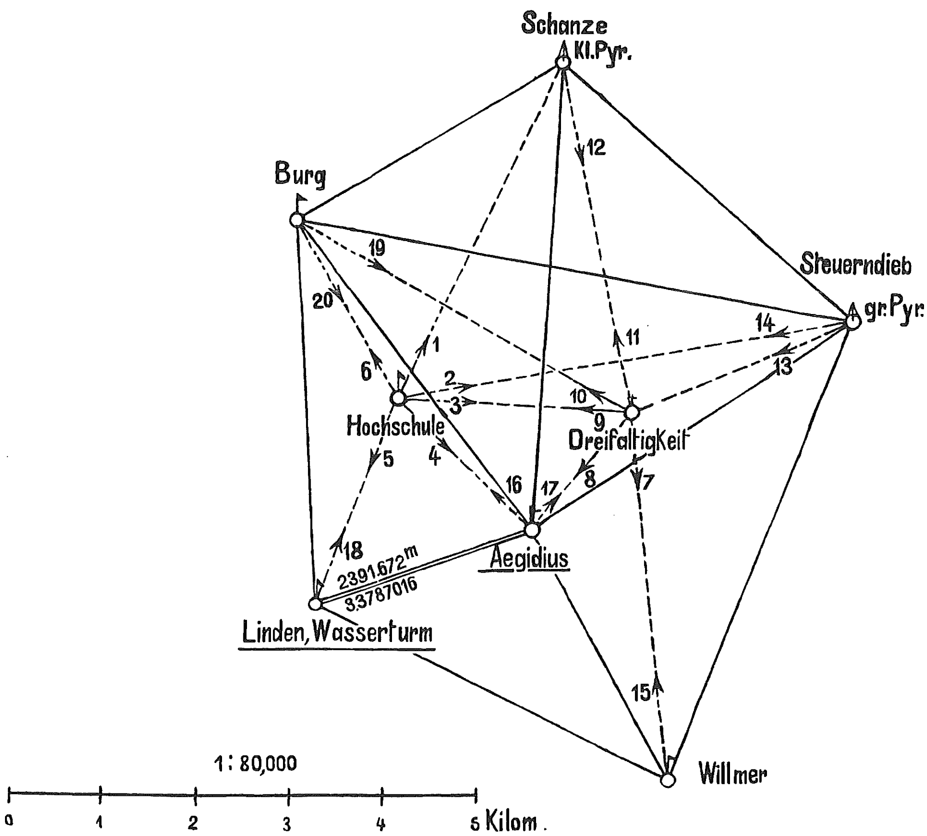


Fig. 2.

In the system of the conformal double projection of the Reichsamt für Landesaufnahme we have the following coordinates of the given points according to section 71, p. 248:

1. Ägidius	$y = -244\,656.090\ m$	$x = -30\,624.971\ m$	
2. Wasserturm	$-246\,956.479$	$-31\,285.875$	
3. Willmer	$-243\,280.909$	$-33\,328.385$	
4. Steuerndieb	$-241\,167.896$	$-28\,421.362$	(12)
5. Schanze	$-244\,244.387$	$-25\,592.941$	
6. Burg	$-247\,076.504$	$-27\,179.218$	

We assume the following approximate coordinates for the two new points:

$$\begin{aligned} &\text{Hochschule} \quad (y_1) = -246\,028.90\ m \quad (x_1) = -29\,120.56\ m \\ &\text{Dreifaltigkeit} \quad (y_2) = -243\,620.76 \quad (x_2) = -29\,282.46 \end{aligned} \qquad (13)$$

The directions measured at the six fixed points are compiled in the Abriss [station data], section 71, p. 247. If we add the reductions $t - T$ to the observed T's, then we obtain the directions α in the plane, which are approximately oriented at the same time. In the last column there are given further the final direction angles t of the individual stations to the other fixed points. The preliminary values (t) were computed from the above coordinates for the direction angles of the stations to the new points.

Then we assemble at first the required numerical values for the stations Ägidius, Steuerndieb, and Burg, from which directions to the two new points are measured.

1. *Ägidius*

	α	t and (t)	$t-\alpha$ $(t)-\alpha$ $=-l$	a	b	a'	b'
1. Wasserturm	253° 58′ 14.1″	253° 58′ 14.1″	0.0″				
2. Hochschule	317 37 8.8	317 37 7.7	−1.1	+6.8	+7.5		
3. Burg	324 54 51.5	324 54 52.2	+0.7				
4. Schanze	4 40 39.0	4 40 38.4	−0.6				
5. Dreifaltigkeit	37 38 23.7	37 38 20.5	−3.2			−7.4	+9.6
6. Steuerndieb	57 43 5.4	57 43 4.8	−0.6				
7. Willmer	153 2 17.5	153 2 18.0	+0.5				

(14)

4. *Steuerndieb*

	α	t and (t)	$t-\alpha$ $(t)-\alpha$ $=-l$	a	b	a'	b'
16. Willmer	203° 17′ 49.5″	203° 17′ 49.9″	+0.4″				
17. Ägidius	237 48 4.8	237 48 4.8	0.0				
18. Dreifaltigkeit	250 39 20.7	250 39 21.4	+0.7			+7.5	−2.6
19. Hochschule	261 48 52.3	261 48 53.4	+1.1	+4.2	−0.6		
20. Burg	281 52 20.9	281 52 19.8	−1.1				
21. Schanze	312 35 39.0	312 35 39.7	+0.7				

(15)

6. *Burg*

	α	t and (t)	$t-\alpha$ $(t)-\alpha$ $=-l$	a	b	a'	b'
26. Schanze	60° 44′ 47.9″	60° 44′ 48.1″	+0.2″				
27. Steuerndieb	101 52 19.0	101 52 19.8	+0.8				
28. Dreifaltigkeit	121 19 30.0	121 19 32.2	+2.2			−4.4	−2.7
29. Ägidius	144 54 53.3	144 54 52.2	−1.1				
30. Hochschule	151 38 50.2	151 38 50.7	+0.5	−4.4	−8.2		
31. Wasserturm	179 19 33.0	179 19 33.2	+0.2				

(16)

In these station data [Abrisse] we have retained the symbols t and (t) of the former section 71, p. 247, instead of the symbols φ and (φ) for the direction angles, which can hardly lead to mistakes. At the same time, the direction coefficients a, b, a', b' have been added for the directions to the two new points.

According to equations (7), p. 380, the following error equations result then for the station Ägidius

$$\left.\begin{array}{llllllr}
v_1 = - & \delta z_1 & . & . & . & . & +0.0 \\
v_2 = - & \delta z_1 & +6.8\,\delta x_1 & +7.5\,\delta y_1 & . & . & -1.1 \\
v_3 = - & \delta z_1 & . & . & . & . & +0.7 \\
v_4 = - & \delta z_1 & . & . & . & . & -0.6 \\
v_5 = - & \delta z_1 & . & . & -7.4\,\delta x_2 & +9.6\,\delta y_2 & -3.2 \\
v_6 = - & \delta z_1 & . & . & . & . & -0.6 \\
v_7 = - & \delta z_1 & . & . & . & . & +0.5
\end{array}\right\} \qquad (17)$$

$$0 = -7\,\delta z_1 + 6.8\,\delta x_1 + 7.5\,\delta y_1 - 7.4\,\delta x_2 + 9.6\,\delta y_2 - 4.3$$
$$0 = +\,\delta z_1 - 1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 + 0.6\,.$$

If the unknown of orientation δz_1 is eliminated with the help of the sum-equation, then we obtain the new error equations

$$\left.\begin{array}{l}
v_1 = -1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 + 0.6 \\
v_2 = +5.8\,\delta x_1 + 6.4\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 - 0.5 \\
v_3 = -1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 + 1.3 \\
v_4 = -1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 \quad . \\
v_5 = -1.0\,\delta x_1 - 1.1\,\delta y_1 - 6.3\,\delta x_2 + 8.2\,\delta y_2 - 2.6 \\
v_6 = -1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 \quad . \\
v_7 = -1.0\,\delta x_1 - 1.1\,\delta y_1 + 1.1\,\delta x_2 - 1.4\,\delta y_2 + 1.1
\end{array}\right\} . \qquad (18)$$

From these we form the portions of the normal equations of the station Ägidius:

$$\left.\begin{array}{rrrrr}
\underline{+39.64\,\delta x_1} & +43.72\,\delta y_1 & +\;\;7.18\,\delta x_2 & -\;\;9.32\,\delta y_2 & -\;\;3.30 \\
 & \underline{+48.22\,\delta y_1} & +\;\;7.92\,\delta x_2 & -10.28\,\delta y_2 & -\;\;3.64 \\
 & & \underline{+46.95\,\delta x_2} & -60.90\,\delta y_2 & +19.13 \\
 & & & \underline{+79.00\,\delta y_2} & -24.82
\end{array}\right\} . \qquad (19)$$

In the same way we treat the stations Steuerndieb and Burg, for which the error equations can likewise be read immediately from the station data (15) and (16). We therefore limit ourselves to specifying immediately the portions of the normal equations for them:

4. *Steuerndieb*

$$\left.\begin{array}{rrrrr}
\underline{+14.70\,\delta x_1} & -\;\;2.10\,\delta y_1 & -\;\;5.25\,\delta x_2 & +\;\;1.82\,\delta y_2 & +3.36 \\
 & \underline{+\;\;0.30\,\delta y_1} & +\;\;0.75\,\delta x_2 & -\;\;0.26\,\delta y_2 & -0.48 \\
 & & \underline{+46.89\,\delta x_2} & -16.26\,\delta y_2 & +3.00 \\
 & & & \underline{+\;\;5.64\,\delta y_2} & -1.04
\end{array}\right\} . \qquad (20)$$

6. *Burg*

$$\left.\begin{array}{rrrrr}
\underline{+16.14\,\delta x_1} & +30.06\,\delta y_1 & -\;\;3.22\,\delta x_2 & -\;\;1.97\,\delta y_2 & -0.14 \\
 & \underline{+56.04\,\delta y_1} & -\;\;6.02\,\delta x_2 & -\;\;3.70\,\delta y_2 & -0.28 \\
 & & \underline{+16.14\,\delta x_2} & +\;\;9.91\,\delta y_2 & -7.62 \\
 & & & \underline{+\;\;6.09\,\delta y_1} & -4.67
\end{array}\right\} . \qquad (21)$$

At the further three stations 2. Wasserturm, 3. Willmer, and 5. Schanze, only the directions to one each of the two new points are measured. Hence, we can determine immediately the final values of the amounts of orientation δz_2, δz_3, and δz_5 for these stations according to section 96 from the station data [Abrisse], and then we have to set up for each station only one error equation with the weight to be apportioned according to section 96, p. 377.

2. *Wasserturm.*

	r	t and (t)	$t-r$	$\alpha = r + \delta z_2$	$(t)-\alpha = -l$
8. Burg	358° 19′ 32.7″	358° 19′ 33.2″	+0.5″		
9. Hochschule . .	23 11 25.8	23 11 21.7		23° 11′ 25.8″	−4.1″
10. Ägidius	73 58 14.6	73 58 14.2	−0.4		
11. Willmer . . .	119 3 39.2	119 3 39.2	0.0		
			$\delta z_2 = 0.0$		

Error equation:

$$v_9 = -3.4\,\delta x_1 + 8.0\,\delta y_1 \ldots -4.1 \text{ wt } 3/4. \tag{22}$$

3. *Willmer*

	r	t and (t)	$t-r$	$\alpha = r + \delta z_3$	$(t)-\alpha = -l$
12. Wasserturm . .	299° 3′ 38.2″	299° 3′ 39.2″	+1.0″		
13. Ägidius	333 2 19.3	333 2 18.0	−1.3		
14. Dreifaltigkeit . .	355 11 56.2	355 11 54.7		355° 11′ 56.2″	−1.5″
15. Steuerndieb . .	23 17 49.5	23 17 49.9	+0.4		
			$\delta z_3 = 0.0$		

Error equation:

$$v_{14} = \ldots + 0.4\,\delta x_2 + 5.1\,\delta y_2 - 1.5 \text{ wt } 3/4. \tag{23}$$

5. *Schanze.*

	r	t and (t)	$t-r$	$\alpha = r + \delta z_5$	$(t)-\alpha = -l$
22. Steuerndieb . .	132° 35′ 39.8″	132° 35′ 39.7″	−0.1″		
23. Dreifaltigkeit . .	170 24 22.4	170 24 22.2		170° 24′ 22.4″	−0.2″
24. Ägidius	184 40 38.4	184 40 38.4	0.0		
25. Burg	240 44 47.9	240 44 48.1	+0.2		
			$\delta z_5 = 0.0$		

Error equation:

$$v_{23} = \ldots - 1.0\,\delta x_2 - 5.7\,\delta y_2 - 0.2 \text{ wt} = 3/4. \tag{24}$$

The portions of normal equations, which correspond to the collected three error equations (22) to (24), are

$$\left.\begin{array}{lllll} \underline{+\ 8.67\,\delta x_1} - 20{,}40\,\delta y_1 & \cdot & \cdot & +10.46 \\ \quad\underline{+48.00\,\delta y_1} & \cdot & \cdot & -24.60 \\ & \underline{+\ 0.87\,\delta x_2} + & 5.80\,\delta y_2 & -\ 0{,}30 \\ & & \underline{+43{,}88\,\delta y_2} & -\ 4{,}88 \end{array}\right\}. \tag{25}$$

If we combine all portions of normal equations, then we obtain the following normal equations

$$\left.\begin{array}{l} +79.2\,\delta x_1 + \;\;51.3\,\delta y_1 - \;\;1.3\,\delta x_2 - \;\;9.5\,\delta y_2 + 10.4 = 0 \\ \qquad\qquad + 152.6\,\delta y_1 + \;\;2.6\,\delta x_2 - 14.2\,\delta y_2 - 29.0 = 0 \\ \qquad\qquad\qquad\qquad + 110.8\,\delta x_2 - 61.4\,\delta y_2 + 14.2 = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad + 134.6\,\delta y_2 - 35.4 = 0 \end{array}\right\} . \tag{26}$$

The solution yields

$$\delta x_1 = -0.303\,\partial\text{m} \qquad \delta x_2 = +0.017\,\partial\text{m}$$
$$\delta y_1 = +0.317 \qquad \delta y_2 = +0.283$$

and with the approximate values (13) we find

$$\left.\begin{array}{llll} \text{Hochschule} & y_1 = -246{,}028.868\text{ m} & x_1 = -29{,}120.590\text{ m} \\ \text{Dreifaltigkeit} & y_2 = -243{,}620.732 & x_2 = -29{,}282.458 \end{array}\right\} . \tag{27}$$

By means of the normal equations (26) we have computed also the weight coefficients and found for them:

$$\left.\begin{array}{llll} [\alpha\alpha] = +0.01624 & [\alpha\beta] = -0.00538 & [\alpha\gamma] = +0.00085 & [\alpha\delta] = +0.00097 \\ & [\beta\beta] = +0.00841 & [\beta\gamma] = +0.00003 & [\beta\delta] = +0.00052 \\ & & [\gamma\gamma] = +0.01214 & [\gamma\delta] = +0.00561 \\ & & & [\delta\delta] = +0.01011 \end{array}\right\} . \tag{28}$$

To determine the corrections v we have to find at first the unknowns of orientation δz for the stations 1, 4, and 6. We find from the sum-equation of the error equations (17) and the corresponding sum-equations of the two other stations

$$\delta z_1 = -0.17'', \qquad \delta z_4 = -0.04'', \qquad \delta z_6 = +0.12'',$$

and the error equations of these three stations yield the corrections:

$v_1 = +0.2''$	$v_{16} = +0.4''$	$v_{26} = +0.1''$
$v_2 = -0.6$	$v_{17} = \;\;0.0$	$v_{27} = +0.7$
$v_3 = -0.9$	$v_{18} = +0.1$	$v_{28} = +1.2$
$v_4 = -0.4$	$v_{19} = -0.3$	$v_{29} = -1.2$
$v_5 = -0.4$	$v_{20} = -1.1$	$v_{30} = -0.9$
$v_6 = -0.4$	$v_{21} = +0.7$	$v_{31} = +0.1$
$v_7 = +0.7$	$[v] = -0.2''$	$[v] = 0$.
$[v] = \;\;0.0$		

Finally we have from the three error equations (22) to (24):

$$v_9 = -0.5'', \qquad v_{14} = 0.0'', \qquad v_{23} = -1.8''.$$

The corrections for the remaining directions of the stations Wasserturm, Willmer, and Schanze do not result hereby. In order to obtain them we would have to treat these three stations in the same manner as the others.

For the sum of squares of the corrections we obtain

Stations 1, 4 and 6:	$[v\,v] =$	8.44
Stations 2, 3 and 5:	$[v\,v\,p] =$	2.62
Together:	$[v\,v\,p] =$	11.06.

The number of the corrections is equal to 22. On the other hand, besides the 4 coordinates, the three unknowns of orientation are available as unknowns so that the following value results for the mean error of an observed direction:

$$m^2 = \frac{11.06}{22-7} = \frac{11.06}{15} = 0.74 \qquad m = \pm 0.86''.$$

With the values (28) of the weight coefficients we obtain also the mean coordinate errors so that the result of the adjustment is

$$\begin{array}{lll} \textbf{Hochschule:} & y_1 = -246{,}028.868 \pm 0.008 \text{ m} & x_1 = -29{,}120.590 \pm 0.011 \text{ m} \\ \textbf{Dreifaltigkeit:} & y_2 = -243{,}620.732 \pm 0.009 & x_2 = -29{,}282.458 \pm 0.009 \end{array} .$$

Section 98. Resection with Angles

The problem of resection by the measurement of individual angles will not occur often in practice as an adjustment problem, since such measurements are combined, as a rule, before the adjustment as sets of directions. Nevertheless we will treat the problem here, partly for the sake of completeness, partly because of the greater simplicity compared with resection by direction measurements, which we will discuss in the next section.

At a point P to be newly determined, let there be measured four angles β_1, β_2, β_3, β_4 between the five fixed points P_1, P_2, . . . , P_5, so that the coordinates of the point P can be found by adjustment. In Fig. 1, let P symbolize the finally adjusted point position; the angles $P_1 P P_2$, $P_2 P P_3$, etc., are then the measured angles β after the adjustment, and hence

$$\left.\begin{array}{l} P_1 P P_2 = \beta_1 + v_1 \\ P_2 P P_3 = \beta_2 + v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} . \tag{1}$$

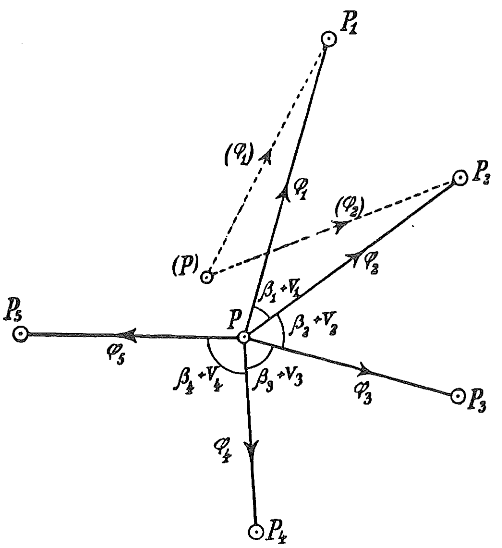


Fig. 1.

But according to the figure, the adjusted angles are equal to the differences of the final direction angles φ_1, φ_2, etc.; we have accordingly

$$\left.\begin{array}{l} \beta_1 + v_1 = \varphi_2 - \varphi_1 \\ \beta_2 + v_2 = \varphi_3 - \varphi_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} . \tag{2}$$

For the further treatment of the problem we must determine again the coordinates (x), (y) for an approximate point (P) in any arbitrary way. From these and the given coordinates of the fixed points we compute then the approximate direction angles (φ_1), (φ_2) . . . , of which only the first two are entered in Fig. 1.

We imagine that between the two points (P) and P there exists the shifting of coordinates δx and δy. According to (11), section 93, p. 359, we can indicate then at once the change of the angles of direction in the case of the passage from (P) to P. We have

$$\varphi = (\varphi) + \frac{\sin(\varphi)}{s} \varrho \, \delta x - \frac{\cos(\varphi)}{s} \varrho \, \delta y,$$

or in simplified notation

$$\varphi = (\varphi) + a \, \delta x + b \, \delta y. \tag{3}$$

If we aim to use the auxiliary quantities $\mathfrak{x}$ and $\mathfrak{y}$ according to the auxiliary table, pp. [10] to [15], of the Appendix, then, because φ is computed from the variable point on this side to the fixed point on the other side, we must compute thus (with s in kilometers):

$$a = -\frac{\mathfrak{x}}{s}, \qquad b = -\frac{\mathfrak{y}}{s}.$$

We apply this to equations (2) and obtain:

$$\left.\begin{array}{l} \beta_1 + v_1 = (\varphi_2) - (\varphi_1) + (a_2 - a_1)\,\delta x + (b_2 - b_1)\,\delta y \\ \beta_2 + v_2 = (\varphi_3) - (\varphi_2) + (a_3 - a_2)\,\delta x + (b_3 - b_2)\,\delta y \\ \dots\dots\dots\dots\dots\dots \end{array}\right\} . \tag{4}$$

Now if we introduce the simplified notation

$$\left.\begin{array}{l} \beta_1 - (\varphi_2) + (\varphi_1) = l_1 \\ \beta_2 - (\varphi_3) + (\varphi_2) = l_2 \\ \dots\dots\dots \end{array}\right\} \tag{5}$$

and set, for instance,

$$\left.\begin{array}{ll} (a_2 - a_1) = A_1 & (b_2 - b_1) = B_1 \\ (a_3 - a_2) = A_2 & (b_3 - b_2) = B_2 \\ \dots\dots & \dots\dots \end{array}\right\} \tag{6}$$

then we obtain the error equations:

$$\left.\begin{array}{l} v_1 = A_1\,\delta x + B_1\,\delta y - l_1 \\ v_2 = A_2\,\delta x + B_2\,\delta y - l_2 \\ \dots\dots\dots\dots \end{array}\right\}, \tag{7}$$

and hence the normal equations:

$$\begin{array}{l} [A\,A]\,\delta x + [A\,B]\,\delta y - [A\,l] = 0 \\ [A\,B]\,\delta x + [B\,B]\,\delta y - [B\,l] = 0 \text{ with } [l\,l]. \end{array}$$

All the rest is as usual.

Now we will illustrate this with a numerical example (Fig. 2, p. 389):

In the system of coordinates of Baden whose $+x$ axis is directed to the south, and whose $+y$ axis is directed to the west, the following five points are given by their coordinates:

	Point	y	x
P_0	St. Michael Turm	$-7,407.582$ m	$+44,332.254$ m
P_1	Durlacher Warte Signalkugel	$-1,892.355$	$+54,452.145$
P_2	Ettlingen Rathaus	$+3,798.300$	$+60,598.479$
P_3	Bulach südl. Turm	$+5,783.457$	$+55,397.802$
P_4	Daxlanden Turm	$+9,738.459$	$+53,469.087$

(8)

At the observatory P of the Polytechnikum [polytechnic college] in Karlsruhe, the following four angles were measured independently:

$$\left.\begin{aligned} P_0PP_1 &= \beta_1 = 53^\circ\,11'\,21.0'' \\ P_0PP_2 &= \beta_2 = 130\;\;48\;\;\;5.0 \\ P_0PP_3 &= \beta_3 = 172\;\;39\;\;17.5 \\ P_0PP_4 &= \beta_4 = 214\;\;43\;\;17.8 \end{aligned}\right\} . \tag{9}$$

In the case of all these four angles, St. Michael, P_0, was taken as target point on the left side. The measurements were done according to the method of repetition; no pointing or reading of an angle is used again for another angle, i.e., the four measured angles are completely independent of one another.

The circumstance that we have taken here a *single* common starting ray PP_0 for all angles facilitates the clear over-all view, but is not essential for such an adjustment of angles.

The approximate coordinates for the base point are:

$$(P) \qquad (y) = +3508.38 \text{ m} \qquad (x) = +53046.42 \text{ m}. \tag{10}$$

With these approximate values there were computed all direction angles (φ) and at the same time also the distances s:

From (P) to P_0	$(\varphi_0) = 231^\circ\,23'\,59.2''$	$\log s_0 = 4.14512$	$s_0 = 13.97$ km
From (P) to P_1	$(\varphi_1) = 284\;\;35\;\;22.0$	$\log s_1 = 3.74669$	$s_1 = 5.58$ km
From (P) to P_2	$(\varphi_2) = 2\;\;11\;\;54.5$	$\log s_2 = 3.87838$	$s_2 = 7.56$ km
From (P) to P_3	$(\varphi_3) = 44\;\;\;3\;\;18.3$	$\log s_3 = 3.51479$	$s_3 = 3.27$ km
From (P) to P_4	$(\varphi_4) = 86\;\;\;7\;\;\;7.8$	$\log s_4 = 3.79549$	$s_4 = 6.24$ km.

From these approximations (φ), with the measured angles β of (9), we compute the absolute terms $-l$ of the error equations:

	(φ)	$(\varphi_n)-(\varphi_0) = \Delta(\varphi)$	β	$\Delta(\varphi)-\beta = -l$
0.	$231^\circ\,23'\,59.2''$	. . . . . .	. . . . . .	. . .
1.	284 35 22.0	$53^\circ\,11'\,22.8''$	$53^\circ\,11'\,21.0''$	$+1.8''$
2.	2 11 54.5	130 47 55.3	130 48 5.0	-9.7
3.	44 3 18.3	172 39 19.1	172 39 17.5	$+1.6$
4.	86 7 7.8	214 43 8.6	214 43 17.8	-9.2 .

In the case of the use of the auxiliary tables of pages [8] to [15] of the Appendix, the computation of the coefficients a and b takes the following form:

Point	(φ)	$\mathfrak{x}$ (Auxiliary Table)	$\mathfrak{y}$ (Auxiliary Table)	s km	$-\frac{\mathfrak{x}}{s} = a$	$-\frac{\mathfrak{y}}{s} = b$	Angle	$a - a_0 = A$	$b - b_0 = B$	$-l$
P_0	231.4°	+16.1	−12.9	13.97	−1.15	+0.92	. . .	. . .	. . .	. . .
P_1	284.6	+20.0	+5.2	5.58	−3.58	−0.93	1.	−2.43	−1.85	+1.8
P_2	2.2	−0.8	+20.6	7.56	+0.11	−2.73	2.	+1.26	−3.65	−9.7
P_3	44.1	−14.3	+14.8	3.27	+4.38	−4.53	3.	+5.53	−5.45	+1.6
P_4	86.1	−20.6	+1.4	6.24	+3.30	−0.22	4.	+4.45	−1.14	−9.2

Therefrom the coefficients of the normal equations:

$$\left.\begin{array}{lll} [A\,A] = +57.9 & [A\,B] = -35.3 & -[A\,l] = -\;\;48.7 \\ & [B\,B] = +47.7 & -[B\,l] = +\;\;33.8 \\ & & [l\,l] = +184.5 \end{array}\right\} .$$

Everything else as on p. 372 so that we obtain:

Approximation	$(P)\,(y) = +3{,}508.380$ m		$(x) = +53{,}046.420$ m	
Corrections	$-\;0.016$	± 0.166 m	$+\;0.075$	± 0.150 m
Result of adjustment	$y = +3{,}508.364$ m	± 0.166 m	$x = +53{,}046.495$ m	± 0.150 m .

(11)

Mean error of a measured angle:

$$m = \sqrt{\frac{[l\,l\cdot 2]}{n-2}} = \sqrt{\frac{143.0}{4-2}} = \pm 8.5'' .$$

With the final coordinates y, x we compute all direction angles φ anew and find:

	φ	$\varphi - \varphi_0 = \Delta\varphi$	β	$\Delta\varphi - \beta = v$	v^2
0.	231° 23′ 58.2″	. . . . .	. . . . .	. .	. .
1.	284 35 19.5	53° 11′ 21.3″	53° 11′ 21.0″	+0.3″	0.09
2.	2 11 55.0	130 47 56.8	130 48 5.0	−8.2	67.24
3.	44 3 22.3	172 39 24.1	172 39 17.5	+6.6	43.56
4.	86 7 10.3	214 43 12.1	214 43 17.8	−5.7	32.49
					$[v^2] = 143.38$.

Since this sum $[v^2] = 143.4$ agrees with the just mentioned value $[l\,l\cdot 2]$ of the elimination, the whole computation is sufficiently checked.

The arrangement of the angles with a common starting point P_0, as is assumed in Fig. 2, has been appropriate in the present case, because that point, St. Michael, was the best with respect to distance and illumination.

Even in the case of five rays we could measure more than four angles, and then we would have to introduce all individually into the adjustment; e.g., if there were measured the seven angles

$$\left.\begin{array}{llll} P_0\,P_1 & P_0\,P_2 & P_0\,P_3 & \cdot\;\cdot \\ & P_1\,P_2 & P_1\,P_3 & P_1\,P_4 \\ & & \cdot\;\cdot & P_3\,P_4 \end{array}\right\} , \qquad (12)$$

then we would obtain seven corresponding error equations with $a_1 - a_0$, $a_2 - a_0$, . . . , $a_4 - a_3$, etc.

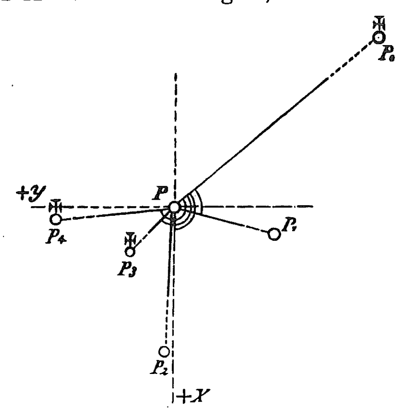


Fig. 2.
Scale 1:400,000.

In the current practice, however, we would hardly proceed thusly. If seven individual angles between five points were actually measured, then we would perhaps offhand adjust the seven angles independently on the station (for instance, according to section 62) and treat the result further as a set of directions according to section 99. Theoretically, this would not be rigorous but with respect to accidental circumstances sufficient in practice. But if we should be induced for any reasons to treat, strictly theoretically, a case as the previous one, (12), with angles, then this would be easy to do according to the above.

Section 99. Resection with Directions

Besides the intersection with direction measurements treated in section 96, the most important problem in practice is resection with direction measurements. Since three directions are necessary for the determination of the base point, then in the case of more than three measured directions excess measurements are present which make an adjustment necessary.

At first we imagine three directions chosen and with the coordinates of the pertinent three points the coordinates of the base point computed in the familiar way (according to the pattern of our volume II, 2nd half-volume, 9th edition, p. 439, or let such coordinates be available from previous measurements; in short, we assume again that we have approximate coordinates (x), (y) of the base point, and we will correct them by the addition of δx, δy in such a way that the measured directions undergo changes v as small as possible.

If we denote the measured directions by r, then the adjusted directions are:

$$\left.\begin{array}{l} r_1 + v_1 \\ r_2 + v_2 \\ \dots \end{array}\right\} . \tag{1}$$

With the coordinates (x), (y) and the given coordinates of the fixed points we can also compute the preliminary direction angles (φ) for the rays from the approximate point (P) to the fixed points. We have:

$$\left.\begin{array}{l} \tan\ (\varphi_1) = \dfrac{y_1 - (y)}{x_1 - (x)} \\ \tan\ (\varphi_2) = \dfrac{y_2 - (y)}{x_2 - (x)} \\ \dots\dots\dots \end{array}\right\} . \tag{2}$$

If in (2) instead of the preliminary coordinates (x), (y) there are introduced their final values $(x) + \delta x$, $(y) + \delta y$, then the preliminary direction angles (φ) change to the final direction angles φ. We can compute the difference $\varphi - (\varphi)$ according to section 93. For according to equation (11), section 93, p. 359, we have:

$$\varphi = (\varphi) + \frac{\sin (\varphi)}{s} \varrho \, \delta x - \frac{\cos (\varphi)}{s} \varrho \, \delta y , \tag{3}$$

or else

$$\varphi = (\varphi) + a \, \delta x + b \, \delta y . \tag{4}$$

Since after the adjustment discrepancies no longer exist, then the differences of the adjusted directions (1) are equal to the differences of the final direction angles φ; e.g., we have

$r_2 + v_2 - r_1 - v_1 = \varphi_2 - \varphi_1$, and so forth.

But hence it follows that the differences between the final direction angles and the adjusted directions are equal for all rays. This constant difference, which we will denote by z, indicates the angle which the zero direction of the measured set of directions makes with the positive direction of the axis of abscissae; hence, the set of directions is oriented in the system of coordinates through the quantity z. Therefore, we have the equations:

$$\left.\begin{array}{l} \varphi_1 - (r_1 + v_1) = z \\ \varphi_2 - (r_2 + v_2) = z \\ \dots\dots\dots \end{array}\right\} . \tag{5}$$

Now if we introduce, in addition, the values from (4) for the individual φ's, then we arrive at once at the error equations:

$$\left.\begin{array}{l} v_1 = (\varphi_1) - r_1 - z + a_1\,\delta x + b_1\,\delta y \\ v_2 = (\varphi_2) - r_2 - z + a_2\,\delta x + b_2\,\delta y \\ \ldots\ldots\ldots\ldots \end{array}\right\} . \tag{6}$$

For the sake of convenience it is advantageous to orient the measured directions by approximation already before setting up the error equations, so that only a small correction of the preliminary orientation is computed by the adjustment. In other words we will set:

$$z = (z) + \delta z \tag{7}$$

and introduce for (z) any approximate value, which we find from any one of the equations (5) if we neglect v and replace φ by (φ). The most accurate is the mean of all values resulting from equations (5). And hence we will introduce:

$$(z) = \frac{[(\varphi) - r]}{n}, \tag{8}$$

where n denotes the number of the measured directions.

We denote the directions oriented approximately by α, as hitherto, and hence we have:

$$\left.\begin{array}{l} \alpha_1 = r_1 + (z) \\ \alpha_2 = r_2 + (z) \\ \ldots\ldots \end{array}\right\} , \tag{9}$$

and if we use, in addition, the abbreviation

$$\left.\begin{array}{l} (\varphi_1) - \alpha_1 = -l_1 \\ (\varphi_2) - \alpha_2 = -l_2 \\ \ldots\ldots \end{array}\right\} , \tag{10}$$

then we have the error equations:

$$\left.\begin{array}{l} v_1 = -\delta z + a_1\,\delta x + b_1\,\delta y - l_1 \\ v_2 = -\delta z + a_2\,\delta x + b_2\,\delta y - l_2 \\ \ldots\ldots\ldots\ldots \\ v_n = -\delta z + a_n\,\delta x + b_n\,\delta y - l_n \end{array}\right\} . \tag{11}$$

The coefficients a and b can again be computed with the auxiliary quantities $\mathfrak{x}$ and $\mathfrak{y}$ of the table, pp. [10] to [15]; it is necessary, however, to set (with s in kilometers)

$$a = -\frac{\mathfrak{x}}{s}, \qquad b = -\frac{\mathfrak{y}}{s}.$$

The following normal equations belong to the error equations (10):

$$\left.\begin{array}{l} \underline{n\,\delta z} - [a]\,\delta x - [b]\,\delta y + [l] = 0 \\ -[a]\,\delta z + \underline{[a\,a]\,\delta x} + [a\,b]\,\delta y - [a\,l] = 0 \\ -[b]\,\delta z + [a\,b]\,\delta x + \underline{[b\,b]\,\delta y} - [b\,l] = 0 \text{ with } [l\,l] \end{array}\right\} . \tag{12}$$

By eliminating the first unknown δz in the usual manner, we obtain:

$$\left.\begin{array}{l}\left([a\,a]-\frac{[a]}{n}[a]\right)\delta x+\left([a\,b]-\frac{[a]}{n}[b]\right)\delta y-\left([a\,l]-\frac{[a]}{n}[l]\right)=0\\ \left([a\,b]-\frac{[b]}{n}[a]\right)\delta x+\left([b\,b]-\frac{[b]}{n}[b]\right)\delta y-\left([b\,l]-\frac{[b]}{n}[l]\right)=0\end{array}\right\}. \qquad (13)$$

For these we will introduce new symbols, namely:

$$\left.\begin{array}{l}[A\,A]\,\delta x+[A\,B]\,\delta y-[A\,L]=0\\ [A\,B]\,\delta x+[B\,B]\,\delta y-[B\,L]=0\end{array}\right\}, \qquad (14)$$

with these we can compute further in the usual manner and determine δx, δy.

But we can also eliminate δz already from the error equations, as we already have done in section 97, p. 383, by forming the sum and the mean from the error equations (11), or, which is the same, dividing the first normal equation of (12) by n:

$$0=-\delta z+\frac{[a]}{n}\delta x+\frac{[b]}{n}\delta y-\frac{[l]}{n}. \qquad (15)$$

If the approximate value (z) is determined according to equation (8), then there is the small advantage that we will have $[l]=0$.

We subtract the equation (15) from each individual one of equations (11):

$$\left.\begin{array}{l}v_1=\left(a_1-\frac{[a]}{n}\right)\delta x+\left(b_1-\frac{[b]}{n}\right)\delta y-\left(l_1-\frac{[l]}{n}\right)\\ v_2=\left(a_2-\frac{[a]}{n}\right)\delta x+\left(b_2-\frac{[b]}{n}\right)\delta y-\left(l_2-\frac{[l]}{n}\right)\\ \cdots\cdots\cdots\cdots\end{array}\right\}. \qquad (16)$$

Let us introduce new symbols for these, namely:

$$\begin{array}{lll}a_1-\frac{[a]}{n}=A_1 & b_1-\frac{[b]}{n}=B_1 & l_1-\frac{[l]}{n}=L_1\\ a_2-\frac{[a]}{n}=A_2 & b_2-\frac{[b]}{n}=B_2 & l_2-\frac{[l]}{n}=L_2,\ \text{etc.,}\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\end{array}$$

with these, we compute further:

$$\begin{aligned}[A\,A]=A_1{}^2+A_2{}^2+\ldots &= a_1{}^2-2\,a_1\frac{[a]}{n}+\frac{[a]^2}{n^2}\\ &\quad+a_2{}^2-2\,a_2\frac{[a]}{n}+\frac{[a]^2}{n^2}+\ldots\\ [A\,A] &= [a\,a]-2\,[a]\frac{[a]}{n}+\frac{[a]}{n}[a]=[a\,a]-\frac{[a]}{n}[a].\end{aligned}$$

The thus computed $[A\,A]$ is therefore identical with the $[A\,A]$ introduced in a different way as a symbol in the case of (13) and (14); and it is likewise with $[A\,B]$, $[B\,B]$, and so forth, as we prove in as simple a manner as just now in the case of $[A\,A]$.

With this, we have found a fundamental theorem which, in connection with the above, reads as follows: In the case of n error equations of the form (11) we form the average values of the coefficients and of the absolute terms, namely with the introduction of the symbols a_0, b_0, l_0:

$$\frac{[a]}{n} = a_0 \qquad \frac{[b]}{n} = b_0 \qquad \frac{[l]}{n} = l_0 \,. \tag{17}$$

Then the deviations from the means:

$$\left.\begin{array}{lll} a_1 - a_0 = A_1 & b_1 - b_0 = B_1 & l_1 - l_0 = L_1 \\ a_2 - a_0 = A_2 & b_2 - b_0 = B_2 & l_2 - l_0 = L_2 \\ \cdots & \cdots & \cdots \end{array}\right\} . \tag{18}$$

With these we compute further as though the unknown δz did not exist at all:

$$\left.\begin{array}{l} v_1 = A_1\,\delta x + B_1\,\delta y - L_1 \\ v_2 = A_2\,\delta x + B_2\,\delta y - L_2 \\ \cdots \end{array}\right\} . \tag{19}$$

Therefrom the normal equations (14) with their solution δx and δy and besides with the final term

$$[L\,L \cdot 2] = [v\,v] . \tag{20}$$

All this is best made clear by a numerical example, which we have calculated with complete thoroughness on pp. 394-395.

As approximate coordinates of the point Hochschule we take again the same as in the example of section 96, and with the inclusion of the coordinates of the five given points of p. 249 we can fill in at once the first part of p. 394 and calculate down to (φ_5). By so doing, we also have calculated the approximate distances s with four places of the logarithm for the case in which we do not have a sufficiently accurate triangulation-net picture, p. 234, in order to determine from it s to 0.1 km. If we so desire, we can also add on p. 394 the accurate calculation of the a's and b's according to the pattern of section 95, p. 371.

There follows the filling in of the upper part of p. 395 with the measured r's according to section 69, p. 233, and with the (φ)'s taken from p. 394, whereupon also (z), $-l$, $-L$, and L^2 result at once. Since (z) is determined according to (8), we have $-l = -L$.

Also the second part of p. 395 hardly needs further explanations after understanding the theory of equations (16) and (17); we can do everything with the slide rule, with the square-table, and almost mentally.

Also the elimination is to be carried out briefly with the slide rule; $\frac{48}{101}\,48 = 23$, etc., down to $[L\,L \cdot 2] = 33$ or 32, and *if needed* the third check with 33.

After $(y) + \delta y = y$ and $(x) + \delta x = x$ are combined, we come to the lower part of p. 394 and with it the lower part of p. 395 with the computation of $\delta z = a_0 \delta x + b_0 \delta y - l_0$ and $v = a\,\delta x + b\,\delta y - l - \delta z$. After $z = (z) + \delta z$ is also determined, there result the final direction angles $= r + v + z$, which must agree with the final direction angles φ of p. 394 below computed from the coordinates. Finally, there are also indicated the v^2's, where we must have $[v\,v] = [L\,L \cdot 2]$.

What is a *sufficient* agreement in the case of the important check $[L\,L \cdot 2] = [v\,v]$, the practical computer must judge according to the circumstances; in our case 33, 32, and 32.6 is in any case a sufficient agreement, but if larger amounts v occur, this check may be off considerably without the computation being wrong; e.g., $v = 8.0$ or 8.2 yield $v^2 = 64.00$ or 67.24, i.e., already different by 3.24 although 0.2 as the uncertainty of a v is still permitted. Not until after some computational exercise will a correct and quick glance be present for the judgment of whether the check agrees sufficiently.

As a check for the $[A\,A]$'s, $[A\,B]$'s, etc., we are to mention here at once also the second method, which is given in the following section 100 on p. 397.

Point A Point P (p. 250)	y_a . . . . . y_p . . . . . $y_a - y_p = \Delta y$	x_a . . . . . x_p . . . . . $x_a - x_p = \Delta x$	$\log \Delta y$ $\log \Delta x$ $\log \tan\ (PA)$	Distance s

I. *Before the Adjustment*

Schanze . . . . Hochschule . .	— 23086.933 — 24709.800 + 1622.867	— 23266.607 — 26868.300 + 3601.693	3.210283 3.556507 9.653776 $(\varphi_1) = 24° 15' 20.0''$	3.5565 9.9599 3.5966 $s_1 = 3950$ m
Steuerndieb . Hochschule . .	— 19888.668 — 24709.800 + 4821.132	— 25951.884 — 26868.300 + 916.416	3.683149 2.962093 0.721056 $(\varphi_2) = 79° 14' 14.7''$	3.6831 9.9923 3.6908 $s_2 = 4907$ m
Ägidius . . . . Hochschule . .	— 23271.813 — 24709.800 + 1437.987	— 28308.395 — 26868.300 — 1440.095	3.157755 3.158391 9.999364 $(\varphi_3) = 135° 2' 31.0''$	3.1584 9.8498 3.3086 $s_3 = 2035$ m
Wasserturm . . Hochschule . .	— 25538.488 — 24709.800 — 828.688	— 29071.474 — 26868.300 — 2203.174	2.918391 3.343049 9.575342 $(\varphi_4) = 200° 36' 46.7''$	3.3430 9.9713 3.3717 $s_4 = 2353$ m
Burg . . . . . Hochschule . .	— 25842.799 — 24709.800 — 1132.999	— 24977.399 — 26868.300 + 1890.901	3.054230 3.276669 9.777561 $(\varphi_5) = 329° 4' 14.2''$	3.2767 9.9334 3.3433 $s_5 = 2204$ m

II. *After the Adjustment*

Schanze . . . . Hochschule . .	— 23086.933 — 24709.762 + 1622.829	— 23266.607 — 26868.280 + 3601.673	3.210273 3.556504 9.653769 $\varphi_1 = 24° 15' 18.8''$	
Steuerndieb . . Hochschule . .	— 19888.668 — 24709.762 + 4821.094	— 25951.884 — 26868.280 + 916.396	3.683145 2.962083 0.721062 $\varphi_2 = 79° 14' 15.3''$	
Ägidius . . . . Hochschule . .	— 23271.813 — 24709.762 + 1437.949	— 28308.395 — 26868.280 — 1440.115	3.157744 3.158397 9.999347 $\varphi_3 = 135° 2' 35.0''$	
Wasserturm . . Hochschule . .	— 25538.488 — 24709.762 — 828.726	— 29071.474 — 26868.280 — 2203.194	2.918411 3.343053 9.575358 $\varphi_4 = 200° 36' 49.2''$	
Burg . . . . . Hochschule . .	— 25842.799 — 24709.762 — 1133.037	— 24977.399 — 26868.280 + 1890.881	3.054244 3.276665 9.777579 $\varphi_5 = 329° 4' 10.4''$	

Intersection Point Hochschule (Fig. 1, p. 234)

No.	Target Point	Measured Direction r (p. 233)	Approximate Direction Angle (φ) (p. 394, I)	$(\varphi) - r$	$(\varphi) - r - (z) = -l = -L$	L^2
1	Schanze. .	249° 12′ 49.4″	24° 15′ 20.0″	135° 02′ 30.6″	− 1.0″	1.00
2	Steuerndieb	304 11 45.1	79 14 14.7	135 02 29.6	− 2.0	4.00
3	Ägidius . .	0 00 00	135 2 31.0	135 02 31.0	− 0.6	0.36
4	Wasserturm	65 34 18.8	200 36 46.7	135 02 27.9	− 3.7	13.69
5	Burg. . .	194 01 35.2	329 4 14.2	135 02 39.0	+ 7.4	54.76
	Sum	00′ 28.5″	13′ 06.6″	12′ 38.1″	+ 0.1″	73.81
	Mean			$(z) = 135°\ 02'\ 31.6''$		

No.	(φ)	$\mathfrak{x}$ Auxiliary Table Appendix pp. [10]-[15]	$\mathfrak{y}$	s km	$-\frac{\mathfrak{x}}{s} = a$	$-\frac{\mathfrak{y}}{s} = b$	$a - a_0 = A$	$b - b_0 = B$	$-L$	A^2	B^2	AB + −	$-AL$ + −	$-BL$ + −
1	24° 20′	− 8.5	+ 18.8	3.95	+ 2.2	− 4.8	+ 1.1	− 5.2	− 1.0	1	27	− 6	− 1	+ 5
2	79 10	− 20.3	+ 3.9	4.91	+ 4.2	− 0.8	+ 3.1	− 1.2	− 2.0	10	1	− 4	− 6	+ 2
3	135 0	− 14.6	− 14.6	2.04	+ 7.2	+ 7.2	+ 6.1	+ 6.8	− 0.6	37	46	+ 41	− 4	− 4
4	200 40	+ 7.3	− 19.3	2.35	− 3.1	+ 8.2	− 4.2	+ 7.8	− 3.7	18	61	− 33	+ 16	− 29
5	329 0	+ 10.6	+ 17.7	2.20	− 4.8	− 8.0	− 5.9	− 8.4	+ 7,4	35	71	+ 50	− 44	− 62
				Sum	+ 5.7	+ 1.8	+ 0.2	− 0.2	+ 0.1	+ 101	+ 206	+ 48	− 39	− 88
				Mean	+ 1.1 $= a_0$	+ 0.4 $= b_0$	Check 0.0							

	$A]$	$B]$	$-L]$
$[A$	+ 101	+ 48	− 39
	$[B$	+ 206	− 88
		− 23	+ 19
		$[-L$	+ 74
			− 15
$\delta y =$ + 0.38 dm	$[BB \cdot 1] =$	+ 183	− 69
$\delta y =$ + 0.038 m			+ 59
$(y) = -24709.800$			− 26
$y = -24709.762$ m		$[LL \cdot 2] =$	+ 33

If needed, 3rd check $[vv] = [LL] - [AL]\,\delta x - [BL]\,\delta y$

$[B$	$[A$	$[-L$	
+ 206	+ 48	− 88	$B]$
	+ 101	− 39	$A]$
	− 11	+ 21	
		+ 74	$-L]$
		− 38	
$[AA \cdot 1] =$	+ 90	− 18	$\delta x =$ + 0.20 dm
		+ 36	$\delta x =$ + 0.020 m
		− 4	$(x) = -26868.300$
	$[LL \cdot 2] =$	+ 32	$x = -26868.280$ m

No.	$a\,\delta x + b\,\delta y$		$-l$	$-\delta z$	v	Measured Direction r (see above)	$r + v + z$	Final Direction Angle φ (p. 394, II)	v^2
1	+ 0.4″	− 1.8″	− 1.0″	− 0.4″	− 2.8″	249° 12′ 49.4″	24° 15′ 18.6″	24° 15′ 18.8″	7.84
2	+ 0.8	− 0.3	− 2.0	− 0.4	− 1.9	304 11 45.1	79 14 15.2	79 14 15.3	3.61
3	+ 1.4	+ 2.7	− 0.6	− 0.4	+ 3.1	0 00 00	135 02 35.1	135 2 35.0	9.61
4	− 0.6	+ 3.1	− 3.7	− 0.4	− 1.6	65 34 18.8	200 36 49.2	200 36 49.2	2.56
5	− 1.0	− 3.0	+ 7.4	− 0.4	+ 3.0	194 01 35.2	329 04 10.2	329 4 10.4	9.00
Sum	+ 1.0″	+ 0.7″	+ 0.1″		− 0.2″ Should be 0.0				32.62
Mean	+ 0.2	+ 0.14	+ 0.0						

$a_0\,\delta x + b_0\,\delta y - l_0 = \delta z = +0.4''$

$z = (z) + \delta z = 135°\ 02'\ 32.0''$

$$[vv] = [ll \cdot 2] = 32.6 \qquad m^2 = \frac{32.6}{2} = 16.3 \qquad m_y^2 = \frac{m^2}{[BB \cdot 1]} = \frac{16.3}{183} \qquad m_x^2 = \frac{m^2}{[AA \cdot 1]} = \frac{16.3}{90}$$

$$n - 3 = 2 \qquad m = \pm 4.0'' \qquad m_y = \pm 0.30\ \partial\mathrm{m} \qquad m_x = \pm 0.42\ \partial\mathrm{m}\,.$$

Final result:

Hochschule $y = -24709.762$ m ± 0.030 m $\qquad x = -26868.280$ m ± 0.042 m.

Section 100. Elimination of the Unknown of Orientation in the Case of Direction Measurements

In the case of intersection (section 97) as well as in the case of resection (section 99) with direction measurements there occurs in the error equations of a station also a correction δz of the unknown of orientation in addition to the corrections δx and δy of the coordinates of the new point. We have already shown in section 97 how we can eliminate δz with the help of the sums of the error equations, a procedure which had already been indicated for such an adjustment by Gauss in 1822 in the *Astronomische Nachrichten*, vol. I, p. 82.

We will discuss this once again in the following and limit ourselves here to the case of resection; the application to other problems with direction measurements on a station follows without difficulties.

The error equations for resection with directions have the following form according to (11) of the previous section 99, p. 391,

$$\left.\begin{aligned} v_1 &= -\delta z + a_1\,\delta x + b_1\,\delta y - l_1 \\ v_2 &= -\delta z + a_2\,\delta x + b_2\,\delta y - l_2 \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} . \tag{1}$$

The normal equations are for n directions in the abbreviated way of writing (section 27, p. 86):

$$\left.\begin{aligned} \underline{n\,\delta z} - [a]\,\delta x - [b]\,\delta y + [l] &= 0 \\ + \underline{[a\,a]\,\delta x} + [a\,b]\,\delta y - [a\,l] &= 0 \\ + \underline{[b\,b]\,\delta y} - [b\,l] &= 0 \\ + \underline{[l\,l]} & \end{aligned}\right\} . \tag{2}$$

We use the first equation for the elimination of δz and substitute

$$\frac{[a]}{n} = a_0\,, \qquad \frac{[b]}{n} = b_0\,, \qquad \frac{[l]}{n} = l_0\,. \tag{3}$$

Then there follows a system reduced for the first time:

$$\left.\begin{aligned} \underline{[A\,A]\,\delta x} + [A\,B]\,\delta y - [A\,L] &= 0 \\ \underline{[B\,B]\,\delta y} - [B\,L] &= 0 \\ \underline{[L\,L]} & \end{aligned}\right\} , \tag{4}$$

where the individual coefficients have the following meaning:

$$\left.\begin{aligned} &[A\,A] = [a\,a] - a_0\,[a] \qquad [A\,B] = [a\,b] - a_0\,[b] \qquad [A\,L] = [a\,l] - a_0\,[l] \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{or } = [a\,b] - [a]\,b_0 \qquad \text{or } = [a\,l] - [a]\,l_0 \\ &\qquad\qquad\qquad\qquad\qquad [B\,B] = [b\,b] - b_0\,[b] \qquad [B\,L] = [b\,l] - b_0\,[l] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{or } = [b\,l] - [b]\,l_0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [L\,L] = [l\,l] - l_0\,[l] \end{aligned}\right\} . \tag{5}$$

Objectively all this is nothing else but that which we have carried out in the previous section 99, p. 392. Nevertheless we will apply the formulae (5) directly now and in them take once again the a's, b's, l's of the previous example of p. 395:

No.	a	b	$-l$	a^2	b^2	l^2	$a\,b$	$-a\,l$	$-b\,l$
1	+ 2.2	− 4.8	+ 0.6	5	23	0.4	− 11	+ 1	− 3
2	+ 4.2	− 0.8	− 0.4	18	1	0.0	− 3	− 2	+ 0
3	+ 7.2	+ 7.2	+ 1.0	52	52	1.0	+ 52	+ 7	+ 7
4	− 3.1	+ 8.2	− 2.1	10	67	4.4	− 25	+ 7	− 17
5	− 4.8	− 8.0	+ 9.0	23	64	81.0	+ 38	− 43	− 72
Sums	+ 5.7	+ 1.8	+ 8.1	+ 108	+ 207	+ 86.8	+ 51	− 30	− 85
Means	+ 1.1	+ 0.4	+ 1.6						
	$= a_0$	$= b_0$	$= l_0$						
Deductions $- a_0\,[a]$, etc.				− 6	− 1	− 13.0	− 2	− 9	− 3
Sums $[A\,A]$, $[B\,B]$, etc.				+ 102	+ 206	+ 73.8	+ 49	− 39	− 88
For comparison, p. 395.				+ 101	+ 206	+ 73.8	+ 48	− 39	− 88
				$[A\,A]$	$[B\,B]$	$[L\,L]$	$[A\,B]$	$-[A\,L]$	$-[B\,L]$

(6)

Within the inevitable uncertainty of rounding off we therefore have again the same sums $[A\,A] = 101$, $[A\,B] = +48$, etc., as in the case of the calculation of section 99, p. 395.

If we compare the elimination of δz by means of the sum-equation in section 99, p. 393, with the above-mentioned computational procedure, then the former procedure has the advantage of being illustrative and of the form of a smooth working rule which would hardly be forgotten again, but nothing is saved on computational work by that procedure of the forming of the mean $A = a - a_0$, etc., in comparison with the second method; on the contrary, the calculation according to the procedure shown in (5) above is more convenient and less extensive (with three columns less), and also has other advantages. If an error is found in a coefficient a, b, or l, then the *whole* calculation in A, B, or L is upset in the case of the first method, not in the second method, or when individual directions forwards *and* backwards occur in adjustments, as, e.g., Steuerndieb, Ägidius, Wasserturm, Burg in pp. 376 and 395, then we can carry over the pertinent a^2, b^2, ab from the one adjustment into the other if we compute according to the second method, but not in the case of the first method; finally, in the case of the second method we can carry out the check (18), p. 366, $a^2 + b^2$ with auxiliary table [18] of the Appendix, but not in the case of the first method. –

The second method even has a small advantage of accuracy insofar as rounding off errors in the formation of differences $A = a - a_0$, etc., do not occur. For if the division $\frac{[a]}{n} = a_0$ leaves a remainder, falling in the neighborhood of 0.5 of the last unit to be retained, then a part of the A's will be too large by nearly 0.5 (absolutely), the others too small and this involves noticeable errors in $[A\,A]$, $[A\,B]$, etc., which do not occur in the case of the second method.

Negative weights. There is still a third form for the formation of the sums $[A\,A]$, etc., which in the numerical calculation, however, hardly varies from the previous one and with which we already have become acquainted in section 34, p. 108, in a more general form. We obtain the sums $[A\,A]$, etc., also by taking in the original error equations, not including δz, the sum of all error equations as a fictitious error equation with the weight $-\frac{1}{n}$, hence with fictitious negative weight; for then we obtain:

$$a_1{}^2 + a_2{}^2 + \ldots + a_n{}^2 + \left(-\frac{1}{n}\right)[a]^2 = [a\,a] - \frac{[a]}{n}\,[a]\,.$$

Essentially this agrees with (3) and (5), and differs only in the manner of expression. This form has been introduced by General Schreiber in the trigonometric section of the Land Survey (cf. Schreiber's rules in the later section 110).

Instead of introducing a negative weight, we can also multiply the sums $[a]$, $[b]$, etc., by the root of the weight $\sqrt{\frac{1}{n}}$, and then simply square and multiply the coefficients, but take the amounts always *negatively* into the calculation.

Or, if for instance the mean values $\frac{[a]}{n} = a_0$, etc., are already formed, we also can multiply these by $\sqrt{n}$ and then treat them further as having *one* weight, but take them negatively into the remaining sum.

To this we have:

$n =$	2	3	4	5	6	7	8	9	10
$\sqrt{n}$	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000	3.162
$\frac{1}{\sqrt{n}}$	0.707	0.577	0.500	0.447	0.408	0.378	0.354	0.333	0.316.

In the previous example we had $[a] = +5.7$, $[b] = +1.8$, $-[l] = +8.1$,

to this $n = 5$, and hence $\frac{1}{\sqrt{n}} = 0.447$ 0.447 0.447

the products: . . . . $a' = 2.55$ $b' = 0.80$ $-l' = 3.62$

therefrom the six squares and products:

$$a'^2 = 6.5 \quad b'^2 = 0.6 \quad l'^2 = 13.1 \quad a'b' = 2.0 \quad -a'l' = 9.2 \quad -b'l' = 2.9.$$

These are the same results as before in the case of (6), p. 397.

After all comparisons we regard the form of computation (6) as the best one; it was used in our more recent practical operations and not put on top into the pattern of p. 395 only because the form $a - a_0 = A$, etc., has been adopted in most cases and is the most illustrative, and because, even if we compute the $[A\,A]$'s, etc., differently, we can hardly evade computing the v's according to p. 395, for we want to have the distribution of errors with the v's right before us for the judgment of the total success of our results.

Different sets of directions

When several complete homogeneous sets of measured directions are present, then they are combined into a mean, which is used as *one* full set for resecting in the manner as is indicated above.

Now we can study further the case in which several *incomplete* sets of directions are measured.

Each set of directions yields a system of error equations with an unknown of orientation z, and each different set has also different unknowns $z_1, z_2, z_3 \ldots$.

Now we can treat the totality of all error equations pertaining to these measurements directly according to the general rules for indirect observations, and in fact, we have the following $2 + n$ unknowns in the case of n sets of directions:

$$\delta x,\ \delta y \text{ and } \delta z_1,\ \delta z_2,\ \delta z_3, \ldots, \delta z_n.$$

In each individual set of directions or, as the case may be, in each group of error equations which belongs to a set of directions, the δz in question can then be eliminated according to section 99 by formation of the mean. This procedure, however, is not convenient. In the case of several incomplete sets we arrive more quickly at the goal if we form the normal equations with the original error equations (each group of which contains a special δz), because the original error equations occurring in different sets have the same coefficients a, b for the same target points, but not the transformed ones.

But all this refers only to a strictly theoretical adjustment.

For practical cases in which the assumption that the given points are free of error is never completely satisfied, however, it will always be sufficient to adjust scattered sets of observations at first independently on the station, for instance according to the approximation method of our former section 64, and to introduce the result of this adjustment into the further adjustment as a *single* set of directions with an unknown of orientation z, i.e., to proceed then according to section 99.

If a new point is to be determined against other given fixed points by means of intersection, according to existing methods one has to distinguish between two kinds of measurements, first, those which send sighting rays from the given points as base points to the new point and, second, such measurements which, made on the new point as base point, furnish sighting rays from the new point to the old points. According to these two kinds of measurements we distinguish between *"outward directions"* and *"inward directions."* Cf. Volume II of this *Handbuch*, first half-volume, ninth edition, 1931, section 91.

In Fig. 1 we have, for example: (18), (6), (17), (25), old given fixed points, and (13), a new point to be determined by intersection. Consequently, we have:

1. and 2., outward directions,
3., 4., 5., 6., inward directions.

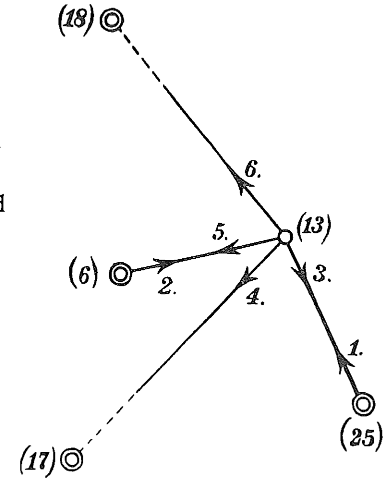


Fig. 1.
1:30,000.

In regard to the use of the outward and the inward directions there is a fundamental difference. Whereas the inward directions can be used forthwith for the determination of the new point, the outward directions must be attached to one or several given fixed points.

Furthermore, it follows therefrom that the inward and the outward directions have different weights as a rule. The weights of the inward directions depend merely on the measuring accuracy; the weights of the outward directions, on the other hand, are conditioned, apart from the measuring accuracy, by the more or less close attachment to the existing fixed rays.

We already have examined these relations in section 96, p. 378, and found that the weight of an outward direction is always smaller than 1 if the *direction measurement* receives the weight 1. If the outward direction is attached to s fixed rays, then its weight is according to (4), section 96, p. 378

$$p = \frac{s}{s+1}. \qquad (1)$$

In the simplest case in which, for example, the outward direction (6), (13) in Fig. 1 is attached by measuring of the angle (18) (6) (13), the weight of this direction would equal $\frac{1}{2}$, and if we wish to exclude a more precise weight distinction for present use, then we may accept as an approximate rule:

$$\text{Weight of an outward direction } p = \frac{1}{2}, \qquad (2)$$

while, at the same time, the weight of an inward direction is set $= 1$.

Even though this rule (2) should appear somewhat rough, in any case it is better than another approximate method occurring whereby the outward and inward directions are to receive all *equal* weight $= 1$.

After these preliminary remarks we pass to the setting up of the error equations for combined intersection and resection of a point.

1. *Outward directions* (section 96). After the direction angle α for the direction from the fixed point to the new point has been taken from the Abriss [station data] and the corresponding preliminary direction angle (φ) computed, we have the absolute term $(\varphi) - \alpha = -l$, and with it, the error equation is

$$v = -\frac{\sin(\varphi)}{s}\varrho\,\delta x + \frac{\cos(\varphi)}{s}\varrho\,\delta y - l. \qquad (3)$$

2. *Inward directions* (section 99). After the computation of the preliminary direction angle (φ) for the direction from the new point to the fixed point we obtain from it and the measured direction r the

preliminary magnitude of orientation

$$(z) = \frac{[(\varphi) - r]}{n} .$$

With it, the absolute term will be $(\varphi) - r - (z) = - l$, and then the error equation reads:

$$v = - \delta z + \frac{\sin(\varphi)}{s} \varrho \, \delta x - \frac{\cos(\varphi)}{s} \varrho \, \delta y - l . \qquad (4)$$

3. *Reciprocal directions.* We consider further the special case in which the outward and the inward direction are referred to the same ray. Let there be measured two opposite directions between the fixed point P_1 and the new point P, as is indicated in Fig. 2. If we set up the error equations (3) and (4) for them, then we note that $(\varphi_1) = (\varphi) \pm 180^\circ$ and consequently,

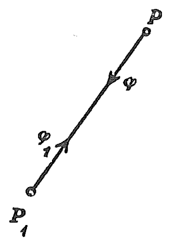


Fig. 2.

$$\begin{aligned} - \sin(\varphi_1) &= + \sin(\varphi) \\ + \cos(\varphi_1) &= - \cos(\varphi) \end{aligned} .$$

Then we obtain two error equations in which the coefficients of the corrections of coordinates δx and δy are equal to one another, i.e.,

$$\left.\begin{aligned} v &= + a \, \delta x + b \, \delta y - l \\ v_1 &= - \delta z_1 + a \, \delta x + b \, \delta y - l_1 \end{aligned}\right\} . \qquad (5)$$

The two equations therefore differ from one another only by the absolute terms and by the occurrence of the unknown of orientation δz_1 for the outward direction.

As a first example we shall take the case of Fig. 1, p. 399. (This example is borrowed from the *Anweisung IX* of 25 October 1881, pp. 187-190, where inward and outward directions are treated as having *equal* weight, however, and δx, δy are counted in meters.)

First we have the given fixed coordinates and the approximate coordinates of the new point (Fig. 1, p. 399);

	Point	Ordinate y			Abscissa x
Given fixed	(25)	X 44,276.21 m	or	= − 55,723.79 m	+ 21,591.03 m
Given fixed	(17)	X 43,114.56	or	= − 56,885.44	+ 21,345.08
Given fixed	(6)	X 43,282.03	or	= − 56,717.97	+ 22,079.51
Given fixed	(18)	X 43,210.38	or	= − 56,789.62	+ 23,094.54
Approximation	(13) (y) =	X 43,949.96	or	= − 56,050.04	(x) = + 22,239.44

(6)

Outward Directions α	Distances s	Inward Directions Already Oriented by Approximation $r + (z)$
1. = 333° 17′ 25″	726 m	3. = 153° 16′ 53″
	1224	4. = 223 2 18
2. = 76 32 6	687	5. = 256 32 31
	1131	6. = 319 9 15

(7)

Approximate direction angles:

For Outward Directions:

$$(\varphi_1) = 333^\circ\,17'\,26'' \text{ with } -l_1 = +1''$$
$$(\varphi_2) = 76\;\;32\;\;4 \text{ with } -l_2 = -2$$

For Inward Directions:

$$\left.\begin{aligned}(\varphi_3) &= 153^\circ\,17'\,26'' \text{ with} - l_3 = +33''\\ (\varphi_4) &= 223\;\;2\;\;52 \text{ with} - l_4 = +34\\ (\varphi_5) &= 256\;\;32\;\;4 \text{ with} - l_5 = -27\\ (\varphi_6) &= 319\;\;8\;\;36 \text{ with} - l_6 = -39\end{aligned}\right\}. \tag{8}$$

Error equations (for δx and δy in decimeters):

$$\begin{aligned}v_1 &= +1 + 13\,\delta x + 25\,\delta y\\ v_2 &= -2 - 29\,\delta x + 7\,\delta y\end{aligned}$$
$$v_1 \text{ and } v_2 \text{ with weight } = \frac{1}{2}$$

$$\left.\begin{aligned}v_3 &= -\delta z + 33 + 13\,\delta x + 25\,\delta y\\ v_4 &= -\delta z + 34 - 12\,\delta x + 12\,\delta y\\ v_5 &= -\delta z - 27 - 29\,\delta x + 7\,\delta y\\ v_6 &= -\delta z - 39 - 12\,\delta x - 14\,\delta y\end{aligned}\right\} \tag{9}$$
$$0 = -\delta z + 0 - 10\,\delta x + 8\,\delta y \quad \text{Mean.}$$

The elimination of δz is carried out according to the method of the formation of the mean of section 99, p. 395, so that we have:

$$\begin{aligned}v_1 &= +13\,\delta x + 25\,\delta y + 1\\ v_2 &= -29\,\delta x + 7\,\delta y - 2\end{aligned}$$
$$\text{with weight } = \frac{1}{2}$$

$$\left.\begin{aligned}v_3 &= +33 + 23\,\delta x + 17\,\delta y \text{ with } p=1\\ v_4 &= +34 - 2\,\delta x + 4\,\delta y \text{ with } p=1\\ v_5 &= -27 - 19\,\delta x - 1\,\delta y \text{ with } p=1\\ v_6 &= -39 - 2\,\delta x - 22\,\delta y \text{ with } p=1\end{aligned}\right\}. \tag{10}$$

Normal equations, first for all $p = 1$:

$$\begin{aligned}\underline{1010}\,\delta x + 122\,\delta y + 71 &= 0\\ +\underline{674}\,\delta y + 11 &= 0\\ +\underline{5} &= [l\,l]\end{aligned}$$

$$\left.\begin{aligned}+\underline{898}\,\delta x + 446\,\delta y + 1282 &= 0\\ +\underline{790}\,\delta y + 1582 &= 0\\ +\underline{4495} &= [L\,L]\end{aligned}\right\}. \tag{11}$$

Now if outward and inward directions are to have *equal* weights (weight = 1), then we simply have to add up, term by term, the previous two systems of normal equations, e.g., $(1010 + 898)\,\delta x = 1908\,\delta x$ and so forth [which corresponds to equation (11), p. 105], and thus we have:

Combined normal equations for equal weights:

$$\left.\begin{aligned}+\underline{1908}\,\delta x + 568\,\delta y + 1353 &= 0\\ +\underline{1464}\,\delta y + 1593 &= 0\\ +\underline{4500}&\end{aligned}\right\}. \tag{12}$$

The solution yields:

$$\left.\begin{aligned}\delta y = -0.92\,\partial\text{m} \qquad \delta x &= -0.44\,\partial\text{m}\\ \text{and } [l\,l\cdot 2] &= 2448\end{aligned}\right\}, \tag{13}$$

the final result for the coordinates rounded off to 1 cm:

Point (13):

$$\left.\begin{aligned}y = -56050.13\text{ m} \qquad x &= +22239.40\text{ m}\\ \pm 0.08\text{ m} \qquad\qquad &\pm 0.07\text{ m}\end{aligned}\right\}. \tag{14}$$

If, however, only half the weight is given to the outward directions, then we have to combine the two groups (11) in such a manner that on the left we only take half, hence, e.g., $(505 + 898)\delta x$ and so forth. This yields:

Combined normal equations for unequal weights:

$$\left.\begin{array}{r} +\underline{1403}\,\delta x + 507\,\delta y + 1318 = 0 \\ +\underline{1127}\,\delta y + 1588 = 0 \\ +\underline{4497} \end{array}\right\} . \tag{15}$$

The solution yields:

$$\left.\begin{array}{c} \delta y = -1.17\,\partial\text{m} \qquad \delta x = -0.52\,\partial\text{m} \\ [ll\cdot 2] = 1952 \end{array}\right\} , \tag{16}$$

the final result for the coordinates rounded off to 1 cm:

Point (13):

$$\left.\begin{array}{cc} y = -56{,}050.16\text{ m} & x = +22{,}239.39\text{ m} \\ \pm 0.08\text{ m} & \pm 0.07\text{ m} \end{array}\right\} . \tag{17}$$

The results (14) and (17) do not greatly deviate from one another; this is a matter of chance, however, not of the principle. The complete termination of the computation includes in both cases also the computation of the final direction angles φ and the corrections v besides $[v^2]$ and so forth. But we shall illustrate this in detail by the next example.

Adjustment of the point Hochschule with intersection and resection

According to the aspect of Fig. 1, p. 234, and the measuring lists, p. 233, the point Hochschule has four outward and five inward directions, namely:

Outward Directions	Inward Directions
. . . .	To Schanze
From Steuerndieb	To Steuerndieb
From Ägidius	To Ägidius
From Wasserturm	To Wasserturm
From Burg	To Burg
With adjustment, section 96, p. 376	With adjustment, section 99, p. 395

Now since the adjustment of the outward directions as well as that of the inward directions have already been taken care of independently from one another, we only need to collect these two adjustments in order to achieve the total adjustment.

We shall note here at the outset that a new special form blank for combined intersection and resection is not necessary since we simply retain the special parts of the forms, pp. 376 and 395, and effect the collection on an additional sheet, in the case of a few directions only, by a somewhat changed usage of the printed columns of p. 376. (In such manner we have adjusted numerous intersections and resections.)

Now we take the coefficients $[aa]$, $[bb]$, etc., from the two separate adjustments:

		$[aa]$	$[bb]$	$[ab]$	$-[al]$	$-[bl]$	$[ll]$
Page 376	Intersection	+103	+184	+62	−13	−54	+17.2
I.e., 1/2	Intersection	+51	+92	+31	−6	−27	+8.6
Page 395	Resection	+101	+206	+48	−39	−88	+73.8
	Total	+152	+298	+79	−45	−115	+82.4.

We will look here at first whether the $-[a\,l]$'s and $-[b\,l]$'s have the same signs by intersection and resection, i.e., are not in direct contrast. Identity of the signs occurs here. After the collection we carry out the elimination as previously on p. 376 or p. 395 (with the slide rule).

		a]	b]	—l]
	[a	+ 152	+ 79	— 45
		[b	+ 298	—115
			— 41	+ 23
			[—l	+ 82
				— 13
$\delta y =$	+ 0,36∂ m	[bb·1] = + 257		— 92
$\delta y =$	+ 0,036 m			+ 69
$(y) =$	— 24,709.800 m			— 33
$y =$	— 24,709.764 m		[ll·2] = + 36	

[b	[a	[—l		
+ 298	+ 79	—115	b]	
	+ 152	— 45	a]	
	— 21	+ 30		
		+ 82	—l]	
		— 44		
[aa·1] = + 131		— 15	$\delta x =$	+ 0.11∂ m
		+ 38	$\delta x =$	+ 0.011 m
		— 2	$(x) =$	— 26,868.300 m
[ll·2] = + 36			$x =$	— 26,868.289 m

With these adjusted coordinates y and x we carry out again the computation of the final direction angles φ (as on the lower part of pp. 376 and 395) and then the calculation of the v's. We list this in the following for the outward directions, omitting the checks for $\delta\varphi$, however (which agree):

Outward Directions (cf. p. 376):

No.	Base Point	Object Point	Measured Direction α	Final Direction Angle φ	$\varphi - \alpha = v$	v^2
1	Steuerndieb	Hochschule	259° 14′ 15.1″	259° 14′ 14.9″	— 0.2″	0,04
2	Ägidius . .	Hochschule	315 2 32.6	315 2 34.5	+ 1.9	3.61
3	Wasserturm	Hochschule	20 36 50.0	20 36 49.4	— 0.6	0.36
4	Burg . . .	Hochschule	149 4 12.3	149 4 11.0	— 1.3	1.69
			110.0	109.8	— 0.2	5.70

For the inward directions we compute at first $\delta z = +0.3''$ and $z = (z) + \delta z = 135°\,02'\,31.9''$, as on the lower part of p. 395, and with these, the individual corrections v with the help of the measured directions r we then set up the final direction angles $r + v + z$ and compare them with the final direction angles φ computed from the coordinates.

Inward Directions (cf. p. 395):

No.	Base Point	Object Point	Corr. v	Measured Direction r	$r + v + z$	Final Direction Angle φ	v^2
1	Hochschule	Schanze . .	— 2.8″	249° 12′ 49.4″	24° 15′ 18.5″	24° 15′ 18.4″	7.84
2	Hochschule	Steuerndieb	— 2.1	304 11 45,1	79 14 14.9	79 14 14.9	4.41
3	Hochschule	Ägidius . .	+ 2,5	0 00 00	135 02 34.4	135 02 34.5	6.25
4	Hochschule	Wasserturm	— 1.4	65 34 18.8	200 36 49.4	200 36 49.4	1.96
5	Hochschule	Burg . . .	+ 3.7	194 01 35.2	329 04 10.8	329 04 11.0	13.69
							34.15

Now we have:

From the four outward directions $[v^2] = 5.7$, i.e., $\frac{1}{2}[v^2] = 2.8$

From the five inward directions $[v^2] = 34.2$

Total sum $[v^2] = 37.0$.

As denominator for the computation of the mean error we have $(4 + 5) - (2 + 1) = 6$, because $4 + 5$ measured directions are present, and δx, δy beside z are together three unknowns. We thus have the mean error of unit of weight:

$$m = \sqrt{\frac{37.0}{6}} = \pm 2.5'' .$$

With this and with the weights $[b\,b \cdot 1] = 257$ and $[a\,a \cdot 1] = 131$ from the previous elimination we also compute the mean coordinate errors themselves by the same scheme as below on p. 376 or p. 395:

$$[v\,v] = 37.0 \qquad m^2 = \frac{37.0}{6} = 6.17 \qquad m_y{}^2 = \frac{m^2}{[b\,b \cdot 1]} = \frac{6.17}{257} \qquad m_x{}^2 = \frac{m^2}{[a\,a \cdot 1]} = \frac{6.17}{131}$$

$$n - u = 9 - 3 = 6 \qquad m = \pm 2.5'' \qquad m_y = \pm 0.16\, \text{dm} \qquad m_x = \pm 0.22\, \text{dm} .$$

Final result:

$$\text{Hochschule} \quad y = -24{,}709.764 \text{ m} \pm 0.016 \text{ m} \qquad x = -26{,}868.289 \text{ m} \pm 0.022 \text{ m} \qquad (m \pm 2.5'') . \tag{18}$$

This holds true for intersection *and* resection. For comparison, we shall list here once again the individual determinations of section 96 and 99:

$$\text{Intersection (p. 376)} \quad -24{,}709.769 \text{ m} \pm 0.007 \text{ m} \; (m = \pm 0.8'') \qquad -26{,}868.306 \text{ m} \pm 0.009 \text{ m} . \tag{19}$$

$$\text{Resection (p. 395)} \quad -24{,}709.762 \text{ m} \pm 0.030 \text{ m} \; (m = \pm 4.0'') \qquad -26{,}868.280 \text{ m} \pm 0.042 \text{ m} . \tag{20}$$

In the case of so few observations, the computed mean errors are more or less accidental values; they prove, however, that the point determination is accurate to a few centimeters. Let there also be noted that the previous is not a prepared school example, but it corresponds exactly to the official measurements made in 1891 for the trigonometric department of the Land Survey.

Every practical trigonometer may have entertained the thought that points located *nearby* must be determined more accurately in their *relative* position than is in general demanded from the determination of points lying far apart. In particular, the later attachment of small surveys, with traverse lines, etc., gives rise to this consideration.

This has been brought into a certain system, by General Schreiber as Chief of the trigonometric section of the Land Survey, with the designations "Leitpunkt" [guiding point] and "Folgepunkt" [following point], by a set of instructions, which has been published in the *Zeitschr. f. Verm.*, 1889, pp. 1-3, as the following excerpt from there shows:

If neighboring points are connected with one another insufficiently or not directly connected at all, then the shiftings which they undergo against one another in the case of the adjustment become quickly inadmissibly large, even if not in themselves, yet in proportion to the reciprocal distance of the points.

Therefore, in doing reconnaissance of the triangulation points, attention is to be paid that such cases are avoided if practicable. But where this is not possible or too troublesome, a decrease of those shiftings can at times be achieved by treating the neighboring point as a *group*, i.e., by applying the following rules to them:

1. One of the points of a group (which, by the way, consists as a rule only of two points) is to be observed and adjusted in the usual manner as a triangulation net point. This is called the *Leitpunkt* [guiding point]; the remaining ones are called *Folgepunkte* [following points].

2. In the choice of the guiding point, at first there is decisive the order or, as the case may be, the class of the point, and then its position in the group, its suitability of being sighted on, visibility, solidity, permanence, etc.

3. All following points are to be intersected from the same base points whereas the guiding point can be determined through any other points. It is by no means excluded that the following points are also intersected from the guiding point; these special regulations, however, do not apply to the intersections starting from the latter.

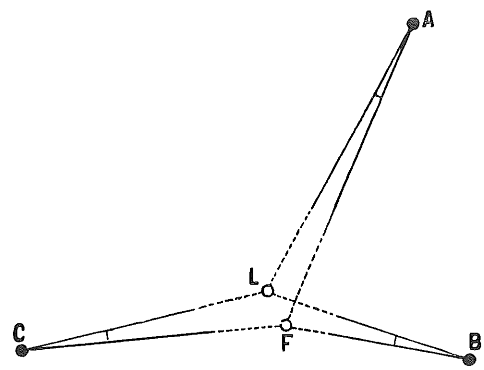


Fig. 1.
L = guiding point; F = following point.

4. Intersecting of a following point is done only by direct measurement of the *parallax*, i.e., of the ordinarily small angle between the guiding and the following point.

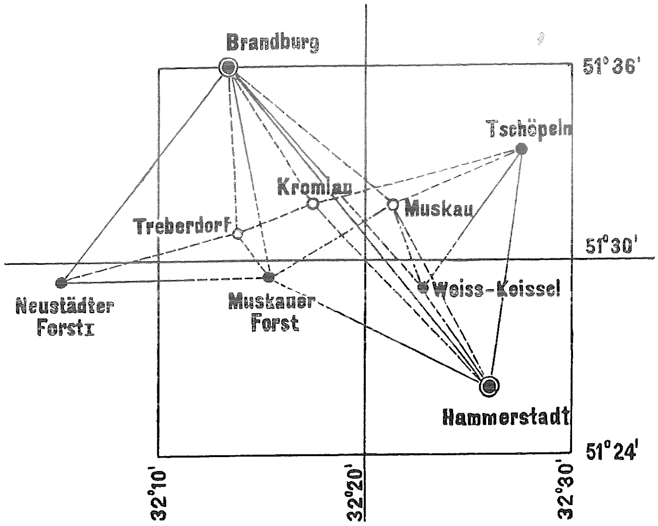
5. The direction angles of the following point to be introduced into the adjustment of the latter are not to be formed, as usual, from the observed, but from the final direction angles of the guiding point by means of the observed parallaxes.

(In the column "Observed" of the station data, however, the values formed from the *observed* direction angles of the guiding point are to be entered with a special notation, as usual.)

Instruction 4 has the purpose of obtaining the parallaxes as accurately as possible whereas the effect of instructions 3 and 5 is supposed to be that the parallaxes are changed as little as possible by the adjustment, and the points of the group thus shifted against one another as little as possible. We may be able to judge approximately, by means of the following remark, as to how far this effect is to be expected in the individual case.

The closer the following point is to the guiding point in comparison to its distance from a base point, the more it is a question of the correct position of the following point with respect to the guiding point (as it is introduced into the adjustment by the final direction angle from one to the other and by the coordinates of both), and the more it is a question of the accuracy of the parallax. Therefore, if all base points from which the following point is intersected are at a distance of several kilometers from the latter, but the latter lies at a distance of only a few hundred meters from the guiding point, then the more or less accurate position of the following point with respect to the guiding point depends almost entirely on the parallaxes. But if the following point is, for instance, just as far away from the guiding point as from the base points, then just as much depends on the accuracy of the final data as on that of the parallaxes. An example of this is given in Fig. 2:

Muskau is observed and adjusted here as the following point of Kromlau. The directions Muskauer Forst-Muskau and Muskau-Weisskeissel are rejected as being harmful.



Explanations to Fig. 2.

◉ Points of first order	———— Old directions
● Old points of second order	- - - - - New directions
O New points of second order	-·-·-·- Harmful directions

Whereas, as a rule, all needed sets of a station are attached to a common starting ray, guiding point and following point are only connected to one another.

The above are the regulations of the Trigonometric Section of 1888.

In the case of city triangulations the mentioned case occurs frequently with two or more high points on a structure, e.g., two symmetrical church towers, etc., and then attention is to be paid to the fact that such points belonging together are not in any case measured in separate sets.

To pay regard to this in the measurements already more than by the just noted arrangement of the sets is in general not possible in the case of the city triangulation. The way the computation may then appear may be shown on an example from the city triangulation of Hannover (cf. section 113) for two points 24 meters away from one another: Oppenheim flagpole and Oppenheim chimney (lightning conductor). The first point was intersected from seven ground points, the second one from nine ground points, among which were the previous seven and two other ones. According to the pure theory of following points, it would be necessary to cancel the other two intersections, but one does this very reluctantly since the intersections are otherwise good. Under these circumstances we have calculated both possibilities; the result was as follows:

Flagpole from seven intersections	$y = -24{,}898.427$ m $(m = \pm 2.1'')\ \pm 0.009$	$x = -25{,}227.035$ m ± 0.007	(1)
Chimney from seven intersections	$y = -24{,}877.051$ m $(m = \pm 1.4'')\ \pm 0.006$	$x = -25{,}237.503$ m ± 0.005	(2)
Chimney from nine intersections	$y = -24{,}877.034$ m $(m = \pm 2.8'')\ \pm 0.010$	$x = -25{,}237.514$ m ± 0.008 .	(3)

According to the theory of following points, (2) would have to be taken and (3) to be rejected; but we have not been able to decide, in regard to all additional circumstances, to sacrifice the two good intersections 8 and 9 (with 1.4 km and 0.9 km length), and have accepted the result (3) as final. (Cf. No. 74 and 73 in the large table of section 113.)

In other cases we have agreed with the requirements of the theory of the following points, and sacrificed the one or the other intersection. Of course, all depends here on the inner comprehension and on the practical eye of the computer who is not supposed to abandon his own judgment in the individual case, but never to disregard the good fundamental idea of guiding point and following point.

In a similar way as *one* point can be determined against a system of fixed points by intersection and resection, also two points can be intercalated together (and not only two but any arbitrary number of points, which we will discuss later in section 106).

Assuming the use of direction measurements, three different kinds of directions will occur, for which also the error equations assume three different forms.

According to the following example, let the two new points be denoted by H and D.

First we have the direction measurements at the fixed points, at which in part other fixed points, in part the two points H and D to be newly determined are sighted on. For the setting up of the error equations we must distinguish here whether only one of the two new points or both new points are sighted on at the fixed point. In the first case, an Abriss [station data] is to be set up for the station according to section 96, from which the direction angle α to the new point results. The weight of this direction angle is to be determined according to equation (4), section 96, p. 378.

If the two new points are sighted on at the station, then we have the case treated in section 97, and the error equations are to be set up by introducing an unknown of orientation according to (7), section 97, p. 380.

If we wish to disregard this rigorous treatment of the outward directions, then we can also set up an Abriss [station data] for each station according to section 96, from which the direction angles α to *both* new points can then be taken. For these direction angles we introduce an approximate weight, for instance $p = \frac{1}{2}$.

The inward directions measured at the two new points H and D to the fixed points form a second class. There result two systems of error equations, which have the form of the error equations in the case of resection. They contain the corrections of coordinates δx_1, δy_1 or, as the case may be, δx_2, δy_2 of the two station points, as well as the corrections δz_1, δz_2 of the preliminary orientations. Nothing needs to be added on the setting up of these error equations, since we have given a complete example of it in section 99.

Finally, there are still to be set up the error equations for the two reciprocal directions between the new points. According to Fig. 1, which corresponds to our following example, let the two measured directions be r_3 and r_9. Since these directions also belong to the inner directions of the two stations just mentioned, then the same quantities of orientation δz_1 and δz_2, which are to be added to the preliminary orientations (z_1) and (z_2), are also valid for them.

Fig. 1.

If the direction r_3 receives the correction v_3, then the final direction angle is, as in the case of resection with directions:

$$\varphi_3 = r_3 + v_3 + (z_1) + \delta z_1 \tag{1}$$

or

$$v_3 = \varphi_3 - r_3 - (z_1) - \delta z_1 . \tag{2}$$

Likewise we have

$$v_9 = \varphi_9 - r_9 - (z_2) - \delta z_2 . \tag{3}$$

For the two points H and D, in order to set up the error equations for the outer and the inner directions, we already need approximate values (x_1), (y_1) and (x_2), (y_2) of the coordinates, from which we also determine now the two approximate direction angles (φ_3) and (φ_9). These change to the final direction angles φ_3 and φ_9 as soon as the corrections δx and δy are added to the approximate coordinates. According to (12), section 93, p. 360, we have, since *both* end points are shifted now,

$$\varphi_3 = (\varphi_3) - \frac{\sin(\varphi_3)}{s}\varrho\,\delta x_2 + \frac{\cos(\varphi_3)}{s}\varrho\,\delta y_2 + \frac{\sin(\varphi_3)}{s}\varrho\,\delta x_1 - \frac{\cos(\varphi_3)}{s}\varrho\,\delta y_1$$

and

$$\varphi_9 = (\varphi_9) - \frac{\sin(\varphi_9)}{s}\varrho\,\delta x_1 + \frac{\cos(\varphi_9)}{s}\varrho\,\delta y_1 + \frac{\sin(\varphi_9)}{s}\varrho\,\delta x_2 - \frac{\cos(\varphi_9)}{s}\varrho\,\delta y_2 .$$

Since we have $\sin(\varphi_9) = -\sin(\varphi_3)$ and $\cos(\varphi_9) = -\cos(\varphi_3)$, then we can also write with simplified coefficients:

$$\left.\begin{aligned} \varphi_3 &= (\varphi_3) + a\,\delta x_1 + b\,\delta y_1 + c\,\delta x_2 + d\,\delta y_2 \\ \varphi_9 &= (\varphi_9) + a\,\delta x_1 + b\,\delta y_1 + c\,\delta x_2 + d\,\delta y_2 \end{aligned}\right\} . \tag{4}$$

If we set further

$$\left.\begin{aligned} r_3 + (z_1) &= \alpha_3 \\ r_9 + (z_2) &= \alpha_9 \end{aligned}\right\} \tag{4a}$$

and

$$\left.\begin{aligned} (\varphi_3) - \alpha_3 &= -l_3 \\ (\varphi_9) - \alpha_9 &= -l_9 \end{aligned}\right\}, \tag{5}$$

then we obtain from (2) to (5) the error equations:

$$\left.\begin{aligned} v_3 &= -\delta z_1 + a\,\delta x_1 + b\,\delta y_1 + c\,\delta x_2 + d\,\delta y_2 - l_3 \\ v_9 &= -\delta z_2 + a\,\delta x_1 + b\,\delta y_1 + c\,\delta x_2 + d\,\delta y_2 - l_9 \end{aligned}\right\}, \tag{6}$$

in which we have $c = -a$ and $d = -b$.

As an example we will treat the intercalation of the pair of points Hochschule-Dreifaltigkeit, which we already have calculated in section 97, p. 381, with limitation to the outer directions. We also set the triangulation net of p. 381 once again here.

For the common determination of the two points we have nine outer directions, further, nine inner directions, and finally the two reciprocal directions between the new points.

We carry out the computation in the system of coordinates of the conformal double projection of the Reichsamt für Landesaufnahme again, but later we will also carry it over into the old cadastral system of Celle.

I. Computation in the system of the conformal double projection of the Landesaufnahme

The coordinates of our fixed points have already been given previously on p. 248; we will set them down again here in order to have everything together, and also add at once the approximate coordinates of the two new points of p. 382:

Fixed	Ägidius	$y = -244{,}656.090\ m$	$x = -30{,}624.971\ m$	
Fixed	Wasserturm	$-246{,}956.479$	$-31{,}285.875$	
Fixed	Willmer	$-243{,}280.909$	$-33{,}328.385$	
Fixed	Steuerndieb	$-241{,}167.896$	$-28{,}421.362$	(7)
Fixed	Burg	$-247{,}076.504$	$-27{,}179.218$	
Fixed	Schanze	$-244{,}244.387$	$-25{,}592.941$	

Approximations	Hochschule	$(y_1) = -246{,}028.90\ m$	$(x_1) = -29{,}120.56\ m$	(8)
Approximations	Dreifaltigkeit	$(y_2) = -243{,}620.76$	$(x_2) = -29{,}282.46$	

With these coordinates (7) and (8), according to the formulae (5), p. 245, there have been computed the reductions $t - T$, which in part have already been given on p. 247, in part are still to be indicated later on.

For the setting up of the error equations for the outer directions, we will use the method of approximation just mentioned by orienting the directions in an Abriss [set of station data]. These oriented outer

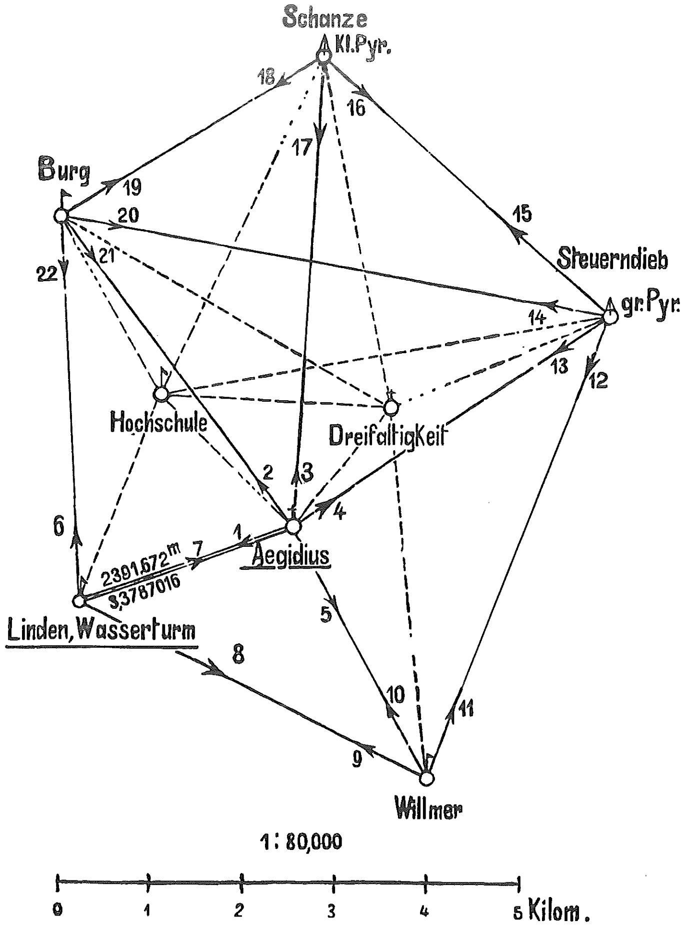


Fig. 2.

directions, e.g. Ägidius-Hochschule, Wasserturm-Hochschule, etc., are already contained on p. 247 in column 1 with the denotation T. In column 4 there are compiled the reductions $t - T$ for the transfer to the plane and with them there were computed the plane directions, which are reproduced in the last column. The denotation "Adjusted t" refers only to the rays of the actual triangulation net, whereas the directions to the two points Hochschule and Dreifaltigkeit in parentheses are only taken over from column 1 with the supplements $t - T$.

We have to introduce these latter values as direction angles α into the adjustment. We assemble them once again:

Base Point	Target	α
Ägidius	Hochschule	317° 37′ 08.8″
Wasserturm	Hochschule	23 11 25.8
Burg	Hochschule	151 38 50.2
Steuerndieb	Hochschule	261 48 52.3
Ägidius	Dreifaltigkeit	37° 38′ 23.7″
Burg	Dreifaltigkeit	121 19 30.0
Schanze	Dreifaltigkeit	170 24 22.4
Steuerndieb	Dreifaltigkeit	250 39 20.7
Willmer	Dreifaltigkeit	355 11 56.2

We also must reduce the inner directions, whose true observations have been given on p. 233, under 7 and 8, to the plane system by means of $t - T$ and at the same time also orient them approximately. These $t - T$'s are computed according to (5) or (6), p. 245, and the following is thereby obtained:

Directions	Observed (p. 233)	Observed and Oriented Approximately		
		Spherical T	$t-T$	Plane t
1. Hochschule				
Schanze	249° 12′ 49.37″	26° 49′ 59.05″	+2.19″	26° 50′ 1.24″
Steuerndieb	304 11 45.10	81 48 54.78	+0.43	81 48 55.21
Dreifaltigkeit	316 13 35.82	93 50 45.50	−0.10	93 50 45.40
Ägidius	0 0 0.00	137 37 9.68	−0.93	137 37 8.75
Linden, Wasserturm	65 34 18.81	203 11 28.49	−1.35	203 11 27.14
Burg	194 1 35.18	331 38 44.86	+1.21	331 38 46.07
2. Dreifaltigkeit				
Willmer	317° 33′ 31.20″	175° 11′ 55.77″	−2.49″	175° 11′ 53.28″
Ägidius	0 0 0.00	217 38 24.57	−0.83	217 38 23.74
Hochschule	56 12 25.93	273 50 50.50	+0.10	273 50 50.60
Burg	83 41 5.14	301 19 29.71	+1.30	301 19 31.01
Schanze	132 45 54.74	350 24 19.31	+2.28	350 24 21.59

The thus obtained t's enter likewise as observed α's into the next table. Besides, there are also inserted the direction coefficients a, b, c, d to be computed according to section 93 (or, as the case may be, according to the auxiliary table, pp. [8] to [15], of the Appendix), and in the last column the weights p. We choose the numbering of the individual directions according to Fig. 2.

All inner directions have a weight $=1$, and we give all the outer directions a weight $=\frac{1}{2}$ (according to section 101); the corrections of coordinates shall be for Hochschule: δx_1, δy_1 and for Dreifaltigkeit δx_2, δy_2; with these, the first three error equations for Hochschule read:

$$
\begin{aligned}
v_1 &= -\delta z_1 + 2.4\,\delta x_1 - 4.7\,\delta y_1 \quad \cdot\ \cdot \qquad \cdot\ \cdot \quad -0.8 \\
v_2 &= -\delta z_1 + 4.2\,\delta x_1 - 0.6\,\delta y_1 \quad \cdot\ \cdot \qquad \cdot\ \cdot \quad -1.8 \\
v_3 &= -\delta z_1 + 8.5\,\delta x_1 + 0.6\,\delta y_1 - 8.5\,\delta x_2 - 0.6\,\delta y_2 + 1.0, \text{ etc.}
\end{aligned}
$$

And for Dreifaltigkeit we obtain similar error equations, but with *another* δz_2:

$$v_7 = -\delta z_2 \quad \cdot\ \cdot \qquad \cdot\ \cdot\ \cdot \quad + 0.4\,\delta x_2 + 5.1\,\delta y_2 + 1.4, \text{ etc.}$$

Direction (cf. Fig. 2, p. 409)			Observed α	Approx (t)	$(t)-\alpha = -l$	a	b	c	d	p
Hochschule—	Schanze	1.	26° 50′ 1.2″	26° 50′ 0.4″	−0.8″	+2.4	−4.7			1
(δx_1, δy_1)	Steuerndieb	2.	81 48 55.2	81 48 53.4	−1.8	+4.2	−0.6			1
"	Dreifaltigkeit	3.	93 50 45.4	93 50 46.4	+1.0	+8.5	+0.6	−8.5	−0.6	1
"	Ägidius	4.	137 37 8.8	137 37 7.7	−1.1	+6.8	+7.5			1
"	Wasserturm	5.	203 11 27.1	203 11 21.7	−5.4	−3.4	+8.0			1
"	Burg	6.	331 38 46.1	331 38 50.7	+4.6	−4.4	−8.2			1
		Sum	3.8″	0.3″	−3.5″	+14.1	+2.6	−8.5	−0.6	$-\frac{1}{6}$
Dreifaltigkeit—	Willmer	7.	175° 11′ 53.3″	175° 11′ 54.7″	+1.4″			+0.4	+5.1	1
(δx_2, δy_2)	Ägidius	8.	217 38 23.7	217 38 20.5	−3.2			−7.4	+9.6	1
"	Hochschule	9.	273 50 50.6	273 50 46.4	−4.2	+8.5	+0.6	−8.5	−0.6	1
"	Burg	10.	301 19 31.0	301 19 32.2	+1.2			−4.4	−2.7	1
"	Schanze	11.	350 24 21.6	350 24 22.2	+0.6			−1.0	−5.7	1
		Sum	60.2″	56.0″	−4.2″	+8.5	+0.6	−20.9	+5.7	$-\frac{1}{5}$
Schanze	—Dreifaltigkeit	12.	170° 24′ 22.4″	170° 24′ 22.2″	−0.2″			−1.0	−5.7	0.5
Steuerndieb	—Dreifaltigkeit	13.	250 39 20.7	250 39 21.4	+0.7			+7.5	−2.6	0.5
Steuerndieb	Hochschule	14.	261 48 52.3	261 48 53.4	+1.1	+4.2	−0.6			0.5
Willmer	—Dreifaltigkeit	15.	355 11 56.2	355 11 54.7	−1.5			+0.4	+5.1	0.5
Ägidius	—Hochschule	16.	317 37 8.8	317 37 7.7	−1.1	+6.8	+7.5			0.5
Ägidius	Dreifaltigkeit	17.	37 38 23.7	37 38 20.5	−3.2			−7.4	+9.6	0.5
Wasserturm	—Hochschule	18.	23 11 25.8	23 11 21.7	−4.1	−3.4	+8.0			0.5
Burg	—Dreifaltigkeit	19.	121 19 30.0	121 19 32.2	+2.2			−4.4	−2.7	0.5
Burg	Hochschule	20.	151 38 50.2	151 38 50.7	+0.5	−4.4	−8.2			0.5

The sum of the error equations for the inner directions yields, besides, the two equations:

$$\left.\begin{array}{l} 0 = -6\,\delta z_1 + 14.1\,\delta x_1 + 2.6\,\delta y_1 - 8.5\,\delta x_2 - 0.6\,\delta y_2 - 3.5 \\ 0 = -5\,\delta z_2 + 8.5\,\delta x_1 + 0.6\,\delta y_1 - 20.9\,\delta x_2 + 5.7\,\delta y_2 - 4.2 \end{array}\right\}. \qquad (9)$$

In order to eliminate the δz_1's and δz_2's from the error equations, we can choose one of the various methods indicated in section 100; this time we will use Schreiber's rule with negative weight $(-1:n$ for n directions) and have added to it the sums in the case of Hochschule and Dreifaltigkeit; the $6+1$ error equations for Hochschule, written in full, are now the following:

$$\left.\begin{array}{lllll} v_1 = +\ 2.4\,\delta x_1 - 4.7\,\delta y_1 & \ldots & \ldots & -0.8 & \text{weight} = 1 \\ v_2 = +\ 4.2\,\delta x_1 - 0.6\,\delta y_1 & \ldots & \ldots & -1.8 & \text{weight} = 1 \\ v_3 = +\ 8.5\,\delta x_1 + 0.6\,\delta y_1 & -8.5\,\delta x_2 & -0.6\,\delta y_2 & +1.0 & \text{weight} = 1 \\ v_4 = +\ 6.8\,\delta x_1 + 7.5\,\delta y_1 & \ldots & \ldots & -1.1 & \text{weight} = 1 \\ v_5 = -\ 3.4\,\delta x_1 + 8.0\,\delta y_1 & \ldots & \ldots & -5.4 & \text{weight} = 1 \\ v_6 = -\ 4.4\,\delta x_1 - 8.2\,\delta y_1 & \ldots & \ldots & +4.6 & \text{weight} = 1 \\ v' = +14.1\,\delta x_1 + 2.6\,\delta y_1 & -8.5\,\delta x_2 & -0.6\,\delta y_2 & -3.5 & \text{weight} - \frac{1}{6} \end{array}\right\}. \qquad (10)$$

If we treat these seven equations further with their weights like common error equations, then we obtain, for instance:

$$[a\,a] = 2.4^2 + 4.2^2 + \ldots + 4.4^2 - \frac{1}{6}\,14.1^2 = 172.81 - 33.14 = +139.67\,,$$

and the whole system of normal equations pertaining to it will be:

$$\text{(Hochschule)} \quad \left.\begin{array}{l} +\underline{139.67\,\delta x_1} + 45.07\,\delta y_1 - 52.28\,\delta x_2 - 3.69\,\delta y_2 - 2.12 = 0 \\ \qquad + \underline{209.17\,\delta y_1} - 1.42\,\delta x_2 - 0.10\,\delta y_2 - 82.21 = 0 \\ \qquad\qquad + \underline{60.21\,\delta x_2} + 4.25\,\delta y_2 - 13.46 = 0 \\ \qquad\qquad\qquad + \underline{0.30\,\delta y_2} - 0.95 = 0 \\ \qquad\qquad\qquad\qquad + \underline{54.37} \end{array}\right\}. \qquad (H)$$

The inner directions of Dreifaltigkeit, treated in the same manner, yield:

$$\text{(Dreifaltigkeit)} \quad \left.\begin{array}{l} +\underline{57.80\,\delta x_1} + 4.08\,\delta y_1 - 36.72\,\delta x_2 - 14.79\,\delta y_2 - 28.56 = 0 \\ \qquad + \underline{0.29\,\delta y_1} - 2.59\,\delta x_2 - 1.04\,\delta y_2 - 2.02 = 0 \\ \qquad\qquad + \underline{60.17\,\delta x_2} - 22.49\,\delta y_2 + 36.50 = 0 \\ \qquad\qquad\qquad + \underline{151.81\,\delta y_2} - 22.93 = 0 \\ \qquad\qquad\qquad\qquad + \underline{28.11} \end{array}\right\}. \qquad (D)$$

Finally, the nine outer directions 12 to 20, likewise as a group independently, with their weights 0.5 yield:

$$\text{(Outer directions)} \quad \left.\begin{array}{l} +\underline{47.40\,\delta x_1} + 28.68\,\delta y_1 \quad \ldots \quad \ldots \quad + 4.44 = 0 \\ \qquad + \underline{93.92\,\delta y_1} \quad \ldots \quad \ldots \quad - 8.14 = 0 \\ \qquad\qquad + \underline{65.76\,\delta x_2} - 35.46\,\delta y_2 + 9.42 = 0 \\ \qquad\qquad\qquad + \underline{82.34\,\delta y_2} - 22.49 = 0 \\ \qquad\qquad\qquad\qquad + \underline{18.64} \end{array}\right\}. \qquad (A)$$

The three groups (H), (D), and (A) are brought into *one* adjustment by adding the coefficients of the three groups term by term; hence, we have to form by addition from (H), (D), (A):

e.g.:

$$139.67 + 57.80 + 47.40 = + 244.87.$$

At the same time we obtain by smoothing in this manner:

$$\left.\begin{array}{rrrrr} +\underline{245\,\delta x_1} + & 78\,\delta y_1 - & 89\,\delta x_2 - & 18\,\delta y_2 - & 26 = 0 \\ & +\underline{303\,\delta y_1} - & 4\,\delta x_2 - & 1\,\delta y_2 - & 107 = 0 \\ & & +\underline{186\,\delta x_2} - & 54\,\delta y_2 + & 32 = 0 \\ & & & +\underline{234\,\delta y_2} - & 46 = 0 \\ & & & & +\underline{101} = 0 \end{array}\right\} . \qquad \text{(S)}$$

These equations are solved in the usual manner and yield:

$$\delta y_1 = + 0.37\,\partial\text{m}, \qquad \delta x_1 = + 0.05\,\partial\text{m}, \qquad \delta y_2 = + 0.16\,\partial\text{m}, \qquad \delta x_2 = - 0.14\,\partial\text{m} \qquad (11)$$

and the term of the sum of the squares of errors $[ll \cdot 5] = 51.$ (12)

(We shall bring some details of the elimination of (S) in the following section 104, p. 416.)

These δx's and δy's added to the approximate coordinates in (8) yield the following composition:

(13)

	Hochschule		Dreifaltigkeit	
Approximations	$(y_1) = -246{,}028.900$,	$(x_1) = -29{,}120.560$	$(y_2) = -243{,}620.760$,	$(x_2) = -29{,}282.460$
Corrections	$+0.037$	-0.005	$+0.016$	-0.014
Results	$y_1 = -246{,}028.863$,	$x_1 = -29{,}120.565$	$y_2 = -243{,}620.744$,	$x_2 = -29{,}282.474$

With these final coordinates and with the coordinates of the fixed points given previously under (7) already, we now compute anew all direction angles t. We determine further the values of δz_1 and δz_2 from the equation (9), and the corrections v from the error equations. With these, we obtain the following second table, corresponding to the previous table of p. 410, to which we remark that the direction angles denoted by t or, as the case may be, (t) here in this section 103 have the same sense as the φ's or (φ)'s in sections 96 and 99.

Direction		Observed α	δz	v	Final t	v^2
Hochschule—Schanze	1.	26° 50′ 1.2″	−0.4″	−2.3″	26° 49′ 58.5″	5.29
Hochschule—Steuerndieb	2.	81 48 55.2	−0.4	−1.9	81 48 52.9	3.61
Hochschule—Dreifaltigkeit	3.	93 50 45.4	−0.4	+2.3	93 50 47.3	5.29
Hochschule—Ägidius	4.	137 37 8.8	−0.4	+1.7	137 37 10.1	2.89
Hochschule—Wasserturm	5.	203 11 27.1	−0.4	−1.9	203 11 24.8	3.61
Hochschule—Burg	6.	331 38 46.1	−0.4	+2.2	331 38 47.9	4.84
Sum				+0.1		25.53
			Should be 0.0			
Dreifaltigkeit—Willmer	7.	175° 11′ 53.3″	−0.1″	+2.2″	175° 11′ 55.4″	4.84
Dreifaltigkeit—Ägidius	8.	217 38 23.7	−0.1	−0.5	217 38 23.1	0.25
Dreifaltigkeit—Hochschule	9.	273 50 50.6	−0.1	−3.2	273 50 47.3	10.24
Dreifaltigkeit—Burg	10.	301 19 31.0	−0.1	+1.5	301 19 32.4	2.25
Dreifaltigkeit—Schanze	11.	350 24 21.6	−0.1	−0.1	350 24 21.5	0.01
Sum				−0.1		17.59
Schanze —Dreifaltigkeit	12.	170° 24′ 22.4″		−1.0″	170° 24′ 21.5″	1.00
Steuerndieb —Dreifaltigkeit	13.	250 39 20.7		−0.8	250 39 20.0	0.64
Steuerndieb —Hochschule	14.	261 48 52.3		+0.7	261 48 52.9	0.49
Willmer —Dreifaltigkeit	15.	355 11 56.2		−0.7	355 11 55.4	0.49
Ägidius —Hochschule	16.	317 37 8.8		+1.3	317 37 10.2	1.69
Ägidius —Dreifaltigkeit	17.	37 38 23.7		−0.6	37 38 23.1	0.36
Wasserturm —Hochschule	18.	23 11 25.8		−1.0	23 11 24.8	1.00
Burg —Dreifaltigkeit	19.	121 19 30.0		+2.4	121 19 32.4	5.76
Burg —Hochschule	20.	151 38 50.2		−2.3	151 38 47.9	5.29
						$\frac{16.72}{2} = 8.36$

Since the outer rays have only half weight, we now obtain the sum total $[p\,v^2]$ thusly:

Of Hochschule. . . .	25.53
Of Dreifaltigkeit . .	17.59
Of the outer rays . .	8.36
	51.48.

This sum must agree with the remaining term $[l\,l\cdot 5]$ of the elimination, which we already have indicated in the case of (4) and $[l\,l\cdot 5] = 51$. Since the agreement is sufficient, we can go further and find the mean error of a direction:

$$m = \sqrt{\frac{51.5}{14}} = \pm 1.9'' \,. \tag{14}$$

The denominator 14 results here from 20 − 6, because there are 20 directions measured and 6 unknowns, namely δx_1, δy_1, δx_2, δy_2 and in addition the two δz_1 and δz_2 for Hochschule and Dreifaltigkeit.

The check indicated in equation (1a) in section 29, p. 90, namely

$$[v\,v] = [l\,l] - [a\,l]\,\delta x_1 - [b\,l]\,\delta y_1 - [c\,l]\,\delta x_2 - [d\,l]\,\delta y_2\,,$$

also gives sufficient agreement, namely:

$$[v\,v] = 101 + 1.3 - 39.5 - 4.5 - 7.3 = 51.0\,.$$

We can make the computation of the mean error in (14) with the same result, but with a different conception and representation, also thusly: the outer directions have received the weight $p = \frac{1}{2}$, because it is assumed there that each outer direction is oriented by *one* second direction each, which was laid at a fixed ray; or we can also say: each outer direction is determined by a measured *angle*, whose one leg is laid at a fixed ray and whose second leg gives the new ray for the determination of the new point. Now since an angle has only half weight in comparison with a direction, the weight computation with $p = \frac{1}{2}$ is justified. But if we consider the two leg directions individually, then, instead of the thus far assumed *single* shifting v with $p = \frac{1}{2}$, we must assume now *two* shiftings $\frac{v}{2}$ with $p = 1$ each; we thus have in the first case $\frac{v^2}{2}$ and in the second case $\left(\frac{v}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \frac{v^2}{2}$, i.e., in both cases the same. There still is a dissimilarity of the two cases, however, in regard to the number of the observations and the number of the unknowns. In the first case we have one observation $\left(\text{with } p = \frac{1}{2}\right)$ and in the second case we have two observations (with $p = 1$), but also one unknown more, namely the amount of shifting at the fixed ray; but since in the denominator of the expression for the mean error there only occurs the number of the *excess* observations, both cases remain again the same.

For we have:

1st Case: $$m^2 = \frac{25.5 + 17.6 + \frac{1.0^2}{2} + \frac{0.8^2}{2} + \ldots + \frac{2.3^2}{2}}{(6-3)+(5-3)+ \qquad +9} = \frac{51.5}{14}\,.$$

2nd Case: $$m^2 = \frac{25.5 + 17.6 + 2\left(\frac{1.0}{2}\right)^2 + 2\left(\frac{0.8}{2}\right)^2 + \ldots 2\left(\frac{2.3}{2}\right)^2}{(6-3)+(5-3) \qquad + \qquad (18-9)} = \frac{51.5}{14}\,.$$

Computation of the distances after the adjustment.

Just as the direction angles t we can also compute the distances s from the adjusted plane rectangular coordinates, and then also convert the $\log s$'s to $\log S$'s [according to (3) or (4), p. 245]. The results are the following in our case.

		log s	log s — log S	log S
Hochschule —Schanze	1.	3.596 9583	3203	3.596 6380
Hochschule —Steuerndieb	2.	3.691 1696	3163	3.690 8533
Hochschule —Dreifaltigkeit	3.	3.382 6573	3195	3.382 3378
Hochschule —Ägidius	4.	3.308 9058	3208	3.308 5850
Hochschule —Wasserturm	5.	3.372 1089	3238	3.372 7851
Hochschule—Burg	6.	3.343 6031	3240	3.343 2791
Dreifaltigkeit —Willmer	7.	3.608 5429	3159	3.608 2270
Dreifaltigkeit —Ägidius	8.	3.229 2614	3177	3.228 9437
Dreifaltigkeit —Hochschule	9.	3.382 6573	3195	3.382 3378
Dreifaltigkeit —Burg	10.	3.606 9708	3208	3.606 6500
Dreifaltigkeit —Schanze	11.	3.573 0886	3171	3.572 7715
Steuerndieb—Dreifaltigkeit	13.	3.414 9084	3132	3.414 5952

The additional ones, 12th, 14th, . . . , 20th, are contained as counter-directions in the above. All other sides are indicated previously already on p. 244.

II. Transfer to the cadastral system Celle

All which has been computed and adjusted, pp. 408-413, in the system of the conformal double projection of the Landesaufnahme and which could also have been computed directly in the cadastral system Celle with the same computational procedure without the $t - T$'s will now be carried over also into this system Celle (System II of section 71). This can be done very simply by inserting the adjusted directions and distances of p. 412 and p. 413 into the Abriss [set of station data] of the section 71, p. 250, whereby the gaps still existing there must be filled. We will give this also in detail with the remark that the terms still carried in the main net and in the Abriss [set of station data], p. 250, of the order $1/r^2$ are no longer taken into account in the case of the intercalation of Hochschule-Dreifaltigkeit.

Supplement of the Abriss, p. 250 (System Celle)

	Observed A	v	Adjusted $\varphi = A + v$	log S
1. Ägidius				
Hochschule	315° 2′ 32.64″	+1.3″	315° 2′ 33.9″	3.308 585
Dreifaltigkeit . . .	35 3 47.73	−0.6	35 3 47.1	3.228 944
2. Wasserturm				
Hochschule	20° 36′ 49.99″	−1.0″	20° 36′ 49.0″	3.710 785
3. Willmer				
Dreifaltigkeit . . .	352° 37′ 21.62″	−0.7″	352° 37′ 20.9″	3.608 227
4. Steuerndieb				
Dreifaltigkeit . . .	248° 4′ 43.63″	−0.8″	248° 4′ 42.8″	3.414 595
Hochschule	259 14 15.09	+0.7	259 14 15.8	3.690 853
5. Schanze				
Dreifaltigkeit . . .	167° 49′ 43.84″	−1.0″	167° 49′ 42.8″	3.572 772
6. Burg				
Dreifaltigkeit . . .	118° 44′ 52.26″	+2.4″	118° 44′ 54.7″	3.606 650
Hochschule	149 4 12.32	−2.3	149 4 10.0	3.343 279

With these final outer directions we now can also orient the directions measured at the two points Hochschule and Dreifaltigkeit after having added the corrections v of p. 412.

	Corr. Obs. (p. 233 and 412) $r+v$	Adjusted φ	Rotation of Set $\varphi-(r+v)$	Oriented Directions $r+v+z$	$\log S$
7. Hochschule					
Schanze	249° 12′ 47.1″			24° 15′ 19.5″	3.596 638
Steuerndieb	304 11 43.2	79° 14′ 15.8″	135° 02′ 32.6″	79 14 15.6	3.690 853
Dreifaltigkeit . . .	316 13 38.1			91 16 10.5	3.382 338
Ägidius	0 00 01.7	135 02 33.9	135 02 32.2	135 02 34.2	3.308 585
Wasserturm. . . .	65 34 16.9	200 36 49.0	135 02 32.1	200 36 49.3	3.372 785
Burg	194 01 37.4	329 04 10.0	135 02 32.6	329 04 09.8	3.343 279
		Mean $z=$	135° 02′ 32.4″		
8. Dreifaltigkeit					
Willmer	317° 33′ 33.4″	172° 37′ 20.9″	215° 03′ 47.5″	172° 37′ 21.2″	3.608 227
Ägidius	359 59 59.5	215 03 47.1	215 03 47.6	215 03 47.3	3.228 944
Hochschule	56 12 22.7			271 16 10.5	3.382 338
Burg	83 41 06.6	298 44 54.7	215 03 48.1	298 44 54.4	3.606 650
Schanze	132 45 54.6	347 49 42.8	215 03 48.2	347 49 42.5	3.572 772
		Mean $z=$	215° 03′ 47.8″		

If we compute the coordinates of Hochschule and Dreifaltigkeit with these adjusted φ's and $\log S$'s, then we find with sufficient agreement the following values (to which we also add the mean errors of coordinates of the following section 104):

Coordinates in the System Celle

Hochschule	$y = -24709.769\ m$ ± 0.012	$x = -26868.278\ m$ ± 0.014	(1)
Dreifaltigkeit	$-22298.580\ m$ ± 0.013	$-26921.716\ m$ ± 0.016 .	(1a)

These are the results of the *double* intercalation of points. For comparison we will also set here from sections 96, 99, and 101:

Hochschule, intersection (p. 376)	$y = -24709.769\ m$ ± 0.007	$x = -26868.306\ m$ ± 0.009	(2)
Hochschule, resection (p. 395)	$-24709.762\ m$ ± 0.030	$-26868.280\ m$ ± 0.042	(3)
Hochschule, intersection and resection [(18), p. 404]	$-24709.764\ m$ ± 0.016	$-26868.289\ m$ ± 0.022 .	(4)

Compared objectively, all these numbers show that it would not matter which one of all these determinations to take; only the pure intersection (2) yields a deviation of 2 cm in x.

Although all these computations had the sole purpose here to serve as examples for the various forms of adjustment which are needed in practice, the practical remark may still be appropriate now that nothing in the whole numerical material is prepared for purposes of school examples, but everything corresponds to the pure measurements of 1891 as it had been handed over in the original to the Trigonometric Section. But, on the other hand, let us admit that in the case of the necessity of a quick conclusion to the computation, the double adjustment of points would not have been made by us at all.

Whereas the intersecting adjustments for *one* point at a time can be made unusually quickly (elimination, etc., with the slide rule), so to speak by machinery, according to the printed forms of p. 376 and p. 395, this is no longer quite true in the case of extension to two and more points which are to be treated together; and whether we will choose, nevertheless, the joint adjustment must depend on various considerations (cf. sections 113 and 114).

 Mean Coordinate Errors and Distance Errors

After the Adjustment

For usual practical purposes, the calculation of our double point intercalation can be concluded with the previous section 103 in I or II; as far as the accuracy is concerned, the mean error $m = \pm 1.9''$ from (14), p. 413, is sufficiently satisfactory.

We shall profit, however, by the opportunity to attach here a few additional calculations for determining accuracy, and in fact, at first we may consider the mean coordinate errors or weights, as the case may be.

If we do not aim to have only the δx's and δy's themselves but also their weights, then we may take care of it already in the case of the solution of the normal equations (S) on p. 412, by inverting completely at least once the order of elimination, proceeding according to (22), p. 98, omitting, however, the terms which control the sums and the terms with $[ll]$, i.e.:

δx_1	δy_1	δx_2	δy_2	$-l$
+245	+78	−89	−18	−26
	+303	−4	−1	−107
		+186	−54	+32
			+234	−46
	+278	+24	+5	−99
		+154	−61	+23
			+233	−48
		+152	−61	+31
			+233	−46
			+209	−34

$p\,y_2 = 209$

$\delta y_2 = \dfrac{+34}{209} = +0.162\ dm$

δy_2	δx_2	δy_1	δx_1	$-l$
+234	−54	−1	−18	−46
	+186	−4	−89	+32
		+303	+78	−107
			+245	−26
	+174	−4	−93	+21
		+303	+78	−107
			+244	−30
		+303	+76	−107
			+194	−19
			+175	+8

$p\,x_1 = 175$

$\delta x_1 = -\dfrac{8}{175} = -0.046\ dm$ (1)

The inversion in the next to the last group at 152 and 303 also yields:

$$p\,x_2 = 136 \qquad \delta x_2 = -0.14\ \partial m \qquad p\,y_1 = 273 \text{ and } \delta y_1 = +0.37\ \partial m. \qquad (2)$$

These δx's and δy's are the same which we already had indicated in section 103 at (11), p. 412, and now we also have the $p\,x$'s, $p\,y$'s. (All this is done smoothly with the slide rule.) The mean coordinate errors are also calculated with these weights and the error of unit of weight $m = \pm 1.9''$ according to the method which was indicated below on p. 376 or p. 395. We find:

For mean errors	y_1	x_1	y_2	x_2	
	$\pm 0.12\ \partial m$	$\pm 0.14\ \partial m$	$\pm 0.13\ \partial m$	$\pm 0.16\ \partial m$.	(3)

Therefore, now the coordinates with their mean errors:

Hochschule		Dreifaltigkeit	
$y_1 = -246028.863$ m	$x_1 = -29120.565$ m	$y_2 = -243620.744$ m	$x_2 = -29282.474$ m
± 0.012	± 0.014	± 0.013	± 0.016 .

Mean distance error

We also compute the distance from the coordinates of the two points, namely:

$$s = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \qquad (4)$$

$$\log s = 3.382\,6573 \qquad s = 2413.556 \text{ m (projection)} .$$

To this, there is further added $\log s - \log S = 0.000\,3195$, as indicated already in the previous section 103, pp. 413-414, and hence now:

$$\log S = 3.382\,3378 \qquad S = 2411.780 \text{ m (true value)}.$$

In order to determine also the weight of s or S, it is necessary to differentiate:

$$ds = -\frac{y_2 - y_1}{s}\,dy_1 + \frac{y_2 - y_1}{s}\,dy_2 - \frac{x_2 - x_1}{s}\,dx_1 + \frac{x_2 - x_1}{s}\,dx_2, \tag{5}$$

or

$$ds = -\sin t\,dy_1 + \sin t\,dy_2 - \cos t\,dx_1 + \cos t\,dx_2.$$

Counting ds in millimeters, but dx, dy, however, in decimeters, then we obtain:

$$ds = +7\,dx_1 - 100\,dy_1 - 7\,dx_2 + 100\,dy_2, \tag{6}$$

i.e., the coefficients of (1), p. 100, are:

$$f_1 = +7, \qquad f_2 = -100, \qquad f_3 = -7, \qquad f_4 = +100. \tag{7}$$

With these, we obtain the following according to the instruction of (10a), p. 101, with reference to the previous normal equations (to be done quickly with the slide rule):

a	b	c	d	f
+245	+ 78	− 89	− 18	+ 7
	+303	− 4	− 1	−100
		+186	− 54	− 7
			+234	+100
				0
	+278	+ 24	+ 5	−102
		+154	− 61	− 4
			+223	+101
				− 37
		+152	− 61	+ 5
			+233	+103
				0
			+209	+105
				− 53
				+ 90 $= +\frac{1}{P}$

(8)

The final term 90 is the reciprocal of the weight required, hence now the mean distance error (for a mean direction error of $\pm 1.91''$):

$$ds = \pm 1.91\sqrt{90} = \pm 18 \text{ mm}. \tag{9}$$

In total, we thus have:

Hochschule-Dreifaltigkeit $S = 2411.780$ m
± 0.018 m.

The line 2.4 km long thus has a mean error of 18 mm or 7.5 mm to 1 km.
All these checks of accuracy are very satisfactory.

We have used a method of approximation for the outward directions in section 103 by orienting these directions in an Abriss [set of station data] and then introducing them with half weight into the adjustment.

In section 97 we already have treated in detail the rigorous treatment of the outward directions. The example there on p. 381 refers likewise to the two new points Hochschule and Dreifaltigkeit, and we therefore could combine the parts of normal equations (19) to (21) and (25) or the normal equations (26), p. 385, directly with the parts of normal equations (H) and (D) of the inward directions of section 103, p. 411, in order to obtain the final normal equations of the double intercalation of points.

For the stations Wasserturm, Willmer, and Schanze we have nothing to add to the former representations in section 97, p. 384; however, we will carry out the elimination of the unknown of orientation by means of the negative weights introduced by Schreiber (section 100, p. 397) for the three stations Ägidius, Steuerndieb, and Burg.

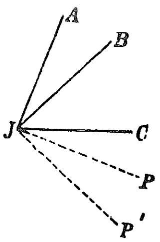


Fig. 1.

In this connection, we consider once again Fig. 1 used before in section 97, p. 379, in the case of which two new rays are determined with respect to several fixed rays, in the present case three.

With reference to section 97 we can indicate at once the error equations for these directions. We have according to (17), section 97, p. 383,

$$\left.\begin{aligned} v_1 &= -\delta z \quad . \quad . \quad . \quad . \quad -l_1 \\ v_2 &= -\delta z \quad . \quad . \quad . \quad . \quad -l_2 \\ v_3 &= -\delta z \quad . \quad . \quad . \quad . \quad -l_3 \\ v_4 &= -\delta z + a\,\delta x + b\,\delta y \quad . \quad . \quad -l_4 \\ v_5 &= -\delta z \quad . \quad . \quad + a'\,\delta x' + b'\,\delta y' - l_5 \end{aligned}\right\}. \tag{1}$$

δx, δy and $\delta x'$, $\delta y'$ denote here again the corrections of the coordinates provisionally assumed for the two new points P and P'. The meaning of the remaining symbols follows from section 97; we note especially that we have

$$l_1 + l_2 + l_3 = 0 \quad \text{and} \quad [l] = l_4 + l_5. \tag{2}$$

To eliminate δz from the error equations (1) we have to add to equations (1), after omitting δz, a new imaginary error equation which is equal to the sum of the error equations and which receives the weight $-\frac{1}{s+2}$. If δz is omitted, then the first three equations (1) are insignificant altogether and then we have only the three error equations:

$$\left.\begin{aligned} v_4 &= a\,\delta x + b\,\delta y \quad . \quad . \quad - l_4 && \text{weight} \quad 1 \\ v_5 &= \quad . \quad . \quad + a'\,\delta x' + b'\,\delta y' - l_5 && \text{weight} \quad 1 \\ v' &= a\,\delta x + b\,\delta y + a'\,\delta x' + b'\,\delta y' - (l_4 + l_5) && \text{weight} \; -\frac{1}{s+2} \end{aligned}\right\}. \tag{3}$$

These equations are to be set up for all stations on which outward directions are measured, and then when the normal equations are formed, these are identical with the normal equations once reduced, which we would have obtained from the original equations (1). All this follows from the previous section 100 and also from section 34, in which we have treated this theory generally.

After this, we now will calculate once again the example of section 103, where we retain the coordinates of the fixed points indicated there and also the approximate coordinates.

The error equations and the parts of the normal equations for the inner directions on the stations Hochschule and Dreifaltigkeit remain the same as formerly in section 103, table, p. 410, and (H) and (D), p. 411.

Also for the outer directions on the stations Wasserturm, Willmer, and Schanze we can take over immediately the error equations (22) to (24), section 97, as well as the pertinent parts of normal equations (25), p. 384.

There still remains the special treatment of the outward directions at the stations Ägidius, Steuerndieb, and Burg.

Instead of the original error equations (17), section 97, p. 383, we obtain for Ägidius, according to (3), the new error equations:

1. Ägidius

$$\left.\begin{aligned} v_2 &= +6.8\,\delta x_1 + 7.5\,\delta y_1 \quad . \quad . \quad -1.1 \quad \text{weight} \quad 1 \\ v_5 &= \quad . \quad . \quad -7.4\,\delta x_2 + 9.6\,\delta y_2 - 3.2 \quad \text{weight} \quad 1 \\ v_1' &= +6.8\,\delta x_1 + 7.5\,\delta y_1 - 7.4\,\delta x_2 + 9.6\,\delta y_2 - 4.3 \quad \text{weight} - \frac{1}{7} \end{aligned}\right\} . \tag{4}$$

We can take likewise from the station data (15) and (16), section 97, p. 382:

4. Steuerndieb

$$\left.\begin{aligned} v_{18} &= \quad . \quad . \quad +7.5\,\delta x_2 - 2.6\,\delta y_2 + 0.7 \quad \text{weight} \quad 1 \\ v_{19} &= +4.2\,\delta x_1 - 0.6\,\delta y_1 \quad . \quad . \quad +1.1 \quad \text{weight} \quad 1 \\ v_4' &= +4.2\,\delta x_1 - 0.6\,\delta y_1 + 7.5\,\delta x_2 - 2.6\,\delta y_2 + 1.8 \quad \text{weight} - \frac{1}{6} \end{aligned}\right\} . \tag{5}$$

6. Burg

$$\left.\begin{aligned} v_{28} &= \quad . \quad . \quad -4.4\,\delta x_2 - 2.7\,\delta y_2 + 2.2 \quad \text{weight} \quad 1 \\ v_{30} &= -4.4\,\delta x_1 - 8.2\,\delta y_1 \quad . \quad . \quad +0.5 \quad \text{weight} \quad 1 \\ v_6' &= -4.4\,\delta x_1 - 8.2\,\delta y_1 - 4.4\,\delta x_2 - 2.7\,\delta y_2 + 2.7 \quad \text{weight} - \frac{1}{6} \end{aligned}\right\} . \tag{6}$$

If we form the parts of normal equations from equations (4), then we obtain

$$\left.\begin{aligned} \underline{+39.63\,\delta x_1} + 43.71\,\delta y_1 + 7.19\,\delta x_2 - 9.33\,\delta y_2 + 3.30 \\ \underline{+48.21\,\delta y_1} + 7.93\,\delta x_2 - 10.29\,\delta y_2 - 3.64 \\ \underline{+46.95\,\delta x_2} - 60.89\,\delta y_2 - 19.13 \\ \underline{+78.99\,\delta y_2} - 24.82 \end{aligned}\right| . \tag{7}$$

We see that these parts of normal equations agree fully with those found previously in (19), section 97, p. 383. We would arrive at the same result also for the other two stations Steuerndieb and Burg, which we will not carry out further here.

Hence, the treatment of all outer directions does not differ in any way from that in the previous section 97 and therefore we now can arrive at the final normal equations for the double adjustment of points if we add up the normal equations (26), section 97, p. 385, with the parts of the inner directions (H) and (D), section 103, p. 411. This yields with omission of the decimal places:

$$\left.\begin{aligned} \underline{+277\,\delta x_1} + 100\,\delta y_1 - 90\,\delta x_2 - 28\,\delta y_2 - 20 &= 0 \\ \underline{+362\,\delta y_1} - 1\,\delta x_2 - 15\,\delta y_2 - 113 &= 0 \\ \underline{+231\,\delta x_2} - 80\,\delta y_2 + 37 &= 0 \\ \underline{+287\,\delta y_2} - 59 &= 0 \\ \underline{+111} \end{aligned}\right\} . \tag{8}$$

The elimination yields the following systems:

$$\left.\begin{aligned} \underline{+325.9\,\delta y_1} + 31.5\,\delta x_2 - 4.9\,\delta y_2 - 105.8 &= 0 \\ \underline{+201.8\,\delta x_2} - 89.1\,\delta y_2 + 30.5 &= 0 \\ \underline{+284.2\,\delta y_2} - 61.0 &= 0 \\ \underline{+109.6} \end{aligned}\right\} . \tag{9}$$

$$\left.\begin{aligned} \underline{+198.8\,\delta x_2} - 88.6\,\delta y_2 + 40.7 &= 0 \\ \underline{+284.1\,\delta y_2} - 62.6 &= 0 \\ \underline{+75.3} \end{aligned}\right\} . \tag{10}$$

$$\left.\begin{aligned} \underline{+244.6\,\delta y_2} - 44.5 &= 0 \\ \underline{+67.0} \end{aligned}\right\} . \tag{11}$$

$$[ll \cdot 4] = +58.9 \,. \tag{12}$$

The values of the unknowns are

$$\delta y_1 = +0.34\,\partial m\,, \quad \delta x_1 = -0.07\,\partial m\,, \quad \delta y_2 = +0.18\,\partial m\,, \quad \delta x_2 = -0.12\,\partial m\,.$$

By adding the δx's, δy's to the approximate coordinates, we obtain:

Hochschule, approximation	$(y_1) = -246{,}028.900$ m	$(x_1) = -29{,}120.560$ m
	$\delta y_1 = +0.034$	$\delta x_1 = -0.007$
Hochschule, adjustment	$y_1 = -246{,}028.866$ m	$x_1 = -29{,}120.567$ m (13)
Dreifaltigkeit, approximation	$(y_2) = -243{,}620.760$ m	$(x_2) = -29{,}282.460$ m
	$\delta y_2 = +0.018$	$\delta x_2 = -0.012$
Dreifaltigkeit, adjustment	$y_2 = -243{,}620.742$ m	$x_2 = -29{,}282.472$ m . (14)

After the solution of the normal equations there still follows the calculation of the weight coefficients, for which we obtain:

$$\left.\begin{array}{llll} [\alpha\alpha] = +0.004\,927 & [\beta\beta] = +0.003\,120 & [\gamma\gamma] = +0.005\,845 & [\delta\delta] = +0.004\,088 \\ [\alpha\beta] = -0.001\,313 & [\beta\gamma] = -0.000\,538 & [\gamma\delta] = +0.001\,831 & \\ [\alpha\gamma] = +0.002\,279 & [\beta\delta] = -0.000\,114 & & \\ [\alpha\delta] = +0.001\,046 & & & \end{array}\right\} \qquad (15)$$

and with these the following weights result:

$$p_{y_1} = 320, \qquad p_{x_1} = 203, \qquad p_{y_2} = 245, \qquad p_{x_2} = 171. \qquad (16)$$

After this, we pass to the calculation of the error distribution, i.e., the calculation of all individual v's and δz's, and then to the calculation of the mean error. The measurements at Hochschule and Dreifaltigkeit are treated in the very same manner as previously on p. 395 and the individual v's yield:

Hochschule $v = -2.2'',\ -1.8'',\ +2.0'',\ +1.5'',\ -2.1'',\ +2.5''$
Dreifaltigkeit $v = +2.5\ ,\ -0.4\ ,\ -3.4\ ,\ +1.5\ ,\ -0.1\ ,$
herefrom **Hochschule**: $[v^2] = 24.99$
Dreifaltigkeit: $[v^2] = 20.23$.

Further we have from (22) to (24), section 97,

Wasserturm	$v_9 = -1.1''$	$v_9^2 = 1.21$	$v_9^2\,p = 0.91$
Willmer	$v_{14} = -0.6$	$v_{14}^2 = 0.36$	$v_{14}^2\,p = 0.29$
Schanze	$v_{23} = -1.3$	$v_{23}^2 = 1.69$	$v_{23}^2\,p = 1.27$

and from the above equations (4) to (6)

Ägidius	$v_2 = +1.0''$	$v_2^2 = 1.00$	$v_2^2\,p = 1.00$
	$v_5 = -0.6$	$v_5^2 = 0.36$	$v_5^2\,p = 0.36$
	$v_1' = +0.4$	$v_1'^2 = 0.16$	$v_1'^2\,p = -0.02$
			$+1.34$
Steuerndieb	$v_{18} = -0.7''$	$v_{18}^2 = 0.49$	$v_{18}^2\,p = 0.49$
	$v_{19} = +0.6$	$v_{19}^2 = 0.36$	$v_{19}^2\,p = 0.36$
	$v_4' = 0$	$v_4'^2 = 0$	$v_4'^2\,p = 0$
			$+0.85$

Burg	$v_{28} = +2.2''$	$v_{28}^2 = 4.84$	$v_{28}^2 p = 4.84$
	$v_{30} = -2.0$	$v_{30}^2 = 4.00$	$v_{30}^2 p = 4.00$
	$v_6' = +0.2$	$v_6'^2 = 0.04$	$v_6'^2 p = -0.01$
			$+8.83$.

Hence, we have for all stations together

Hochschule	$[v\,v\,p] =$	24.99 with 6 directions
Dreifaltigkeit		20.23 with 5 directions
Wasserturm		0.91 with 1 directions
Willmer		0.29 with 1 directions
Schanze		1.27 with 1 directions
Ägidius		1.34 with 2 directions
Steuerndieb		0.85 with 2 directions
Burg		8.83 with 2 directions
	$[v\,v\,p] =$	58.73 with 20 directions .

This agrees sufficiently with $[l\,l \cdot 4] = 58.9$ of (12) (see above).
Now as far as the mean error is concerned, we compute:

$$m = \sqrt{\frac{59.14}{20 - 4 - 2}} = \pm 2.1''. \tag{17}$$

The denominator $20 - 4 - 2 = 14$ results from the fact that 20 directions are entered as measured and in addition to the four main unknowns δx_1, δy_1, δx_2, δy_2 there still occur the two δz_1 and δz_2 for the entirely free points Hochschule and Dreifaltigkeit as unknowns. Originally, it is true, there were existing unknowns of orientation on the fixed points Schanze, Steuerndieb, etc., but these are not only eliminated but they are taken into account in another way by the weights 0.75 and the negative weights $-\frac{1}{7}$, etc.

Section 106. Intercalation of an Arbitrary Number of Points

Hitherto we have treated the intercalation of individual points and the simultaneous intercalation of two points into a given net. But with the latter problem there is shown at the same time the way for the intercalation of an arbitrary number of points. As in the case of the intercalation of two points we have to distinguish three different kinds of directions:

1. Outward directions of fixed points to new points which are attached to directions to other fixed points (section 97).
2. Inward directions of new points to fixed points (section 99).
3. Reciprocal directions between new points (section 103, p. 407).

The setting up of the error equations for the outer directions is done according to (1), section 105, p. 418, where in addition to the points P and P' still further new points may occur, however. One of the methods treated in section 100, p. 396, can then be used for the elimination of the quantity of orientation δz, the most appropriate being the introduction of the sum-equation with negative weight according to (3), section 105, p. 418.

For the inward directions and the reciprocal directions the error equations are set up together as in the example of section 103, p. 410, and the quantities of orientation δz are eliminated also here by means of the sum-equations by introducing negative weights [equation (10), section 103, p. 411].

From all error equations together there are then set up the normal equations whose number is equal to double the number of the new points.

In this way the problem of the intercalation of points can be solved in the most general form. Whether to use the method of the net adjustment according to conditioned observations instead of the above described intercalation of points for a rather large number of new points if only a few fixed points are given must be decided in the individual case according to the number of the normal equations required for the two methods. The method of the intercalation of points can even be applied for an independent triangulation net by regarding two points as given fixed points, all remaining ones as new points. An example for this is given by the reviser in his *Einführung in die Geodäsie*, Leipzig, 1907, pp. 430-437.

Using Fig. 1, p. 423, we will now examine a net of the trigonometric division of the Reichsamt für Landesaufnahme which contains the large number of ten points to be newly determined with attachment to eight old fixed points.

This case is reported in the official work, *Die Kgl. Preuss. Landestriangulation*, Hauptdreiecke, V. Teil, Berlin, 1893, pp. 95-165.

In the following we will indicate the page numbers of this official publication (with V, p. . . .). The rectangular coordinates are referred to the system of the conformal double projection of the Landesaufnahme.

Of the 18 points of the net, eight points are already finally established from former measurements, namely the points lying west and north which, in our net picture, are connected by double lines and denoted with numbers in brackets. The rectangular coordinates of these points are:

Fixed Points (V, p. 96)

	y	x
[11] Lubetzko	+ 375,385.440 m	— 209,143.102 m
[12] Annaberg	+ 343,473.234	— 238,364.429
[13] Lossen	+ 297,880.169	— 203,614.224
[14] Goy	+ 274,702.823	— 190,391.109
[15] Tschelertnig	+ 270,135.349	— 149,357.096
[16] Kröben	+ 252,033.953	— 96,473.717
[17] Schwarze Berg	+ 260,049.649	— 73,766.737
[18] Lissagora	+ 290,483.939	— 60,132.087

(1)

The remaining ten points are to be newly determined with attachment to the old fixed points. We give their table at once with the assumed approximate coordinates and with the symbols for the corrections of coordinates:

Table (2)

Point	Approx Coordinates (V, p. 141)		Corrections of Coordinates	
	(y)	(x)		
(1) Sacrau . . .	+ 345,508.3 m	— 202,211.5 m	$\delta x_1 = \mathrm{I}$	$\delta y_1 = \mathrm{II}$
(2) Skronskau . .	+ 361,465.0	— 180,675.2	$\delta x_2 = \mathrm{III}$	$\delta y_2 = \mathrm{IV}$
(3) Eckersdorf . .	+ 311,632.1	— 178,192.6	$\delta x_3 = \mathrm{V}$	$\delta y_3 = \mathrm{VI}$
(4) Rosen . . .	+ 337,588.0	— 171,408.6	$\delta x_4 = \mathrm{VII}$	$\delta y_4 = \mathrm{VIII}$
(5) Laski . . .	+ 326,170.3	— 155,739.3	$\delta x_5 = \mathrm{IX}$	$\delta y_5 = \mathrm{X}$
(6) Ossen . . .	+ 298,827.9	— 137,625.2	$\delta x_6 = \mathrm{XI}$	$\delta y_6 = \mathrm{XII}$
(7) Johannas Höhe	+ 273,033.9	— 129,893.6	$\delta x_7 = \mathrm{XIII}$	$\delta y_7 = \mathrm{XIV}$
(8) Kotlow . . .	+ 321,735.4	— 117,815.1	$\delta x_8 = \mathrm{XV}$	$\delta y_8 = \mathrm{XVI}$
(9) Dzielitz . . .	+ 286,627.9	— 95,566.9	$\delta x_9 = \mathrm{XVII}$	$\delta y_9 = \mathrm{XVIII}$
(10) Baranowik . .	+ 303,288.6	— 81,761.8	$\delta x_{10} = \mathrm{XIX}$	$\delta y_{10} = \mathrm{XX}$.

These ten points are determined by 76 directions, which are numbered 1, 2, 3, . . . , 70 in the net picture, p. 423, while 76 sights are actually measured. The difference 76 – 70 = 6 results from the fact that at the fixed points with two old connecting directions, in each case (on [12] to [17]) only *one* correction to each direction is introduced for each of the two attachments, which is in connection with the station adjustments on these points (cf. section 87 and attachment *Peer*, p. 349).

Now before we go to our adjustment of coordinates itself, which offers 20 unknowns and 20 normal equations in the case of ten new points, we will reflect how many equations we *would* get according to the correlate method.

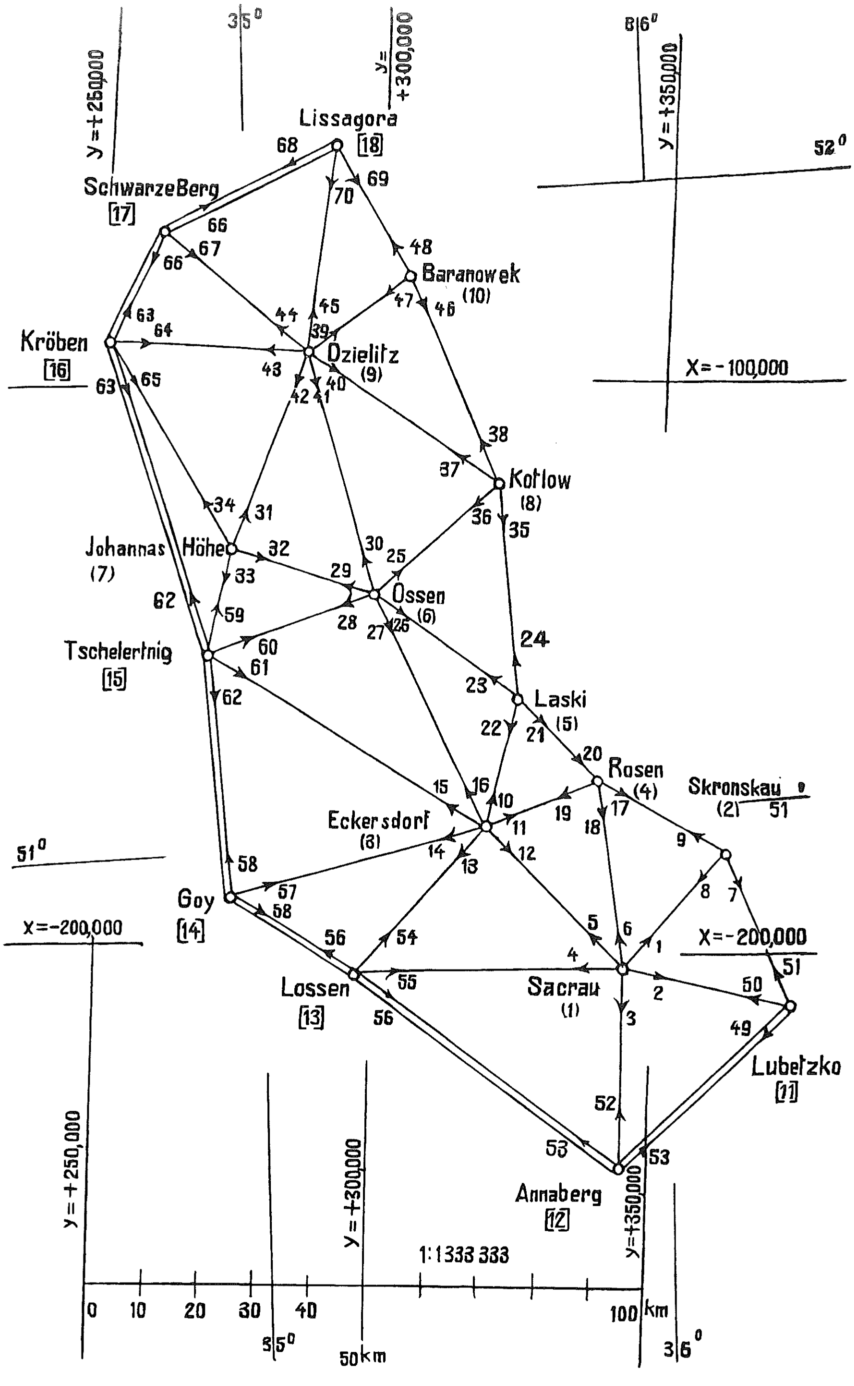


Fig. 1.

We count here $p = 18$ points and $l = 38$ lines, both measured back and forth, hence $R = 76$ measured directions. According to the rules of (19), p. 226, this yields:

$$l - 2p + 3 = 38 - 36 + 3 = 5 \quad \text{side equations,}$$
$$l - p + 1 = 38 - 18 + 1 = 21 \quad \text{triangle equations,}$$
$$R - 3p + 4 = 76 - 54 + 4 = 26 \quad \text{total number of equations.}$$

But this holds true for the *free* net, and for the forced attachments there would still be added six side equations, so that we would get a total of $26 + 6 = 32$ normal equations, while the adjustment of coordinates has only 20 equations, hence is at an advantage by 12 equations.

Passing to this adjustment of coordinates, we have at first to report about the measurements in general that they are done according to the Schreiber method with angles in all combinations, as is indicated previously in section 89.

In order to obtain nearly equal weight for all adjusted directions, the Schreiber method states that the angle measurements are to be repeated in themselves (according to p. 330):

In the case of	2	3	4	5	6	7	8	directions,
In each case	24	16	12	10	8	8	6	times.

For instance, at Sacrau with six free directions there is measured (according to V, pp. 99-100):

Angle 1,2	8 times	Angle 2,3	8 times
Angle 1,3	8 times	Angle 2,4	8 times
Angle 1,4	8 times	Angle 2,5	8 times
Angle 1,5	8 times	Angle 2,6	8 times
Angle 1,6	8 times		
Angle 3,4	8 times	Angle 4,5	8 times
Angle 3,5	8 times	Angle 4,6	8 times
Angle 3,6	8 times		
Angle 5,6	8 times		

i.e., each of the 15 possible angles between the six free rays is measured 8 times.

On the other hand, at a station with two fixed connecting rays, e.g., at [13] Lossen, there is measured (according to V, p. 129):

Angle 54,55	16 times	Angle 55,56 A	8 times
Angle 54,56 A	8 times	Angle 55,56 G	8 times
Angle 54,56 G	8 times		
Angle 56 A, 56 G	0 times,		

i.e., the two fixed rays 56 A and 56 G count like *one* free ray (cf. section 87).

The adjustment is carried out at a station with only free rays in the usual manner (section 86) and at a station with two fixed connecting rays in such a way that the angle between these two receives the value already established by the coordinates (section 87). The retention of this angle in the following net adjustment is simply achieved by assigning the two fixed rays *the same* correction to the direction, in the case of the previous example 56 A = 56 G.

The setting up of the error equations is done according to the fundamental principles indicated at the beginning of this section 106. Before that we still have to determine the direction angles, which we will denote by t here instead of φ, for the plane. For the reduction of the directions we use the formulae for $t - T$, (5), section 71, p. 245, for which we will treat the direction Sacrau-Skronskau as an example, using the approximate coordinates of these two points indicated above under (2):

	(y)	(x)
(2) Skronskau	+ 361,465.0	− 180,675.2
(1) Sacrau	+ 345,508.3	− 202,211.5
	+ 15,956.7	+ 21,536.3

log 15956.7	4.202 9430·8		4.33317
log 21536.3	4.333 1710·9	*log cos* t_1	9.90498
log tan (t_1)	9.869 7719·9	*log s*	4.42819
$(t_1) = 36° \, 32' \, 7.904''$		$s = 26803$ m	

(3)

$$
\begin{array}{r|l}
\log \varrho & 5.31443 \\
\log s & 4.42819 \\
\hline
\log(\varrho : s) & 0.88624
\end{array}
\qquad
\begin{array}{r|l}
 & 0.88624_n \\
\sin t\,.. & 9.77474 \\
\hline
a\,.. & 0.66098_n \\
\multicolumn{2}{c}{a = -4.581}
\end{array}
\qquad
\begin{array}{r|l}
 & 0.88624 \\
\cos t\,.. & 9.90498 \\
\hline
b\,.. & 0.79122 \\
\multicolumn{2}{c}{b = +6.183}
\end{array}
\qquad (4)
$$

$$
\begin{array}{rl}
y_2 & = +\ 361{,}465 \\
y_2 & = +\ 361{,}465 \\
y_1 & = +\ 345{,}508 \\
y_1 & = +\ 345{,}508 \\
\hline
2y_1 + y_2 = & 1{,}052{,}481 \\
2y_2 + y_1 = & 1{,}068{,}438
\end{array}
\qquad
\begin{array}{r|l}
\multicolumn{2}{l}{\text{from (5), p. 245,}} \\
x_2 - x_1 & 4.33317 \\
2y_1 + y_2 & 6.02222 \\
\varrho : 6r^2 & 0.92622 \\
\hline
 & 1.28161 \\
\multicolumn{2}{l}{T_1 - t_1 = +19.125''}
\end{array}
\qquad
\begin{array}{r|l}
 & \\
x_1 - x_2 & 4.33317_n \\
2y_2 + y_1 & 6.02875 \\
\varrho : 6r^2 & 0.92622 \\
\hline
 & 1.28814_n \\
\multicolumn{2}{l}{T_2 - t_2 = -19.415''\,.}
\end{array}
\qquad (5)
$$

These thus computed reductions $T - t$ are accurate to approximately 0.01'', which is usually sufficient; but if we want to have it rigorous even to 0.01'', a fourth-order term is further added, i.e., according to a communication by Oberstleutnant v. S c h m i d t in *Zeitschr. f. Verm.*, 1894, p. 400, the more accurate reduction which is called there $U_1 - t_1$, is:

$$
U_1 - t_1 = \frac{\varrho}{4A^2}(y_1 + y_2)(x_2 - x_1) - \frac{\varrho}{12A^2}(y_2 - y_1)\ (x_2 - x_1)
$$
$$
- \frac{\varrho}{48A^4}(y_1 + y_2)^3 (x_2 - x_1)\,.
$$

The first two terms of this formula are algebraically identical with our formula for $T_1 - t_1$ in (5), section 71, p. 245, and the third term yields the coefficient:

$$
\log \frac{\varrho}{48A^4} = 6.41307\,,
$$

and with this we compute the amount $-0.0197''$, in the case of our example Sacrau-Skronskau, hence together to four places:

$$
\begin{array}{rl}
T_1 - t_1 = & +19.1248'' \\
 & -\ \ 0.0197 \\
\hline
T_1 - t_1 = & +19.1051''
\end{array}
\qquad\qquad
\begin{array}{rl}
T_2 - t_2 = & -19.4149'' \\
 & +\ \ 0.0197 \\
\hline
T_2 - t_2 = & -19.3952''\,.
\end{array}
$$

Later we will find this $T_1 - t_1$ again rounded off $= 19.11''$ (cf. Vol III, 1923, section 109).

By computing also all other $T - t$'s for the Station Sacrau in the same manner, we obtain the following station data of measured directions, approximately oriented, and then transferred to the plane, according to V, p. 101:

Station 1. Sacrau Table (6)

Target Point	Observed and Approx Oriented T'	Reduction $t - T = t' - T'$	Obs. and Transf. to the Plane t'
1. Skronskau . . .	36° 32′ 28.78″	− 19.11″	36° 32′ 9.67″
2. Lubetzko	103 3 44.68	+ 6.23	103 3 50.91
3. Annaberg . . .	183 12 49.69	+ 31.54	183 13 21.23
4. Lossen	268 18 41.99	+ 1.18	268 18 43.17
5. Eckersdorf . . .	305 20 30.57	− 20.30	305 20 10.27
6. Rosen	345 34 17.13	+ 33.29	345 34 50.42
	Sums 212.84	+ 32.83	185.67

These t''s now play the role of observations of entirely the same kind as, e.g., the measured and approximately oriented directions α in the adjustment of resection, section 99, p. 391, equation (9), and by adding the direction angle computed from approximate coordinates from the previous (3) with $(t_1) = 36° 32' 7.90''$, in the meaning of the first (φ) on p. 395, we also have the absolute term of the first error equation, namely $(t_1) - t_1 = 36° 32' 7.90'' - 36° 32' 9.67'' = -1.77''$. We have the coefficients $a = -4.581$ and $b = +6.183$ of this error equation from the previous (4); and hence, according to the view of the net picture, p. 423, the first error equation will be written directly as follows:

$$v_1 = +1.77'' - \delta z - 4.581\,(\delta x_2 - \delta x_1) + 6.183\,(\delta y_2 - \delta y_1).$$

We find the previous error equation again as the first of the following group (according to V, p. 101):

Error Equations of the Station Sacrau Table (7)

		I	III	V	VII	II	IV	VI	VIII	
	δz	δx_1	δx_2	δx_3	δx_4	δy_1	δy_2	δy_3	δy_4	$-l$
(1)	−1	+4.580	−4.580	. .	. .	−6.183	+6.183	. .	. .	−1.76
(2)	−1	+6.551	. .	. .	. .	+1.520	. .	. .	. .	−8.63
(3)	−1	−0.320	. .	. .	. .	+5.686	. .	. .	. .	−2.72
(4)	−1	−4.327	. .	. .	. .	+0.127	. .	. .	. .	+3.76
(5)	−1	−4.051	. .	+4.051	. .	−2.873	. .	+2.873	. .	+4.36
(6)	−1	−1.615	. .	. .	+1.615	−6.281	. .	. .	+6,281	−2.40
Sum	−6	+0.818	−4.580	+4.051	+1.615	−8.004	+6.183	+2.873	+6.281	−7.39

We have here $4.580^2 + 6.551^2 + 0.320^2 + \ldots = +101.7361$

$+4.580\,(-4.580)\ldots = -20.9764$

. . . .

$6.183^2 + 1.520^2 + 5.686^2 + \ldots = +120.5917$

. . . ., etc.

δz could be eliminated therefrom; but we also can set up the parts of normal equations directly from (7) and reduce then once, whereby δz is likewise eliminated.

The parts of normal equations are in an abbreviated manner of writing:

Table (8)

δz	δx_1	δx_2	δx_3	δx_4	δy_1	δy_2	δy_3	δy_4	$-l$
+6.000	− 0.818	+ 4.580	− 4.051	−1.615	+ 8.004	− 6.183	− 2.873	− 6.281	+ 7.390
	+101.736	−20.976	−16.411	−2.608	+ 1.053	+28.318	−11.638	−10.144	− 93.781
		+20.976	. .	. .	+ 28.318	−28.318	. .	. .	+ 8.061
			+16.411	. .	− 11.638	. .	+11.638	. .	+ 17.662
				+2.608	− 10.144	. .	. .	+10.144	− 3.876
					+120.592	−38.230	− 8.254	−39.451	− 14.676
						+38.230	. .	. .	− 10.882
							+ 8.254	. .	+ 12.526
								+39.451	− 15.074
									+123.880.

After a single reduction we obtain (cf. V, pp. 101-102):

Table (9)

I	III	V	VII	II	IV	VI	VIII	
δx_1	δx_2	δx_3	δx_4	δy_1	δy_2	δy_3	δy_4	$-l$
+101.63	−20.36	−16.96	−2.83	+ 2.14	+27.48	−12.03	−11.00	− 92.77
	+17.47	+ 3.09	+1.23	+ 22.21	−23.60	+ 2.19	+ 4.80	+ 2.42
		+13.67	−1.09	− 6.24	− 4.17	+ 9.70	− 4.24	+ 22.65
			+2.17	− 7.99	− 1.66	− 0.77	+ 8.45	− 1.89
				+109.91	−29.98	− 4.42	−31.07	− 24.54
					+31.86	− 2.96	− 6.47	− 3.26
						+ 6.87	− 3.01	+ 16.06
							+32.87	− 7.33
								+114.78.

Each station yields such a system of station normal equations or, as the case may be, reduced normal equations, while δz is eliminated in each case, and finally all these "contributions to the normal equations of the system" must be added up term by term.

If we wish to carry this out for the first four unknowns as an example $\delta x_1 = \text{I}$, $\delta x_2 = \text{III}$, $\delta y_1 = \text{II}$, $\delta y_2 = \text{IV}$, then the net picture, p. 423, shows us that to do so we need the contributions of the following stations: (1) = Sacrau, (2) = Skronskau, (3) = Eckersdorf, (4) = Rosen, (11) = Lubetzko, (12) = Annaberg, (13) = Lossen, i.e. we have:

	$\delta x_1 = \text{I}$	$\delta x_2 = \text{III}$	$\delta y_1 = \text{II}$	$\delta y_2 = \text{IV}$	
1. Sacrau (V, p. 101—102) . .	+ 101.63	— 20.36 + 17.47	+ 2.14 + 22.21 + 109.91	+ 27.48 — 23.60 — 29.93 + 31.86	I III II IV
2. Skronskau (V, p. 103) . . .	+ 13.99	— 6.89 + 57.14	— 18.88 + 9.30 + 25.49	+ 14.40 + 38.32 — 19.44 + 53.21	I III II IV
3. Eckersdorf (V, p. 106—107) .	+ 12.31		+ 8.73 + 6.19		I II
4. Rosen (V, p. 109)	+ 1.96	— 3,04 + 42.28	+ 7.60 — 11.79 + 29.59	— 1.18 + 16.40 — 4.57 + 6.37	I III II IV
11. Lubetzko (V, p. 127) . . .	+ 28.61	— 6.24 + 5.45	+ 6.64 — 1.45 + 1.54	— 12.77 + 11.15 — 2.96 — 22.80	I III II IV
12. Annaberg (V, p. 129) . . .	+ 0,05		— 0.91 + 16.17		I II
13. Lossen (V, p. 131)	+ 12,48		— 0.37 + 0.01		I II
Sum . . . . (V, p. 141—142)	+ 171.03	— 36.53 + 122.34	+ 4.95 + 18.27 + 188.90	+ 27.93 + 42.27 — 56.95 + 114.24	I III II IV

Therefore in an abbreviated manner of writing the normal equations must begin [from (3), p. 86] thusly:

$$\begin{array}{rrrrl}
+\underline{171.03}\ \text{I} & -\ 36.53\ \text{III} & +\ 4.95\ \text{II} & +\ 27.93\ \text{IV} & +\ldots = 0 \\
 & +\underline{122.34}\ \text{III} & +\ 18.27\ \text{II} & +\ 42.27\ \text{IV} & +\ldots = 0 \\
 & & +\underline{188.90}\ \text{II} & -\ 56.95\ \text{IV} & +\ldots = 0 \\
 & & & +\underline{114.24}\ \text{IV} & +\ldots = 0 \\
 & & & & \ldots\ldots
\end{array}$$

This order I, III, II, IV corresponds to the meaning $\text{I} = \delta x_1$, $\text{III} = \delta x_2$, $\text{II} = \delta y_1$, $\text{IV} = \delta y_2 \ldots$; on the other hand, if we aim to produce the series I, II, III, IV . . . , then we have to write the above normal equations thusly:

$$\begin{array}{rrrrl}
+\underline{171.03}\ \text{I} & +\ 4.95\ \text{II} & -\ 36.53\ \text{III} & +\ 27.93\ \text{IV} & +\ldots = 0 \\
 & +\underline{188.90}\ \text{II} & +\ 18.27\ \text{III} & -\ 56.95\ \text{IV} & +\ldots = 0 \\
 & & +\underline{122.34}\ \text{III} & +\ 42.27\ \text{IV} & +\ldots = 0 \\
 & & & +\underline{114.24}\ \text{IV} & +\ldots = 0 \\
 & & & & \ldots\ldots
\end{array}$$

This is the beginning of the normal equations on V, pp. 141-142, whose solution yields the 20 unknowns (V, p. 143):

$$\left.\begin{array}{llll}
\text{I} = \delta x_1 = +1.145\ \text{m} & & \text{II} = \delta y_1 = +0.710\ \text{m} \\
\text{III} = \delta x_2 = +1.997 & & \text{IV} = \delta y_2 = +1.665 \\
\text{V} = \delta x_3 = +0.609 & & \text{VI} = \delta y_3 = +0.306 \\
\text{VII} = \delta x_4 = +2{,}863 & & \text{VIII} = \delta y_4 = +0.759 \\
\text{IX} = \delta x_5 = +0.828 & & \text{X} = \delta y_5 = +0.065 \\
\text{XI} = \delta x_6 = +0.232 & & \text{XII} = \delta y_6 = -0.485 \\
\text{XIII} = \delta x_7 = -0.291 & & \text{XIV} = \delta y_7 = -0.247 \\
\text{XV} = \delta x_8 = +0.520 & & \text{XVI} = \delta y_8 = +0.421 \\
\text{XVII} = \delta x_9 = -0.542 & & \text{XVIII} = \delta y_9 = +0.648 \\
\text{XIX} = \delta x_{10} = +0.097 & & \text{XX} = \delta y_{10} = +0.852
\end{array}\right\} . \qquad (10)$$

By adding these corrections to the approximate values given previously in table (2) we obtain the following table of the final coordinates:

1. Sacrau	+ 345,509.010 m	— 202,210.355 m	
2. Skronskau	+ 361,466.665	— 180,673.203	
3. Eckersdorf	+ 311,632.406	— 178,191.991	
4. Rosen	+ 337,588.759	— 171,405,737	
5. Laski	+ 326,170.365	— 155,738.472	
6. Ossen	+ 298,827.415	— 137,624.968	(11)
7. Johannas Höhe	+ 273,033.653	— 129,893.891	
8. Kotlow	+ 321,735.821	— 117,814.580	
9. Dzielitz	+ 286,628.548	— 95,567.142	
10. Baranowek	+ 303,289.452	— 81,761.703	

With this, the adjustment is essentially completed; but in addition, we also can produce the adjusted station data [Abrisse], similarly as in the case of the simple adjustments of intersection and resection, on pp. 376 and 395. As an example, we will calculate the adjusted direction angle (1,2) = Sacrau-Skronskau with the above final coordinates:

Skronskau	+ 361,466.665	— 180,673.203
Sacrau	+ 345,509.010	— 202,210.355
	+ 15,957.655	+ 21,537.152

log 15957.655	4.202 9690·7
log 21537.152	4.333 1882·7
log tan t	9.869 7808·0

$$t_1 = 36^\circ\, 32'\, 9.906'' . \qquad (12)$$

This corresponds to the previous observed $t' = 36^\circ\, 32'\, 9.67''$ in (6), p. 425.

If we calculate all these adjusted t's, then we obtain a comparison for each station as, for example, in the following for the first Station Sacrau:

Station Sacrau

Target Point	Observed from (6), p. 425 t'	Adjusted t	$t - t'$ $= V$	$V - z$ $= v$
1. Skronskau . . .	36° 32′ 9.67″	36° 32′ 9.91″	+ 0.24″	— 0,05″
2. Lubetzko . . .	103 3 50.91	103 3 50.86	— 0.05	— 0.34
3. Annaberg . . .	183 13 21.23	183 13 22.18	+ 0.95	+ 0.66
4. Lossen	268 18 43.17	268 17 42.06	— 1.11	— 1.40
5. Eckersdorf . .	305 20 10.27	305 20 11.32	+ 1.05	+ 0.76
6. Rosen	345 34 50.42	345 34 51.10	+ 0.68	+ 0.39
	Sum 185.67	187.43	+ 1.76	+ 0.02
	Mean		$z = +0.29$.	

This corresponds to the first group of V, p. 144.

By adding also the $T - t'$s of p. 425 again, then we obtain the final station data [Abriss]:

Target Point	Observed $t' + (T - t) + z = T'$	Adjusted $t + (T - t) = T$	$T - T' = v$
1. **Skronskau** . . .	36° 32′ 29.07″	36° 32′ 29.02″	− 0.05″
2. **Lubetzko** . . .	103 3 44.97	103 3 44.63	− 0.34
3. **Annaberg** . . .	183 12 49.98	183 12 50.64	+ 0.66
4. **Lossen**	268 18 42.28	268 18 40.88	− 1.40
5. **Eckersdorf** . .	305 20 30.86	305 20 31.62	+ 0.76
6. **Rosen**	345 34 17.42	345 34 17.81	+ 0.39
	214.58	214.60	+ 0.02

This corresponds to the Abriss of Station Sacrau in V, p. 147, where the meridian convergence $-3° 48' 1.30''$ and log S are also added.

(In the case of Rosen we have 345° 34' whereas in V, p. 147, there is 345° 35', as it seems due to a printing error.)

In the same way as for the Station Sacrau, the official work, *Kgl. Preuss. Landestriangulation*, V. Teil, Berlin, 1893, from which on pp. 95-165 we have borrowed the above, gives the station data [Abrisse] also for all other stations on p. 147 and following. This Silesia-Posen triangulation net is one of the most elegant examples for net intercalation with adjustment of coordinates, which was introduced by General Schreiber since about 1876 at the trigonometric division of the Prussian Land Survey and thence has been retained for the adjustment of the filling nets and intermediate points.

Section 107. Double Intercalation of Points with Measured Distance Between the Points

In the case of the simultaneous intercalation of two points, there will sometimes occur the case in which the two points lie very closely beside one another and consequently the distance between them can be measured with greater accuracy than it can be computed from the adjustment. In such cases we can introduce the measured distance as free of error and now we have to carry out the adjustment in such a way that the given distance between the points is not changed by it. In the case of the computation of the approximate coordinates we proceed in such a way that we compute at first one point from favorable rays, then find the direction angle to the second point, and with it and the measured distance determine the coordinates of the second point.

In Fig. 1 let (P_1) and (P_2) be the points given by the approximate coordinates with the measured distance s, while P_1 and P_2 shall denote the final coordinates. We can imagine the displacement of the points from (P_1) (P_2) to $P_1 P_2$ divided into a parallel displacement to $P_1 P_2'$ and a rotation by the angle ω. If δx_1, δy_1 and δx_2, δy_2 are the corrections of coordinates by the adjustment so that therefore we have

$$\left.\begin{aligned}(y_1) + \delta y_1 &= y_1 \qquad (x_1) + \delta x_1 = x_1 \\ (y_2) + \delta y_2 &= y_2 \qquad (x_2) + \delta x_2 = x_2\end{aligned}\right\} \quad (1)$$

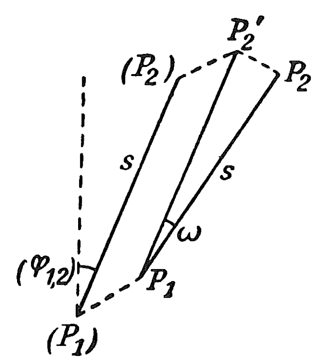


Fig. 1.

then from Fig. 1 there follows:

$$\left.\begin{aligned}\delta x_2 &= \delta x_1 - s\,\frac{\omega}{\varrho}\,sin\,(\varphi_{1,2}) \\ \delta y_2 &= \delta y_1 + s\,\frac{\omega}{\varrho}\,cos\,(\varphi_{1,2})\end{aligned}\right\} \quad (2)$$

or, since

$$s \sin(\varphi_{1,2}) = (y_2) - (y_1)$$
$$s \cos(\varphi_{1,2}) = (x_2) - (x_1)$$

then we have also

$$\left.\begin{aligned} \delta x_2 &= \delta x_1 - ((y_2) - (y_1)) \frac{\omega}{\varrho} \\ \delta y_2 &= \delta y_1 + ((x_2) - (x_1)) \frac{\omega}{\varrho} \end{aligned}\right\}. \qquad (3)$$

The two equations (3) represent the conditions which must exist between the corrections of coordinates of the two points if the given distance s is to remain unchanged by the adjustment.

When we have set up the error equations for all inward and outward directions of the two points, then we can eliminate the two unknowns δx_2 and δy_2 with the help of the equations (3), whereby the angle ω then occurs as a new unknown.

But the introduction of the angle ω yields at the same time an essential simplification for the two error equations of the reciprocal directions from P_1 to P_2 and from P_2 to P_1. If we denote the measured directions by r, their corrections by v, the preliminary unknowns of orientation by (z), and their corrections by δz, then the finally oriented and corrected directions are:

$$\left.\begin{aligned} r_{1,2} + (z_1) + \delta z_1 + v_{1,2} \\ r_{2,1} + (z_2) + \delta z_2 + v_{2,1} \end{aligned}\right\}. \qquad (4)$$

But on the other hand, the final direction angles are according to Fig. 1:

$$(\varphi_{1,2}) + \omega \qquad \text{and} \qquad (\varphi_{2,1}) + \omega. \qquad (5)$$

Hence we have the equations

$$\left.\begin{aligned} r_{1,2} + (z_1) + \delta z_1 + v_{1,2} &= (\varphi_{1,2}) + \omega \\ r_{2,1} + (z_2) + \delta z_2 + v_{2,1} &= (\varphi_{2,1}) + \omega \end{aligned}\right\}. \qquad (6)$$

Now since we have according to (9) and (10), section 99, p. 391,

$$(\varphi) - r - (z) = -l, \qquad (7)$$

we obtain the error equations:

$$\left.\begin{aligned} v_{1,2} &= -\delta z_1 + \omega - l_{1,2} \\ v_{2,1} &= -\delta z_2 + \omega - l_{2,1} \end{aligned}\right\}. \qquad (8)$$

Therefore, in the two error equations for the reciprocal directions between the two new points, the corrections of coordinates do not occur at all but only the angle of rotation ω, whereby further computation is greatly simplified.

Accordingly, we have the following computational procedure: At first there are set up all error equations with the exception of those for the directions P_1P_2 and P_2P_1, for which latter only the absolute terms can be found. Then the unknowns δx_2 and δy_2 are eliminated from the error equations for the base point P_2 with the help of (3), and finally the two error equations (8) are set up. After, in addition to this, the two unknowns of orientation δz_1 and δz_2 are eliminated, station by station, as in the case of resection according to section 99, we can set up the three normal equations with the unknowns δx_1, δy_1, and ω from all the equations.

For the computation of the mean error of direction there are to be computed as unknowns the two corrections of coordinates δx_1, δy_1, the angle of rotation ω, and the two unknowns of orientation δz_1 and δz_2 so that the formula for the computation of the mean error reads:

$$m = \sqrt{\frac{[v\,v]}{n-5}}. \tag{9}$$

The computation of the mean error of coordinates is somewhat more complicated. For x_1 and y_1 we could determine the weights directly from the normal equations according to section 30. The coordinates x_2 and y_2, however, are functions of the unknowns of the adjustment, since the corrections δx_2 and δy_2 are represented according to the equations (3) in the form

$$\delta x_2 = \delta x_1 + p\,\omega \qquad \delta y_2 = \delta y_1 + q\,\omega \tag{10}$$

or generally in the form

$$F = f_1 \delta x_1 + f_2 \delta y_1 + f_3 \omega. \tag{11}$$

Hence, their weights must be determined with the help of equation (13), section 31, p. 102,

$$\frac{1}{P} = \frac{f_1^2}{[a\,a]} + \frac{[f_2 \cdot 1]^2}{[b\,b \cdot 1]} + \frac{[f_3 \cdot 2]^2}{[c\,c \cdot 2]}. \tag{12}$$

If we examine at first δx_2 according to (10), then we have for it

$$f_1 = 1 \qquad f_2 = 0 \qquad f_3 = p$$

and with this we will have

$$\left.\begin{aligned} [f_2 \cdot 1] &= -\frac{[a\,b]}{[a\,a]} \qquad [f_3 \cdot 1] = p - \frac{[a\,c]}{[a\,a]} \\ [f_3 \cdot 2] &= p - \frac{[a\,c]}{[a\,a]} + \frac{[b\,c \cdot 1]\,[a\,b]}{[b\,b \cdot 1]\,[a\,a]} \end{aligned}\right\}. \tag{13}$$

By introducing these values into (12) we obtain:

$$\frac{1}{P_x} = \frac{1}{[a\,a]} + \frac{[a\,b]^2}{[a\,a]^2\,[b\,b \cdot 1]} + \frac{\left(p - \frac{[a\,c]}{[a\,a]} + \frac{[b\,c \cdot 1]\,[a\,b]}{[b\,b \cdot 1]\,[a\,a]}\right)^2}{[c\,c \cdot 2]}. \tag{14}$$

Accordingly, we have for δy_2 from (10) and (11)

$$f_1 = 0 \qquad f_2 = 1 \qquad f_3 = q$$

$$[f_2 \cdot 1] = 1 \qquad [f_3 \cdot 1] = q$$

$$[f_3 \cdot 2] = q - \frac{[b\,c \cdot 1]}{[b\,b \cdot 1]}$$

and consequently

$$\frac{1}{P_y} = \frac{1}{[b\,b \cdot 1]} + \frac{\left(q - \frac{[b\,c \cdot 1]}{[b\,b \cdot 1]}\right)^2}{[c\,c \cdot 2]}. \tag{15}$$

These two formulae (14) and (15) can also be used at the same time for the computation of the mean coordinate error of the point P_1, for which we only have to set $p = 0$ and $q = 0$.

The values of the various coefficients in (14) and (15) are to be taken directly from the normal equations and the reduced normal equations.

Intercalation of a Group of Points with Given Distances

The above theory is readily applicable also to the problem of intercalating into a triangulation net, by an arbitrary number of inward and outward directions, a whole group of points whose reciprocal position is fixed. Such a problem occurred in the case of the city survey of Hannover, as shown in Fig. 2.

The plan of the site (Fig. 2) shows the Königsworther Platz in Hannover, from which seven trigonometric points can be seen, but never more than four from *one* point, because the buildings, trees, etc., disturb the sights. After much searching there were chosen four base points A, B, D, E, which were measured

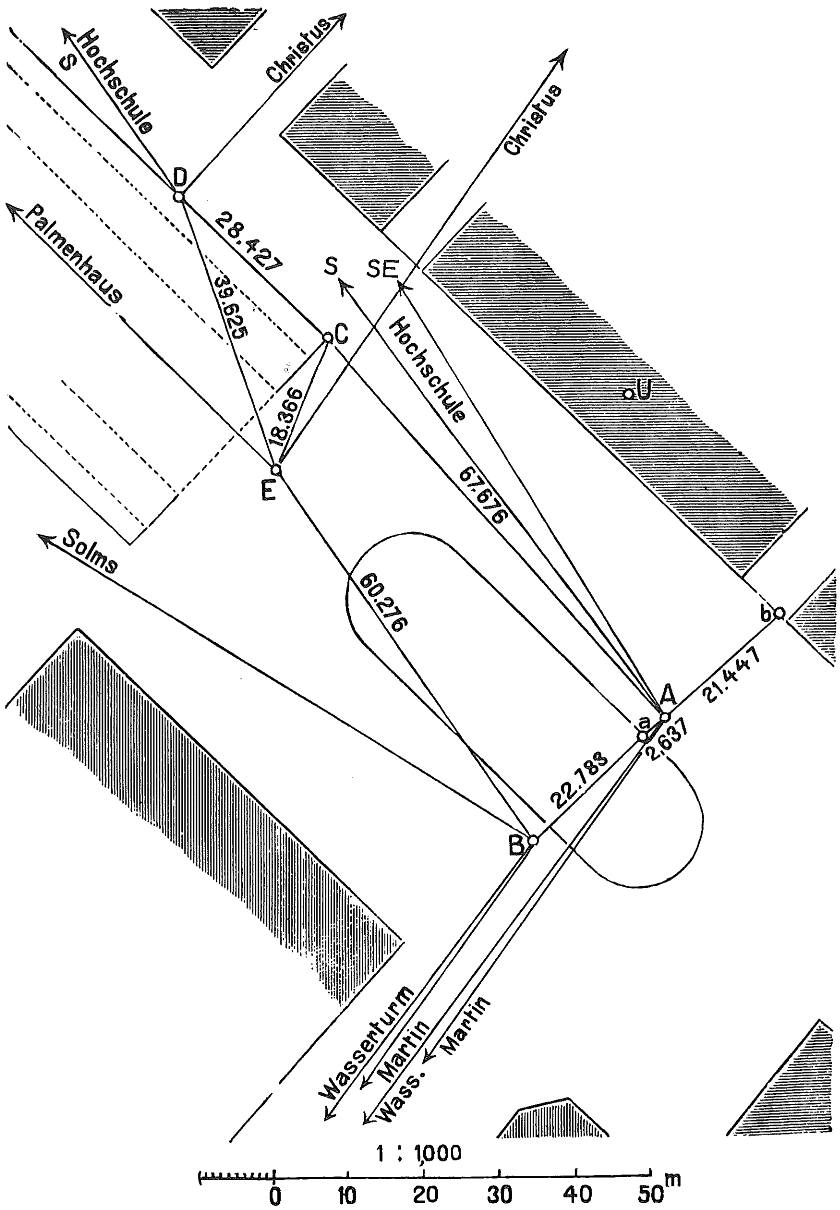


Fig. 2.
Königsworther Platz in Hannover.

together as a polygon with an auxiliary point C and brought thereby into the necessary centering connections. At the same time there also were set two securing bolts a and b, which, it is true, are not related to the following adjustment but are to be listed also in order to mention everything related of a technical nature. (The markings of A, B, C, D, E are made with iron bolts according to *Handb. d. Verm.*, II. Band, 1. Halbbd., 9. Aufl., 1931, p. 406, and the points a and b according to p. 536, ibid.) The six distances $AB = 22.783$ m, $BE = 60.276$ m, etc., are obtained accurately to a few millimeters with common measuring rods. The best point is A with four sights: Martin, Wasserturm, Hochschule S and SE; also B still has three sights: Martin, Wasserturm, Solms, while E and D have only two sights each: Palmenhaus, Christus and Hochschule, Christus.

At first we will give the coordinates of all given points and the approximately oriented measuring data [Messungsabrisse]:

Given points:

	y	x
Martin, Turm	$y = -25273.930$ m	$x = -28710.901$ m
Wasserturm	−25538.488	−29071.474
Solms, Turm	−24695.660	−27176.634
Palmenhaus	−25977.983	−25706.108
Hochschule S Turm	−24709.769	−26868.278
Hochschule SE Turm	−24667,066	−26851.965
Christus, Turm	−24158.271	−26989.625

(16)

Measuring data [Messungsabrisse] of the approximately oriented directions

Base Point A	
Martin	214° 32′ 50.5″
Wasserturm	214 55 2.9
B	227 12 7.9
C	319 32 58.2
Hochschule S	323 30 43.0
Hochschule SE	327 42 9.2

Base Point B	
Martin	214° 22′ 18.7″
Wasserturm	214 47 6.0
Solms	300 56 40.3
E	325 49 32.9
A	47 11 44.0

Base Point E	
Palmenhaus	315° 41′ 25.8″
D	340 20 5.3
C	21 16 49.6
Christus	35 7 49.8
B	145 49 25.8

Base Point D	
Hochschule S	324° 28′ 52.9″
Christus	39 53 29.3
C	135 16 52.9
E	160 20 11.1

Base Point C	
A	139° 33′ 0.0″
E	201 17 5.2
D	315 16 41.2

(17)

By means of the measurements on A there were computed approximate coordinates of A as resected point, and the whole pentagon $ABEDC$ was attached to it in the usual manner with the distances entered on p. 432:

Approximate coordinates:

	(y)	(x)
A	$(y) = -24345.004$ m	$(x) = -27361.323$ m
B	−24361.720	−27376.805
E	−24395.581	−27326.940
D	−24408.920	−27289.629
C	−24388.916	−27309.827

(18)

At first we have computed the approximate direction angles for all rays of the station data [Messungsabrisse] from the coordinates (16) and (18):

Approximate direction angles (φ):

Base Point A	
Martin	214° 32′ 23.8″
Wasserturm	214 54 37.6
B	227 11 41.2
C	319 32 41.6
Hochschule S	323 30 18.6
Hochschule SE	327 41 42.8

Base Point B	
Martin	214° 21′ 46.9″
Wasserturm	214 46 33.1
Solms	300 56 21.9
E	325 49 17.1
A	47 11 41.2

Base Point E	
Palmenhaus	315° 41′ 14.5″
D	340 19 39.4
C	21 16 46.0
Christus	35 07 38.4
B	145 19 17.1

Base Point D	
Hochschule S	324° 28′ 21.8″
Christus	39 52 41.9
C	135 16 35.2
E	160 19 39.4

Base Point C	
A	139° 32′ 41.6″
E	201 16 46.0
D	315 16 35.2

(19)

For the setting up of the error equations there are assumed, moreover, approximate quantities of orientation (z), which, for each base point, were set equal to the arithmetic mean of the differences $(\varphi) - r$ from (19) and (17):

Approximate orientations (z):

$$\left.\begin{array}{l} (z_a) = -24.4'' \\ (z_b) = -20.3 \\ (z_e) = -12.2 \\ (z_d) = -32.0 \\ (z_c) = -14.5 \end{array}\right\} . \tag{20}$$

Now while the absolute terms $-l$ were computed according to (7), the following error equations, in which δx and δy are computed in decimeters, resulted for the one-sided directions to the triangulation points and for the reciprocal directions between the polygon points, computed according to (8).

Base point A.

Error equations:

$$\left.\begin{array}{llllll} v_1 = - & \delta z_a - & 7.1\,\delta x_a + & 10.4\,\delta y_a & . & -2.3 \\ v_2 = - & \delta z_a - & 5.7\,\delta x_a + & 8.1\,\delta y_a & . & -0.9 \\ v_3 = - & \delta z_a & . & . & +\omega & -2.3 \\ v_4 = - & \delta z_a & . & . & +\omega & +7.8 \\ v_5 = - & \delta z_a - & 20.0\,\delta x_a - & 27.0\,\delta y_a & . & 0.0 \\ v_6 = - & \delta z_a - & 18.3\,\delta x_a - & 28.9\,\delta y_a & . & -2.0 \\ \hline 0 = - & 6\,\delta z_a - & 51.1\,\delta x_a - & 37.4\,\delta y_a & +2\omega & +0 \end{array}\right\} \tag{21}$$

$$0 = +\delta z_a + 8.5\,\delta x_a + 6.2\,\delta y_a - 0.333\,\omega + 0. \tag{22}$$

Reduced error equations:

$$\left.\begin{array}{lllll} v_1 = +1.4\,\delta x_a & +16.6\,\delta y_a & -0.33\,\omega & -2.3 \\ v_2 = +2.8 & +14.3 & -0.33 & -0.9 \\ v_3 = +8.5 & +6.2 & +0.66 & -2.3 \\ v_4 = +8.5 & +6.2 & +0.66 & +7.8 \\ v_5 = -11.5 & -20.8 & -0.33 & 0.0 \\ v_6 = -9.8 & -22.7 & -0.33 & -2.0 \end{array}\right\} . \tag{23}$$

Parts of normal equations:

$$\left.\begin{array}{rrrr} +\underline{382.6}\,\delta x_a + & 630.3\,\delta y_a + & 17.02\,\omega + & 60.61 = 0 \\ & +\underline{1504.9}\,\delta y_a + & 12.45\,\omega + & 28.45 = 0 \\ & & +\underline{1.33}\,\omega + & 5.39 = 0 \end{array}\right\} . \tag{24}$$

Base point B.

Error equations:

$$\left.\begin{array}{llllll} v_1 = -\delta z_b - & 7.2\,\delta x_b + & 10.5\,\delta y_b & . & & -11.5 \\ v_2 = -\delta z_b - & 5.7\,\delta x_b + & 8.2\,\delta y_b & . & & -12.6 \\ v_3 = -\delta z_b - & 45.4\,\delta x_b - & 27.2\,\delta y_b & . & & +1.9 \\ v_4 = -\delta z_b & . & . & +\omega & & +4.9 \\ v_5 = -\delta z_b & . & . & +\omega & & +17.5 \end{array}\right\} . \tag{25}$$

Elimination equations according to (3):

$$\left.\begin{array}{l} \delta x_b = \delta x_a + 167.16\,\dfrac{\omega}{\varrho} \\ \delta y_b = \delta y_a - 154.82\,\dfrac{\omega}{\varrho} \end{array}\right\} \tag{26}$$

where the differences of coordinates $(y_b) - (y_a) = -167.16$ and $(x_b) - (x_a) = -154.82$ are likewise computed in decimeters.

Transformed error equations:

$$\left.\begin{array}{l} v_1 = -\delta z_b - 7{,}2\,\delta x_a + 10.5\,\delta y_a - 0.01\,\omega - 11.5 \\ v_2 = -\delta z_b - 5.7\,\delta x_a + 8.2\,\delta y_a - 0.01 \quad - 12.6 \\ v_3 = -\delta z_b - 45.4\,\delta x_a - 27.2\,\delta y_a - 0.02 \quad + 1.9 \\ v_4 = -\delta z_b \quad . \quad . \quad + 1.00 \quad + 4.5 \\ v_5 = -\delta z_b \quad . \quad . \quad + 1.00 \quad + 17.5 \end{array}\right\} \tag{27}$$

$$\begin{array}{l} 0 = -5\,\delta z_b - 58.3\,\delta x_a - 8.5\,\delta y_a + 1.96\,\omega + 0 \\ 0 = +\,\delta z_b + 11.7\,\delta x_a + 1.7\,\delta y_a - 0.39\,\omega + 0\,. \end{array} \tag{28}$$

Reduced error equations:

$$\left.\begin{array}{lllll} v_1 = +\ 4{,}5\ \delta x_a & + 12.2\,\delta y_a & - 0.40\,\omega & - 11.5 \\ v_2 = +\ 6.0 & + 9.9 & - 0.40 & - 12.6 \\ v_3 = -33.7 & - 25.5 & - 0.41 & + 1.9 \\ v_4 = +11.7 & + 1.7 & + 0.61 & + 4.5 \\ v_5 = +11.7 & + 1.7 & + 0.61 & + 17.5 \end{array}\right\}. \tag{29}$$

Parts of normal equations:

$$\left.\begin{array}{r} +\underline{1465.7}\,\delta x_a + 1013.4\,\delta y_a + 23.85\,\omega + 66.02 = \\ +\underline{902.9}\,\delta y_a + 3.57\,\omega - 276.09 = 0 \\ +\underline{1.24}\,\omega + 22.40 = 0 \end{array}\right\}. \tag{30}$$

After having shown all details of the numerical computation for the two base points A and B we will limit ourselves for the further points to communicate only the most essential results:

Base point E.

Error equations:

$$\left.\begin{array}{l} v_1 = -\delta z_e - 6.4\,\delta x_e - 6.5\,\delta y_e \quad . \quad + 0.9 \\ v_2 = -\delta z_e \quad . \quad . \quad + \omega - 13.7 \\ v_3 = -\delta z_e \quad .. \quad . \quad + \omega + 8.6 \\ v_4 = -\delta z_e + 28.8\,\delta x_e - 40.9\,\delta y_e \quad . \quad + 0.8 \\ v_5 = -\delta z_e \quad . \quad . \quad + \omega + 3.5 \end{array}\right\}. \tag{31}$$

Elimination equations:

$$\left.\begin{array}{l} \delta x_e = \delta x_a + 505.8\,\dfrac{\omega}{\varrho} \\ \delta y_e = \delta y_a + 343.8\,\dfrac{\omega}{\varrho} \end{array}\right\}. \tag{32}$$

Sum-equation:

$$0 = +\,\delta z_e - 4.5\,\delta x_a + 9.5\,\delta y_a - 0.59\,\omega + 0. \tag{33}$$

Parts of normal equations:

$$\left.\begin{array}{r} +\underline{770.0}\,\delta x_a - 924.0\,\delta y_a - 13.11\,\omega + 16.83 = 0 \\ +\underline{1265.7}\,\delta y_a + 28.35\,\omega - 37.62 = 0 \\ \underline{1.24}\,\omega - 1.69 = 0 \end{array}\right\}. \tag{34}$$

Error equations:

$$\left.\begin{array}{l} v_1 = -\delta z_d - 23.1\,\delta x_d - 32.4\,\delta y_d \quad . \quad + 0.9 \\ v_2 = -\delta z_d + 33.8\,\delta x_d - 40.5\,\delta y_d \quad . \quad - 15.4 \\ v_3 = -\delta z_d \quad . \quad . \quad + \omega \quad + 14.3 \\ v_4 = -\delta z_d \quad . \quad . \quad + \omega \quad + 0.3 \end{array}\right\} . \tag{35}$$

Elimination equations:

$$\left.\begin{array}{l} \delta x_d = \delta x_a + 639.2 \dfrac{\omega}{\varrho} \\ \delta y_d = \delta y_a + 716.9 \dfrac{\omega}{\varrho} \end{array}\right\} . \tag{36}$$

Sum-equation:

$$0 = + \delta z_d - 2.7\,\delta x_a + 18.2\,\delta y_a - 0.44\,\omega + 0. \tag{37}$$

Parts of normal equations:

$$\left.\begin{array}{r} + \underline{1647.4}\,\delta x_a - 425.4\,\delta y_a - 1.70\,\omega - 541.58 = 0 \\ + \underline{1361.4}\,\delta y_a + 40.03\,\omega + 596.36 = 0 \\ + \underline{1.25}\,\omega + 15.00 = 0 \end{array}\right\} . \tag{38}$$

The direction measurements on point C do not enter into the adjustment, since directions to triangulation points do not exist here; however, we will indicate the elimination equations for the later computation of the mean error of coordinates.

Base point C.

Elimination equations:

$$\left.\begin{array}{l} \delta x_c = \delta x_a + 439.1 \dfrac{\omega}{\varrho} \\ \delta y_c = \delta y_a + 515.0 \dfrac{\omega}{\varrho} \end{array}\right\} . \tag{39}$$

The collection of the parts of normal equations (24), (30), (34), and (38) of the four base points A, B, E, and D yields the following normal equations and reduced normal equations:

$$\left.\begin{array}{r} + \underline{4265.7}\,\delta x_a + 294.3\,\delta y_a + 26.1\,\omega - 247.92 = 0 \\ + \underline{5034.9}\,\delta y_a + 84.4\,\omega + 311.10 = 0 \\ + \underline{5.1}\,\omega + 41.10 = 0 \\ \\ + \underline{5014.6}\,\delta y_a + 82.6\,\omega + 328.20 = 0 \\ + \underline{4.9}\,\omega + 42.62 = 0 \\ \\ + \underline{4.7}\,\omega + 41.10 = 0 \end{array}\right\} \tag{40}$$

$$\left.\begin{array}{ll} \omega = -8.74'' & p_\omega = 4.7 \\ \delta x_a = +0.1062\,\partial\mathrm{m} & \\ \delta y_a = +0.0786\,\partial\mathrm{m} & \end{array}\right\} . \tag{41}$$

With these values of the unknowns we can compute all corrections of coordinates by means of the elimination equations (26), (32), (36), and (39). We obtain:

$$
\left.\begin{array}{llll}
A & \delta y = +0.079\, \partial m & & \delta x = +0.106\, \partial m \\
B & \quad +0.085 & & \quad +0.099 \\
E & \quad +0.064 & & \quad +0.085 \\
D & \quad +0.048 & & \quad +0.079 \\
C & \quad +0.057 & & \quad +0.088
\end{array}\right\} . \tag{42}
$$

The following values result for the unknowns of orientation from the sum-equations (22), (28), (33), and (37):

$$
\left.\begin{array}{ll}
\delta z_a = -4.34'' & z_a = -28.74'' \\
\delta z_b = -4.80 & z_b = -25.10 \\
\delta z_e = -5.43 & z_e = -17.63 \\
\delta z_d = -5.03 & z_d = -37.03 \\
\delta z_c = -8.74 & z_c = -23.24
\end{array}\right\} . \tag{43}
$$

The value of δz_c is directly equal to the angle of rotation ω, since the direction measurements on C are not influenced, as to the rest, by the adjustment.

After the computation of the individual corrections v we obtain the sum of squares

$$[v\,v] = 969.5$$

and the mean errors:

$$m = \sqrt{\frac{969.5}{20-7}} = \pm 8.64'' \tag{44}$$

$$m_\omega = \pm \frac{8.64}{\sqrt{4.7}} = \pm 3.98''. \tag{45}$$

The denominator $20 - 7$ in m is composed of the 20 corrections v and the 7 unknowns, namely the two corrections of coordinates δx_a and δy_a, the four corrections to orientation δz_a, δz_b, δz_e, δz_d, and the angle of rotation ω.

For the computation of the weights of coordinates we compute at first the two general equations (14) and (15) and obtain:

$$
\left.\begin{array}{l}
\dfrac{1}{P_x} = 0.000\,2354 + \dfrac{(p - 0.004\,9822)^2}{4.7} \\
\dfrac{1}{P_y} = 0.000\,1994 + \dfrac{(q - 0.016\,4726)^2}{4.7}
\end{array}\right\} \tag{46}
$$

in which the coefficients p and q for the individual points are to be introduced. According to the elimination equations we have:

$$
\left.\begin{array}{lll}
A & p = 0.0 & q = 0.0 \\
B & \quad +0.000\,8104 & \quad -0.000\,7506 \\
E & \quad +0.002\,4522 & \quad +0.001\,6668 \\
D & \quad +0.003\,0987 & \quad +0.003\,4758 \\
C & \quad +0.002\,1288 & \quad +0.002\,4968
\end{array}\right\} \tag{47}
$$

and with these we obtain the following reciprocal weights:

$$
\left.\begin{array}{lll}
A & \dfrac{1}{P_x} = 0.000\,2407 & \dfrac{1}{P_y} = 0.000\,2572 \\
B & \quad 0.000\,2391 & \quad 0.000\,2625 \\
E & \quad 0.000\,2367 & \quad 0.000\,2461 \\
D & \quad 0.000\,2361 & \quad 0.000\,2354 \\
C & \quad 0.000\,2371 & \quad 0.000\,2410
\end{array}\right\} . \tag{48}
$$

With the mean direction error computed above we also find the mean errors m_x and m_y and have then as result of the adjustment:

$$\left.\begin{array}{llll} A & y = -24{,}344.996 \pm 0.014 \text{ m} & & x = -27{,}361.312 \pm 0.013 \text{ m} \\ B & \quad -24{,}361.711 \pm 0.014 & & \quad -27{,}376.795 \pm 0.013 \\ E & \quad -24{,}395.575 \pm 0.014 & & \quad -27{,}326.932 \pm 0.013 \\ D & \quad -24{,}408.915 \pm 0.013 & & \quad -27{,}289.621 \pm 0.013 \\ C & \quad -24{,}388.910 \pm 0.013 & & \quad -27{,}309.818 \pm 0.013 \end{array}\right\} . \qquad (49)$$

The small size of the angle of rotation ω results in the corrections of coordinates of all points deviating only a little from one another. This will always be the case when the approximate coordinates are accurate to some extent, and when the new points lie close to one another, so that the angle of rotation ω cannot exert a noticeable influence. This circumstance can be taken into account for the simplification of the adjustment by assuming from the outset the angle ω equal to zero and with this setting all corrections of abscissae and of ordinates equal to one another. In this form, Jordan has computed the group of points at hand for the Hannover city survey and published it in *Zeitschr. f. Verm.*, 1895, pp. 273-276. With the consideration that the directions between the points of the polygon are less accurate because of the short sightings than the directions to the triangulation points, these first-mentioned directions were not used for the adjustment. The result of this simple adjustment of approximation agrees very well with the rigorous adjustment while the largest deviation in the ordinates and in the abscissae amounts to only 4 mm or 15 mm, as the case may be. We can use this simple method to advantage in the case of angle measurements on several eccentric base points of a station if the points are connected with one another.

Section 108. The Mean Error of a Point and the Error Ellipse

For the judgment of the accuracy of the point determination, thus far we have learned, in addition to the mean errors of the measured angles or directions, as the case may be, only the mean errors m_x and m_y of the coordinates of points. But since these latter ones depend on the direction of the axes of coordinates, i.e., assume other values in the case of a rotation of the system of coordinates, they do not form a single-valued measure of accuracy. Therefore, we will study the matter further.

Let there exist the following error equations for the determination of a point:

$$\left.\begin{array}{l} v_1 = +a_1 x + b_1 y - l_1 \\ v_2 = +a_2 x + b_2 y - l_2 \\ v_3 = +a_3 x + b_3 y - l_3 \\ \cdots\cdots\cdots \end{array}\right\} . \qquad (1)$$

To these there belong the normal equations:

$$\left.\begin{array}{l} [a\,a]\,x + [a\,b]\,y - [a\,l] = 0 \\ [a\,b]\,x + [b\,b]\,y - [b\,l] = 0 \end{array}\right\} , \qquad (2)$$

from which the unknowns receive the values:

$$x = \frac{[b\,b]\,[a\,l] - [a\,b]\,[b\,l]}{D} \qquad y = \frac{[a\,a]\,[b\,l] - [a\,b]\,[a\,l]}{D} \qquad (3)$$

with

$$D = [a\,a]\,[b\,b] - [a\,b]^2. \qquad (4)$$

Finally, we have the mean error of an angle or direction measurement, as the case may be:

$$m = \pm \sqrt{\frac{[v\,v]}{n-u}}\,. \tag{5}$$

Let the true errors of the observations be $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$; the following errors result then for the coordinates x and y from (3):

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{D}\Big\{(a_1\,[b\,b] - b_1\,[a\,b])\,\varepsilon_1 + (a_2\,[b\,b] - b_2\,[a\,b])\,\varepsilon_2 + \ldots\Big\} \\ \varepsilon_y &= \frac{1}{D}\Big\{(b_1\,[a\,a] - a_1\,[a\,b])\,\varepsilon_1 + (b_2\,[a\,a] - a_2\,[a\,b])\,\varepsilon_2 + \ldots\Big\} \end{aligned}\right\}. \tag{6}$$

If we change from the true errors to the mean errors, then we have according to the law of the propagation of errors (11), section 5, p. 14:

$$m_x^2 = \frac{1}{D^2}\Big\{[a\,a]\,[b\,b]^2 + [b\,b]\,[a\,b]^2 - 2\,[a\,b]^2\,[b\,b]\Big\}m^2$$

$$m_y^2 = \frac{1}{D^2}\Big\{[b\,b]\,[a\,a]^2 + [a\,a]\,[a\,b]^2 - 2\,[a\,b]^2\,[a\,a]\Big\}m^2$$

or

$$m_x^2 = \frac{[b\,b]}{D}\,m^2 \qquad m_y^2 = \frac{[a\,a]}{D}\,m^2\,. \tag{7}$$

The mean point error

To the errors ε_x and ε_y of the coordinates there corresponds a displacement of the point by the amount of

$$\varepsilon = \sqrt{\varepsilon_x^{\,2} + \varepsilon_y^{\,2}}\,, \tag{8}$$

which indicates the true error of the point. If we introduce here the values of ε_x and ε_y from (6), then we will have

$$\begin{aligned} \varepsilon^2 = \frac{1}{D^2}\Big\{ & (a_1^2\,([b\,b]^2 + [a\,b]^2) + b_1^2\,([a\,a]^2 + [a\,b]^2) - 2\,a_1\,b_1\,[a\,b]\,([b\,b] + [a\,a]))\,\varepsilon_1^2 \\ & + (a_2^2\,([b\,b]^2 + [a\,b]^2) + b_2^2\,([a\,a]^2 + [a\,b]^2) - 2\,a_2\,b_2\,[a\,b]\,([b\,b] + [a\,a]))\,\varepsilon_2^2 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + f_{1,2}\,\varepsilon_1\,\varepsilon_2 + f_{1,3}\,\varepsilon_1\,\varepsilon_3 + \ldots\Big\}, \end{aligned}$$

where the coefficients of the products $\varepsilon_1\,\varepsilon_2$, $\varepsilon_1\,\varepsilon_3$, etc., are only indicated.

If we change in the last equation to average values of the individual terms, then let us suppose the average value of ε^2 is denoted by M^2 while the average values of $\varepsilon_1^{\,2}$, $\varepsilon_2^{\,2}$ . . . are equal to m^2; the average values of the products $\varepsilon_1\,\varepsilon_2$, $\varepsilon_1\,\varepsilon_3$, etc., are to be assumed equal to zero as in section 5, p. 13. And hence we have:

$$M^2 = \frac{1}{D^2}\Big\{[a\,a]\,([b\,b]^2 + [a\,b]^2) + [b\,b]\,([a\,a]^2 + [a\,b]^2) - 2\,[a\,b]^2\,([b\,b] + [a\,a])\Big\}\,m^2\,,$$

or after a few reductions:

$$M^2 = \frac{[a\,a] + [b\,b]}{D}\,m^2. \tag{9}$$

From (7) there also follows

$$M^2 = m_x^2 + m_y^2. \tag{10}$$

It is easy to prove that the expression (9) for the mean point error is independent of the direction of the axes of coordinates. According to (14), section 93, p. 360, we have

$$a = -\frac{\varrho}{s} \sin \varphi, \qquad b = +\frac{\varrho}{s} \cos \varphi. \tag{11}$$

If we introduce this in the numerator of (9), then we will have

$$[a\,a] + [b\,b] = \varrho^2 \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} + \dots\right),$$

which remains unchanged for any arbitrary direction of the axes of coordinates.

If we write the denominator of (9) in the form:

$$D = (a_1^2 + a_2^2 + \dots)(b_1^2 + b_2^2 + \dots) - (a_1 b_1 + a_2 b_2 + \dots)^2,$$

then we can easily transform it into

$$D = (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 + \dots, \tag{12}$$

and if the values of the a's and b's are introduced according to (11), then (12) changes to

$$D = \varrho^4 \left\{\frac{\sin^2(\varphi_2 - \varphi_1)}{s_1^2 s_2^2} + \frac{\sin^2(\varphi_3 - \varphi_1)}{s_1^2 s_3^2} + \frac{\sin^2(\varphi_3 - \varphi_2)}{s_2^2 s_3^2} + \dots\right\}.$$

Since only the differences of the direction angles occur here, D is likewise independent of the direction of the coordinates.

The error ellipse

The quantities m_x and m_y indicate the mean displacements of the point in the directions of the two axes of coordinates to be expected due to the measuring errors. Now it still is to be examined how large the mean displacements in other directions are, and in which directions the largest or smallest displacement is to be expected. To do this we start from the two equations (7).

Since we already have recognized that D is independent of the system of coordinates, we need to examine only the two numerators $[a\,a]$ and $[b\,b]$ in (7). If we assume a new system of coordinates which is rotated clockwise with respect to the original one by the angle ψ, then we have according to (11):

$$a' = -\frac{\varrho}{s} \sin(\varphi - \psi) \qquad b' = +\frac{\varrho}{s} \cos(\varphi - \psi) \tag{13}$$

and $[a\,a]$ changes to

$$[a'\,a'] = \varrho^2 \left\{\frac{\sin^2(\varphi_1 - \psi)}{s_1^2} + \frac{\sin^2(\varphi_2 - \psi)}{s_2^2} + \dots\right\}$$

or

$$[a'\,a'] = \varrho^2 \left\{\left[\frac{\sin^2 \varphi}{s^2}\right] \cos^2 \psi + \left[\frac{\cos^2 \varphi}{s^2}\right] \sin^2 \psi - \left[\frac{\sin \varphi \cos \varphi}{s^2}\right] \sin 2\psi\right\}$$

and this is according to (11):

$$[a' a'] = [a\,a] \cos^2 \psi + [b\,b] \sin^2 \psi + [a\,b] \sin 2\psi . \tag{14}$$

Likewise we will have

$$[b' b'] = [b\,b] \cos^2 \psi + [a\,a] \sin^2 \psi - [a\,b] \sin 2\psi . \tag{15}$$

In order to determine for which values of ψ the expressions (14) and (15) reach their maximum or minimum, as the case may be, we have to differentiate with respect to ψ and obtain

$$\left.\begin{aligned} \frac{d\,[a' a']}{d\,\psi} &= -([a\,a] - [b\,b]) \sin 2\psi + 2\,[a\,b] \cos 2\psi = 0 \\ \frac{d\,[b' b']}{d\,\psi} &= +([a\,a] - [b\,b]) \sin 2\psi - 2\,[a\,b] \cos 2\psi = 0 \end{aligned}\right\}, \tag{16}$$

whence there follows

$$\tan\, 2\psi = \frac{2\,[a\,b]}{[a\,a] - [b\,b]} . \tag{17}$$

This equation has two roots of the form 2ψ and $2\psi \pm 180°$, and hence ψ and $\psi \pm 90°$, i.e., the equation (17) determines two directions perpendicular to one another, which correspond to the maximum or the minimum of m_x or m_y, as the case may be. We will determine in the following which one of the two values ψ is to belong to the maximum or minimum of m_x or m_y, as the case may be, and for the present compute further with (17).

We denote the values corresponding to this direction, i.e., the maximum or minimum values of m_x and m_y by μ_1 and μ_2 (hence, without deciding for the time being which one corresponds to the maximum or minimum, as the case may be), and thus we have from (7) and (14) or (15), as the case may be, with respect to the invariance of the denominator D:

$$\left.\begin{aligned} &\text{for } \; x')\quad \mu_1{}^2 = m^2 \frac{[b' b']}{D} = m^2 \frac{[a\,a] \sin^2 \psi + [b\,b] \cos^2 \psi - 2\,[a\,b] \sin \psi \cos \psi}{D}, \\ &\text{for } \; y')\quad \mu_2{}^2 = m^2 \frac{[a' a']}{D} = m^2 \frac{[a\,a] \cos^2 \psi + [b\,b] \sin^2 \psi + 2\,[a\,b] \sin \psi \cos \psi}{D} \end{aligned}\right\}. \tag{18}$$

To substitute in here we write at first with respect to (17):

$$\sin^2 \psi = \frac{1 - \cos 2\psi}{2}, \qquad \cos^2 \psi = \frac{1 + \cos 2\psi}{2} \tag{19}$$

and

$$\cos 2\psi = \frac{1}{\sqrt{1 + \tan^2 2\psi}} = \frac{1}{\sqrt{1 + \dfrac{4\,[a\,b]^2}{([a\,a] - [b\,b])^2}}} = \frac{1}{\sqrt{\dfrac{([a\,a] - [b\,b])^2 + 4\,[a\,b]^2}{([a\,a] - [b\,b])^2}}};$$

by setting

$$\sqrt{([a\,a] - [b\,b])^2 + 4\,[a\,b]^2} = W, \tag{20}$$

we obtain:

$$\cos 2\psi = \frac{[a\,a] - [b\,b]}{W} \tag{21}$$

and hence with (17):

$$\sin 2\psi = \frac{2\,[a\,b]}{W}. \tag{22}$$

Further from (19):

$$\left.\begin{aligned} \sin^2\psi &= \frac{1-\cos 2\psi}{2} = \frac{W-[a\,a]+[b\,b]}{2\,W} \\ \cos^2\psi &= \frac{1+\cos 2\psi}{2} = \frac{W+[a\,a]-[b\,b]}{2\,W} \end{aligned}\right\}. \tag{23}$$

If we set this into the numerators of (18) and collect the expressions, then we find ultimately:

$$[a'\,a'] = \frac{[a\,a]+[b\,b]+W}{2} \qquad [b'\,b'] = \frac{[a\,a]+[b\,b]-W}{2} \tag{24}$$

$$\mu_1{}^2 = m^2\frac{[a\,a]+[b\,b]-W}{2\,D} \qquad \mu_2{}^2 = m^2\frac{[a\,a]+[b\,b]+W}{2\,D}. \tag{25}$$

The sum yields again:

$$\mu_1{}^2 + \mu_2{}^2 = m^2\frac{[a\,a]+[b\,b]}{D} = M^2$$

in agreement with (9), as it is supposed to be.

Now we are to take care of the question of the signs of the roots left unanswered above or, as the case may be, the decision about the direction angle for the maximum and minimum.– If we establish that W in (20) shall always be taken positively and absolutely, then by this the quadrant of 2ψ in (21) and (22) is decided on while we assume that in (17) the quadrant of 2ψ is determined according to the sign of the numerator and denominator as in the case of the common computation of coordinates (Vol. II, 1, 9th Edition, section 35, p. 149). Accordingly, in (25) μ_1 corresponds furthermore to the minimum with the direction angle ψ, and μ_2 to the maximum with the direction angle $\psi \pm 90°$.

In addition, however, we will carry out a change of notation by setting

$$\psi = \Theta - 90° \qquad 2\psi = 2\Theta - 180° \tag{26}$$

$$\tan\ 2\Theta = \frac{-2\,[a\,b]}{-([a\,a]-[b\,b])}, \tag{27}$$

i.e., if ψ was the direction angle of the minimum μ_1, then Θ becomes now the direction angle of the maximum μ_2. If the absolute value of W is considered again, then we have in this connection the following formulae:

$$\tan\ 2\Theta = \frac{-2\,[a\,b]}{-([a\,a]-[b\,b])} \tag{28}$$

$$W = \sqrt{([a\,a]-[b\,b])^2 + 4\,[a\,b]^2} = \frac{-2\,[a\,b]}{\sin 2\Theta} = \frac{-([a\,a]-[b\,b])}{\cos 2\Theta} \tag{29}$$

$$\left.\begin{aligned} m^2_{max} &= m^2 \frac{[a\,a] + [b\,b] + W}{2\,D} = A^2 \text{ direction angle } \Theta \\ m^2_{min} &= m^2 \frac{[a\,a] + [b\,b] - W}{2\,D} = B^2 \text{ direction angle } \Theta \pm 90^\circ \end{aligned}\right\}. \tag{30}$$

Conversely, we can find again therefrom the errors of coordinates m_x and m_y by assuming that the axis of abscissae makes with the maximum value m_{max} the angle $\Theta = \psi + 90°$.

From (7) we have

$$m_x^{\,2} = \frac{[b\,b]}{D} m^2,$$

or in expanded form

$$m_x^2 = \left\{\frac{[a\,a] + [b\,b] + W + [a\,a] + [b\,b] - W}{4\,D} + \frac{[a\,a] - [b\,b]}{2\,W} \;\; \frac{[a\,a] + [b\,b] - W - [a\,a] - [b\,b] - W}{2\,D}\right\} m^2.$$

According to (21) we then have

$$m_x^2 = \left\{\frac{[a\,a] + [b\,b] + W}{2\,D} \;\; \frac{1 - \cos 2\,\psi}{2} + \frac{[a\,a] + [b\,b] - W}{2\,D} \;\; \frac{1 + \cos 2\,\psi}{2}\right\} m^2$$

and according to (23) and (30):

$$m_x^2 = A^2 \sin^2 \psi + B^2 \cos^2 \psi \,.$$

If we introduce in addition $\psi = \Theta - 90°$, then we will have

$$m_x^2 = A^2 \cos^2 \Theta + B^2 \sin^2 \Theta \,. \tag{31}$$

Likewise we obtain:

$$m_y^2 = A^2 \sin^2 \Theta + B^2 \cos^2 \Theta \,. \tag{32}$$

Hence there follows again:

$$A^2 + B^2 = M^2 = m_y^{\,2} + m_x^{\,2}.$$

If we enter the values m_x or m_y resulting from (31) or (32) for the various directions of the axes, then we obtain a closed curve, which is in close relationship to a definite ellipse.

We will pursue this further with Fig. 1. A tangent is laid here on an ellipse with the semiaxes a and b at the point Q, whose coordinates are u and v. The equation of this tangent is from the properties of the ellipse

$$\frac{u}{a^2} x + \frac{v}{b^2} y = 1 \qquad \text{or} \qquad u\,b^2 x + v\,a^2 y = a^2 b^2$$

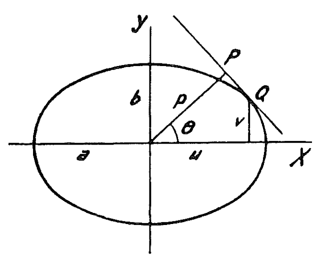


Fig. 1.

and written in a different form

$$\frac{u\,b^2}{\sqrt{u^2 b^4 + v^2 a^4}} x + \frac{v\,a^2}{\sqrt{u^2 b^4 + v^2 a^4}} y = \frac{a^2 b^2}{\sqrt{u^2 b^4 + v^2 a^4}},$$

then we have

$$\frac{a^2 b^2}{\sqrt{u^2 b^4 + v^2 a^4}} = p \tag{33}$$

the perpendicular distance of the tangent from the center of the ellipse.

We have further:

$$\frac{u\, b^2}{\sqrt{u^2 b^4 + v^2 a^4}} = \cos \Theta \,, \qquad \frac{v\, a^2}{\sqrt{u^2 b^4 + v^2 a^4}} = \sin \Theta \,,$$

if Θ denotes the direction angle of p. Hence there follows:

$$\tan \Theta = \frac{a^2}{b^2} \frac{v}{u} . \tag{34}$$

If we add the ellipse equation, which must be satisfied by the coordinates u and v, i.e., the equation

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 , \tag{35}$$

then we can eliminate the coordinates u and v from (33) by means of the equations (34) and (35) and then we obtain an equation between p and Θ. This equation represents the geometrical locus of all points P, i.e., the foot points of the perpendiculars dropped from the center O to arbitrary tangents to the ellipse.

Further computation yields very easily from (34) and (35):

$$u^2 = \frac{a^4}{a^2 + b^2 \tan^2 \Theta} \qquad v^2 = \frac{b^4 \tan^2 \Theta}{a^2 + b^2 \tan^2 \Theta} ,$$

and if we introduce this into (33), then we obtain after simple transformations

$$p^2 = a^2 \cos^2 \Theta + b^2 \sin^2 \Theta . \tag{36}$$

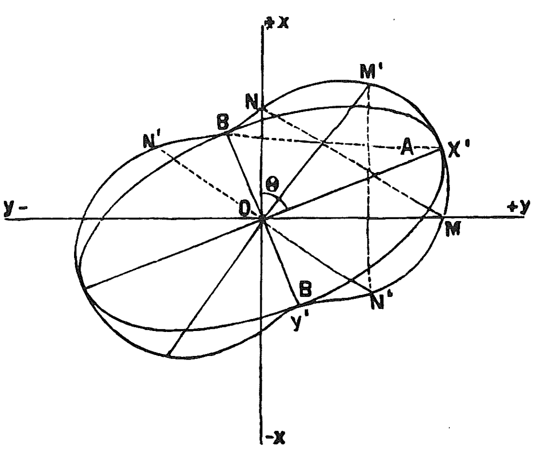


Fig. 2.
Curve of the mean errors and error ellipse.

We refer to the curve corresponding to equation (36) as the foot-point curve of the ellipse in relation to the center of the ellipse.

Now if we replace in (36), p by m_x and a and b by A and B, then equation (36) changes to our previous equation (31). Accordingly, we obtain for $\Theta + 90°$ instead of Θ the equation (32) for m_y. Hence there follows that the mean errors of the point for the various directions are shown by the foot-point curve of an ellipse, and the ellipse itself, whose semiaxes are equal to the maximum or, as the case may be, minimum value of the mean error, is called the *mean error ellipse*.

These relations are represented in Fig. 2; $+x$ and $+y$ correspond to the original system of coordinates, in which the mean errors OM and ON are determined for the point O. If we did have other directions of coordinates, the mean errors would have been computed in these, e.g., OM' and ON'.

Now among all positions of the system of coordinates there is a prominent position $x'y'$, to which the maximum A and the minimum B is assigned for the mean errors.

For a numerical example we use the determination of the point Hochschule by resection of section 99, p. 395.

There we have

$$m = \pm 4.0'' \qquad [a\,a] = +101 \qquad [a\,b] = +48 \qquad [b\,b] = +206.$$

If we introduce this into equations (28) to (30), then we obtain:

$$\tan\ 2\,\Theta = \frac{-2 \times 48}{-(101-206)} = \frac{-96}{+105} = -0.9143\,,$$

and hence

$$2\,\Theta = 317^\circ\,34' \qquad \Theta = 158^\circ\,47'\,.$$

We have further:

$$W^2 = (101-206)^2 + 4 \times 48^2 = +20{,}241 \qquad W = +142.3$$

$$D = +101 \times 206 - 48 \times 48 = +18{,}500 \qquad 2\,D = +37{,}000\,.$$

$$\frac{[a\,a]+[b\,b]+W}{2\,D} = \frac{+449}{37{,}000} = 0.01214$$

$$\frac{[a\,a]+[b\,b]-W}{2\,D} = \frac{+165}{37{,}000} = 0.00446$$

$$A = 4.0\sqrt{0.01214} = \pm\,0.440\ \partial\text{m}$$

$$B = 4.0\sqrt{0.00446} = \pm\,0.267\ \partial\text{m}\,.$$

Therefore, the error ellipse for the point Hochschule is determined by the following values:

$$A = \pm\,0.044 \text{ m}, \qquad B = \pm\,0.027 \text{ m}, \qquad \Theta = 158^\circ\,47'.$$

With these numerical values we have designed Fig. 3 which now offers an illustrative picture of the

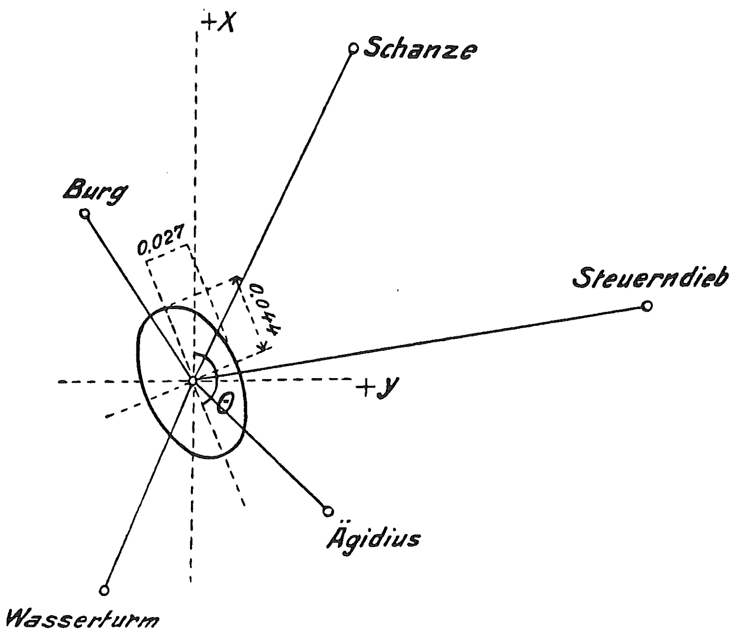


Fig. 3. Error ellipse for the point Hochschule.
Scale for the aiming rays 1:80,000.
Scale for the error ellipse 1:4.

accuracy of the determination of the point Hochschule by resection. For comparison we also have entered the fixed points to which the directions are measured.

Introduction of the weight coefficients

Instead of the coefficients of normal equations $[a\,a]\,[a\,b]\,[b\,b]$ the weight coefficients $[\alpha\alpha]\,[\alpha\beta]\,[\beta\beta]$ can also be used for the determination of the error ellipse. For the transformation we have according to the equations (18) to (22) of section 18, pp. 58-59:

$$[a\,a] = \frac{[\beta\,\beta]}{\Delta}, \qquad [b\,b] = \frac{[\alpha\,\alpha]}{\Delta}, \qquad -[a\,b] = \frac{[\alpha\,\beta]}{\Delta}, \tag{37}$$

where

$$\Delta = [\alpha\,\alpha]\,[\beta\,\beta] - [\alpha\,\beta]\,[\alpha\,\beta] = \frac{1}{D}. \tag{38}$$

If we set this into (28), then there results:

$$\tan\, 2\,\Theta = \frac{2\,[\alpha\,\beta]}{[\alpha\,\alpha] - [\beta\,\beta]} \tag{39}$$

and in the case of setting into W^2 from (29):

$$W^2 = \frac{([\alpha\,\alpha] - [\beta\,\beta])^2 + 4\,[\alpha\,\beta]^2}{\Delta^2} = \frac{Q^2}{\Delta^2}, \tag{40}$$

where Q^2 expresses the numerator, and therefore we have now

$$Q = W\Delta \quad \text{or} \quad Q = \frac{W}{D}. \tag{41}$$

Now with these we have in accordance with (28), (29), (30):

$$\left.\begin{array}{c} \tan\, 2\,\Theta = \dfrac{2\,[\alpha\,\beta]}{[\alpha\,\alpha] - [\beta\,\beta]} \\ Q = \sqrt{([\alpha\,\alpha] - [\beta\,\beta])^2 + 4\,[\alpha\,\beta]^2} = \dfrac{2\,[\alpha\,\beta]}{\sin\, 2\,\Theta} = \dfrac{[\alpha\,\alpha] - [\beta\,\beta]}{\cos\, 2\,\Theta} \\ A^2 = m^2\,\dfrac{[\alpha\,\alpha] + [\beta\,\beta] + Q}{2}; \quad B^2 = m^2\,\dfrac{[\alpha\alpha] + [\beta\beta] - Q}{2} \\ A^2 + B^2 = m^2\,([\alpha\,\alpha] + [\beta\,\beta]) = M^2 \end{array}\right\}. \tag{42}$$

The application to the numerical example of p. 445 yields:

$$[\alpha\alpha] = \frac{1}{[a\,a \cdot 1]} = \frac{1}{p_x} = +0.0111\,; \quad [\beta\,\beta] = \frac{1}{[b\,b \cdot 1]} = \frac{1}{p_y} = +0.00546$$

$$[\alpha\,\beta] = \frac{1}{[a\,b \cdot 1]} = \frac{-[a\,b]}{D} = -0.00259$$

$$\tan\, 2\,\Theta = \frac{-0.00518}{+0.00565} = -0.916\,; \quad 2\,\Theta = 317^\circ\,31'\,; \quad \Theta = 158^\circ\,46'\,; \quad Q = 0.00768$$

$A = 0.440\,\partial$m; $B = 0.267\,\partial$m in agreement with the result of p. 445.

If in (28) to (30) or (42) $[a\,b] = 0$ or, as the case may be, $[\alpha\beta] = 0$, then we will have $2\Theta = 0°$ or $= 180°$, $\Theta = 0°$ or $= 90°$, i.e., the maximum or, as the case may be, minimum directions or the axes of the ellipse coincide with the axes of coordinates X or Y. If $[a\,b] = 0$ and $[a\,a] - [b\,b] = 0$, or, as the case may be, if $[\alpha\beta] = 0$ and $[\alpha\alpha] - [\beta\beta] = 0$, then we will have $\tan 2\Theta = \frac{0}{0}$, i.e., indeterminate, at the same time we will have $W = 0$ and $Q = 0$. In this case the error ellipse changes to a circle with the radius

or

$$\left.\begin{aligned} A = B &= m\sqrt{\frac{1}{[a\,a]}} = m\sqrt{\frac{1}{[b\,b]}} \\ &= m\sqrt{[\beta\beta]} = m\sqrt{[\alpha\,\alpha]} = m_y = m_x \end{aligned}\right\}. \tag{43}$$

This circle is at the same time the foot-point curve, and the mean error is the same in all directions.

In the above we have become acquainted with the error ellipse only by its relationship to the curve of the mean errors, which proved to be the foot-point curve of the error ellipse.

We are led directly to the error ellipse by the consideration of the probability of different positions of points. If we assume that the greatest probability belongs to the point found by the adjustment, then we find that the points of the same probability are grouped around the adjustment point on similar and similarly located ellipses. One of these ellipses, whose semiaxes agree with the maximum or, as the case may be, minimum value of the mean error, forms the *mean error ellipse*, which we have found above in another way. We will return to this in Chapter V.

Auxiliary tables for the computation of the error ellipse

Although the computation of the error ellipse does not give too great difficulties according to the above systems of formulae (28) to (30), pp. 442-443, or (42), p. 446, it can be substantially simplified by the use of auxiliary tables.

If we set in the formulae (28) to (30), pp. 442 and 443,

$$\frac{[b\,b]}{[a\,a]} = \alpha \quad \text{and} \quad \frac{[a\,b]}{[a\,a]} = \beta,$$

then we will have according to (29), p. 442,

$$\frac{W}{[a\,a]} = \sqrt{(\alpha - 1)^2 + 4\beta^2}$$

and according to (4), p. 438,

$$\frac{2D}{[a\,a]} = 2[a\,a]\,(\alpha - \beta^2).$$

Consequently we obtain from (30), p. 443,

$$A^2[a\,a] = \frac{\alpha + 1 + \sqrt{(\alpha - 1)^2 + 4\beta^2}}{2(\alpha - \beta^2)}\,m^2 \tag{44}$$

and in the same way we find

$$B^2[a\,a] = \frac{\alpha + 1 - \sqrt{(\alpha - 1)^2 + 4\beta^2}}{2(\alpha - \beta^2)}\,m^2 \tag{45}$$

and ultimately, equation (28), p. 442, yields

$$\tan\ 2\Theta = \frac{-2\beta}{-(1-\alpha)}. \tag{46}$$

The expressions on the right-hand side in the equations (44) to (46), with the exception of m^2, depend only on the two auxiliary quantities α and β, so that they can be represented in numerical tables or graphical tables. Such a table is indicated by the author in *Zeitschr. f. Math. u. Phys.*, vol. 49, 1903, in Fig. 1 to p. 149, and a table proposed by the Prussian Cadaster Administration is based on the same idea.

The graphical determination of the error ellipse is treated further by Fr. Reinhold in *Zeitschr. f. Verm.*, 1929, pp. 609-614, and by E. Emschermann in *Mitt. a. d. Markscheidewesen*, 1934, pp. 74-78.

Such a table can also be used if the weight coefficients instead of the coefficients of normal equations are to be used for the computation of the error ellipse.

If we set again

$$\frac{[\beta\beta]}{[\alpha\alpha]} = \alpha \quad \text{and} \quad \frac{[\alpha\beta]}{[\alpha\alpha]} = \beta,$$

then we will have for the system of equations (42), p. 446.

$$\left.\begin{aligned} Q &= [\alpha\alpha]\sqrt{(\alpha-1)^2+4\beta^2} \\ \frac{A^2}{[\alpha\alpha]} &= \left(\alpha+1+\sqrt{(\alpha-1)^2+4\beta^2}\right)\frac{m^2}{2} \\ \frac{B^2}{[\alpha\alpha]} &= \left(\alpha+1-\sqrt{(\alpha-1)^2+4\beta^2}\right)\frac{m^2}{2} \\ \tan\ 2\Theta &= \frac{2\beta}{1-\alpha} \end{aligned}\right\}. \tag{47}$$

Section 109. Determination of the Error Ellipse in the Case of Multiple Intercalation of Points

In the case of the double intercalation of points as in section 97 and section 105 we obtain a system of two equations with the two unknowns δx_2 and δy_2 after reducing twice the normal equations. The error ellipse for the point P_2 can then be computed from the coefficients of these two equations according to the above equations (28) to (30), pp. 442 and 443. For the point P_1 we have the simple method of the inversion of the system of the normal equations so that the corrections δx_1 and δy_1 occur as the last unknowns. The reduction of the normal equations done twice yields then two equations for δx_1 and δy_1, whose coefficients can again be used for the determination of the error ellipse for the point P_1.

A second method is opened by the use of the weight coefficients $[\alpha\alpha]$, $[\alpha\beta]$, etc., as we have indicated them also for the two examples in section 97 and in section 105 on p. 385 and on p. 420. For P_1 the values of $[\alpha\alpha]$, $[\beta\beta]$, and $[\alpha\beta]$ can then be used immediately for the above formulae (42), p. 446; for P_2 the coefficients $[\gamma\gamma]$, $[\delta\delta]$, and $[\gamma\delta]$ take their place.

In the case of two points the computation of the weight coefficients does not give great difficulties although the values of $[\alpha\gamma]$, $[\alpha\delta]$, $[\beta\gamma]$, and $[\beta\delta]$ must be computed here also, which are not needed for the remaining part. This is felt to be still more inconvenient if a rather large number of points is to be intercalated at the same time, as we have explained in section 106. Therefore, we will show in addition how we can determine the coefficients $[\alpha\alpha]$, $[\beta\beta]$, and $[\alpha\beta]$ or, as the case may be, the corresponding coefficients of other points independently of the weight equations.

We will limit ourselves to four unknowns $x_1 y_1$, $x_2 y_2$ although the following development holds for an arbitrary number of unknowns.

Using the theories of section 31, p. 100, we compute the weight of a function

$$F = x_1 + y_1 ,$$

for which we obtain according to (3), section 31, p. 100,

$$\frac{1}{P_{x_1+y_1}} = [\alpha\alpha] + 2[\alpha\beta] + [\beta\beta] \,. \tag{1}$$

But on the other hand we can compute the weight $P_{x_1+y_1}$ also according to section 32, p. 102, by setting in (1), p. 102:

$$\left.\begin{matrix} f_1 = 1 & f_2 = 0 & f_3 = 0 & f_4 = 0 \\ f_1' = 0 & f_2' = 1 & f_3' = 0 & f_4' = 0 \end{matrix}\right\} . \tag{2}$$

If we form then, according to (3), p. 102, the function

$$F = X + Y = x_1 + y_1 , \tag{3}$$

then we obtain

$$\left.\begin{aligned} \frac{1}{P_{x_1+y_1}} &= \frac{f_1^2}{[aa]} + \frac{[f_2\cdot 1]^2}{[bb\cdot 1]} + \frac{[f_3\cdot 2]^2}{[cc\cdot 2]} + \frac{[f_4\cdot 3]^2}{[dd\cdot 3]} \\ &+ \frac{f_1'^2}{[aa]} + \frac{[f_2'\cdot 1]^2}{[bb\cdot 1]} + \frac{[f_3'\cdot 2]^2}{[cc\cdot 2]} + \frac{[f_4'\cdot 3]^2}{[dd\cdot 3]} \\ &+ 2\left(\frac{f_1 f_1'}{[aa]} + \frac{[f_2\cdot 1][f_2'\cdot 1]}{[bb\cdot 1]} + \frac{[f_3\cdot 2][f_3'\cdot 2]}{[cc\cdot 2]} + \frac{[f_4\cdot 3][f_4'\cdot 3]}{[dd\cdot 3]}\right) \end{aligned}\right\} . \tag{4}$$

But according to (13), section 31, p. 102, we have

$$\left.\begin{aligned} \frac{1}{P_{x_1}} &= [\alpha\alpha] = \frac{f_1^2}{[aa]} + \frac{[f_2\cdot 1]^2}{[bb\cdot 1]} + \frac{[f_3\cdot 2]^2}{[cc\cdot 2]} + \frac{[f_4\cdot 3]^2}{[dd\cdot 3]} \\ \frac{1}{P_{y_1}} &= [\beta\beta] = \frac{f_1'^2}{[aa]} + \frac{[f_2'\cdot 1]^2}{[bb\cdot 1]} + \frac{[f_3'\cdot 2]^2}{[cc\cdot 2]} + \frac{[f_4'\cdot 3]^2}{[dd\cdot 3]} \end{aligned}\right\} . \tag{5}$$

Therefore it follows by means of (1) and (4) that we must have

$$[\alpha\beta] = \frac{f_1 f_1'}{[aa]} + \frac{[f_2\cdot 1][f_2'\cdot 1]}{[bb\cdot 1]} + \frac{[f_3\cdot 2][f_3'\cdot 2]}{[cc\cdot 2]} + \frac{[f_4\cdot 3][f_4'\cdot 3]}{[dd\cdot 3]} . \tag{6}$$

It is to be noted here that, it is true, according to (2), individual ones of the coefficients $f_1 f_2 \ldots$ or, as the case may be, $f_1' f_2' \ldots$ are equal to zero, but that the reduced coefficients $[f_2 \cdot 1]$, $[f_3 \cdot 2]$, etc., obtain nevertheless definite numerical values.

Therefore, if the weight coefficients $[\alpha\alpha]$ and $[\beta\beta]$ are set up in the form of the equations (5), then we can determine accordingly $[\alpha\beta]$ from (6).

For a second pair of coordinates $x_2 y_2$ we obtain from the equations (2), (5), and (6) accordingly $[\gamma\gamma]$, $[\delta\delta]$, and $[\gamma\delta]$, and we can set up in this way the individual terms of the expressions (5) and (6) for an arbitrary number of points as soon as the reduction of the normal equations is carried out.

As a numerical example for this we will use the intercalation of two points of the previous section 105.

For this we have according to equations (2) and (3)

$$X = x_1\,, \qquad f_1 = 1\,, \qquad f_2 = 0\,, \qquad f_3 = 0\,, \qquad f_4 = 0$$

$$[f_2 \cdot 1] = f_2 - \frac{[a\,b]}{[a\,a]} f_1 \qquad [f_3 \cdot 1] = f_3 - \frac{[a\,c]}{[a\,a]} f_1 \qquad [f_4 \cdot 1] = f_4 - \frac{[a\,d]}{[a\,a]} f_1$$

$$[f_3 \cdot 2] = [f_3 \cdot 1] - \frac{[b\,c \cdot 1]}{[b\,b \cdot 1]} [f_2 \cdot 1] \qquad [f_4 \cdot 2] = [f_4 \cdot 1] - \frac{[b\,d \cdot 1]}{[b\,b \cdot 1]} [f_2 \cdot 1]$$

$$[f_4 \cdot 3] = [f_4 \cdot 2] - \frac{[c\,d \cdot 2]}{[c\,c \cdot 2]} [f_3 \cdot 2]$$

and with the numerical values of section 105

$$[f_2 \cdot 1] = -0.361 \qquad [f_3 \cdot 1] = +0.325 \qquad [f_4 \cdot 1] = +0.101$$
$$[f_3 \cdot 2] = +0.360 \qquad [f_4 \cdot 2] = +0.096$$
$$[f_4 \cdot 3] = +0.256\,.$$

With these we will have

$$[\alpha\,\alpha] = \frac{1}{277} + \frac{(-0.361)^2}{326} + \frac{0.360^2}{199} + \frac{0.256^2}{245} \tag{7}$$

$$[\alpha\alpha] = 0.004928. \tag{8}$$

We have likewise

$$Y = y_1\,, \qquad f_1' = 0\,, \qquad f_2' = 1\,, \qquad f_3' = 0\,, \qquad f_4' = 0$$

$$[f_2' \cdot 1] = 1 \qquad [f_3' \cdot 1] = 0 \qquad [f_4' \cdot 1] = 0$$
$$[f_3' \cdot 2] = -0.096 \qquad [f_4' \cdot 2] = +0.015$$
$$[f_4' \cdot 3] = -0.028$$

$$[\beta\,\beta] = 0 + \frac{1}{326} + \frac{(-0.096)^2}{199} + \frac{(-0.028)^2}{245} \tag{9}$$

$$[\beta\beta] = 0.003117. \tag{10}$$

From (7) and (9) there follows according to (6)

$$[\alpha\,\beta] = 0 - \frac{0.361}{326} - \frac{0.360 \times 0.096}{199} - \frac{0.256 \times 0.028}{245} \tag{11}$$

$$[\alpha\beta] = -\,0.001310. \tag{12}$$

The results (8), (10), and (12) agree sufficiently with the numerical values (15), section 105, p. 420. We take from there also

$$\left.\begin{aligned} [\gamma\gamma] &= +\,0.005845 \\ [\delta\,\delta] &= +\,0.004088 \\ [\gamma\,\delta] &= +\,0.001831 \end{aligned}\right\} \tag{13}$$

and then obtain from the formulae (42), section 108, p. 446,

$$Q_1 = 0.003188 \qquad Q_2 = 0.004062$$
$$A_1{}^2 = 0.005618\, m^2 \qquad A_2{}^2 = 0.006998\, m^2$$
$$B_1{}^2 = 0.002430\, m^2 \qquad B_2{}^2 = 0.002936\, m^2$$

and with $m = \pm 2.1''$

$$A_1 = 0.157\, \partial \mathrm{m} \qquad A_2 = 0.176\, \partial \mathrm{m}$$
$$B_2 = 0.104\, \partial \mathrm{m} \qquad B_2 = 0.114\, \partial \mathrm{m}$$
$$2\,\Theta_1 = 304^\circ\, 31' \qquad 2\,\Theta_2 = 64^\circ\, 22' \; .$$

The two error ellipses are thus determined by the values:

$$\left.\begin{array}{ll} A_1 = 0.016 \text{ m} & A_2 = 0.018 \text{ m} \\ B_1 = 0.010 \text{ m} & B_2 = 0.011 \text{ m} \\ \Theta_1 = 152^\circ\, 16' & \Theta_2 = 32^\circ\, 11' \end{array}\right\} . \tag{14}$$

According to these, the two error ellipses in Fig. 1 have been drawn. If we aim to compare the error ellipse for P_1, Hochschule, with the error ellipse resulting from the resection of Fig. 3, section 108, p. 445, then it is to be noted that there the mean error of a direction was equal to ± 4.0" while the present Fig. 1 is based on the mean error ± 2.1".

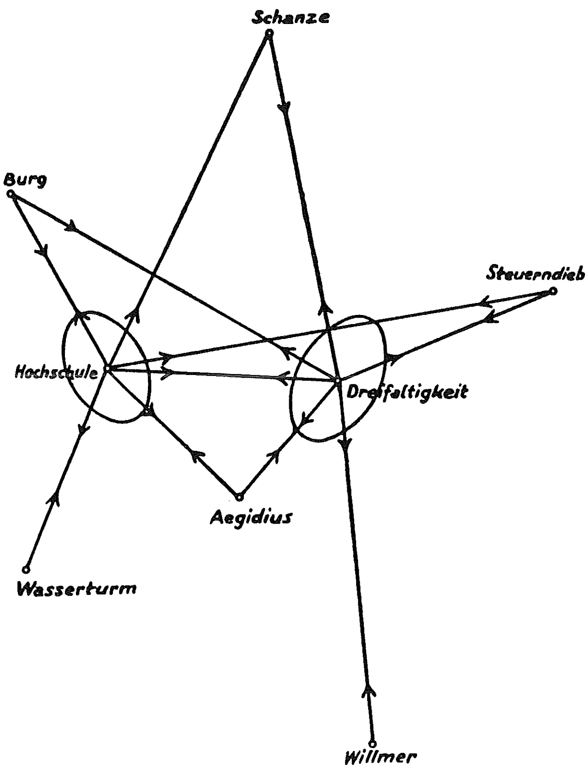


Fig. 1.
Scale for the aiming rays 1:80,000.
Scale for the error ellipses 1:2.

Therefore, we must imagine the dimensions of the error ellipse of p. 445 to be reduced by about one half. If we take this into account, then we see that by adding the outer directions and by the connection with the point P_2, Dreifaltigkeit, the error ellipse for Hochschule, in substance, has retained the same form and position, but that its two axes are reduced by about two thirds.

Error ellipse in the case of adjustment according to conditioned observations

In addition, we will consider the case that the computation of coordinates of triangulation points is carried out following a net adjustment according to conditioned observations. In this connection, we can refer to the general theory of section 51.

Let the elements adjusted according to conditioned observations be x_1, x_2, x_3 . . . and let the coordinates X and Y for a point be obtained therefrom in the form

$$X = f_1 x_1 + f_2 x_2 + f_3 x_3 + \dots, \qquad Y = f_1' x_1 + f_2' x_2 + f_3' x_3 + \dots. \tag{15}$$

Then we have according to section 46, p. 145,

$$\left.\begin{aligned} \frac{1}{P_x} &= [f f] - \frac{[a f]^2}{[a a]} - \frac{[b f \cdot 1]^2}{[b b \cdot 1]} - \dots \\ \frac{1}{P_y} &= [f' f'] - \frac{[a f']^2}{[a a]} - \frac{[b f' \cdot 1]^2}{[b b \cdot 1]} - \dots \end{aligned}\right\}. \tag{16}$$

Now we form the weight of the function

$$F = X + Y$$

and have for this according to (5), section 51, p. 163,

$$\frac{1}{P_{x+y}} = [(f + f')^2] - \frac{[a f + a f']^2}{[a a]} - \frac{[(b f + b f') \cdot 1]^2}{[b b \cdot 1]} - \dots. \tag{17}$$

According to (6), section 51, p. 163, we have in here

$$[(f + f')^2] = [f f] + 2 [f f'] + [f' f']$$

and likewise

$$\begin{aligned} [a f + a f']^2 &= [a f]^2 + 2 [a f] [a f'] + [a f']^2 \\ [(b f + b f') \cdot 1]^2 &= [b f \cdot 1]^2 + 2 [b f \cdot 1] [b f' \cdot 1] + [b f']^2 . \end{aligned}$$

Thus (17) changes to

$$\begin{aligned} \frac{1}{P_{x+y}} = [f f] &- \frac{[a f]^2}{[a a]} - \frac{[b f \cdot 1]^2}{[b b \cdot 1]} - \dots \\ + [f' f'] &- \frac{[a f']^2}{[a a]} - \frac{[b f' \cdot 1]^2}{[b b \cdot 1]} - \dots \\ + 2 \Big([f f'] &- \frac{[a f] [a f']}{[a a]} - \frac{[b f \cdot 1] [b f' \cdot 1]}{[b b \cdot 1]} - \dots \Big). \end{aligned} \tag{18}$$

On the other hand, we can imagine the two coordinates X and Y to be expressed directly by the original measured quantities l_1, l_2, l_3 . . . and then we would obtain

$$X = \alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \dots$$
$$Y = \beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3 + \dots$$

so that

$$\frac{1}{P_x} = [\alpha\alpha], \qquad \frac{1}{P_y} = [\beta\beta]. \tag{19}$$

Furthermore we have according to (3), section 31, p. 100,

$$\frac{1}{P_{x+y}} = [\alpha\alpha] + [\beta\beta] + 2[\alpha\beta]. \tag{20}$$

From (16), (18), (19), and (20) there follows then

$$[\alpha\beta] = [ff'] - \frac{[af][af']}{[aa]} - \frac{[bf\cdot 1][bf'\cdot 1]}{[bb\cdot 1]} - \dots. \tag{21}$$

Hence, we can again compute the coefficient $[\alpha\beta]$ according to (21) if the weight reciprocals $[\alpha\alpha]$ and $[\beta\beta]$ are set up in the form of the expressions (16), and the quantities required for the determination of the error ellipse are thus known.

An application of the above theory is carried out in connection with the adjustment of the base net of Schwerd according to conditioned observations in the former 3rd Edition of this Volume 1888 in section 125, pp. 358-361.

Literature on the error ellipse

Andrä, "Fehlerbestimmung bei der Auflösung der Pothenotschen Aufgabe mit dem Messtische," *Astronom. Nachr.*, Nr. 1117, Bd. 47, 1858, pp. 193-202.

Andrä, *Danske Gradmaaling*, Band I, Kopenhagen, 1867 (cf. Helmert, *Vierteljahrsschrift der Astronomischen Gesellschaft*, 12. Jahrg., 1877, pp. 184-209).

Helmert, "Studien über rationelle Vermessungen im Gebiete der höheren Geodäsie," *Zeitschr. f. Mathematik und Physik* (Schlömilch), 1868, pp. 73 and following.

Helmert, *Ausgleichungsrechnung nach der M. d. kl. Q.*, 2. Aufl., Leipzig, 1907, pp. 303-327.

Czuber, *Theorie der Beobachtungsfehler*, Leipzig, 1891, pp. 382-399.

Section 110. The Rules of Schreiber

The method of least squares was first applied to triangulation of lower order (second and third order) in a wholesale manner, i.e., to hundreds and thousands of cases of the current trigonometrical practice of the state by General Schreiber, as the Chief of the trigonometrical division of the Prussian Land Survey in 1876. In so doing, the question of the measurements attached to fixed directions with unknowns of orientation z was also treated mathematically. An autographed treatise designed for the internal use of the trigonometrical division of 8 Sept. 1877 *Rechnungsvorschriften für die trigonometrische Abteilung der Landesaufnahme, Ausgleichung und Berechnung der Triangulation zweiter Ordnung* was handed to us by the author Schreiber for publication in the work, Jordan-Steppe's *Deutsches Vermessungswesen*, 1882, I, pp. 151-164. On pp. 157-159 there is treated the adjustment according to indirect observations with reduced error equations in short theorems and rules, as we give in the following in a brief excerpt:

The reduced error equations are by no means true error equations with a simple geometrical meaning, but they are to be understood merely as assumed computational expressions whose meaning is limited to the indicated property, and which have no further purpose than that of a shortening of the computation. For this reason, there are no objections in introducing *negative* (imaginary) weights.

The formation of the reduced error equations rests on the following theorems:

Theorem 1. If the unknown z occurs in each of the ν error equations

$$(1) = -z + a_1 x + b_1 y + \ldots + n_1 \quad \text{weight} = 1$$
$$(2) = -z + a_2 x + b_2 y + \ldots + n_2 \quad \text{weight} = 1, \text{ etc.},$$

but in none of the remaining ones, then we can set down the following reduced error equations instead of those ν error equations which contain z:

$$0 = a_1 x + b_1 y + \ldots + n_1 \quad \text{weight} = 1$$
$$0 = a_2 x + b_2 y + \ldots + n_2 \quad \text{weight} = 1, \text{ etc.}$$
$$0 = (a)x + (b)y + \ldots + (n) \quad \text{weight} = -\frac{1}{\nu},$$

where the last equation means the sum of the ones above it.

From the given ν error equations there results the complete normal equation for z, namely:

$$0 = \nu z - (a)x - (b)y - \ldots - (n).$$

Since the right-hand side of this equation is equal to the sum of the ν corrections (1), (2) . . . with opposite signs, hence there follows the theorem:

At every station the sum of the corrections $(1) + (2) + \ldots$ is equal to zero, of which we shall make applications later on. One proves this theorem, just as the following ones, simply by a check; for the second group of equations must yield the same normal equations as the first one after elimination of z.

Theorem 2. The error equations

$$(1) = ax + by + \ldots + n_1 \quad \text{weight} = p_1$$
$$(2) = ax + by + \ldots + n_2 \quad \text{weight} = p_2, \text{ etc.},$$

which vary from one another only in the constant terms and the weights, can be replaced by the one:

$$0 = ax + by + \ldots + \frac{p_1 n_1 + p_2 n_2 + \ldots}{p_1 + p_2 + \ldots} \quad \text{weight} = p_1 + p_2 + \ldots$$

Theorem 3. Instead of the error equation

$$(1) = ax + by + \ldots + n \quad \text{weight} = p$$

the following can be set down:

$$0 = qax + qby + \ldots + qn \quad \text{weight} = \frac{p}{q^2}.$$

From these general theorems there result the following special ones, namely: theorem 4 from 1 and 2, theorem 5 and 6 from 2 and 3.

Theorem 4. If the unknown z occurs only in the two error equations

$$(1) = -z \qquad\qquad \text{weight} = 1$$
$$(2) = -z + ax + by + \ldots + n \quad \text{weight} = 1$$

then we can put for them the reduced one:

$$0 = ax + by + \ldots + n \quad \text{weight} = \frac{1}{2}.$$

Theorem 5. The two error equations

$$(1) = ax + by + \ldots + n_1 \quad \text{weight} = 1$$
$$(2) = ax + by + \ldots + n_2 \quad \text{weight} = 1,$$

which differ from one another only in the constant term, can be replaced by the one:

$$0 = 2ax + 2by + \ldots + n_1 + n_2 \quad \text{weight} = \frac{1}{2}.$$

Theorem 6. The two error equations

$$(1) = ax + by + \ldots + n_1 \quad \text{weight} = 1$$
$$(2) = ax + by + \ldots + n_1 \quad \text{weight} = \frac{1}{2},$$

which differ from the previous ones only in the weight of the second equation, can be replaced by the one:

$$0 = 3ax + 3by + \ldots + 2n_1 + n_2 \quad \text{weight} = \frac{1}{6}.$$

But instead of applying these six theorems one after the other, we can carry out at once what they require gradually; i.e., the theorems yield also the following ones:

Mechanical rules for the formation of the reduced error equations

Rule 1. Two directions 1 and 2 opposite one another yield:

$$0 = (1) + (2) \ldots \quad \text{weight} = \frac{1}{2},$$

except when 1 or 2 is the only indefinite direction of a definitive station, in which case we have (2 regarded as only indefinite direction of its station):

Rule 1a: $$0 = 2(1) + (2) \ldots \quad \text{weight} = \frac{1}{6}.$$

Rule 2: Every one-sided direction 1 yields:

$$0 = (1) \ldots \quad \text{weight} = 1,$$

except when it is the only indefinite direction of the station, in which case we have:

Rule 2a: $$0 = (1) \ldots \quad \text{weight} = \frac{1}{2}.$$

Rule 3: Besides, each station of ν directions 1, 2, 3 . . . yields

$$0 = (1) + (2) + (3) \ldots \quad \text{weight} = -\frac{1}{\nu}.$$

On the definitive stations the number ν is to be understood inclusive of the direction of attachment. If, in addition to the latter, only just *one* direction exists here, then the above equation is omitted.

As for the proofs of these theorems and rules, we can take them from our various sections 97, 101, and 105 besides the earlier sections 28 and 33; there is to be noted, however, that the various questions about the computation of the mean errors are not touched on in the theorems of Schreiber.

A numerical example for the application and practice of these rules was given in connection with Fig. 1, where G and N mean a pair of points to be newly determined and the remaining signs mean fixed points; thus we have:

	Inward directions	Outward directions
(G)	1, 2, 3, 4,	8, 10,
(N)	5, 6, 7,	9, 11, 12,

together 12 measured directions.

The adjustment may in substance have to take the same course as our double intercalation of points Hochschule-Dreifaltigkeit in section 103 or section 105.

We have applied rule 3 in section 103 on the Stations Hochschule and Dreifaltigkeit, but rule 1 for reciprocal measurements, which can be applied 5 times in Fig. 1, was not used in sections 103 and 105 because the connection of the station measurements is thereby broken. In general, these theorems and rules of Schreiber are not instructions which *must* be applied, but they are *forms* of mathematical ideas, which can be expressed and applied also in other ways.

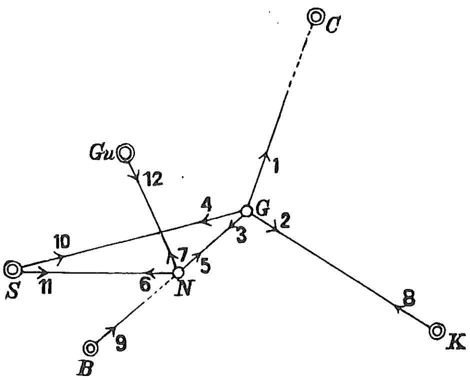


Fig. 1.
Scale 1:800,000.

Especially there is to be noted that the weight $\frac{1}{2}$ in the case of rule 2a presupposes that the new ray is tied only to *one* old fixed ray. The rays 10 and 11 have *one* common z with a connecting direction at S, whereby the two error equations for 10 and 11 become *dependent* on one another (whereas in section 103 we assumed such two rays independent).

The example of Fig. 1 of Schreiber is treated in the original treatise and in the reprint (Jordan-Steppe, 1882, pp. 153-163) precisely according to Schreiber's rules, but in the former edition of this book, *Handb. d. Verm.*, 3. Aufl., 1888, I, pp. 167-171, the same example is computed partly in a different way; namely in order not to have to break up the groups of stations, rule 1 is left unused, which has changed nothing in the final result, however.

Also our example section 103 or, as the case may be, 105 has been computed on several occasions according to Schreiber's rules; but it is to be noted at once that these rules rest on partly *different assumptions* than our two forms of sections 103 and 105. For in section 103 we have introduced at Steuerndieb, Ägidius, and Burg the new directions to Hochschule and Dreifaltigkeit as *independent* (with the weight $\frac{1}{2}$ as in the case of Schreiber), which neglects rule 3 for simplification. On the other hand, our adjustment section 105 is formally just as rigorous as Schreiber's rules, but it makes the assumption that only full sets with the inclusion of *all* fixed attaching rays are measured whereas Schreiber's rules assume only *one* fixed attaching ray in each case.

The computational execution of our example of section 105 according to Schreiber's rules would be fashioned in the following way:

We put down all error equations for the 20 directions measured, without any regard to the z's. Then we form:

$$\left.\begin{array}{r} v_1 \text{ with } p = 1 \\ v_{13} \text{ with } p = 1 \end{array}\right\} \quad \text{One-sided directions (rule 2),}$$

$$\left.\begin{array}{r} 2v_7 + v_{15} \text{ with } p = 1{:}6 \\ 2v_{11} + v_{12} \text{ with } p = 1{:}6 \\ 2v_5 + v_{18} \text{ with } p = 1{:}6 \end{array}\right\} \quad \text{Reciprocal inward and outward } \textit{single} \text{ direction (rule 1a),}$$

$v_8 + v_{17}$ with $p = 1:2$ $v_9 + v_3$ with $p = 1:2$ $v_{10} + v_{19}$ with $p = 1:2$ $v_2 + v_{14}$ with $p = 1:2$ $v_4 + v_{16}$ with $p = 1:2$ $v_6 + v_{20}$ with $p = 1:2$	Reciprocal directions, inward and inward, or inward and outward multiple direction (rule 1),
$v_1 + v_2 + v_3 + v_4 + v_5 + v_6$ with $p = -1:6$ $v_7 + v_8 + v_9 + v_{10} + v_{11}$ with $p = -1:5$	Inward sets of directions (rule 3),
$v_{13} + v_{14}$ with $p = -1:3$ $v_{19} + v_{20}$ with $p = -1:3$ $v_{16} + v_{17}$ with $p = -1:3$	Sets of outward directions with more than *one* new ray (rule 3 with assumption of *one* fixed ray).

These 16 equations were put down (without the z's) as if they were independent error equations with the sole unknowns δx_1, δy_1, δx_2, δy_2 and with the indicated (partially negative) weights. The pertinent normal equations $[p\,a\,a]\,\delta x_1 \ldots$ are formed and solved whereby the coordinates are determined.

The computational execution, which is not given here because of the space available, furnishes the final coordinates with a deviation of only a few millimeters from the coordinates of sections 103 and 105. It is certainly not the question of the most rigorous computational procedure which occupies us here, but the question whether in the case of one or the other arrangement the unavoidable omissions are the result of the inner comprehension of all circumstances, or only of practical necessity. The former holds true in the case of Schreiber's rules of 1877 and also in the case of our adjustments of sections 103 and 105.

Section 111. Various Numerical Questions

The numerical calculation plays always an important role in the case of extended computations of adjustment, and in the case of intercalations of points with adjustment of coordinates we can greatly facilitate the computation by some small artifices, or conversely, render the computation difficult by awkward arrangement, as in general in the method of least squares.

The main point is always to arrange the error equations in such a way that we have to deal with *small* numerical values only and furthermore with numerical values as *equal* as possible; then in the case of only *two* unknowns δx, δy the whole adjustment, inclusive of elimination, etc., can always be done easily with the *slide rule*, which is quite an extraordinary facilitation in comparison to logarithmic computation, to the computation with Crelle's Table of products [Produktentafel] and the like.

We already have referred to this several times and in the case of all practical examples, especially p. 376 and p. 395, we always have computed in the meantime in this manner.

Concerning this there belongs, above all, the introduction of the corrections of coordinates in decimeters, where we can often write down directly the squares and products of the coefficients; e.g., let us consider the error equations (9), p. 401, the first of which has the coefficients + 13 and + 25, while another corresponding computation, mentioned on p. 401, has the coefficients + 128 and + 254.

Now we may believe at first glance that the rounding off of 128 to 13 and 254 to 25 is at the expense of the necessary accuracy, but this is not the case for two reasons. First, experience teaches us in hundreds of cases (cf. section 113 following) that in the third to fourth order in general one- to two-place coefficients suffice, and second, there exists a theoretical justification in the case of such adjustments to treat the error equations and normal equations even *less* rigorously than we will later carry on the coordinates, direction angles, and distances.

For according to the investigations of section 40, p. 126, the accuracy of the unknowns changes only very little in the neighborhood of the maximum even though the coordinates deviate 0.1 to 0.2 of the mean error from their most favorable values.

So if we aim to compute more rigorously, nevertheless, for *formal* reasons, e.g., to an accuracy of 1 mm in the case of precise city surveys, or to an accuracy of 1 cm in the field, then this is a question independent to a certain degree of the above one.

By this, these intersection adjustments according to indirect observations differ quite considerably from the net adjustments according to conditioned observations (e.g., section 70), for in the case of those nets it is necessary to carry the adjustment itself *at least* as rigorously (usually even more rigorously because of rounding off errors) as we wish to have the results in agreement for formal reasons later, whereas this is not necessary in the case of intersection adjustments.

In short, in the case of adjustment computations of the kind considered here (especially also in sections 113 and 114), we regard the putting down of the coefficients a, b with only one- to two-place values, as they appear by themselves in the case of the computation of δx, δy in decimeters without decimals at the right-hand side of the decimal point, as fully sufficient and therefore recommendable.

Let us refer here to a discussion by Fr. Schulze in *Zeitschr. f. Verm.*, 1904, p. 20, treating also this question.

We would like to touch here briefly still another question of form, namely the checks referring to the sum $[v\,v]$, which are also admitted into the official Prussian Survey instruction under the name "Sigmaproben" [sigma checks].

In the case of this check, it is a question of computing the difference

$$[l\,l] - [v\,v] = \Sigma \tag{1}$$

in different ways, for which we have according to (23) and (24), section 95, p. 373,

$$\Sigma = \frac{[a\,l]^2}{[a\,a]} + \frac{[b\,l\cdot 1]^2}{[b\,b\cdot 1]} \tag{2}$$

and

$$\Sigma = [a\,l]\,\delta x + [b\,l]\,\delta y\,. \tag{3}$$

If we compute the quantity Σ according to (2) and (3) and compare these values with the value computed from the corrections v according to (1), then the agreement yields a check for the correctness of the adjustment.

However, the check in this form is not always absolutely reliable; e.g., if we have good measurements but bad approximate values of the unknowns, then $[v\,v]$ will become very small, but $[l\,l]$ and thereby also Σ very large, and from the relative agreement of the different Σ's we cannot conclude as yet to the correctness of $[v\,v]$. If we have, e.g., $[v\,v] = 65$ and $[l\,l] = 2239$, then we will have according to (1) $\Sigma_1 = 2174$. On the other hand, let there be found according to (2) $\Sigma_2 = 2114$, so that the difference of the two Σ's is equal to 60 or approximately 2.8% of Σ, and we could be satisfied with this agreement. However, if we use the value Σ_2 found according to (2) in connection with $[l\,l]$ for the computation of $[v\,v]$, then we obtain in the present case $[v\,v] = 125$, i.e., a difference of 60 or of more than 90% of the directly found value $[v\,v] = 65$, which is not satisfactory. In the case of this proof, it is thus necessary that we pay attention repeatedly to the agreement of the different values of $[v\,v]$.

Intersection adjustments in the new division

If direction measurements in the *new* angle division are present, then the coefficients a, b and also the absolute terms l become approximately three times as large as for the old [sexagesimal] division, i.e., at the ratio $1{:}0.324 = 3.0864$, which yields the square $1{:}0.324^2 = 1{:}0.1050 = 9.526$, so that the a^2's, $a\,b$'s, etc., become almost ten times larger than in the case of the old division.

Therefore, various questions arise in regard to the most convenient choice of computational units, which we will approach by converting the former example of p. 376 into the new division, however with the small change that from the outset the angles are rounded off to 1 cc; and hence 0.1 cc shall no longer be carried. Thus we obtain from p. 376:

α	(φ)	$(\varphi) - \alpha = -l$	l^2
288 g 04 c 17 cc	288 g 04 c 16 cc	− 1 cc	1
350 04 71	350 04 66	− 5	25
22 90 43	22 90 33	− 10	100
165 63 34	165 63 40	+ 6	36
165 cc	155 cc	− 10 cc	162

Then by means of the auxiliary table, pp. [16] to [17] of the Appendix:

(φ)	ξ	η	s	a	b	ab	a^2	b^2	a^2+b^2
288.0	+ 62.5	− 11.9	4.91 *km*	+ 12.7	− 2.4	− 30	161	6	167
350.0	+ 45.0	+ 45.0	2.04	+ 22.0	+ 22.0	+ 484	484	484	968
22.9	− 22.4	+ 59.6	2.35	− 9.5	+ 25.3	− 240	90	640	730
165.6	− 32.7	− 54.6	2.20	− 14.8	− 24.8	+ 367	219	615	834
						+ 581	954	1745	2699

We can check the $a^2 + b^2$'s by multiplying the data of the auxiliary table, p. [18], of the Appendix, which holds for the old division, by 9.526 (it would be better to compute such a table as p. [18] also for the new division).

Then by computing also the $-al$'s and $-bl$'s (which we do not put down here individually), we obtain the elimination, corresponding to p. 376 (or corresponding to the general instruction of pp. 63 and 64):

	a	b	$-l$	b	a	$-l$	
	+ 954	+ 581	− 116	+ 1745	+ 581	− 509	
		+ 1745	− 509		+ 954	− 116	
		− 354	+ 71		− 193	+ 170	
			+ 162			+ 162	
			− 14			− 148	
$\delta y = +0.31\ \partial$m		+ 1391	− 438		+ 761	+ 54	$\delta x = -0.07\ \partial$m
$= +0.031$ m			+ 148			+ 14	$= -0.007$ m
			− 138			− 4	
		$[ll \cdot 2] =$	+ 10		$[ll \cdot 2] =$	+ 10	

$$m = \sqrt{\frac{10}{2}} = \pm 2.2 \text{ cc}.$$

The thus obtained δy's and δx's are in agreement within 1 mm with the former ones of p. 376, and also the whole remaining part agrees within the rounding off limits observed.

In the above, the coefficients are still unnecessarily large, however; we will diminish them by assuming now, in the case of the angles, with 10 cc as unit instead of 1 cc as unit; by so doing, also the a's and b's become accordingly smaller, and the elimination is formed thusly:

	a	b	$-l$	b	a	$-l$	
	+ 9.5	+ 5.8	− 1.16	+ 17.5	+ 5.8	− 5.09	
		+ 17.5	− 5.09		+ 9.5	− 1.16	
		− 3.5	+ 0.71		− 1.9	+ 1.69	
			+ 1.62			+ 1.62	
			− 0.14			− 1.48	
$\delta y = +0.31\ \partial$m		+ 14.0	− 4.38		+ 7.6	+ 0.53	$\delta x = -0.07\ \partial$m
			+ 1.48			+ 0.14	
			− 1.37			− 0.04	
		$[ll \cdot 2] =$	+ 0.11		$[ll \cdot 2] =$	+ 0.10	

$$m = \sqrt{\frac{0.11}{2}} = \pm 0.23 \quad \text{units of} \quad 10 \text{ cc} = \pm 2.3 \text{ cc}.$$

These test-and-comparison computations are to be made if we aim to determine the most convenient units for such small adjustments. Incidentally, the above example is not convenient as an illustrating example, because the remainder term $[ll \cdot 2]$ becomes very small.

Whereas in the case of computations in the *old* division, 1" and 1 ∂m have resulted as the best measuring units from our computational experiences for third to fourth order, i.e., with ray lengths, say, 1-5 km (forms, pp. 376 and 395), the matter is somewhat different, at any rate, in the case of computations in the *new* division; we would regard here 10 cc and 1 ∂m as suitable, but we cannot speak here from long experience. The "Vorschriften, betreffend die Erhaltung und Fortfuhrung des Katasters" [instructions regarding the maintenance and continuation of the cadaster] of Württemberg of 1895 contain computational forms which hold in the old and the new division for 1 ∂m or, as the case may be, 1'' and 1 cc as units (cf. *Zeitschr. f. Verm.*, 1895, p. 284).

He who has some practice with the slide rule, can use the computational checks of pp. 376 and 395 or 397; if this should not suffice, for instance, then we could compute the a's and b's according to p. 371, once logarithmically and then check them according to the method of p. 376; we can always apply the check with $a^2 + b^2$ according to the auxiliary table, p. [18] of the Appendix besides, without setting up a special column in the forms for it.

At the end of all these discussions about small questions of form, we must emphasize expressly that only he who himself has computed in practice for a long time will attach importance to such small matters, but that, conversely, these small matters by all means play a role in the case of setting up computation forms which are to be used many hundred and thousand times.

Section 112. Errors of Coordinates of the Given Points

In the case of the considerations to date we have assumed that the coordinates of the given points are *free of errors*, which is not the case in reality.— Ordinarily, we can say approximately, however, that the errors of the coordinates obtained from repeated measurements and an adjustment based on them are small in comparison to the errors of the direct measurements. Now such an assumption may even prove true more or less, yet, as a rule, one computes in this way; but from a practical viewpoint we must bear in mind here the thereby changed sense of such adjustments, or better: intercalations.

In the following we will show how we can take into account the errors of the given points in the case of the adjustment by the introduction of weights.

1. *Resection with directions*

The mean errors of the given coordinates will show in the preliminary direction angles, in the case of the adjustment. If we assume at first that the coordinates of the given points contain the true errors f_x and f_y, then an error f_φ results hence according to (11), section 93, p. 359, for the preliminary direction angle (φ), and we have

$$f_\varphi = + \frac{\sin\varphi}{s} \varrho f_x - \frac{\cos\varphi}{s} \varrho f_y .$$

For the measured direction r we assume further the true error f_r. Now since the following expression holds for the absolute term $-l$ of the error equation according to (9) and (10), section 99, p. 391:

$$-l = (\varphi) - r - (z),$$

then the true error of the absolute term is

$$-f_l = + \frac{\sin\varphi}{s} \varrho f_x - \frac{\cos\varphi}{s} \varrho f_y - f_r . \tag{1}$$

If we aim to pass to mean errors, then there is to be noted that the coordinates x and y of the given point are not independent of one another. And hence we would have to go back to the measurements from which these coordinates have been computed. Such a procedure, however, can never come up in practice;

therefore, we will limit ourselves to an approximation and treat the coordinates x and y as though they are quantities measured directly. In this case, nothing remains to be done but to assume the same mean errors for both coordinates x and y, which we will denote by $\pm c$. This assumption corresponds at the same time to a circular error ellipse of the point. If we assume further the mean direction error equal to $\pm \delta$, then the mean error of the absolute term $-l$ will be

$$M^2 = \frac{\sin^2 \varphi}{s^2} \varrho^2 c^2 + \frac{\cos^2 \varphi}{s^2} \varrho^2 c^2 + \delta^2$$

or

$$M^2 = \left(\frac{c}{s} \varrho\right)^2 + \delta^2. \tag{2}$$

The direction weights to be introduced into the adjustment would have to be assumed inversely proportional to the thus determined M^2's; e.g., if we set simply $P = \frac{1}{M^2}$ and after the adjustment compute the mean error of the unit of weight from the sum of squares $[p v v]$, then we must have $M^2 = 1$ if c and δ are chosen correctly. Otherwise, we would have to repeat the adjustment with new values of c and δ.

2. *Intersection with directions*

In the case of the determination of a point by intersection we have to distinguish between the directions measured at the station to other fixed points and to new points.

Retaining the denotations of the previous section 97, p. 379, we have the following error equation for the direction to a fixed point P_1:

$$v_1 = -r_1 - (z) + \varphi_1 - \delta z,$$

or if we set $-r_1 - (z) + \varphi_1 = -l_1$,

$$v_1 = -l_1 - \delta z.$$

If the true errors in the coordinates of the fixed point P_1 are equal to f_{x_1} and f_{y_1} and those of the station point equal to f_x and f_y, then we have according to (13), section 93, p. 360,

$$f_{\varphi_1} = + \frac{\sin \varphi_1}{s_1} \varrho f_x - \frac{\cos \varphi_1}{s_1} \varrho f_y - \frac{\sin \varphi_1}{s_1} \varrho f_{x_1} + \frac{\cos \varphi_1}{s_1} \varrho f_{y_1} \tag{3}$$

and consequently, the error of the absolute term $-l_1$ will be

$$-f_{l_1} = + \frac{\sin \varphi_1}{s_1} \varrho f_x - \frac{\cos \varphi_1}{s_1} \varrho f_y - \frac{\sin \varphi_1}{s_1} \varrho f_{x_1} + \frac{\cos \varphi_1}{s_1} \varrho f_{y_1} - f_{r_1} .$$

Then we obtain for the mean error

$$M^2 = 2 \left(\frac{c}{s_1} \varrho\right)^2 + \delta^2. \tag{4}$$

For the directions to new points only the errors of coordinates of the station point is involved, and since the signs of the coefficients of f_x and f_y do not affect the results of the squaring in the case of the passage to the mean error, then we obtain for the mean error the same expression as in the case of resection, and hence

$$M^2 = \left(\frac{c}{s}\varrho\right)^2 + \delta^2 . \tag{5}$$

With (4) and (5) the mean errors and with these also the weights for all directions of a station can be set up.

Now we will in addition treat the case in which at a station only *one* new point has been aimed at so that the final orientation can be determined from the station data [Abriss] (section 96). If n connecting directions are present, then the oriented direction to the new point is according to (1), section 96, p. 377,

$$\alpha = \frac{1}{n}\left(\varphi_1 + \varphi_2 + \ldots + \varphi_n\right) - \frac{1}{n}\left(r_1 + r_2 + \ldots + r_n\right) + r$$

and if the preliminary direction angle to the new point is denoted by (φ), then we have

$$-l = (\varphi) - \alpha$$

or

$$\begin{aligned} -l = &-\frac{1}{n}\left(\varphi_1 + \varphi_2 + \ldots + \varphi_n\right) + (\varphi) \\ &+ \frac{1}{n}\left(r_1 + r_2 + \ldots + r_n\right) - r . \end{aligned} \tag{6}$$

The true errors of the direction angles $\varphi_1\, \varphi_2 \ldots \varphi_n$ have the form (3), while the true error of (φ) is expressed by

$$f_{(\varphi)} = + \frac{\sin\varphi}{s}\varrho f_x - \frac{\cos\varphi}{s}\varrho f_y .$$

With this, the true error of the absolute term $-l$ in (6) will be

$$\begin{aligned} -f_l = -\frac{1}{n}\Bigg\{ &\frac{\sin\varphi_1}{s_1}\varrho f_x - \frac{\cos\varphi_1}{s_1}\varrho f_y - \frac{\sin\varphi_1}{s_1}\varrho f_{x_1} + \frac{\cos\varphi_1}{s_1}\varrho f_{y_1} \\ &+ \frac{\sin\varphi_2}{s_2}\varrho f_x - \frac{\cos\varphi_2}{s_2}\varrho f_y - \frac{\sin\varphi_2}{s_2}\varrho f_{x_2} + \frac{\cos\varphi_2}{s_2}\varrho f_{y_2} \\ &\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ &+ \frac{\sin\varphi_n}{s_n}\varrho f_x - \frac{\cos\varphi_n}{s_n}\varrho f_y - \frac{\sin\varphi_n}{s_n}\varrho f_{x_n} + \frac{\cos\varphi_n}{s_n}\varrho f_{y_n} \Bigg\} \\ &+ \frac{\sin\varphi}{s}\varrho f_x - \frac{\cos\varphi}{s}\varrho f_y \\ &+ \frac{1}{n}\left(f_{r_1} + f_{r_2} + \ldots + f_{r_n}\right) - f_r \end{aligned}$$

and if we pass to mean errors, then we obtain

$$\begin{aligned} M^2 = &\varrho^2\left(\frac{1}{n}\left[\frac{\sin\varphi}{s}\right] + \frac{\sin\varphi}{s}\right)^2 c^2 \\ &+ \varrho^2\left(\frac{1}{n}\left[\frac{\cos\varphi}{s}\right] + \frac{\cos\varphi}{s}\right)^2 c^2 \\ &+ \varrho^2\left[\frac{1}{s^2}\right] c^2 + \frac{n+1}{n}\delta^2 . \end{aligned} \tag{7}$$

The sums refer here to the terms with the indices $1 \ldots n$.

According to (7), a mean error and hence a weight is to be computed for each station.

We have treated the above questions more out of a theoretical interest, without thinking of a practical application of the formulae found. Such an application will always meet with difficulties, because rarely there is known anything about the inaccuracies lying in the coordinates of the given points. Here it is not a question of the mean errors of coordinates which result for the given points from all the measurements of the triangulation net in its various orders with the inclusion of the astronomical orientation, but only of the mean errors in the reciprocal position of the given points. Because of the difficulties arising here we forego the calculation of a numerical example.

From the literature about the consideration of the errors of coordinates of the given points we mention:

W. Láska, "Über den Einfluss der Ungenauigkeit gegebener Punkte auf das Resultat des Vorwärtseinschneidens," *Zeitschr. f. Verm.*, 1900, p. 557.

Fr. Schulze, "Über die Grösse des mittleren Punktfehlers bei den drei Methoden des Einschneidens," *Zeitschr. f. Verm.*, 1906, pp. 585 and 601.

P. Werkmeister, "Einfluss von Fehlern in den Koordinaten der Festpunkte auf die Koordinaten des Neupunktes bei trigonometrischen Punktbestimmungen durch Einschneiden," *Österr. Zeitschr. f. Verm.*, 1915, p. 165.

F. Ackerl, "Über den Einfluss fehlerhafter Festpunkte auf das Ergebnis des Vorwärtseinschneidens," *Zeitschr. f. Verm.*, 1930, p. 41.

J. M. Tienstra, "Over den invloed van de fouten in de gegeven punten op het gewicht van de richtingen bij de puntsbepaling," *Tijdschr. v. Kad. en Landmeetk.*, 1934.

Section 113. Example of a Triangulation. Triangulation of the City of Hannover

In connection with the sections 105 and 106 of Volume II, 1 of this *Handbuch*, 9th Edition, 1931, in the following we shall show the triangulation of the city of Hannover carried out by the author as an example for the computational procedure of a triangulation.

After the basis for the city triangulation was created with the triangulation net of p. 236, with the station data [Abriss], p. 250, and the double intercalation of points of pp. 414 and 415, it was only a question of determining a great number of further points by intersection with outward and inward directions.

We will show this further on the basis of the picture of the triangulation net of p. 469 by showing again in it at first the previous net of p. 236 and p. 409 at a larger scale. This may hold as of third order, while the further points to be intercalated, which are drawn on p. 469 only as points without sighting connections, shall hold further as of fourth order.

However, we must add to it what is not included in the system discussed thus far and what is not discussed as yet.

First, on p. 469 as a central system we have around Linden Wasserturm a small, special net which was already laid out for the city survey of Linden (left of the Leine River) in 1887 and has already been described in our volume II, 1, 9th Ed., 1931, section 91.

Second, the net of p. 409, it is true, has sufficed for the city proper of Hannover, but not for the northwestern suburbs Leinhausen, etc., and further points of the land survey had to be brought in there, namely Velber, Vinnhorst, Stelingen, and Bothfeld I, of which only Velber and Bothfeld I, however, are represented on p. 469 itself. Nevertheless it is easily understood that with this the connection Kunst, Leinhausen, Tannenkamp was established and everything else was secured.

In the system Celle of the cadastral survey now we thus have the six main points with their coordinates on p. 250, then the points Hochschule and Dreifaltigkeit intercalated in conjunction in (1) and (1a), p. 415, and the four points, borrowed from the Land Survey:

Velber Pyr. of second order	$L = 27° 18' 45.5791''$	$B = 52° 22' 47.0583''$	(1)
Vinnhorst flagpole	27 22 38.3058	52 25 30.5902	
Stelingen Pyr. of second order	27 18 32.3673	52 26 53.6458	
Bothfeld I	27 25 55.0686	52 24 14.8752	

(Cf. in this connection the picture of the net in the following section 114.)

The point Bothfeld I (in the northeast) would not have been needed, but since it falls into our territory and was already determined by the Land Survey, it also was taken as a fixed connection. Now the above geographic coordinates were converted into the cadastral system (p. 235) with the zero point Celle and thereby in the case of Vinnhorst an auxiliary point was brought in, in addition:

$$\left.\begin{array}{lrr} \text{Velber Pyr. of second order} & y = -29679.509 \text{ m} & x = -27281.934 \text{ m} \\ \text{Vinnhorst flagpole} & -25252.053 & -22252.373 \\ \text{Vinnhorst chimney lightning rod} & -25221.908 & -22208.093 \\ \text{Stelingen Pyr. of second order} & -29883.047 & -19659.388 \\ \text{Bothfeld I} & -21544.773 & -24610.116 \end{array}\right\} \qquad (2)$$

In the case of the pyramids these coordinates hold true for the stone center below; the apex of the pyramid was determined separately in each case by a plumb line, which will not be indicated here (cf. Vol. II, 1, 9th Ed., 1931, section 87).

The following table yields the computational procedure and the proofs of accuracy for all points which have been determined by intersection in the years 1892 and 1893. These are 117 − 3 = 114 points, which are

Intersected Point	Number of Outward Directions 1	Inward 2	$N = n - u$ 1	2	$[v^2]$ 1	2	$m = \sqrt{\frac{[v^2]}{N}}$ 1	2	m_y	m_x	Mean Distance
1. Hoftheater	7	7		11		53		± 2.2″	± 6*mm*	± 12*mm*	2.6
2. Marktturm	8		6		14		± 1.5″		4	3	2.3
3. Kreuzturm	7		5		80		4.0		27	10	2.6
4. Neustädterturm	8		6		32		2.3		12	6	2.5
5. Waterloosäule	6		4		135		5.8		31	9	1.8
6. Christusturm	8		6		16		1.8		11	5	3.6
7. Schlachthaus	6		4		61		3.9		29	26	3.1
8. Gartenturm	8		6		35		2.4		10	7	2.4
9. Paulusturm	7		5		16		1.8		9	17	2.8
10. Apostelturm	7		5		68		3.7		15	13	2.2
11. Marienturm	8		6		50		2.9		15	11	2.7
12. Langelaube	5		3		39		3.6		19	14	1.3
13. Synagoge	5		3		39		3.6		20	10	1.4
14. Lyzeum II	3		1		13		3.6		20	16	1.3
15. Welfenkaserne	7		5		39		2.8		13	12	2.2
16. Kaserne VIII	6		4		117		5.4		30	20	2.4
17. Leibniz-Realgymn.	7		5		45		3.0		10	9	2.1
19. Brandkasse		9		6		156		5.1	11	5	1.0
20. Ägid. Torplatz	1	4		2		50		5.0	5	5	0.3
21. Hildesh. Rampe		5		2		11		2.3	13	14	1.9
22. Schneller Graben		7		4		19		2.2	9	12	1.7
23. Wiesenweg		9		6		162		5.2	11	23	1.4
24. Masch		9		6		188		5.6	19	30	0.9
25. Engesohde		10		7		86		3.5	18	13	1.0
26. Schleuse		9		6		82		3.7	29	25	1.6
27. Landwehrwiesen		7		4		16		2.0	6	6	1.9
28. Langensalzastr.	9		7		86		5.3		4	8	1.2
29. Emmerberg	6		4		67		4.1		6	2	1.0
30. Haarstrasse	4		2		233		10.8		18	10	0.6
31. Hildesheimerstr.	5		3		300		10.0		5	4	0.8
32. Bischofsholer Damm		6		3		20		2.6	23	5	1.6
33. Grosse Bult	1	6		4		100		5.0	10	17	1.2
34. Eilenriede		5		2		24		3.5	16	32	2.0
35. Kasseler Eisenbahn	1	6		4		25		2.5	16	29	1.7
36. Steinweg	2	8		7		96		3.7	7	11	1.5
37. Haspelfeld		7		4		58		3.8	10	22	1.7
38. Henriettenstift I.	9		7		40		2.4		10	6	1.4
39. Henriettenstift II.	13		11		99		3.0		10	7	1.6
40. Blank Schornstein	11		9		234		5.1		13	7	1.3
41. Dieterich.	9		7		235		5.8		14	11	1.3
42. Dieterich. Schorn.	7		5		106		4.6		12	9	1.2
43. Bürgerl. Bräuhaus	8		6		182		5.5		5	17	1.2
44. Städt. Brauerei I	13		11		286		5.1		4	8	1.2
45. Städt. Brauerei II	6		4		52		3.6		3	12	1.1
46. Ziegeleischornstein	14		12		539		6.7		10	8	1.4
47. Geibelstrasse	9		7		161		4.8		10	13	1.4
48. Friedhof		5		2		39		4.4	11	16	1.0
49. Döhrener Mühlweg		7		4		121		5.5	7	8	0.9
50. Geibelstrasse Stein		6		3		9		1.7	1	3	0.5
51. Grosse Barlinge	1	6		4		276		9.6	13	24	0.6
52. Questenhorst		9		6		434		8.5	25	30	0.8
53. Lehmweg		8		5		65		3.6	8	13	1.3
54. Kinderheilanstalt	5		3		35		3.4		12	16	1.0
55. Wasserstation NW	7		5		54		3.3		10	17	1.5
56. Wasserstation SE	7		5		22		2.1		6	10	1.5
58. Heidornstrasse		10		7		530		8.7	6	5	0.8
59. Bethlehem Kap.	4		2		1		0.7		10	5	1.1
60. Kleine Düvelstrasse		7		4		100		5.0	6	3	0.4
61. Kunst	5	8		10		78		2.8	17	16	2.9
62. Palmenhaus (18)	5	9		11		44		2.0	5	14	3.0
63. Zentralheizung	6	12		15		163		3.3	14	13	2.9
64. Leinhausen Pfeiler		11		8		72		3.0	40	24	3.7
65. Tannenkamp Pyr.	4	11		12		183		3.9	15	26	2.8

Intersected Point	Number of Outward Directions 1	Number of Inward Directions 2	$N = n - u$ 1	$N = n - u$ 2	$[v^2]$ 1	$[v^2]$ 2	$m = \sqrt{\frac{[v^2]}{N}}$ 1	$m = \sqrt{\frac{[v^2]}{N}}$ 2	m_y	m_z	Mean Distance
66. Hainholzturm	9		7		296		± 6.5″		± 23*mm*	± 24 *mm*	2.7
67. Fuchsberg		9		6		106		± 4.2″	22	19	2.5
68. Heide		7		4		190		6.9	26	28	1.9
69. Stüh		7		4		27		2.6	8	6	1.5
70. Lokomotivschuppen	6		4		27		2.6		14	6	1.5
71. **Melanchtonstrasse**		9		6		144		4.9	15	16	2.8
72. Rübekamp	1	9		7		123		4.2	8	5	1.4
73. **Oppen. Fahne**	7		5		22		2.1		9	7	1.9
74. **Oppen. Schornst.**	9		7		55		2.8		10	8	1.7
75. Hainhölzer Rampe		7		4		18		2.1	4	3	1.2
76. Neues Land	1	9		7		235		5.8	20	19	1.6
77. Sandgrube	1	10		8		251		5.6	21	11	1.7
78. Tasch Schornstein	10		8		162		4.5		10	8	1.4
79. Vorderschönworth	2	7		6		294		7.0	21	39	0.9
80. Fuchsgarten		8		5		146		5.4	43	16	2.0
81. Kolonnenbrücke	3	10		10		116		3.4	9	11	1.5
82. **Fuhrenkamp**	1	7		5		130		5.1	5	12	1.9
83. **Kolonnenweg**	1	8		6		50		2.9	12	6	1.9
84. Mühlenfeld		8		5		174		5.9	25	22	1.8
85. **Biesterstrasse**	1	6		4		41		3.2	7	40	1.4
86. Lister Kirchweg	1	6		4		433		10.4	20	78	1.3
87. Feldhagen	2	8		7		175		5.0	13	9	1.5
88. **Langfeld II**		8		5		68		3.7	13	14	1.8
89. **Langfeld I**	1	10		8		242		5.5	8	11	1.1
90. VahrenwalderTurm	8		6		294		7.0		11	7	0.6
91. Listerfeld	1	5		3		39		3.6	8	19	1.3
92. Herrman, Giebel	6		4		184		6.8		6	20	0.8
93. Schiedsmann	5		3		294		9.9		6	16	0.5
94. **Grosse Roh.**	1	6		4		14		1.9	4	9	1.8
95. Rangierrampe		8		5		274		7.4	35	23	1.3
96. Wasserbehälter		6		3		23		2.8	10	6	1.8
97. Eisenb. Vinnhorst		7		4		46		3.4	9	9	1.5
98. Burgbrücke	1	12		10		302		5.5	11	10	1.5
100. Strangriede	1	8		6		156		5.1	18	11	1.5
101. Simonturm	6		4		64		4.0		9	13	1.3
102. Herrenh. Allee		5		2		125		7.9	16	12	1.6
103. Entenfang	1	11		9		795		9.4	20	17	1.2
104. Kirchhof Herrenh.		5		2		58		5.4	37	60	2.5
105. Asphaltschornstein	5		3		63		4.6		52	20	2.4
106. Spargelpflanzung		9		6		127		4.6	14	37	2.3
107. Brauereischornstein	11		9		604		8.2		19	14	1.4
108. Westerfeld	1	9		7		426		7.8	12	12	2.0
109. Leinestein 25 km	2	9		8		328		6.4	14	19	1.6
110. Friedhofkuppel	10		8		269		5.8		16	11	1.9
111. Stöckener Friedhof		7		4		180		6.7	44	13	2.2
112. Ziegeleischornstein	11		9		590		8.1		7	15	1.4
113. Blutegelanstalt		8		5		361		8.5	28	48	2.3
114. Dunstmoor	1	6		4		21		2.3	5	4	1.4
115. Bahnhof Herrenh.	10		8		722		9.5		11	11	0.9
116. Zementschornstein	8		6		195		5.7		21	32	2.8
117. Aufsatzschornstein Stöcken	7		5		266		7.5		30	14	1.3
Sums from 1 to 117	437	497	280	362	7638	9498	232.2	302.4	1653	1711	1871
18, 57, 99 are missing; therefore, there are altogether 114 points $n = 114$.	934		642		17136		534.6		3364		

also drawn on the net picture, p. 469, and, in fact, the high points (in general only intersected) with names and numbers, the ground points (in general only resected) only with numbers. To draw the sighting lines themselves at the small scale 1:40,000 of our net picture for all these points would be completely impossible (there are 934 sighting lines), because by so doing only a confused cobweblike formation would be created.

The numbers indicate the succession of the computations of intersection so that, for instance, the point 15, Welfenkaserne, can only be computed through points whose numbers are smaller than 15. Moreover, the eight third-order points have a number in brackets in front here, e.g., [1] Ägidius, [2] Wasserturm, etc., and the point 1, Hoftheater, for its part, is thus adjusted only through such points numbered in brackets, namely through all seven third-order points, with the exception of Burg, and hence with seven outward and seven inward directions.

All these 114 intersections of points are computed according to the forms on pp. 376 and 395 (all with the slide rule); the adjustments for intersection *and* resection (outward *and* inward directions) are made according to the pattern of section 101, where the outer directions are introduced with half the weight. A special pattern is not needed for such combined intersections; we took for this simply a sheet of the pattern on p. 376 *and* a sheet on p. 395, which latter, with a small change of the printed form, serves at the same time for the purpose of combining.

According to this, the table will be comprehensible, and now we will combine its final sums as mean values.

In all we have 114 points since of the numbering 1-117 the numbers 18, 57, 99 are omitted. To this there belong 934 directions; and hence, on the average there are approximately eight directions to one point.

Distributed according to forward directions and backward directions we have:

Only intersected	50 points with 388 directions	
Only resected	35 points with 256 directions	(3)
Intersected and resected	29 points with 290 directions	
Total	114 points with 934 directions	

The number of the directions to one point is therefore for these three kinds 7.8, 7.3, and 10.0, or fairly equal; the intersected *and* resected points have the most directions, as is easily understood. Point 63, Zentralheizung, with $6 + 12 = 18$ directions (p. 464) has the largest number of rays.

In 1894 the triangulation experienced another continuation with 24 points which, with a total of 23 outward and 180 inward directions, are adjusted in the same manner as are the former 114 points. And hence we have together:

From 1891 to 1893	114 points with 437 + 497 = 934 directions
From 1894 to 1893	24 points with 23 + 180 = 203 directions
Total 1891 to 1894	138 points with 460 + 677 = 1137 directions.

Now a point has 8.2 directions on the average.

Then if we count to this the 20 points of Linden (left of the Ihme River, p. 469, then we have a total of 158 points with approximately 1300 directions.

The sum of the squares $[v^2]$, which have remained in the case of the individual adjustments are not equivalent because half the weight is given all outward directions, and for this reason in the column of the $[v^2]$'s, the amounts originating from the outward directions under 1 and the amounts originating from the inward directions under 2 are kept apart; in the case of intersection *and* resection, half the weight is already taken into account according to the example of p. 404, and hence the result $[v^2]$ is to be put under 2. The case is similar with the individual denominators $n - u = N$, which lead in each individual case to the determination of the mean direction error m which, however, has again a different meaning in the fifth column whereas the mean errors of coordinates m_y and m_x are homogeneous.

In order to determine a mean error m from all 934 measured outward and inward directions of the table (pp. 464-465), now we have to proceed thusly (cf. p. 465):

Outward directions	$[v^2] = 7638$ with $[N] = 280$		(4)
Inward directions	$[v^2] = 9498$ with $[N] = 362$		(5)
Outward directions with half the weight	$[v^2] = 3819$ with $[N] = 280$		(6)
Total from (5) and (6)	13317	642,	

and hence, mean error of a direction of the weight 1, i.e., of an inward direction here:

$$m = \sqrt{\frac{13317}{642}} = \pm 4.56'' . \tag{7}$$

But now we can also compute the mean error for the three kinds of intersections separately, and in fact we have made for this purpose the following special arrangement from the table, pp. 464-465:

Outward Directions			Outward and Inward Directions			Inward Directions			
n	N	$[v^2]$	n	N	$[v^2]$	n	N	$[v^2]$	
50	280	7638	29	203	5239	35	159	4259	(8)
$m_1 = \sqrt{\frac{7638}{280}}$			$m_{1.2} = \sqrt{\frac{5239}{203}}$			$m_2 = \sqrt{\frac{4259}{159}}$			
$m_1 = \pm 5.22''$			$m_{1.2} = \pm 5.08''$			$m_2 = \pm 5.18''$.			

The thus obtained values m_1 and m_2 are nearly equal, while according to our assumption m_1 pertains to half the weight and m_1 should therefore have $\sqrt{2} = 1.4$ times the amount of m_2.— Or in short, our assumption of (2), p. 399, that all outward directions shall have *half* the weight in comparison with the inward directions is *not* confirmed in the sum of these 114 cases according to the practical execution.

In many cases, as, e.g., Nos. 61, 62, 63, 65, etc., the outward directions were tied by far to *more* than *one* each old fixed direction, and if we set on the average $s = 7$ for such points with numerous attachments, then we have according to section 96, p. 378, $P = \frac{7}{8} = 0.88$, which is nearer to 1 than to 0.5. And hence, the rounding off to 1 would have been better for such outward directions. However, in the many cases with numerous inward and only one to two outward directions, which, e.g., are frequent under Nos. 82-87, p. 465 (there are in all about 24 such cases), the outward directions, usually only about 0.5 km long and measured for the connection with a neighboring point, were of low value. For these directions and also in other cases (cf., e.g., the net in the following section 114, p. 476) the rounding off of the outward weight to 0.5 would thus be appropriate.

Therefore, if there are numerous attachment rays for the outward directions, then it is to be preferred to round off their weight to $P = 1$; if this is not the case and the orientation is uncertain, then the rounding off to 0.5 is appropriate.— However, we are to refer here again to the theoretical considerations in sections 96, 101, and 105.

And hence, since we have the mean error of a direction as simply $m = \pm 5''$ according to (7) and (8), must we also consider how much of it is to be accounted to pure measuring errors and how much to centerings and to the continued accumulation of errors in the case of the numerous intercalation and arbitrary insertion of eight points up to ultimately 114 points? The measurements for the 114 points are all made with a small Wanschaff 13 cm microscope theodolite which has been illustrated and described in our Volume II, *Handb. d. Verm.*, 8th Ed., 1914, p. 255. The performance can be taken, approximately, corresponding to the computation (1), p. 207, for a similar Bamberg instrument, according to which the mean error of a direction in a set would be taken equal to 2''; and hence, for four sets only equal to 1''. However, first, one does not measure so exactly in fourth order for city points, morally, so to speak, as in the case of third order for a fundamental net as p. 236; then the lightning conductors, flagpoles, etc., produce already such considerable disturbances at short distances that the measuring error of the individual stations is not to be estimated below 2''–3''. Then if by the hundreds of attachments to old fixed points, which count as *free of error* in the adjustments of intersection while they are affected, however, with the sum of errors of all previous intercalations (cf. section 112) – if all this has not raised the mean remaining error a posteriori higher than 5'', then the work of our 114 base points can be considered good.

We can compute a mean angle error also in a rougher way; i.e., according to the summation in the table on pp. 464-465 the rough average value of all values m is equal to $534.6 : 114 = 4.7''$, and if we multiply this by the coefficient 1.25 (for Gauss' law of errors), then we have 5.9'', and hence again nearly the same as before.

We will collect, at first, also the mean coordinate errors m_y and m_x in this rough average computation, namely according to the tabular summation of pp. 464-465:

$$\frac{1653}{114} = 14.5 \text{ mm} \qquad \text{and} \qquad \frac{1711}{114} = 15.0 \text{ mm},$$

in the mean 14.8 mm for y and x, and hence we have the mean error of coordinates

$$m_k = m_y = m_x = 1.25 \cdot 14.8 = 18.6 \text{ mm}. \tag{9}$$

If we form the square mean for m_y and m_x, then we obtain $[m_y^2] = 34{,}076$ and $[m_x^2] = 40{,}763$, and with these the better values

$$m_y = \pm\, 17.3 \text{ mm}, \qquad m_x = \pm\, 18.8 \text{ mm},$$

and hence the mean error of coordinates

$$m_k = m_y = m_x = \pm\, 18 \text{ mm} \tag{9*}$$

and the mean error of points

$$M_p = 18\sqrt{2} = \pm 25.4 \text{ mm.} \tag{10}$$

If we take further the limit error equal to three times the mean error, then with $3 \times 25.4 = 76$ mm we can assume the limit error in any case within $= 1\ \partial$m and the traverse lines must join on in the worst case *within* this amount, disregarding the errors of their own.

Finally our table, pp. 464-465, gives also the mean distances of the individual adjustments, whose total mean becomes $1871 : 114 = 1.65$ km, while the points among themselves have much smaller distances according to the view of the picture of the triangulation net, p. 469, namely below 1 km. If, nevertheless, the aiming range is considerably larger on the average, then this is caused by the peculiar form of a *city* triangulation net, where very closely neighboring points among themselves can often not be seen together, but each one must be resected independently from long sightings to distant towers (cf. about the layout in general, our Volume II, 1, 9th Ed., 1931, sections 85-91, where also the coordinates of a part of our points are indicated). The number of the points is very large especially on the southeastern part of the picture of p. 469, e.g., the points 51, 52, 53 have intervals of only 300 meters approximately. (We cannot discuss here the special reasons of the close accumulation of points in this section.)

To this, we will make some further considerations.

If we introduce mean errors in equation (5), section 104, p. 417, and, as before, take the errors of coordinates all equal, and hence $m_y = m_x = m_k$, then we have:

$$m_s = \pm \frac{y_2 - y_1}{s} m_k \pm \frac{y_2 - y_1}{s} m_k \pm \frac{x_2 - x_1}{s} m_k \pm \frac{x_2 - x_1}{s} m_k$$

and according to the law of errors of section 5

$$m_s = \pm m_k \sqrt{\frac{2\,(y_2 - y_1)^2}{s^2} + \frac{2\,(x_2 - x_1)^2}{s^2}} = m_k \sqrt{2}\,. \tag{11}$$

According to this, we have as a final consideration for the mean ray length of 1.65 km with the application to (10):

$$m_s = M_p = m_k \sqrt{2} = \pm 25.4 \text{ mm.}$$

For the main net Fig. 1, p. 236, there resulted in section 72, p. 255, as mean error for a length of 1 km: ± 7 mm, and hence for 1.65 km: ± 12 mm. The intercalation of two points of Fig. 2, p. 409, yielded in (9), section 104, p. 417, for 2.41 km ± 18 mm, or for 1.65 km likewise ± 12 mm. According to (11), this would yield $m_k = \frac{12}{\sqrt{2}} = \pm 8.5$ mm. The mean error of coordinates for the intercalation of two points is on the average ± 15 mm according to (3), section 104, p. 416. According to this, we will have to assume the uncertainty of coordinates of the fundamental net for the mean ray length of the intercalations of points at approximately ± 1 cm.

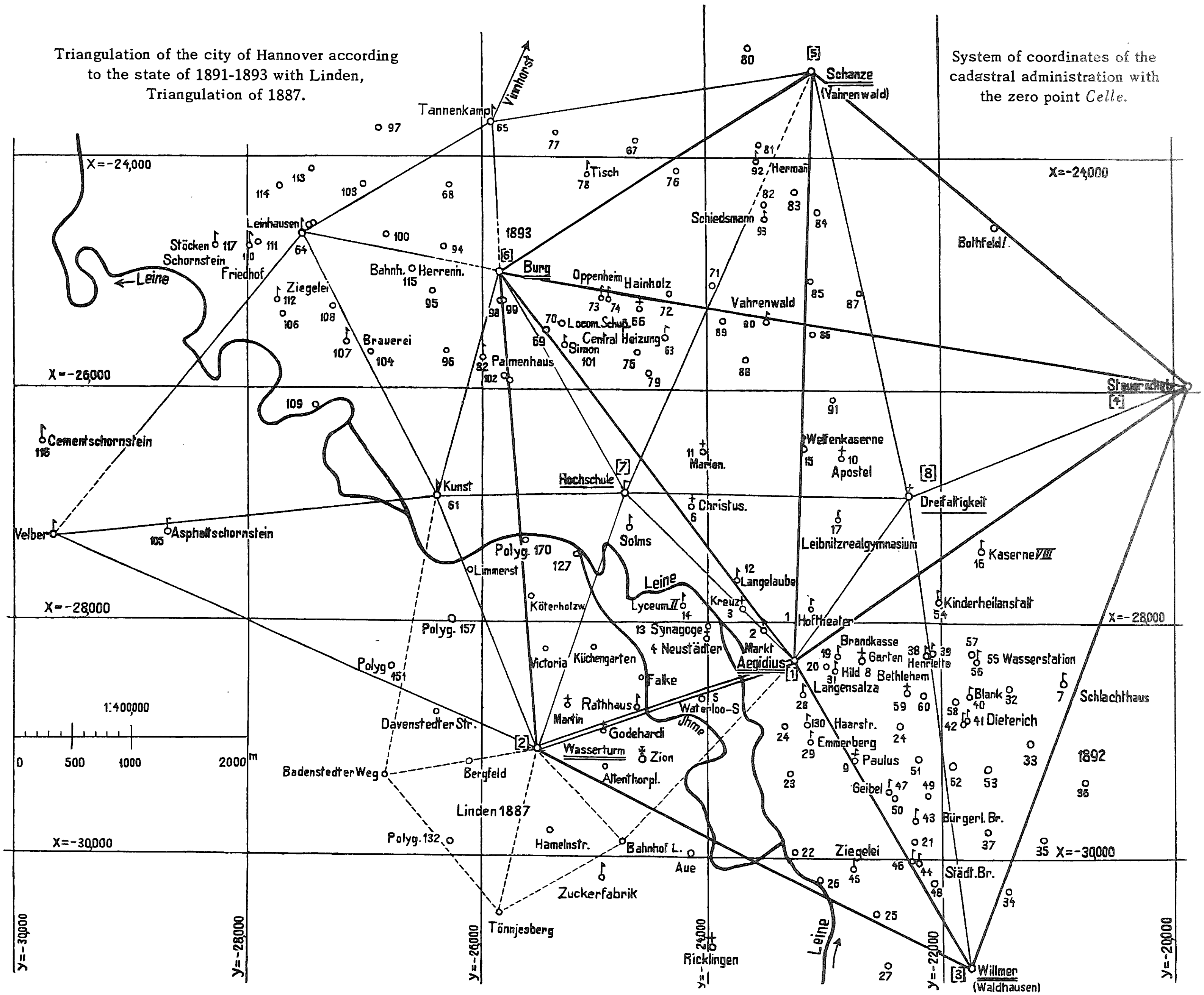
The following share in the total direction error which appears in the case of the intercalation of points, according to (8), p. 466, $m = \pm 5''$: the pure measuring errors (according to p. 467 estimated at $2'' - 3''$), the errors arising by centering uncertainties (variations of the designation of the point, also by regular pointing errors, etc.), and the part added in the case of the continuous intercalation of points.

If a trigonometric net offers a sufficient number of closed triangles (or polygons), then a mean angle error and hence a mean direction error can be formed from their closing errors, as already shown in section 10 and as we shall treat later again in section 116. The latter is independent of the errors of coordinates or intercalations; but it contains the pure measuring errors and the uncertainty of centering, so that, if the measuring errors, for instance, were determined according to section 63, a conclusion can hence be drawn as to the portion of the individual errors in the total error.

In *Zeitschr. f. Verm.*, 1892, p. 452 (Bemerkungen über Kleintriangulierungen [Notations on small triangulations]), some small triangulations have been investigated in this manner and it was found that in the

Triangulation of the city of Hannover according to the state of 1891-1893 with Linden, Triangulation of 1887.

System of coordinates of the cadastral administration with the zero point *Celle*.



(Text to this, cf. section 113, p. 469.)

case of these, the measuring errors with the centering uncertainties on the one hand and the coordinate or, as the case may be, intercalation errors on the other, share in the same manner in the occurrence of the total error.

For instance, if we assumed the same ratio for the city net of Hannover, then of the error squared $5^2 = 25$ indicated in (8), p. 466, half with 12.5 and $\sqrt{12.5} = \pm 3.5''$ would therefore fall to intercalation or, as the case may be, coordinate errors, and 6.25 each with $\sqrt{6.25} = \pm 2.5''$ to the centering uncertainty and pure measuring error, which latter corresponds to the estimate $2''$-$3''$ of p. 467.

Instead of computing the mean direction error of the net for each point intercalation individually (e.g., section 96, p. 376, section 99, p. 395, section 101, p. 403) we can derive the v^2's also in summary for a whole net after completion of the computational operations by comparison of the adjusted and of the observed directions for all base points, where $[v]$ becomes equal to zero for each base point. The divisor N for the thus obtained $[v^2]$'s is then $N =$ the number of all observed directions minus the number of the base points (orientations z) minus $2 \times$ the number of the new points (δx, δy). (Cf. to this Reinhertz, *Die Verbindungs-triangulation zwischen dem Rheinischen Dreiecksnetze der Europäischen Gradmessung und der Triangulierung des Dortmunder Kohlenreviers der Landesaufnahme* [The connecting triangulation between the Rhinish triangulation net of the European degree-measurement and the triangulation of the coal region of Dortmund of the Land Survey], Stuttgart, 1889, p. 43.)

For comparison with the triangulation of Hannover treated above, let us mention also some results of other city triangulations:

Berlin, Zeitschr. f. Verm., 1891, pp. 393-394, communication by v. Höegh, in addition also the first-order net, *Zeitschr. f. Verm.*, 1881, p. 12.

Strassburg, Zeitschr. f. Verm., 1893, p. 131, communication by Rodenbusch.

Leipzig, Zeitschr. f. Verm., 1895, p. 108, communication by Händel.

All these undertakings which knew nothing of one another have yielded nearly the same final results, where also the conditions of instruments, etc., referred to at the places indicated, are to be taken into account.

As for the forms of adjustment, Berlin and Strassburg are done according to the forms of the Prussian Instruction IX. Leipzig used the calculating machine with corresponding forms. Hannover is treated according to pp. 376 and 395 with the use of the slide rule.

From all these data we have formed the following summary table:

City	Order	Number of Points	Mean No. of Rays for One Point	Mean Length of a Ray	Mean Error of a Direction	Mean Point Error
Hannover 1891—1893	III	114	8	1.6 km	± 5″	± 25 mm
Berlin	II	40	22	3.4	3.5	24
	III	304	8	1.8	4.8	25
	IV	211	6	0.9	6.1	15
Strassburg	II	12	10	7.5	1.0	23
	III	57	8	2.6	2.5	20
	IV	250	—	0.7	6.3	16
Leipzig	II	37	5	4.4	0.6	11
	III	179	9	1.4	2.5	7
	IV	298	6	1.0	2.3	10
	V	243	6	0.9	4.2	9
Sum		1745	88	26.2 km	34.3″	185 mm
Mean			9	2.4 km	± 3″	± 17 mm

We shall add some further remarks about the working procedure in connection with Volume II, 1, 9th Ed., 1931, sections 105 and 106:

We already have described sufficiently the choice of the points in the field, ground points and high points, in Volume II, 1, 9th Ed., 1931, section 87, likewise the centerings, sections 89 and 90. In 1892 to

1894, there was used as an instrument the small Wanschaff 13 cm microscope theodolite, mentioned already on p. 467. As a rule, four sets were measured with the turning of the circle by 45° from set to set. In addition to the author, a second observer was always active also, so that, alternatingly, one measured and the other wrote (*Feldbuch* [field book], Volume II, 1, section 88).

The sets are computed in the field book with the auxiliary table of Volume II, 1, p. [7], and with check sums (Volume II, 1, p. 393). We usually orient the measurements on the side in the field already in such a way that zero of the readings comes about to the north (Volume II, 1, p. 395), because this is desirable later in the case of the calculation.

Taking the mean from the four sets of each station requires the writing out of these sets from the field book to a special sheet (for instance, form 2 of the Prussian Instruction IX); all four sets are brought to *one* value of the starting ray here, in order that we can see from the deviations at the remaining directions whether everything agrees sufficiently. The taking of the mean from the four sets is done again with check sums.

The calculated means are compiled as an Abriss [station data collected according to a systematic way] station by station, section 91, Volume II, 1.

Then there follows the calculation of the whole group worked according to the approximate coordinates, in the case of resection according to the form of Volume II, 1, section 95, which contains directly the direction angles $(A\,P)$, $(M\,P)$, $(B\,P)$ or, as the case may be, their inversions $(P\,A)$, $(P\,M)$, $(P\,B)$. One to two more direction angles to other points are calculated to this for insurance and in order that the mean of the sets in question is immediately turned and oriented approximately so that later in the adjustment (p. 395), the measured directions α are available approximately oriented, and the further twisting of orientation δz becomes but small.

In the case of the numerous calculations of the intersections and resections the computing machine can serve well according to Volume II, 1, 1931, section 92, p. 426, and section 96, p. 444.

After all intersections and resections were thus calculated through for obtaining approximate coordinates, the plotting of the net image at 1:10,000 was carried out and only *after that* the setting up of a good adjustment plan was approached, whereby in the succession of the intersections and resections a wide range and opportunity for appropriate choice is given (Volume II, 1, 1931, sections 105 and 106).

To do so, we make ourselves for this purpose an outline for the outward and inward directions to be used and entered into the computation forms from the Abriss [station data] according to the example given in Volume II, 1, section 105, p. 491; for instance, for the point 61 of the table, p. 464,

Point to be determined: *61 Kunst*

Determining directions	Outward:	Burg, Hochschule, Ägidius, Wasserturm, Velber.
	Inward:	Vinnhorst, Fahnenstange [flagpole], the same, Schornstein [chimney], Burg, Marienkirche, Hochschule, Ägidius, Wasserturm, Velber.

The net picture at 1:10,000 also gives the distances s sufficiently which we need on pp. 376 and 395; when the plan was not on hand, we calculated the s's on the occasion of the calculation of the direction angles (φ) on pp. 375 and 394 along with it. These direction angles (φ) must be computed *twice* independently before the adjustment; this is the most painful part of the whole work, for which also the computing machine can be used to advantage in the case of operations in large numbers. If we have an auxiliary computer, then it is the most natural thing to do that two computers check one another on independent sheets from the writing out of the coordinates to the angular values; *one* computer must always compute the (φ)'s for a few days ahead and then make up on a new sheet until all agrees. But now if the (φ)'s are available, computed twice independently (say, the s's are obtained at one time incidentally with four-place logarithms and at a second time from the picture of the net), then the further computation according to the forms, pp. 376 and 395, is an easy matter; a few places can be sufficient and we can do all with the slide rule. In the case of resection, the arrangement on p. 397 is to be preferred to the usual formation of the mean of p. 395. It is not necessary to compute twice the direction angles φ *after* the adjustment, on pp. 375 and 394, as is needed for the (φ)'s before the adjustment; but a few words can still be said about the φ's. As already mentioned, since the (φ)'s *before* the adjustment must moreover be computed on two separate sheets, we can write one of them in pencil and use it later for the final φ's with ink by changing the last places. At the trigonometric section of the Reichsamt für Landesaufnahme, one proceeds even in such a way that the φ's are no longer computed on a separate sheet, but are produced only by red corrections of the final figures in the former sheet of the (φ)'s.

In the case of combined intersection and resection, *two* forms, pp. 376 and 395, are taken together.

After the adjustment, the Abriss [station data] for each base point is set up, which contains once again the lower part of p. 395 and thereby co-orients the still free rays for the further intersection, which appear then again as measured outer directions α in a new scheme, p. 376 (cf. Volume II, 1, 1931, section 91).

About 4-5 forms (appropriately with red print on white paper) are sufficient for these trigonometric computations which forming a small pamphlet for each point are then arranged and bound according to the succession of the computations of points. (According to the Prussian Instruction IX, the forms 2, 5, perhaps also 7 and 8, then 9, 10, and 11 apply to the case.)

Section 114. Example of a Third-Order Triangulation of the Trigonometric Section of the Reichsamt für Landesaufnahme

In our Volume III, 7th Ed., 1923, section 23, as well as Volume II, 1, 9th Ed., 1931, section 107, we have treated several things about the total layout of the triangulations of our land survey. Concerning the progress of these triangulations there have repeatedly been communicated reports with outline maps in the *Zeitschr. f. Verm.;* more recently such reports appear regularly in the *Mitt. d. Reichsamts f. Landesaufn.*

In the general plan, p. 475, which we have taken from *Zeitschr. f. Verm.*, 1892, p. 195, there are represented 63 plane-table sheets with two diagonals on each of which the third-order triangulation in 1891 has been carried out. Since a plane-table sheet in that region contains approximately 126 square kilometers, then altogether 7938 square kilometers have been triangulated in third order in 1891.

A sector therefrom, namely the territory which the triangulation engineer Hauptmann Messner worked up in 1891, is reproduced on p. 476, on which the area of six plane-table sheets between 52° 12' and 52° 30' latitude and 27° 20' and 27° 40' longitude is represented. According to the table in Volume III, 7th Ed., 1923, p. [41], these six plane-table sheets contain an area of 758.04 square kilometers.

Within this mathematical frame there are 140 points of third to fourth order, and hence approximately one point to 5-6 square kilometers. Not all sighting lines are drawn on p. 476, but mainly only the sights of third order, and in the city territory of Hannover there are only the points of the partial net, which we have already drawn and treated on p. 234 and p. 409, while the many points of p. 469 are of course absent on p. 476. This will be under further discussion later in the case of the "adjustment plan."

In addition to the frame given by the meridians and parallels, the picture of the net on p. 476 contains also the system of the rectangular coordinates in the conformal double projection and also the cadastral system Celle.

Since we had to compute for the city survey of Hannover partly in one, and partly in the other of these systems, we have indicated both systems on p. 476 by a few main lines.

Since 1922 the conformal double projection has been abandoned by the Reichsamt für Landesaufnahme; the conformal plane Gauss-Krüger coordinates have taken its place (Vol. II, 1, 9th Ed., 1931, section 34, and Vol. III, 7th Ed., 1923, Chapter VIII). For the net of Hannover cf. in this connection section 71, p. 252.

Now we are to pass over to the "adjustment plan" on which the triangulation of 1891 which we studied was based.

As *given* there were available the following first-order points which are denoted on p. 476 by numbers and, insofar as they fall outside the frame, by letters:

1. Wettmar, main point of the Hannover-Saxon chain,
2. Burgdorf, intermediate point of the Hannover-Saxon chain,
3. Lühnde I, — intermediate points of the Weser net,
4. Hannover, Ägidius, — intermediate points of the Weser net,
5. Linden, Wasserturm, — intermediate points of the Weser net,

6. Deister west,
7. Sauberg southeast,
8. Kahnstein southwest, no longer on the sheet,
9. Brelingerberg northwest.

Concerning the introduction of the point Linden Wasserturm as the "following point" of Ägidius we have reported previously in all thoroughness in *Zeitschr. f. Verm.*, 1889, pp. 1-14, where further details about these first-order points are indicated.

Further points of first order for the attachment were:

a) Stelingen on the western margin of the sheet,
b) Ronnenberg on the western margin of the sheet,
c) Velber on the western margin of the sheet,
d) Hüddessum southeast,
e) Osterberg southeast,
f) Immensen east,
g) Obershagen northeast,
h) Scherenbostel northwest,
i) Bründeln southeast,
k) Evern southeast,
l) Klein Lopke southeast,
m) Schulenburg southwest.

All these 21 points in the territory under examination and in its neighborhood were fixed at the beginning of 1891, and the new determinations were carried out according to the following adjustment plan which accompanies in print the autographed net picture (at 1:100,000):

1) 10 Kronsberg I and 11 Ilten I as double points adjusted together through 4 Ägidius, 2 Burgdorf, 3 Lühnde I.
2) 12 Wolfsberg and 13 Eschersberg I, adjusted likewise together as double points through 6 Deister, 4 Ägidius, 3 Lühnde, 7 Sauberg, 8 Kahnstein.
3) 14 Isernhagen, adjusted through 1 Wettmar, 2 Burgdorf, 4 Ägidius, 9 Brelingerberg.
4) 15 Pattensen I, 16 Gödringen, 17 Marienburg adjusted together as a three-point system through 3 Lühnde, 13 Escherberg, 8 Kahnstein, 12 Wolfsberg.
5) 18 Bothfeld adjusted through 14 Isernhagen, 10 Kronsberg I, 4 Ägidius, a) Stelingen.
6) 19 Stelle adjusted through 1 Wettmar, 2 Burgdorf, 14 Isernhagen.
7) 20 Arnum adjusted through 4 Ägidius, 10 Kronsberg, 15 Pattensen I, 12 Wolfsberg, b) Ronnenberg, 5 Linden, Wasserturm.
8) 21 Rethen I, adjusted through 10 Kronsberg, 3 Lühnde I, 16 Gödringen, 15 Pattensen I.
9) 22 Gestorf I, adjusted through 15 Pattensen, 13 Escherberg I, c) Osterwald, 12 Wolfsberg.
10) 23 Vinnhorst, adjusted through 14 Isernhagen, 18 Bothfeld, 4 Ägidius, c) Velber, a) Stelingen.
11) Four-point system, 24 Waldhausen Willmer, 25 Steuerndieb, 26 Burg, 27 Vahrenwald Schanze, adjusted through 4 Ägidius, and 5 Linden Wasserturm.

This is our adjustment of section 70 with Fig. 1, p. 236, and in fact an adjustment according to *conditioned* observations (with correlates), while all the rest is adjusted according to indirect observations. Our intercalation of double points Hochschule-Dreifaltigkeit of section 103 also appears now in this connection, and in fact the official computation is carried out according to Schreiber's rules whose application to our case we could indicate but briefly on pp. 456 and 457, this yields

12) 35 Hochschule and 36 Dreifaltigkeit, adjusted together through 4 Ägidius, 5 Linden Wasserturm, 26 Burg, 27 Schanze, 25 Steuerndieb, 24 Willmer.

In these 12 steps there are thus adjusted now together 20 points, in five cases more than one, in the remaining cases only one each.

All the rest are only *one*-point intersections, which we will reproduce, however, in the following table also for an over-all illustration:

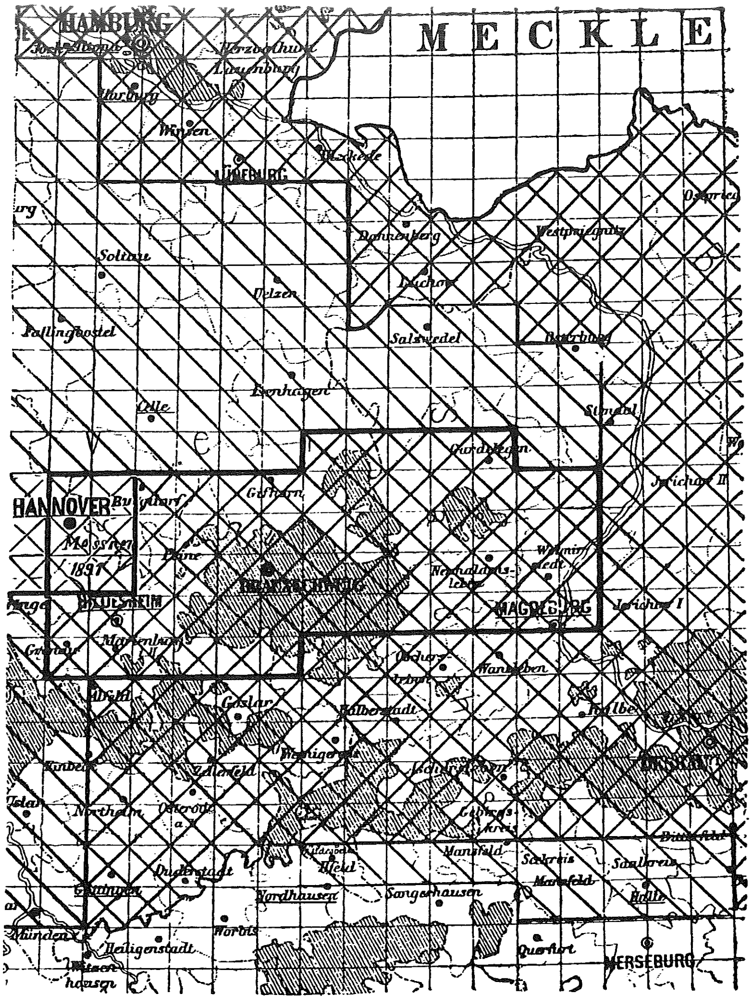
Newly Determined Point		Adjusted Through the Points Previously Determined
28.	Harsum	d Hüddessum, e Osterberg, 16 Gödringen, 3 Lühnde I
29.	Gleidingen . .	3 Lühnde I, 16 Gödringen, 21 Rethen I
30.	Aligse	2 Burgdorf, f Immensen I, 11 Ilten I
31.	Isernhagen I . .	1 Wettmar, 19 Stelle, 14 Isernhagen
32.	Otze I	g Obershagen, 2 Burgdorf, 1 Wettmar
33.	Schlickum . . .	16 Gödringen, 17 Marienburg, 15 Pattensen I
34.	Kaltenweide I .	h Scherenbostel, 14 Isernhagen, 23 Vinnhorst
37.	Ahrbergen . .	28 Harsum, e Osterberg, 16 Gödringen
38.	Wassel	11 Ilten I, 3 Lühnde I, 29 Gleidingen, 10 Kronsberg I
39.	Misburg . . .	10 Kronsberg I, 25 Steuerndieb, 18 Bothfeld
40.	Lahe	2 Burgdorf, 25 Steuerndieb, 14 Isernhagen
41.	Kolshorn . . .	2 Burgdorf, 30 Aligse, 19 Stelle
42.	Gr. Burgwedel .	1 Wettmar, 31 Isernhagen I, 14 Isernhagen, h Scherenbostel
43.	Koldingen . . .	21 Rethen I, 16 Gödringen, 33 Schlickum, 15 Pattensen I
44.	Laatzen . . .	21 Rethen I, 20 Arnum, 24 Waldhausen
45.	Wettbergen . .	5 Linden, 24 Waldhausen, b Ronnenberg, c Velber
46.	Langenhagen .	14 Isernhagen, 27 Vahrenwald, 23 Vinnhorst, h Scherenbostel
47.	Ickhorst . . .	h Scherenbostel, 14 Isernhagen, 34 Kaltenweide I
48.	Borsum I . . .	i Bründeln, d Hüddessum, 28 Harsum
49.	Ummeln . . .	k Evern, l Kl. Lopke, 3 Lühnde I, 38 Wassel
50.	Bilm	11 Ilten I, 38 Wassel, 10 Kronsberg I
51.	Röddensen . .	2 Burgdorf, 30 Aligse, 40 Kolshorn
52.	Gr. Horst I . .	19 Stelle, 18 Bothfeld, 14 Isernhagen, 42 Burgwedel
53.	Oldhorst . . .	1 Wettmar, 19 Stelle, 42 Gr. Burgwedel
54.	Örie	33 Schlickum, m Schulenburg, 15 Pattensen I
55.	Reden	44 Laatzen, 43 Koldingen, 12 Wolfsberg, 20 Arnum
56.	Bothfeld I . . .	18 Bothfeld, 25 Steuerndieb, 36 Hannover Dreifaltigkeitskirche, 27 Vahrenwald
57.	Godshorn . . .	34 Kaltenweide I, 46 Langenhagen, 23 Vinnhorst, a Stelingen
58.	Isernhagen II .	14 Isernhagen, 18 Bothfeld, 46 Langenhagen, h Scherenbostel
59.	Isernhagen III .	42 Gr. Burgwedel, 14 Isernhagen, h Scherenbostel
60.	Kronsberg II .	10 Kronsberg I, 21 Rethen I, 55 Reden, 24 Waldhausen
61.	Lehrte II . . .	n Lehrte I, 49 Ummeln, 11 Ilten I
62.	Altwarmbüchen .	52 Gr. Horst I, 18 Bothfeld, 14 Isernhagen
63.	Heessel	2 Burgdorf, 51 Röddensen, 40 Kolshorn
64.	Gestorf II . . .	54 Örie, m Schulenburg, 22 Gestorf I
65.	Ahlten I . . .	30 Aligse, 61 Lehrte II, 11 Ilten I
66.	Bennigsen . . .	15 Pattensen I, 64 Gestorf II, 12 Wolfsberg
67.	Ahlten II . . .	65 Ahlten I, 11 Ilten I, 10 Kronsberg I.

With these, we have thus shown the computational procedure for 58 points; the remaining, approximately 80 points, are drawn on p. 476 without connections of direction.

All this (i.e., with elimination of the part of Hannover City 11 and 12) was measured in *one* summer, 1891, and computed in the following winter by Hauptmann Messner. The measurement was carried out with the Bamberg 14 cm theodolite, which is illustrated and described in our Volume II, 1, 9th Ed., 1931, p. 312. This theodolite also has a vertical circle of a 12-cm diameter, which was used according to the example of Volume II, 2, 9th Ed., 1933, sections 26 and 28, while in the time indicated all trigonometric heights were also measured and computed at the same time.

The measurements of horizontal angles were carried out in *three* sets only; following is an example measured by Hauptmann Messner on the water tower of Linden [Lindener Wasserturm] on 21 August 1891:

Aiming Point	Set 1	Set 2	Set 3	Mean
Ägidius	0° 0′ 0.0″	0.0″	0.0″	0° 0′ 0.0″
Ricklingen . .	66 15 14.5	12.0	14.0	66 15 13.5
Hemmingen .	77 41 29.5	26.0	28.5	77 41 28.0
Wettbergen .	125 43 4.5	4.5	5.5	125 43 4.8
Badenstedt . .	194 17 36.5	42.5	37.5	194 17 38.8



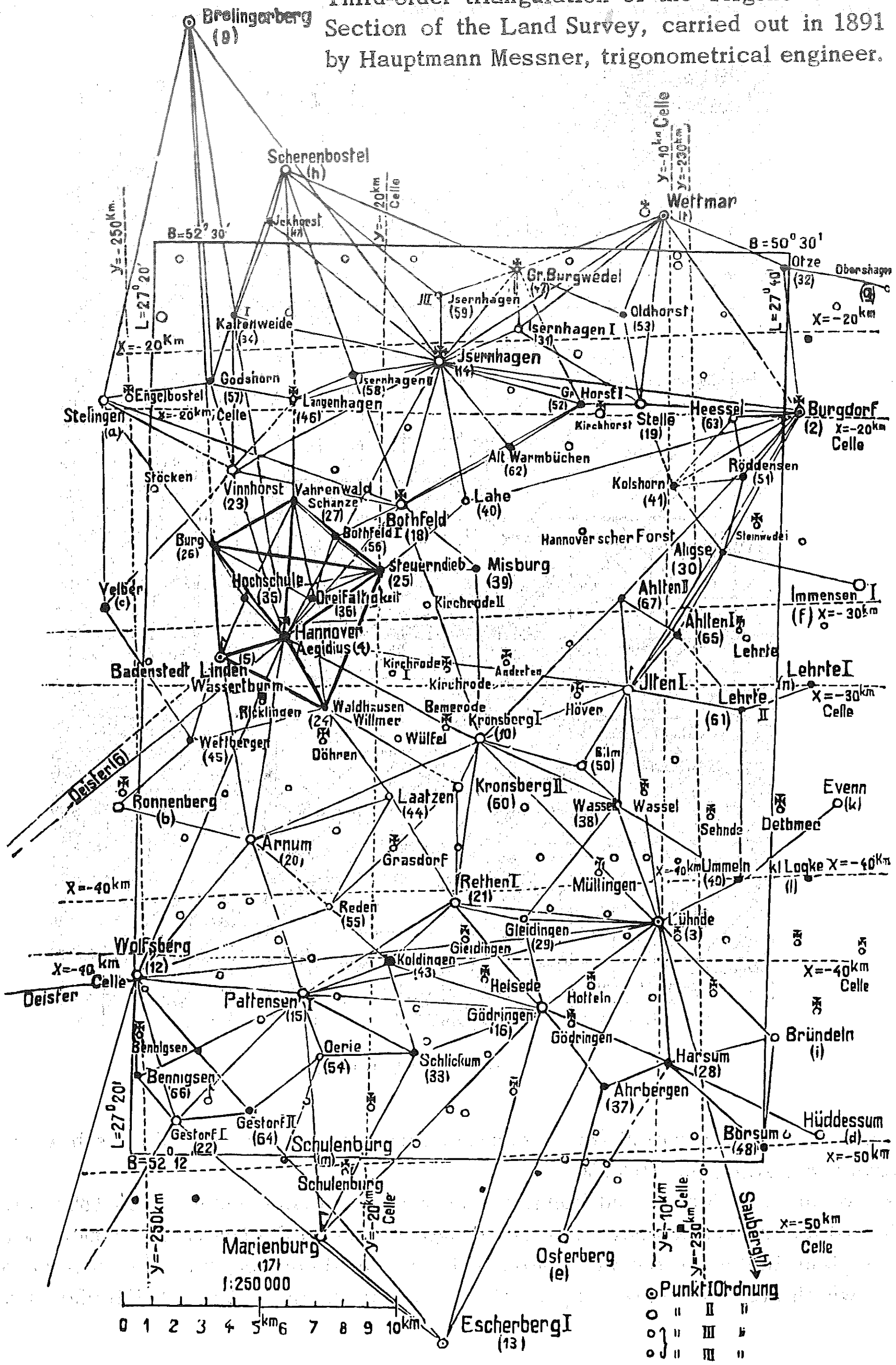
Scale 1:2,000,000.

This general plan is a cutout from a communication by Oberst Morsbach in the *Zeitschrift f. Vermessungswesen,* 1892, pp. 193-196.

In the provinces of Saxony and Hannover, as well as in Braunschweig, in 1891, there were worked up 63 plane tables which are marked here with two diagonals and collectively are framed by a heavy line.

The six plane tables, to the west, around the city of Hannover, framed specially, are the work by Messner, 1891, which are represented separately on the following p. 476 at 1:250,000 as a triangulation-net picture.

Third-order triangulation of the Trigonometrical Section of the Land Survey, carried out in 1891 by Hauptmann Messner, trigonometrical engineer.



[Cf. in this connection pp. 234, 472, 473, and 474.]

We compute hence a mean error of a direction of ± 1.8" according to the method of section 63.

The adjustments of the measurements described here were carried out since 1876 according to indirect observations (cf. p. 453) reducing all directions and distances from the actual surface of the Earth to the plane system of the conformal double projection. Therefore, the $t - T$ reductions are always required to all directions involved according to the formulae (5) or (6), p. 245, as is put in, for example, on p. 410. These $t - T$'s were computed with four-place logarithms in a printed scheme, as well as the coefficients a and b of the error equations according to the formulae (14) to (16), p. 360.

In the case of a set of inner directions, the unknown of orientation z was eliminated by a sum-equation with negative weight according to Schreiber's rule 3, p. 455. In the case of the combined effect of outward and inward directions, the outward directions were given half the weight according to Schreiber's rules (p. 455). Mean errors were not computed, which seems to be connected with the words reproduced on pp. 4 and 5.

We have given more detailed indications about the publication of the triangulation results of the former Prussian Land Survey and the Reichsamt für Landesaufnahme in our Volume II, 1, 9th Ed., 1931, section 107.

Chapter IV

ACCURACY OF TRIANGULATIONS
HISTORICAL SKETCHES

Section 115. General

The question about the accuracy of measurements and the answer to this question is the beginning and the end of all finer geodetic investigations, and in order to obtain an opinion about such questions it is of special importance to study the works of the past with respect to the desired and the achieved accuracy.

As we have already noted in the introduction on p. 5, investigations of accuracy have not been cultivated much in earlier geodetic literature.

About 70 years ago, at the beginning of the European degree-measurement, an incidental remark by General Baeyer in his *Messen auf der sphäroidischen Oberfläche*, 1862, p. 79, namely: [translated] "We cannot assume the probable error of the best angle measurements to be below ¼ of a second," served as the basis for geodetic evaluation of accuracy. The degree-measurement in East Prussia did not offer a computation of accuracy at all, and the "determination of the mean error of the angle measurements" in Baeyer's famous *Küstenvermessung* at first sight proved incorrect in the sense of the method of least squares.

Nothing was known about the classical measurements by Gauss in Hannover.

It was likewise in South Germany. Concerning the Bavarian triangulation, the main geodetic work in Germany at the beginning of the nineteenth century, the Bavarian literature which was accessible to the public, gave only indefinite information about "satisfactory agreement" or the like. The official land survey work of Württemberg, also, either evaded intentionally all questions of accuracy or it offered numbers which, evidently, left much to be desired in objectivity. The excellent triangulation of Baden was completely unknown in geodetic literature. At that time, one man made an honorable exception in such a silence: Gerling, whom we already had to mention with praise on p. 4 gave information, by means of mean errors, about his measurements in the electorate of Hesse, likewise also Schleiermacher in Darmstadt.

In 1872 the author collected everything which could then be found on geodetic data of accuracy and published it in *Astronomische Nachrichten*, Volume 80, 1873, No. 1898, pp. 17-22; this was the first work of this kind. After that, numerous individual communications have been given in *Zeitschrift für Vermessungswesen.*

From a later time we have accuracy material which the Internationale Erdmessung has collected and published.

For triangulation we have first the reports of the Italian General Ferrero, of which there have been published in *Verhandlungen der internationalen Erdmessung*

of the 10th General Conference in Brussels, 1892, Enclosure A I
of the 11th General Conference in Berne, 1895, Enclosure A III
of the 12th General Conference in Stuttgart, 1898, Enclosure A VII.

After Ferrero's death, the Zentralbureau der internationalen Erdmessung took care of bringing a sketch about the progress of triangulation in the same manner from time to time. Such reports were published by Helmert and Krüger in *Verhandlungen der 14. Allgemeinen Konferenz in Kopenhagen*, 1903, Part II, pp. 216-292, by Helmert and Galle in *Verhandlungen der 16. Allgemeinen Konferenz in London*, 1909, Part II, pp. 68-104, by Galle in *Verhandlungen der 17. Allgemeinen Konferenz in Hamburg*, 1912, Part II, pp. 75-117.

Section 116. The International Approximation Formula for the Mean Angle Error

At the Conference of the Permanent Commission of the Internationale Erdmessung in Nizza, 1887, an agreement was reached on a formula for the computation of the mean angle error of triangulations, concerning which the *Verhandlungen* published about this conference, Berlin, 1888, pp. 54-55 and respectively 56-58, give the following:

In accordance with the proposal of General Ferrero, in cooperation with Helmert and F o e r s t e r, the reports about triangulations shall in future contain a numerical value m for each system of triangles, to be computed according to the formula:

$$m^2 = \frac{\Sigma w^2}{3 \cdot n} \quad \text{or} \quad m = \sqrt{\frac{\Sigma w^2}{3 \cdot n}}. \tag{1}$$

In this formula we denote by w the discrepancy between 180° + spherical excess and the sum of angles of each triangle, and by n the number of the triangles of the net.

If a triangulation cannot be divided only into triangles, i.e. if also quadrilaterals, pentagons, and so forth occur, then the latter are of course to be treated accordingly, i.e., from a quadrilateral we would compute $m = w : \sqrt{4}$. In order to obtain smooth tables, we then proceed in the simplest manner so that we compute first w^2 of a quadrilateral, but take into account only $\frac{3}{4}$ of its value for the triangles, likewise $\frac{3}{5}$ in the case of a pentagon, etc.

In general, the mean error computed according to equation (1) will not agree with the rigorous value which results from the net adjustment. Only in a few special cases do both computations of the mean error lead to the same value.

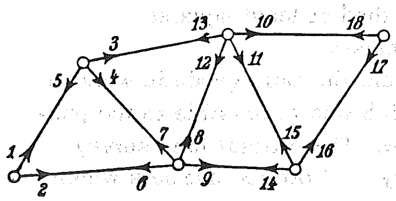


Fig. 1.

This is visible offhand for an individual triangle in which the three angles or the six directions are measured.

The case of a simple chain of triangles with *angle* measurement likewise needs no further explanation.

It is different with a simple chain of triangles in which the *directions* are measured. We will consider this case more closely in connection with Fig. 1. From the four triangles there result at once the four condition equations:

$$\left.\begin{aligned}
-v_1 + v_2 - v_4 + v_5 - v_6 + v_7 + w_1 &= 0 \\
-v_3 + v_4 - v_7 + v_8 - v_{12} + v_{13} + w_2 &= 0 \\
-v_8 + v_9 - v_{11} + v_{12} - v_{14} + v_{15} + w_3 &= 0 \\
-v_{10} + v_{11} - v_{15} + v_{16} - v_{17} + v_{18} + w_4 &= 0
\end{aligned}\right\}. \tag{2}$$

The normal equations to these are:

$$\left.\begin{aligned}
6k_1 - 2k_2 \quad . \; . \quad\quad . \; . \quad + w_1 &= 0 \\
-2k_1 + 6k_2 - 2k_3 \quad . \; . \quad + w_2 &= 0 \\
. \; . \quad - 2k_2 + 6k_3 - 2k_4 + w_3 &= 0 \\
. \; . \quad\quad . \; . \quad - 2k_3 + 6k_4 + w_4 &= 0
\end{aligned}\right\}. \tag{3}$$

Hence there follows:

$$\begin{aligned}
-110\,k_1 &= 21\,w_1 + 8\,w_2 + 3\,w_3 + w_4 \\
-110\,k_2 &= 8\,w_1 + 24\,w_2 + 9\,w_3 + 3\,w_4 \\
-110\,k_3 &= 3\,w_1 + 9\,w_2 + 24\,w_3 + 8\,w_4 \\
-110\,k_4 &= w_1 + 3\,w_2 + 8\,w_3 + 21\,w_4 .
\end{aligned}$$

Since four condition equations exist, then

$$-[wk] = 4m^2$$

and we obtain:

$$\left.\begin{aligned} 440\,m^2 = {} & 21\,(w_1^2 + w_4^2) + 24\,(w_2^2 + w_3^2) \\ & + 16\,(w_1 w_2 + w_3 w_4) + 6\,(w_1 w_3 + w_2 w_4) \\ & + 2\,w_1 w_4 + 18\,w_2 w_3 \end{aligned}\right\} . \tag{4}$$

On the other hand, Ferrero's formula (1) yields as the mean error of a direction

$$24\,m^2 = w_1^2 + w_2^2 + w_3^2 + w_4^2 . \tag{5}$$

These two values will agree only when

$$\left.\begin{aligned} & 8\,(w_1^2 + w_4^2) + 17\,(w_2^2 + w_3^2) + 48\,(w_1 w_2 + w_3 w_4) \\ & + 18\,(w_1 w_3 + w_2 w_4) + 6\,w_1 w_4 + 18\,w_2 w_3 = 0 \end{aligned}\right\} . \tag{6}$$

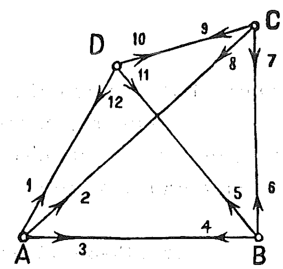


Fig. 2.

Ferrero's formula thus yields in this case, for the mean error, only an approximate value, which will be more or less accurate, depending on the values of w.

Here we adjoin, in addition, the simplest case of a central system, that of a quadrilateral with both diagonals, while we assume again direction measurements.

In Fig. 2 we consider the three triangles

$$ADC \qquad ADB \qquad ABC$$

and form, corresponding to these, the following three condition equations:

$$\left.\begin{array}{l} -v_1 + v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad -v_8 + v_9 - v_{10} \quad . \quad + v_{12} + w_1 = 0 \\ -v_1 \quad . \quad + v_3 - v_4 + v_5 \quad . \quad . \quad . \quad . \quad . \quad -v_{11} + v_{12} + w_2 = 0 \\ . \quad -v_2 + v_3 - v_4 \quad . \quad + v_6 - v_7 + v_8 \quad . \quad . \quad . \quad . \quad + w_3 = 0 \end{array}\right\} . \tag{7}$$

Neglecting the side equation referring to these, we form from (7) the normal equations:

$$\left.\begin{aligned} +6\,k_1 + 2\,k_2 - 2\,k_3 + w_1 &= 0 \\ +2\,k_1 + 6\,k_2 + 2\,k_3 + w_2 &= 0 \\ -2\,k_1 + 2\,k_2 + 6\,k_3 + w_3 &= 0 \end{aligned}\right\} . \tag{8}$$

The solution yields:

$$k_1 = \frac{-2\,w_1 + w_2 - w_3}{8}, \quad k_2 = \frac{w_1 - 2\,w_2 + w_3}{8}, \quad k_3 = \frac{-w_1 + w_2 - 2\,w_3}{8} .$$

Now we have $[vv] = -[wk]$, and the calculation yields according to this:

$$-8\,[wk] = 2\,w_1^2 + 2\,w_2^2 + 2\,w_3^2 - 2\,w_1 w_2 + 2\,w_1 w_3 - 2\,w_2 w_3 . \tag{9}$$

However, we can bring this into a more perspicuous form by introducing a *fourth* sum discrepancy w_4 which corresponds to the fourth triangle BCD as a supplement to (1), namely:

$$w_4 = w_1 - w_2 + w_3. \tag{10}$$

By this introduction we can bring the sum (9) algebraically into the following form:

$$-8[w\,k] = w_1^2 + w_2^2 + w_3^2 + w_4^2,$$

and we obtain the mean direction error μ, since three independent condition equations were used:

$$\mu^2 = \frac{[v^2]}{3} = \frac{-[w\,k]}{3} = \frac{w_1^2 + w_2^2 + w_3^2 + w_4^2}{3 \cdot 8}.$$

The mean angle error m, which is related to the direction error $\mu : m = \mu\sqrt{2}$, is thus obtained:

$$\left.\begin{aligned} m^2 = 2\,\mu^2 &= \frac{w_1^2 + w_2^2 + w_3^2 + w_4^2}{12} \\ \text{or} \qquad m &= \sqrt{\frac{[w^2]}{12}} \end{aligned}\right\}, \tag{11}$$

where $[w^2]$ is now the sum of all *four* w^2's.

However, Ferrero's formula (1), also, yields the same value (11). There results therefrom the following theorem: If in a quadrilateral with both diagonals all directions are measured, then the adjustment of the three angle equations yields, for the mean error, the same value as Ferrero's formula.

This theorem holds generally for any arbitrary polygon, in which, in addition to the sides, all diagonals are observed with equal weight. The proof for this, as well as a thorough investigation of the validity of Ferrero's formula in general, is given by L. Krüger in *Zeitschrift für Mathematik und Physik*, vol. 47, 1902, pp. 157-196.

If the four triangles which belong to the quadrilateral (Fig. 2) shall now be taken together with *other* triangles, then we can introduce the value m^2 from (11) as computed only from *three* triangles; e.g., if the following six values w thus exist:

$$w_0,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5, \tag{12}$$

where w_0 and w_5 are obtained independently from *one* triangle each, but on the other hand, w_1, w_2, w_3, w_4 are obtained together from a quadrilateral with two diagonals, then we have to compute:

$$m^2 = \frac{w_0^2 + \frac{3}{4}(w_1^2 + w_2^2 + w_3^2 + w_4^2) + w_5^2}{3\,(1 + 3 + 1)}. \tag{13}$$

If we do not make this distinction, but compute briefly thus:

$$m^2 = \frac{w_0^2 + w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2}{3 \cdot 6}, \tag{14}$$

then the 12 directions of the quadrilateral are introduced with too strong weights.

We will take the city triangulation net of Hannover of p. 236 as a simple numerical example for this.

According to pp. 237-238 this net yields the following seven triangle closures:

$w_1 = -1.02''$	$w^2 = 1.04$	$w_4 = -0.76''$	$w^2 = 0.58$
$w_2 = +2.22$	$w^2 = 4.93$	$w_5 = +2.30$	$w^2 = 5.29$
$w_3 = -2.36$	$w^2 = 5.57$	$w_6 = +4.30$	$w^2 = 18.49$
	11.54	$w_7 = -2.76$	$w^2 = 7.62$
	31.98		31.98
	$[w^2] = 43.52$		$\frac{3}{4}[w^2] = 23.99$.

If we compute offhand according to formula (1) or (14), then we have:

$$m = \sqrt{\frac{43.52}{3 \cdot 7}} = \pm 1.44'', \tag{15}$$

however, according to the somewhat more rigorous formula (13):

$$m = \sqrt{\frac{11.54 + 23.99}{3 \cdot 6}} = \pm 1.40''. \tag{16}$$

These two values do not vary much from one another; we therefore see that in this case the smooth international formula yields nearly the same as the small improvement with the treatment of the quadrilateral.

The two values (15) and (16) are not rigorous, for the rigorous computation with all checks yielded according to p. 242 the mean error of a *direction* (denoted there, likewise, by m) $= +1.04''$; therefore, the mean angle error is to be taken accordingly:

$$m = 1.04\sqrt{2} = \pm 1.47''; \tag{17}$$

the fact that this agrees somewhat better with (15) than with (16) is accidental. –

We will add a supplement to the international error formula in the sense that we estimate the reliability of an m computed according to it. Since we treat the w's as true independent errors, we must further apply formula (16) of the later section 144, which, transferred to the case in question, yields:

$$m = \sqrt{\frac{[w^2]}{3\,n}}\left(1 \pm \frac{0.7071}{\sqrt{n}}\right),$$

hence, according to (15):

$$m = 1.44''\left(1 \pm \frac{0.7071}{\sqrt{7}}\right) = 1.44''(1 \pm 0.267),$$

or

$$m = \pm 1.44'' \pm 0.38'',$$

i.e. the statement, that the mean error $= 1.44''$, itself is affected with an uncertainty of 27% of its own value or $\pm 0.38''$.

For the angle adjustment with r condition equations we have in general the mean angle error

$$m = \sqrt{\frac{[v^2]}{r}} \text{ or, as the case may be, } = \sqrt{\frac{[p\,v^2]}{r}}\,. \tag{1}$$

As an example for this we can take Schwerd's base net of section 75, which, according to this, has given the mean error of an angle of weight 1 on p. 269: $m = \pm\, 4.77''$. However, we have also seen here at once (p. 270) that this error of the *unit* of weight is not that which we wish to have as the characteristic of a triangulation; therefore, on p. 270, there was also computed, in addition, the mean error for the mean weight g, which we will now denote by m':

$$m' = \pm\, 0.99''. \tag{2}$$

But this, also, is not yet a value comparable with the results from other nets, because at one of the stations of the net on p. 265, a redundant angle for the station itself has been measured. (Station Mannheim with three angles (7), (8), (9) between three rays.) We will therefore set up the further problem of computing the mean error for the mean weight of an angle adjusted at the station.

For the three angles measured at Station Mannheim there exists the condition that their sum after the adjustment must have a definite value, namely 360°. Therefore, the same problem of adjustment as in section 10, p. 30, exists, only with the difference that there the sum of the angles had to amount to 180°.

If the three angles α, β, γ are measured, then, in order to obtain the weight P_α of the angle α after the adjustment, we have to sum up, according to p. 30, the weight of the measured angle α and the weight of the sum of the other two angles $\beta + \gamma$. According to (16), section 10, p. 30, these two weights are

$$p_1 = \frac{1}{\dfrac{1}{p_\alpha}} \qquad p = \frac{1}{\dfrac{1}{p_\beta} + \dfrac{1}{p_\gamma}}$$

and then we will have

$$p_1 + p_2 = P_\alpha = \frac{\left[\dfrac{1}{p}\right]}{\dfrac{1}{p_\alpha}\left(\dfrac{1}{p_\beta} + \dfrac{1}{p_\gamma}\right)}\,.$$

By applying this to the last three values in (2), p. 266, we obtain:

$$\frac{1}{P_7} = 0.1000 - \frac{0.1000^2}{0.1690} = 0.0408$$
$$\frac{1}{P_8} = 0.0357 - \frac{0.0357^2}{0.1690} = 0.0282$$
$$\frac{1}{P_9} = 0.0333 - \frac{0.0333^2}{0.1690} = 0.0267\,.$$

Adding to this the sum of the first six $\frac{1}{p}$'s of p. 266	0.2177
Total	0.3134
Mean for nine values	0.0348.

This is the mean weight reciprocal for an angle adjusted at the station; we thus compute the mean error m of an angle adjusted at the station with a mean weight by bringing in the number 4.77" of (15), p. 269:

$$m = 4.77 \sqrt{0.0348} = \pm 0.89''. \tag{3}$$

This is somewhat smaller than the above 0.99" in (2), as it must be, because the weight of the angle adjusted at the station must in general be larger than the weight of the unadjusted angle.

This small example has served us to form the rigorous concept for that which we will always compute in the future, namely the *mean error* m *of the mean weight of an angle adjusted at the station.*

If we have adjusted a triangulation according to *directions*, as, e.g., the city net of Hannover in section 70 with Fig. 1, p. 236, then the adjustment yields a mean *direction error*, which we will now denote by μ, hence with r condition equations:

$$\mu = \sqrt{\frac{[v^2]}{r}}. \tag{4}$$

To this there corresponds a mean angle error

$$m = \mu\sqrt{2}, \tag{5}$$

as was already given as an example in (17), section 116, p. 483.

If we have here unequal weights of measured directions p in the number n, then the mean weight g is to be computed from the reciprocals, i.e.:

$$\frac{1}{g} = \frac{1}{n}\left[\frac{1}{p}\right]. \tag{6}$$

E.g., in the Belgo-German connecting net of pp. 349-350 we have $\left[\frac{1}{p}\right] = 57.8$ and with $n = 34$ (without taking into account the peculiarities of the connection):

$$\frac{1}{g} = \frac{57.8}{34} = 1.70,$$

the mean direction error for the unit of weight is according to p. 350 (denoted there by m):

$$\mu_1 = \pm 0.62'';$$

therefore, the mean direction error for the mean weight of a station direction:

$$\mu = \mu_1\sqrt{\frac{1}{g}} = 0.62\ \sqrt{1.7} = \pm 0.81''. \tag{7}$$

Convenient computations with the average error

In all formulae hitherto used there occur sums of squares which are to be computed for the purpose in question. Although this is not an important work, we feel sometimes, in the case of rough calculations, critical comparisons, etc., the need of forming quickly a mean error by using only the absolute sum $[\pm\delta]$ of any error elements δ. Since we shall treat the theories needed for this later in section 141, we form here at once, with reference to this, the applications:

If n individual triangles are present and the sum $[\pm w]$ of their discrepancies taken together without taking into account the sign, then we have the mean angle error:

$$m = 1.2533 \frac{[\pm w]}{n\sqrt{3}} . \tag{8}$$

This formula, for instance, has been applied by us to 464 triangle closures of the English land survey in section 121 I.

For Bessel's adjustment and angle corrections (1), (2) . . . we have by absolute summation (1) + (2) + (3) + . . . according to (12), section 141:

$$m = 1.2533 \frac{(1) + (2) + (3) + \ldots (n)}{\sqrt{n r}} . \tag{9}$$

Here we are also to present the three formulae for the error of the unit of weight, which we have learned earlier on p. 183, taken together with brief designations:

$$\text{Stations} \qquad \mu_1 = \sqrt{\frac{[v' v']}{n'}} \tag{10}$$

$$\text{Net} \qquad \mu_2 = \sqrt{\frac{[v'' v'']}{r}} \tag{11}$$

$$\text{Total adjustment} \qquad \mu = \sqrt{\frac{[v' v'] + [v'' v'']}{n' + r}} . \tag{12}$$

The ratio $\mu_2 : \mu_1$ gives information for a triangulation whether and to what extent the net adjustment has brought to light error influences, which remained hidden at the stations.

The net adjustment is on the whole decisive for the judgment of accuracy, and our mean angle error m hitherto treated refers only to net adjustment. If we have computations of accuracy, which refer in a triangulation to μ_1 or μ, then, if all three values μ_1, μ_2, μ are known, we can thereby reduce them proportionately to μ_2.

Section 118. Triangulation by Snell, 1610

We owe the first triangulation in the present sense, with angle measurement in degrees and trigonometric calculation, to the Dutchman Willebrord Snel van Roien (latinized Snellius), born 1580 in Leiden, died 1626.

His famous work (existing in the Staatsbibliothek in Berlin) has the title:

> „Eratosthenes Batavus, de terrae ambitus vera qantitate, a Willebrordo Snellio *Διὰ τῶν ἐξ ἀποσχημάτων μετρυσῶν διοπτρῶν.* Suscitatus. (O quam contemta res est homo, nisi supra humana se erexerit) Lugduni Batavorum apud Jodocum à Colster. Ann. CIↃIↃCXVII (1617).

In addition, there is further to be mentioned a valuable historical treatise in *Tijdschrift voor Kadaster en Landmeetkunde*, onder redactie van I. Boer, Hz. te Utrecht, Jaargang V, 1889, 1e Aflevering: "Overzicht van de graadmetingen in Nederland (met plaat)" door Dr. J. D. van der Plaats, and: "Graadmeting, Geschiedkundig overzicht" door G. B. H. de Balbian.

According to this, a calculation of accuracy was given by Jordan in *Tijdschrift voor Kadaster en Landmeetkunde*, XV, 1899, pp. 3-15, from which we give the following:

Fig. 1 first shows the base net with the different base lines. In 1615 the base net consisted of five stations, in 1622 of six stations. The measured base lines are given in Rhineland rods, which are converted here to meters with one Rhineland rod = 3.7 662 420 m (log = 0.5 759 082).

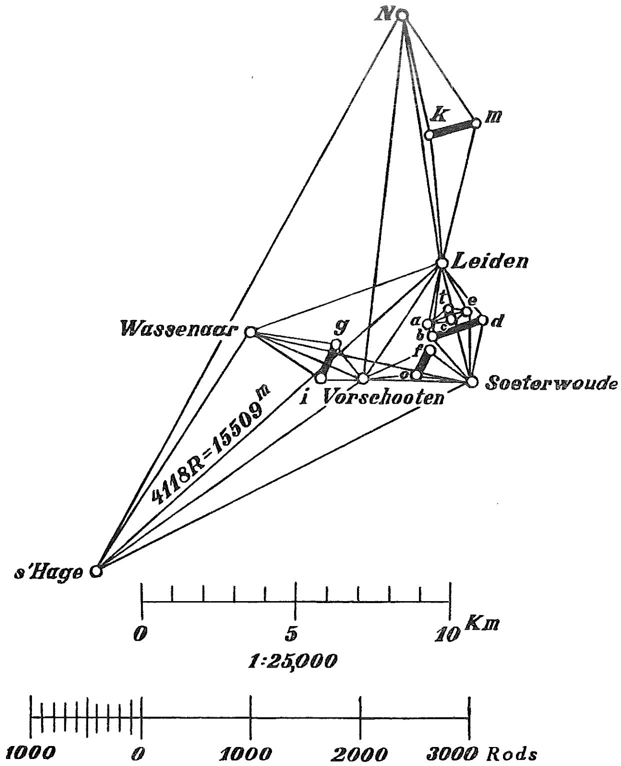


Fig. 1. Base net of the triangulation by Snell.

Year 1615	1st Base	tc	= 87.05	Rods	=	327.85 m
Year 1616	2nd Base	ig	= 348.1	Rods	=	1311.03
Year 1622	3rd Base	bd	= 475.0	Rods	=	1788.96
Year 1622	4th Base	fo	= 250.0	Rods	=	941.56
Year 1622	5th Base	km	= 471.5	Rods	=	1775.78

All these base lines show the rhombic net connection, which today is considered as the best.

The first base of Snell was the small length tc on the straight line Leiden-Soeterwoude, namely $tc = 87.05$ rods (= 327.85 m). By means of two triangles there was derived from this $ae = 326.43$ rods and measured directly 326.90 rods. The trigonometric result $ae = 326.43$ rods was retained, and derived therefrom again by means of two triangles: Leiden-Soeterwoude = 1092.35 rods (= 4114.06 m). With this, the triangulation was carried out as far as Wassenaar and Vorschooten, and between these two points a base line $ig = 348.1$ rods was again measured. The trigonometric error of the connection is not given here. However, Snell has a connection to a third base line, 166 rods long, between Oudewater and Montfort about 30 km east of the first base line; the trigonometric connection is $2923.3 - 2934.6 = -11.3$ rods or 1:260, to which Snell remarked (p. 181) that he hardly dared to hope for such an accuracy.

Musschenbroek (Snell's successor), who edited Snell's work, considers the 3rd base bd as the foundation of his work; it was measured three times and yielded the side LN (Leiden-Noordwyk) = 2338.22 rods, while the 5th base km yielded 2338 rods. The same base yielded for the side Leiden-Soeterwoude 1097.117 rods against 1092.35 rods for the first determination.

Fig. 2, p. 488, gives a diagram of Snell's chain of triangles, for the connection of the points Alkmaar and Bergen op Zoom. The arrows and numbers refer to the investigation under discussion. The whole triangulation by Snell comprises 33 triangles, which, for the most part, were used, in a manner still customary today, for the measurement of a degree of latitude between Alkmaar and Bergen op Zoom.

For the judgment of accuracy of Snell's triangulation, the adjustment for the Station Leiden, which yields 13 angles between eight rays, was first carried out from the angle measurements indicated by van der Plaats (pp. 17-18). This yielded for the mean error of a direction:

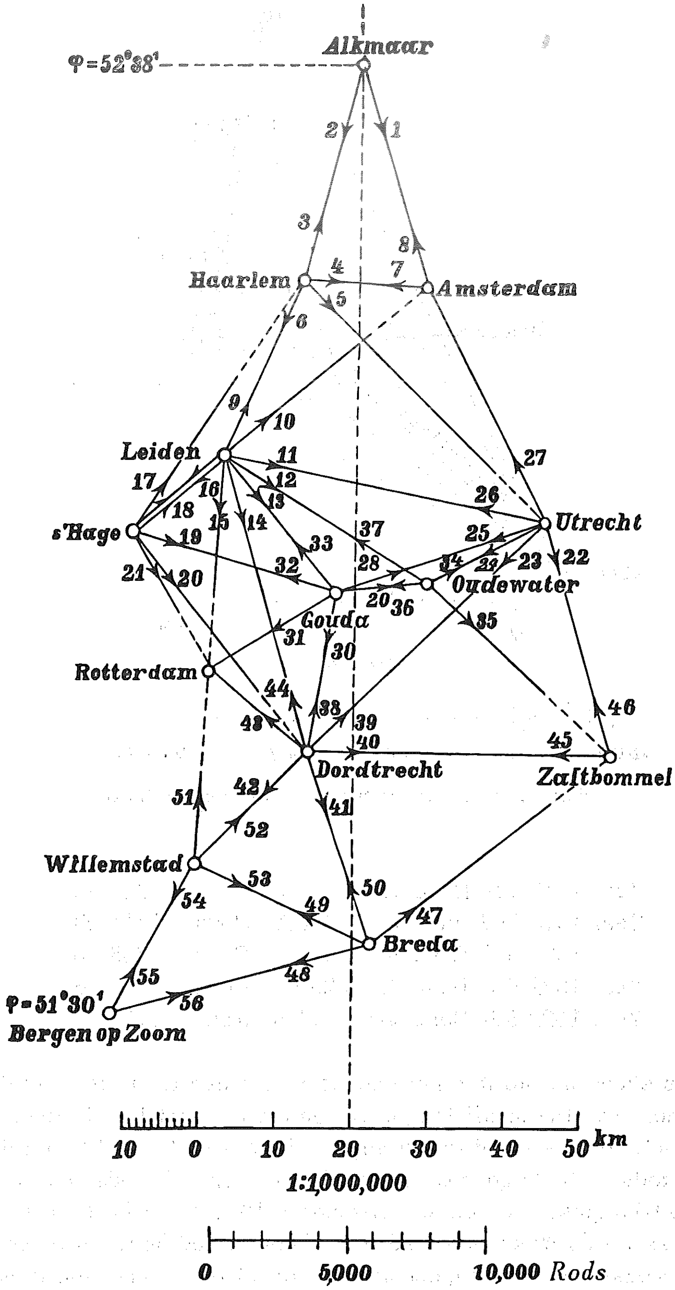


Fig. 2.

$$m_r = \sqrt{\frac{47.34}{6}} = \pm\ 2'\ 48'',$$

hence, the mean error of an angle measured by Snell at the Station Leiden

$$m_w = m_r \sqrt{2} = \pm\ 3'\ 58''.$$

For the remaining stations the number of the excessive measurements is not so large that it was worthwhile to compute mean errors. With the measurements adjusted at the stations, 12 triangle closures were now computed:

$-2'\,25''$, $-7'\,21''$, $+1'\,2''$, $+6'\,3''$, $+0'\,3''$, $-4'\,22''$, $+1'\,59''$, $+3'\,55''$, $-3'\,49''$, $-1'\,31''$, $+12'\,22''$, $-5'\,21''$, with the sum of squares $[w^2] = 120\,3485$, hence, according to (1), p. 480

$$m = \sqrt{\frac{120\,3485}{3 \cdot 12}} = \pm 183'' = \pm 3'\,03''.$$

Furthermore, a net adjustment was carried out for a part of the net, namely from Fig. 2 the group of points Leiden, s'Hage, Dordtrecht, Utrecht, Oudewater, Gouda with six triangle closures and three side equations on the basis of the side s'Hage-Leiden (see Figs. 1 and 2). The mean direction error was found at:

$$m'_r = \sqrt{\frac{246\,371}{6+3}} = \pm 165'' = \pm 2'\,45'', \text{ and } m'_w = m'_r \sqrt{2} = \pm 3'\,53''.$$

For judging Snell's triangulation we thus have for the mean angle error:

1. Station adjustment Leiden . . . . $m = \pm$ 3' 58''
2. From 12 triangle closures . . . . $m = \pm$ 3 03
3. Net adjustment with nine equations . . . $m = \pm$ 3 53 .

It follows with sufficient agreement that the mean angle error of Snell in 1615 was about three to four minutes.

Section 119. Triangulation by Schickhart in Württemberg in 1620

While Snell used his beautiful triangulation net for a degree-measurement, we now have to mention the Swabian Professor Schickhart in Tübingen as the originator of a land triangulation, probably the first in Germany. In 1629 he published a small paper, which, after his death, has been printed once more with the title:

Kurtze Anweisung, wie Künstliche Land-Tafeln aus rechtem Grund zu machen, und die bissher begangene Irrthumb zu verbessern, sampt etlich New erfundenen Vörtheln, die Polus Höhin auffs leichtest, und doch scharpff gnug zu forschen. Durch Herrn Wilhelm Schickharten Seel. gewesenen Professorn in Tübingen. Emendationis primus est gradus, Errorem detexisse. Tübingen. Verlegts Johann Georg Cotta. Im Jahr 1669. (22 Seiten 4° und 1 Kupfertafel.)

Researches in the Stuttgart State Archive led to the discovery of all angle measurements of Schickhart; this small book of 216 pages has the title: *Pinax observationum chorographicarum.*

Regelmann has given a detailed report, with interesting facsimile illustrations, about the contents of this find and the *Kurtze Anweisung*:

"Abriss einer Geschichte der württembergischen Topographie, und nähere Angaben über die Schickhartsche Landesaufnahme Württembergs, zum X. Geographentag in Stuttgart," von Inspektor C. Regelmann, from the *Württemb. Jahrbücher für Statistik und Landeskunde*, 1893, in abstract also *Zeitschr. f. Verm. 1893*, pp. 289-296.

Steiff submitted the found material to a thorough investigation and compiled its result in a report: Wilhelm Schickhart und seine Landesaufnahme Württembergs, 1624-1635, *Zeitschr. f. Verm. 1899*, pp. 401, 537.

Wilhelm Schickhart, born 1592, mathematician and astronomer, Professor in Tübingen, died 1635, made his trigonometric measurements from 1624 to 1635 completely at his own initiative as a private enterprise. As a scholarch of the State, on his inspection trips he climbed all mountains and towers and measured the angles to all the most important target points, made also notations about watercourses, etc.

According to Steiff's investigations it is assumed that Schickhart knew Snell's degree-measurement and that it was suggestive to him in his measurements.

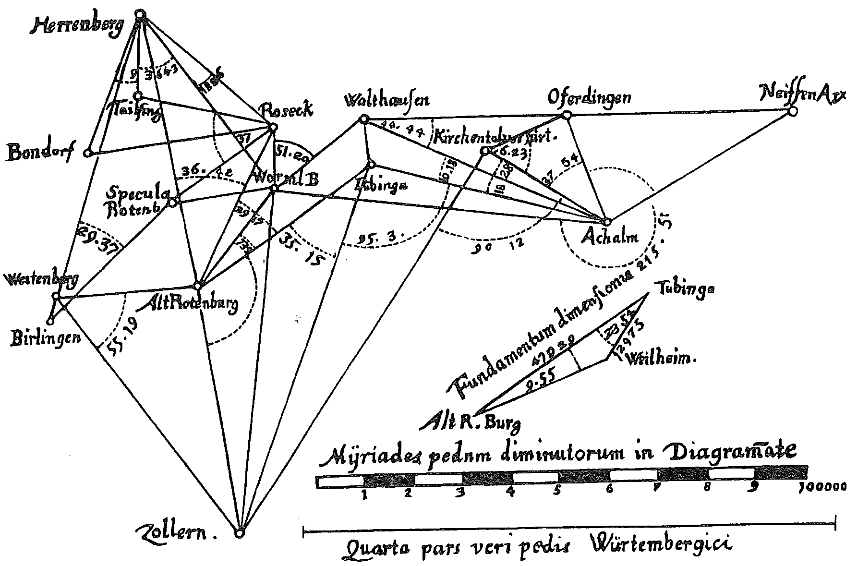
About his trigonometric measuring procedure Schickhart says in the *Kurtze Anweisung*:

[Translated] To do so, we must have a suitable instrument, climb, with it, here and there, high mountains and church-towers, sight carefully the angles to the surrounding places, record their number on a writing tablet, and form therefrom, afterwards, the map. For this, others usually need a disc divided into 360 degrees, equipped with a pointer and sight, as is seen in Fig. 1. I, however, do not think much of it, since we must act quite precisely: if it is small, it does not give the result finely enough, but if it is large, it becomes inconvenient to take it to the field. The metal is heavy, but wood is changeable. Therefore, I only put together 3 equal sticks in the form of an equilateral △, graduate them from the tables of tangents, give them at the corners a fixed vane, but at the sides a sliding vane, and with this, I observe; then it gives me all minutes carefully. Although the sticks are long, they do not give the traveller any trouble, because we can take them apart; thus their stability can also well be trusted, since wood does not shrink in length. With such sticks, the places, as traced in the other example, have been sighted.

Then he gives a numerical example for the measurement of the angles at a full horizon for the Station Achalm, as is given in the following station data [Abriss] under α.

Fig. 1 shows a portion from the trigonometric net, which Schickhart gives in the *Kurtze Anweisung* as an example.

According to Steiff (*Zeitschrift für Vermessungswesen*, 1899, p. 538), Schickhart apparently did not carry out trigonometric computations from the measured lines, apart from the computation of a few triangle sides, but constructed the map from the angle measurements.



(Scale 1:440,000)

Fig. 1. Part of the trigonometric net by Schickhart, about 1620.
Exemplum Regionis circa Tubingam ex concatenatis angulis accuratissime delineatae.

The following data [Abriss] of the Station Achalm has been formed according to the numerical example given in the *Kurtze Anweisung*. The error of the horizon closure is $-2'$.

Data [Abriss] of the Station Achalm, according to Schickhart, 1629

Target Point	Dist. s	Single Angles α	Single Angles β	Direction Angles Obs. B	Direction Angles Trig. φ	$v = \varphi - B$	v^2
Neuffen . . .	12.9 km			57° 56.1′	57° 59.4′	+3.3′	11
		67° 7′	67° 6.7′				
Eningen . . .	1.45			125 2.8	125 2.8	0.0	0
		148 46	148 45.7				
Wurmlingen . .	19.39			273 48.5	273 51.8	+3.3	11
		7 45	7 44.8				
Tübingen . . .	14.19			281 33.3	281 31.0	−2.3	5
		18 28	18 27.7				
Kirchentellinsf. .	8.16			300 1.0	300 1.4	+0.4	0
		37 54	37 53.7				
Oferdingen . .	6.79			337 54.7	337 49.0	−5.7	32
		55 13	55 12.7				
Metzingen . .	5.64			33 7.4	33 8.7	+1.3	2
		24 49	24 48.7	204.0′	204.1′	+0.3′	61
Neuffen . .	12.93						
		360° 2′	360° 0.0′				

$$m = \sqrt{\frac{61}{7-3}} = \pm 3.9'$$

According to the individual angles corrected to the horizon, the set of directions B (see table) was set up, and with this, a resection was calculated according to section 99, since Schickhart's points were recovered by different means and their coordinates were determined in the present-day Württemberg system.

The table shows the individual error and the mean direction error ±4', in which the uncertainty of the determination of coordinates for the target points is contained further. Steiff continued such investigations (*Zeitschrift für Vermessungswesen*, 1899, p. 540) and estimated, according to this, the mean direction error of Schickhart at ±9'.

Section 120. The French Degree-Measurements of the Eighteenth Century

From the Dutch (Snell, 1615) the geodetic leadership passed over to the French, to whom we owe a series of important degree-measurements from the 17th and 18th centuries. For the measurement of the chain of triangles which Picard determined in the years 1669 and 1670 for the determination of the earth's radius, there was used a quadrant of 38-inch radius, and a telescope with cross hairs was employed as a sighting device for the first time. Unfortunately, there exist only scarce data about the accuracy of the measurements, which are not sufficient for the determination of a mean error.

The work is published in the work: "Mesure de la terre par Monsieur Picard" (*Memoires de l'académie royale des sciences contenant les ouvrages adoptés par cette académie avant renouvellement en 1699*, Tome IV, La Haye, 1731, pp. 1-59).

Picard's degree-measurement extended from Paris to Amiens. In the years 1683-1701 this measurement was continued under the direction of J. D. Cassini southward to the Pyrenees and in 1718 northward to Dunkirk, so that it passed through the whole of France with an amplitude of 8-1/2°. Also about the accuracy of these measurements, which were carried out entirely according to the pattern of Picard, there are no data.

Cassini's degree-measurement is described in the work: *Traité de la grandeur et de la figure de la terre*, par M. Cassini, Amsterdam, 1723.

Only from the further French degree-measurements of the eighteenth century in Peru and Lapland, by means of which the flattening of the earth has geodetically been settled, there exist investigations of accuracy.

The French degree-measurement in Peru of 1736 is described in the work: *Mesure des trois premiers degrés dans l'hémisphère australe*, par M. de La Condamine, Paris, 1751. On pp. 22-39 of this work, the angles of 43 triangles are given. The largest discrepancy is 13" and the sum of squares of all discrepancies is 1718, hence, the mean angle error:

$$m = \sqrt{\frac{1718}{43 \cdot 3}} = \pm 3.65'' \; . \qquad (1)$$

On p. 85 of this work the base connection is also given. The triangulation rested on the base of Yarouqui, 6273 toises long, and derived from it, by means of 43 triangles, the base of Tarqui with the result of 5260.03 toises, while the direct measurement yielded 5258.949 toises. The difference is:

1.081 toises or 205 millimeters per 1 kilometer.

The distance of the two base-lines from each other is about 3° or approximately 330 km.

The triangulation of the French degree-measurement in Lapland from 1736 between Torneâ and Kittis comprises 21 triangle points with one base and is described in the work: *La figure de la terre*, par M. de Maupertuis, Paris, 1738. This triangulation has repeatedly been treated, on the part of the Germans, by the method of least squares. In 1827 it was adjusted by Rosenberger, where the probable error of an angle was computed =6.0" (*Astronom. Nachr.*, vol. 6, p. 18). In 1831 Hansen undertook, likewise, an adjustment of this triangulation and found the mean angle error = 10.99'' (*Astronom. Nachr.*, vol. 9, p. 243). The difference results from the fact that Rosenberger among 24 condition equations had six which were already contained in the remaining ones, as Hansen states in *Astronom. Nachr.*, vol. 9, p. 216.

According to a note by Nagel in *Zivilingenieur*, 1890, p. 412, we find from 16 triangles according to the international formula:

$$m = \sqrt{\frac{8744}{16 \cdot 3}} = \pm 13.50'' . \tag{2}$$

The French degree-measurement by Méchain and Delambre between Dunkirk and Barcelona from 1792 was published in three volumes, 1806-1810, in the famous work: *Base du système métrique*, etc.

According to Ferrero's report, from 93 triangles:

$$m = \pm 1.08''. \tag{3}$$

A thorough representation of the degree-measurements from Cassini to Delambre and Méchain is found in the work: J. B. J. Delambre, *Grandeur et Figure de la Terre*, Ouvrage augmenté de notes, de cartes, et publié par les soins de G. Bigourdan, Membre de l'Institut, Paris, 1912.

Section 121. The Older Degree-Measurements in England, Russia, Denmark

I. The British land survey

The triangulation, already started in 1783 under General Roy, was completed in 1858 under James and Clarke.

A large work was published about it:

Ordnance trigonometrical survey of Great Britain and Ireland, Account of the observations and calculations of the principal triangulation and of the figure, dimensions and mean specific gravity of the earth as derived from, etc., by Captain Alexander Ross Clarke under the direction of Colonel H. James, Superintendent of the Ordnance survey, London, 1858.

The adjustment of the triangulation has been broken down into 21 partial nets, with a total of 202 points.

The adjustment was carried out by using bearings (*ord. trig. surv.*, pp. 273-277). The net adjustment was preceded by an approximate station adjustment, which we have already described in section 64, in the table of p. 211. Approximate weight determinations were attached to these station adjustments (*ord. trig. surv.*, p. 66), where partly the deviations of the individual direction observations from their mean, and partly the number of settings were used as a measure of accuracy. The results of the station adjustments entered in the net adjustments with these weights like directly observed directions.

Therefore, in each of the 21 partial nets there takes place an adjustment of directions with unequal weights, about which a few remarks were already made in section 91, p. 341.

The number of all directions is 1554; therefore, on an average there are 74 directions in each partial net.

Treated as a whole, the adjustment would have yielded 920 equations, while the number of equations in the groups varies between 12 and 64 (*ord. trig. survey*, p. 277). With respect to the mutual group connections the following is reported on pp. 272-273: After one group was adjusted independently of all others, the corrections obtained therefrom were introduced into the condition equations of the next group and the sum of squares of the errors in this second group was made a minimum; in the same manner, a third group was attached, and so forth. Four of the triangulation groups have an independent beginning (or five according to pp. 276 and 277, namely the 1st, 6th, 7th, 12th, 14th), without extraneous conditions; conditions were rather transferred from these starting groups to the adjacent groups. For the connections of the measured base lines, forcing conditions were not introduced, for two base lines never came together in one group. It is true that the accumulation of errors could be decreased by means of conditions on the base connection, but otherwise we obtain a good check for the theodolite work in the unconstrained base connections.

Computations on the accuracy of angles have not been carried out, however the triangle closures are all given; from these 464 triangle closures (*ord. trig. surv.*, pp. 426-495) we have formed the following table, where there are denoted by n the number of triangles in each group, by $[\pm w]$ the absolute sum of triangle discrepancies and by $\frac{[\pm w]}{n}$ their average values.

Group No.	n	$[\pm w]$	$\frac{[\pm w]}{n}$	Group No.	n	$[\pm w]$	$\frac{[\pm w]}{n}$
1.	21	34.65″	1.6″	11.	35	116.67″	3.3″
2.	24	94.58	3.9	12.	16	32.49	2.0
3.	14	66.31	4.7	13.	16	69.23	4.3
4.	19	65.68	3.5	14	22	43.03	2.0
5.	40	104.31	2.6	15.	45	202.14	4.5
6.	14	78.69	5.6	16.	29	63.72	2.2
7.	18	42.70	2.4	17.	19	57.17	3.0
8.	20	39.54	2.0	18.	12	33.35	2.8
9.	21	27.08	1.3	19.	19	45.42	2.4
10.	14	38.77	2.8	20.	19	53.25	2.8
				21.	27	105.79	3.9
				Sum from 1 to 21	464	1414.57″	

(1)

From all 464 triangles we thus have according to the convenient formula (8), section 117, p. 486:

$$m = 1.2533\,\frac{1414.57}{464\sqrt{3}} = \pm 2.21''.$$

In the report by Ferrero of 1898, on p. 134 there is computed according to the international formula from 552 triangles:

$$m = \sqrt{\frac{5548}{552\cdot 3}} = \pm 1.83'' \; . \tag{2}$$

II. Russia

The first Russian degree-measurement, undertaken at the instigation of the director of the Dorpat observatory F. G. W. Struve, is treated in the work:

Beschreibung der unter Allerhöchstem Kaiserlichen Schutze von der Universität veranstalteten Breitengradmessung in den Ostseeprovinzen Russlands, ausgeführt und bearbeitet 1821-1831 mit Beihilfe von B. W. v. Wrangel und anderen, von F. G. W. Struve, Direktor der Dorpater Sternwarte. Erster und zweiter Teil, Dorpat, 1831.

The mean error of an angle of a triangle after the repetitions and comparisons at the stations was found to be (in the place cited pp. 137-138 and pp. 148-149):

$$m = \pm 0.60''.$$

According to Ferrero's report of 1898, on p. 414 the sum of the squares of the discrepancies for 38 closed triangles =38.03, therefore, the mean error of an angle from this:

$$m = \sqrt{\frac{38.03}{38\cdot 3}} = \pm 0.58'' \; . \tag{3}$$

This Baltic degree-measurement, until 1852, was extended under the direction of Struve and General Tenner northward as far as Torneå and southward as far as the mouth of the Danube, covering a difference of latitude of 20-1/2°. The results of these measurements are published in *L'arc du Méridien entre le Danube* *et la Mer Glaciale*, par J. G. W. Struve, St. Pétersbourg, 1860. According to Ferrero's report 1898, p. 414, we form by combination of the individual parts of this meridional arc 214 triangles with the sum of the squares of the discrepancies =379.6, with which we have:

$$m = \pm \sqrt{\frac{379.6}{214 \times 3}} = \pm 0.77'' . \tag{4}$$

This degree-measurement was connected, in the years 1832-34, near Memel with Bessel's degree-measurement in East Prussia, from 1852-54 near Thorn and Tarnowitz with the Weichsel chain or with the Silesian-Posen triangulation net of the Prussian land triangulation.

Russia has furnished a further important contribution for the determination of the shape of the earth in the measurement of one degree of longitude on the 52nd degree of latitude, a chain of triangles which begins with the above-mentioned connection to the Prussian land triangulation near Thorn, penetrates the Struve-Tenner latitude degree-measurement and extends as far as Orsk. The whole chain, which was measured in the years 1827-61, with some additions from the years 1863, 1864, and 1872, contains 364 triangles with a difference of longitude of 39-1/2°. For the mean error, according to Ferrero's report (1898), there follows:

$$m = \pm 1.05''.$$

The measurement of one degree of longitude at 47-1/2° latitude between Kischinew and Astrachan, which was measured from 1847 to 1856, forms a third main triangulation chain. It starts from the Struve-Tenner chain and comprises 196 triangles with a difference of longitude of 19° 12'. The mean error of this chain is according to Ferrero:

$$m = \pm 1.06''.$$

The two longitude degree-measurements are connected near Charkow and Saratow by means of meridional chains.

A report on the Russian degree-measurements is given by S. Truck in *Zeitschrift für Vermessungswesen*, 1903, pp. 193-204.

A diagrammatical map of Struve's arc of the degree-measurement and his connections with newer triangulations are contained in the report on the 7th session of the Baltic-Geodetic Commission in Moscow, Vol. I, 1935, p. 102.

III. Denmark

The Danish degree-measurement was started in 1816 under the direction of Schumacher and carried out until 1824 from Altona to Lysabbel on Alsen; at the same time there was effected the connection to the side Hamburg-Hohenhorn of the degree-measurement planned by Gauss (cf. Fig. 3, p. 499). During the years 1837 to 1848 the measurements were resumed. A chain was carried from Copenhagen westward to the continent and then southward as far as the attachment to the older chain. Finally, by means of a meridional chain, Copenhagen was connected with the "Küstenvermessung" of the Prussian land triangulation, and at the same time, the connection between Copenhagen and the Swedish net was also produced.

The measurements from the years 1867 to 1870, in which the triangulation was continued on the continent as far as Skagen, form a third epoch.

After Schumacher's death (1850), the direction of the operations passed over, in 1852, to Andrae, who published the results in the work:

Den Danske Gradmaaling, udgived af C. G. Andrae, Geheime Etatsraad og Directeur for Gradmaalingen Kjobenhavn, Vols. I-IV, 1867-1884. To this, there also belongs:

Problèmes de haute géodésie, extraits de l'ouvrage danois: Den danske Gradmaaling, 1er cahier: Formation et calcul des triangulations géodésiques, Copenhague, 1881; 2e cahier; calcul des latitudes des longitudes et des azimuts sur le sphéroïde terrestre, etc., 1883.

An interesting literary review of the work: *Den Danske Gradmaaling* is given by Helmert in *Vierteljahrsschrift der Astronomischen Gesellschaft*, 1877, pp. 184-239, and 1878, pp. 57-80.

On the occasion of the hundredth anniversary of the day of the foundation of the Danish land survey, there was issued a report on the activity of the Danish land survey for the previous hundred years, which has been published as No. 16 of the work, *Den danske Gradmaaling* under the title "Le service géodésique du Danemark 1816-1916," Copenhague, 1916. Cf. in this connection *Zeitschrift für Vermessungswesen*, 1916, pp. 299-306.

The computation of the Danish triangulation distinguishes itself by fine theories and by rigorous investigations of accuracy. Fig. 1 shows a portion of the Danish triangles, which are computed on the basis of the base of Copenhagen. The connection of the base line at Copenhagen, 2701 meters long, with the first main side Copenhagen-Snoldelev is achieved by means of a base net of five triangles (which is not shown in Fig. 1 because of the smallness of the scale). For the seven triangle sides, which are drawn heavily in Fig. 1, the mean errors have been computed and, in fact, with the separation of the influence of the errors of the measurement of angles from the influence of the base error. The following table gives the error quotients for this, expressed in millionths of the sides themselves, or in millimeters for 1 kilometer.

We notice that the influence of the base error in comparison to the influence of the angle error almost vanishes already in the fifth triangle.

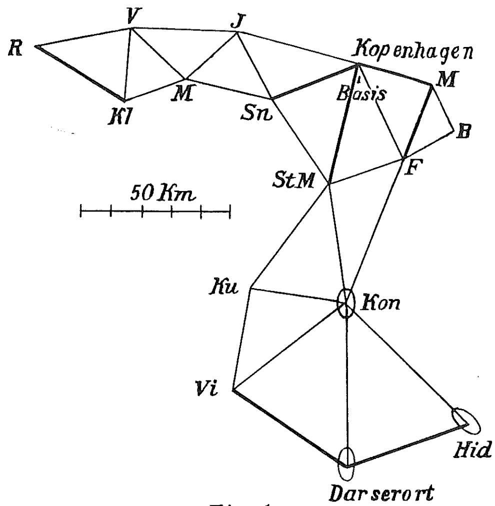


Fig. 1.
Danish main triangulation net.
(Scale 1:2,500,000).

Triangle Sides s	Number of Triangles	Errors Originating from Angle Measurement	Errors Originating from Base Measurement	Total Mean Error
Base Line	..	0.0	1.7	1.7 Millionths
Kopenhagen-Snoldelev	5	4.4	1.7	4.7 Millionths
Kopenhagen-St. Möllehöi	6	4.5	1.7	4.8 Millionths
Kopenhagen-Malmö	8	4.9	1.7	5.2 Millionths
Malmö-Falsterbo	8	5.0	1.7	5.3 Millionths
Refsnaes-Klöveshöi	11	6.8	1.7	6.9 Millionths
Vigerlöse-Darserort	11	7.1	1.7	7.3 Millionths
Darserort-Hiddensö	12	7.3	1.7	7.5 Millionths

For the points Kongsbjerg, Darserort, and Hiddensö, the error ellipses were also computed. By assuming the base of Copenhagen as free of error, and taking into account all angle errors from the base (Andrae Case II), we have the major semiaxis a, the minor semiaxis b, and the azimuth z of the major semiaxis for the three error ellipses mentioned:

Kongsbjerg	$a = 0.56\ m$	$b = 0.12\ m$	$z = 1°$
Darserort	$a = 1.06$	$b = 0.29$	$z = 178$
Hiddensö	$a = 1.00$	$b = 0.34$	$z = 156$

The angle measurements, which were used for these error ellipses, are very accurate, e.g., from the 32 corrections to the angles of p. 296 of the first volume of the *Danish degree-measurement* we find the sum of squares 3.57 with seven condition equations, therefore the mean error of an angle adjusted at the station:

$$m = \sqrt{\frac{3.57}{7}} = \pm 0.71'' .$$

But the accuracy, which is illustrated by these error ellipses, is still given too great, because the error of the unit of weight thereby used [μ according to formula (12), section 117, p. 486] does not take into account that μ_1 at the stations and μ_2 in the net are not equal.

The values μ_1 and μ_2 are the following for five adjustments of the Danish degree-measurement (according to a summary given already previously in *Zeitschrift für Vermessungswesen*, 1877, p. 393):

1.	$\mu_1 = \pm 1.00$	$\mu_2 = \pm 1.63$	$\mu_2 : \mu_1 = 1.63$
2.	1.00	1.65	1.65
3.	1.00	3.08	3.08
4.	1.64	2.99	1.82
5.	0.96	1.75	1.82 .

The value 3. $\mu_2 : \mu_1 = 3.08$ is dependent on special circumstances; as to the rest, we have on the average approximately:

$$\mu_2 : \mu_1 = 1.7,$$

i.e., nearly the same value as in the case of the Prussian land triangulation.

According to the report of Ferrero of 1898, pp. 44 and 45, Denmark has according to the international formula:

	1817—1824	$[w^2] = 37.25$,	$n = 20$,	$m = \pm 0.79''$
	1837—1847	$[w^2] = 152.19$,	$n = 51$,	$m = \pm 1.00$
and for the later measurements	1867—1870	$[w^2] = 9.39$,	$n = 16$,	$m = \pm 0.44$
		$[w^2] = 198.83$,	$n = 87$,	

hence from all 87 triangles:

$$m = \sqrt{\frac{198.83}{87 \cdot 3}} = \pm 0.87''. \tag{5}$$

A sketch of the history of the Danish triangulations since the middle of the 18th century is contained in the report on the 5th session of the Baltic-Geodetic Commission in Copenhagen, 1930, p. 45.

Section 122. The Classical Work of Gauss

The theory of the triangulation adjustment of whole nets according to conditioned observations (with correlates) was first published in 1826 by Gauss in "Supplementum theoriae combinationis" (cf. p. 3 or Gauss' *Werke*, IV. Band, pp. 82-93).

It is true that the first trigonometric adjustment in general took place somewhat earlier, namely in 1821, a resection for a station point with six observations of directions (cf. pp. 3 and 306), however the triangulation nets occur in 1826 in "Supplementun . . ." for the first time with two examples, first for angle measurements, second for direction measurements.

The first example of Art. 23 of "Supplementum . . ." is drawn in Fig. 1; the measurements are taken from Krayenhof, *Précis historique des opérations trigonométriques en Hollande* (cf. also *Allgemeine geographische Ephemeriden*, Vol. 4, 1799, p. 80, and *Zeitschrift für Vermessungswesen*, 1885, pp. 172-173). The 27 angles of the triangles, which are numbered in Fig. 1 by 0, 1, . . . , 26, are considered as measured individually independently (whether this is actually true was probably of no great concern in a computational example of that treatise). There exist nine triangles with the following closure errors:

		w	w^2
121.	c	$-3.958''$	15.6658
122.	d	$+0.722$	0.5213
123.	e	-0.753	0.5670
124.	f	$+2.355$	5.5460
125.	g	-1.201	1.4424
			23.7425

		w	w^2
127.	h	$-0.461''$	0.2125
128.	i	$+2.596$	6.7392
131.	k	$+0.043$	0.0018
132.	l	-0.616	0.3795
			23.7425
			31.0755 .

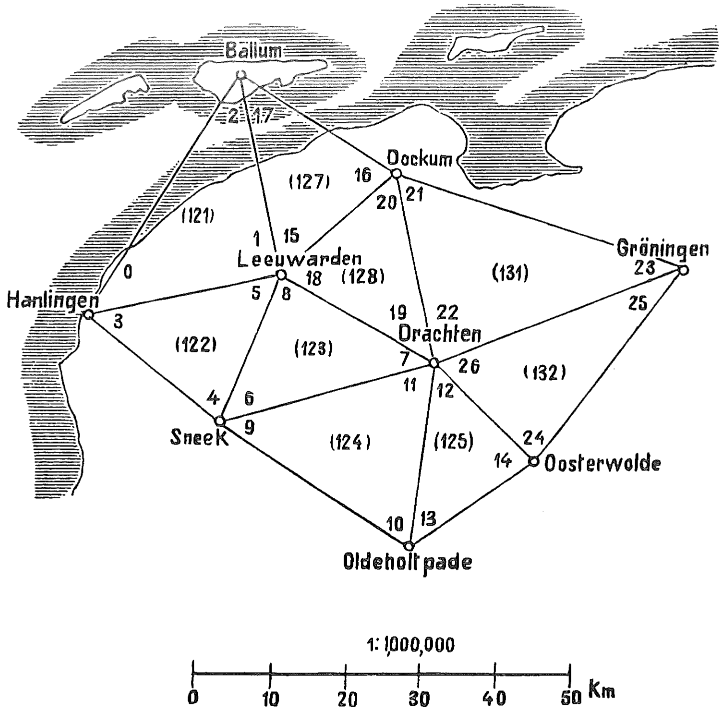


Fig. 1. Dutch triangulation net.

According to the international formula (1), section 116, p. 480, we would therefore compute:

$$m = \sqrt{\frac{31.0755}{9 \cdot 3}} = \pm 1.073'' . \tag{1}$$

However, we do not discuss this now. We rather will pursue Gauss' adjustment and we can be brief insofar as a quite similar example, likewise with nine triangles and 27 angles with two horizon checks and two side equations, whose net is drawn on p. 225, has been calculated with all details in our previous editions (for the first time in *Astronomische Nachrichten*, volume 75, 1870, pp. 299-302) for many years. It is also not accidental that this Baden net on p. 225 has quite closely the form of the above Gaussian net of Fig. 1, since the author, at that time, 1869, had in view to combine, from the angles of Baden, a net which was to correspond exactly to the classical pattern of "Supplementum theor. comb." This is carried out in such a way that also the computation in units of the seventh place of logarithms for the linear side equations was originally borrowed from the classical pattern, and was not abandoned until the third edition, volume I, 1888, section 70, in favor of a better computation, in units of the sixth decimal.

Since we can thus dispense with a further demonstration of details from Art. 23 of "Suppl. theor. comb." by referring to the net on p. 225, we will give from there only just the sum of the squares of the corrections and the computation of the mean error belonging to this, namely:

$$m = \sqrt{\frac{97.8845}{13}} = \pm 2.7440'' , \tag{2}$$

while the corrections to the angles applied by K r a y e n h o f yield 341.42 for the sum of the squares and $m = \pm 5.12''$ for the mean error.

We will also compare the actual mean error $m = 2.74''$ according to (2) with the value 1.07'' of (1), obtained only from the triangle closures, and notice that the correct value is approximately 2-1/2 times the approximate value; this is perhaps connected with the conditions already mentioned in *Zeitschrift für Vermessungswesen*, 1885, pp. 172-173 below.

The second example from "Suppl. theor. comb." is the pentagon of the degree-measurement of Hannover, drawn in Fig. 2, whose net has been given by Gauss as an enclosure to *Astronomische Nachrichten* 1st volume, No. 24, as we shall see in Fig. 3, p. 499 following.

If we stop at this pentagon cut out as an example, then we can say briefly about it that it contains an adjustment according to equally weighted *directions*, similar to our two examples of section 68, p. 226, and section 70, p. 236; and our computations given there are in fact nothing else but imitations of the example from the Luneburg Heath, already more than 100 years old now, only with the formal difference that the computations were no longer carried out in units of the seventh place of logarithms in the linear side equations. The classical pentagon has already repeatedly been explained, for the first time in Helmert's *Adjustment Computation According to the Method of Least Squares*, 1872, pp. 185-195 (2nd edition, 1907, pp. 251-261), where, on p. 189, the inconvenience with the unequally large coefficients is already recognized and eliminated by a change of the units of measure (cf. p. 281), further in Jordan-Steppe's *Deutsches Vermessungswesen*, 1882, pp. 1-10, so that we can content ourselves here to count, according to Fig. 1, with the rules of p. 225 for five points and 18 directions, the number $18 - 15 + 4 = 7$ condition equations, among which are $9 - 10 + 3 = 2$ side equations and $9 - 5 + 1 = 5$ independent triangle closures. The 18 corrections to the directions yield the sum of squares 1.2288, hence, the mean direction error is:

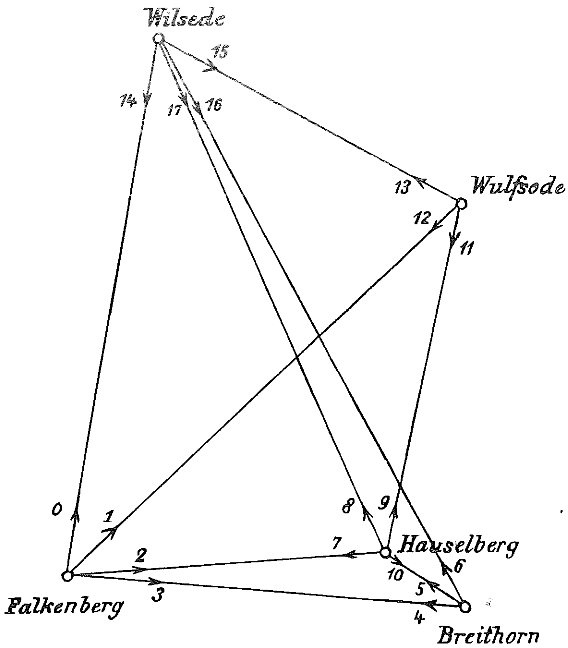


Fig. 2. (1:500,000.)

$$\mu = \sqrt{\frac{1.2288}{7}} = \pm 0.419'', \tag{3}$$

or the mean angle error is:

$$m = \mu\sqrt{2} = \pm 0.593''. \tag{4}$$

Let us make also use, in this connection, of the international error formula of section 116; seven triangles are possible, from which we obtain:

w	$-1.368''$	$-1.139''$	$+1.773''$	$+1.042''$	$-1.481''$	$-0.813''$	$-0.750''$
w^2	1.8714	1.2973	3.1435	1.0858	2.1934	0.6610	0.5625

$$m = \sqrt{\frac{10.8149}{7 \cdot 3}} = \pm 0.718''. \tag{5}$$

In Art. 25 of "Supplementum . . ." Gauss then computed, in addition, the function weight of a side (hence, in a similar way as we have computed the accuracy of a net diagonal for our pentagon of Hannover in section 72). The northern side Wilsede-Wulfsode = 22877.94 m was assumed as an error-free base and the southern side Falkenberg-Breithorn = 26766.68 m was derived therefrom with a mean error ± 0.12 m, and, besides, the assumption was made that the point Hauselberg is omitted so that the southern side is attached to the northern side only by means of the western side. The accuracy of transfer, of course, is thereby decreased and the southern side becomes 26766.63 ± 0.15 m.

The degree-measurement and land survey of Hannover

The degree-measurement of Hannover was undertaken by Gauss, connecting onto the Danish degree-measurement by Schumacher, see above section 121 III, p. 494, in order to extend the latter to the south, whereupon it had its continuation later by Gerling (see further below, p. 502) in the triangulation of the electorate of Hesse.

In this connection, let us mention, from a letter by Gauss to Bohnenberger of 16 November 1823, the following passage which now, after one century, shines in the brightest light:

> [Translated] "How beautiful would it be if at one time all measurements extending over Europe from Scotland to the Banat and from Copenhagen to Genoa and Formentera could be brought into o n e connecting system. I would like to prepare for this to the best of my powers, however, if one is beyond the middle of one's life, so extended an object should be begun the sooner the better."

We have presented the net of the actual degree-measurement of Hannover in Fig. 3 below.

This is the form as it has been given by Gauss himself in two enclosures to *Astronomische Nachrichten*, 1st volume, No. 7 (Altona, February 1822) and No. 24 (December 1822), the latter, essentially, like our following Fig. 3, however, without the connection between Timpendorf and Lüneburg, which is found with the point

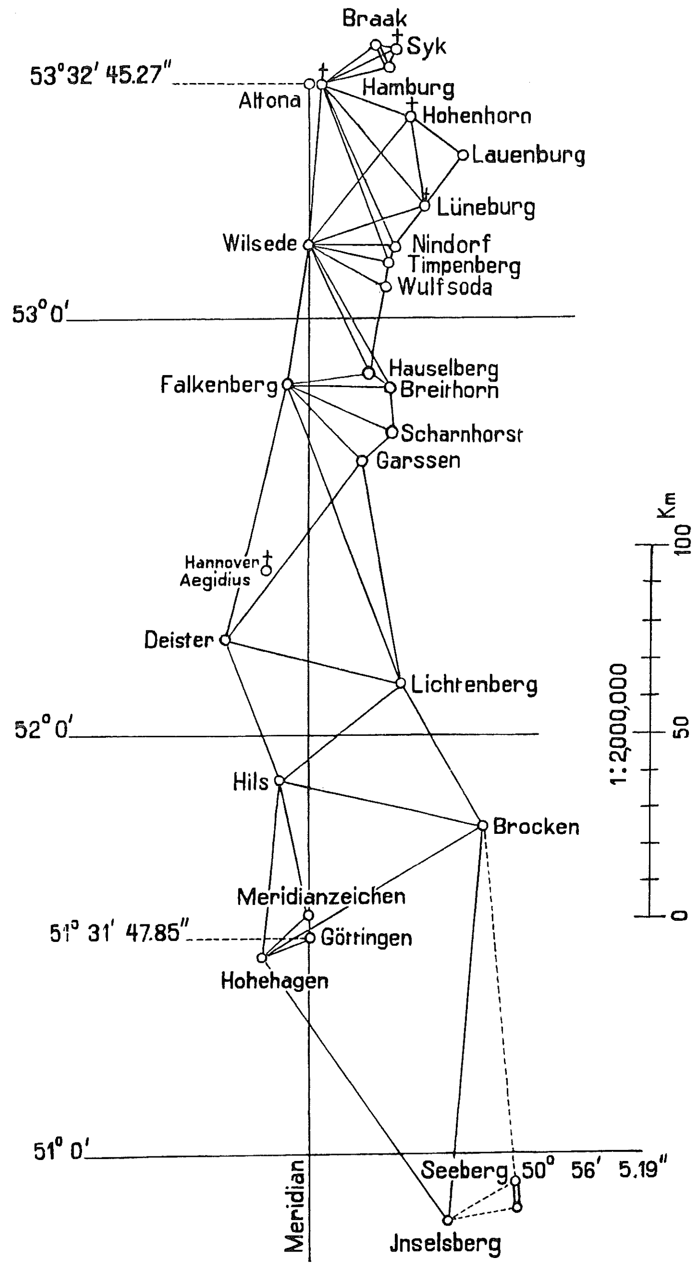


Fig. 3.
Degree-measurement of Hannover between Göttingen and Altona by Gauss, 1820-1825.

Nindorf in the work; *Die Kgl. Preuss. Landestriangulation,* Hauptdreiecke, Part VI, Berlin, 1894, Sketch 3.

The base lines north near Braak and south near Seeberg (Gotha) indicated in Fig. 3, also, are not contained in the triangulation net picture of Gauss himself of 1822, but have been added according to other data.

The latitudes 51°31' 47.85" for Göttingen and 53°32' 45.27" for Altona entered in Fig. 3 yield the difference of latitude 2°0' 57.42" according to p. 64 of the work, *Bestimmung des Breitenunterschiedes zwischen den Sternwarten von Göttingen und Altona,* by Gauss, Göttingen, 1823.

A connected adjustment, a presentation in a public work in general, has unfortunately never been published by Gauss concerning that degree-measurement. Original files of Gauss' measurements were found, after 1866, in the Royal Archives at Hannover, but at the request of Field-Marshal General v. Moltke, they were then delivered to the Prussian General Staff in Berlin, where they were taken over by the trigonometric section of the Land Survey.

In this way, information has been given about the history of Gauss' surveys in Hannover in a paper: "Beiträge zur Kenntnis von Gauss' praktisch geodätischen Arbeiten, nach Originalmaterialien bearbeitet von Gaede, Hauptmann bei der trigonometrischen Abteilung der Landesaufnahme." (*Zeitschrift für Vermessungswesen,* 1885, pp. 113, 145, 161, 177, 193, 225.) There is to be mentioned further: *Gedächtnisrede auf Carl Friedrich Gauss, zur Feier des 30. April 1877, by* Theodor Wittstein, Dr. phil. and Professor, Hannover, Hahnsche Buchhandlung, 1877, and finally Band IX von Gauss' *Werke* published in 1903, which, especially (also in continuation to Band IV) contains the geodetic operations. It is edited by Krüger with the assistance of Börsch and gives valuable information and, to a great extent, hitherto unknown contributions from the handwritten posthumous works, which are preserved in the Gauss Archives in Göttingen. While we must refer to the numerous individual expositions and examples of this valuable source, only a few general features can be mentioned here.

Concerning the measuring procedure it should first be mentioned that there was used a twelve-inch repeating theodolite, which is described by Ambronn in *Zeitschrift für Vermessungswesen,* 1900, pp. 177-180. In his degree-measurement itself Gauss did not rigorously apply the method of measuring angles in all combinations of directions, originating with him and already mentioned on p. 204 (see also section 86), but according to circumstances, he measured a series of angles by repetition with various repetition numbers. The one-sided repetition errors were already recognized here (cf. this *Handbook,* Volume II, 1, 9th Edition, 1931, p. 374). Gauss measured the individual angles until he believed that each one was given its due. The repetition numbers served as weights.

For signaling there was used the heliotrope, cf. this *Handbook,* Volume III, 7th Edition, 1923, p. 33, for whose invention for angle observations in Lüneburg, 1818, a window of the Michaelis Tower in Hamburg flashing in the sun gave the inspiration.

In the case of the station adjustment the corrections to the directions were taken as unknowns; from the measured angles and the system of the approximate azimuths there followed the normal equations, which were solved indirectly by taking up the unknowns successively according to their amount. The thus obtained values were introduced in the net adjustment as equally weighted and independent.

For the net adjustment, which no longer exists connectedly, and explained by Krüger on the basis of examples according to sheets found, Gauss used a successive partial elimination method, peculiar to him, by setting up first, alternately, normal equations for the angle conditions and taking these then together with side equations transformed according to them. We must refer here to the mentioned volume IX, Gauss' *Werke,* p. 297, and the explanations given to this by Krüger, p. 327, and the quoted passages from letters.

About the accuracy of Gauss' measurements let us mention first the following passages from letters, in which Gauss expresses himself about his results.

Gauss wrote to Bessel on 5 November 1823 (*Briefwechsel mit Bessel,* p. 423; cf. also *Zeitschrift für Vermessungswesen,* 1885, p. 205): [Translated] "I have carefully adjusted the system of my main triangles these last few days . . . . There are together 26 triangles in which all angles are observed by myself. The largest sum of the errors is 2.2", where pointing was very difficult in the case of one side; the next-to-the-largest is 1.8". None of the 76 occurring directions is changed by a whole second in the adjustment; the largest change amounts to 0.813"."

Further, Gauss to Bohnenberger on 16 November 1823 (cf. *Zeitschrift für Vermessungswesen,* 1882, p. 431, and 1885, p. 205):

[Translated] "In the case of my measurements I have found that that which I call the mean error in my treatise 'Theoria combinationis etc.' is about $3.5'' : \sqrt{n}$ (n = number of repetitions) from several stations, good and less good measurements calculated together. With a very firm setup, very favorable (i.e. not trembling) air and exclusively heliotropic target points, the mean error is considerably smaller, however. All of my measurements yield thus far 76 main directions (38 forward and 38 backward), and from the adjustment of the errors it was found that the *mean* error of a main direction was = 0.47"."

Then in Gauss' *Werke*, Band IX, pp. 300 and 301, there are given 51 triangle closures, which belong to the net Fig. 3, p. 499, in addition to its northwest extension from the sides Falkenberg, Wilsede, Hamburg along the coast through Bremen, Varel as far as Jever; according to the international formula of triangle closure we form from this:

$$\text{The mean angle error} = \sqrt{\frac{125.907}{3\times 51}} = \pm 0.909'' \; . \; . \; . \; . \qquad (6)$$

Finally, on p. 314, there is also given the mean direction error of the net from the adjustment treated there as

$$\pm 0.7548'' \; . \; . \; . \; . \qquad (7)$$

The degree-measurement and its mentioned northwesterly extension gave rise to lay a main triangulation net over the whole of Hannover, which extended between 52° and 54° latitude (cf. Fig. 3, also the later section 131) about 200 km west and 100 km east of the meridian of Göttingen (cf. the map in Gauss' *Werke*, Volume IX). This land triangulation of Hannover was carried out during the time from 1828 to 1844 by Gauss' technical sides in the degree-measurement, namely his son, Lieutenant Gauss (later Baurat [government surveyor of buildings]), Captains Müller and Hartmann (the latter was teacher of practical geometry in Hannover from 1831 until his death 1834). On the main net there were based a net of second order and a detail net.

In the above-mentioned (p. 500) commemoration speech [Gedächtnisrede] Wittstein says on p. 12:

[Translated] The result of the whole survey, which lasted 24 years, and caused the moderate expenditure of 42,000 thalers, amounted finally to a list of coordinates of 2578 fixed points of the Land or, on an average, three points to a square mile (hence, one point = 50 marks).

Gauss' coordinates, in the conformal system, with the zero point Göttingen, was published by Wittstein with an introduction in the small volume:

Grundsteuerveranlagung. Allgemeines Koordinatenverzeichnis als Ergebnis der hannoverschen Landesvermessung aus den Jahren 1821-1844. Abgedruckt zum Zweck der Benützung bei den Vermessungsarbeiten zur Vorbereitung der anderweitigen Regelung der Grundsteuer. Hannover, 1868. Druck von Wilh. Riemschneider. (This work was at that time sent to all members of the European degree-measurement, and has in the meantime become very rare in Hannover itself.-)

Gauss' coordinates are also printed in *Karl Friedrich Gauss' Werke*, IV. Band, Göttingen, 1873, pp. 415-445. Wittstein first changed the signs of the original coordinates, which counted $+x$ south and $+y$ west, and second, he reduced them by a constant reduction of 1:62900; e.g., Hannover: Ägidius has

according to Gauss' IV. Band, p. 428	$y = +$ 13,880.010 m	$x = -$ 93,577.384 m,
according to Wittstein, p. 36	− 13,879.79 m	+ 93,575.89 m.

We can further give a few characteristic figures of the Hannover land triangulation by comparing, in the city of Hannover, the five old trigonometric points, namely the four old church towers and the Waterloo Column, with the new survey of 1891-1892. With everything referred to Ägidius as the center point, we have computed from the coordinates the direction angles to the other four points, reduced them to north by means of the meridian convergences, and also inserted the distances computed from the coordinates, whereby the following comparison resulted:

Data [Abrisse] of Station Ägidius

Direction to	1840		1892		Differences	
	Azimuth	Distance	Azimuth	Distance		
Waterloosäule	247° 23′ 15″	841.78 *m*	247° 22′ 40″	841.26 *m*	− 35″	− 0.52 *m*
Neustädter Turm	284 29 22	761.64	284 29 24	761.98	+ 2	+ 0.34
Marktturm	313 25 10	378.56	313 24 54	378.58	− 16	+ 0.02
Kreuzturm	314 8 0	631.85	314 9 11	631.64	+ 71	− 0.21
	85′ 47″	2613.83 *m*	86′ 09″	2613.46 *m*	+ 22″	− 0.37 *m*

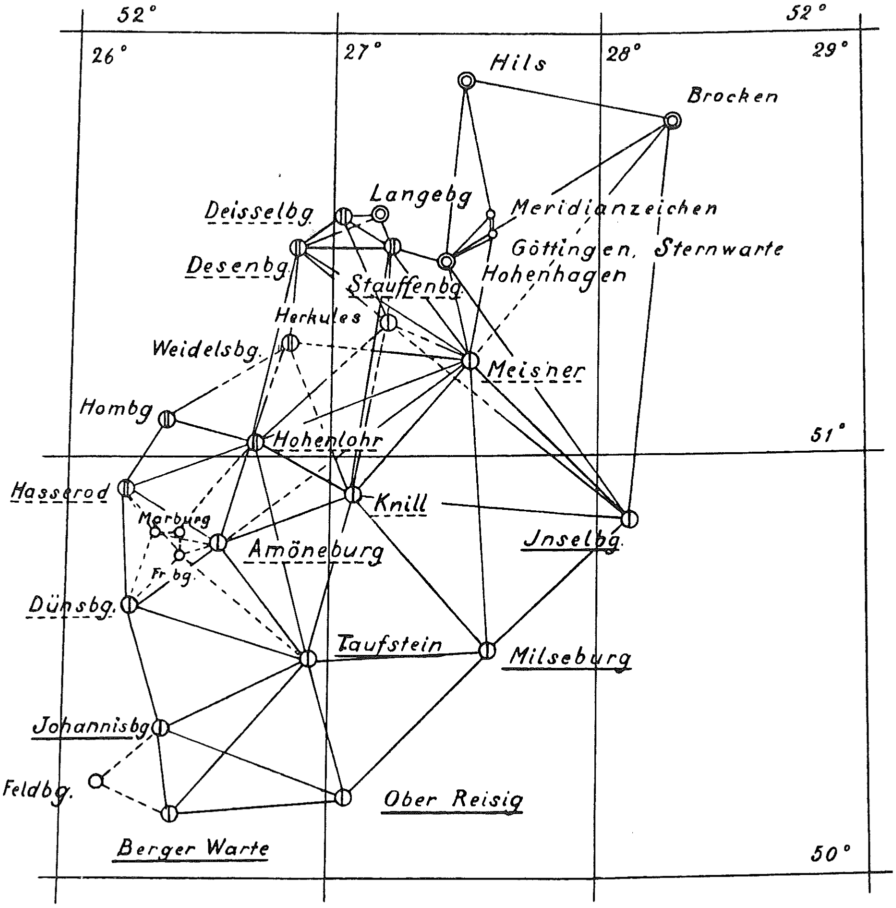
Although a part of these differences are perhaps due to point displacements in the course of 50 years (*Zeitschrift d. Hann. Arch.- u. Ing.-Ver.*, 1889, p. 156), this comparison shows at one glance that those measurements from 1828 to 1844 did not furnish and also would not furnish what we call today an accurate special triangulation.

Gauss expressed himself on this also in a report of 21 November 1827 (Gauss' *Werke*, Band IX, p. 414):

[Translated] "This statement of the position of a great number of fixed points in numbers (i.e. how much north or south, west or east, of an arbitraty starting point, e.g. the Göttingen Observatory) *accurate to a few feet*, must be considered as the main result of the operation from the topographical point of view."

The land survey of Hannover has become of outstanding significance also through the conformal coordinate system introduced for it by Gauss, which is discussed in detail in this *Handbook*, Volume III, 7th Edition, 1923, sections 48-50, 54-55, 57, 89-94 (cf. also the previous section 71, p. 251). After 1866, this old classical Gauss system with the zero point Göttingen was dissolved, for the cadastral administration, into 31 partial systems, which by themselves were still conformal. In 1881, they were replaced by the new Soldner coordinates, which were introduced with 40 systems for the whole of Prussia.

To conclude the work of Gauss there is further to be mentioned the triangulation of the electorate of Hesse, yielding a continuation of the degree-measurement of Hannover, by a student of Gauss, Professor Gerling of Marburg.



◎ *Points observed by Gauss.*

⦶ *Points of the net of the Ist period.*

⦶ *Points of the net of the IInd period.*

——— *Observed by Gerling himself in the Ist period.*

------ *Observed by Gerling himself in the IInd period.*

Fig. 4.

Gerling's triangulation of the electorate of Hesse, 1822-1824 and 1835-1837.

Gerling published his triangulation in the work, *Beiträge zur Geographie Kurhessens und der umliegenden Gegenden aus der kurhessischen Triangulierung der Jahre 1822-1837*, Kassel, 1839. In this publication as well as in his textbook *Die Ausgleichungsrechnungen der praktischen Geometrie*, Hamburg and Gotha, 1843, Gerling gives some information about Gauss' measuring and computing method (concerning the angle measurement in all combinations, setting up the condition equations, the indirect elimination) according to the verbal and written instructions which he had received from Gauss, and according to which he had arranged his method; cf. concerning this, "Chr. L. Gerling's geodätische Tätigkeit," *Zeitschrift für Vermessungswesen*, 1901, pp. 1-56.

Gerling's triangulation of the electorate of Hesse, which was carried out in two periods, from 1822 to 1824, and from 1835 to 1837, connected onto Gauss' triangle Hohehagen, Brocken, Inselsberg (see Fig. 3, p. 499) and continued the chain of the degree-measurement south close to the 50th degree of latitude. The net is represented in Fig. 4, p. 502; it contains 24 main points and, in addition, 17 auxiliary points. The net of the main points yielded 45 condition equations, which were treated according to Gauss' procedure by indirect elimination. The mean error of his directions is given by Gerling, *Beiträge*, p. 182, = ± 0.88".

Furthermore, in the *Gen. Bericht d. europ. Gradmessung für 1865*, p. 47, there were made statements for the mean direction error of the triangulation of the electorate of Hesse by Börsch and Kaupert, namely for the first part 0.95" and 0.99", a mean of 0.97", and for the second part 1.37". From the 25 triangle closures of the main net there results, according to the international formula, ± 1.20", or therefrom for comparison with the above direction error, correspondingly, ± 0.85".

Section 123. The Operations of Bessel and Baeyer

I. The degree-measurement in East Prussia

The second classical geodetic work in Germany is published in the work, *Gradmessung in Ostpreussen und ihre Verbindung mit preussischen und russischen Dreieckskettten*, ausgeführt von F. W. Bessel, Direktor der Königsberger Sternwarte, Baeyer, Major im Generalstabe, Berlin, 1838.

This degree-measurement, whose net Fig. 1 represents, is, just as the degree-measurement of Hannover, a highly important work in the history of our science, in which triangulation adjustment and the

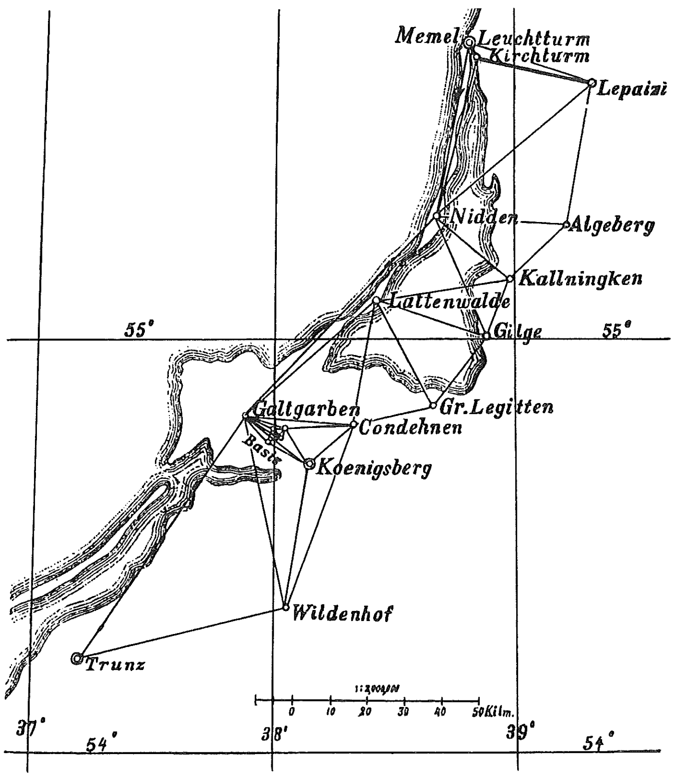


Fig. 1.

determination of the earth, in general, have in their theoretical development considerably been promoted.

On page IV of the Preface, the new triangulation is labeled as a component of an uninterrupted trigonometric connection from Formentera and from northern England to the Russian degree-measurements, which, connected to the main observatories of Europe, was to give "a foundation for the determination of the figure of the earth at least in the extent of this continent."

We already see herein the basic idea of the international unification, which, 25 years later, was in fact realized by the co-worker of Bessel, General Baeyer.

Bessel concluded his operations without any investigations of accuracy, although he had already set up, in 1816, a formula for the computation of the mean error (cf. a note from Helmert to Hagen *Vierteljahrsschrift der Astronomischen Gesellschaft*, 1877, p. 192).

We will profit by this opportunity to apply some of our previous formulae.

The triangulation of the degree-measurement in East Prussia has 17 base points and one point which is only cut in (Memel-Turm), hence, $p = 18$ points, in addition, 87 intersected directions, to which correspond 70 angles at 17 stations. Of the 87 directions, 76 (= 2 times 38) are reciprocal, in addition, four one-sided net directions and seven directions which do not belong to the net, hence, $= 38 + 4 = 42$ net lines. Therefore, according to (20), p. 226, we have the number $42 - 36 + 3 = 9$ side equations and $38 - 17 + 1 = 22$ triangle equations, together $84 - 4 - 54 + 1 + 4 = 31$ condition equations, as is also indicated on p. 140 of *Gradmessung in Ostpreussen* according to another method of counting.

Without entering further on the question what significance would have to be assigned to the seven directions which do not belong to the net at all, we have computed the following for 31 condition equations with 70 angle corrections (1), (2), (3), . . . , (70) or 87 corresponding corrections to the directions according to the formulae of section 117:

The mean angle error from the squares of (1), (2), . . . , (70) becomes according to (1), section 117, p. 484:

$$m = \sqrt{\frac{(1)^2 + (2)^2 + (3)^2 + \ldots (70)^2}{r}} = \sqrt{\frac{17.868}{31}} = \pm 0.76'' ,$$

or from the first powers, according to (9), section 117, p. 486:

$$m = 1.2533 \frac{(1) + (2) + (3) + \ldots (70)}{\sqrt{n \cdot r}} = 1.2533 \frac{24.785}{\sqrt{70 \cdot 31}} = \pm 0.67'' .$$

For the computation according to the international formula, in the work *Die preussische Landestriangulation*, Part VII, p. 153 (cf. the following section 124, p. 531 [no longer translated]), the excesses of the triangles have been submitted to a new computation; with this, there follows for 29 triangles:

$$m = \sqrt{\frac{41.154}{3 \cdot 29}} = \pm 0.688'' . \tag{1}$$

II. The Küstenvermessung [survey of the coast]

The co-worker of Bessel in the degree-measurement in East Prussia, Major Baeyer, had in this enterprise also gained knowledge of Bessel's theory and, after Bessel's death (1846), utilized it independently in publishing the work:

Die Küstenvermessung und ihre Verbindung mit der Berliner Grundlinie. Ausgeführt von der trigonometrischen Abteilung des Generalstabes. Herausgegeben von J. J. Baeyer, Oberst und Abteilungsvorsteher im Generalstabe und Dirigent der trigonometrischen Abteilung, Berlin, 1849.

In the Küstenvermessung we recognize the most painstaking endeavour to follow the pattern of the degree-measurement in East Prussia.

The Küstenvermessung consists of two nets (northern and southern part). The first net has 30 stations, 47 condition equations, 113 angle corrections and, correspondingly, $113 + 30 = 143$ corrections to the directions; however, on pp. 290-294 there are indicated 145 corrections to the directions, because in Trunz there occurs an increase by two.

The second net has 25 stations, 86 condition equations, 141 angle corrections, and $141 + 25 = 166$ corrections to the directions.

According to section 117 we compute therefrom, just as in the above, the mean angle error m in a different manner:

For the First Net:

From the Angle Corrections

$$\left.\begin{aligned} m &= \sqrt{\frac{30.15}{47}} = \pm\, 0.80'' \\ m &= 1.2533\,\frac{45.327}{\sqrt{113 \cdot 47}} = \pm\, 0.78'' \,, \end{aligned}\right\} \qquad (2)$$

For the Second Net:

From the Angle Corrections

$$\left.\begin{aligned} m &= \sqrt{\frac{43.86}{86}} = \pm\, 0.71'' \\ m &= 1.2533\,\frac{72.63}{\sqrt{141 \cdot 86}} = \pm\, 0.83'' \,, \end{aligned}\right\} \qquad (3)$$

$$\text{Average value} \quad m = \pm\, 0.78''. \qquad (4)$$

The work "Hauptdreiecke" already mentioned previously in Part VII, p. 186, gives as a computation from the corrections to the directions:

1. For the first net (northern part) $\quad m^2 = 3.1416\,\frac{34.3764^2}{145 \times 47} = 0.545$

   $m = \pm\, 0.738''$

2. For the second net (southern part) $\quad m^2 = 3.1416\,\frac{49.7174^2}{166 \times 86} = 0.544$

   $m = \pm\, 0.738''$

These values agree with ours, indicated above under (2) and (3) (apart from rounding off).

On p. 187, Part VII of "Hauptdreiecke" there are also given the computations of the mean angle error according to the international formula:

$$\text{For the northern part} \quad \sqrt{\frac{39.021}{3 \cdot 45}} = \pm\, 0.538'' \qquad (5)$$

$$\text{For the southern part} \quad \sqrt{\frac{100.638}{3 \cdot 103}} = \pm\, 0.571'' \qquad (6)$$

$$\text{For both parts} \quad \sqrt{\frac{139.659}{3 \cdot 148}} = \pm\, 0.561''. \qquad (7)$$

Earlier, the author has given a few further details to the *Küstenvermessung* in Jordan-Steppe's *Deutsches Vermessungswesen* I, pp. 38-44, in addition to an explanation of a further work by Baeyer, *Verbindung der preussischen und russischen Dreieckskette n.*

* * * *

Chapter V.

THEORY OF THE PROBABILITY OF ERRORS

Section 136. Main Propositions of the Calculus of Probabilities

The calculus of probabilities deals with accidental happenings, that is, with happenings whose causes are unknown in the individual case. Such happenings, for instance, are the drawing from an urn, the falling of a die, in the case of insurance companies the occurrence of a case of fire or death, in the case of measurements of any kind the occurrence of a measuring or observational error.

Even though the causes of such happenings are not entirely unknown, yet these causes are not further investigated in the individual case when it is a question of calculation of probabilities.

Mathematical probability

The *probability* w of an occurrence is the proportion of the number of cases favorable to the occurrence to the number of cases at all possible.

For instance, the probability of throwing the number 6 with an ordinary die is $= 1:6$, because one case is favorable and six cases are possible.

The mathematical probability is always a proper fraction, with the limits of *zero* for impossibility, and *one* for certainty.

If we are able to count the possible and the favorable cases, we can always indicate the mathematical probability accordingly. For instance, we ask for the probability of throwing the visible sum $= 7$ with two ordinary dice; we reflect here that 36 cases are at all possible because each casting of the one die can be combined with 6 cases of the other die; we count furthermore that the throw seven can occur in all 6 times, namely:

$$1 + 6, \quad 2 + 5, \quad 3 + 4, \quad 4 + 3, \quad 5 + 2, \quad 6 + 1;$$

it follows therefrom that the throw seven in the case of two dice has the probability $\frac{6}{36} = \frac{1}{6}$.

In the same fashion we can also count directly that the throw two has the probability $\frac{1}{36}$, etc.

Instead of direct counting, however, we can also apply in such cases the propositions about sums and products of probabilities, to which we now turn.

Sum of probabilities

If w and w' are the probabilities of two events which are exclusive of one another, the probability w of the occurrence of one *or* the other equals the sum of the two individual probabilities.

$$W = w + w'. \tag{1}$$

Let us imagine that in an urn there are a black, b white, and c other balls; the probability of obtaining a black ball in one drawing then is:

$$w = \frac{a}{a + b + c},$$

likewise for a white ball:

$$w' = \frac{b}{a+b+c}$$

and for a black *or* a white ball, since a and b are favorable now:

$$W = \frac{a+b}{a+b+c} \quad \text{or} \quad = \frac{a}{a+b+c} + \frac{b}{a+b+c} = w + w' .$$

Equation (1) written on p. 507 is thus demonstrated, for all cases which in regard to probability correspond to the chosen example of the urn.

This proposition on the sum of the probabilities can also be applied to more than two events and then yields the following: If w, w', w'' . . . are the probabilities of any desired number of events, the probability of the occurrence of the first *or* the second *or* the third, etc., equals the sum of all individual probabilities.

Later we will apply this proposition to observational errors, in some such form as this: Let the probability of committing an error ε in any measurement be $= w$, the probability of committing an error $\varepsilon' = w'$, the probability of committing an error $\varepsilon'' = w''$, etc., the probability of committing an error ε or ε' or ε'', etc., is then the sum of the individual probabilities:

$$W = w + w' + w'' + \ldots . \tag{2}$$

Product of probabilities

If w' and w'' are the probabilities of two events independent of one another, the probability W of the two occurring together equals the product of the two individual probabilities, that is:

$$W = w \cdot w' . \tag{3}$$

Let us imagine, in this connection, two urns with a and a' black balls, in addition to b and b' other ones; the probabilities for black in the case of these urns are as follows:

$$w = \frac{a}{a+b} \quad \text{or, as the case may be,} \quad w' = \frac{a'}{a'+b'} .$$

Now we inquire about the probability that in two drawings from the two urns black appears both times. $a\,a'$ cases are favorable here because each ball a can be joined by a ball a', and $(a+b)\,(a'+b')$ cases are possible; we thus have: the probability for black and black:

$$W = \frac{a\,a'}{(a+b)\,(a'+b')} \quad \text{or} \quad = \frac{a}{a+b} \cdot \frac{a'}{a'+b'} = w \cdot w' .$$

Equation (3) written above is thus demonstrated for the probabilities corresponding to the example of the urn; and just as it does for two events, the product also holds for the probability of the simultaneous occurrence of more than two events; for instance, in the theory of errors we will later have an application of the following form: Let the probability of an error ε be w, the probability of an error ε' be w', furthermore ε'' be w'', etc. The probability of obtaining just the errors ε, ε', ε'', and no others, in three observations, or more generally, the probability of obtaining in a series of observations the errors ε, ε', ε'', etc., is then expressed by the product:

$$W = w \cdot w' \cdot w'' \ldots . \tag{4}$$

The law of the large numbers. If we cannot calculate a probability from the causes, a priori, then we draw a conclusion from the frequency of the occurrence a posteriori as to the probability.

For instance, if of 1000 born boys only 250 reach the fiftieth year of life, we conclude therefrom that for any male child born today there exists the probability $= \frac{250}{1000}$ or $= \frac{1}{4}$ of reaching the fiftieth year of life.

Or, if among 491 comparisons, in the case of a double leveling, 209 comparisons have yielded the difference 1 mm, and if differences other than 0 mm, 1 mm, 2 mm cannot occur at all, the probability of the difference 1 mm is the fraction $\frac{209}{491} = 0.43$.

The probability theory as an auxiliary science of the method of the least squares is given by Hagen, *Grundzüge der Wahrscheinlichkeitsrechnung* [fundamental principles of the calculus of probabilities], second edition, Berlin, 1867.

Section 137. Probability of Observational Errors

In order to apply the calculus of probabilities to observational errors it is necessary to assume that observational errors are accidental occurrences in the sense explained at the beginning of previous section 136. (Gross errors should be excluded, c.f. p. 7.)

We will further assume that we have procured a rather long series of observational errors. It is often difficult, however, to discover real observational errors, yet there are some perceptions which are not pure errors but have fully their character, for instance, the triangle closures of triangulations (cf. p. 8), the differences of double measurements, etc., or we are satisfied with apparent errors instead of real ones, for instance, the differences of a large number of homogeneous measurements of a quantity and their arithmetic mean (cf. p. 18).

We will assume here that the observations are free of one-sided effects of error, and the distribution of the errors to the positive and negative side is therefore nearly equal.

If we arrange such errors now according to their *magnitude*, we find that their distribution follows a certain law whose investigation is our task now. We find that small errors are more frequent than large ones, and that the errors are most numerous especially around the value zero whereas larger errors which exceed twice to three times the mean error occur very rarely.

Even the small number of 18 observations which we discussed previously in section 7, p. 19, shows this; for we have ten positive and eight negative errors; and if we arrange according to the size, regardless of the sign, we obtain the following series:

$$\left.\begin{matrix} 0.10'' & 0.12'' & 0.12'' & 0.13'' & 0.30'' & 0.38'' & 0.62'' & 0.83'' & 1.12'' \\ 1.13 & 1.17 & 1.27 & 1.38 & 1.63 & 1.71 & 2.09 & 2.63 & 4.62\ . \end{matrix}\right\} \quad (1)$$

The distribution according to the size yields:

between	0''	and	1''	. . .	8 errors
between	1	and	2	. . .	7 errors
between	2	and	3	. . .	2 errors
between	3	and	4	. . .	0 error
between	4	and	5	. . .	1 error
				Total	18 errors .

It is quite evident that in this series small errors are more frequent than large ones, for between 0'' and 1'', for instance, we have eight cases, between 4'' and 5'', however, only one case.

We have this experience in all rather long observation series and can draw the conclusion therefrom that the frequency of the occurrence or the probability of an error depends on the size of the error. We can bring this into a mathematical form by regarding the probability of an error ε as a function of ε, which we will denote by $\varphi(\varepsilon)$.

In the small series mentioned above we have eight errors between 0'' and 1'', and since there are 18 errors in all we can say that the probability of an error between 0'' and 1'' is $8:18 = 0.444$.

The function $\varphi(\varepsilon)$ gives the probability of a definite error ε. If we aim to express the fact that the error is supposed to lie within a small interval $d\varepsilon$ between the limits $\varepsilon - \frac{d\varepsilon}{2}$ and $\varepsilon + \frac{d\varepsilon}{2}$ or ε and $\varepsilon + d\varepsilon$ or else ε and $\varepsilon - d\varepsilon$, we can assume that the probability for it is proportional to the interval $d\varepsilon$. We can express it therefore by

$$W = \varphi(\varepsilon)\, d\varepsilon\,. \tag{2}$$

If we imagine that W is set up for a series of consecutive intervals, the probability of the error lying in any one of these intervals will be equal to the sum of the individual probabilities, according to (2), section 136, p. 508, and since $d\varepsilon$ is infinitesimally small, the sum is replaced by a definite integral. Therefore, we can say that the probability of committing an error between two given limits a and b is

$$W_a^b = \int_a^b \varphi(\varepsilon)\, d\varepsilon\,. \tag{3}$$

If we take the limits $-\infty$ and $+\infty$ instead of the limits a and b, every error *must* lie within these limits, that is, the probability of an error lying between the limits $-\infty$ and $+\infty$ is equal to certainty, therefore equal to 1.

We obtain therefore

$$\int_{-\infty}^{+\infty} \varphi(\varepsilon)\, d\varepsilon = 1\,. \tag{4}$$

The average and the mean error

The function $\varphi(\varepsilon)$ is the probability of an error ε, that is, in a series of errors it is the quotient of the number of errors of the magnitude ε divided by the total number of errors. The expression $\varepsilon\varphi(\varepsilon)$ thus gives the ratio of the sum of the errors of the magnitude ε to the total number of errors. If we extend this to the small interval $d\varepsilon$, $\varepsilon\varphi(\varepsilon)\, d\varepsilon$ is nothing else but the sum of all ε's within this interval divided by the number of all errors. Finally, the integral

$$\int \varepsilon\varphi(\varepsilon)\, d\varepsilon\,,$$

which covers all errors which occur, yields the sum of all errors divided by their number, that is, the average error, with which we became acquainted on p. 8.

But one more consideration is to be made in this connection, because of the signs of the ε's and because of the integration limits. If the integral is supposed to include all positive ε's, the limits 0 and $+\infty$ must be taken, and the limits $-\infty$ and 0 hold for all negative ε's. If we aim to include the positive *and* the negative ε's regardless of the sign, one must not take the limits $-\infty$ and $+\infty$, because the integral would become equal to zero. One obtains, however, the correct value for all positive and negative ε's regardless of the sign if he takes the integral between 0 and ∞ and then doubles it. The average error is thus

$$t = 2\int_0^\infty \varepsilon\varphi(\varepsilon)\, d\varepsilon\,. \tag{5}$$

The same study leads also to an expression for the mean errors. According to this, the magnitude $\varepsilon^2\varphi(\varepsilon)\, d\varepsilon$ indicates for an interval $d\varepsilon$ the sum of the squares of the ε's divided by the total number of all ε's, and if we add up this quantity for all $d\varepsilon$'s, we have the average value of all ε^2's. Thus, the square of the mean error (equation (1), section 4, p. 10) is

$$m^2 = \int_{-\infty}^{+\infty} \varepsilon^2 \varphi(\varepsilon)\, d\varepsilon\,. \tag{6}$$

Since the difference of the signs of the individual ε's is not important here, the limits $-\infty$ and $+\infty$ for the integral can be retained without further consideration.

Section 138. The Probability Function

In order to arrive at the determination of the function $\varphi(\varepsilon)$, we study first the limiting values of ε.

As already stated in the case of the series of observations (1), section 137, p. 509, experience has shown that small errors are more probable than larger ones, from which we conclude that the function $\varphi(\varepsilon)$ must have its maximum at $\varepsilon = 0$, that is:

$$\varphi(0) = \text{maximum}\,. \tag{1}$$

In addition to the lower limit $\varepsilon = 0$, let us study the upper limit. No doubt, for every method of observation there is a certain limit which cannot be exceeded by the errors; but since this limit cannot in general be given, we assume in the general theory of errors the extreme limit imaginable, namely infinity, in both directions, $-\infty$ and $+\infty$.

For these limits, the probability of errors must become equal to zero, thus:

$$\varphi(\pm\infty) = 0\,. \tag{2}$$

Now let us furthermore assume that the arithmetic mean of several equally accurate observations is the most probable value of the observed quantity. If thus several observations of an unknown have yielded the values $l_1, l_2, l_3, \ldots, l_n$,

$$x = \frac{[l]}{n} \tag{3}$$

is supposed to be the most probable value of the unknown. A rigorous proof cannot be given for this assumption; in section 7, p. 16, however, we have already introduced the arithmetic mean as most suitable value, and it is not thinkable to specify any other quantity to be computed from the observations in which we could place the same confidence as we do in the arithmetic mean.

From the above assumption there follows also the fact that the corrections referred to (3)

$$\left.\begin{aligned} v_1 &= x - l_1 \\ v_2 &= x - l_2 \\ &\cdots\cdots \\ v_n &= x - l_n \end{aligned}\right\} \tag{4}$$

present likewise the most probable system of corrections.

The corrections v mentioned are naturally not identical with the real errors ε of the observations. On the previously established basis of the arithmetic mean, we can therefore develop only a law of error for the most probable corrections v, and not for the real errors ε; we will however drop this distinction and at once regard the law which thus results as a general law of error.

By introducing now in (4) the real errors ε instead of the corrections v, we begin to set up the probability of the system of errors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. The probability of all ε's occurring simultaneously is equal to the product of all individual probabilities $\varphi(\varepsilon)$, and is thus equal to

$$\varphi(\varepsilon_1)\,\varphi(\varepsilon_2)\ldots\varphi(\varepsilon_n) \tag{5}$$

and this expression is to be a maximum. For a more convenient further treatment we bring this into logarithmic form and have

$$log\, \varphi(\varepsilon_1) + log\, \varphi(\varepsilon_2) + \ldots + log\, \varphi(\varepsilon_n) = \text{maximum} . \tag{6}$$

The maximum is obtained by a suitable choice of x; therefore, we have to differentiate (6) with respect to x

$$\frac{d\, log\, \varphi(\varepsilon_1)}{d\, \varepsilon_1} \frac{d\, \varepsilon_1}{d\, x} + \frac{d\, log\, \varphi(\varepsilon_2)}{d\, \varepsilon_2} \frac{d\, \varepsilon_2}{d\, x} + \ldots + \frac{d\, log\, \varphi(\varepsilon_n)}{d\, \varepsilon_n} \frac{d\, \varepsilon_n}{d\, x} = 0 . \tag{7}$$

According to equations (4) however in which we must imagine the v's replaced by ε's,

$$\frac{d\, \varepsilon_1}{d\, x} = \frac{d\, \varepsilon_2}{d\, x} = \ldots = \frac{d\, \varepsilon_n}{d\, x} = 1 ,$$

thus we have

$$\frac{d\, log\, \varphi(\varepsilon_1)}{d\, \varepsilon_1} + \frac{d\, log\, \varphi(\varepsilon_2)}{d\, \varepsilon_2} + \ldots + \frac{d\, log\, \varphi(\varepsilon_n)}{d\, \varepsilon_n} = 0 . \tag{8}$$

Since the ε's are referred to the arithmetic mean, their sum must be equal to zero; thus

$$\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n = 0 . \tag{9}$$

If we write equation (8) in the form

$$\frac{d\, log\, \varphi(\varepsilon_1)}{\varepsilon_1\, d\, \varepsilon_1} \varepsilon_1 + \frac{d\, log\, \varphi(\varepsilon_2)}{\varepsilon_2\, d\, \varepsilon_2} \varepsilon_2 + \ldots + \frac{d\, log\, \varphi(\varepsilon_n)}{\varepsilon_n\, d\, \varepsilon_n} \varepsilon_n = 0 , \tag{10}$$

then from the two equations (9) and (10) there follows

$$\frac{1}{\varepsilon_1} \frac{d\, log\, \varphi(\varepsilon_1)}{d\, \varphi_1} = k , \quad \frac{1}{\varepsilon_2} \frac{d\, log\, \varphi(\varepsilon_2)}{d\, \varphi_2} = k , \quad \ldots \frac{1}{\varepsilon_n} \frac{d\, log\, \varphi(\varepsilon_n)}{d\, \varphi_n} = k ,$$

where k denotes a constant still to be defined. In general we must thus have

$$\frac{d\, log\, \varphi(\varepsilon)}{d\, \varepsilon} = k\, \varepsilon ,$$

or, integrating

$$log\, \varphi(\varepsilon) = \frac{1}{2} k\, \varepsilon^2 + C ,$$

where C is the constant of integration. If *log* means the natural logarithm to the base e, there follows herefrom

$$\varphi(\varepsilon) = e^{\left(\frac{1}{2} k\, \varepsilon^2 + C\right)} \quad \text{or else:} \quad \varphi(\varepsilon) = e^{C} \cdot e^{k\, \varepsilon^2 : 2} .$$

The constant k must necessarily be negative, because according to (1) the function $\varphi(\varepsilon)$ has its largest value at $\varepsilon = 0$, and from there constantly decreases; we write therefore:

$$\frac{1}{2} k = -h^2 .$$

At the same time, we can indicate the constant of integration e^C more briefly as $= A$ and have therefore:

$$\varphi(\varepsilon) = A e^{-h^2 \varepsilon^2} . \tag{11}$$

Now we still have the question of determining the constant of integration A for which we use equation (4), section 137, p. 510, for the definite integral between the limits $-\infty$ and $+\infty$, namely:

$$A \int_{-\infty}^{+\infty} e^{-h^2 \varepsilon^2} d\varepsilon = 1 ; \tag{12}$$

with $h\varepsilon = t$, and hence $d\varepsilon = \frac{dt}{h}$, which leaves the limits $-\infty$ and $+\infty$ unchanged, we will have the following:

$$\frac{A}{h} \int_{-\infty}^{+\infty} e^{-t^2} dt = 1 . \tag{13}$$

In order to evaluate this definite integral, let us have recourse to a geometric study:

Let a surface of rotation whose equation is

$$z = e^{-(x^2 + y^2)} \quad \text{or} \quad z = e^{-r^2}$$

be represented by Fig. 1.

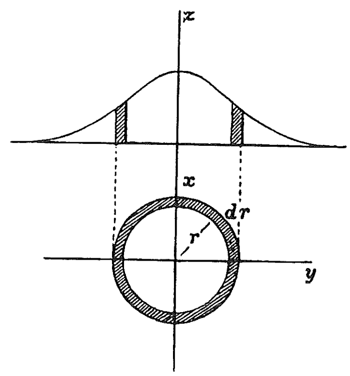


Fig. 1.

In order to determine the volume which is between the curved surface and the xy-plane, we go two different ways:

Firstly, the breakdown according to dx and dy yields the surface differential:

$$dV = dx\, dy\, z ,$$

hence

$$V = \iint_{-\infty}^{+\infty} e^{-(x^2 + y^2)} dx\, dy = \iint_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} dx\, dy ,$$

where the limits $-\infty$ and $+\infty$ refer to both integrations.

By integrating first with respect to x and then with respect to y we have:

$$V = \int e^{-x^2} \left(\int e^{-y^2} dy \right) dx \quad \text{or} \quad V = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy .$$

Since the denotation of the original variables by x, y, or t is unessential in a definite integral, we can also write instead:

$$V = \left(\int_{-\infty}^{+\infty} e^{-t^2} \, dt \right)^2 . \tag{14}$$

Secondly, we can also make a breakdown according to cylinder elements, and for this, according to Fig. 1, the surface of the circular ring in the xy-plane $= 2 r \pi \, dr$, thus the volume of a hollow cylinder of the radius r, the thickness dr, and the altitude z is

$$dV' = 2 r \pi \, dr \, z = 2 r \pi \, dr \, e^{-r^2} ,$$

the total volume is hence:

$$V = \int dV' = 2 \pi \int_0^{\infty} r e^{-r^2} \, dr . \tag{15}$$

The general integral is here

$$\int r e^{-r^2} \, dr = - \frac{1}{2} e^{-r^2} ;$$

the definite integral can thus also be given, that is, with the upper limit $r = \infty$ and the lower limit $r = 0$:

$$\int_0^{\infty} r e^{-r^2} \, dr = \left(-0 - \left[-\frac{1}{2} \right] \right) = + \frac{1}{2} , \tag{16}$$

consequently because of (15):

$$V = \pi . \tag{17}$$

The comparison of (14) and (17) yields:

$$\int_{-\infty}^{+\infty} e^{-t^2} \, dt = \sqrt{\pi} , \tag{18}$$

thus because of (13):

$$A = \frac{h}{\sqrt{\pi}} , \tag{19}$$

(11) becomes herewith:

$$\varphi(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2} . \tag{20}$$

We have defined herewith the analytic form of the desired probability function, and we can also specify by means of it the probability of the occurrence of an error between certain limits a and b, namely according to (3), section 137, p. 510:

$$W_a^b = \int_a^b \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2} d\varepsilon ,$$

or with $h\varepsilon = t$, and hence $d\varepsilon = \frac{dt}{h}$, by which the limits $\varepsilon = a$ and $\varepsilon = b$ turn into $t = ah$ and $t = bh$:

$$W_a^b = \frac{1}{\sqrt{\pi}} \int_{ah}^{bh} e^{-t^2} dt . \tag{21}$$

In our final formulae (20) and (21) there still remained a constant h which cannot in general be defined by any auxiliary condition, as can the constant of integration A, but must remain in the general formulae as a kind of parameter. For the constant h depends on the method of the observation in the individual case, that is, on the *accuracy*, and it is intuitively quite obvious that the probability of an error ε can by no means depend only on the quantity ε (and the interval $d\varepsilon$) alone, but must also take into account the accuracy of the method of measurement. The constant h which depends on the accuracy in the way mentioned is called the accuracy number.

Definition of the constant h

We obtain from equation (6), section 137, p. 511, for the error function after introducing the expression (20):

$$m^2 = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \varepsilon^2 e^{-h^2 \varepsilon^2} d\varepsilon . \tag{22}$$

The integral occurring here can be easily evaluated. To do this, we introduce the new variable t by setting:

$$h\varepsilon = t ,$$

with which we will have

$$d\varepsilon = \frac{dt}{h}$$

while the limits of the integral remain unchanged. If we denote the indefinite integral by J, we will have

$$J = \frac{1}{h^3} \int t^2 e^{-t^2} dt , \tag{23}$$

and if we write this in the form

$$J = \frac{1}{2h^3} \int -t \cdot e^{-t^2} (-2t)\, dt$$

we find by partial integration

$$J = -\frac{1}{2h^3} t e^{-t^2} + \frac{1}{2h^3} \int e^{-t^2} dt + \mathrm{C} . \tag{24}$$

If we turn now to the limits $-\infty$ and $+\infty$, the first term in (24) vanishes, and (22) then turns into:

$$m^2 = \frac{1}{2h^2\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt \,,$$

and according to (18) we will have:

$$m^2 = \frac{1}{2\,h^2} \,, \tag{25}$$

$$\text{thus} \quad m = \frac{1}{h\sqrt{2}} \quad \text{or} \quad h = \frac{1}{m\sqrt{2}} . \tag{26}$$

Introducing this into (20), we obtain

$$\varphi(\varepsilon) = \frac{1}{m\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2m^2}} . \tag{27}$$

The probability of an error ε is

$$W(\varepsilon) = \varphi(\varepsilon)\, d\varepsilon = \frac{1}{m\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2m^2}} d\varepsilon \,. \tag{28}$$

Now let us introduce further

$$\frac{\varepsilon}{m} = x \quad \text{and} \quad \frac{d\varepsilon}{m} = dx \,, \tag{29}$$

that is, we count the errors as ratios $\varepsilon : m$ or in units of the mean error m; we have herewith:

$$W(\varepsilon) = W(m\,x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \,, \tag{30}$$

$$\text{or} \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . \tag{31}$$

In the interest of clear and immediate comprehension and also of the subject it is best to carry all further developments and calculations in this latter form with $\frac{\varepsilon}{m} = x$, where x is thus a pure number and also $\varphi(x)$ is a pure number; in the following, however, we have applied this form only partially and elsewhere introduced ε itself besides h, in order not to lose the connection of our computations with the forms in use since Gauss [and partly because of difficulties of setting up the forms (27) to (31)].

In analogy to equations (22) to (25), the constant h can also be related to the average error. We start here from equation (5), section 137, p. 510, by replacing the general function of error $\varphi(\varepsilon)$ by Gauss' law. Then we have

$$t = \frac{2h}{\sqrt{\pi}} \int_0^{\infty} \varepsilon e^{-h^2 \varepsilon^2} d\varepsilon . \tag{32}$$

If we set up again

$$h\varepsilon = t , \quad \text{and hence} \quad d\varepsilon = \frac{dt}{h} ,$$

we have

$$t = \frac{2}{h\sqrt{\pi}} \int_0^{\infty} t e^{-t^2} dt . \tag{33}$$

The general integral is here

$$\int t e^{-t^2} dt = -\frac{1}{2} e^{-t^2} \tag{34}$$

and with the introduction of the limits 0 and ∞ the definite integral becomes [just as earlier in the case of (16), p. 514]:

$$\int_0^{\infty} t e^{-t^2} dt = +\frac{1}{2} ; \tag{35}$$

this yields according to (33)

$$t = \frac{1}{h\sqrt{\pi}}$$

or

$$h = \frac{1}{t\sqrt{\pi}} . \tag{36}$$

From (26) and (36) we find furthermore

$$t = \sqrt{\frac{2}{\pi}}\, m = 0{,}79788\, m \tag{37}$$

and

$$m = \sqrt{\frac{\pi}{2}}\, t = 1.2533\, t . \tag{38}$$

Thus we have found that the average error t and the mean error m are in a constant ratio, that the one can thus always be computed from the other.

We can make practical use of it insofar as, with given elements ε, it is more convenient to compute the average error t than it is to compute the mean error m, since in the case of t we save the squaring of the ε; when the average value t is thus computed, it must only be multiplied by 1.2533.

As already indicated on p. 3, Hagen has proceeded another way for the development of the law of error in his *Grundzüge der Wahrscheinlichkeitsrechnung* [fundamental principles of the calculus of probabilities] in the year 1837. If we assume that every observational error ε consists of a number of small elementary errors δ, individually not known, which are all of the same magnitude, but can be positive or negative, then, according to the number of the positive and negative signs occurring, the elementary errors will in part be added, in part cancelled, and observational errors of different magnitudes arise in this way.

If we assume, for instance, 10 elementary errors of the magnitude δ, the maximum error $+10\delta$ or -10δ can occur only when all elementary errors have the same sign; hence, the error $+10\delta$ or -10δ can come about only by *one* combination of the elementary errors δ.

If we have 9 positive δ's and 1 negative δ or vice versa, an error $\varepsilon = +8\delta$ or $\varepsilon = -8\delta$ arises. Since the one positive or negative sign is possible for all 10 errors δ, the error $+8\delta$ or -8δ is possible in 10 different combinations of the δ.

If 8 elementary errors δ have the one sign, 2 the other, the number of the errors $+6\delta$ or -6δ which result from this equals the number of the combinations of the 10 elements two at a time, and hence is equal to $\frac{10 \cdot 9}{1 \cdot 2} = 45$.

We obtain, likewise, the error $+4\delta$ or -4δ in $\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$ combinations.

Hence, in the case of $n = 10$ elementary errors we obtain:

Errors ε	Number
+10 or −10	1
+ 8 or − 8	10
+ 6 or − 6	45
+ 4 or − 4	120
+ 2 or − 2	210
+ 0 or − 0	252 .

It appears that the number of the individual errors ε is represented by the binomial coefficients. We thus have in general for the individual errors and their number the following summary:

$$\left.\begin{array}{ll} \varepsilon_0 = n\delta & \text{Number } \binom{n}{0} \\ \varepsilon_1 = (n - 1 \times 2)\,\delta & \text{Number } \binom{n}{1} \\ \varepsilon_2 = (n - 2 \times 2)\,\delta & \text{Number } \binom{n}{2} \\ \varepsilon_3 = (n - 3 \times 2)\,\delta & \text{Number } \binom{n}{3} \\ \cdots & \cdots \\ \varepsilon_i = (n - i \times 2)\,\delta & \text{Number } \binom{n}{i} \\ \varepsilon_{i+1} = (n - (i+1) \times 2) & \text{Number } \binom{n}{i+1} \end{array}\right\} . \tag{1}$$

Since the last factor in $\binom{n}{i+1}$ is equal to $\frac{n-i}{i+1}$, we have

$$\binom{n}{i+1} = \frac{n-i}{i+1}\binom{n}{i} . \tag{2}$$

Now we introduce further the relative frequency $\varphi(\varepsilon)$ of the errors ε. If N is the total number of all errors, we have

$$\varphi(\varepsilon_i) = \frac{\binom{n}{i}}{N} \qquad \varphi(\varepsilon_{i+1}) = \frac{\binom{n}{i+1}}{N} \tag{3}$$

and if we set

$$\frac{\varphi(\varepsilon_i) + \varphi(\varepsilon_{i+1})}{2} = \varphi(\varepsilon)\,,$$

we obtain with the help of (2)

$$N\varphi(\varepsilon) = \frac{1}{2}\left(\binom{n}{i} + \binom{n}{i+1}\right) = \binom{n}{i}\frac{n+1}{2(i+1)} \tag{4}$$

and, on the other hand, with respect to (1)

$$\frac{\varepsilon_i + \varepsilon_{i+1}}{2} = \varepsilon = (n - 2i - 1)\,\delta\,. \tag{5}$$

We set further

$$\varepsilon_{i+1} - \varepsilon_i = \Delta\,\varepsilon \text{ and } \varphi(\varepsilon_{i+1}) - \varphi(\varepsilon_i) = \Delta\,\varphi(\varepsilon)\,, \tag{6}$$

where, according to (1)

$$\Delta\,\varepsilon = -2\,\delta \tag{7}$$

and from (3) and (2)

$$N\,\Delta\,\varphi(\varepsilon) = \frac{n-2i-1}{i+1}\binom{n}{i}\,. \tag{8}$$

We thus obtain from (8) and (4)

$$\frac{\Delta\,\varphi(\varepsilon)}{\varphi(\varepsilon)} = \frac{2\,(n-2i-1)}{n+1}$$

and with (5)

$$\frac{\Delta\,\varphi(\varepsilon)}{\varphi(\varepsilon)} = \frac{2}{n+1}\,\frac{\varepsilon}{\delta}\,.$$

With (7), this turns finally into

$$\frac{\Delta\,\varphi(\varepsilon)}{\varphi(\varepsilon)} = -\frac{\varepsilon\,\Delta\,\varepsilon}{(n+1)\,\delta^2}\,. \tag{9}$$

The previously established theory holds for small *finite* values of the elementary errors δ. Now if we assume the values of the δ infinitesimally small, $\Delta\,\varepsilon$ and $\Delta\,\varphi(\varepsilon)$ also become infinitesimally small, and we can replace them by $d\,\varepsilon$ and $d\varphi(\varepsilon)$. We then have

$$\frac{d\,\varphi(\varepsilon)}{\varphi(\varepsilon)} = -\frac{\varepsilon\,d\,\varepsilon}{(n+1)\,\delta^2}\,. \tag{10}$$

Now we turn to the limiting case in which n becomes infinitely large and δ infinitely small. The product $(n+1)\,\delta^2$ receives then a value which at first is still indefinite, and which we will denote by k. Instead of (10) we then have

$$\frac{d\varphi(\varepsilon)}{\varphi(\varepsilon)} = -\frac{\varepsilon\, d\,\varepsilon}{k}$$

and after integration, this yields

$$\log \text{nat.}\ \varphi(\varepsilon) = -\frac{1}{2\,k}\,\varepsilon^2 + C$$

or

$$\varphi(\varepsilon) = e^{-\frac{\varepsilon^2}{2k} + C} \qquad e^{C} \cdot e^{-\varepsilon:2k} .$$

We thus see that, if we set $2\,k = \frac{1}{h^2}$, the above development leads again to the Gauss law of error

$$\varphi(\varepsilon) = e^{C} \cdot e^{-h^2 \varepsilon^2} . \tag{11}$$

In the way indicated in section 138, p. 514, the constant e^C yields likewise $\frac{h}{\sqrt{\pi}}$.

The hypothesis of the elementary errors introduced by Hagen has been followed further several times later, as by Encke in the treatise, "Über die Anwendung der Wahrscheinlichkeitsrechnung auf Beobachtungen" [On the application of the calculus of probability to observations] in the *Berlin Astronomical Yearbook for 1853*. An elementary treatment of the subject is found in Koppe, *Die Ausgleichsrechnung nach der Methode der kleinsten Quadrate in der praktischen Geometrie* [Calculus of adjustment according to the method of the least squares in practical geometry], Nordhausen, 1885, pp. 1-7. Further there is to be mentioned Czuber, *Theorie der Beobachtungsfehler* [Theory of observational errors], Leipzig, 1891, pp. 61-99. In the case of the previously established development we have followed, on the whole, Helmert's line of thought in *Die Ausgleichsrechnung nach der Methode der kleinsten Quadrate* [Calculus of adjustment according to the method of least squares], Second edition, Leipzig and Berlin, 1907, pp. 13-15.

Justification of the method of the least squares

If a system of n error equations with the unknowns x, y, z and the corrections $v_1, v_2, \ldots, v_n$ of the observations is present, we can answer also the question as to the most favorable values of the unknowns in the form that we seek to determine those values which include the greatest probability. The system of corrections v connected with these values of the unknowns will then be at the same time the most probable of all possible ones.

If we assume Gauss' law of errors for the corrections as valid, then the probability of the individual v is

$$W_1 = \frac{h_1}{\sqrt{\pi}} e^{-h_1^2 v_1^2}, \quad W_2 = \frac{h_2}{\sqrt{\pi}} e^{-h_2^2 v_2^2}, \ \ldots, \ W_n = \frac{h_n}{\sqrt{\pi}} e^{-h_n^2 v_n^2} . \tag{12}$$

It is assumed here at the same time that the precision constants $h_1, h_2, \ldots, h_n$ are assigned to the individual observations.

The probability W of the simultaneous occurrence of all these corrections $v_1, v_2, \ldots, v_n$ is then according to (4), section 136, equal to the product of the individual probabilities, and hence

$$W = \frac{h_1, h_2, \ldots, h_n}{(\sqrt{\pi})^n} \left(e^{-h_1^2 v_1^2}\, e^{-h_2^2 v_2^2} \ldots e^{-h_n^2 v_n^2} \right)$$

or

$$W = \frac{h_1, h_2, \ldots, h_n}{(\sqrt{\pi})^n} e^{-(h_1^2 v_1^2 + h_2^2 v_2^2 + \ldots + h_n^2 v_n^2)} . \qquad (13)$$

If this probability W is to be a maximum, it is necessary that

$$h_1^2 v_1^2 + h_2^2 v_2^2 + \ldots + h_n^2 v_n^2 = \text{minimum} .$$

Since according to (26), p. 516, the precision constant h, on the contrary, is proportional to the mean error, we can set h^2 proportional to the weight, and we have then

$$p_1 v_1^2 + p_2 v_2^2 + \ldots + p_n v_n^2 = [p\,v\,v] = \text{minimum} . \qquad (14)$$

This is again the fundamental theorem of the method of least squares.

If we have previously found in section 40, p. 124, that the adjustment by the least squares method yields for the unknowns those values which have the smallest mean errors, we have proved now that the least squares method yields the most probable values of the unknowns, presupposing the validity of Gauss' law of errors.

Section 140. Developments of Series and Curves of Error

The probability function is according to (20), section 138, p. 514:

$$\varphi(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2} . \qquad (1)$$

This function can be computed directly for any value of h and of ε, that is, logarithmically since $\log\ e = \log\ 2.71828 \ldots = 0.4342945 \ldots = \mu$ and $\log \sqrt{\pi} = 0.2485749$,

$$\log \varphi(\varepsilon) = 9.7514251 + \log h - \mu h^2 \varepsilon^2 . \qquad (2)$$

The following numerical values are computed herewith:

ε	$\varphi(\varepsilon)$ For $h = 1$	$\varphi(\varepsilon)$ For $h = 2$	ε	$\varphi(\varepsilon)$ For $h = 1$	$\varphi(\varepsilon)$ For $h = 2$
0.0	0.56419	1.12838	1.0	0.20755	0.02067
0.1	0.55858	1.08413	1.1	0.16824	0.00892
0.2	0.54207	0.96514	1.2	0.13367	0.00356
0.3	0.51563	0.78724	1.3	0.10410	0.00131
0.4	0.48077	0.59499	1.4	0.07947	0.00044
0.5	0.43939	0.41511	1.5	0.04946	0.00014
0.6	0.39362	0.26734	1.6	0.04361	0.00004
0.7	0.34564	0.15894	1.7	0.03136	0.00001
0.8	0.29749	0.08723	1.8	0.02210	0.000003
0.9	0.25098	0.04419	1.9	0.01526	0.0000006
1.0	0.20755	0.02067	2.0	0.01033	0.0000001
			3.0	0.00007	0.0 . .
			∞	0	0

We can represent such a function also graphically by plotting the values $\varphi(\varepsilon)$ as ordinates to the values ε as abscissae. The curve for $h = 1$ thus obtained is given in the following Fig. 1 at the end of this section on p. 524.

Let us represent the integral function also by numerical values and graphically, that is, according to (21), section 138, p. 515, the function

$$W_a^b = \frac{1}{\sqrt{\pi}} \int\limits_{t=ah}^{t=bh} e^{-t^2} dt \,. \tag{3}$$

This is the probability for the occurrence of an error between the limits a and b, and hence we obtain also the probability for the occurrence between the limits $-a$ and $+a$:

$$W_{-a}^{+a} = \frac{1}{\sqrt{\pi}} \int\limits_{t=-ah}^{t=+ah} e^{-t^2} dt \,.$$

But because of the fact that equally large positive or negative errors are equally probable, and hence the probabilities between $-a$ and zero on the one hand, and zero and $+a$ on the other, are equal to one another, we have:

$$W_{-a}^{+a} = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{ah} e^{-t^2} dt \,. \tag{4}$$

In order to compute this integral, we must apply a development in series. We use here the exponential series:

$$e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots \,.$$

This yields with $x = -t^2$:

$$e^{-t^2} = 1 - \frac{t^2}{1\,!} + \frac{t^4}{2\,!} - \frac{t^6}{3\,!} + \frac{t^8}{4\,!} - \dots$$

$$\int e^{-t^2} dt = t - \frac{t^3}{3 \cdot 1\,!} + \frac{t^5}{5 \cdot 2\,!} - \frac{t^7}{7 \cdot 3\,!} + \frac{t^9}{9 \cdot 4\,!} - \dots$$

$$\int\limits_0^T e^{-t^2} dt = T - \frac{T^3}{3 \cdot 1\,!} + \frac{T^5}{5 \cdot 2\,!} - \frac{T^7}{7 \cdot 3\,!} + \frac{T^9}{9 \cdot 4\,!} \cdot \tag{5}$$

This series (5) converges for finite values of T in any case, for the quotient q of two successive terms is, if we proceed as far as $n\,!$ in the denominator:

$$q = \frac{(2\,n - 1)\,T^2}{(2\,n + 1)\,n} \,.$$

Since the series has alternately positive and negative terms, it is sufficient for the convergence if from any point on we have always $q < 1$; this is the case, however, for every finite value of T, if we continue the series long enough.

For rather great values of T we must take a great many terms, however, in order to reach only the beginning of the convergence; the theoretical possibility of the calculation according to series (5), however, is sufficient for us here.

(As for the rest, we refer in this connection to B r ü n n o w, *Lehrbuch der sphärischen Astronomie* [Textbook of spherical astronomy], Fourth edition, p. 34, and *Theorie der Beobachtungsfehler* [Theory of observational errors] by Emanuel Czuber, Leipzig, 1891, pp. 115-120.)

From (4) and (5) we have now the probability that an error lies between the limits $-a$ and $+a$:

$$W_{-a}^{+a} = \frac{2}{\sqrt{\pi}} \left(ah - \frac{(ah)^3}{3 \cdot 1!} + \frac{(ah)^5}{5 \cdot 2!} - \frac{(ah)^7}{7 \cdot 3!} + \frac{(ah)^9}{9 \cdot 4!} - \dots \right) . \qquad (6)$$

For instance, with $ah = 0.1$ we obtain thus:

$$W_{-a}^{+a} = 1.12838\,(0.100000 - 0.000333 + 0.000001) = 0.112\,463 \ .$$

Following are some numerical values for the function W_{-a}^{+a} according to equation (6):

ah	W_{-a}^{+a}	ah	W_{-a}^{+a}	ah	W_{-a}^{+a}	ah	W_{-a}^{+a}
0.0	0.00000	1.0	0.84270	2.0	0.99532	3.0	0.99997 79
0.1	0.11246	1.1	0.88020	2.1	0.99702	3.1	0.99998 84
0.2	0.22270	1.2	0.91031	2.2	0.99814	3.2	0.99999 40
0.3	0.32863	1.3	0.93401	2.3	0.99886	3.3	0.99999 69
0.4	0.42839	1.4	0.95229	2.4	0.99931	3.4	0.99999 85
0.47694	0.5						
0.5	0.52050	1.5	0.96611	2.5	0.99959	3.5	0.99999 26
0.6	0.60386	1.6	0.97635	2.6	0.99976	3.6	0.99999 96
0.7	0.67780	1.7	0.98379	2.7	0.99987	3.7	0.99999 98
0.8	0.74210	1.8	0.98909	2.8	0.99992	3.8	0.99999 99
0.9	0.79691	1.9	0.99279	2.9	0.99996	∞	1.0

A more detailed table of this kind was given by Encke in the *Berliner Astronomisches Jahrbuch für 1834*, pp. 305-308, and by Czuber, *Theorie der Beobachtungsfehler*, Leipzig, 1891, p. 121 and pp. 411-413, and by Ferrero, *Esposizione del metodo dei minimi quadrati*, Firenze, 1876, pp. 223-225.

In the above small table a value $ah = 0.47694$ is especially emphasized; it is the one which corresponds to $W_{-a}^{+a} = \frac{1}{2}$. We will deal more thoroughly with this value 0.47694, or more accurately, 0.4769363 . . . in the next section 141; meanwhile we only say that, if introduced into series (6), it permits the computation of the pertinent $W\binom{a}{0}$, as does every other, and the corresponding calculation will yield $W_{-a}^{+a} = 0.50000$.

If we introduce in the above instead of the modulus of precision h the mean error m, namely according to (26), section 138, p. 516,

$$h = \frac{1}{m\sqrt{2}}, \quad \text{and hence} \quad ah = \frac{a}{m\sqrt{2}},$$

we obtain

$$W_{-a}^{+a} = \frac{2}{\sqrt{\pi}} \left[\left(\frac{a}{m\sqrt{2}}\right) - \left(\frac{a}{m\sqrt{2}}\right)^3 \frac{1}{3 \cdot 1!} + \left(\frac{a}{m\sqrt{2}}\right)^5 \frac{1}{5 \cdot 2!} - \left(\frac{a}{m\sqrt{2}}\right)^7 \frac{1}{7 \cdot 3!} + \dots \right] . \qquad (7)$$

This is only another form for (6); if we take out the denominator $\sqrt{2}$, however, and at the same time represent the ratio $\frac{a}{m}$ as variable, we obtain yet another somewhat different form:

$$W_{-a}^{+a} = \sqrt{\frac{2}{\pi}}\left[\left(\frac{a}{m}\right) - \frac{1}{6}\left(\frac{a}{m}\right)^3 + \frac{1}{40}\left(\frac{a}{m}\right)^5 - \frac{1}{336}\left(\frac{a}{m}\right)^7 + \frac{1}{3456}\left(\frac{a}{m}\right)^9 - \dots\right]. \quad (8)$$

Or by setting the ratio $\frac{a}{m} = n$, we can write (8) also in the following form:

$$W\binom{n\,m}{0} = \sqrt{\frac{2}{\pi}}\left(n - \frac{n^3}{6} + \frac{n^5}{40} - \frac{n^7}{336} + \frac{n^9}{3456} - \dots\right). \quad (9)$$

This is the probability for the occurrence of an error between the limits zero and n times the mean error, as function of n.

Our table in the Appendix, p. [21], is calculated on this base.

Graphic representations

The following Fig. 1 gives curves for the functions (1), (2) and (6), (8) together with a curve for W_0^{nr}, which we will not discuss until section 141, in (6), p. 528, in the case of the probable error.

The curve denoted by $\varphi(\varepsilon)$ corresponds to function $\varphi(\varepsilon)$ and the pertinent table on p. 521 with $h = 1$.

The curve W corresponds to function (6) or (8) with the table on p. 523 where again $h = 1$ is assumed.

The graphic representations lead further to the following geometric considerations:

The probability of the occurrence of an error between the limits ε and $\varepsilon + d\varepsilon$ is

$$W = \varphi(\varepsilon)\, d\varepsilon,$$

and its geometric representation is a narrow, infinitely small rectangle whose height equals the ordinate $= \varphi(\varepsilon)$ of the first curve, and whose width $= d\varepsilon$ is measured in the direction OX. Accordingly, the probability of the occurrence of an error between the limits a and b is

$$W_a^b = \int_a^b \varphi(\varepsilon)\, d\varepsilon. \quad (10)$$

This is represented geometrically by a surface which is limited by the axis of abscissa, by the curve $\varphi(\varepsilon)$ itself, and by the two ordinates which correspond to the abscissae $\varepsilon = a$ and $\varepsilon = b$. For instance, the probability of the occurrence of an error between the limits $\varepsilon = 0.477$ and $\varepsilon = 0.707$ is represented in Fig. 1 by the surface $ACDB$.

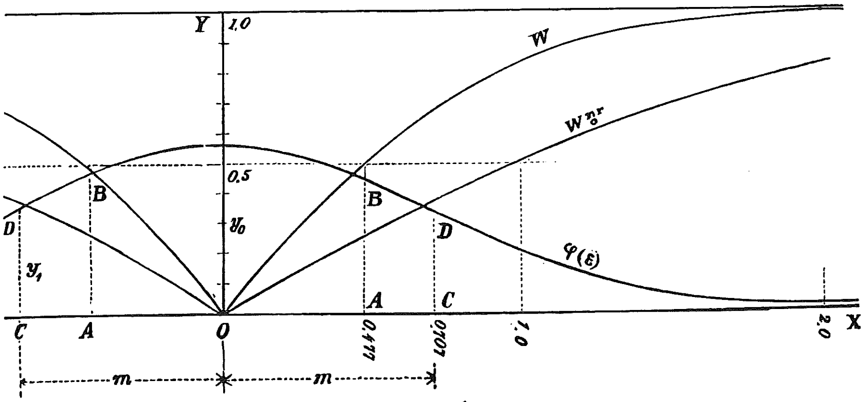


Fig. 1.

The total surface between the axis of abscissa (asymptote) and the curve $\varphi(\varepsilon)$ is $=1$, which corresponds to equation (4), section 137, p. 510, namely:

$$\int_{-\infty}^{+\infty}\varphi(\varepsilon)\,d\varepsilon \quad \text{or} \quad 2\int_{0}^{\infty}\varphi(\varepsilon)\,d\varepsilon = 1\,. \tag{11}$$

Point of inflection of the curve $\varphi(\varepsilon)$

Another remarkable relation exists between the point of inflection of the curve $\varphi(\varepsilon)$ and the mean error m; that is, the abscissa of this point of inflection is equal to the mean error.

At the point of inflection of a curve, the second differential quotient equals zero, as we know. We have therefore:

$$\varphi(\varepsilon) = \frac{h}{\sqrt{\pi}}e^{-h^2\varepsilon^2} \tag{12}$$

$$\frac{d\varphi(\varepsilon)}{d\varepsilon} = \frac{h}{\sqrt{\pi}}e^{-h^2\varepsilon^2}\left(-2h^2\varepsilon\right) = -\frac{2h^3}{\sqrt{\pi}}\varepsilon e^{-h^2\varepsilon^2}$$

$$\frac{d^2\varphi(\varepsilon)}{d\varepsilon^2} = -\frac{2h^3}{\sqrt{\pi}}\left\{e^{-h^2\varepsilon^2} - \varepsilon e^{-h^2\varepsilon^2}\,2h^2\varepsilon\right\}.$$

This can become zero only by setting the brace term equal to zero, whence follows:

$$1-2h^2\varepsilon^2 = 0$$

$$\varepsilon = \frac{1}{h\sqrt{2}}\,. \tag{13}$$

According to (26), section 138, p. 516, this is $=m$; hence, that special value of ε which causes the second derivative of function $\varphi(\varepsilon)$ to vanish is equal to the mean error m. Or in Fig. 1 we have $OC = m$ for the point of inflection D, and especially with $h=1$, we will have $OC = \frac{1}{\sqrt{2}} = 0.707\ldots$, as is entered in Fig. 1.

Error function for $m = 1$

In the table, p. 521, and the curve, Fig. 1, p. 524, which is referred to it, there is made the simple assumption which first presents itself $h=1$ or $h=2$; this, however, is simple from the mathematical viewpoint but not useful in the sense of intuitive and ready perception of errors; in the latter sense it is better to reduce everything to the mean error $m=1$, as we already hinted in reference to (27) to (31) in section 138, p. 516, with the expression

$$\frac{\varepsilon}{m} = x \tag{14}$$

$$y = \varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{15}$$

$$x = 0 \quad \text{yields} \quad \varphi_0 = \frac{1}{\sqrt{2\pi}} = 0.39894 \tag{16}$$

$x = 1$, or $\varepsilon = m$ yields $\varphi_1 = \frac{1}{\sqrt{2e\pi}} = 0.24197$ (17)

and the rest of the calculation yields

$$\log y = 9.6009106 - 0.21714724\, x^2 ,$$

according to which the following table was computed:

$\frac{\varepsilon}{m} = x$	$\varphi(x)$	$\frac{\varepsilon}{m} = x$	$\varphi(x)$
0.0	0.39894	0.7	0.31225
0.1	0.39695	0.8	0.28969
0.2	0.39104	0.9	0.26609
0.3	0.38139	1.0	0.24197
0.4	0.36827	1.5	0.12951
0.5	0.35206	2.0	0.05399
0.6	0.33322	3.0	0.00443
0.7	0.31225	∞	0.00000

According to this, the following Fig. 2 has been plotted, which represents now the error function in natural proportions, so to speak, in comparison to the previous distorted Fig. 1, p. 524. (Unfortunately, the tangent of the point of inflection turned out bad on the wood engraving.)

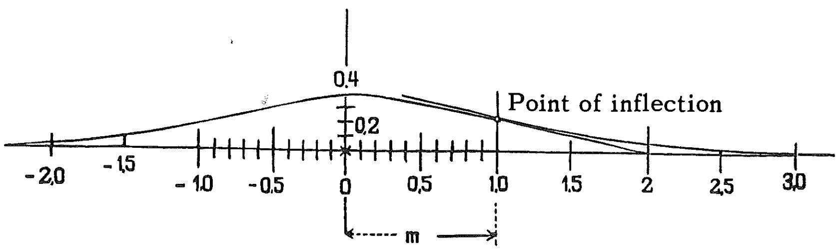


Fig. 2.

Section 141. The Probable Error

The special value $W_{-a}^{+a} = 0.5$, which we studied in the previous section 140 in the case of the table on p. 523, leads to the introduction of a new concept, namely the probable error.

For, if the probability of the occurrence of an error between the limits 0 and a has the value 0.5, the probability of its occurrence between the limits a and ∞ must likewise be 0.5 because the total probability of occurrence between the extreme limits 0 and ∞ is $= 1$. If we thus denote the special value of a, for which this takes place, by r, then we have:

$$W_0^r = \frac{1}{2} = W_r^\infty .$$

The value r thus defined is called the *probable error* [wahrscheinlicher Fehler]. It represents the limit below which just as many smaller errors are to be expected as are larger ones above it.

The probable error r is determined with the help of series (6), section 140, p. 523, by the fact that this series must yield the value $\frac{1}{2}$ if $a = r$, that is:

$$W_0^r = \frac{1}{2} = \frac{2}{\sqrt{\pi}}\left(rh - \frac{(rh)^3}{3\cdot 1!} + \frac{(rh)^5}{5\cdot 2!} - \frac{(rh)^7}{7\cdot 3!} + \frac{(rh)^9}{9\cdot 4!} - \ldots\right), \quad (1)$$

since r always occurs together with h here, the product rh is specially symbolized, namely

$$rh = \rho \tag{2}$$

and with this, the above equation yields the following determination for ρ:

$$\rho - \frac{\rho^3}{3} + \frac{\rho^5}{10} - \frac{\rho^7}{42} + \frac{\rho^9}{216} - \ldots = \frac{\sqrt{\pi}}{4} = 0.443113 \ldots .$$

In order to solve this equation for ρ, we have in the first approximation:

$$\rho = \frac{\sqrt{\pi}}{4} = 0.443113 \ldots = p ,$$

where the symbol p is temporarily introduced only for the sake of abbreviation. By setting then $\rho^3 = p^3$ also in the first approximation, we have in the second approximation:

$$\rho = p + \frac{p^3}{3} ,$$

further $$\rho^3 = p^3 + 3p^2 \frac{p^3}{3} + \ldots = p^3 + p^5 \quad \text{and} \quad \rho^5 = p^5 + \ldots$$

and hence $$\rho - \left(\frac{p^3}{3} + \frac{p^5}{3} \right) + \frac{p^5}{10} = p$$

$$\rho = p + \frac{p^3}{3} + \frac{7}{30} p^5 + \ldots .$$

Proceeding in this way, we obtain in the next step of approximation:

$$\rho = p + \frac{p^3}{3} + \frac{7}{30} p^5 + \frac{127}{630} p^7 + \ldots$$
$$\rho = 0.4431 + 0.0290 + 0.0040 + 0.0007$$
$$\rho = 0.4768 .$$

By means of such and similar approximations — to which also the use of a detailed table such as section 140, p. 523, especially belongs — we can calculate the value ρ carrying on the approximation as far as desired. As specified by Gauss, the value is:

$$rh = \rho = 0.476\,9363 \qquad \log \rho = 9.678\,4604 \tag{3}$$

(from Gauss, "Bestimmung der Genauigkeit der Beobachtungen," *Zeitschrift für Astronomie und verwandte Wissenschaften* [determination of the precision of observations, periodical for astronomy and related sciences], published by Lindenau and Bohnenberger, Tübingen, 1816, p. 194).

After the numerical value of ρ is thus precisely determined, we follow its importance further and take up once more equation (2), namely:

$$rh = \rho \quad \text{or} \quad h = \frac{\rho}{r} . \tag{4}$$

We see from this that h is inversely proportional to the probable error r; h is therefore called the *modulus of precision.* If we set $h = 1$, then $r = \rho$; this means that ρ is the probable error for the modulus of precision 1.

By introducing $\frac{\rho}{r}$ instead of h into series (6), section 140, p. 523, we can bring it into a more illustrative form. Thus we have:

$$W_{-a}^{+a} = \frac{2}{\sqrt{\pi}}\left(\frac{a}{r}\varrho - \frac{1}{3\cdot 1!}\left(\frac{a}{r}\varrho\right)^3 + \frac{1}{5\cdot 2!}\left(\frac{a}{r}\varrho\right)^5 - \frac{1}{7\cdot 3!}\left(\frac{a}{r}\varrho\right)^7 + \frac{1}{9\cdot 4!}\left(\frac{a}{r}\varrho\right)^9 - \ldots\right). \tag{5}$$

We denote the ratio $\frac{a}{r}$ of any error a to the probable error r by n, and we call now accordingly the probability that an error lies between the limits 0 and n times the probable error, $W\binom{nr}{0}$; with this, (5) receives the following form:

$$W_{0}^{nr} = \frac{2}{\sqrt{\pi}}\left(n\varrho - \frac{(n\varrho)^3}{3\cdot 1!} + \frac{(n\varrho)^5}{5\cdot 2!} - \frac{(n\varrho)^7}{7\cdot 3!} + \frac{(n\varrho)^9}{9\cdot 4!} - \frac{(n\varrho)^{11}}{11\cdot 5!} + \ldots\right) \tag{6}$$

or, calculating the coefficients:

$$\left.\begin{aligned} W_{0}^{nr} = 0.538\,1650\,n - 0.040\,8051\,n^3 + 0.002\,7846\,n^5 - 0.000\,1508\,n^7 \\ + 0.000\,0067\,n^9 - 0.000\,0002\,n^{11} + \ldots \end{aligned}\right\}. \tag{7}$$

A check for this equation exists in the special case $n = 1$, for with $n = 1$ it must yield the value 0.5, which it does as far as seven decimal places.

Following are the more precise coefficient logarithms for formula (7):

$$\left.\begin{aligned} [9.730\,9154]\,n - [8.610\,7149]\,n^3 + [7.444\,7570]\,n^5 - [6.178\,428]\,n^7 \\ + [4.824\,14]\,n^9 - [3.3949]\,n^{11} + \ldots \end{aligned}\right\}. \tag{8}$$

In the Appendix, p. [20], we compiled a table of the function values (6), (7), that is, of the probability that the error of an observation lies between the limits zero and n times the probable error. This table has been formed from Table II by Encke in the *Berliner Astronomisches Jahrbuch, 1834*, pp. 309-312, omitting the fifth decimal place. A part of the tabular values was directly recomputed according to formulas (6) and (7), and individual values were determined from Bessel's original table in the work, *Fundamenta astronomiae*, pp. 36 and 37.

Encke's table is reprinted and explained also by Czuber, *Theorie der Beobachtungsfehler*, Leipzig, 1891, pp. 414-416 and 189.

Determination of the probable error from the mean error

Connecting equation (4), p. 527, with the previous equation (26), section 138, p. 516, we have:

$$rh = \rho \quad \text{and} \quad \frac{1}{h} = m\sqrt{2},$$

$$\text{and hence} \qquad r = \rho\sqrt{2}\,m\,. \tag{9}$$

Here $\rho\sqrt{2}$ is a constant factor whose calculation yields:

$$\rho\sqrt{2} = 0.674\,4898 \qquad \log\,\rho\sqrt{2} = 9.828\,9754\,. \tag{10}$$

If we apply this to our example of section 7, p. 18, then we have found there, p. 19, $m = \pm 1.66''$; consequently, we have according to the preceding equations (9) and (10)

$$r = \pm 1.12'' . \tag{11}$$

Determination of the probable error by counting

If we have procured, in any way, a rather large number of homogeneous real observational errors, and arranged them according to their absolute magnitude, we can take that limiting value as probable error which divides the series of errors in such a way that the values of the one half are smaller and those of the other half larger than it. In the case of an odd number of errors we take the error lying in the middle, and in the case of an even number we take the mean of the two errors lying in the middle as probable error. If we arrange, for instance, the 18 values v of section 7, p. 19, according to their size, as has been done on p. 509, we have:

	First half		Middle	Second half		
. . .	0.83	1.12	|	1.13	1.17	. . .

The two errors lying in the middle are 1.12 and 1.13; hence, according to the given rule we take 1.125 as probable error.

This determination is obviously a very unreliable one, for here it is not a question of the absolute size of all other than just those errors lying in the middle. If the value 1.125 which has been found here is in good agreement with the value (11), this is rather owing to chance. The determination of the probable error by the way of the mean error m, according to (9), will always be the best one.

In *Mitteilungen über Gegenstände des Artillerie- und Geniewesens* [Information bulletins on subjects of artillery and engineering], Year 1908, Nos. 11 and 12, W e l l i s c h specifies for the calculation of the probable error the formula

$$\sqrt{r} = \frac{[\sqrt{\varepsilon}]}{n},$$

where the absolute values for the ε as well as for the radicals are to be introduced. It is to be noted here that this is but an empirical formula which has nothing to do with the theory of the probable error, which, however, comes very close to the theoretical value. (Cf. *Zeitschr. f. Verm.*, 1909, pp. 727-730.)

The probable error cannot be determined without the law of the probability of errors, whereas the mean error exists completely independently of the probabilities of errors, as it appears from its introduction in section 4, pp. 10-11. In earlier times, the probable error was frequently indicated as the measure of precision (for instance, in astronomy); in geodesy, however, it was mostly the mean error, according to instruction by Gauss himself who says in the *Briefwechsel zwischen Gauss und Schumacher* [Exchange of letters between Gauss and Schumacher] I, p. 435: "Die sogenannten wahrscheinlichen Fehler wünsche ich eigentlich, als von Hypothese abhängig, ganz proskribiert; man mag sie aber berechnen, indem man die mittleren Fehler mit 0.674 4897 multipliziert." [I actually would like the so-called probable errors, as dependent on hypothesis, to be entirely prescribed; one may calculate them, however, by multiplying the mean errors by 0.674 4897.]

Section 142. Relations Between the Mean, Probable, and Average Error

After the average and the mean error, and then the probable and the mean error, have been put into relationship to one another in the previously established theory, we can form the reciprocal relations of all these three errors among themselves.

The following symbolizations hold:

$$\text{Mean error} = m\,,$$
$$\text{Probable error} = r\,,$$
$$\text{Average error} = t\,.$$

If we have n real errors ε, then

$$m = \sqrt{\frac{[\varepsilon^2]}{n}} \qquad t = \frac{[\pm\varepsilon]}{n}\,. \tag{1}$$

We have the following numerical values for the reciprocal conversion of m, r, and t:

$$\left.\begin{aligned}
\varrho &= 0.476\,9363 & \log &= 9.678\,4603.8\\
r = \varrho\sqrt{2}\,m &= 0.674\,4898\,m & \log &= 9.828\,9753.8\\
r = \varrho\sqrt{\pi}\,t &= 0.845\,3476\,t & \log &= 9.927\,0353.1\\
m = \frac{1}{\varrho\sqrt{2}}\,r &= 1.482\,6021\,r & \log &= 0.171\,0246.2\\
m = \sqrt{\frac{\pi}{2}}\,t &= 1.253\,3141\,t & \log &= 0.098\,0599.4\\
t = \sqrt{\frac{2}{\pi}}\,m &= 0.797\,8846\,m & \log &= 9.901\,9400.6\\
t = \frac{1}{\varrho\sqrt{\pi}}\,r &= 1.182\,9454\,r & \log &= 0.072\,9646.9
\end{aligned}\right\}. \tag{2}$$

The calculations are based on $\rho = 0.476\,9363$ here, according to Gauss, "Bestimmung der Genauigkeit der Beobachtungen," Art. 1 (see preceding section 141, p. 527).

If, instead of the real errors ε, only apparent errors or corrections v are available, then we reckon in the case of *one* unknown, according to section 7, p. 17, or section 11, p. 36:

$$\frac{[\varepsilon^2]}{n} = \frac{[v^2]}{n-1} \tag{3}$$

and therefore:

$$m = \sqrt{\frac{[v^2]}{n-1}}\,. \tag{4}$$

In order to introduce the apparent errors or corrections in the calculation of t also, we use the following calculation:

Assuming, as in section 138, p. 511, that the v's follow the same error law as do the ε's, we can conclude that the individual ε^2's are larger, on the average, than the corresponding v^2's in the ratio $n:(n-1)$, or that the individual ε's are larger, on the average, than the corresponding v's in the ratio $\sqrt{n}:\sqrt{n-1}$; consequently, we can conclude further that

$$[\pm\varepsilon] : [\pm v] = \sqrt{n} : \sqrt{n-1}\,. \tag{5}$$

And hence the average error:

$$t = \frac{[\pm \varepsilon]}{n} = \frac{[\pm v]}{\sqrt{n\,(n-1)}} \tag{6}$$

and the mean error from (2):

$$m = \sqrt{\frac{\pi}{2}}\, t = 1.2533\, \frac{[\pm v]}{\sqrt{n\,(n-1)}}\,. \tag{7}$$

This is the formula by Peters.

In the case of more than *one* unknown – let us assume u unknowns and n error equations – then, according to (19), section 29, p. 93:

$$m = \sqrt{\frac{[\varepsilon\varepsilon]}{n}} = \sqrt{\frac{[v^2]}{n-u}}\,, \tag{8}$$

and in order to obtain a corresponding formula also for t, we assume in connection with (8) [similarly as in the case of (5)]:

$$\varepsilon : v = \sqrt{n} : \sqrt{n-u}$$

$$[\pm \varepsilon] : [\pm v] = \sqrt{n} : \sqrt{n-u}$$

and hence

$$t = \frac{[\pm \varepsilon]}{n} = \frac{[\pm v]}{\sqrt{n\,(n-u)}} \tag{9}$$

and the mean error:

$$m = \sqrt{\frac{\pi}{2}}\, t = 1.2533\, \frac{[\pm v]}{\sqrt{n\,(n-u)}}\,. \tag{10}$$

This formula is by Lüroth.

In the case of conditioned observations with r conditioned equations, we are to set $n - u = r$, and hence, according to (8), section 42, p. 137:

$$m = \sqrt{\frac{[v^2]}{r}} \tag{11}$$

and, accordingly, for (10):

$$m = 1.25331\, \frac{[\pm v]}{\sqrt{n\,r}}\,. \tag{12}$$

The formulas (7), (10), and (12) are the most customary ones for the calculation with the first powers v of the corrections instead of the squares v^2.

Peters' formula (7), however, has the disadvantage that it does not agree with the rigorous formula (4) for the simple case of two observations, that is, $n = 2$, for, if we assume two observations with the difference d, then we have according to (4), with $n = 2$:

$$m_2 = \sqrt{\frac{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}{2-1}} = \frac{d}{\sqrt{2}} = 0.70711\, d\,, \tag{13}$$

according to (7), however:

$$m_2 = \sqrt{\frac{\pi}{2}} \frac{\frac{d}{2}+\frac{d}{2}}{\sqrt{2\cdot 1}} = \sqrt{\frac{\pi}{2}} \frac{d}{\sqrt{2}} = 0.88623\, d\,. \tag{14}$$

We can now arrive at the thought of bringing formulas (7) and (4) to agree in the case $n = 2$ by setting in the denominator of (7), instead of $n - 1$, a value $n - x$, which at first we leave undetermined and then determine x in such a way that for $n = 2$ both formulas (7) and (4) coincide. Hence, we take at first, according to (7):

$$m = \sqrt{\frac{\pi}{2}} \frac{[\pm v]}{\sqrt{n(n-x)}} \tag{15}$$

and for the special case $n = 2$:

$$m_2 = \sqrt{\frac{\pi}{2}} \frac{\frac{d}{2}+\frac{d}{2}}{\sqrt{2(2-x)}} = \sqrt{\frac{\pi}{2}} \frac{d}{\sqrt{2(2-x)}}\,,$$

and this yields, if set equal to the rigorous formula (13):

$$\sqrt{\frac{\pi}{2(2-x)}} = 1\,.$$

This equation yields, if solved for x:

$$x = \frac{4-\pi}{2}\,.$$

If further we set approximately $\pi = 3$, then $x = \frac{1}{2}$, and, with this, we have for (15):

$$m = \sqrt{\pi} \frac{[\pm v]}{\sqrt{n(2n-1)}} = 1.77245 \frac{[\pm v]}{\sqrt{n(2n-1)}}\,. \tag{16}$$

F e c h n e r ' s formula.

What has been mentioned in the preceding section, from (5) to (16), for the most part does not contain rigorous developments, but only makes the corresponding formulas plausible. The corresponding, more detailed, source indications follow:

Formula (7) or (6), $t = \frac{[\pm v]}{\sqrt{n(n-1)}}$, was first established by Peters in the 44th volume of the *Astronomische Nachrichten*, p. 29 (1856).

The generalized formula (9), $t = \frac{[\pm v]}{\sqrt{n(n-u)}}$, was specified by Lüroth in the 73rd volume (1869), p. 187 of the *Astronomische Nachrichten*.

Helmert treated this matter more precisely in *Astronomische Nachrichten*, 85th volume (1875), pp. 353-366, and 88th volume (1876), pp. 113-132, and found here that Peters' formula (7) is rigorously correct only for *one* unknown, whereas, formula (10) for u unknowns, with the denominator $\sqrt{n(n-u)}$, is to be considered only as an approximate formula.

Two more treatises by Helmert are to be mentioned in this connection, in S c h l ö m i l c h ' s *Zeitschr. f. Mathematik und Physik*, 1875, pp. 300-303: "Über die Berechnung des wahrscheinlichen Fehlers aus einer endlichen Zahl wahrer Beobachtungsfehler" [On the calculation of the probable error from a finite number of real observational errors], and 1876, pp. 192-218: "Über die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler und über einige damit im Zusammenhange stehende Fragen" [On the probability of sums of powers of observational errors and on some questions connected with it].

Finally, as far as Fechner's formula (16) is concerned, it was developed by Fechner in Poggendorff's *Annalen der Physik*, Jubelband [anniversary volume], 1874, pp. 66-81, and a critical investigation of this formula was given by Helmert in the 88th volume of the *Astronom. Nachr.* (1876), pp. 120-127. According to this, Fechner's formula is the best of the formulas which calculate the probable or mean error from the absolute sum $[\pm v]$ of the residual errors v instead of from the square sum $[v^2]$.

All formulas with $[\pm v]$ are dependent on the assumption about the law of the probability of errors and, apart from this, are also less accurate than the formulas with the squares, that is, formulas (3) and (8); this, however, will not be proved until later in section 144, but it may already now (together with the question of the dependence on the error law) lead to the remark that all the formulas $[\pm v]$ only deserve interest insofar as the calculation of the squares, i.e. a comparatively minor labor, can be eliminated by using them.

Section 143. Various Sums of Powers of Errors

The same reflection which has led to formulas (5) and (6), pp. 510-511, with the sums $[\pm \varepsilon]$ and $[\varepsilon^2]$ of the first and second powers of real errors at the end of section 137, can also be applied to other powers and yields with Gauss' error function:

$$m^2 = \frac{[\varepsilon^2]}{n} = \frac{2h}{\sqrt{\pi}} \int_0^\infty \varepsilon^2 e^{-h^2 \varepsilon^2} d\varepsilon \tag{1}$$

$$\frac{[\varepsilon^4]}{n} = \frac{2h}{\sqrt{\pi}} \int_0^\infty \varepsilon^4 e^{-h^2 \varepsilon^2} d\varepsilon \tag{2}$$

in general:

$$\frac{[\varepsilon^p]}{n} = \frac{2h}{\sqrt{\pi}} \int_0^\infty \varepsilon^p e^{-h^2 \varepsilon^2} d\varepsilon . \tag{3}$$

Of these let us first treat once again the mean square, according to (1), although we cannot find anything new here (for it must yield $m^2 = m^2$). We have first with $h\varepsilon = t$ from (1):

$$m^2 = \frac{2}{h^2\sqrt{\pi}} \int_0^\infty t^2 e^{-t^2} dt . \tag{4}$$

Partial integration yields

$$\int t\, t e^{-t^2} dt = -\frac{1}{2} \quad {}^{-t^2} + \frac{1}{2}\int e^{-t^2} dt . \tag{5}$$

From (18), section 138, p. 514, we know:

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi} \qquad \text{or} \qquad \int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi} . \tag{6}$$

If we insert, therefore, the limits 0 and ∞ into (5), the first part of (5), namely $\frac{t}{2} e^{-t^2} = \frac{t}{2e^{t^2}}$, yields both times 0, which we can demonstrate, if necessary, for $t = \infty$ by applying the series for e^{t^2}; on the whole, we thus have from (5) and (6)

$$\int_0^\infty t^2 e^{-t^2} dt = 0 + \frac{1}{4}\sqrt{\pi}, \tag{7}$$

and hence, according to (4):

$$m^2 = \frac{2}{h^2\sqrt{\pi}}\frac{1}{4}\sqrt{\pi} = \frac{1}{2h^2}. \tag{8}$$

According to (25), section 138, p. 516, however,

$$m^2 = \frac{1}{2h^2} = m^2, \tag{9}$$

which means that we did not find anything new here, but only a proof of our development.

To turn to the fourth powers, we take from (2) with $\varepsilon h = t$:

$$\frac{[\varepsilon^4]}{n} = \frac{2}{h^4\sqrt{\pi}}\int_0^\infty t^4 e^{-t^2} dt; \tag{10}$$

by partial integration, we have:

$$\int t^3 t e^{-t^2} dt = -\frac{t^3}{2} e^{-t^2} + \frac{3}{2}\int t^2 e^{-t^2} dt. \tag{11}$$

If we insert the limits 0 and ∞ here, we have for the first part $t^3 e^{-t^2}$ both times $= 0$ [of which we are to be convinced as above in the case of (5) and (7)], and the second part of (11), with the limits 0 and ∞, yields the integral which we already found on the preceding page in (5); hence, we have from (10) and (11) in all:

$$\frac{[\varepsilon^4]}{n} = \frac{2}{h^4\sqrt{\pi}}\frac{3}{2}\cdot\frac{1}{4}\sqrt{\pi} = \frac{3}{4h^4}. \tag{12}$$

With respect to $h^2 = 1{:}2\,m^2$ in (9), we thus have now the mean values of the first, second, and fourth powers as a summary:

$$\frac{[\pm\varepsilon]}{n} = t = \frac{1}{h\sqrt{\pi}} \quad \text{or} \quad = \sqrt{\frac{2}{\pi}}\,m \tag{13}$$

$$\frac{[\varepsilon^2]}{n} = m^2 = \frac{1}{2h^2} \quad \text{or} \quad = m^2 \tag{14}$$

$$\frac{[\varepsilon^4]}{n} = \nu^4 = \frac{3}{4h^4} \quad \text{or} \quad = 3\,m^4. \tag{15}$$

All sums of powers can be evaluated in the same way, for a gradual recursion, as in the case of (11), leads for even powers to the integral (6) and for odd powers to the integral (35), section 138, p. 517, which can easily be obtained; any desired power sum can thus be reduced gradually to $[\varepsilon^2]$ or $[\pm\varepsilon]$.

The general recursion formula is

$$\frac{[\varepsilon^{p+2}]}{n} = \frac{p+1}{2h^2}\frac{[\varepsilon^p]}{n} \quad \text{or} \quad = (p+1)\,m^2\,\frac{[\varepsilon^p]}{n}\,.$$

The general expression for the pth powers will be,

If p odd:

$$\frac{[\pm\varepsilon^p]}{n} = \frac{1\cdot 2\cdot 3\ldots\frac{p-1}{2}}{h^p}\,\frac{1}{\sqrt{\pi}}$$

$$\frac{[\pm\varepsilon^p]}{n} = \frac{(m\sqrt{2})^p}{\sqrt{\pi}}\,1\cdot 2\cdot 3\ldots\frac{p-1}{2}$$

If p even:

$$\frac{[\varepsilon^p]}{n} = \frac{1\cdot 3\cdot 5\ldots(p-1)}{(h\sqrt{2})^p} \qquad (16)$$

$$\frac{[\varepsilon^p]}{n} = m^p\,1\cdot 3\cdot 5\ldots(p-1)\,. \qquad (17)$$

The general formulas (16), (17) include the particular cases (13) to (15), and yield, applied still further, the following:

$$\left.\begin{array}{ll}
\dfrac{[\pm\varepsilon]}{n} = \sqrt{\dfrac{2}{\pi}}\,m & \dfrac{[\varepsilon^2]}{\hbar} = m^2 \\
\dfrac{[\pm\varepsilon^3]}{n} = 2\sqrt{\dfrac{2}{\pi}}\,m^3 & \dfrac{[\varepsilon^4]}{n} = 3\,m^4 \\
\dfrac{[\pm\varepsilon^5]}{n} = 8\sqrt{\dfrac{2}{\pi}}\,m^5 & \dfrac{[\varepsilon^6]}{n} = 15\,m^6 \\
\dfrac{[\pm\varepsilon^7]}{n} = 48\sqrt{\dfrac{2}{\pi}}\,m^7 & \dfrac{[\varepsilon^8]}{n} = 105\,m^8 \\
\ldots\ldots\ldots & \ldots\ldots\ldots
\end{array}\right\}. \qquad (18)$$

Inversely, these formulas can be used to calculate the mean error in a roundabout way from any sums of powers, which would yield for the fourth powers, for instance:

$$m = \sqrt[4]{\frac{[\varepsilon^4]}{3n}}\,. \qquad (19)$$

For practical use, however, for any kind of calculations of errors, only the first and the second powers (ε and ε^2) are suited.

Section 144. The Mean Error of Measures of Accuracy

In section 3, p. 8, and in section 4, p. 10, we defined the average error as the limiting value of the expression $t = \frac{[\pm \varepsilon]}{n}$ and the square of the mean error as the limiting value of the expression $m^2 = \frac{[\varepsilon^2]}{n}$ in the case of an infinity of real errors ε. There, too, we already indicated that from a small number of errors we can only find approximate values of t and m^2. The greater the number of errors is, the greater will be the certainty in the calculation of t and m^2. Therefore, we can ask the question with what mean error can t and m be calculated if a definite number of observations is present.

There is a second question involved here, concerning the ways and means of determining the mean error. For we have seen in section 138 that we can also calculate the mean error m conveniently by the way of the average error t, according to formula (38), p. 517, $m = \sqrt{\frac{\pi}{2}}\, t = 1.2533\, t$; indeed, we have even seen on p. 535, at the end of section 143, that we could calculate the mean error m also in a roundabout way through the third, fourth, etc., powers of real errors ε, and therefore, the question arises as to the reliability of these various kinds of determinations of m.

As we approach this question, let us retain at first the assumption that we are in possession of a certain number n of *real* errors ε, as not until later, in section 145, is the transition to the corrections v to be discussed.

Let us use for the symbolization of mean errors the symbol m as a functional symbol, i.e., the mean error of any magnitude x, for instance, is to be denoted by $m(x)$, just as $f(x)$ means a function of x, hence:

$$m(x) = \text{mean error of } x\,. \tag{1}$$

After these preliminary remarks, we begin with the method of calculation of the first powers ε. The average error is calculated from the equation:

$$t = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots \varepsilon_n}{n} = \frac{[\pm \varepsilon]}{n}\,. \tag{2}$$

If we inquire about the mean error of this t, it is a question of the deviation of the t from that theoretical value t_0 which we would obtain if not only n accidental values $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, but an infinite number of values ε were present. Let us call such a value t_0 the real value, hence $\varepsilon(t) = t - t_0$ will be the real error of the calculated value t.

If we set for t the value (2), we have:

$$\varepsilon(t) = \frac{1}{n}\left\{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots \varepsilon_n - n\, t_0\right\}. \tag{3}$$

Thence we obtain by squaring:

$$\left.\begin{aligned}\left(\varepsilon(t)\right)^2 = \frac{1}{n^2}\Big\{&\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \ldots + \varepsilon_n^2 \\ &+ 2\,\varepsilon_1\varepsilon_2 + 2\,\varepsilon_1\varepsilon_3 + \ldots + 2\,\varepsilon_{n-1}\varepsilon_n \\ &- 2\,n\,t_0(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_n) + n^2 t_0^2\Big\}.\end{aligned}\right\} \tag{4}$$

We have to take into account here that all ε's must be counted only as absolute values (for instance, all ε's positive). For the individual right-hand terms in which the real errors ε occur, we shall introduce average values which they would assume if the observations were repeated an infinite number of times.

According to the definition of p. 10, the average value of the terms $\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_n^2$ equals the square of the mean error; thus we can replace each of these terms by m^2.

In the products $2\varepsilon_1\varepsilon_2$, $2\varepsilon_1\varepsilon_3$, etc., whose number equals $\frac{n(n-1)}{2}$ we set instead of the individual errors ε_1, ε_2, etc., their average value t_0; therefore, each product will be $2t_0^2$.
Likewise, we also replace in the last line of (4) the individual ε's by their average value t_0.
The average value of $(\varepsilon(t))^2$ is the square of the mean error of t, hence $(m(t))^2$.
Instead of equation (4), we thus obtain:

$$\left(m(t)\right)^2 = \frac{1}{n^2}\left\{n\,m^2 + n(n-1)\,t_0^2 - 2\,n^2\,t_0^2 + n^2\,t_0^2\right\}$$

and after collecting the individual terms:

$$\left(m(t)\right)^2 = \frac{m^2 - t_0^2}{n}. \tag{5}$$

According to (38), section 138, p. 517, however,

$$\frac{m^2}{t_0^2} = \frac{\pi}{2} \quad \text{or} \quad m^2 = t_0^2\,\frac{\pi}{2};$$

we thus have from (5)

$$\left(m(t)\right)^2 = \frac{t_0^2}{n}\left(\frac{\pi}{2} - 1\right)$$

and hence

$$m(t) = \pm\frac{t_0}{\sqrt{n}}\sqrt{\frac{\pi}{2} - 1}. \tag{6}$$

This is the mean error of the determination of t from the mean of all $\pm\varepsilon$'s present; hence, we can write:

$$t = \frac{[\pm\varepsilon]}{n} \pm \frac{t_0}{\sqrt{n}}\sqrt{\frac{\pi}{2} - 1}. \tag{7}$$

The numerical calculation of the mean error of t is not possible in all rigorousness, as the true value t_0 of t would have to be known for it. If n is not too small, the calculated t will not deviate too much from t_0, and then we can give with t, instead of t_0, an approximate value of $m(t)$. In this case we have:

$$t = \frac{[\pm\varepsilon]}{n}\left(1 \pm \frac{1}{\sqrt{n}}\sqrt{\frac{\pi}{2} - 1}\right). \tag{8}$$

If we aim to calculate the mean error from the average error, then we have according to (38), section 138, p. 517:

$$m = \sqrt{\frac{\pi}{2}}\,t$$

and thus obtain from (8):

$$m = \sqrt{\frac{\pi}{2}}\,\frac{[\pm\varepsilon]}{n}\left(1 \pm \frac{1}{\sqrt{n}}\sqrt{\frac{\pi}{2} - 1}\right) \tag{9}$$

or

$$m = 1.2533 \frac{[\pm \varepsilon]}{n} \left(1 \pm \frac{0.7555}{\sqrt{n}}\right) . \tag{10}$$

In the same way as we have determined the mean error of the average error here, we also can determine the mean error of the mean error itself, for which we have:

$$m^2 = \frac{\varepsilon_1^{\ 2} + \varepsilon_2^{\ 2} + \varepsilon_3^{\ 2} + \ldots \varepsilon_n^{\ 2}}{n} = \frac{[\varepsilon^2]}{n} .$$

Let the true value of m^2, i.e. that which we would obtain in the case of an infinite number of ε $(n = \infty)$'s, be $m_0^{\ 2}$, then the true error of m^2 will be:

$$\varepsilon (m^2) = m^2 - m_0^{\ 2}$$

or

$$\varepsilon (m^2) = \frac{1}{n} \left(\varepsilon_1^{\ 2} + \varepsilon_2^{\ 2} + \varepsilon_3^{\ 2} + \ldots \varepsilon_n^{\ 2} - n\, m_0^{\ 2}\right) .$$

By squaring on both sides, we obtain:

$$\left.\begin{aligned}\left(\varepsilon (m^2)\right)^2 = \frac{1}{n^2} \Big\{ & \varepsilon_1^4 + \varepsilon_2^4 + \varepsilon_3^4 + \ldots + \varepsilon_n^4 \\ & + 2\, \varepsilon_1^2\, \varepsilon_2^2 + 2\, \varepsilon_1^2\, \varepsilon_3^2 + \ldots + 2\, \varepsilon^2_{\ n-1}\, \varepsilon_n^2 \\ & - 2\, n\, m_0^2 (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \ldots + \varepsilon_n^2) + n^2 m_0^4 \Big\}\end{aligned}\right\} . \tag{11}$$

If we turn from the true values of the right-hand terms to average values, then we have for $\varepsilon_1^{\ 4}$, $\varepsilon_2^{\ 4}$, etc., according to (15), section 143, p. 534, the average value $3\, m_0^{\ 4}$. We have to assume $m_0^{\ 2}$ as average value for $\varepsilon_1^{\ 2}$, $\varepsilon_2^{\ 2}$, etc., and the average value for $(\varepsilon (m^2))^2$ equals $(m (m^2))^2$. Then we find from (11):

$$\begin{aligned}\left(m (m^2)\right)^2 &= \frac{1}{n^2} \left(3\, n\, m_0^4 + n\, (n - 1)\, m_0^4 - 2\, n^2\, m_0^4 + n^2\, m_0^4\right) \\ &= m_0^4 \frac{2}{n}\end{aligned}$$

or

$$m (m^2) = m_0^2 \sqrt{\frac{2}{n}} . \tag{12}$$

This is the mean error of the determination of m^2 from the sum of the squares of the ε's present.

As the true value m_0 of the mean error of the observations is not known, there remains nothing else to do but to replace m_0 by m calculated from n errors, which will be admissible if n is a rather large number. Then we have

$$m (m^2) = m^2 \sqrt{\frac{2}{n}} . \tag{13}$$

In order to obtain $m (m)$, the mean error of m from the above, we apply the law of propagation of errors (13), section 5, p. 14. According to this, we have

$$m (m) = \frac{m (m^2)}{2\, m} ,$$

or $$m\,(m) = \pm\, m\sqrt{\frac{1}{2\,n}}\,. \tag{14}$$

We thus obtain as result for rather large values of n:

$$m = \sqrt{\frac{[\varepsilon^2]}{n}}\left(1 \pm \sqrt{\frac{1}{2\,n}}\right), \tag{15}$$

or else $$m = \sqrt{\frac{[\varepsilon^2]}{n}}\left(1 \pm \frac{0.70710}{\sqrt{n}}\right). \tag{16}$$

Comparing these with (9) or (10), we find that the calculation of m from the squares, ε^2, is more favorable than is the calculation from the ε's themselves, and in fact, in the ratio 0.7555:0.7071.

The continuation of these investigations to higher powers, which no longer have a practical significance, yields the following series in which, because they are related, the two values (10) and (16) for the first and second powers are given once again:

$$\left.\begin{aligned}
&1.\quad m = 1.2533\,\frac{[\pm\varepsilon]}{n}\left(1 \pm \frac{0.7555}{\sqrt{n}}\right)\\
&2.\quad m = \sqrt{\frac{[\varepsilon^2]}{n}}\left(1 \pm \frac{0.7071}{\sqrt{n}}\right)\\
&3.\quad m = 0.8557\,\sqrt[3]{\frac{[\pm\varepsilon^3]}{n}}\left(1 \pm \frac{0.7371}{\sqrt{n}}\right)\\
&4.\quad m = 0.7598\,\sqrt[4]{\frac{[\varepsilon^4]}{n}}\left(1 \pm \frac{0.8165}{\sqrt{n}}\right)\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\\
&9.\quad m = 0.5293\,\sqrt[9]{\frac{[\pm\varepsilon^9]}{n}}\left(1 \pm \frac{2.1408}{\sqrt{n}}\right)\\
&10.\quad m = 0.5040\,\sqrt[10]{\frac{[\varepsilon^{10}]}{n}}\left(1 \pm \frac{2.7058}{\sqrt{n}}\right)
\end{aligned}\right\}. \tag{17}$$

The most favorable determination of m is the determination from the squares, ε^2, for this determination 2 has the smallest mean error, namely 0.7071 m. The determination from the tenth powers of the ε's has, according to 10, the nearly four times greater error 2.7058 m.

In this connection, we refer to the citation of the source paper by Gauss, Bestimmung der Genauigkeit der Beobachtungen [Determination of the accuracy of observations] (see Introduction, p. 2); also to Encke, *Berl. Astronom. Jahrb. für 1834*, pp. 289-293, where we must not by any means overlook, however, that Gauss and Encke calculate throughout with *probable* errors, in the case of the means of powers as well as their uncertainties; hence, they always give the probable error with its probable error, whereas our calculation in the previously established theory always refers to *mean* errors and mean errors of the mean errors; all our coefficients are therefore greater than the coefficients according to Gauss-Encke in the ratio $1:\rho\sqrt{2}$ or 1:0.6745.

A further question is now to what extent the reliability of a determination of the mean error decreases if real errors ε are not available, as was previously assumed, but only the corrections v (which result in the case of an adjustment).

If the adjustment had n observations and u unknowns, we calculate, as is known, from the sum of squares $[v^2]$ of the corrections, the mean square of error according to the formula:

$$m^2 = \frac{[\varepsilon^2]}{n} = \frac{[v^2]}{n-u}, \tag{1}$$

which we have developed in (19), section 29, p. 93.

According to (12), section 29, p. 92, the relation between $[v^2]$ and $[\varepsilon^2]$ is as follows:

$$[v^2] = [\varepsilon^2] - \frac{[a\,\varepsilon]^2}{[a\,a]} - \frac{[b\,\varepsilon \cdot 1]^2}{[b\,b \cdot 1]} - \frac{[c\,\varepsilon \cdot 2]^2}{[c\,c \cdot 2]} - \dots \,. \tag{2}$$

Let us retain at first *one* unknown, then we obtain with $u = 1$ from (1) and (2):

$$m^2 = \frac{[v^2]}{n-1} = \frac{[\varepsilon^2] - \dfrac{[\varepsilon]^2}{n}}{n-1} \text{ for } u = 1\,. \tag{3}$$

If now $m_0{}^2$ is again the true value of m^2, i.e. that value which would be obtained in the case of an infinite number of ε's (hence, $n = \infty$), then $m^2 - m_0{}^2$ represents the real error of m^2 in an individual case. We then have

$$m^2 - m_0{}^2 = \frac{[\varepsilon^2] - \dfrac{[\varepsilon]^2}{n}}{n-1} - m_0{}^2 = \frac{1}{n\,(n-1)}\left(n\,[\varepsilon^2] - [\varepsilon]^2 - n\,(n-1)\,m_0{}^2\right) \tag{4}$$

and squaring

$$\left(m^2 - m_0{}^2\right)^2 = \frac{1}{n^2\,(n-1)^2}\Big(n^2\,[\varepsilon^2]^2 + [\varepsilon]^4 + n^2\,(n-1)^2\,m_0{}^4 - 2\,n\,[\varepsilon^2]\,[\varepsilon]^2$$
$$- 2\,n^2\,(n-1)\,[\varepsilon^2]\,m_0{}^2 + 2\,n\,(n-1)\,[\varepsilon]^2\,m_0{}^2\Big)\,. \tag{5}$$

If we take at the left-hand side the average value from an infinite number of cases, the latter equals the square of the mean error of m^2, or $(m\,(m^2))^2$.

Of the six terms at the right-hand side in (5), we must now determine individually the average values, for which we already have some preparations. In any case, algebraically

$$[\varepsilon^2]^2 = (\varepsilon_1{}^2 + \varepsilon_2{}^2 + \dots)^2 = (\varepsilon_1{}^4 + \varepsilon_2{}^4 + \dots) + 2\,(\varepsilon_1{}^2\,\varepsilon_2{}^2 + \varepsilon_1{}^2\,\varepsilon_3{}^2 + \dots)\,. \tag{6}$$

The average value of the fourth powers of the errors has been found in (15), section 143, p. 534, to equal $3\,m_0{}^4$. In the terms $\varepsilon_1{}^2\varepsilon_2{}^2$, $\varepsilon_1{}^2\varepsilon_3{}^2$, and so on, whose number equals $\dfrac{n\,(n-1)}{2}$, we replace $\varepsilon_1{}^2$, $\varepsilon_2{}^2$, and so on, by the mean value $m_0{}^2$. Hence, (6) turns into

$$[\varepsilon^2]^2 = 3\,n\,m_0{}^4 + n\,(n-1)\,m_0{}^4 = n\,(n+2)\,m_0{}^4\,; \tag{7}$$

with this, we have the first term of the parenthesis of (5).

Similarly, we also develop the fourth term of (5), namely:

$$[\varepsilon^2]\,[\varepsilon]^2 = (\varepsilon_1{}^2 + \varepsilon_2{}^2 + \ldots)\;(\varepsilon_1 + \varepsilon_2 + \ldots)^2$$
$$= (\varepsilon_1{}^2 + \varepsilon_2{}^2 + \ldots)\;(\varepsilon_1{}^2 + \varepsilon_2{}^2 + \ldots + 2\,\varepsilon_1\,\varepsilon_2 + 2\,\varepsilon_1\,\varepsilon_3 \ldots)\;.$$

But since the average value of a group of odd powers vanishes, because of the changing sign of the ε's, we will have just as in the case of (7):

$$[\varepsilon^2]\,[\varepsilon]^2 = [\varepsilon^2]\,[\varepsilon^2] = [\varepsilon^2]^2 = (n^2 + 2\,n)\,m_0{}^4\;. \tag{8}$$

Of the remaining terms in (5), we shall explain furthermore in detail only $[\varepsilon]^4$, namely:

$$[\varepsilon]^4 = (\varepsilon_1 + \varepsilon_2 + \ldots)^4 = (\varepsilon_1{}^4 + \varepsilon_2{}^4 + \ldots) + (6\,\varepsilon_1{}^2\,\varepsilon_2{}^2 + \ldots) + (4\,\varepsilon_1\,\varepsilon_2{}^3 + \ldots)\;;$$

but since the odd powers vanish, there remains only

$$[\varepsilon^4] = 3\,n\,m_0{}^4 + 3\,n\,(n-1)\,m_0{}^4 = 3\,n^2\,m_0{}^4\;. \tag{9}$$

If we treat all terms of (5) in this way, we obtain:

$$\left(m\,(m^2)\right)^2 = \left(\frac{1}{n\,(n-1)}\right)^2 \left\{\begin{array}{r} n^2\,(n^2+2\,n)\,m_0{}^4 - 2\,n\,(n^2+2\,n)\,m_0{}^4 - 2\,n^3\,(n-1)\,m_0{}^4 \\ + 3\,n^2 m_0{}^4 + 2\,n^2\,(n-1)\,m_0{}^4 \\ + n^2\,(n-1)^2\,m_0{}^4 \end{array}\right\} \tag{10}$$

$$\left(m\,(m^2)\right)^2 = \left(\frac{1}{n\,(n-1)}\right)^2 \left(2\,n^3\,m_0{}^4 - 2\,n^2\,m_0{}^4\right) = \frac{2\,m_0{}^4}{n-1}$$

$$m\,(m^2) = m_0{}^2 \sqrt{\frac{2}{n-1}}\;. \tag{11}$$

This is the mean error of m^2 insofar as m^2 has been calculated according to (3).

We shall assume again that for not too small values of n, the unknown mean error m_0 can be replaced by m, and obtain

$$m\,(m^2) = m^2 \sqrt{\frac{2}{n-1}}\;. \tag{12}$$

It follows from the law of propagation of errors, according to the procedure on p. 538, that

$$m\,(m) = \frac{m\,(m^2)}{2\,m}\,,$$

hence

$$m\,(m) = m \sqrt{\frac{1}{2\,(n-1)}}\;. \tag{13}$$

Summarizing, we can also write

$$m = \sqrt{\frac{[v^2]}{n-1}}\left(1 \pm \sqrt{\frac{1}{2(n-1)}}\right) = \sqrt{\frac{[v^2]}{n-1}}\left(1 \pm \frac{0.7071}{\sqrt{n-1}}\right). \tag{14}$$

This differs from the previous equation (16), section 144, p. 539, only by the fact that we have now everywhere v instead of ε, and $n-1$ instead of n.

Transition to two and more unknowns

The previous study from (3) to (14) referred to *one* unknown, as it occurs in the arithmetical mean, where the case of unequally accurate individual observations (section 8) and the case of coefficients a, as in the case of (5), section 13, p. 40, are also included, insofar as we assume here reduction to equal weights, hence $\varepsilon\sqrt{p}$, etc.

The previous way of treating can still be retained, on the whole, even in the case of *two* unknowns. According to (12), section 29, p. 92, we have for two unknowns:

$$[v^2] = [\varepsilon^2] - \frac{[a\,\varepsilon]^2}{[a\,a]} - \frac{[b\,\varepsilon\cdot 1]^2}{[b\,b\cdot 1]} \quad \text{and} \quad m^2 = \frac{[v^2]}{n-2}. \tag{15}$$

Hence, if m_0^2 is the real mean square of error, and the mean error of m^2 is denoted by $m\,(m^2)$, then we have

$$m^2 - m_0^2 = \frac{[\varepsilon^2] - \dfrac{[a\,\varepsilon]^2}{[a\,a]} - \dfrac{[b\,\varepsilon\cdot 1]^2}{[b\,b\cdot 1]}}{n-2} - m_0^2$$

$$\left(m^2 - m_0^2\right)^2 = \left(\frac{1}{n-2}\right)^2 \left([\varepsilon^2] - \frac{[a\,\varepsilon]^2}{[a\,a]} - \frac{[b\,\varepsilon\cdot 1]^2}{[b\,b\cdot 1]} - (n-2)\,m_0^2\right)^2 \tag{16}$$

$$\left(m^2 - m_0^2\right)^2 = \left(\frac{1}{n-2}\right)^2 \left\{\begin{aligned} &[\varepsilon^2]^2 - 2\frac{[a\,\varepsilon]^2}{[a\,a]}[\varepsilon^2] - 2\frac{[b\,\varepsilon\cdot 1]^2}{[b\,b\cdot 1]}[\varepsilon^2] - 2\,(n-2)\,[\varepsilon^2]\,m_0^2 \\ &\quad + \frac{[a\,\varepsilon]^4}{[a\,a]^2} + 2\frac{[a\,\varepsilon]^2\,[b\,\varepsilon\cdot 1]^2}{[a\,a]\,[b\,b\cdot 1]} + 2\,(n-2)\frac{[a\,\varepsilon]^2}{[a\,a]}m_0^2 \\ &\qquad + \frac{[b\,\varepsilon\cdot 1]^4}{[b\,b\cdot 1]^2} + 2\,(n-2)\frac{[b\,\varepsilon\cdot 1]^2}{[b\,b\cdot 1]}m_0^2 \\ &\qquad\qquad + (n-2)^2\,m_0^4 \end{aligned}\right\}. \tag{17}$$

Of the ten terms which occur in this brace, the mean values are to be formed individually, and we soon see that the previous considerations (6) to (8) as well as the former equation (16), p. 92, etc., can be used again for all those terms which do not contain a $[b\,\varepsilon\cdot 1]$; and as far as the terms with $[b\,\varepsilon\cdot 1]$ are concerned, we only need to solve them to $[b\,\varepsilon] - \dfrac{[a\,b]}{[a\,a]}[a\,\varepsilon]$, in order to reduce everything to $[a\,\varepsilon]\cdot[b\,\varepsilon]$, etc.

The execution is complicated; it yields

$$\left(m\,(m^2)\right)^2 = \left(\frac{m_0^2}{n-2}\right)^2 \left\{\begin{matrix} +\,n\,(n+2) & -\,2\,(n+2) & -\,2\,(n+2) & -\,2\,(n-2)\,n \\ & +\,3 & +\,2 & +\,2\,(n-2) \\ & & +\,3 & +\,2\,(n-2) \\ & & & +\,(n-2)^2 \end{matrix}\right\}$$

$$\left(m\,(m^2)\right)^2 = \left(\frac{m_0^2}{n-2}\right)^2 \left\{2\,n - 4\right\} = 2\,\frac{m_0^4}{n-2}. \tag{18}$$

If we replace again m_0 by m in (18), assuming large values of n, we have

$$m\,(m^2) = m^2 \sqrt{\frac{2}{n-2}},$$

or in a different form

$$m^2 = \frac{[v^2]}{n-2}\left(1 \pm \sqrt{\frac{2}{n-2}}\right)$$

$$m = \sqrt{\frac{[v^2]}{n-2}}\left(1 \pm \sqrt{\frac{1}{2\,(n-2)}}\right). \tag{19}$$

Extension to more than two unknowns, in general u unknowns, will yield

$$m = \sqrt{\frac{[v^2]}{n-u}}\left(1 \pm \sqrt{\frac{1}{2\,(n-u)}}\right). \tag{20}$$

Without carrying through this further, we shall only note that we can proceed here different ways, say by introducing $b'\,\varepsilon'$ into m, as in the case of (14), section 29, p. 92, where the formation of the mean values, however, is not as simple as it is there.

The original source of the matter is art. 39-40 of Gauss' Theoria combinationis (cf. p. 2), where the coefficients $[\alpha\,\alpha]$, etc., of our section 30, p. 96, are used. Gauss treats the task at first without a definite assumption about the law of errors and finds all occurring mean values expressible by $m_0^{\,2}$ and by ν^4 in the sense of (15), section 143, p. 534. Hence, in the case of an indefinite law of error, we do *not* set at first $\nu^4 = 3\,m^4$, but we leave ν^4 undetermined. The desired $m\,(m^2)$ can nevertheless be included in certain limits which lead to a plausible estimate of $m\,(m^2)$ even without the knowledge of ν^4 and then, with $\nu^4 = 3\,m^4$, lead to the final result which is contained in our equation (20). The corresponding paper from Helmert, *Ausgleichsrechnung nach der M.d.kl.Q.* [Calculus of adjustment according to the method of least squares], second edition, 1907, pp. 139-144, is to be mentioned also.

We mention further F. R. Helmert, "Zur Ableitung der Formel von C. F. Gauss für den mittleren Beobachtungsfehler und ihrer Genauigkeit" [On the derivation of the formula by C. F. Gauss for the mean observational error and its accuracy], 1904, pp. 950-964. *Sitzungsber. d. Kgl. Preuss. Akademie d. Wiss.*

If we apply equation (20) to the case of conditioned observations with r independent condition equations, for instance, a triangulation with r condition equations, then $n - u = r$, hence:

$$m = \sqrt{\frac{[v^2]}{r}}\left\{1 \pm \sqrt{\frac{1}{2r}}\right\}. \tag{21}$$

If we take in this connection, for example, the Belgian-German junction net of section 92, then we have, according to p. 350, the sum $[v^2p] = 4.2$, hence within the meaning of our (21).

$$[v^2] = 4.2 \quad \text{and} \quad r = 11\,,$$

hence: $$m = \sqrt{\frac{4{,}2}{11}}\left(1 \pm \sqrt{\frac{1}{22}}\right)$$

$$m = \pm\, 0.618''(1 \pm 0.213) = \pm\,(0.618'' \pm 0.132''),$$

i.e. the mean error 0.618" has resulted from that triangulation with an uncertainty of approximately 21% of its own amount, or 0.13".

In order to compare with experience the law of the distribution of errors of section 138, namely $\varphi(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2}$, which we found in a purely theoretical way, it is necessary that we procure a series of real observational errors or a series of accidental events such that we can assume about them that they are subject to the same conditions as are observational errors.

As such an accidental series we shall take the distribution of the zeros in a table of logarithms.

We have made counts in Vega-Hülsse (Leipzig, 1840) and, in fact, on pp. 2-185, where the seven-place logarithms of the numbers from 1 to 99999 are. We find the same logarithms, of course, also in Vega-Bremiker and in Schrön in the same arrangement, only with the small difference in the case of Vega-Bremiker that there are not 50, but 51 values in each column because the number of the following column is repeated at the bottom. Apart from this, we have 50 logarithms in each column, and the counting was extended over the number of the zeros in the sixth place, as the following example of the counting of a page shows:

Counting of the Zeros in the Sixth Place of Logarithms

	0	1	2	3	4	5	6	7	8	9
2000	0300	0517	0734	0951	1168	1386	1603	1820	2037	2254
2001	2471	2688	2905	3122	3339	3556	3773	3990	4207	4424
2002	4641	4858	5075	5291	5508	5725	5942	6159	6376	6593
2003	6809	7026	7243	7460	7677	7893	8110	8327	8544	8760
2004	8977	9194	9411	9627	9844	0061	0277	0494	0711	0927
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
2045	6933	7145	7358	7570	7783	7995	8207	8419	8632	8845
2046	9056	9269	9481	9693	9905	0117	0330	0542	0754	0966
2047	1178	1391	1603	1815	2027	2239	2451	2663	2875	3087
2048	3300	3512	3724	3936	4148	4360	4572	4784	4996	5208
2049	5420	5632	5843	6055	6267	6479	6691	6903	7115	7327
Total	9	1	7	6	6	4	8	5	6	3

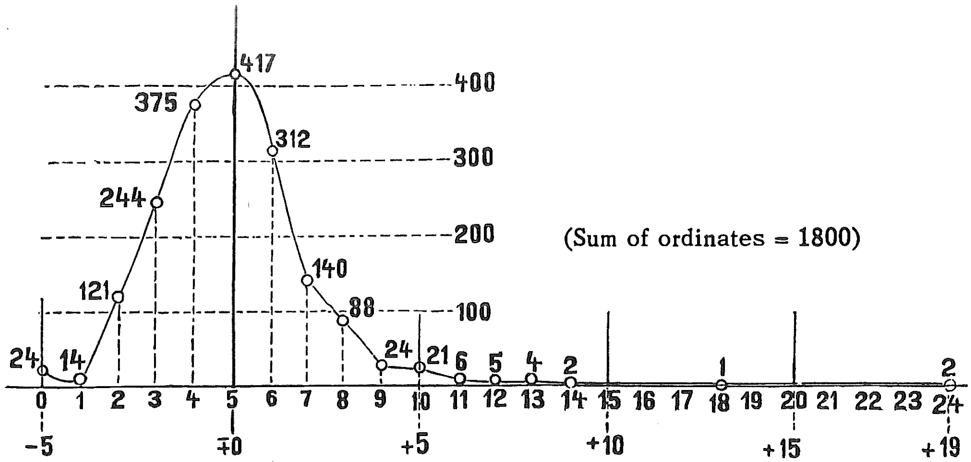
Eighteen hundred such columns were counted through with a total of 90,000 numerals with respect to zeros.

We gave some details about this in the *Zeitschr. f. Verm.*, 1890, pp. 560-564; let us also note, in reference to the corresponding example in the previous editions of this book (for instance, 3rd edition, Vol. I, 1888, pp. 282-285), that the former example was a different one in two respects: First, only 600 columns were treated instead of the present 1800 columns, and secondly, the columns were not treated individually but two at a time.

In detail, the counting yielded the following:

Number of the Zeros in a Column l	Occurring p	Product pl	Error $l-5=\varepsilon$	ε^2	$p\varepsilon^2$
0	24 times	0	− 5	25	600
1	14	14	− 4	16	224
2	121	242	− 3	9	1089
3	244	732	− 2	4	976
4	375	1500	− 1	1	375
5 (mean value)	417 (maximum)	2085	0	0	0
6	312	1872	+ 1	1	312
7	140	980	+ 2	4	560
8	88	704	+ 3	9	792
9	24	216	+ 4	16	384
10	21	210	+ 5	25	525
11	6	66	+ 6	36	216
12	5	60	+ 7	49	245
13	4	52	+ 8	64	256
14	2	28	+ 9	81	162
18	1	18	+ 13	169	169
24	2	48	+ 19	361	722
Total	1800	8827			7607

In this connection, the following graphic representation was also made:



The abscissae 0, 1, 2, . . . , 24 refer to the number l of the zeros which were found in each fifty-term column. The ordinates $p = 24$, $p = 14$ . . . , $p = 417$ . . . , $p = 2$ specify the number of cases in which the number of zeros l was found; for instance, four zeros were found in one column 375 times, or five zeros in one column 417 times, etc.

Now we calculate further as if our l's were the results of the independent homogeneous observations of an unknown x, which will then be computed as the arithmetic mean of all l's:

$$x = \frac{[p\,l]}{[p]} = \frac{8827}{1800} = 4.9033 \ .$$

This value $x = 4.9033$ is smaller than the value $x = 5$ which was to be expected in the case of equal distribution of the numerals in the counted columns. We shall not base the further computations on the observed mean value $x = 4.9033$ but on the theoretical value $x = 5$ by treating further the differences $l - x = \varepsilon$ as real observational errors.

Before doing so, let us, however, look at the graphic representation once again, where upon it strikes the eye that this curve is not symmetrical about the axis. Such an absence of symmetry, or unequal probabilities for $-\varepsilon$ and for $+\varepsilon$, was to be expected from the outset, especially in the case of the limiting values; for the error $\varepsilon = -5$, or 0 zeros, is absolutely the lower *limit* because $\varepsilon = -6$, or less than 0 zeros, is unthinkable in a column, whereas on the positive side with $\varepsilon = +5$ or 10 zeros in a column, the matter is not finished at all, and beyond that, 11, 12, 13, 14 zeros still occur quite constantly, upon which two gaps follow, and the series terminates with 24 zeros in a column twice.

In spite of this lack of symmetry on the whole, we can calculate a mean error, for which the squares, $p\,\varepsilon^2$, are already contained above in the main listing of p. 544, i.e. $[p\,\varepsilon^2] = 7607$, hence the mean error $m = \sqrt{\frac{7607}{1800}} = \pm 2.056$.

Besides, we shall treat separately the two parts, $-\varepsilon$ and $+\varepsilon$, and in fact, the important case $\varepsilon = 0$ with $n = 417$ is to be counted half to the negative side and half to the positive side here, so that we obtain:

Negative Errors			
ε	ε^2	p	$p\varepsilon^2$
0	0	208.5	0
— 1	1	375	375
— 2	4	244	976
— 3	9	121	1089
— 4	16	14	224
— 5	25	24	600
		986.5	3264

$$m^2 = \frac{3264}{986.5} = 3.3087$$

$$m = \pm 1.8190$$

Positive Errors			
ε	ε^2	p	$p\varepsilon^2$
0	0	208.5	0
+ 1	1	312	312
+ 2	4	140	560
+ 3	9	88	792
+ 4	16	24	384
+ 5	25	21	525
+ 6	36	6	216
+ 7	49	5	245
+ 8	64	4	256
+ 9	81	2	162
+13	169	1	169
+19	361	2	722
		813.5	4343

$$m^2 = \frac{4343}{813.5} = 5.3387$$

$$m = \pm 2.3106$$

The next study is to be concerned with the negative errors alone, and we shall compare the distribution with the theoretical law of errors of section 138, and therefore it is necessary at first that we express the interval $d\varepsilon = 1$ as a fraction of mean error m. From $m = 1.8190$ follows

$$d\varepsilon = 1 = 0.54976\, m$$

and

$$\frac{d\varepsilon}{2} = 0.5 = 0.27488\, m\,.$$

Now we define, according to the table on p. [21] of the Appendix, the probabilities W for the occurrence of an error between the limits zero and n times the mean error and calculate at the same time further, as the following table shows:

Limits				W from Page [21]	986.5 W	Differences p'
0 and ε		0 and n				
0	0.5	0	0.27488	0.2166	213.7	213.7
0	1.5	0	0.82464	0.5904	582.4	368.7
0	2.5	0	1.37440	0.8307	819.5	237.1
0	3.5	0	1.92416	0.9456	932.8	113.3
0	4.5	0	2.47392	0.9867	973.4	40.6
0	5.5	0	3.02368	0.9975	984.0	10.6
0	∞	0	∞	1.0000	986.5	2.5
						986.5

The differences p' of the last column are now the theoretical numbers which correspond to the observed p's of the previous table, and therefore we have the following comparison for the negative ε's:

Limits	Number of Errors from		Deviation
	Theory p'	Experience p	
0 and 0.5	214	209	+ 5
0.5 and 1.5	369	375	— 6
1.5 and 2.5	237	244	— 7
2.5 and 3.5	113	121	— 8
3.5 and 4.5	41	14	+27
4.5 and 5.5	11	24	—13
5.5 and ∞	2	0	+ 2
Total	987	987	0

The agreement between theory and experience is such that one can say that the distribution of the zeros follows approximately the law of section 138. The corresponding distribution of the positive ε's agrees with theory considerably worse yet; we shall not present this distribution also; it becomes distorted especially by the isolated *large* ε's (cf. the Fig. on p. 545, with the values running to the right to 24).

Such comparisons between the distribution of errors from experience and from the theory of the Gauss error law have been undertaken first by Bessel in "Fundamenta astronomiae," pp. 19-20, with three error series, the third of which, comprising 470 right-ascension observations, is treated also by Encke in the *Berl. Astronom. Jahrbuch* for 1834, pp. 274-275; it also was included in the second edition of our *Handb. d. Verm.*, 1st vol., 1877, pp. 101 and 104.

An extensive investigation of this kind refers to the errors of closure of 2238 triangles of the Italian triangulation, treated at the suggestion of General Ferrero by the engineer Guarducci, published in the Italian periodical *Rivista di topografia e catasto*, Roma, 1889, Vol. II, pp. 1-12. There are two groups: First, 661 triangles with the mean error of closure 14.08", and secondly, 1577 triangles with the error of closure 17.13". A report on this is given also by Czuber in *Theorie der Beobachtungsfehler* [Theory of observational errors], Leipzig, 1891, pp. 193-194, and furthermore a series of 40 determinations of graduation marks from A. R. Clarke's *Geodesy*, discussed by Faye in *Comptes rendus*, 106th vol., 1888, pp. 783-786.

In the "Rapport sur les triangulations, présenté à la dixième Conférence générale à Bruxelles en 1892," General Ferrero gave furthermore a table with 12 graphic representations on 18,085 errors of closure, and reports, on p. 4 of this "Rapport," that already at the Paris meeting in 1889, he distributed a treatise which proved strikingly the agreement between theory and observation on more than 5000 errors of triangle closure.

For the main triangulation of the Austrian Military-Geographic Institute in Vienna, which contains a total of 1285 triangles, there exists also an investigation of the distribution of errors, which was discussed by Wilhelm Tinter in "Die Schlussfehler der Dreiecke der Triangulierung erster Ordnung in der k. undk. österreichisch-ungarischen Monarchie" [The errors of closure of triangles of the first-order triangulation in the Royal and Imperial Austro-Hungarian Monarchy], *Veröffentlichung der k. k. österreichischen Kommission der internationalen Erdmessung*, Wien, 1904 and 1905.

Section 147. Observations with Different Moduli of Precision h

The example treated in the previous section for the testing of the Gauss law of errors is based on assumptions which will not always prove true in the case of real measurements. When, for instance, an angle is measured repeatedly with the same instrument by the same observer, the measuring results cannot be looked upon as perfectly equal in precision, since the illumination of the targets, the stability of the instrument, the attention of the observer, etc., will constantly change in the course of the measurement. Consequently, to one part of the observations a modulus of precision h_1 will be assigned, to a second, a modulus h_2, to a third, h_3, etc., or in other words, the whole series of observations can be broken down into several individual series with the moduli of precision h_1, h_2, h_3, etc. In fact, most series of observations show deviations from the Gauss law of errors which can perhaps be explained by the assumption of different moduli of precision. This is more thoroughly investigated by S. Newcomb in *American Journal of Mathematics*, vol. VIII, 1886, pp. 343-366. Newcomb suggests, for the case in which rather large deviations from the Gauss law of errors occur, assuming two, three, or more values of h and distributing the observations over these different classes. If the distribution takes place in the ratio $p_1 : p_2 : p_3 : \ldots$, then the law of errors is as follows:

$$\Phi(\varepsilon) = \frac{1}{\sqrt{\pi}} \left(p_1 h_1 e^{-h_1^2 \varepsilon^2} + p_2 h_2 e^{-h_2^2 \varepsilon^2} + \ldots \right). \tag{1}$$

Since this expression must reduce again to the Gauss law for equal values of h, we must have:

$$p_1 + p_2 + p_3 + \ldots = 1 .$$

We shall assume now that the modulus of precision h corresponds to the mean error m calculated from the total number of measurements, so that, according to (26), section 138, p. 516,

$$h = \frac{1}{m\sqrt{2}} . \tag{2}$$

Let

$$h_1 = h + \Delta_1 \qquad h_2 = h + \Delta_2 \ \ldots \ , \tag{3}$$

so that in (1) each term can be developed according to the Taylor series. Limiting ourselves to terms of second order, we obtain:

$$h_1 e^{-h_1^2 \varepsilon^2} = h e^{-h^2 \varepsilon^2} + e^{-h^2 \varepsilon^2} (1 - 2 h^2 \varepsilon^2) \Delta_1 - h \varepsilon^2 e^{-h^2 \varepsilon^2} (3 - 2 h^2 \varepsilon^2) \Delta_1^2 + \ldots$$

and then we will have:

$$\Phi(\varepsilon) = \frac{1}{\sqrt{\pi}} \left\{ h e^{-h^2 \varepsilon^2} + e^{-h^2 \varepsilon^2} (1 - 2h^2 \varepsilon^2) [p\Delta] - h \varepsilon^2 e^{-h^2 \varepsilon^2} (3 - 2h^2 \varepsilon^2) [p\Delta^2] + \ldots \right\}. \quad (4)$$

If

$$\varphi(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2}$$

symbolizes the Gauss law of errors, then we have also:

$$\Phi(\varepsilon) - \varphi(\varepsilon) = \frac{1}{\sqrt{\pi}} \left\{ e^{-h^2 \varepsilon^2} (1 - 2h^2 \varepsilon^2) [p\Delta] - h \varepsilon^2 e^{-h^2 \varepsilon^2} (3 - 2h^2 \varepsilon^2) [p\Delta^2] + \ldots \right\}. \quad (5)$$

For this difference of the two laws of error, a special relation results if we introduce, instead of the quantities Δ, the corresponding changes of m^2, which we shall denote by δ, so that we will have

$$m_1^2 = m^2 + \delta_1 \qquad m_2^2 = m^2 + \delta_2 \qquad \ldots .$$

Since

$$m^2 = \frac{[\varepsilon\varepsilon]}{n} \qquad m_1^2 = \frac{[\varepsilon\varepsilon]_1}{n_1} \qquad m_2^2 = \frac{[\varepsilon\varepsilon]_2}{n_2} \qquad \ldots$$

we have also

$$m^2 = \frac{[\varepsilon\varepsilon]}{n} = \frac{n_1}{n} m_1^2 + \frac{n_2}{n} m_2^2 + \ldots$$

$$m^2 = p_1 m_1^2 + p_2 m_2^2 + \ldots$$

and hence it follows that

$$[p\delta] = 0 . \quad (6)$$

For the transition from Δ to δ and vice versa, we have

$$\frac{1}{m^2 + \delta} = 2(h + \Delta)^2$$

or

$$\frac{1}{m^2} - \frac{1}{m^4}\delta + \frac{1}{m^6}\delta^2 + \ldots = 2h^2 + 4h\Delta + 2\Delta^2 ,$$

and according to (2)

$$-\frac{1}{m^4}\delta + \frac{1}{m^6}\delta^2 + \ldots = 4h\Delta + 2\Delta^2 . \quad (7)$$

Hence we can easily find a value for Δ by gradual approximation. For in the first approximation we have

$$\Delta = -\frac{1}{4\,h\,m^4}\,\delta + \ldots = -h^3\,\delta + \ldots .$$

When brought into (7), this yields

$$4\,h\,\Delta + 2\,h^6\,\delta^2 = -\frac{1}{m^4}\,\delta + \frac{1}{m^6}\,\delta^2 + \ldots$$

or

$$4\,h\,\Delta + 2\,h^6\,\delta^2 = -4\,h^4\,\delta + 8\,h^6\,\delta^2 + \ldots ,$$

and hence it follows

$$\Delta = -h^3\,\delta + \frac{3}{2}\,h^5\,\delta^2 + \ldots .$$

If we introduce this value of Δ in (5), bearing in mind (6), we obtain easily

$$\Phi(\varepsilon) - \varphi(\varepsilon) = \frac{h^5}{2\sqrt{\pi}}\,e^{-h^2\varepsilon^2}(3 - 12\,h^2\,\varepsilon^2 + 4\,h^4\,\varepsilon^4)\,[p\,\delta^2] + \ldots . \tag{8}$$

We see from this that the difference of the two laws of errors vanishes if

$$3 - 12\,h^2\varepsilon^2 + 4\,h^4\varepsilon^4 = 0$$

or

$$\left.\begin{array}{lll} h\,\varepsilon = 1.651 & \text{and} & 0.525 \\ \dfrac{\varepsilon}{m} = 2.335 & \text{and} & 0.742 \end{array}\right\} . \tag{9}$$

The errors

$$\varepsilon = 2.335\ m \qquad \text{and} \qquad \varepsilon = 0.742\ m \tag{10}$$

thus have the same probability according to both laws.

Both expressions (10) hold for arbitrarily assumed values $h_1, h_2 \ldots$ in (1) provided that these values do not deviate too greatly from h. If no details are known about the differences of precision of the individual measurements, we can make the assumption that the occurrence of the different h's is merely subject to accident. The values $h_1, h_2 \ldots$ will in any case lie in the neighborhood of a mean value h_0, for which

$$h_0 = [p\,h] \tag{11}$$

holds, and to the deviations

$$h_1 - h_0 = \Delta_1' , \quad h_2 - h_0 = \Delta_2' \ldots \tag{12}$$

the Gauss law of errors can then be applied. Hence we have

$$\psi(h) = \frac{k}{\sqrt{\pi}}\,e^{-k^2\Delta'^2} \tag{13}$$

where k symbolizes the modulus of precision for Δ'.

Now we return again to equation (5) where we have to replace the symbols Δ by the symbols Δ'. From (3) and (12) follows

$$\Delta_1 = \Delta_1' + (h_0 - h) , \qquad \Delta_2 = \Delta_2' + (h_0 - h) \quad \ldots \tag{14}$$

and from (11) it follows that

$$[p\,\Delta'] = 0\,. \tag{15}$$

We can further make the assumption that h_0 lies in the immediate neighborhood of h so that the second and higher powers of $(h_0 - h)$ can be neglected, which need not be true either for the differences Δ' or for the differences Δ.

If we then set the equations (14) into (5), then we obtain, taking into account (15),

$$\Phi(\varepsilon) - \varphi(\varepsilon) = \frac{1}{\sqrt{\pi}}\left\{ e^{-h^2\varepsilon^2}(1-2h^2\varepsilon^2)(h_0-h) - h\varepsilon^2 e^{-h^2\varepsilon^2}(3-2h^2\varepsilon^2)[p\,\Delta'^2] \right\}. \tag{16}$$

Since the differences Δ' are to be treated as accidental errors, we can set also

$$\frac{[p\,\Delta'^2]}{[p]} = [p\,\Delta'^2] = \mu^2\,,$$

where μ^2 symbolizes the mean square of error of the quantities $h_1, h_2 \ldots$. Then we have according to (2) and (13)

$$\mu^2 = \frac{1}{2\,k^2} \quad \text{or} \quad [p\,\Delta'^2] = \frac{1}{2\,k^2} \tag{17}$$

and (16) thus turns into

$$\Phi(\varepsilon) - \varphi(\varepsilon) = \frac{1}{\sqrt{\pi}} e^{-h^2\varepsilon^2}(1-2h^2\varepsilon^2)(h_0-h) - \frac{h\,\varepsilon^2}{2\sqrt{\pi}} e^{-h^2\varepsilon^2}(3-2h^2\varepsilon^2)\frac{1}{k^2}\,. \tag{18}$$

Now we shall show further how the two unknowns h_0 and k can be determined from a given series of real observational errors.

The probability that an error lies within the interval ε and $\varepsilon + \Delta\varepsilon$ is, according to the two laws of errors,

$$\Phi(\varepsilon)\,\Delta\varepsilon \quad \text{and} \quad \varphi(\varepsilon)\,\Delta\varepsilon$$

and the number of errors within this interval is

$$N\,\Phi(\varepsilon)\,\Delta\varepsilon \quad \text{and} \quad N\,\varphi(\varepsilon)\,\Delta\varepsilon\,,$$

if N is the total number of errors. Thus we obtain from (18):

$$N(\Phi(\varepsilon) - \varphi(\varepsilon))\Delta\varepsilon = \frac{N}{\sqrt{\pi}} e^{-h^2\varepsilon^2}(1-2h^2\varepsilon^2)\,\Delta\varepsilon\,(h_0-h) - \frac{N\,h\,\varepsilon^2}{2\sqrt{\pi}} e^{-h^2\varepsilon^2}(3-2h^2\varepsilon^2)\Delta\varepsilon\frac{1}{k^2}\,. \tag{19}$$

If we assume that the distribution of errors actually corresponds to the law $\Phi(\varepsilon)$, the expression at the left-hand side in (19) represents the deviation of the errors actually occurring in the interval from the Gauss law $\varphi(\varepsilon)$ and this expression is thus known for every interval. Hence we only need to set up equation (19) for two different intervals $\Delta\varepsilon$ and then we can calculate the unknowns h_0 and $\frac{1}{k}$ from these two equations. It is better, however, to use a rather large number of intervals $\Delta\varepsilon$ and then to ascertain the most favorable values of the unknowns by adjustment.

The thought of applying the Gauss law of errors to the different moduli of precision has its origin in Lehmann-Filhés and is represented in *Astr. Nachr.*, Vol. 117, No. 2792 (1887). In the above we have developed this theory in a substantially different way in connection with the previous discussions.

For the testing of the theory developed in the previous section we use the angular errors in the main triangles of the Saxon triangulation which we already discussed in section 133.* In all there are 197 errors of closure with the mean error ±0.607" for a triangle closure.

We divided the errors into groups with the limits 0.00-0.05", 0.05-0.10", 0.10-0.15", etc., and calculated for each group the theoretical number according to the Gauss law by following the way by which we have already proceeded in section 146. This yielded the following summary:

	Experience	Theory	Difference
0.00—0.05″	13	12.9	+ 0.1
0.05—0.15	26	25.4	+ 0.6
0.15—0.25	30	24.8	+ 5.2
0.25—0.35	28	22.8	+ 5.2
0.35—0.45	17	20.8	— 3.8
0.45—0.55	18	18.5	— 0.5
0.55—0.65	15	15.8	— 0.8
0.65—0.75	12	13.3	— 1.3
0.75—0.85	9	10.9	— 1.9
0.85—0.95	8	8.6	— 0.6
0.95—1.05	6	6.8	— 0.8
1.05—1.15	2	5.0	— 3.0
1.15—1.25	1	3.6	— 2.6
1.25—1.35	2	2.7	— 0.7
1.35—1.45	2	1.8	+ 0.2
1.45—1.55	3	1.2	+ 1.8
1.55—1.65	3	0.8	+ 2.2
1.65—1.75	1	0.5	+ 0.5
1.75—1.85	0	0.3	— 0.3
1.85—1.95	1	0.2	+ 0.8

(1)

For the setting up of equation (19) of the previous section we have from (26), section 138, p. 516:

$$h = \frac{1}{m\sqrt{2}} = 1.165$$

and further

$$N = 197 \qquad \Delta\varepsilon = 0.1'' \ .$$

If we bring these numerical values into (19), section 147, p. 550, and, at the same time, symbolize the difference at the left-hand side by d, we obtain:

$$d = 11.11 \cdot 0.25743^{\varepsilon^2} (1 - 2.714\,\varepsilon^2)\,(h_0 - h) - 6.47\,\varepsilon^2 \cdot 0.25743^{\varepsilon^2} (3 - 2.714\,\varepsilon^2) \frac{1}{k^2}\,. \tag{2}$$

This equation refers to the number of errors within an interval ε to $\varepsilon + \Delta\varepsilon$, whereas the above table (1) gives the number of errors in the two intervals ε to $\varepsilon + \Delta\varepsilon$ and $-\varepsilon$ to $-\varepsilon - \Delta\varepsilon$ together. Hence, we must double the coefficients in the above equation and we shall also replace the difference d by $2d$. Then we have

$$2d = 22.22 \cdot 0.25743^{\varepsilon^2} (1 - 2.714\,\varepsilon^2)\,x - 6.47\,\varepsilon^2 \cdot 0.25743^{\varepsilon^2} (3 - 2.714\,\varepsilon^2)\,y\,, \tag{3}$$

where we set $h_0 - h = x$ and $\frac{1}{k^2} = y$.

* Section 133 not included in this translation.

By calculating the coefficients of x and y now and adding the corrections v to the differences $2d$ from table (1), we obtain the error equations:

$$
\begin{aligned}
v_1 &= +21.3\,x - 0.4\,y - 0.6 & v_{11} &= -9.8\,x + 0.8\,y + 3.0\\
v_2 &= +18.7\,x - 1.4\,y - 5.2 & v_{12} &= -9.2\,x + 2.4\,y + 2.6\\
v_3 &= +14.8\,x - 2.8\,y - 5.2 & v_{13} &= -8.0\,x + 3.5\,y + 0.7\\
v_4 &= +10.1\,x - 4.3\,y + 3.8 & v_{14} &= -6.7\,x + 4.1\,y - 0.2\\
v_5 &= +5.1\,x - 5.3\,y + 0.5 & v_{15} &= -5.4\,x + 4.3\,y - 1.8\\
v_6 &= +0.4\,x - 5.7\,y + 0.8 & v_{16} &= -4.1\,x + 4.0\,y - 2.2\\
v_7 &= -3.8\,x - 5.4\,y + 1.3 & v_{17} &= -3.0\,x + 3.6\,y - 0.5\\
v_8 &= -6.8\,x - 4.4\,y + 1.9 & v_{18} &= -2.1\,x + 3.0\,y + 0.3\\
v_9 &= -8.8\,x - 2.8\,y + 0.6 & v_{19} &= -1.5\,x + 2.4\,y - 0.8\\
v_{10} &= -9.8\,x - 1.0\,y + 0.8 & &
\end{aligned}
$$

The normal equations are:

$$
\begin{aligned}
+1736.0\,x - 209.7\,y &= 213.48\\
-209.7\,x + 244.1\,y &= 28.42\,,
\end{aligned}
$$

from which we obtain the following values for the unknowns:

$$x = h_0 - h = +0.1530$$

$$y = \frac{1}{k^2} = +0.2478$$

or

$$h_0 = 1.318 \qquad k = 2.01\,.$$

With these values we calculate further the corrections v and obtain:

$$
\begin{array}{llll}
v_1 = +2.6 & v_6 = -0.6 & v_{11} = +1.7 & v_{16} = -1.8\\
v_2 = -2.7 & v_7 = -0.6 & v_{12} = +1.8 & v_{17} = -0.1\\
v_3 = -3.6 & v_8 = -0.2 & v_{13} = +0.3 & v_{18} = +0.7\\
v_4 = +4.3 & v_9 = -1.4 & v_{14} = -0.2 & v_{19} = -0.4\,.\\
v_5 = 0.0 & v_{10} = -1.0 & v_{15} = -1.6 &
\end{array} \tag{4}
$$

We find as the mean error

$$m = \pm\sqrt{\frac{61.94}{17}} = \pm 1.91\,,$$

with which we obtain

$$m_x = \pm 0.048 \qquad m_y = \pm 0.134\,.$$

In order to obtain from $m_y = m_{\frac{1}{k^2}}$ the value of m_k, we still need a small conversion. For we have according to the law of propagation of errors, section 5, p. 14:

$$m_{\frac{1}{k^2}} = \frac{2}{k^3}\,m_k\,, \qquad \text{hence} \qquad m_k = \frac{k^3}{2}\,m_{\frac{1}{k^2}} = \pm 0.54\,.$$

Hence the result of the adjustment is

$$
\begin{aligned}
h_0 &= 1.318 \pm 0.048\\
k &= 2.01 \pm 0.54\,.
\end{aligned}
$$

In Fig. 1 we represented the differences between experience and the Gauss law of errors, according to the last column of summary (1) of p. 551, as ordinates. From the points thus found, the corrections v of (4), p. 552, were set off

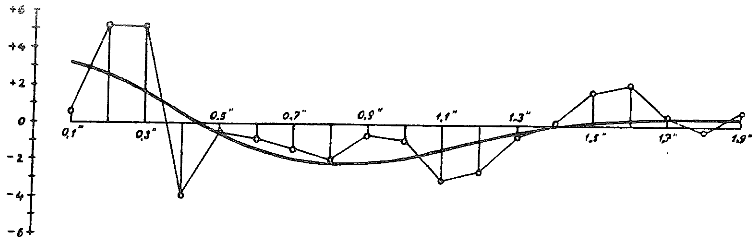


Fig. 1.

which yielded the curve of the law of errors $\Phi(\varepsilon)$ of (18), p. 550. We see that this curve is in very good agreement with the values given.

As a check we can use further the relation (10) of the previous section 147, p. 549, according to which the curve for the values

$$\varepsilon = 2.335\, m = 1.41'' \qquad \text{and for} \qquad \varepsilon = 0.742\, m = 0.45''$$

must go through the axis of abscissae. If we set in equation (3) $2\,d = 0$, we obtain easily the equation:

$$\varepsilon^4 - 2.16\,\varepsilon^2 + 0.39 = 0\,,$$

and thence we find

$$\varepsilon = 1.40'' \qquad \text{and} \qquad \varepsilon = 0.45''\,,$$

which agrees with theory.

Finally we shall investigate within what limits of precision the observations are to be assumed. With the help of k we can indicate the mean deviation m_h of the individual h's from their mean value h_0, according to equation (26), section 138, p. 516. We have

$$m_h = \frac{1}{k\sqrt{2}} = \pm 0.35\,,$$

and if we assume that the maximum deviation of the individual h's does not go beyond $3\,m_h$, we have the following rounded values:

$$\text{Maximum of } h = 2.3$$
$$\text{Minimum of } h = 0.3\,.$$

In this connection we refer, according to (26), section 138, p. 516, to the two mean errors

$$m_{min} = \pm 0.3''$$
$$m_{max} = \pm 2.3''$$

within which the mean errors of the individual measurements will lie.

Section 149. The Maximum Error

The question of the maximum error of an observation, i.e. that limiting value which the observational errors can reach without falling in the category of gross errors has been treated several times, as has also been thoroughly done in former editions of this volume. Since the practical significance of the maximum error is very small, we shall limit ourselves to summarizing the most essential in a few words.

The Gauss law of errors fails completely in regard to the maximum error. Since for the determination of the constants of this law (section 138, p. 516), only the limiting values $-\infty$ and $+\infty$ were introduced, it is not to be expected that for any finite error value the probability yields zero. For large errors, however, the probability becomes so small that the occurrence of such an error is to be expected only in the case of a very large number of observations. For comparison, we shall list a few numerical data, by assuming for the maximum error definite multiples of the mean error. The probability that an observational error is larger than twice the mean error becomes equal to 0.0455 according to p. [21] of the Appendix. Hence, an error would have to be expected beyond that limit in the case of 22 observations. The probability drops to 0.0027 for an observational error which exceeds three times the mean error; therefore, not until the case of 370 observations is reached will such an error be probable. If we extend this further to three times and four times the mean error, we obtain the following table:

Multiple of the Mean Error	Number of the Observations
2	22
3	370
4	15,800
5	1,740,000

We can summarize the result thus: that in the case of a not too large number of observations, three times the mean error may be assumed as the limiting error; and that five times the mean error may hold as the limiting error in any case.

The practical significance of the maximum error is inconsiderable. In land survey, when it is a question of setting up limits of error for the usability of measurements, it may be not so much the maximum error but rather the accuracy required for the purposes of the survey which has to be taken into account.

Another field in which the maximum error plays a part is formed by the many attempts of reaching a criterion for taking out deviating observations in those cases in which it may be doubtful whether gross errors are present. It would lead too far if we should discuss the numerous suggestions made for it, all the more since none of them has found a more general acceptance. In any case, we hardly go beyond an arbitrary elimination of deviating observations with these more or less complicated methods.

Further material on this subject is contained in E. Czuber, *Theorie der Beobachtungsfehler* [Theory of observational errors], Leipzig, 1891, sections 10 and 11.

The retention of all observations remains, in any case, the best method unless during the measurement itself, there exist reasons sufficient to justify an immediate rejection, before any recomputation.

Let the following error equations be laid down for the determination of a point:

$$\left.\begin{aligned} v_1 &= a_1 x + b_1 y - l_1 \\ v_2 &= a_2 x + b_2 y - l_2 \\ &\cdots\cdots\cdots \\ v_n &= a_n x + b_n y - l_n \end{aligned}\right\}, \qquad (1)$$

where we have to regard the unknowns x and y as corrections of given coordinates of approximation.

If the coordinates x and y are to be determined according to the method of least squares, we are to use for it the two normal equations

$$\left.\begin{aligned} [a\,a]\,x + [a\,b]\,y - [a\,l] &= 0 \\ [a\,b]\,x + [b\,b]\,y - [b\,l] &= 0 \end{aligned}\right\} \qquad (2)$$

which we can also write briefly in the form

$$[a\,v] = 0 \qquad [b\,v] = 0\,. \qquad (3)$$

Aside from the values x and y corresponding to the method of least squares, we introduce now other arbitrary values ξ and η which thus do not satisfy the normal equations (2); and let the corrections $v'_1, v'_2, \ldots, v'_n$, of the observations belong to them. Hence we have in general

$$v' = a\,\xi + b\,\eta - l \qquad (4)$$

and from (1) and (4) we obtain

$$v' = a\,(\xi - x) + b\,(\eta - y) + v\,. \qquad (5)$$

If we form the sum of the squares of the v', we have

$$\left.\begin{aligned} [v'\,v'] = [a\,a]\,(\xi - x)^2 + 2\,[a\,b]\,(\xi - x)\,(\eta - y) &+ 2\,[a\,v]\,(\xi - x) \\ + [b\,b]\,(\eta - y)^2 \qquad &+ 2\,[b\,v]\,(\eta - y) \\ &+ [v\,v] \end{aligned}\right\} \qquad (6)$$

or, according to (3),

$$[v'\,v'] = [v\,v] + [a\,a]\,(\xi - x)^2 + 2\,[a\,b]\,(\xi - x)\,(\eta - y) + [b\,b]\,(\eta - y)^2\,. \qquad (7)$$

According to (13), section 139, p. 521, the probability of the sum of squares $[v'\,v']$ is proportional to the quantity

$$e^{-h^2\,[v'\,v']}\,. \qquad (8)$$

From (26), section 138, p. 516, however, we have

$$h^2 = \frac{1}{2\,m^2}\,. \qquad (9)$$

Now we are to bear in mind that the sum of the squares $[v\,v]$ of the corrections calculated according to the method of least squares represents a constant numerical value, so that we can write for (8) also

$$e^{-h^2\,([v'\,v'] - [v\,v])} \qquad (10)$$

or, in short, the probability for the sum of the squares $[v' v']$ is proportional to the quantity

$$e^{-s^2} \tag{11}$$

where we set $s^2 = h^2 ([v' v'] - [v v])$.

For s^2 there follows from (7) and (9)

$$s^2 = \frac{[a a] (\xi - x)^2 + 2 [a b] (\xi - x) (\eta - y) + [b b] (\eta - y)^2}{2 m^2} . \tag{12}$$

Now this is the equation of an ellipse with the coordinates $\xi - x$ and $\eta - y$ whose zero point lies at the center point of the ellipse. Thus we have found that all points which lie on this ellipse have the same probability.

In order to bring the ellipse equation to its basic form, we carry out a rotation of the system of coordinates by an angle ψ and denote the new coordinates by x', y'. The equations of conversion are

$$\left.\begin{aligned} \xi - x &= x' \cos \psi - y' \sin \psi \\ \eta - y &= x' \sin \psi + y' \cos \psi \end{aligned}\right\} . \tag{13}$$

If we bring this into (12), we obtain an equation of the form

$$[a' a'] x'^2 + 2 [a' b'] x' y' + [b' b'] y'^2 - 2 s^2 m^2 = 0 \tag{14}$$

in which the coefficients are as follows:

$$\left.\begin{aligned} [a' a'] &= [a a] \cos^2 \psi + [b b] \sin^2 \psi + 2 [a b] \sin \psi \cos \psi \\ [b' b'] &= [a a] \sin^2 \psi + [b b] \cos^2 \psi - 2 [a b] \sin \psi \cos \psi \end{aligned}\right\} \tag{15}$$

$$[a' b'] = [a b] (\cos^2 \psi - \sin^2 \psi) - ([a a] - [b b]) \sin \psi \cos \psi . \tag{16}$$

If the term $[a' b'] x' y'$ is to vanish in the new system of coordinates, we must have, according to (16),

$$2 [a b] \cos 2 \psi - ([a a] - [b b]) \sin 2 \psi = 0$$

or

$$\tan 2 \psi = \frac{2 [a b]}{[a a] - [b b]} . \tag{17}$$

This is the same equation as found in equation (17), section 108, p. 441; hence, the error ellipse found on the base of the calculus of probabilities has the same position as the error ellipse treated in the previous section 108.

For the value of ψ defined by (17), the ellipse equation (14) turns into

$$\frac{[a' a']}{2 s^2 m^2} x'^2 + \frac{[b' b']}{2 s^2 m^2} y'^2 = 1 . \tag{18}$$

Hence the squares of the semiaxes A and B for the error ellipse are the following:

$$A^2 = \frac{2\,s^2}{[a'\,a']}\,m^2 \qquad \text{and} \qquad B^2 = \frac{2\,s^2}{[b'\,b']}\,m^2\,. \tag{19}$$

If we introduce now for s^2 a special value by setting

$$s^2 = \frac{1}{2}\,, \tag{20}$$

then the squares of the semiaxes will be

$$A^2 = \frac{m^2}{[a'\,a']} \qquad B^2 = \frac{m^2}{[b'\,b']}\,. \tag{21}$$

We thus obtain the same values which we have found in section 108, p. 443, for the mean error ellipse, as is easily shown. With this, the remaining formulae of section 108 for the error ellipse treated above, which is based on the fundamental principles of the theory of probabilities, hold also.

Probability of the location of a point within the error ellipse

Now we shall further investigate the question how great the probability is that a point lies within the error ellipse. In this connection, we look first at a narrow elliptic ring which is joined to the error ellipse (19), and whose width in the direction of the two axes equals dA and dB. We denote the surface of the ellipse by f and that of the ring by df.

If the probability for the position of a point on the error ellipse itself is proportional to e^{-s^2}, according to (11), p. 556, then we have to assume the probability for the occurrence of the point within the surface of the elliptic ring to be proportional to the quantity

$$e^{-s^2}\,df\,, \tag{22}$$

according to the previous equation (2) in section 137, p. 510.

We begin with the determination of the surface df of the ring. The surface of an ellipse with the semiaxes A and B is

$$f = A\,B\,\pi\,;$$

hence the surface of the elliptic ring is

$$df = (A\,dB + B\,dA)\,\pi\,. \tag{23}$$

From (19) we obtain

$$dA = \sqrt{\frac{2}{[a'a']}}\,m\,ds \qquad dB = \sqrt{\frac{2}{[b'b']}}\,m\,ds\,;$$

hence (23) turns into

$$df = \frac{4\,m^2\,\pi}{\sqrt{[a'a']\,[b'b']}}\,s\,ds$$

or, if we symbolize the constant part briefly by $2\,k$,

$$df = 2\,k\,s\,ds\,. \tag{24}$$

If we join this with (22), the probability for the position of the point within the elliptic ring will be proportional to the term

$$2k\, e^{-s^2}\, s\, ds$$

and the probability for the position of the point within the whole elliptic surface is proportional to the integral

$$2k \int_0^s e^{-s^2} s\, ds\,. \tag{25}$$

The general integral is

$$\int e^{-s^2} s\, ds = -\frac{1}{2} e^{-s^2}\,;$$

hence we will have

$$2k \int_0^s e^{-s^2} s\, ds = k\left(-e^{-s^2} + 1\right) = k\left(1 - e^{-s^2}\right).$$

If we denote the probability for the position of the point within the elliptic surface by W, then W is proportional to the term we just found. But if we assume $s = \infty$, then the point must lie within this ellipse which now comprises the whole plane, and for this, the probability $W = 1$. Consequently, the factor of proportionality, together with the constant k, must have the value 1, and we have for any arbitrary ellipse

$$W = 1 - e^{-s^2}\,. \tag{26}$$

For the numerical calculation we shall set

$$1 - W = e^{-s^2} \qquad \log(1 - W) = -0.4342945\, s^2\,. \tag{27}$$

The probability that the point lies within the mean error ellipse, for which according to (20), we have $s^2 = \frac{1}{2}$, will be

$$W = 0.39348$$

and for this, the mean error of the point is according to p. 443

$$M = \sqrt{A^2 + B^2}\,.$$

Aside from the mean error ellipse, there is still another error ellipse of interest for which the probability of the position of the point within and beyond its surface is equally great, hence $= \frac{1}{2}$. We have for this, according to (26)

$$W = 1 - e^{-s^2} = \frac{1}{2}; \qquad \text{hence} \qquad e^{-s^2} = \frac{1}{2}$$

and hence it follows

$$s^2 = 0.69315 \qquad s = 0.83255\,.$$

This ellipse is called the probable error ellipse.

Finally, we mention further the error ellipse which corresponds to the value $s = 1$, and for which we will have $W = 0.63212$. This error ellipse was used by Andrae in the working up of the Danish geodetic survey.

We add further to the literature about the error ellipse already mentioned at the end of section 109, p. 453:

Ch. M. Scholz, "Théorie des erreurs dans le plan et dans l'espace," *Ann. de l'école polyt. de Delft*, t. II, 1886.

APPENDIX

I. Square Numbers

a	a^2	d	a	a^2	d	a	a^2	d	a	a^2	d
0.00	0.0000	1	0.50	0.2500	101	1.00	1.0000	201	1.50	2.2500	301
0.01	0.0001	3	0.51	0.2601	103	1.01	1.0201	203	1.51	2.2801	303
0.02	0.0004	5	0.52	0.2704	105	1.02	1.0404	205	1.52	2.3104	305
0.03	0.0009	7	0.53	0.2809	107	1.03	1.0609	207	1.53	2.3409	307
0.04	0.0016	9	0.54	0.2916	109	1.04	1.0816	209	1.54	2.3716	309
0.05	0.0025	11	0.55	0.3025	111	1.05	1.1025	211	1.55	2.4025	311
0.06	0.0036	13	0.56	0.3136	113	1.06	1.1236	213	1.56	2.4336	313
0.07	0.0049	15	0.57	0.3249	115	1.07	1.1449	215	1.57	2.4649	315
0.08	0.0064	17	0.58	0.3364	117	1.08	1.1664	217	1.58	2.4964	317
0.09	0.0081	19	0.59	0.3481	119	1.09	1.1881	219	1.59	2.5281	319
0.10	0.0100	21	0.60	0.3600	121	1.10	1.2100	221	1.60	2.5600	321
0.11	0.0121	23	0.61	0.3721	123	1.11	1.2321	223	1.61	2.5921	323
0.12	0.0144	25	0.62	0.3844	125	1.12	1.2544	225	1.62	2.6244	325
0.13	0.0169	27	0.63	0.3969	127	1.13	1.2769	227	1.63	2.6569	327
0.14	0.0196	29	0.64	0.4096	129	1.14	1.2996	229	1.64	2.6896	329
0.15	0.0225	31	0.65	0.4225	131	1.15	1.3225	231	1.65	2.7225	331
0.16	0.0256	33	0.66	0.4356	133	1.16	1.3456	233	1.66	2.7556	333
0.17	0.0289	35	0.67	0.4489	135	1.17	1.3689	235	1.67	2.7889	335
0.18	0.0324	37	0.68	0.4624	137	1.18	1.3924	237	1.68	2.8224	337
0.19	0.0361	39	0.69	0.4761	139	1.19	1.4161	239	1.69	2.8561	339
0.20	0.0400	41	0.70	0.4900	141	1.20	1.4400	241	1.70	2.8900	341
0.21	0.0441	43	0.71	0.5041	143	1.21	1.4641	243	1.71	2.9241	343
0.22	0.0484	45	0.72	0.5184	145	1.22	1.4884	245	1.72	2.9584	345
0.23	0.0529	47	0.73	0.5329	147	1.23	1.5129	247	1.73	2.9929	347
0.24	0.0576	49	0.74	0.5476	149	1.24	1.5376	249	1.74	3.0276	349
0.25	0.0625	51	0.75	0.5625	151	1.25	1.5625	251	1.75	3.0625	351
0.26	0.0676	53	0.76	0.5776	153	1.26	1.5876	253	1.76	3.0976	353
0.27	0.0729	55	0.77	0.5929	155	1.27	1.6129	255	1.77	3.1329	355
0.28	0.0784	57	0.78	0.6084	157	1.28	1.6384	257	1.78	3.1684	357
0.29	0.0841	59	0.79	0.6241	159	1.29	1.6641	259	1.79	3.2041	359
0.30	0.0900	61	0.80	0.6400	161	1.30	1.6900	261	1.80	3.2400	361
0.31	0.0961	63	0.81	0.6561	163	1.31	1.7161	263	1.81	3.2761	363
0.32	0.1024	65	0.82	0.6724	165	1.32	1.7424	265	1.82	3.3124	365
0.33	0.1089	67	0.83	0.6889	167	1.33	1.7689	267	1.83	3.3489	367
0.34	0.1156	69	0.84	0.7056	169	1.34	1.7956	269	1.84	3.3856	369
0.35	0.1225	71	0.85	0.7225	171	1.35	1.8225	271	1.85	3.4225	371
0.36	0.1296	73	0.86	0.7396	173	1.36	1.8496	273	1.86	3.4596	373
0.37	0.1369	75	0.87	0.7569	175	1.37	1.8769	275	1.87	3.4969	375
0.38	0.1444	77	0.88	0.7744	177	1.38	1.9044	277	1.88	3.5344	377
0.39	0.1521	79	0.89	0.7921	179	1.39	1.9321	279	1.89	3.5721	379
0.40	0.1600	81	0.90	0.8100	181	1.40	1.9600	281	1.90	3.6100	381
0.41	0.1681	83	0.91	0.8281	183	1.41	1.9881	283	1.91	3.6481	383
0.42	0.1764	85	0.92	0.8464	185	1.42	2.0164	285	1.92	3.6864	385
0.43	0.1849	87	0.93	0.8649	187	1.43	2.0449	287	1.93	3.7249	387
0.44	0.1936	89	0.94	0.8836	189	1.44	2.0736	289	1.94	3.7636	389
0.45	0.2025	91	0.95	0.9025	191	1.45	2.1025	291	1.95	3.8025	391
0.46	0.2116	93	0.96	0.9216	193	1.46	2.1316	293	1.96	3.8416	393
0.47	0.2209	95	0.97	0.9409	195	1.47	2.1609	295	1.97	3.8809	395
0.48	0.2304	97	0.98	0.9604	197	1.48	2.1904	297	1.98	3.9204	397
0.49	0.2401	99	0.99	0.9801	199	1.49	2.2201	299	1.99	3.9601	399
0.50	0.2500		1.00	1.0000		1.50	2.2500		2.00	4.0000	

I. Square Numbers

a	a^2	d	a	a^2	d	a	a^2	d	a	a^2	d
2.00	4.0000	401	2.50	6.2500	501	3.00	9.0000	601	3.50	12.2500	701
2.01	4.0401	403	2.51	6.3001	503	3.01	9.0601	603	3.51	12.3201	703
2.02	4.0804	405	2.52	6.3504	505	3.02	9.1204	605	3.52	12.3904	705
2.03	4.1209	407	2.53	6.4009	507	3.03	9.1809	607	3.53	12.4609	707
2.04	4.1616	409	2.54	6.4516	509	3.04	9.2416	609	3.54	12.5316	709
2.05	4.2025	411	2.55	6.5025	511	3.05	9.3025	611	3.55	12.6025	711
2.06	4.2436	413	2.56	6.5536	513	3.06	9.3636	613	3.56	12.6736	713
2.07	4.2849	415	2.57	6.6049	515	3.07	9.4249	615	3.57	12.7449	715
2.08	4.3264	417	2.58	6.6564	517	3.08	9.4864	617	3.58	12.8164	717
2.09	4.3681	419	2.59	6.7081	519	3.09	9.5481	619	3.59	12.8881	719
2.10	4.4100	421	2.60	6.7600	521	3.10	9.6100	621	3.60	12.9600	721
2.11	4.4521	423	2.61	6.8121	523	3.11	9.6721	623	3.61	13.0321	723
2.12	4.4944	425	2.62	6.8644	525	3.12	9.7344	625	3.62	13.1044	725
2.13	4.5369	427	2.63	6.9169	527	3.13	9.7969	627	3.63	13.1769	727
2.14	4.5796	429	2.64	6.9696	529	3.14	9.8596	629	3.64	13.2496	729
2.15	4.6225	431	2.65	7.0225	531	3.15	9.9225	631	3.65	13.3225	731
2.16	4.6656	433	2.66	7.0756	533	3.16	9.9856	633	3.66	13.3956	733
2.17	4.7089	435	2.67	7.1289	535	3.17	10.0489	635	3.67	13.4689	735
2.18	4.7524	437	2.68	7.1824	537	3.18	10.1124	637	3.68	13.5424	737
2.19	4.7961	439	2.69	7.2361	539	3.19	10.1761	639	3.69	13.6161	739
2.20	4.8400	441	2.70	7.2900	541	3.20	10.2400	641	3.70	13.6900	741
2.21	4.8841	443	2.71	7.3441	543	3.21	10.3041	643	3.71	13.7641	743
2.22	4.9284	445	2.72	7.3984	545	3.22	10.3684	645	3.72	13.8384	745
2.23	4.9729	447	2.73	7.4529	547	3.23	10.4329	647	3.73	13.9129	747
2.24	5.0176	449	2.74	7.5076	549	3.24	10.4976	649	3.74	13.9876	749
2.25	5.0625	451	2.75	7.5625	551	3.25	10.5625	651	3.75	14.0625	751
2.26	5.1076	453	2.76	7.6176	553	3.26	10.6276	653	3.76	14.1376	753
2.27	5.1529	455	2.77	7.6729	555	3.27	10.6929	655	3.77	14.2129	755
2.28	5.1984	457	2.78	7.7284	557	3.28	10.7584	657	3.78	14.2884	757
2.29	5.2441	459	2.79	7.7841	559	3.29	10.8241	659	3.79	14.3641	759
2.30	5.2900	461	2.80	7.8400	561	3.30	10.8900	661	3.80	14.4400	761
2.31	5.3361	463	2.81	7.8961	563	3.31	10.9561	663	3.81	14.5161	763
2.32	5.3824	465	2.82	7.9524	565	3.32	11.0224	665	3.82	14.5924	765
2.33	5.4289	467	2.83	8.0089	567	3.33	11.0889	667	3.83	14.6689	767
2.34	5.4756	469	2.84	8.0656	569	3.34	11.1556	669	3.84	14.7456	769
2.35	5.5225	471	2.85	8.1225	571	3.35	11.2225	671	3.85	14.8225	771
2.36	5.5696	473	2.86	8.1796	573	3.36	11.2896	673	3.86	14.8996	773
2.37	5.6169	475	2.87	8.2369	575	3.37	11.3569	675	3.87	14.9769	775
2.38	5.6644	477	2.88	8.2944	577	3.38	11.4244	677	3.88	15.0544	777
2.39	5.7121	479	2.89	8.3521	579	3.39	11.4921	679	3.89	15.1321	779
2.40	5.7600	481	2.90	8.4100	581	3.40	11.5600	681	3.90	15.2100	781
2.41	5.8081	483	2.91	8.4681	583	3.41	11.6281	683	3.91	15.2881	783
2.42	5.8564	485	2.92	8.5264	585	3.42	11.6964	685	3.92	15.3664	785
2.43	5.9049	487	2.93	8.5849	587	3.43	11.7649	687	3.93	15.4449	787
2.44	5.9536	489	2.94	8.6436	589	3.44	11.8336	689	3.94	15.5236	789
2.45	6.0025	491	2.95	8.7025	591	3.45	11.9025	691	3.95	15.6025	791
2.46	6.0516	493	2.96	8.7616	593	3.46	11.9716	693	3.96	15.6816	793
2.47	6.1009	495	2.97	8.8209	595	3.47	12.0409	695	3.97	15.7609	795
2.48	6.1504	497	2.98	8.8804	597	3.48	12.1104	697	3.98	15.8404	797
2.49	6.2001	499	2.99	8.9401	599	3.49	12.1801	699	3.99	15.9201	799
2.50	6.2500		3.00	9.0000		3.50	12.2500		4.00	16.0000	

I. Square Numbers

a	a^2	d	a	a^2	d	a	a^2	d	a	a^2	d
4.00	16.0000	801	4.50	20.2500	901	5.00	25.0000	1001	5.50	30.2500	1101
4.01	16.0801	803	4.51	20.3401	903	5.01	25.1001	1003	5.51	30.3601	1103
4.02	16.1604	805	4.52	20.4304	905	5.02	25.2004	1005	5.52	30.4704	1105
4.03	16.2409	807	4.53	20.5209	907	5.03	25.3009	1007	5.53	30.5809	1107
4.04	16.3216	809	4.54	20.6116	909	5.04	25.4016	1009	5.54	30.6916	1109
4.05	16.4025	811	4.55	20.7025	911	5.05	25.5025	1011	5.55	30.8025	1111
4.06	16.4836	813	4.56	20.7936	913	5.06	25.6036	1013	5.56	30.9136	1113
4.07	16.5649	815	4.57	20.8849	915	5.07	25.7049	1015	5.57	31.0249	1115
4.08	16.6464	817	4.58	20.9764	917	5.08	25.8064	1017	5.58	31.1364	1117
4.09	16.7281	819	4.59	21.0681	919	5.09	25.9081	1019	5.59	31.2481	1119
4.10	16.8100	821	4.60	21.1600	921	5.10	26.0100	1021	5.60	31.3600	1121
4.11	16.8921	823	4.61	21.2521	923	5.11	26.1121	1023	5.61	31.4721	1123
4.12	16.9744	825	4.62	21.3444	925	5.12	26.2144	1025	5.62	31.5844	1125
4.13	17.0569	827	4.63	21.4369	927	5.13	26.3169	1027	5.63	31.6969	1127
4.14	17.1396	829	4.64	21.5296	929	5.14	26.4196	1029	5.64	31.8096	1129
4.15	17.2225	831	4.65	21.6225	931	5.15	26.5225	1031	5.65	31.9225	1131
4.16	17.3056	833	4.66	21.7156	933	5.16	26.6256	1033	5.66	32.0356	1133
4.17	17.3889	835	4.67	21.8089	935	5.17	26.7289	1035	5.67	32.1489	1135
4.18	17.4724	837	4.68	21.9024	937	5.18	26.8324	1037	5.68	32.2624	1137
4.19	17.5561	839	4.69	21.9961	939	5.19	26.9361	1039	5.69	32.3761	1139
4.20	17.6400	841	4.70	22.0900	941	5.20	27.0400	1041	5.70	32.4900	1141
4.21	17.7241	843	4.71	22.1841	943	5.21	27.1441	1043	5.71	32.6041	1143
4.22	17.8084	845	4.72	22.2784	945	5.22	27.2484	1045	5.72	32.7184	1145
4.23	17.8929	847	4.73	22.3729	947	5.23	27.3529	1047	5.73	32.8329	1147
4.24	17.9776	849	4.74	22.4676	949	5.24	27.4576	1049	5.74	32.9476	1149
4.25	18.0625	851	4.75	22.5625	951	5.25	27.5625	1051	5.75	33.0625	1151
4.26	18.1476	853	4.76	22.6576	953	5.26	27.6676	1053	5.76	33.1776	1153
4.27	18.2329	855	4.77	22.7529	955	5.27	27.7729	1055	5.77	33.2929	1155
4.28	18.3184	857	4.78	22.8484	957	5.28	27.8784	1057	5.78	33.4084	1157
4.29	18.4041	859	4.79	22.9441	959	5.29	27.9841	1059	5.79	33.5241	1159
4.30	18.4900	861	4.80	23.0400	961	5.30	28.0900	1061	5.80	33.6400	1161
4.31	18.5761	863	4.81	23.1361	963	5.31	28.1961	1063	5.81	33.7561	1163
4.32	18.6624	865	4.82	23.2324	965	5.32	28.3024	1065	5.82	33.8724	1165
4.33	18.7489	867	4.83	23.3289	967	5.33	28.4089	1067	5.83	33.9889	1167
4.34	18.8356	869	4.84	23.4256	969	5.34	28.5156	1069	5.84	34.1056	1169
4.35	18.9225	871	4.85	23.5225	971	5.35	28.6225	1071	5.85	34.2225	1171
4.36	19.0096	873	4.86	23.6196	973	5.36	28.7296	1073	5.86	34.3396	1173
4.37	19.0969	875	4.87	23.7169	975	5.37	28.8369	1075	5.87	34.4569	1175
4.38	19.1844	877	4.88	23.8144	977	5.38	28.9444	1077	5.88	34.5744	1177
4.39	19.2721	879	4.89	23.9121	979	5.39	29.0521	1079	5.89	34.6921	1179
4.40	19.3600	881	4.90	24.0100	981	5.40	29.1600	1081	5.90	34.8100	1181
4.41	19.4481	883	4.91	24.1081	983	5.41	29.2681	1083	5.91	34.9281	1183
4.42	19.5364	885	4.92	24.2064	985	5.42	29.3764	1085	5.92	35.0464	1185
4.43	19.6249	887	4.93	24.3049	987	5.43	29.4849	1087	5.93	35.1649	1187
4.44	19.7136	889	4.94	24.4036	989	5.44	29.5936	1089	5.94	35.2836	1189
4.45	19.8025	891	4.95	24.5025	991	5.45	29.7025	1091	5.95	35.4025	1191
4.46	19.8916	893	4.96	24.6016	993	5.46	29.8116	1093	5.96	35.5216	1193
4.47	19.9809	895	4.97	24.7009	995	5.47	29.9209	1095	5.97	35.6409	1195
4.48	20.0704	897	4.98	24.8004	997	5.48	30.0304	1097	5.98	35.7604	1197
4.49	20.1601	899	4.99	24.9001	999	5.49	30.1401	1099	5.99	35.8801	1199
4.50	20.2500		5.00	25.0000		5.50	30.2500		6.00	36.0000	

I. Square Numbers

a	a^2	d	a	a^2	d	a	a^2	d	a	a^2	d
6.00	36.0000	1201	6.50	42.2500	1301	7.00	49.0000	1401	7.50	56.2500	1501
6.01	36.1201	1203	6.51	42.3801	1303	7.01	49.1401	1403	7.51	56.4001	1503
6.02	36.2404	1205	6.52	42.5104	1305	7.02	49.2804	1405	7.52	56.5504	1505
6.03	36.3609	1207	6.53	42.6409	1307	7.03	49.4209	1407	7.53	56.7009	1507
6.04	36.4816	1209	6.54	42.7716	1309	7.04	49.5616	1409	7.54	56.8516	1509
6.05	36.6025	1211	6.55	42.9025	1311	7.05	49.7025	1411	7.55	57.0025	1511
6.06	36.7236	1213	6.56	43.0336	1313	7.06	49.8436	1413	7.56	57.1536	1513
6.07	36.8449	1215	6.57	43.1649	1315	7.07	49.9849	1415	7.57	57.3049	1515
6.08	36.9664	1217	6.58	43.2964	1317	7.08	50.1264	1417	7.58	57.4564	1517
6.09	37.0881	1219	6.59	43.4281	1319	7.09	50.2681	1419	7.59	57.6081	1519
6.10	37.2100	1221	6.60	43.5600	1321	7.10	50.4100	1421	7.60	57.7600	1521
6.11	37.3321	1223	6.61	43.6921	1323	7.11	50.5521	1423	7.61	57.9121	1523
6.12	37.4544	1225	6.62	43.8244	1325	7.12	50.6944	1425	7.62	58.0644	1525
6.13	37.5769	1227	6.63	43.9569	1327	7.13	50.8369	1427	7.63	58.2169	1527
6.14	37.6996	1229	6.64	44.0896	1329	7.14	50.9796	1429	7.64	58.3696	1529
6.15	37.8225	1231	6.65	44.2225	1331	7.15	51.1225	1431	7.65	58.5225	1531
6.16	37.9456	1233	6.66	44.3556	1333	7.16	51.2656	1433	7.66	58.6756	1533
6.17	38.0689	1235	6.67	44.4889	1335	7.17	51.4089	1435	7.67	58.8289	1535
6.18	38.1924	1237	6.68	44.6224	1337	7.18	51.5524	1437	7.68	58.9824	1537
6.19	38.3161	1239	6.69	44.7561	1339	7.19	51.6961	1439	7.69	59.1361	1539
6.20	38.4400	1241	6.70	44.8900	1341	7.20	51.8400	1441	7.70	59.2900	1541
6.21	38.5641	1243	6.71	45.0241	1343	7.21	51.9841	1443	7.71	59.4441	1543
6.22	38.6884	1245	6.72	45.1584	1345	7.22	52.1284	1445	7.72	59.5984	1545
6.23	38.8129	1247	6.73	45.2929	1347	7.23	52.2729	1447	7.73	59.7529	1547
6.24	38.9376	1249	6.74	45.4276	1349	7.24	52.4176	1449	7.74	59.9076	1549
6.25	39.0625	1251	6.75	45.5625	1351	7.25	52.5625	1451	7.75	60.0625	1551
6.26	39.1876	1253	6.76	45.6976	1353	7.26	52.7076	1453	7.76	60.2176	1553
6.27	39.3129	1255	6.77	45.8329	1355	7.27	52.8529	1455	7.77	60.3729	1555
6.28	39.4384	1257	6.78	45.9684	1357	7.28	52.9984	1457	7.78	60.5284	1557
6.29	39.5641	1259	6.79	46.1041	1359	7.29	53.1441	1459	7.79	60.6841	1559
6.30	39.6900	1261	6.80	46.2400	1361	7.30	53.2900	1461	7.80	60.8400	1561
6.31	39.8161	1263	6.81	46.3761	1363	7.31	53.4361	1463	7.81	60.9961	1563
6.32	39.9424	1265	6.82	46.5124	1365	7.32	53.5824	1465	7.82	61.1524	1565
6.33	40.0689	1267	6.83	46.6489	1367	7.33	53.7289	1467	7.83	61.3089	1567
6.34	40.1956	1269	6.84	46.7856	1369	7.34	53.8756	1469	7.84	61.4656	1569
6.35	40.3225	1271	6.85	46.9225	1371	7.35	54.0225	1471	7.85	61.6225	1571
6.36	40.4496	1273	6.86	47.0596	1373	7.36	54.1696	1473	7.86	61.7796	1573
6.37	40.5769	1275	6.87	47.1969	1375	7.37	54.3169	1475	7.87	61.9369	1575
6.38	40.7044	1277	6.88	47.3344	1377	7.38	54.4644	1477	7.88	62.0944	1577
6.39	40.8321	1279	6.89	47.4721	1379	7.39	54.6121	1479	7.89	62.2521	1579
6.40	40.9600	1281	6.90	47.6100	1381	7.40	54.7600	1481	7.90	62.4100	1581
6.41	41.0881	1283	6.91	47.7481	1383	7.41	54.9081	1483	7.91	62.5681	1583
6.42	41.2164	1285	6.92	47.8864	1385	7.42	55.0564	1485	7.92	62.7264	1585
6.43	41.3449	1287	6.93	48.0249	1387	7.43	55.2049	1487	7.93	62.8849	1587
6.44	41.4736	1289	6.94	48.1636	1389	7.44	55.3536	1489	7.94	63.0436	1589
6.45	41.6025	1291	6.95	48.3025	1391	7.45	55.5025	1491	7.95	63.2025	1591
6.46	41.7316	1293	6.96	48.4416	1393	7.46	55.6516	1493	7.96	63.3616	1593
6.47	41.8609	1295	6.97	48.5809	1395	7.47	55.8009	1495	7.97	63.5209	1595
6.48	41.9904	1297	6.98	48.7204	1397	7.48	55.9504	1497	7.98	63.6804	1597
6.49	42.1201	1299	6.99	48.8601	1399	7.49	56.1001	1499	7.99	63.8401	1599
6.50	42.2500		7.00	49.0000		7.50	56.2500		8.00	64.0000	

I. Square Numbers

a	a²	d	a	a²	d	a	a²	d	a	a²	d
8.00	64.0000	1601	8.50	72.2500	1701	9.00	81.0000	1801	9.50	90.2500	1901
8.01	64.1601	1603	8.51	72.4201	1703	9.01	81.1801	1803	9.51	90.4401	1903
8.02	64.3204	1605	8.52	72.5904	1705	9.02	81.3604	1805	9.52	90.6304	1905
8.03	64.4809	1607	8.53	72.7609	1707	9.03	81.5409	1807	9.53	90.8209	1907
8.04	64.6416	1609	8.54	72.9316	1709	9.04	81.7216	1809	9.54	91.0116	1909
8.05	64.8025	1611	8.55	73.1025	1711	9.05	81.9025	1811	9.55	91.2025	1911
8.06	64.9636	1613	8.56	73.2736	1713	9.06	82.0836	1813	9.56	91.3936	1913
8.07	65.1249	1615	8.57	73.4449	1715	9.07	82.2649	1815	9.57	91.5849	1915
8.08	65.2864	1617	8.58	73.6164	1717	9.08	82.4464	1817	9.58	91.7764	1917
8.09	65.4481	1619	8.59	73.7881	1719	9.09	82.6281	1819	9.59	91.9681	1919
8.10	65.6100	1621	8.60	73.9600	1721	9.10	82.8100	1821	9.60	92.1600	1921
8.11	65.7721	1623	8.61	74.1321	1723	9.11	82.9921	1823	9.61	92.3521	1923
8.12	65.9344	1625	8.62	74.3044	1725	9.12	83.1744	1825	9.62	92.5444	1925
8.13	66.0969	1627	8.63	74.4769	1727	9.13	83.3569	1827	9.63	92.7369	1927
8.14	66.2596	1629	8.64	74.6496	1729	9.14	83.5396	1829	9.64	92.9296	1929
8.15	66.4225	1631	8.65	74.8225	1731	9.15	83.7225	1831	9.65	93.1225	1931
8.16	66.5856	1633	8.66	74.9956	1733	9.16	83.9056	1833	9.66	93.3156	1933
8.17	66.7489	1635	8.67	75.1689	1735	9.17	84.0889	1835	9.67	93.5089	1935
8.18	66.9124	1637	8.68	75.3424	1737	9.18	84.2724	1837	9.68	93.7024	1937
8.19	67.0761	1639	8.69	75.5161	1739	9.19	84.4561	1839	9.69	93.8961	1939
8.20	67.2400	1641	8.70	75.6900	1741	9.20	84.6400	1841	9.70	94.0900	1941
8.21	67.4041	1643	8.71	75.8641	1743	9.21	84.8241	1843	9.71	94.2841	1943
8.22	67.5684	1645	8.72	76.0384	1745	9.22	85.0084	1845	9.72	94.4784	1945
8.23	67.7329	1647	8.73	76.2129	1747	9.23	85.1929	1847	9.73	94.6729	1947
8.24	67.8976	1649	8.74	76.3876	1749	9.24	85.3776	1849	9.74	94.8676	1949
8.25	68.0625	1651	8.75	76.5625	1751	9.25	85.5625	1851	9.75	95.0625	1951
8.26	68.2276	1653	8.76	76.7376	1753	9.26	85.7476	1853	9.76	95.2576	1953
8.27	68.3929	1655	8.77	76.9129	1755	9.27	85.9329	1855	9.77	95.4529	1955
8.28	68.5584	1657	8.78	77.0884	1757	9.28	86.1184	1857	9.78	95.6484	1957
8.29	68.7241	1659	8.79	77.2641	1759	9.29	86.3041	1859	9.79	95.8441	1959
8.30	68.8900	1661	8.80	77.4400	1761	9.30	86.4900	1861	9.80	96.0400	1961
8.31	69.0561	1663	8.81	77.6161	1763	9.31	86.6761	1863	9.81	96.2361	1963
8.32	69.2224	1665	8.82	77.7924	1765	9.32	86.8624	1865	9.82	96.4324	1965
8.33	69.3889	1667	8.83	77.9689	1767	9.33	87.0489	1867	9.83	96.6289	1967
8.34	69.5556	1669	8.84	78.1456	1769	9.34	87.2356	1869	9.84	96.8256	1969
8.35	69.7225	1671	8.85	78.3225	1771	9.35	87.4225	1871	9.85	97.0225	1971
8.36	69.8896	1673	8.86	78.4996	1773	9.36	87.6096	1873	9.86	97.2196	1973
8.37	70.0569	1675	8.87	78.6769	1775	9.37	87.7969	1875	9.87	97.4169	1975
8.38	70.2244	1677	8.88	78.8544	1777	9.38	87.9844	1877	9.88	97.6144	1977
8.39	70.3921	1679	8.89	79.0321	1779	9.39	88.1721	1879	9.89	97.8121	1979
8.40	70.5600	1681	8.90	79.2100	1781	9.40	88.3600	1881	9.90	98.0100	1981
8.41	70.7281	1683	8.91	79.3881	1783	9.41	88.5481	1883	9.91	98.2081	1983
8.42	70.8964	1685	8.92	79.5664	1785	9.42	88.7364	1885	9.92	98.4064	1985
8.43	71.0649	1687	8.93	79.7449	1787	9.43	88.9249	1887	9.93	98.6049	1987
8.44	71.2336	1689	8.94	79.9236	1789	9.44	89.1136	1889	9.94	98.8036	1989
8.45	71.4025	1691	8.95	80.1025	1791	9.45	89.3025	1891	9.95	99.0025	1991
8.46	71.5716	1693	8.96	80.2816	1793	9.46	89.4916	1893	9.96	99.2016	1993
8.47	71.7409	1695	8.97	80.4609	1795	9.47	89.6809	1895	9.97	99.4009	1995
8.48	71.9104	1697	8.98	80.6404	1797	9.48	89.8704	1897	9.98	99.6004	1997
8.49	72.0801	1699	8.99	80.8201	1799	9.49	90.0601	1899	9.99	99.8001	1999
8.50	72.2500		9.00	81.0000		9.50	90.2500		10.00	100.0000	

II. Reciprocals of Squares

Weight $p = \frac{1}{m^2}$

m	*p*	*m*	*p*	*m*	*p*	*m*	*p*	*m*	*p*
0.0	∞	2.0	0.250	4.0	0.062	6.0	0.028	8.0	0.016
0.1	100.000	2.1	0.227	4.1	0.059	6.1	0.027	8.1	0.015
0.2	25.000	2.2	0.207	4.2	0.057	6.2	0.026	8.2	0.015
0.3	11.111	2.3	0.189	4.3	0.054	6.3	0.025	8.3	0.015
0.4	6.250	2.4	0.174	4.4	0.052	6.4	0.024	8.4	0.014
0.5	4.000	2.5	0.160	4.5	0.049	6.5	0.024	8.5	0.014
0.6	2.778	2.6	0.148	4.6	0.047	6.6	0.023	8.6	0.014
0.7	2.041	2.7	0.137	4.7	0.045	6.7	0.022	8.7	0.013
0.8	1.562	2.8	0.128	4.8	0.043	6.8	0.022	8.8	0.013
0.9	1.235	2.9	0.119	4.9	0.042	6.9	0.021	8.9	0.013
1.0	1.000	3.0	0.111	5.0	0.040	7.0	0.020	9.0	0.012
1.1	0.826	3.1	0.104	5.1	0.038	7.1	0.020	9.1	0.012
1.2	0.694	3.2	0.098	5.2	0.037	7.2	0.019	9.2	0.012
1.3	0.592	3.3	0.092	5.3	0.036	7.3	0.019	9.3	0.012
1.4	0.510	3.4	0.087	5.4	0.034	7.4	0.018	9.4	0.011
1.5	0.444	3.5	0.082	5.5	0.033	7.5	0.018	9.5	0.011
1.6	0.391	3.6	0.077	5.6	0.032	7.6	0.017	9.6	0.011
1.7	0.346	3.7	0.073	5.7	0.031	7.7	0.017	9.7	0.011
1.8	0.309	3.8	0.069	5.8	0.030	7.8	0.016	9.8	0.010
1.9	0.277	3.9	0.066	5.9	0.029	7.9	0.016	9.9	0.010

m	*p*	*m*	*p*	*m*	*p*	*m*	*p*	*m*	*p*
0.10	100.00	0.30	11.11	0.50	4.00	0.70	2.04	0.90	1.23
0.11	82.64	0.31	10.41	0.51	3.84	0.71	1.98	0.91	1.21
0.12	69.44	0.32	9.77	0.52	3.70	0.72	1.93	0.92	1.18
0.13	59.17	0.33	9.18	0.53	3.56	0.73	1.88	0.93	1.16
0.14	51.02	0.34	8.65	0.54	3.43	0.74	1.83	0.94	1.13
0.15	44.44	0.35	8.16	0.55	3.31	0.75	1.78	0.95	1.11
0.16	39.06	0.36	7.72	0.56	3.19	0.76	1.73	0.96	1.09
0.17	34.60	0.37	7.30	0.57	3.08	0.77	1.69	0.97	1.06
0.18	30.86	0.38	6.93	0.58	2.97	0.78	1.64	0.98	1.04
0.19	27.70	0.39	6.57	0.59	2.87	0.79	1.60	0.99	1.02
0.20	25.00	0.40	6.25	0.60	2.78	0.80	1.56	1.00	1.00
0.21	22.68	0.41	5.95	0.61	2.69	0.81	1.52	1.01	0.98
0.22	20.66	0.42	5.67	0.62	2.60	0.82	1.49	1.02	0.96
0.23	18.90	0.43	5.41	0.63	2.52	0.83	1.45	1.03	0.94
0.24	17.36	0.44	5.17	0.64	2.44	0.84	1.42	1.04	0.92
0.25	16.00	0.45	4.94	0.65	2.37	0.85	1.38	1.05	0.91
0.26	14.79	0.46	4.73	0.66	2.30	0.86	1.35	1.06	0.89
0.27	13.72	0.47	4.53	0.67	2.23	0.87	1.32	1.07	0.87
0.28	12.76	0.48	4.34	0.68	2.16	0.88	1.29	1.08	0.86
0.29	11.89	0.49	4.16	0.69	2.10	0.89	1.26	1.09	0.84

III. Direction Coefficients

φ	𝔵	Δ	𝔶	Δ
0°	– 0.00	36	+20.63	01
1	0.36	36	20.62	01
2	0.72	36	20.61	01
3	1.08	36	20.60	02
4	1.44	36	20.58	03
5°	– 1.80	36	+20.55	04
6	2.16	35	20.51	04
7	2.51	36	20.47	04
8	2.87	36	20.43	06
9	3.23	35	20.37	06
10°	– 3.58	36	+20.31	06
11	3.94	35	20.25	07
12	4.29	35	20.18	08
13	4.64	35	20.10	09
14	4.99	35	20.01	09
15°	– 5.34	35	+19.92	09
16	5.69	34	19.83	10
17	6.03	34	19.73	11
18	6.37	35	19.62	12
19	6.72	33	19.50	12
20°	– 7.05	34	+19.38	12
21	7.39	34	19.26	14
22	7.73	33	19.12	13
23	8.06	33	18.99	15
24	8.39	33	18.84	15
25°	– 8.72	32	+18.69	15
26	9.04	32	18.54	16
27	9.36	32	18.38	17
28	9.68	32	18.21	17
29	10.00	31	18.04	18
30°	–10.31	31	+17.86	18
31	10.62	31	17.68	19
32	10.93	30	17.49	19
33	11.23	30	17.30	20
34	11.53	30	17.10	20
35°	–11.83	29	+16.90	21
36	12.12	29	16.69	22
37	12.41	29	16.47	22
38	12.70	28	16.25	22
39	12.98	28	16.03	23
40°	–13.26	27	+15.80	23
41	13.53	27	15.57	24
42	13.80	27	15.33	24
43	14.07	26	15.09	25
44	14.33	26	14.84	25
45	14.59		14.59	

φ	𝔵	Δ	𝔶	Δ
45°	–14.59	25	+14.59	26
46	14.84	25	14.33	26
47	15.09	24	14.07	27
48	15.33	24	13.80	27
49	15.57	23	13.53	27
50°	–15.80	23	+13.26	28
51	16.03	22	12.98	28
52	16.25	22	12.70	29
53	16.47	22	12.41	29
54	16.69	21	12.12	29
55°	–16.90	20	+11.83	30
56	17.10	20	11.53	30
57	17.30	19	11.23	30
58	17.49	19	10.93	31
59	17.68	18	10.62	31
60°	–17.86	18	+10.31	31
61	18.04	17	10.00	32
62	18.21	17	9.68	32
63	18.38	16	9.36	32
64	18.54	15	9.04	32
65°	–18.69	15	+ 8.72	33
66	18.84	15	8.39	33
67	18.99	13	8.06	33
68	19.12	14	7.73	34
69	19.26	12	7.39	34
70°	–19.38	12	+ 7.05	33
71	19.50	12	6.72	35
72	19.62	11	6.37	34
73	19.73	10	6.03	34
74	19.83	09	5.69	35
75°	–19.92	09	+ 5.34	35
76	20.01	09	4.99	35
77	20.10	08	4.64	35
78	20.18	07	4.29	35
79	20.25	06	3.94	36
80°	–20.31	06	+ 3.58	35
81	20.37	06	3.23	36
82	20.43	04	2.87	36
83	20.47	04	2.51	35
84	20.51	04	2.16	36
85°	–20.55	03	+ 1.80	36
86	20.58	02	1.44	36
87	20.60	01	1.08	36
88	20.61	01	0.72	36
89	20.62	01	0.36	36
90	20.63		0.00	

φ	𝔵	Δ	𝔶	Δ
90°	–20.63	01	– 0.00	36
91	20.62	01	0.36	36
92	20.61	01	0.72	36
93	20.60	02	1.08	36
94	20.58	03	1.44	36
95°	–20.55	04	– 1.80	36
96	20.51	04	2.16	35
97	20.47	04	2.51	36
98	20.43	06	2.87	36
99	20.37	06	3.23	35
100°	–20.31	06	– 3.58	36
101	20.25	07	3.94	35
102	20.18	08	4.29	35
103	20.10	09	4.64	35
104	20.01	09	4.99	35
105°	–19.92	09	– 5.34	35
106	19.83	10	5.69	34
107	19.73	11	6.03	34
108	19.62	12	6.37	35
109	19.50	12	6.72	33
110°	–19.38	12	– 7.05	34
111	19.26	14	7.39	34
112	19.12	13	7.73	33
113	18.99	15	8.06	33
114	18.84	15	8.39	33
115°	–18.69	15	– 8.72	32
116	18.54	16	9.04	32
117	18.38	17	9.36	32
118	18.21	17	9.68	32
119	18.04	18	10.00	31
120°	–17.86	18	–10.31	31
121	17.68	19	10.62	31
122	17.49	19	10.93	30
123	17.30	20	11.23	30
124	17.10	20	11.53	30
125°	–16.90	21	–11.83	29
126	16.69	22	12.12	29
127	16.47	22	12.41	29
128	16.25	22	12.70	28
129	16.03	23	12.98	28
130°	–15.80	23	–13.26	27
131	15.57	24	13.53	27
132	15.33	24	13.80	27
133	15.09	25	14.07	26
134	14.84	25	14.33	26
135	14.59		14.59	

φ	𝔵	Δ	𝔶	Δ
135°	–14.59	26	–14.59	25
136	14.33	26	14.84	25
137	14.07	27	15.09	24
138	13.80	27	15.33	24
139	13.53	27	15.57	23
140°	–13.26	28	–15.80	23
141	12.98	28	16.03	22
142	12.70	29	16.25	22
143	12.41	29	16.47	22
144	12.12	29	16.69	21
145°	–11.83	30	–16.90	20
146	11.53	30	17.10	20
147	11.23	30	17.30	19
148	10.93	31	17.49	19
149	10.62	31	17.68	18
150°	–10.31	31	–17.86	18
151	10.00	32	18.04	17
152	9.68	32	18.21	17
153	9.36	32	18.38	16
154	9.04	32	18.54	15
155°	– 8.72	33	–18.69	15
156	8.39	33	18.84	15
157	8.06	33	18.99	13
158	7.73	34	19.12	14
159	7.39	34	19.26	12
160°	– 7.05	33	–19.38	12
161	6.72	35	19.50	12
162	6.37	34	19.62	11
163	6.03	34	19.73	10
164	5.69	35	19.83	09
165°	– 5.34	35	–19.92	09
166	4.99	35	20.01	09
167	4.64	35	20.10	08
168	4.29	35	20.18	07
169	3.94	36	20.25	06
170°	– 3.58	35	–20.31	06
171	3.23	36	20.37	06
172	2.87	36	20.43	04
173	2.51	35	20.47	04
174	2.16	36	20.51	04
175°	– 1.80	36	–20.55	03
176	1.44	36	20.58	02
177	1.08	36	20.60	01
178	0.72	36	20.61	01
179	0.36	36	20.62	01
180	0.00		20.63	

(cf. pp. 363, 376, 395)

$\mathfrak{x} = -20.6265 \sin\varphi$, $\mathfrak{y} = +20.6265 \cos\varphi$.

φ	$\mathfrak{x}$		$\mathfrak{y}$		φ	$\mathfrak{x}$		$\mathfrak{y}$		φ	$\mathfrak{x}$		$\mathfrak{y}$		φ	$\mathfrak{x}$		$\mathfrak{y}$	
180°	+ 0.00	36	−20.63	01	225°	+14.59	25	−14.59	26	270°	+20.63	01	+ 0.00	36	315°	+14.59	26	+14.59	25
181	0.36	36	20.62	01	226	14.84	25	14.33	26	271	20.62	01	0.36	36	316	14.33	26	14.84	25
182	0.72	36	20.61	01	227	15.09	24	14.07	27	272	20.61	01	0.72	36	317	14.07	27	14.09	24
183	1.08	36	20.60	02	228	15.33	24	13.80	27	273	20.60	02	1.08	36	318	13.80	27	15.33	24
184	1.44	36	20.58	03	229	15.57	23	13.53	27	274	20.58	03	1.44	36	319	13.53	27	15.57	23
185°	+ 1.80	36	−20.55	04	230°	+15.80	23	−13.26	28	275°	+20.55	04	+ 1.80	36	320°	+13.26	28	+15.80	23
186	2.16	35	20.51	04	231	16.03	22	12.98	28	276	20.51	04	2.16	35	321	12.98	28	16.03	22
187	2.51	36	20.47	04	232	16.25	22	12.70	29	277	20.47	04	2.51	36	322	12.70	29	16.25	22
188	2.87	36	20.43	06	233	16.47	22	12.41	29	278	20.43	06	2.87	36	323	12.41	29	16.47	22
189	3.23	35	20.37	06	234	16.69	21	12.12	29	279	20.37	06	3.23	35	324	12.12	29	16.69	21
190°	+ 3.58	36	−20.31	06	235°	+16.90	20	−11.83	30	280°	+20.31	06	+ 3.58	36	325°	+11.83	30	+16.90	20
191	3.94	35	20.25	07	236	17.10	20	11.53	30	281	20.25	07	3.94	35	326	11.53	30	17.10	20
192	4.29	35	20.18	08	237	17.30	19	11.23	30	282	20.18	08	4.29	35	327	11.23	30	17.30	19
193	4.64	35	20.10	09	238	17.49	19	10.93	31	283	20.10	09	4.64	35	328	10.93	31	17.49	19
194	4.99	35	20.01	09	239	17.68	18	10.62	31	284	20.01	09	4.99	35	329	10.62	31	17.68	18
195°	+ 5.34	35	−19.92	09	240°	+17.86	18	−10.31	31	285°	+19.92	09	+ 5.34	35	330°	+10.31	31	+17.86	18
196	5.69	34	19.83	10	241	18.04	17	10.00	32	286	19.83	10	5.69	34	331	10.00	32	18.04	17
197	6.03	34	19.73	11	242	18.21	17	9.68	32	287	19.73	11	6.03	34	332	9.68	32	18.21	17
198	6.37	35	19.62	12	243	18.38	16	9.36	32	288	19.62	12	6.37	35	333	9.36	32	18.38	16
199	6.72	33	19.50	12	244	18.54	15	9.04	32	289	19.50	12	6.72	33	334	9.04	32	18.54	15
200°	+ 7.05	34	−19.38	12	245°	+18.69	15	− 8.72	33	290°	+19.38	12	+ 7.05	34	335°	+ 8.72	33	+18.69	15
201	7.39	34	19.26	14	246	18.84	15	8.39	33	291	19.26	14	7.39	34	336	8.39	33	18.84	15
202	7.73	33	19.12	13	247	18.99	13	8.06	33	292	19.12	13	7.73	33	337	8.06	33	18.99	13
203	8.06	33	18.99	15	248	19.12	14	7.73	34	293	18.99	15	8.06	33	338	7.73	34	19.12	14
204	8.39	33	18.84	15	249	19.26	12	7.39	34	294	18.84	15	8.39	33	339	7.39	34	19.26	12
205°	+ 8.72	32	−18.69	15	250°	+19.38	12	− 7.05	33	295°	+18.69	15	+ 8.72	32	340°	+ 7.05	33	+19.38	12
206	9.04	32	18.54	16	251	19.50	12	6.72	35	296	18.54	16	9.04	32	341	6.72	35	19.50	12
207	9.36	32	18.38	17	252	19.62	11	6.37	34	297	18.38	17	9.36	32	342	6.37	34	19.62	11
208	9.68	32	18.21	17	253	19.73	10	6.03	34	298	18.21	17	9.68	32	343	6.03	34	19.73	10
209	10.00	31	18.04	18	254	19.83	09	5.69	35	299	18.04	18	10.00	31	344	5.69	35	19.83	09
210°	+10.31	31	−17.86	18	255°	+19.92	09	− 5.34	35	300°	+17.86	18	+10.31	31	345°	+ 5.34	35	+19.92	09
211	10.62	31	17.68	19	256	20.01	09	4.99	35	301	17.68	19	10.62	31	346	4.99	35	20.01	09
212	10.93	30	17.49	19	257	20.10	08	4.64	35	302	17.49	19	10.93	30	347	4.64	35	20.10	08
213	11.23	30	17.30	20	258	20.18	07	4.29	35	303	17.30	20	11.23	30	348	4.29	35	20.18	07
214	11.53	30	17.10	20	259	20.25	06	3.94	36	304	17.10	20	11.53	30	349	3.94	36	20.25	06
215°	+11.83	29	−16.90	21	260°	+20.31	06	− 3.58	35	305°	+16.90	21	+11.83	29	350°	+ 3.58	35	+20.31	06
216	12.12	29	16.69	22	261	20.37	06	3.23	36	306	16.69	22	12.12	29	351	3.23	36	20.37	06
217	12.41	29	16.47	22	262	20.43	04	2.87	36	307	16.47	22	12.41	29	352	2.87	36	20.43	04
218	12.70	28	16.25	22	263	20.47	04	2.51	35	308	16.25	22	12.70	28	353	2.51	35	20.47	04
219	12.98	28	16.03	23	264	20.51	04	2.16	36	309	16.03	23	12.98	28	354	2.16	36	20.51	03
220°	+13.26	27	−15.80	23	265°	+20.55	03	− 1.80	36	310°	+15.80	23	+13.26	27	355°	+ 1.80	36	+20.55	03
221	13.53	27	15.57	24	266	20.58	02	1.44	36	311	15.57	24	13.53	27	356	1.44	36	20.58	02
222	13.80	27	15.33	24	267	20.60	01	1.08	36	312	15.33	24	13.80	27	357	1.08	36	20.60	01
223	14.07	26	15.09	25	268	20.61	01	0.72	36	313	15.09	25	14.07	26	358	0.72	36	20.61	01
224	14.33	26	14.84	25	269	20.62	01	0.36	36	314	14.84	25	14.33	26	359	0.36	36	20.62	01
225	14.59		14.59		270	20.63		0.00		315	14.59		14.59		360	0.00		20.63	

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance

φ		$\mathfrak{x}$	$\mathfrak{y}$	φ		$\mathfrak{x}$	$\mathfrak{y}$	φ		$\mathfrak{x}$	$\mathfrak{y}$
*180°	0° 0′	− 0,00	+ 20,63	*190°	10° 0′	− 3,58	+ 20,31	*200°	20° 0′	− 7,05	+ 19,38
	10	0,06	20,63		10	3,64	20,30		10	7,11	19,36
	20	0,12	20,63		20	3,70	20,29		20	7,17	19,34
	30	0.18	20,63		30	3,75	20,28		30	7,22	19,32
	40	0,24	20,63		40	3,82	20,27		40	7,28	19,30
	50	0,30	20,62		50	3,88	20,26		50	7,34	19,28
*181°	1° 0′	− 0,36	+ 20,62	*191°	11° 0′	− 3,94	+ 20,25	*201°	21° 0′	− 7,39	+ 19,26
	10	0,42	20,62		10	3,99	20,24		10	7,45	19,23
	20	0,48	20,62		20	4,05	20,22		20	7,50	19,21
	30	0,54	20,62		30	4,11	20,21		30	7,56	19,19
	40	0,60	20,62		40	4,17	20,20		40	7,62	19,17
	50	0,66	20,62		50	4,23	20,19		50	7,67	19,15
*182°	2° 0′	− 0,72	+ 20,61	*192°	12° 0′	− 4,29	+ 20,18	*202°	22° 0′	− 7,73	+ 19,12
	10	0,78	20,61		10	4,35	20,16		10	7,78	19,10
	20	0,84	20,61		20	4,41	20,15		20	7,84	19,08
	30	0,90	20,61		30	4,46	20,14		30	7,89	19,06
	40	0,96	20,60		40	4,52	20,12		40	7,95	19,03
	50	1,02	20,60		50	4,58	20,11		50	8,00	19,01
*183°	3° 0′	− 1,08	+ 20,60	*193°	13° 0′	− 4,64	+ 20,10	*203°	23° 0′	− 8,06	+ 18,99
	10	1,14	20,59		10	4,70	20,08		10	8,11	18,96
	20	1.20	20,59		20	4,76	20,07		20	8,17	18,94
	30	1.26	20,59		30	4,82	20,06		30	8,22	18,92
	40	1.32	20,58		40	4,87	20,04		40	8,28	18,89
	50	1.38	20,58		50	4,93	20,03		50	8,33	18,87
*184°	4° 0′	− 1,44	+ 20,58	*194°	14° 0′	− 4,99	+ 20,01	*204°	24° 0′	− 8,39	+ 18,84
	10	1,50	20,57		10	5,05	20,00		10	8,44	18,82
	20	1,56	20,57		20	5,11	19,98		20	8,50	18,79
	30	1,62	20,56		30	5,16	19,97		30	8,55	18,77
	40	1,68	20,56		40	5,22	19,95		40	8,61	18,74
	50	1,74	20,55		50	5,28	19,94		50	8,66	18,72
*185°	5° 0′	− 1,80	+ 20,55	*195°	15° 0′	− 5,34	+ 19,92	*205°	25° 0′	− 8,72	+ 18,69
	10	1,86	20,54		10	5,40	19,91		10	8,77	18,67
	20	1,92	20,54		20	5,45	19,89		20	8,83	18,64
	30	1,98	20,53		30	5,51	19,88		30	8,88	18,62
	40	2,04	20,53		40	5,57	19.86		40	8,93	18,59
	50	2.10	20,52		50	5,63	19,84		50	8,99	18,57
*186°	6° 0′	− 2,16	+ 20,51	*196°	16° 0′	− 5,69	+ 19,83	*206°	26° 0′	− 9,04	+ 18,54
	10	2,22	20,51		10	5,74	19,81		10	9,10	18,51
	20	2,28	20,50		20	5,80	19,79		20	9,15	18,49
	30	2,33	20,49		30	5,86	19,78		30	9,20	18,46
	40	2,39	20,49		40	5,92	19,76		40	9,26	18,43
	50	2,45	20,48		50	5,97	19,74		50	9,31	18,41
*187°	7° 0′	− 2,51	+ 20,47	*197°	17° 0′	− 6,03	+ 19,73	*207°	27° 0′	− 9,36	+ 18,38
	10	2.57	20,47		10	6,09	19,71		10	9,42	18,35
	20	2.63	20,46		20	6,15	19,69		20	9,47	18,32
	30	2.69	20,45		30	6,20	19,67		30	9,52	18,30
	40	2.75	20,44		40	6,26	19,65		40	9,58	18.27
	50	2,81	20,43		50	6,32	19,64		50	9,63	18.24
*188°	8° 0′	− 2,87	+ 20,43	*198°	18° 0′	− 6,37	+ 19,62	*208°	28° 0′	− 9,68	+ 18,21
	10	2,93	20,42		10	6,43	19,60		10	9,74	18.18
	20	2,99	20,41		20	6,49	19,58		20	9,79	18,16
	30	3,05	20,40		30	6,54	19,56		30	9,84	18,13
	40	3,11	20,39		40	6,60	19,54		40	9,89	18,10
	50	3,17	20,38		50	6,66	19,52		50	9,95	18,07
*189°	9° 0′	− 3,23	+ 20,37	*199°	19° 0′	− 6,72	+ 19,50	*209°	29° 0′	−10,00	+ 18.04
	10	3,29	20,36		10	6,77	19,48		10	10,05	18,01
	20	3,35	20,35		20	6,83	19,46		20	10,10	17,98
	30	3,40	20,34		30	6.89	19,44		30	10,16	17,95
	40	3,46	20,33		40	6,94	19,42		40	10,21	17,92
	50	3,52	20,32		50	7,00	19,40		50	10,26	17,89
*190°	10° 0′	− 3,58	+ 20,31	*200°	20° 0′	− 7,05	+ 19,38	*210°	30° 0′	−10.31	+ 17.86

* For the values φ marked by * $\mathfrak{x}$ and $\mathfrak{y}$ are taken with *inverse* sign.

Direction Coefficients for 1 Kilometer Distance

φ		𝔵	𝔶
*210°	30° 0′	− 10.31	+ 17.86
	10	10.37	17.83
	20	10.42	17.80
	30	10.47	17.77
	40	10.52	17.74
	50	10.57	17.71
*211°	31° 0′	− 10.62	+ 17.68
	10	10.67	17.65
	20	10.73	17.62
	30	10.78	17.59
	40	10.83	17.56
	50	10.88	17.52
*212°	32° 0′	− 10.93	+ 17.49
	10	10.98	17.46
	20	11.03	17.43
	30	11.08	17.40
	40	11.13	17.37
	50	11.18	17.33
*213°	33° 0′	− 11.23	+ 17.30
	10	11.28	17.27
	20	11.33	17.23
	30	11.38	17.20
	40	11.43	17.17
	50	11.48	17.13
*214°	34° 0′	− 11.53	+ 17.10
	10	11.58	17.07
	20	11.63	17.03
	30	11.68	17.00
	40	11.73	16.96
	50	11.78	16.93
*215°	35° 0′	− 11.83	+ 16.90
	10	11.88	16.86
	20	11.93	16.83
	30	11.98	16.79
	40	12.03	16.76
	50	12.08	16.72
*216°	36° 0′	− 12.12	+ 16.69
	10	12.17	16.65
	20	12.22	16.62
	30	12.27	16.58
	40	12.32	16.54
	50	12.37	16.51
*217°	37° 0′	− 12.41	+ 16.47
	10	12.46	16.44
	20	12.51	16.40
	30	12.56	16.36
	40	12.60	16.33
	50	12.65	16.29
*218°	38° 0′	− 12.70	+ 16.25
	10	12.75	16.22
	20	12.79	16.18
	30	12.84	16.14
	40	12.89	16.10
	50	12.93	16.07
*219°	39° 0′	− 12.98	+ 16.03
	10	13.03	15.99
	20	13.07	15.95
	30	13.12	15.92
	40	13.17	15.88
	50	13.21	15.84
*220°	40° 0′	− 13.26	+ 15.80

φ		𝔵	𝔶
*220°	40° 0′	− 13.26	+ 15.80
	10	13.30	15.76
	20	13.35	15.72
	30	13.40	15.68
	40	13.44	15.65
	50	13.49	15.61
*221°	41° 0′	− 13.53	+ 15.57
	10	13.58	15.53
	20	13.62	15.49
	30	13.67	15.45
	40	13.71	15.41
	50	13.76	15.37
*222°	42° 0′	− 13.80	+ 15.33
	10	13.85	15.29
	20	13.89	15.25
	30	13.94	15.21
	40	13.98	15.17
	50	14.02	15.13
*223°	43° 0′	− 14.07	+ 15.09
	10	14.11	15.04
	20	14.15	15.00
	30	14.20	14.96
	40	14.24	14.92
	50	14.28	14.88
*224°	44° 0′	− 14.33	+ 14.84
	10	14.37	14.80
	20	14.41	14.75
	30	14.46	14.71
	40	14.50	14.67
	50	14.54	14.63
*225°	45° 0′	− 14.59	+ 14.59
	10	14.63	14.54
	20	14.67	14.50
	30	14.71	14.46
	40	14.75	14.41
	50	14.80	14.37
*226°	46° 0′	− 14.84	+ 14.33
	10	14.88	14.28
	20	14.92	14.24
	30	14.96	14.20
	40	15.00	14.15
	50	15.04	14.11
*227°	47° 0′	− 15.09	+ 14.07
	10	15.13	14.02
	20	15.17	13.98
	30	15.21	13.94
	40	15.25	13.89
	50	15.29	13.85
*228°	48° 0′	− 15.33	+ 13.80
	10	15.37	13.76
	20	15.41	13.71
	30	15.45	13.67
	40	15.49	13.62
	50	15.53	13.58
*229°	49° 0′	− 15.57	+ 13.53
	10	15.61	13.49
	20	15.65	13.44
	30	15.68	13.40
	40	15.72	13.35
	50	15.76	13.30
*230°	50° 0′	− 15.80	+ 13.26

φ		𝔵	𝔶
*230°	50° 0′	− 15.80	+ 13.26
	10	15.84	13.21
	20	15.88	13.17
	30	15.92	13.12
	40	15.95	13.07
	50	15.99	13.03
*231°	51° 0′	− 16.03	+ 12.98
	10	16.07	12.93
	20	16.10	12.89
	30	16.14	12.84
	40	16.18	12.79
	50	16.22	12.75
*232°	52° 0′	− 16.25	+ 12.70
	10	16.29	12.65
	20	16.33	12.60
	30	16.36	12.56
	40	16.40	12.51
	50	16.44	12.46
*233°	53° 0′	− 16.47	+ 12.41
	10	16.51	12.37
	20	16.54	12.32
	30	16.58	12.27
	40	16.62	12.22
	50	16.65	12.17
*234°	54° 0′	− 16.69	+ 12.12
	10	16.72	12.08
	20	16.76	12.03
	30	16.79	11.98
	40	16.83	11.93
	50	16.86	11.88
*235°	55° 0,	− 16.90	+ 11.83
	10	16.93	11.78
	20	16.96	11.73
	30	17.00	11.68
	40	17.03	11.63
	50	17.07	11.58
*236°	56° 0′	− 17.10	+ 11.53
	10	17.13	11.48
	20	17.17	11.43
	30	17.20	11.38
	40	17.23	11.33
	50	17.27	11.28
*237°	57° 0′	− 17.30	+ 11.23
	10	17.33	11.18
	20	17.37	11.13
	30	17.40	11.08
	40	17.43	11.03
	50	17.46	10.98
*238°	58° 0′	− 17.49	+ 10.93
	10	17.52	10.88
	20	17.56	10.83
	30	17.59	10.78
	40	17.62	10.73
	50	17.65	10.67
*239°	59° 0′	− 17.68	+ 10.62
	10	17.71	10.57
	20	17.74	10.52
	80	17.77	10.47
	40	17.80	10.42
	50	17.83	10.37
*240°	60° 0′	− 17.86	+ 10.31

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance

φ		𝔵	𝔶
*240°	60° 0′	— 17.86	+ 10.31
	10	17.89	10.26
	20	17.92	10.21
	30	17.95	10.16
	40	17.98	10.10
	50	18.01	10.05
*241°	61° 0′	— 18.04	+ 10.00
	10	18.07	9.95
	20	18.10	9.89
	30	18.13	9.84
	40	18.16	9.79
	50	18.18	9.74
*242°	62° 0′	— 18.21	+ 9.68
	10	18.24	9.63
	20	18.27	9.58
	30	18.30	9.52
	40	18.32	9.47
	50	18.35	9.42
*243°	63° 0′	— 18.38	+ 9.36
	10	18.41	9.31
	20	18.43	9.26
	30	18.46	9.20
	40	18.49	9.15
	50	18.51	9.10
*244°	64° 0′	— 18.54	+ 9.04
	10	18.57	8.99
	20	18.59	8.93
	30	18.62	8.88
	40	18.64	8.83
	50	18.67	8.77
*245°	65° 0′	— 18.69	+ 8.72
	10	18.72	8.66
	20	18.74	8.61
	30	18.77	8.55
	40	18.79	8.50
	50	18.82	8.44
*246°	66° 0′	— 18.84	+ 8.39
	10	18.87	8.33
	20	18.89	8.28
	30	18.92	8.22
	40	18.94	8.17
	50	18.96	8.11
*247°	67° 0′	— 18.99	+ 8.06
	10	19.01	8.00
	20	19.03	7.95
	30	19.06	7.89
	40	19.08	7.84
	50	19.10	7.78
*248°	68° 0′	— 19.12	+ 7.73
	10	19.15	7.67
	20	19.17	7.62
	30	19.19	7.56
	40	19.21	7.50
	50	19.23	7.45
*249°	69° 0′	— 19.26	+ 7.39
	10	19.28	7.34
	20	19.30	7.28
	30	19.32	7.22
	40	19.34	7.17
	50	19.36	7.11
*250°	70° 0′	— 19.38	+ 7.05

φ		𝔵	𝔶
*250°	70° 0′	— 19.38	+ 7.05
	10	19.40	7.00
	20	19.42	6.94
	30	19.44	6.89
	40	19.46	6.83
	50	19.48	6.77
*251°	71° 0′	— 19.50	+ 6.72
	10	19.52	6.66
	20	19.54	6.60
	30	19.56	6.54
	40	19.58	6.49
	50	19.60	6.43
*252°	72° 0′	— 19.62	+ 6.37
	10	19.64	6.32
	20	19.65	6.26
	30	19.67	6.20
	40	19.69	6.15
	50	19.71	6.09
*253°	73° 0′	— 19.73	+ 6.03
	10	19.74	5.97
	20	19.76	5.92
	30	19.78	5.86
	40	19.79	5.80
	50	19.81	5.74
*254°	74° 0′	— 19.83	+ 5.69
	10	19.84	5.63
	20	19.86	5.57
	30	19.88	5.51
	40	19.89	5.45
	50	19.91	5.40
*255°	75° 0′	— 19.92	+ 5.34
	10	19.94	5.28
	20	19.95	5.22
	30	19.97	5.16
	40	19.98	5.11
	50	20.00	5.05
*256°	76° 0′	— 20.01	+ 4.99
	10	20.03	4.93
	20	20.04	4.87
	30	20.06	4.82
	40	20.07	4.76
	50	20.08	4.70
*257°	77° 0′	— 20.10	+ 4.64
	10	20.11	4.58
	20	20.12	4.52
	30	20.14	4.46
	40	20.15	4.41
	50	20.16	4.35
*258°	78° 0′	— 20.18	+ 4.29
	10	20.19	4.23
	20	20.20	4.17
	30	20.21	4.11
	40	20.22	4.05
	50	20.24	3.99
*259°	79° 0′	— 20.25	+ 3.94
	10	20.26	3.88
	20	20.27	3.82
	30	20.28	3.75
	40	20.29	3.70
	50	20.30	3.64
*260°	80° 0′	— 20.31	+ 3.58

φ		𝔵	𝔶
*260°	80° 0′	— 20.31	+ 3.58
	10	20.32	3.52
	20	20.33	3.46
	30	20.34	3.40
	40	20.35	3.35
	50	20.36	3.29
*261°	81° 0′	— 20.37	+ 3.23
	10	20.38	3.17
	20	20.39	3.11
	30	20.40	3.05
	40	20.41	2.99
	50	20.42	2.93
*262°	82° 0′	— 20.43	+ 2.87
	10	20.43	2.81
	20	20.44	2.75
	30	20.45	2.69
	40	20.46	2.63
	50	20.47	2.57
*263°	83° 0′	— 20.47	+ 2.51
	10	20.48	2.45
	20	20.49	2.39
	30	20.49	2.33
	40	20.50	2.28
	50	20.51	2.22
*264°	84° 0′	— 20.51	+ 2.16
	10	20.52	2.10
	20	20.53	2.04
	30	20.53	1.98
	40	20.54	1.92
	50	20.54	1.86
*265°	85° 0′	— 20.55	+ 1.80
	10	20.55	1.74
	20	20.56	1.68
	30	20.56	1.62
	40	20.57	1.56
	50	20.57	1.50
*266°	86° 0′	— 20.58	+ 1.44
	10	20.58	1.38
	20	20.58	1.32
	30	20.59	1.26
	40	20.59	1.20
	50	20.59	1.14
*267°	87° 0′	— 20.60	+ 1.08
	10	20.60	1.02
	20	20.60	0.96
	30	20.61	0.90
	40	20.61	0.84
	50	20.61	0.78
*268°	88° 0′	— 20.61	+ 0.72
	10	20.62	0.66
	20	20.62	0.60
	30	20.62	0.54
	40	20.62	0.48
	50	20.62	0.42
*269°	89° 0′	— 20.62	+ 0.36
	10	20.62	0.30
	20	20.63	0.24
	30	20.63	0.18
	40	20.63	0.12
	50	20.63	0.06
*270°	90° 0′	— 20.63	+ 0.00

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance

φ		𝔵	𝔶
*270°	90° 0′	— 20,63	— 0,00
	10	20,63	0,06
	20	20,63	0,12
	30	20,63	0,18
	40	20,63	0,24
	50	20,62	0,30
*271°	91° 0′	— 20,62	— 0,36
	10	20,62	0,42
	20	20,62	0,48
	30	20,62	0,54
	40	20,62	0,60
	50	20,62	0,66
*272°	92° 0′	— 20,61	— 0,72
	10	20,61	0,78
	20	20,61	0,84
	30	20,61	0,90
	40	20,60	0,96
	50	20,60	1,02
*273°	93° 0′	— 20,60	— 1,08
	10	20,59	1,14
	20	20,59	1,20
	30	20,59	1,26
	40	20,58	1,32
	50	20,58	1,38
*274°	94° 0′	— 20,58	— 1,44
	10	20,57	1,50
	20	20,57	1,56
	30	20,56	1,62
	40	20,56	1,68
	50	20,55	1,74
*275°	95° 0′	— 20,55	— 1,80
	10	20,54	1,86
	20	20,54	1,92
	30	20,53	1,98
	40	20,53	2,04
	50	20,52	2,10
*276°	96° 0′	— 20,51	— 2,16
	10	20,51	2,22
	20	20,50	2,28
	30	20,49	2,33
	40	20,49	2,39
	50	20,48	2,45
*277°	97° 0′	— 20,47	— 2,51
	10	20,47	2,57
	20	20,46	2,63
	30	20,45	2,69
	40	20,44	2,75
	50	20,43	2,81
*278°	98° 0′	— 20,43	— 2,87
	10	20,42	2,93
	20	20,41	2,99
	30	20,40	3,05
	40	20,39	3,11
	50	20,38	3,17
*279°	99° 0′	— 20,37	— 3,23
	10	20,36	3,29
	20	20,35	3,35
	30	20,34	3,40
	40	20,33	3,46
	50	20,32	3,52
*280°	100° 0′	— 20,31	— 3,58

φ		𝔵	𝔶
*280°	100° 0′	— 20,31	— 3,58
	10	20,30	3,64
	20	20,29	3,70
	30	20,28	3,76
	40	20,27	3,82
	50	20,26	3,88
*281°	101° 0′	— 20,25	— 3,94
	10	20,24	3,99
	20	20,22	4,05
	30	20,21	4,11
	40	20,20	4,17
	50	20,19	4,23
*282°	102° 0′	— 20,18	— 4,29
	10	20,16	4,35
	20	20,15	4,41
	30	20,14	4,46
	40	20,12	4,52
	50	20,11	4,58
*283°	103° 0′	— 20,10	— 4,64
	10	20,08	4,70
	20	20,07	4,76
	30	20,06	4,82
	40	20,04	4,87
	50	20,03	4,93
*284°	104° 0′	— 20,01	— 4,99
	10	20,00	5,05
	20	19,98	5,11
	30	19,97	5,16
	40	19,95	5,22
	50	19,94	5,28
*285°	105° 0′	— 19,92	— 5,34
	10	19,91	5,40
	20	19,89	5,45
	30	19,88	5,51
	40	19,86	5,57
	50	19,84	5,63
*286°	106° 0′	— 19,83	— 5,69
	10	19,81	5,74
	20	19,79	5,80
	30	19,78	5,86
	40	19,76	5,92
	50	19,74	5,97
*287°	107° 0′	— 19,73	— 6,03
	10	19,71	6,09
	20	19,69	6,15
	30	19,67	6,20
	40	19,65	6,26
	50	19,64	6,32
*288°	108° 0′	— 19,62	— 6,37
	10	19,60	6,43
	20	19,58	6,49
	30	19,56	6,54
	40	19,54	6,60
	50	19,52	6,66
*289°	109° 0′	— 19,50	— 6,72
	10	19,48	6,77
	20	19,46	6,83
	30	19,44	6,89
	40	19,42	6,94
	50	19,40	7,00
*290°	110° 0′	— 19,38	— 7,05

φ		𝔵	𝔶
*290°	110° 0′	— 19,38	— 7,05
	10	19,36	7,11
	20	19,34	7,17
	30	19,32	7,22
	40	19,30	7,28
	50	19,28	7,34
*291°	111° 0′	— 19,26	— 7,39
	10	19,23	7,45
	20	19,21	7,50
	30	19,19	7,56
	40	19,17	7,62
	50	19,15	7,67
*292°	112° 0′	— 19,12	— 7,73
	10	19,10	7,78
	20	19,08	7,84
	30	19,06	7,89
	40	19,03	7,95
	50	19,01	8,00
*293°	113° 0′	— 18,99	— 8,06
	10	18,96	8,11
	20	18,94	8,17
	30	18,92	8,22
	40	18,89	8,28
	50	18,87	8,33
*294°	114° 0′	— 18,84	— 8,39
	10	18,82	8,44
	20	18,79	8,50
	30	18,77	8,55
	40	18,74	8,61
	50	18,72	8,66
*295°	115° 0′	— 18,69	— 8,72
	10	18,67	8,77
	20	18,64	8,83
	30	18,62	8,88
	40	18,59	8,93
	50	18,57	8,99
*296°	116° 0′	— 18,54	— 9,04
	10	18,51	9,10
	20	18,49	9,15
	30	18,46	9,20
	40	18,43	9,26
	50	18,41	9,31
*297°	117° 0′	— 18,38	— 9,36
	10	18,35	9,42
	20	18,32	9,47
	30	18,30	9,52
	40	18,27	9,58
	50	18,24	9,63
*298°	118° 0′	— 18,21	— 9,68
	10	18,18	9,74
	20	18,16	9,79
	30	18,13	9,84
	40	18,10	9,89
	50	18,07	9,95
*299°	119° 0′	— 18,04	— 10,00
	10	18,01	10,05
	20	17,98	10,10
	30	17,95	10,16
	40	17,92	10,21
	50	17,89	10,25
*300°	120° 0′	— 17,86	— 10,31

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance

φ		𝔵	𝔶
*300°	120° 0′	— 17,86	— 10,31
	10	17,83	10,37
	20	17,80	10,42
	30	17,77	10,47
	40	17,74	10,52
	50	17,71	10,57
*301°	121° 0′	— 17,68	— 10,62
	10	17,65	10,67
	20	17,62	10,73
	30	17,59	10,78
	40	17,56	10,83
	50	17,52	10,88
*302°	122° 0′	— 17,49	— 10,93
	10	17,46	10,98
	20	17,43	11,03
	30	17,40	11,08
	40	17,37	11,13
	50	17,33	11,18
*303°	123° 0′	— 17,30	— 11,23
	10	17,27	11,28
	20	17,23	11,33
	30	17,20	11,38
	40	17,17	11,43
	50	17,13	11,48
*304°	124° 0′	— 17,10	— 11,53
	10	17,07	11,58
	20	17,03	11,63
	30	17,00	11,68
	40	16,96	11,73
	50	16,93	11,78
*305°	125° 0′	— 16,90	— 11,83
	10	16,86	11,88
	20	16,83	11,93
	30	16,79	11,98
	40	16,76	12,03
	50	16,72	12,08
*306°	126° 0′	— 16,69	— 12,12
	10	16,65	12,17
	20	16,62	12,22
	30	16,58	12,27
	40	16,54	12,32
	50	16,51	12,37
*307°	127° 0′	— 16,47	— 12,41
	10	16,44	12,46
	20	16,40	12,51
	30	16,36	12,56
	40	16,33	12,60
	50	16,29	12,65
*308°	128° 0′	— 16,25	— 12,70
	10	16,22	12,75
	20	16,18	12,79
	30	16,14	12,84
	40	16,10	12,89
	50	16,07	12,93
*309°	129° 0′	— 16,03	— 12,98
	10	15,99	13,03
	20	15,95	13,07
	30	15,92	13,12
	40	15,88	13,17
	50	15,84	13,21
*310°	130° 0′	— 15,80	— 13,26

φ		𝔵	𝔶
*310°	130° 0′	— 15,80	— 13,26
	10	15,76	13,30
	20	15,72	13,35
	30	15,68	13,40
	40	15,65	13,44
	50	15,61	13,49
*311°	131° 0′	— 15,57	— 13,53
	10	15,53	13,58
	20	15,49	13,62
	30	15,45	13,67
	40	15,41	13,71
	50	15,37	13,76
*312°	132° 0′	— 15,33	— 13,80
	10	15,29	13,85
	20	15,25	13,89
	30	15,21	13,94
	40	15,17	13,98
	50	15,13	14,02
*313°	133° 0′	— 15,09	— 14,07
	10	15,04	14,11
	20	15,00	14,15
	30	14,96	14,20
	40	14,92	14,24
	50	14,88	14,28
*314°	134° 0′	— 14,84	— 14,33
	10	14,80	14,37
	20	14,75	14,41
	30	14,71	14,46
	40	44,67	14,50
	50	14,63	14,54
*315°	135° 0′	— 14,59	— 14,59
	10	14,54	14,63
	20	14,50	14,67
	30	14,46	14,71
	40	14,41	14,75
	50	14,37	14,80
*316°	136° 0′	— 14,33	— 14,84
	10	14,28	14,88
	20	14,24	14,92
	30	14,20	14,96
	40	14,15	15,00
	50	14,11	15,04
*317°	137° 0′	— 14,07	— 15,09
	10	14,02	15,13
	20	13,98	15,17
	30	13,94	15,21
	40	13,89	15,25
	50	13,85	15,29
*318°	138° 0′	— 13,80	— 15,33
	10	13,76	15,37
	20	13,71	15,41
	30	13,67	15,45
	40	13,62	15,49
	50	13,58	15,53
*319°	139° 0′	— 13,53	— 15,57
	10	13,49	15,61
	20	13,44	15,65
	30	13,40	15,68
	40	13,35	15,72
	50	13,30	15,76
*320°	140° 0′	— 13,26	— 15,80

φ		𝔵	𝔶
*320°	140° 0′	— 13,26	— 15,80
	10	13,21	15,84
	20	13,17	15,88
	30	13,12	15,92
	40	13,07	15,95
	50	13,03	15,99
*321°	141° 0′	— 12,98	— 16,03
	10	12,93	16,07
	20	12,89	16,10
	30	12,84	16,14
	40	12,79	16,18
	50	12,75	16,22
*322°	142° 0′	— 12,70	— 16,25
	10	12,65	16,29
	20	12,60	16,33
	30	12,56	16,36
	40	12,51	16,40
	50	12,46	16,44
*323°	143° 0′	— 12,41	— 16,47
	10	12,37	16,51
	20	12,32	16,54
	30	12,27	16,58
	40	12,22	16,62
	50	12,17	16,65
*324°	144° 0′	— 12,12	— 16,69
	10	12,08	16,72
	20	12,03	16,76
	30	11,98	16,79
	40	11,93	16,83
	50	11,88	16,86
*325°	145° 0′	— 11,83	— 16,90
	10	11,78	16,93
	20	11,73	16,96
	30	11,68	17,00
	40	11,63	17,03
	50	11,58	17,07
*326°	146° 0′	— 11,53	— 17,10
	10	11,48	17,13
	20	11,43	17,17
	30	11,38	17,20
	40	11,33	17,23
	50	11,28	17,27
*327°	147° 0′	— 11,23	— 17,30
	10	11,18	17,33
	20	11,13	17,37
	30	11,08	17,40
	40	11,03	17,43
	50	10,98	17,46
*328°	148° 0′	— 10,93	— 17,49
	10	10,88	17,52
	20	10,83	17,56
	30	10,78	17,59
	40	10,73	17,62
	50	10,67	17,65
*329°	149° 0′	— 10,62	— 17,68
	10	10,57	17,71
	20	10,52	17,74
	30	10,47	17,77
	40	10,42	17,80
	50	10,37	17,83
*330°	150° 0′	— 10,31	— 17,86

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance

φ		𝔵	𝔶
*330°	160° 0′	— 10,31	— 17,86
	10	10,26	17,89
	20	10,21	17,92
	30	10,16	17,95
	40	10,10	17,98
	50	10,05	18,01
*331°	151° 0′	— 10,00	— 18,04
	10	9,95	18,07
	20	9,89	18,10
	30	9,84	18,13
	40	9,79	18,16
	50	9,74	18,18
*332°	152° 0′	— 9,68	— 18,21
	10	9,63	18,24
	20	9,58	18,27
	30	9,52	18,30
	40	9,47	18,32
	50	9,42	18,35
*333°	153° 0′	— 9,36	— 18,38
	10	9,31	18,41
	20	9,26	18,43
	30	9,20	18,46
	40	9,15	18,49
	50	9,10	18,51
*334°	154° 0′	— 9,04	— 18,54
	10	8,99	18,57
	20	8,93	18,59
	30	8,88	18,62
	40	8,83	18,64
	50	8,77	18,67
*335°	155° 0′	— 8,72	— 18,69
	10	8,66	18,72
	20	8,61	18,74
	30	8,55	18,77
	40	8,50	18,79
	50	8,44	18,82
*336°	156° 0′	— 8,39	— 18,84
	10	8,33	18,87
	20	8,28	18,89
	30	8,22	18,92
	40	8,17	18,94
	50	8,11	18,96
*337°	157° 0′	— 8,06	— 18,99
	10	8,00	19,01
	20	7,95	19,03
	30	7,89	19,06
	40	7,84	19,08
	50	7,78	19,10
*338°	158° 0′	— 7,73	— 19,12
	10	7,67	19,15
	20	7,62	19,17
	30	7,56	19,19
	40	7,50	19,21
	50	7,45	19,23
*339°	159° 0′	— 7,39	— 19,26
	10	7,34	19,28
	20	7,28	19,30
	30	7,22	19,32
	40	7,17	19,34
	50	7,11	19,36
*340°	160° 0′	— 7,05	— 19,38

φ		𝔵	𝔶
*340°	160° 0′	— 7,05	— 19,38
	10	7,00	19,40
	20	6,94	19,42
	30	6,89	19,44
	40	6,83	19,46
	50	6,77	19,48
*341°	161° 0′	— 6,72	— 19,50
	10	6,66	19,52
	20	6,60	19,54
	30	6,54	19,56
	40	6,49	19,58
	50	6,43	19,60
*342°	162° 0′	— 6,37	— 19,62
	10	6,32	19,64
	20	6,26	19,65
	30	6,20	19,67
	40	6,15	19,69
	50	6,09	19,71
*343°	163° 0′	— 6,03	— 19,73
	10	5,97	19,74
	20	5,92	19,76
	30	5,86	19,78
	40	5,80	19,79
	50	5,74	19,81
*344°	164° 0′	— 5,69	— 19,83
	10	5,63	19,84
	20	5,57	19,86
	30	5,51	19,88
	40	5,45	19,89
	50	5,40	19,91
*345°	165° 0′	— 5,34	— 19,92
	10	5,28	19,94
	20	5,22	19,95
	30	5,16	19,97
	40	5,11	19,98
	50	5,05	20,00
*346°	166° 0′	— 4,99	— 20,01
	10	4,93	20,03
	20	4,87	20,04
	30	4,82	20,06
	40	4,76	20,07
	50	4,70	20,08
*347°	167° 0′	— 4,64	— 20,10
	10	4,58	20,11
	20	4,52	20,12
	30	4,46	20,14
	40	4,41	20,15
	50	4,35	20,16
*348°	168° 0′	— 4,29	— 20,18
	10	4,23	20,19
	20	4,17	20,20
	30	4,11	20,21
	40	4,05	20,22
	50	3,99	20,24
*349°	169° 0′	— 3,94	— 20,25
	10	3,88	20,26
	20	3,82	20,27
	30	3,76	20,28
	40	3,70	20,29
	50	3,64	20,30
*350°	170° 0′	— 3,58	— 20,31

φ		𝔵	𝔶
*350°	170° 0′	— 3,58	— 20,31
	10	3,52	20,32
	20	3,46	20,33
	30	3,40	20,34
	40	3,35	20,35
	50	3,29	20,36
*351°	171° 0′	— 3,23	— 20,37
	10	3,17	20,38
	20	3,11	20,39
	30	3,05	20,40
	40	2,99	20,41
	50	2,93	20,42
*352°	172° 0′	— 2,87	— 20,43
	10	2,81	20,43
	20	2,75	20,44
	30	2,69	20,45
	40	2,63	20,46
	50	2,57	20,47
*353°	173° 0′	— 2,51	— 20,47
	10	2,45	20,48
	20	2,39	20,49
	30	2,33	20,49
	40	2,28	20,50
	50	2,22	20,51
*354°	174° 0′	— 2,16	— 20,51
	10	2,10	20,52
	20	2,04	20,53
	30	1,98	20,53
	40	1,92	20,54
	50	1,86	20,54
*355°	175° 0′	— 1,80	— 20,55
	10	1,74	20,55
	20	1,68	20,56
	30	1,62	20,56
	40	1,56	20,57
	50	1,50	20,57
*356°	176° 0′	— 1,44	— 20,58
	10	1,38	20,58
	20	1,32	20,58
	30	1,26	20,59
	40	1,20	20,59
	50	1,14	20,59
*357°	177° 0′	— 1,08	— 20,60
	10	1,02	20,60
	20	0,96	20,60
	30	0,90	20,61
	40	0,84	20,61
	50	0,78	20,61
*358°	178° 0′	— 0,72	— 20,61
	10	0,66	20,62
	20	0,60	20,62
	30	0,54	20,62
	40	0,48	20,62
	50	0,42	20,62
*359°	179° 0′	— 0,36	— 20,62
	10	0,30	20,62
	20	0,24	20,63
	30	0,18	20,63
	40	0,12	20,63
	50	0,06	20,63
*360°	180° 0′	— 0,00	— 20,63

(cf. pp. 363, 376, 395)

Direction Coefficients for 1 Kilometer Distance for the *New* [Centesimal] Division of the Circle

𝜑		𝔵		𝔶	
* 200 g	0 g	− 0,00	1,00	+ 63,66	0,01
201	1	1,00	1,00	63,65	0,02
202	2	2,00	1,00	63,63	0,04
203	3	3,00	1,00	63,59	0,05
204	4	4,00		63,54	
			0,99		0,07
* 205 g	5 g	− 4,99	1,00	+ 63,47	0,09
206	6	5,99	1,00	63,38	0,10
207	7	6,99	0,99	63,28	0,12
208	8	7,98	0,99	63,16	0,13
209	9	8,97		63,03	
			0,99		0,15
* 210 g	10 g	− 9,96	0,99	+ 62,88	0,17
211	11	10,95	0,98	62,71	0,18
212	12	11,93	0,98	62,53	0,19
213	13	12,91	0,98	62,34	0,21
214	14	13,89		62,13	
			0,97		0,23
* 215 g	15 g	− 14,86	0,97	+ 61,90	0,24
216	16	15,83	0,97	61,66	0,25
217	17	16,80	0,96	61,41	0,28
218	18	17,76	0,96	61,13	0,28
219	19	18,72		60,85	
			0,95		0,30
* 220 g	20 g	− 19,67	0,95	+ 60,55	0,32
221	21	20,62	0,94	60,23	0,33
222	22	21,56	0,94	59,90	0,35
223	23	22,50	0,94	59,55	0,36
224	24	23,44		59,19	
			0,92		0,37
* 225 g	25 g	− 24,36	0,92	+ 58,82	0,39
226	26	25,28	0,92	58,43	0,41
227	27	26,20	0,91	58,02	0,42
228	28	27,11	0,90	57,60	0,43
229	29	28,01		57,17	
			0,89		0,45
* 230 g	30 g	− 28,90	0,89	+ 56,72	0,46
231	31	29,79	0,88	56,26	0,47
232	32	30,67	0,87	55,79	0,49
233	33	31,54	0,87	55,30	0,50
234	34	32,41		54,80	
			0,85		0,52
* 235 g	35 g	− 33,26	0,85	+ 54,28	0,53
236	36	34,11	0,84	53,75	0,54
237	37	34,95	0,83	53,21	0,56
238	38	35,78	0,83	52,65	0,56
239	39	36,61		52,09	
			0,81		0,59
* 240 g	40 g	− 37,42	0,80	+ 51,50	0,59
241	41	38,22	0,80	50,91	0,61
242	42	39,02	0,78	50,30	0,62
243	43	39,80	0,78	49,68	0,63
244	44	40,58		49,05	
			0,76		0,64
* 245 g	45 g	− 41,34	0,76	+ 48,41	0,66
246	46	42,10	0,75	47,75	0,66
247	47	42,85	0,73	47,09	0,68
248	48	43,58	0,72	46,41	0,69
249	49	44,30	0,72	45,72	0,70
* 250 g	50 g	− 45,02		+ 45,02	

𝜑		𝔵		𝔶	
* 250 g	50 g	− 45,02	0,60	+ 45,02	0,72
251	51	45,72	0,69	44,30	0,72
252	52	46,41	0,68	43,58	0,73
253	53	47,09	0,66	42,85	0,75
254	54	47,75		42,10	
			0,66		0,76
* 255 g	55 g	− 48,41	0,64	+ 41,34	0,76
256	56	49,05	0,63	40,58	0,78
257	57	49,68	0,62	39,80	0,78
258	58	50,30	0,61	39,02	0,80
259	59	50,91		38,22	
			0,59		0,80
* 260 g	60 g	− 51,50	0,59	+ 37,42	0,81
261	61	52,09	0,56	36,61	0,83
262	62	52,65	0,56	35,78	0,83
263	63	53,21	0,54	34,95	0,84
264	64	53,75		34,11	
			0,53		0,85
* 265 g	65 g	− 54,28	0,52	+ 33,26	0,85
266	66	54,80	0,50	32,41	0,87
267	67	55,30	0,49	31,54	0,87
268	68	55,79	0,47	30,67	0,88
269	69	56,26		29,79	
			0,46		0,89
* 270 g	70 g	− 56,72	0,45	+ 28,90	0,89
271	71	57,17	0,43	28,01	0,90
272	72	57,60	0,42	27,11	0,91
273	73	58,02	0,41	26,20	0,92
274	74	58,43		25,28	
			0,39		0,92
* 275 g	75 g	− 58,82	0,37	+ 24,36	0,92
276	76	59,19	0,36	23,44	0,94
277	77	59,56	0,35	22,50	0,94
278	78	59,90	0,33	21,56	0,94
279	79	60,23		20,62	
			0,32		0,95
* 280 g	80 g	− 60,55	0,30	+ 19,67	0,95
281	81	60,85	0,28	18,72	0,96
282	82	61,13	0,28	17,76	0,96
283	83	61,41	0,25	16,80	0,97
284	84	61,66		15,83	
			0,24		0,97
* 285 g	85 g	− 61,90	0,23	+ 14,86	0,97
286	86	62,13	0,21	13,89	0,98
287	87	62,34	0,19	12,91	0,98
288	88	62,53	0,18	11,93	0,98
289	89	62,71		10,95	
			0,17		0,99
* 290 g	90 g	− 62,88	0,15	+ 9,96	0,99
291	91	63,03	0,13	8,97	0,99
292	92	63,16	0,12	7,98	0,99
293	93	63,28	0,10	6,99	1,00
294	94	63,38		5,99	
			0,09		1,00
* 295 g	95 g	− 63,47	0,07	+ 4,99	0,99
296	96	63,54	0,05	4,00	1,00
297	97	63,59	0,04	3,00	1,00
298	98	63,63	0,02	2,00	1,00
299	99	63,65	0,01	1,00	1,00
* 300 g	100 g	− 63,66		+ 0,00	

* For the values 𝜑 marked by * 𝔵 and 𝔶 are taken with *inverse* sign.

(cf. p. 363)

$$\mathfrak{x} = -\,63.6620\, \sin\varphi \;,\; \mathfrak{y} = +\,63.6620\, \cos\varphi$$

for the *New* [Centesimal] Division of the Circle

φ		$\mathfrak{x}$		$\mathfrak{y}$	
* 300 g	100 g	— 63,66	0,01	— 0,00	1,00
301	101	63,65	0,02	1,00	1,00
302	102	63,63	0,04	2,00	1,00
303	103	63,59	0,05	3,00	1,00
304	104	63,54	0,07	4,00	0,99
* 305 g	105 g	— 63,47	0,09	— 4,99	1,00
306	106	63,38	0,10	5,99	1,00
307	107	63,28	0,12	6,99	0,99
308	108	63,16	0,13	7,98	0,99
309	109	63,03	0,15	8,97	0,99
* 310 g	110 g	— 62,88	0,17	— 9,96	0,99
311	111	62,71	0,18	10,95	0,98
312	112	62,53	0,19	11,93	0,98
313	113	62,34	0,21	12,91	0,98
314	114	62,13	0,23	13,89	0,97
* 315 g	115 g	— 61,90	0,24	— 14,86	0,97
316	116	61,66	0,25	15,83	0,97
317	117	61,41	0,28	16,80	0,96
318	118	61,13	0,28	17,76	0,96
319	119	60,85	0,30	18,72	0,95
* 320 g	120 g	— 60,55	0,32	— 19,67	0,95
321	121	60,23	0,33	20,62	0,94
322	122	59,90	0,35	21,56	0,94
323	123	59,55	0,36	22,50	0,94
324	124	59,19	0,37	23,44	0,92
* 325 g	125 g	— 58,82	0,39	— 24,36	0,92
326	126	58,43	0,41	25,28	0,92
327	127	58,02	0,42	26,20	0,91
328	128	57,60	0,43	27,11	0,90
329	129	57,17	0,45	28,01	0,89
* 330 g	130 g	— 56,72	0,46	— 28,90	0,89
331	131	56,26	0,47	29,79	0,88
332	132	55,79	0,49	30,67	0,87
333	133	55,30	0,50	31,54	0,87
334	134	54,80	0,52	32,41	0,85
* 335 g	135 g	— 54,28	0,53	— 33,26	0,85
336	136	53,75	0,54	34,11	0,84
337	137	53,21	0,56	34,95	0,83
338	138	52,65	0,56	35,78	0,83
339	139	52,09	0,59	36,61	0,81
* 340 g	140 g	— 51,50	0,59	— 37,42	0,80
341	141	50,91	0,61	38,22	0,80
342	142	50,30	0,62	39,02	0,78
343	143	49,68	0,63	39,80	0,78
344	144	49,05	0,64	40,58	0,76
* 345 g	145 g	— 48,41	0,66	— 41,34	0,76
346	146	47,75	0,66	42,10	0,75
347	147	47,09	0,68	42,85	0,73
348	148	46,41	0,69	43,58	0,72
349	149	45,72	0,70	44,30	0,72
* 350 g	150 g	— 45,02		— 45,02	

φ		$\mathfrak{x}$		$\mathfrak{y}$	
* 350 g	150 g	— 45,02	0,72	— 45,02	0,70
351	151	44,30	0,72	45,72	0,69
352	152	43,58	0,73	46,41	0,68
353	153	42,85	0,75	47,09	0,66
354	154	42,10	0,76	47,75	0,66
* 355 g	155 g	— 41,34	0,76	— 48,41	0,64
356	156	40,58	0,78	49,05	0,63
357	157	39,80	0,78	49,68	0,62
358	158	39,02	0,80	50,30	0,61
359	159	38,22	0,80	50,91	0,59
* 360 g	160 g	— 37,42	0,81	— 51,50	0,59
361	161	36,61	0,83	52,09	0,56
362	162	35,78	0,83	52,65	0,56
363	163	34,95	0,84	53,21	0,54
364	164	34,11	0,85	53,75	0,53
* 365 g	165 g	— 33,26	0,85	— 54,28	0,52
366	166	32,41	0,87	54,80	0,50
367	167	31,54	0,87	55,30	0,49
368	168	30,67	0,88	55,79	0,47
369	169	29,79	0,89	56,26	0,46
* 370 g	170 g	— 28,90	0,89	— 56,72	0,45
371	171	28,01	0,90	57,17	0,43
372	172	27,11	0,91	57,60	0,42
373	173	26,20	0,92	58,02	0,41
374	174	25,28	0,92	58,43	0,39
* 375 g	175 g	— 24,36	0,92	— 58,82	0,37
376	176	23,44	0,94	59,19	0,36
377	177	22,50	0,94	59,55	0,35
378	178	21,56	0,94	59,90	0,33
379	179	20,62	0,95	60,23	0,32
* 380 g	180 g	— 19,67	0,95	— 60,55	0,30
381	181	18,72	0,96	60,85	0,28
382	182	17,76	0,96	61,13	0,28
383	183	16,80	0,97	61,41	0,25
384	184	15,83	0,97	61,66	0,24
* 385 g	185 g	— 14,86	0,97	— 61,90	0,23
386	186	13,89	0,98	62,13	0,21
387	187	12,91	0,98	62,34	0,19
388	188	11,93	0,98	62,53	0,18
389	189	10,95	0,99	62,71	0,17
* 390 g	190 g	— 9,96	0,99	— 62,88	0,15
391	191	8,97	0,99	63,03	0,13
392	192	7,98	0,99	63,16	0,12
393	193	6,99	1,00	63,28	0,10
394	194	5,99	1,00	63,38	0,09
* 395 g	195 g	— 4,99	0,99	— 63,47	0,07
396	196	4,00	1,00	63,54	0,05
397	197	3,00	1,00	63,59	0,04
398	198	2,00	1,00	63,63	0,02
399	199	1,00	1,00	63,65	0,01
* 400 g	200 g	— 0,00		— 63,66	

(cf. p. 363)

$$a^2 + b^2 = \frac{\varrho^2}{100\, s^2} \qquad \left(\text{for } a = \frac{\varrho}{10\, s} \sin\varphi \qquad b = \frac{\varrho}{10\, s} \cos\varphi\right)$$

s	00^m	10^m	20^m	30^m	40^m	50^m	60^m	70^m	80^m	90^m
0.1 km	42545	35161	29545	25174	21706	18909	16619	14722	13131	11785
0.2	10635	9648	8790	8043	7386	6807	6294	5836	5427	5059
0.3	4727	4427	4155	3907	3680	3473	3283	3108	2946	2797
0.4	2659	2531	2412	2301	2198	2101	2011	1926	1847	1772
0.5	1702	1636	1573	1515	1459	1406	1357	1309	1265	1222
0.6	1182	1143	1108	1072	1039	1007	977	927	920	894
0.7	868	844	821	798	777	756	737	718	699	682
0.8	665	648	633	618	603	589	575	562	549	537
0.9	525	514	503	492	481	471	462	452	443	434
1.0	425	417	409	401	393	386	379	372	365	358
1.1	352	345	339	333	327	322	316	311	306	300
1.2	295	291	286	281	277	272	268	264	260	256
1.3	252	248	244	241	237	233	230	227	223	220
1.4	217	214	211	208	205	202	200	196	194	192
1.5	189	187	184	182	179	177	175	173	170	168
1.6	166	164	162	160	158	156	154	153	151	149
1.7	147	145	144	142	141	139	137	136	134	133
1.8	131	130	128	127	126	124	123	122	120	119
1.9	118	117	115	114	113	112	111	110	109	107
2.0	106	105	104	103	102	101	100	99	98	97
2.1	96	96	95	94	93	92	91	90	89	89
2.2	88	87	86	86	85	84	83	83	82	81
2.3	80	80	79	78	78	77	76	76	75	74
2.4	74	73	73	72	72	71	70	70	69	69
2.5	68.1	67.5	67.0	66.5	66.0	65.4	65.0	64.4	63.9	63.4
2.6	62.9	62.5	62.0	61.5	61.0	60.6	60.1	59.7	59.2	58.8
2.7	58.4	57.9	57.5	57.1	56.7	56.3	55.8	55.4	55.0	54.6
2.8	54.3	53.9	53.5	53.1	52.7	52.4	52.0	51.7	51.3	50.9
2.9	50.6	50.3	49.9	49.6	49.2	48.9	48.6	48.2	47.9	47.6
3.0	47.3	47.0	46.6	46.3	46.0	45.7	45.4	45.1	44.9	44.6
3.1	44.3	44.0	43.7	43.4	43.2	42.9	42.6	42.3	42.1	41.8
3.2	41.5	41.3	41.0	40.8	40.5	40.3	40.0	39.8	39.5	39.3
3.3	39.1	38.8	38.6	38.4	38.1	37.9	37.7	37.5	37.2	37.0
3.4	36.8	36.6	36.4	36.2	36.0	35.7	35.5	35.3	35.1	34.9
3.5	34.7	34.5	34.3	34.1	33.9	33.8	33.6	33.4	33.2	33.0
3.6	32.8	32.6	32.5	32.3	32.1	31.9	31.7	31.6	31.4	31.2
3.7	31.1	30.9	30.7	30.6	30.4	30.3	30.1	29.9	29.8	29.6
3.8	29.5	29.3	29.2	29.0	28.9	28.7	28.6	28.4	28.3	28.1
3.9	28.0	27.8	27.7	27.5	27.4	27.3	27.1	27.0	26.9	26.7
4.0	26.6	26.5	26.3	26.2	26.1	25.9	25.8	25.7	25.6	25.4
4.1	25.3	25.2	25.1	24.9	24.8	24.7	24.6	24.5	24.3	24.2
4.2	24.1	24.0	23.9	23.8	23.7	23.6	23.4	23.3	23.2	23.1
4.3	23.0	22.9	22.8	22.7	22.6	22.5	22.4	22.3	22.2	22.1
4.4	22.0	21.9	21.8	21.7	21.6	21.5	21.4	21.3	21.2	21.1
4.5	21.0	20.9	20.8	20.7	20.6	20.6	20.5	20.4	20.3	20.2
4.6	20.1	20.0	19.9	19.8	19.8	19.7	19.6	19.5	19.4	19.3
4.7	19.3	19.2	19.1	19.0	18.9	18.9	18.8	18.7	18.6	18.5
4.8	18.5	18.4	18.3	18.2	18.2	18.1	18.0	17.9	17.9	17.8
4.9	17.7	17.6	17.6	17.5	17.4	17.4	17.3	17.2	17.2	17.1
5.0	17.0	17.0	16.9	16.9	16.8	16.7	16.7	16.6	16.5	16.4

(cf. pp. 366, 367)

$$A = \frac{\varrho}{10} \sin \varphi \cos \varphi \qquad B = \frac{\varrho^2}{100} \sin \varphi \cos \varphi$$

φ				log A	log B	φ			
270°	180°	90°	0°	— ∞	— ∞	90°	180°	270°	360°
271	181	91	1	2.5562	6.8706	89	179	269	359
272	182	92	2	2.8570	7.1714	88	178	268	358
273	183	93	3	3.0326	7.3471	87	177	267	357
274	184	94	4	3.1570	7.4714	86	176	266	356
275°	185°	95°	5°	3.2531	7.5675	85°	175°	265°	355°
276	186	96	6	3.3313	7.6457	84	174	264	354
277	187	97	7	3.3971	7.7115	83	173	263	353
278	188	98	8	3.4537	7.7682	82	172	262	352
279	189	99	9	3.5034	7.8178	81	171	261	351
280°	190°	100°	10°	3.5474	7.8619	80°	170°	260°	350°
281	191	101	11	3.5870	7.9014	79	169	259	349
282	192	102	12	3.6227	7.9371	78	168	258	348
283	193	103	13	3.6552	7.9697	77	167	257	347
284	194	104	14	3.6850	7.9994	76	166	256	346
285°	195°	105°	15°	3.7124	8.0268	75°	165°	255°	355°
286	196	106	16	3.7376	8.0520	74	164	254	344
287	197	107	17	3.7610	8.0754	73	163	253	343
288	198	108	18	3.7826	8.0970	72	162	252	342
289	199	109	19	3.8027	8.1172	71	161	251	341
290°	200°	110°	20°	3.8215	8.1359	70°	160°	250°	340°
291	201	111	21	3.8389	8.1533	69	159	249	339
292	202	112	22	3.8552	8.1696	68	158	248	338
293	203	113	23	3.8703	8.1848	67	157	247	337
294	204	114	24	3.8845	8.1989	66	156	246	336
295°	205°	115°	25°	3.8976	8.2121	65°	155°	245°	335°
296	206	116	26	3.9099	8.2244	64	154	244	334
297	207	117	27	3.9214	8.2358	63	153	243	333
298	208	118	28	3.9320	8.2464	62	152	242	332
299	209	119	29	3.9418	8.2562	61	151	241	331
300°	210°	120°	30°	3.9509	8.2654	60°	150°	240°	330°
301	211	121	31	3.9593	8.2738	59	149	239	329
302	212	122	32	3.9671	8.2815	58	148	238	328
303	213	123	33	3.9741	8.2886	57	147	237	327
304	214	124	34	3.9806	8.2950	56	146	236	326
305°	215°	125°	35°	3.9864	8.3018	55°	145°	235°	325°
306	216	126	36	3.9916	8.3060	54	144	234	324
307	217	127	37	3.9962	8.3107	53	143	233	323
308	218	128	38	4.0003	8.3147	52	142	232	322
309	219	129	39	4.0038	8.3182	51	141	231	321
310°	220°	130°	40°	4.0067	8.3212	50°	140°	230°	320°
311	221	131	41	4.0091	8.3236	49	139	229	319
312	222	132	42	4.0110	8.3254	48	138	228	318
313	223	133	43	4.0123	8.3268	47	137	227	317
314	224	134	44	4.0131	8.3276	46	136	226	316
315	225	135	45	4.0134	8.3278	45	135	225	315

$$a = -\frac{A}{\Delta x}\ , \quad b = +\frac{A}{\Delta y}\ , \quad a\,b = -\frac{B}{s^2}$$

(cf. p. 365)

Probability for the Occurrence of an Error Between the Limits Zero and n Times the Probable Error

n	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	diff.
0.0	0.0000	0.0054	0.0108	0.0161	0.0215	0.0269	0.0323	0.0377	0.0430	0.0484	54
0.1	0538	0591	0645	0699	0752	0806	0859	0913	0966	1020	53
0.2	1073	1126	1180	1233	1286	1339	1392	1445	1498	1551	52
0.3	1603	1656	1709	1761	1814	1866	1918	1971	2023	2075	52
0.4	2127	2179	2230	2282	2334	2385	2436	2488	2539	2590	51
0.5	0.2641	0.2691	0.2742	0.2793	0.2843	0.2893	0.2944	0.2994	0.3043	0.3093	50
0.6	3143	3192	3242	3291	3340	3389	3438	3487	3535	3583	49
0.7	3632	3680	3728	3775	3823	3870	3918	3965	4012	4059	46
0.8	4105	4152	4198	4244	4290	4336	4381	4427	4472	4517	45
0.9	4562	4606	4651	4695	4739	4783	4827	4870	4914	4957	43
1.0	0.5000	0.5043	0.5085	0.5128	0.5170	0.5212	0.5254	0.5295	0.5337	0.5378	41
1.1	5419	5460	5500	5540	5581	5620	5660	5700	5739	5778	39
1.2	5817	5856	5894	5932	5970	6008	6046	6083	6120	6157	37
1.3	6194	6231	6267	6303	6339	6375	6410	6445	6480	6515	35
1.4	6550	6584	6618	6652	6686	6719	6753	6786	6818	6851	32
1.5	0.6883	0.6915	0.6947	0.6979	0.7011	0.7042	0.7073	0.7104	0.7134	0.7165	30
1.6	7195	7225	7255	7284	7313	7342	7371	7400	7428	7457	28
1.7	7485	7512	7540	7567	7594	7621	7648	7675	7701	7727	26
1.8	7753	7778	7804	7829	7854	7879	7904	7928	7952	7976	24
1.9	8000	8023	8047	8070	8093	8116	8138	8161	8183	8205	22
2.0	0.8227	0.8248	0.8270	0.8291	0.8312	0.8332	0.8353	0.8373	0.8394	0.8414	19
2.1	8433	8453	8473	8492	8511	8530	8549	8567	8585	8604	18
2.2	8622	8639	8657	8674	8692	8709	8726	8742	8759	8775	17
2.3	8792	8808	8824	8840	8855	8870	8886	8901	8916	8930	15
2.4	8945	8960	8974	8988	9002	9016	9029	9043	9056	9069	13
2.5	0.9082	0.9095	0.9108	0.9121	0.9133	0.9146	0.9158	0.9170	0.9182	0.9193	12
2.6	9205	9217	9228	9239	9250	9261	9272	9283	9293	9304	10
2.7	9314	9324	9334	9344	9354	9364	9373	9383	9392	9401	9
2.8	9410	9419	9428	9437	9446	9454	9463	9471	9479	9487	8
2.9	9495	9503	9511	9519	9526	9534	9541	9548	9556	9563	7
3.0	0.9570	0.9577	0.9583	0.9590	0.9597	0.9603	0.9610	0.9616	0.9622	0.9629	6
3.1	9635	9641	9647	9652	9658	9664	9669	9675	9680	9686	5
3.2	9691	9696	9701	9706	9711	9716	9721	9726	9731	9735	5
3.3	9740	9744	9749	9753	9757	9761	9766	9770	9774	9778	4
3.4	9782	9786	9789	9793	9797	9800	9804	9807	9811	9814	4
3.5	0.9818	0.9821	0.9824	0.9827	0.9830	0.9833	0.9837	0.9840	0.9842	0.9845	3
3.6	9848	9851	9854	9856	9859	9862	9864	9867	9869	9872	2
3.7	9874	9877	9879	9881	9884	9886	9888	9890	9892	9894	2
3.8	9896	9898	9900	9902	9904	9906	9908	9909	9911	9913	2
3.9	9915	9916	9918	9920	9921	9923	9924	9926	9927	9929	1
4.0	9930	9932	9933	9934	9936	9937	9938	9939	9941	9942	1
4.1	9943	9944	9945	9947	9948	9949	9950	9951	9952	9953	1
$n=$	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0	
$W=$	0.9943	0.9954	0.9963	0.9970	0.9976	0.9981	0.9985	0.9988	0.9991	0.9993	

(cf. p. 528)

Probability for the Occurrence of an Error Between the Limits Zero and n Times the Mean Error

n	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	Diff.
0.0	0.0000	0.0080	0.0159	0.0239	0.0319	0.0399	0.0478	0.0558	0.0637	0.0717	80
0.1	0797	0876	0955	1034	1113	1192	1271	1350	1428	1507	78
0.2	1585	1663	1741	1819	1897	1974	2051	2128	2205	2282	76
0.3	2358	2434	2510	2586	2661	2737	2812	2886	2960	3035	73
0.4	3108	3182	3255	3328	3401	3473	3545	3616	3688	3759	70
0.5	0.3829	0.3900	0.3969	0.4039	0.4108	0.4177	0.4245	0.4313	0.4381	0.4448	67
0.6	4515	4581	4647	4713	4778	4843	4908	4971	5035	5098	63
0.7	5161	5223	5285	5346	5407	5467	5527	5587	5746	5705	58
0.8	5763	5821	5878	5935	5991	6047	6090	6157	6211	6265	54
0.9	6319	6372	6424	6476	6528	6579	6629	6680	6729	6778	49
1.0	0.6827	0.6875	0.6923	0.6970	0.7016	0.7063	0.7109	0.7154	0.7198	0.7243	44
1.1	7287	7330	7373	7415	7457	7498	7539	7580	7620	7660	39
1.2	7699	7737	7775	7813	7850	7887	7923	7959	7994	8030	34
1.3	8064	8098	8132	8165	8197	8229	8262	8293	8324	8355	30
1.4	8385	8415	8444	8473	8501	8529	8557	8584	8611	8638	26
1.5	0.8664	0.8689	0.8715	0.8740	0.8764	0.8789	0.8812	0.8836	0.8859	0.8882	22
1.6	8904	8926	8948	8969	8990	9011	9031	9051	9070	9090	19
1.7	9109	9127	9146	9164	9181	9199	9216	9233	9249	9266	15
1.8	9281	9297	9312	9328	9342	9357	9371	9385	9399	9412	14
1.9	9426	9439	9451	9464	9476	9488	9500	9518	9533	9534	11
2.0	0.9545	0.9556	0.9566	0.9576	0.9586	0.9596	0.9606	0.9616	0.9625	0.9634	9
2.1	9643	9651	9660	9668	9676	9684	9692	9700	9707	9715	7
2.2	9722	9729	9736	9742	9749	9756	9762	9768	9774	9780	5
2.3	9785	9791	9797	9802	9807	9812	9817	9822	9827	9832	4
2.4	9836	9840	9845	9849	9853	9857	9861	9865	9869	9872	4
2.5	0.9876	0.9879	0.9883	0.9886	0.9889	0.9892	0.9895	0.9898	0.9901	0.9904	3
2.6	9907	9909	9912	9915	9917	9920	9922	9924	9926	9928	3
2.7	9931	9933	9935	9937	9939	9940	9942	9944	9946	9947	2
2.8	9949	9950	9952	9953	9955	9956	9958	9959	9960	9961	2
2.9	9963	9964	9965	9966	9967	9968	9969	9970	9971	9972	1
$n=$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	
$W=$	0.9973	0.9981	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9999	0.9999	
$n=$	4.0	4.1								$n=\infty$	
$W=$	0.9999	1.0000								$W=1$	

(cf. pp. 523 and 524)