

## An inverse for the Prime Counting function

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Given a real number  $x$ , the prime counting function  $\pi(x)$  estimates the number of primes less than or equal to  $x$ :

$$\pi(x) = \#\{\text{all primes } p \leq x\}$$

Conversely, for a given counting function  $\pi(x)$ , what  $x$  yields this counter?. The stepwise nature of  $\pi(x)$  [1] obviously prevents an unique answer.

At first glance  $\pi(x)\ln(\pi(x))$  might provide a close approximation, for sufficiently large  $x$ . However, for  $x = 100,000$ , the MATLAB 'primes' function returns

$$\pi(100,000) = 9,592$$

so that rounded off to the nearest integer

$$\pi(100,000)\ln(\pi(100,000)) = 87,946$$

a gross underestimate of  $x$ .

Gauss suggested that

$$\pi(x) \approx x/\ln x$$

and Newton iteration offers an opportunity to do better. Let  $p = x/\ln x$

so that

$$p \ln p = x(1 - \ln \ln x / \ln x)$$

showing more clearly why  $p \ln p$  is an underestimate of  $x$ , though a good starting point for the iteration

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

on the function

$$f(x) = p \ln x - x$$

with derivative

$$f'(x) = p/x - 1$$

After some algebra, the iteration becomes

$$(1) \quad x_{n+1} = px_n(1 - \ln(x_n))/(p - x_n), \quad x_1 = p \ln p$$

To test this numerically, for

$$p = 10,000, \quad x_1 = 92,103$$

the iteration converges very quickly:

$$x_3 = 116,671, \quad \pi(116,671) = 11,045$$

a good deal better than  $x_1$ .

An even closer result follows by assuming (with Chebyshev) that

$$p = x/(\ln x - 1)$$

and using Newton's method on the function

$$f(x) = p \ln x - x - p$$

The iteration becomes

$$(2) \quad x_{n+1} = px_n(2 - \ln(x_n))/(p - x_n), \quad x_1 = p \ln p$$

only a slight change from scheme (1). Once again the iteration converges very quickly:

For  $p = 10,000$  we get

$$x_3 = 105,793, \quad \pi(105,793) = 10,090$$

an order of magnitude improvement over scheme (1).

The iteration in scheme 2) can be easily implemented in MATLAB, producing errors around 1% for sufficiently large entries.

## Reference

[1] Manfred R. Schroeder, 'Number Theory in Science and Communication', Springer (2009), p. 48.