# Computing Linear Restrictions of Neural Networks

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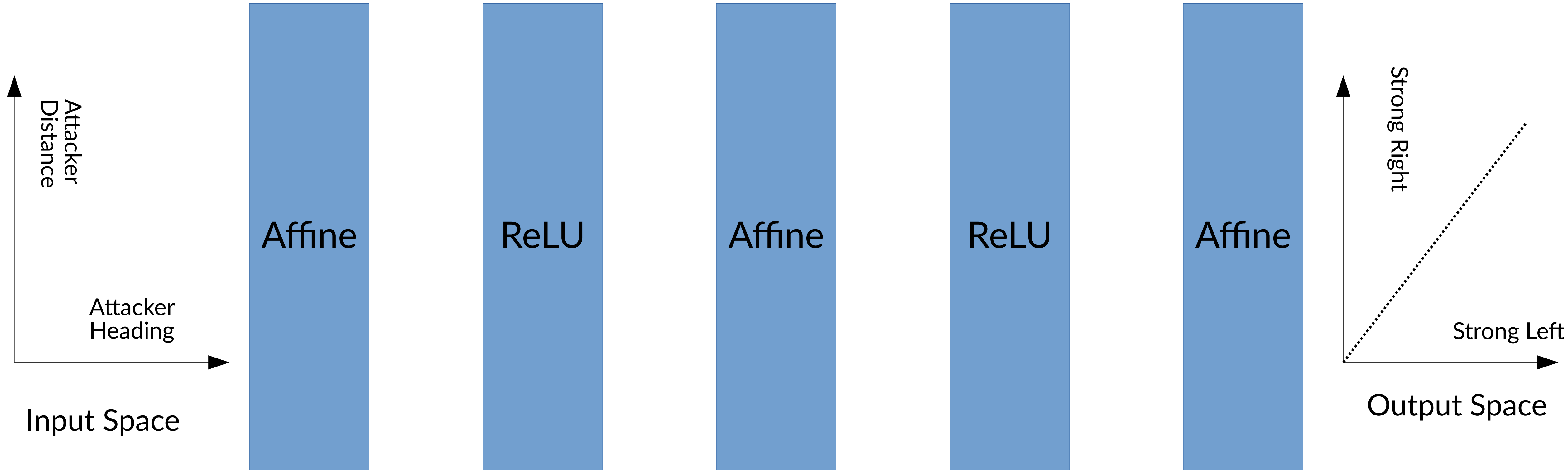
# **Our Work:** A Technique for Examining Trained Neural Networks

Specifically, computing *succinct representation of the network restricted to a line.*

# Overview: Neural Networks

Neural Networks: sequential composition of other functions. Can transform individual points through each layer to find the output of the network.



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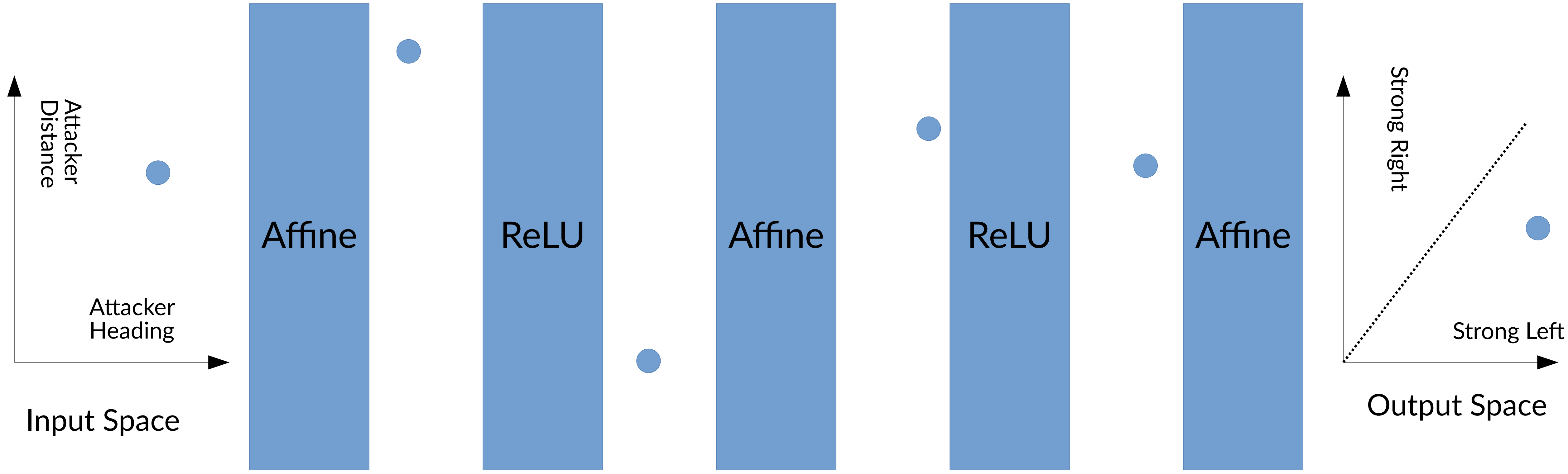

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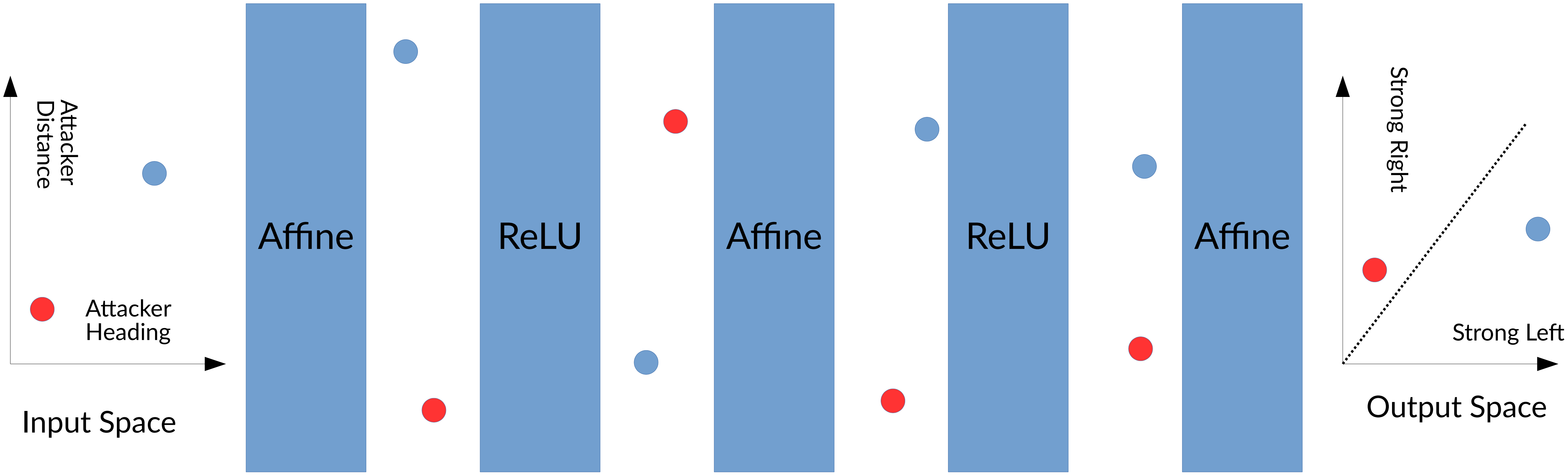

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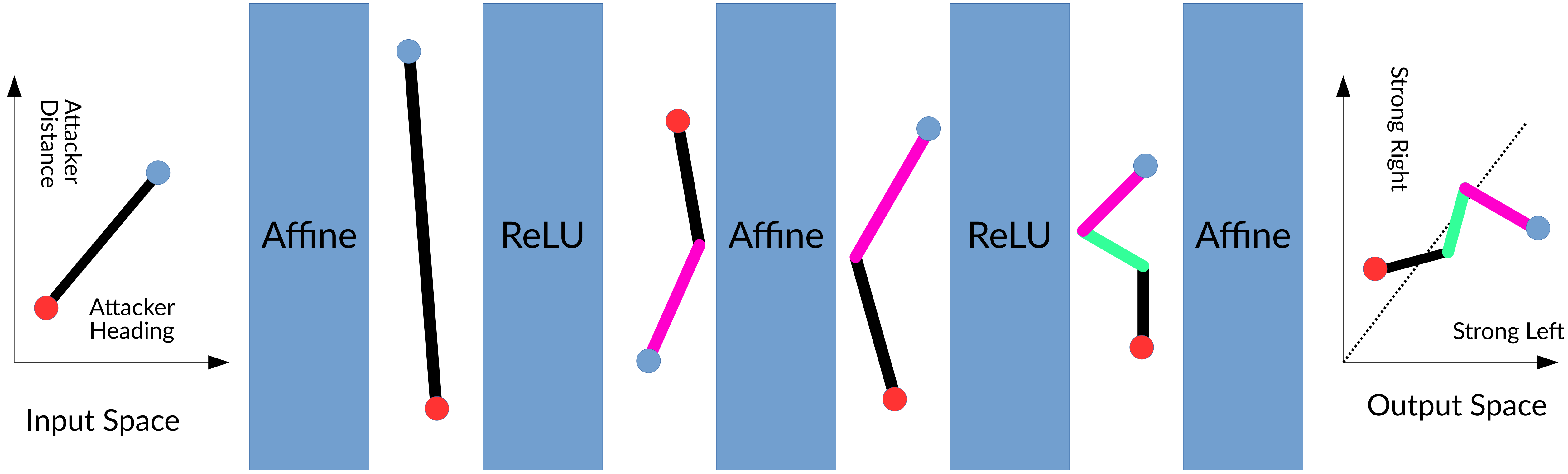
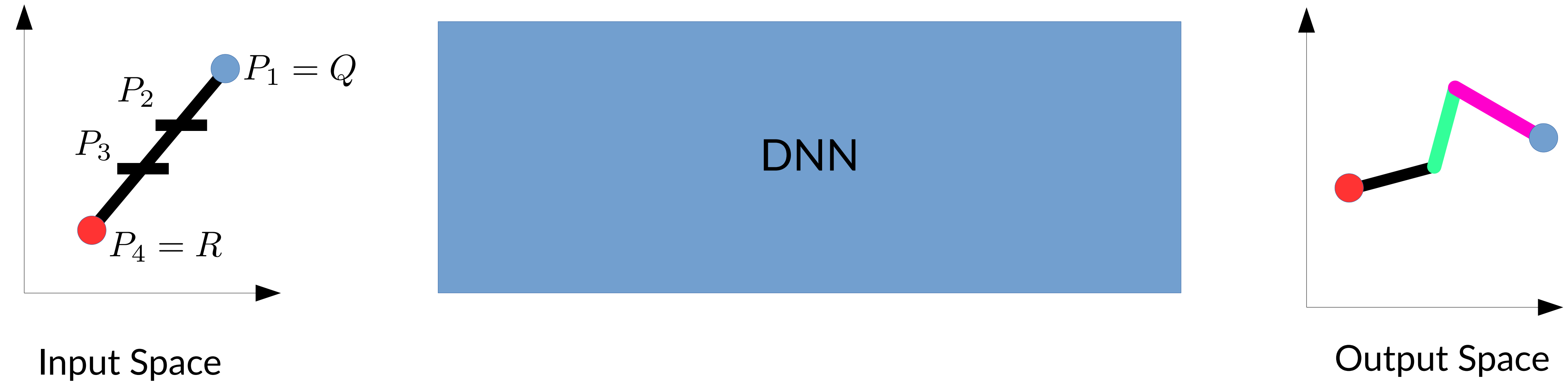



However, when analyzing the network we would like to understand its behavior over infinitely-many points, eg. a line.

# ExactLine Formal Definition

**Definition 1.** *Given a function $f : A \to B$ and line segment $\overline{QR} \subseteq A$, a tuple $(P_1, P_2, P_3, \ldots, P_n)$ is a linear partitioning of $f_{\restriction \overline{QR}}$, denoted $\mathcal{P}(f_{\restriction \overline{QR}})$ and referred to as "*ExactLine *of $f$ over $\overline{QR}$," if:*



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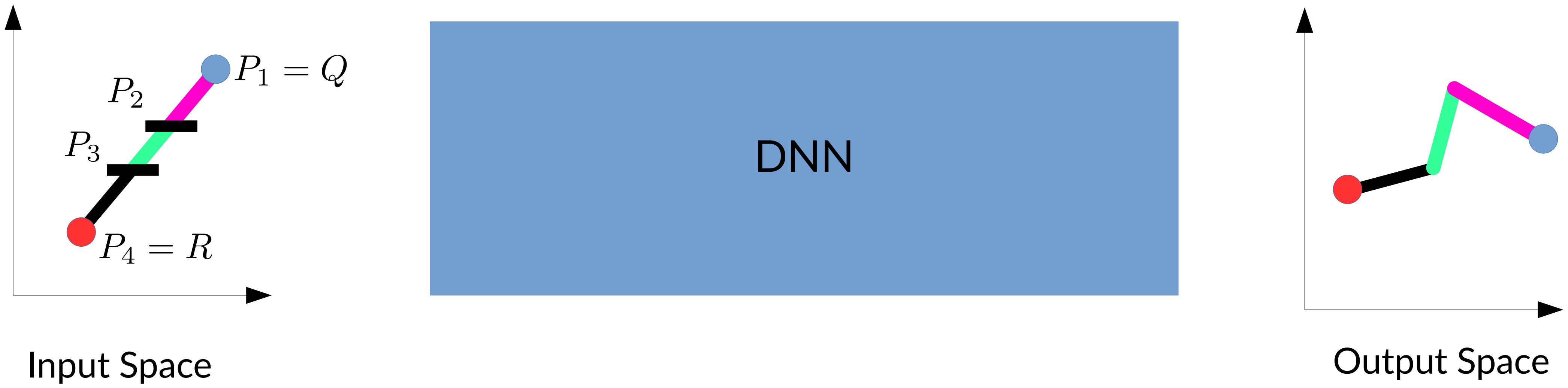

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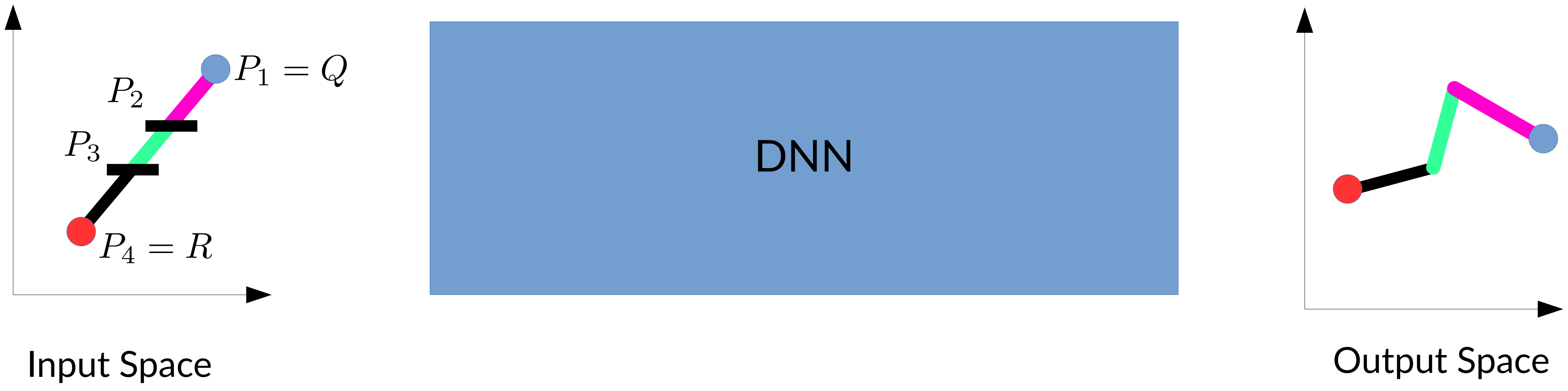

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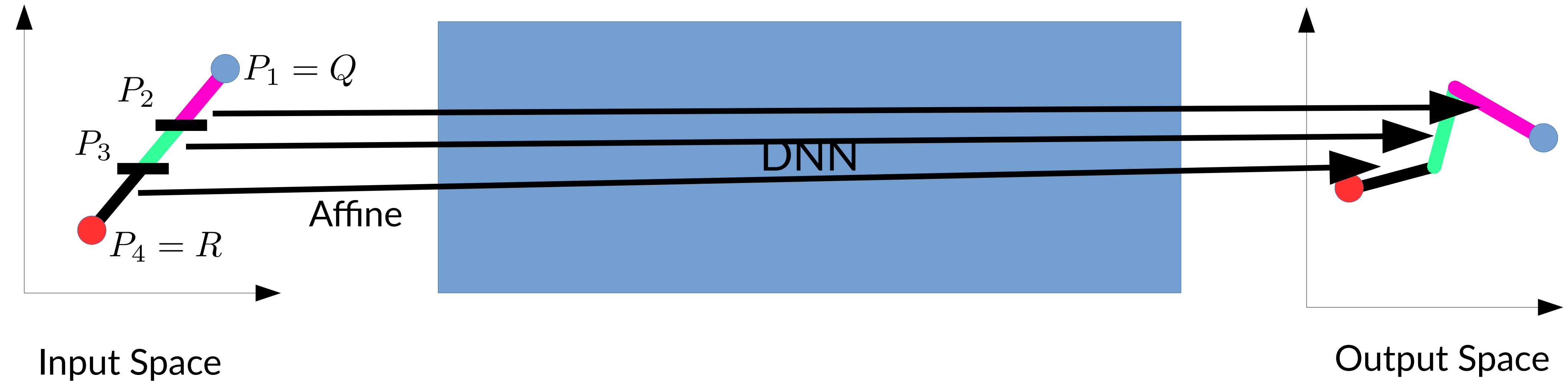
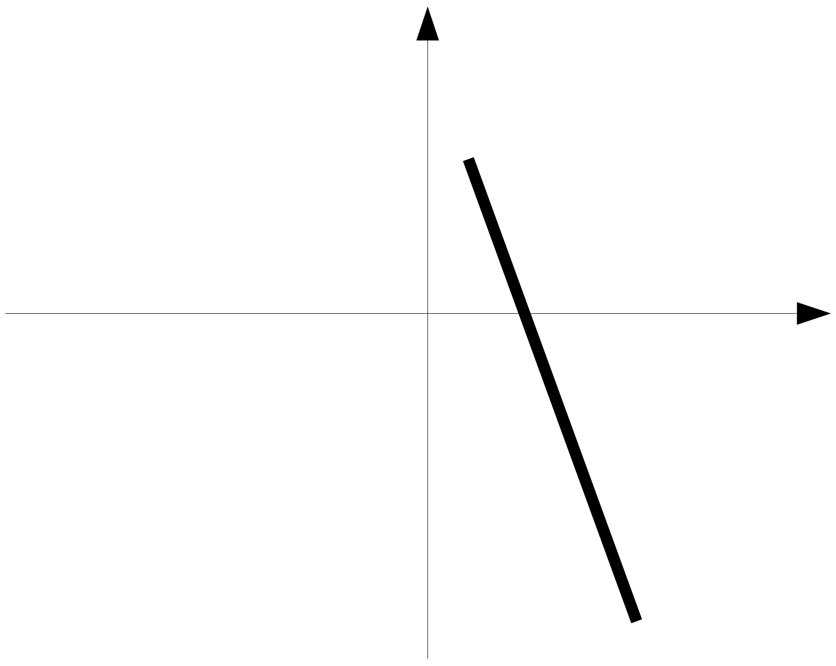


# Computing ExactLine: Single Layer




$$\mathcal{P}(f_{\restriction \overline{QR}})$$

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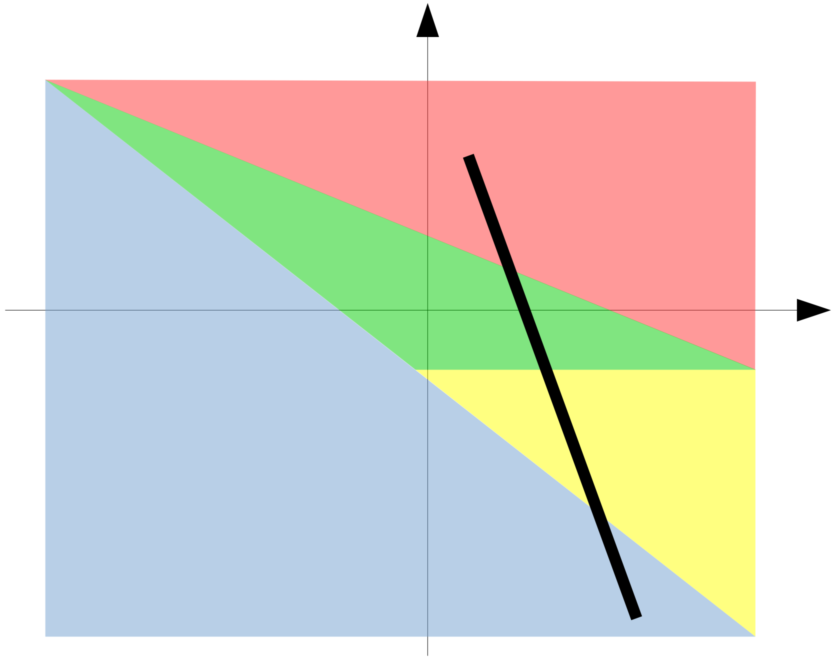

Input Space

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1. Partition input space according to PWL function.

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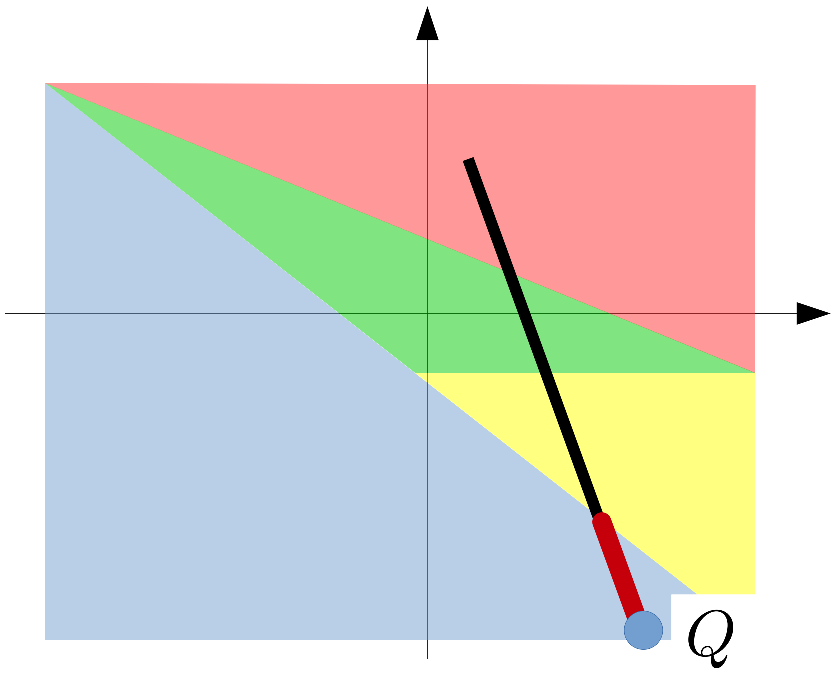


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$$\mathcal{P}(f_{\restriction \overline{QR}}) = (Q,$$

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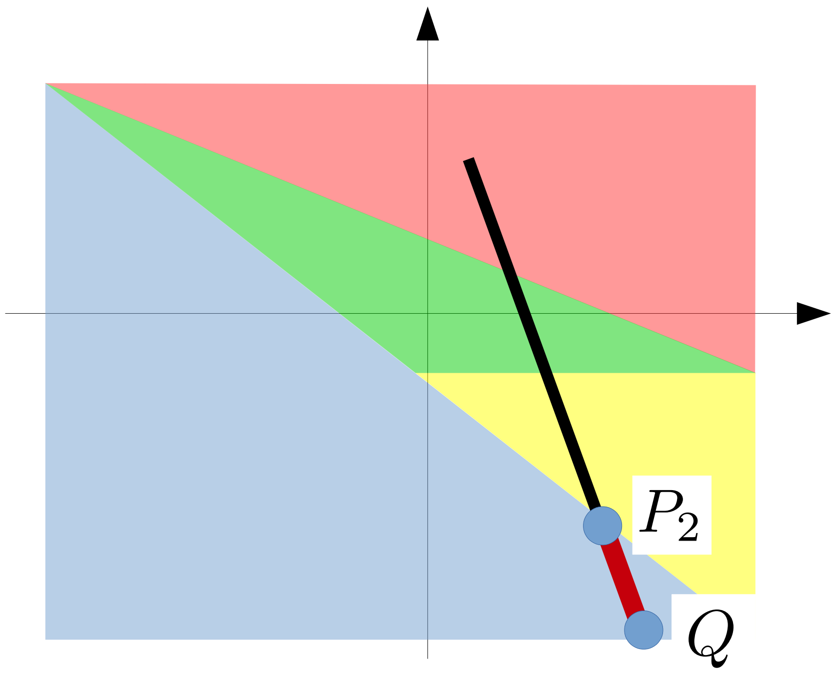


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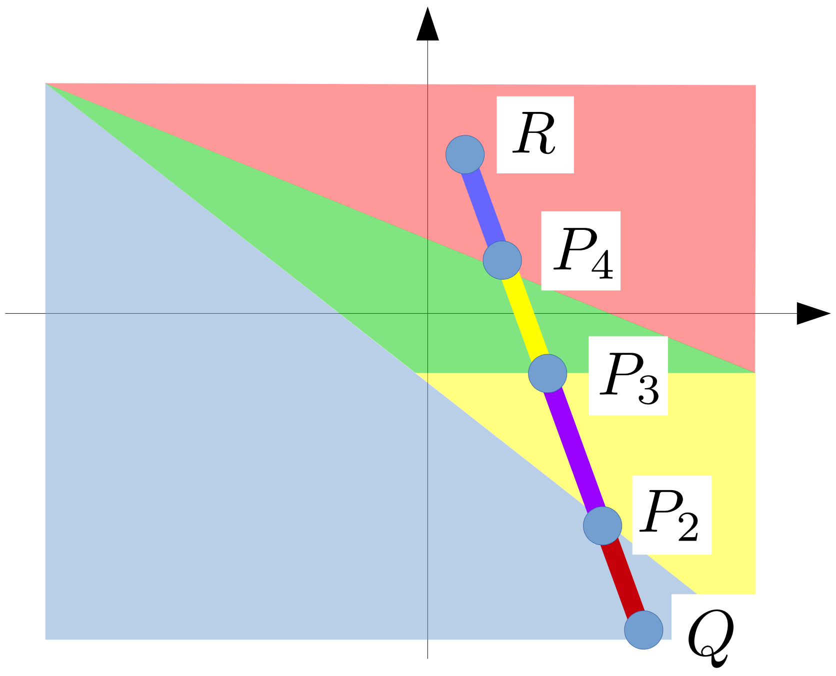


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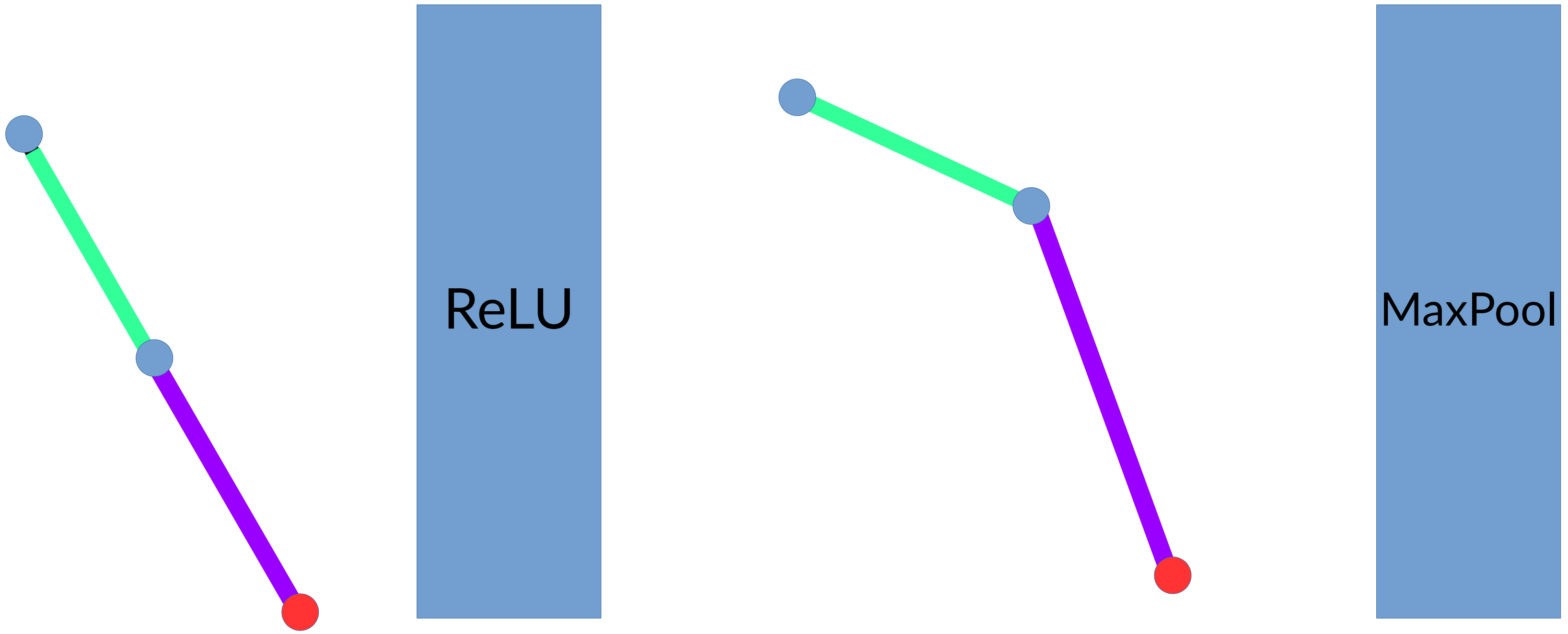


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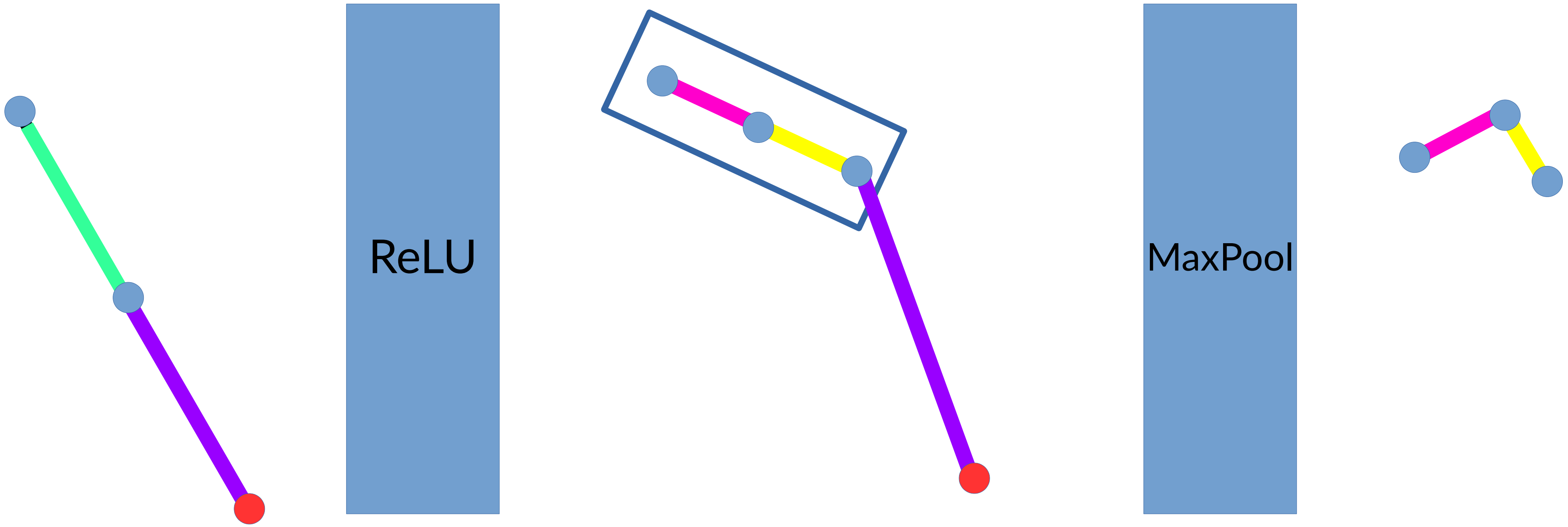


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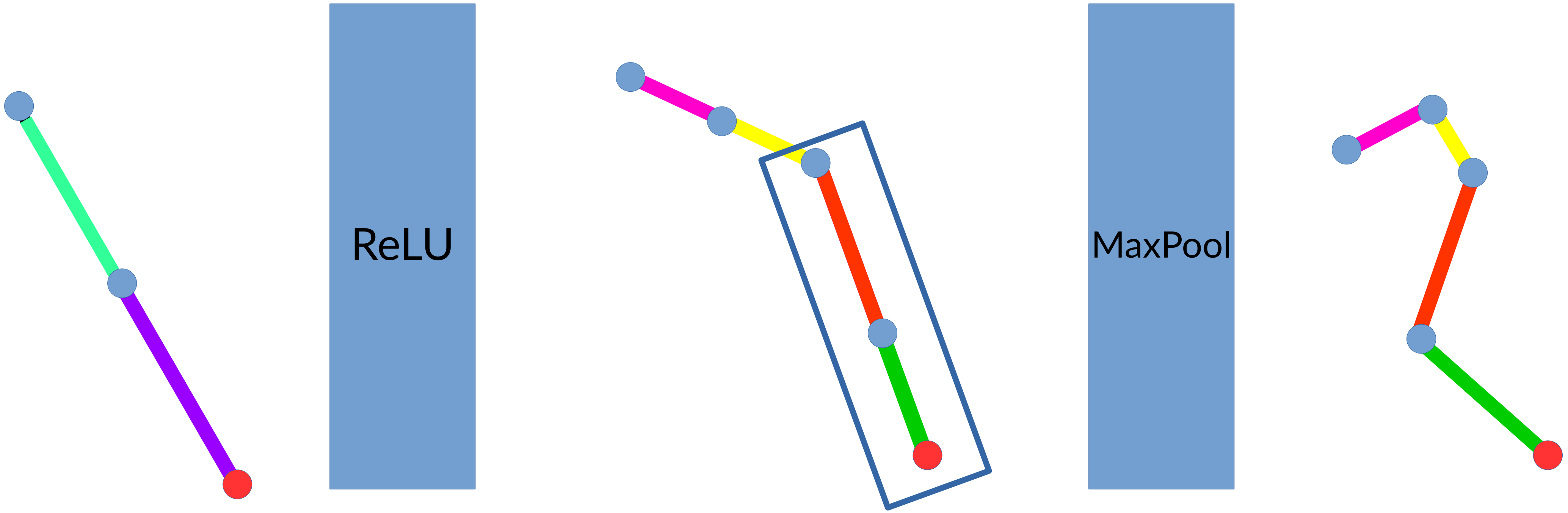


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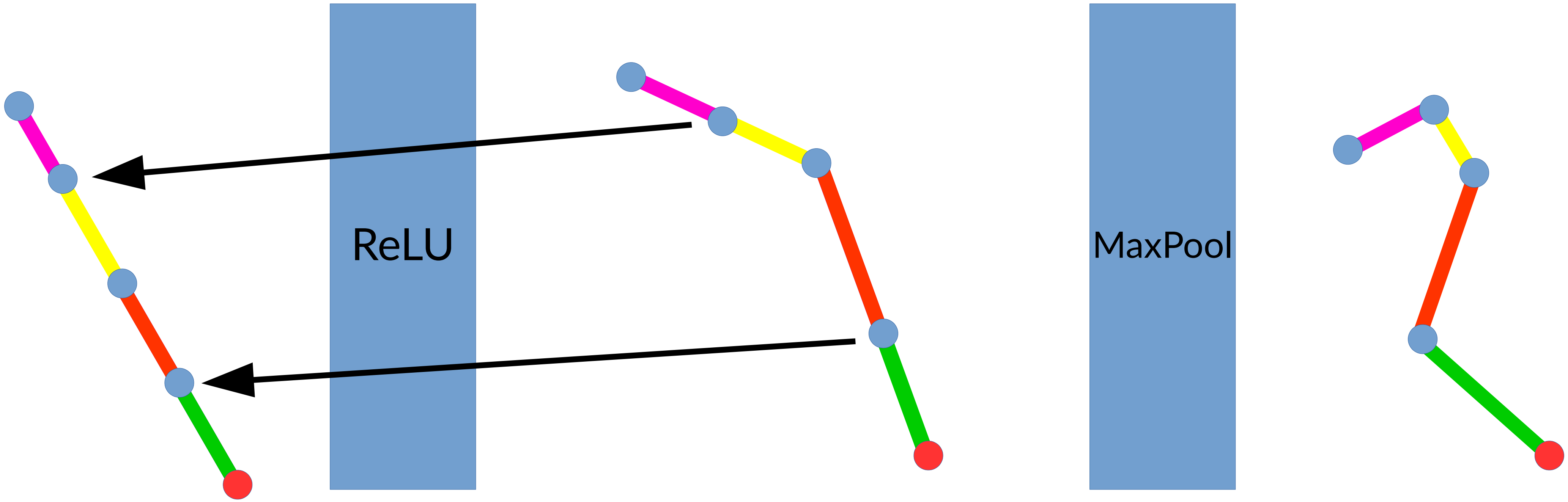
3. Project the new endpoints (and partitions) back onto the input space.

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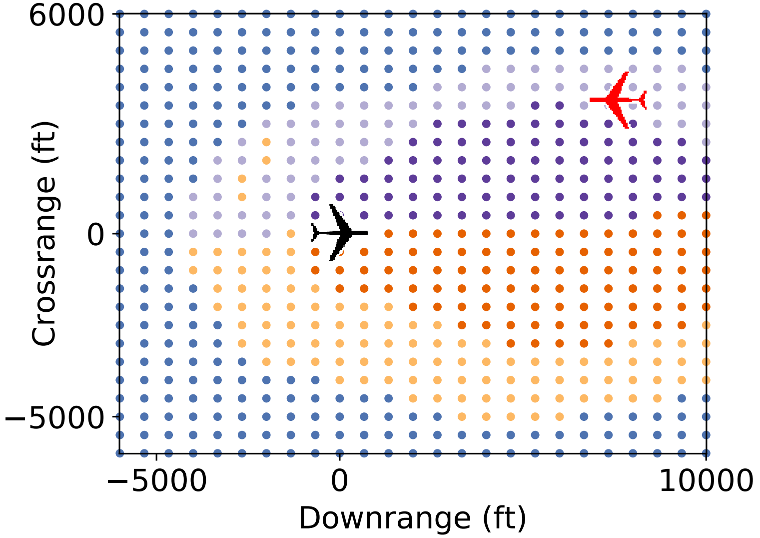
# Three Initial Applications

1. Understanding Decision Boundaries

# Visualizing ACAS Xu Decision Boundaries

ACAS Xu network:
Attacker Position (polar) → Advisory

Prior work: sampling individual points.




Legend: Clear-of-Conflict Weak Right Strong Right Strong Left Weak Left

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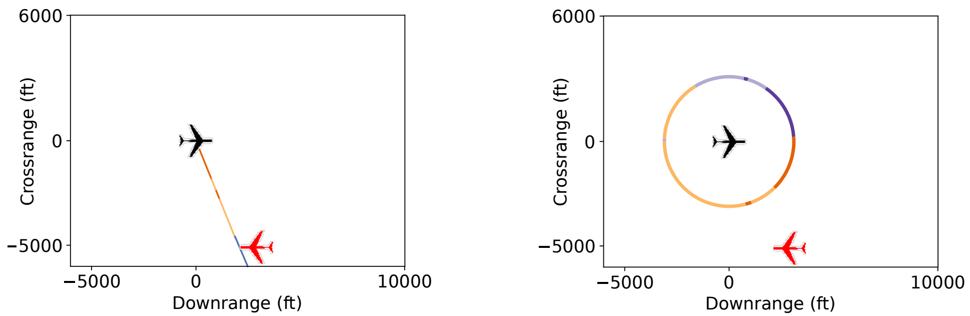


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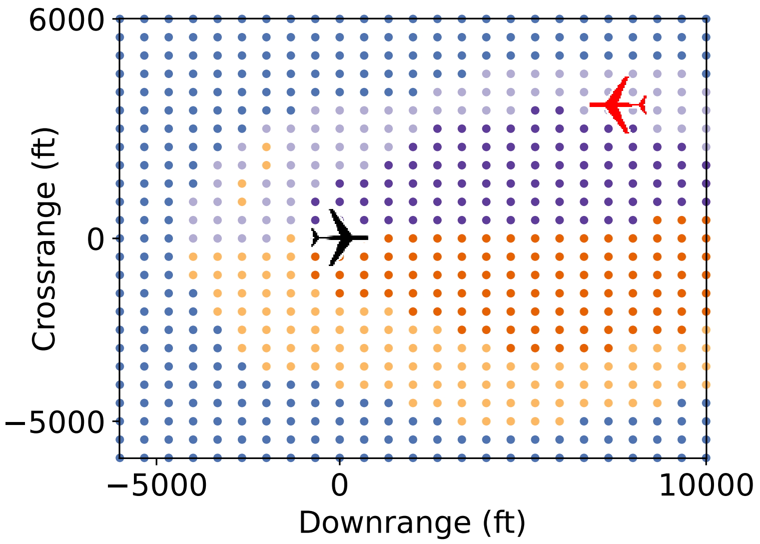


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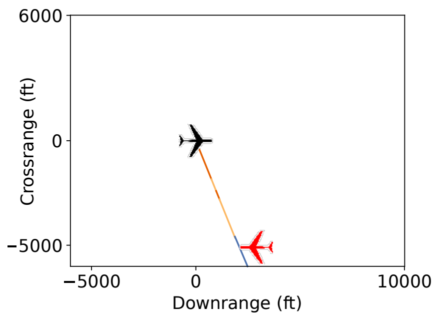


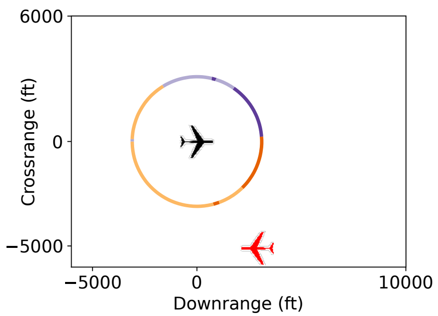


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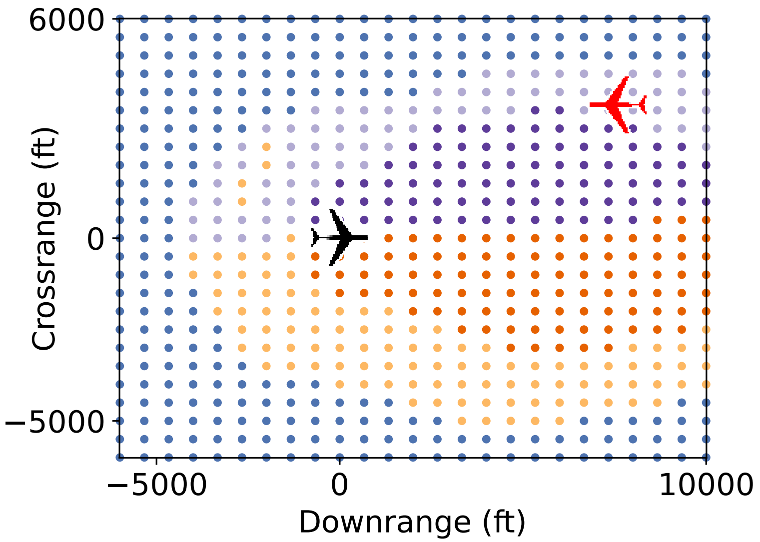


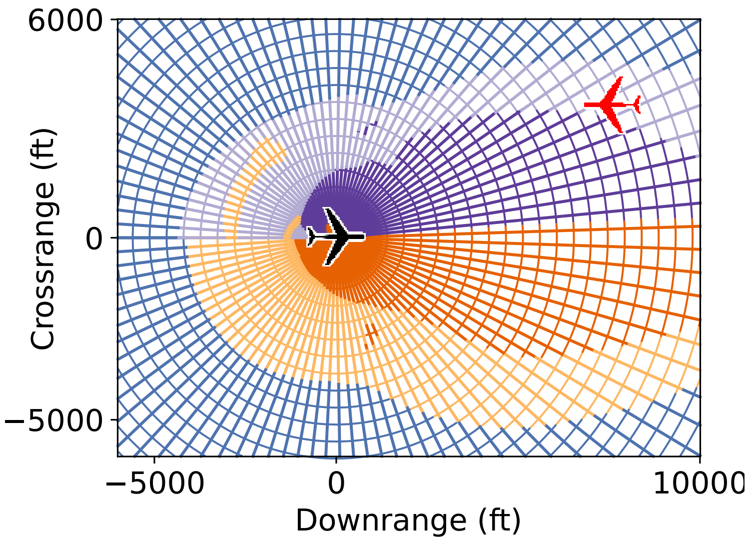


Overlapping multiple line segments creates a classification grid with much higher confidence than finite sampling.

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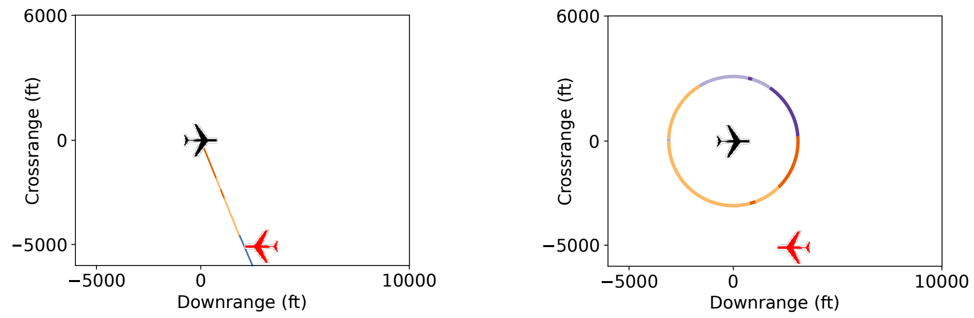


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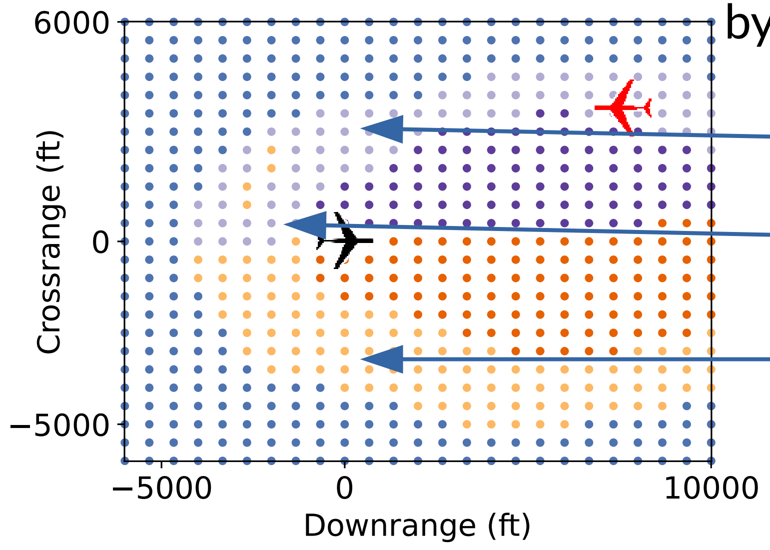


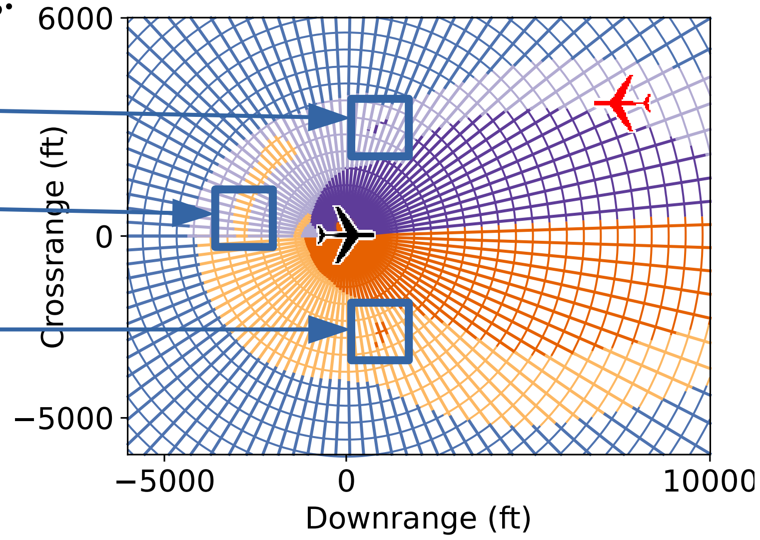


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# Three Initial Applications

1. Understanding Decision Boundaries
2. Exact Computation of Integrated Gradients Attribution Method

# Integrated Gradients

Popular DNN attribution method ("why did the network call this a fireboat?").

Relies on computing a path integral between a *baseline* and the image.

$$IG_i(x) = (x_i - x'_i) \times \int_{\alpha=0}^{1} \frac{\partial F(x' + \alpha \times (x - x'))}{\partial x_i} d\alpha$$

"Fireboat"

Black Baseline

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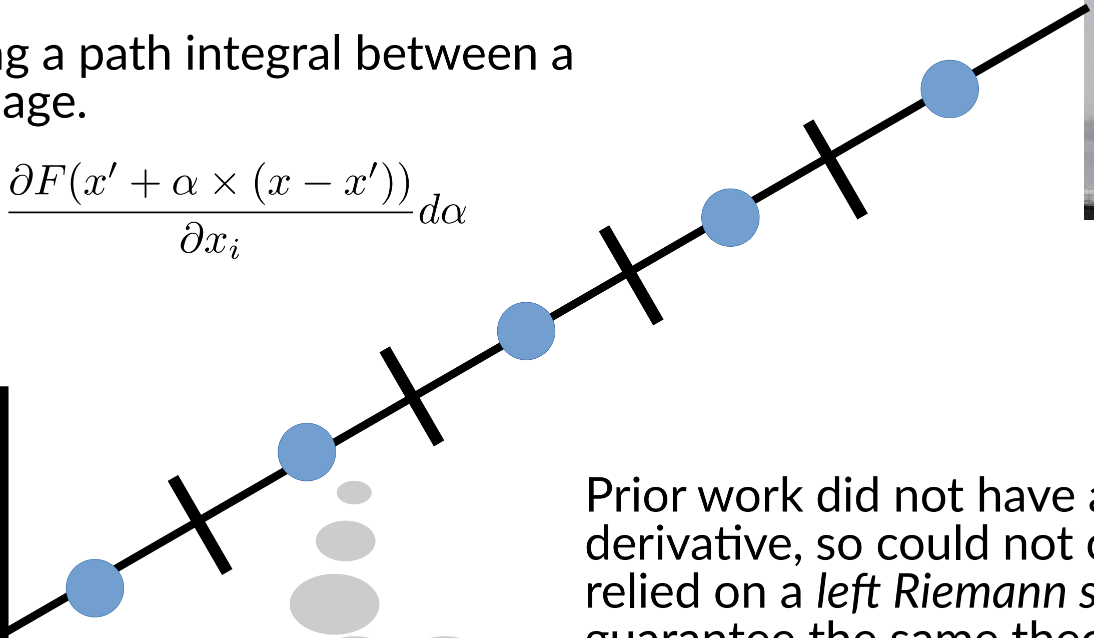

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Would be exact if gradient were constant within partitions!

Prior work did not have an analytic form for the partial derivative, so could not compute this integral. Instead, relied on a *left Riemann sum approximation* that does not guarantee the same theoretical properties.

$$\widetilde{IG}_i^m(x) = (x_i - x'_i) \times \sum_{0 \le k < m} \frac{\partial F(x' + \frac{k}{m} \times (x - x'))}{\partial x_i} \times \frac{1}{m}$$

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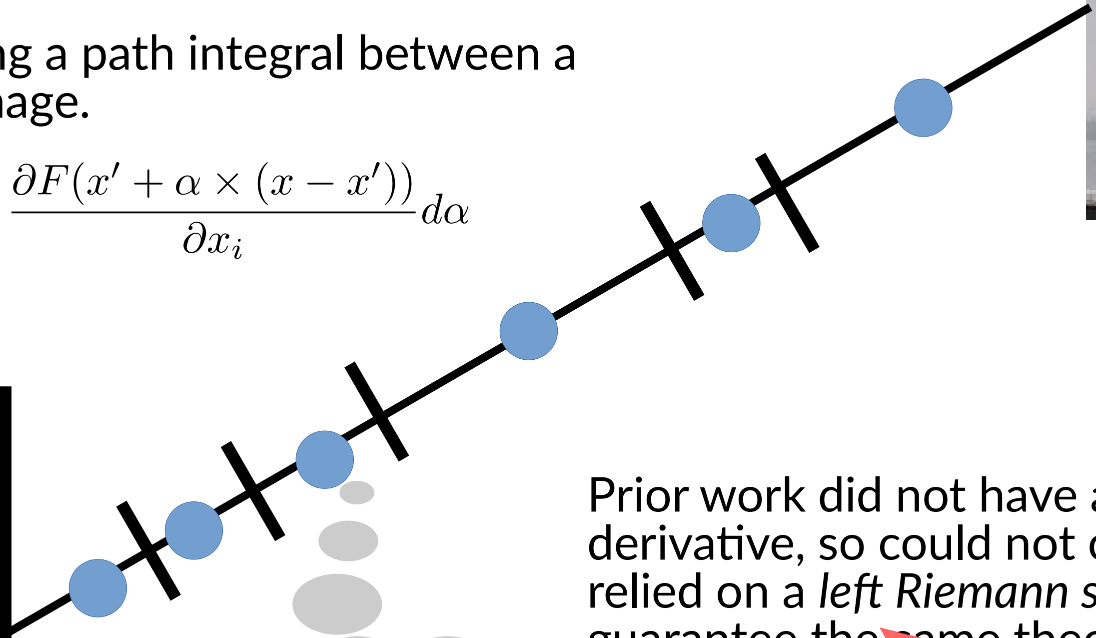

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ExactLine guarantees a partitioning with constant gradients!

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# Integrated Gradients: Results

- How accurate is the prior best-practice approximation?
  - 25-45% error
- How many samples are needed to get to 5% error?
  - Usually about 100-300
- Do different sampling methods perform better/worse?
  - Trapezoidal rule is 20-40% more sample-efficient than left/right approximations.

# Three Initial Applications

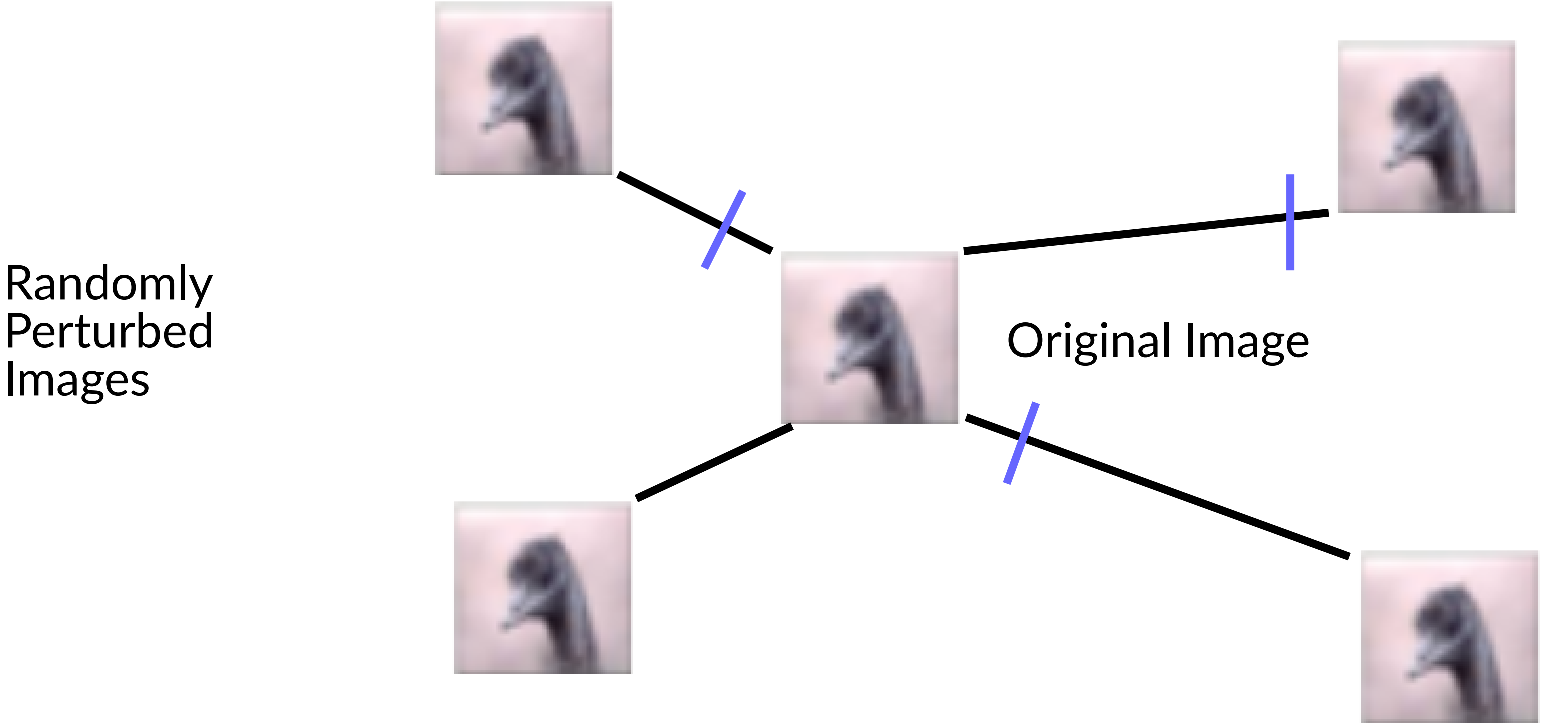
1. Understanding Decision Boundaries
2. Exact Computation of Integrated Gradients Attribution Method
3. Investigating Adversarial Examples, and Falsifying the Linearity Hypothesis

# Adversarial Examples and the Linear Explanation

- Adversarial examples: small perturbations cause big classification changes.
- Goodfellow *et al.* introduce influential "Linear Explanation"
  - **Linearity Assumption**: around 'natural' input images, the network behaves *linearly* (i.e., tangent plane at point matches output).
  - **Theoretical Claim**: classification boundaries of linear classifiers become closer with higher dimensionality.
  - **Conclusion**: adversarial examples are natural consequence of linearity hypothesis, so we need more non-linear neural networks.
- Theoretical discussion of claim, but (until now) underlying assumption untested.

# Investigating the Linearity Hypothesis

- Q1: Is the area around input points linear?
- A1: No!




We will draw blue lines to delineates different linear partitions (i.e. show where non-linearities are introduced).

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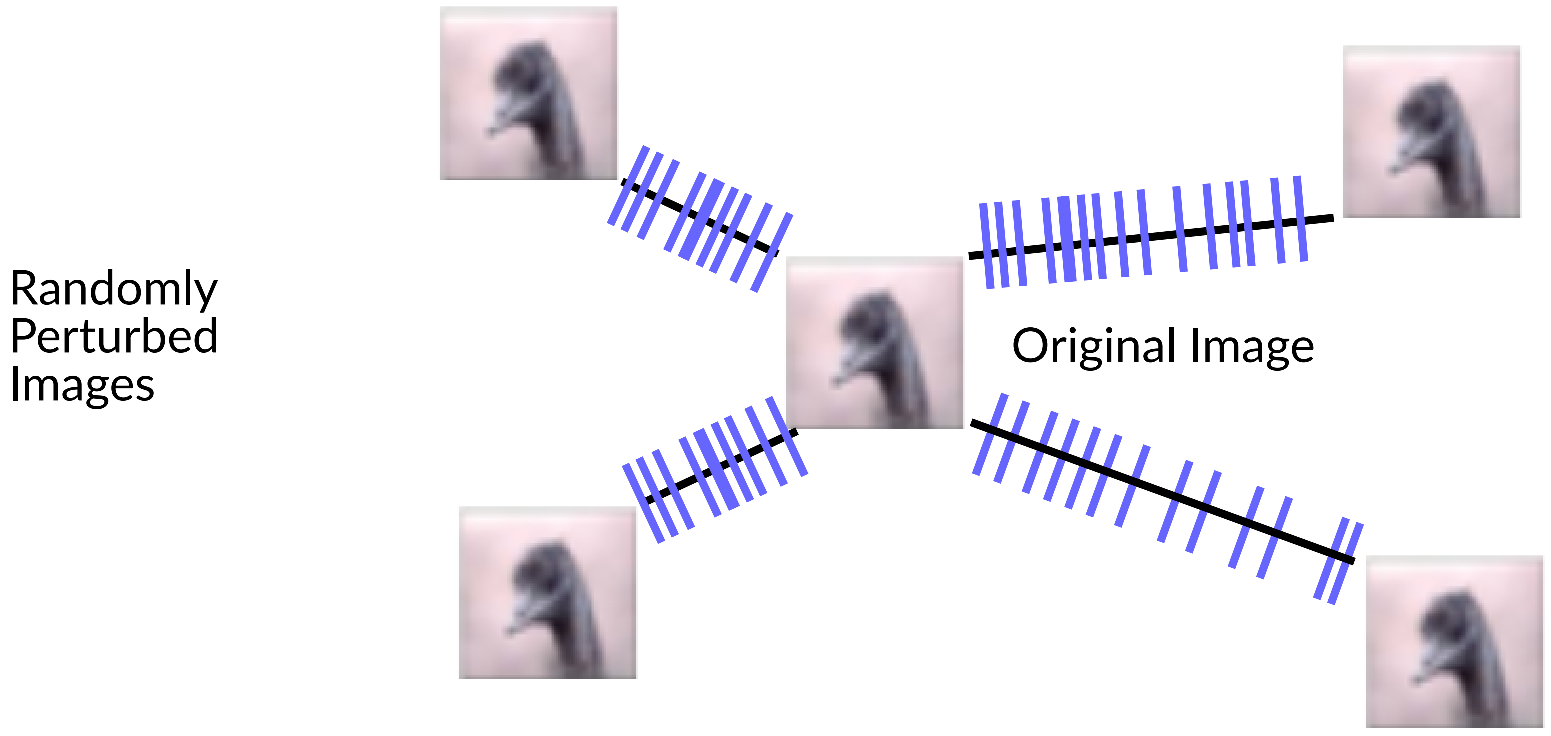


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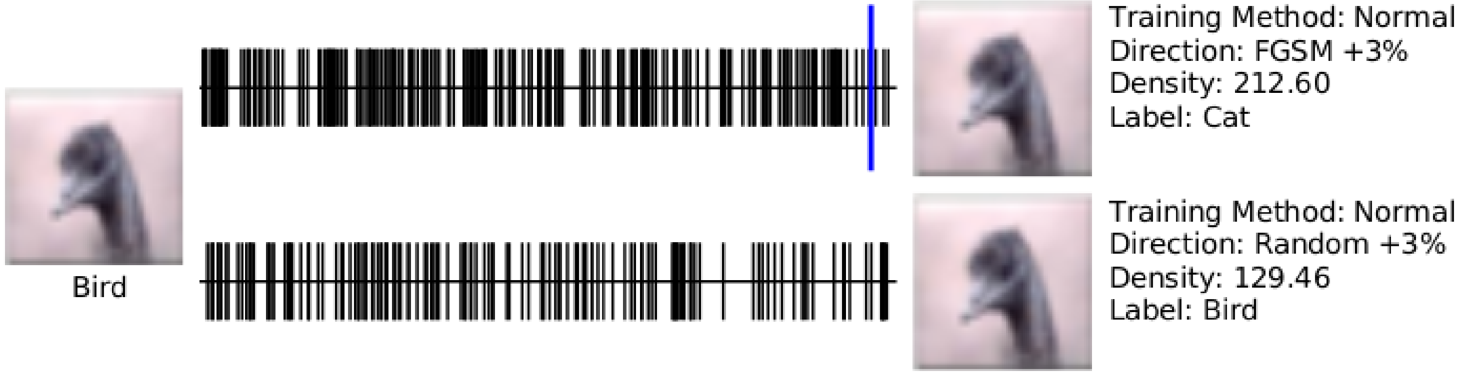
**Prediction:** network is “mostly-linear,” so should have few linear partitions.

**Reality:** network is extremely non-linear, with often thousands of different partitions.

# Investigating the Linearity Hypothesis

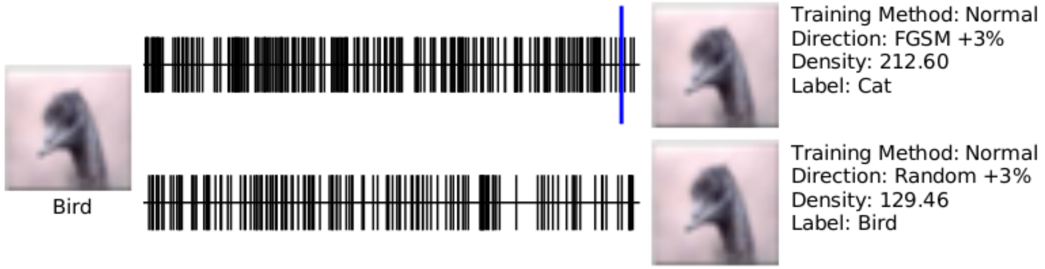
- Perhaps only the adversarial direction lies on the same linear partition.
- Q2: Are adversarial directions particularly linear?
- A2: No!




Adversarial perturbations in fact lie in a *more non-linear* direction than random perturbations.

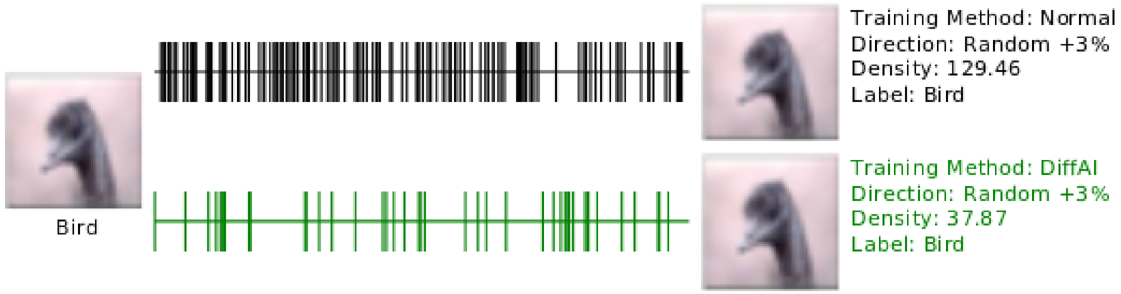
# Investigating the Linearity Hypothesis

- Perhaps the gradients in each partition are "relatively close" to that around the natural point.
- Q3: Are the gradients in each partition close to that of the natural point?
- A3: No!
- Experiment:
  - On each partition, find relative error between that partition's gradient and the gradient at the natural point.
  - Average the relative errors, weighted by width of partition.
  - Result: >250% relative error.



# Investigating the Linearity Hypothesis

- Q4: Are all models this non-linear?
- A4: Surprisingly, no! DiffAI- and PGD-trained models show less non-linearity.
- Interesting direction for future work.



# Linearity Hypothesis: Takeaways and Future Work

- Linearity Hypothesis (and surrogate assumptions) empirically falsified -> Linear Explanation rejected.
  - Need to find new explanations for adversarial examples and *tools (like ExactLine!) to empirically verify/falsify them.*
- Eg., “A Boundary Tilting Perspective on the Phenomenon of Adversarial Examples” (Tanay and Griffin):

# Conclusion

- ExactLine efficiently and precisely decomposes a neural network into affine partitions.
  - When restricted to a line in the input domain.
- Wide variety of uses, we tried three:
  - Decision boundary understanding.
  - Exact computation of IG
  - Investigating adversarial examples

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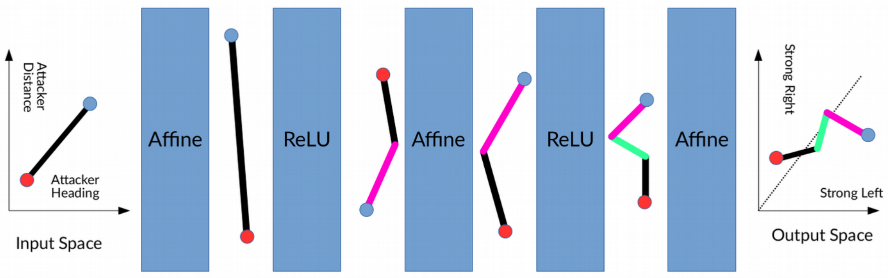


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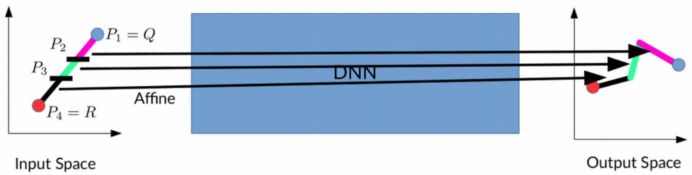


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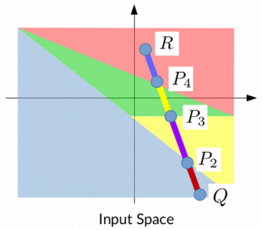


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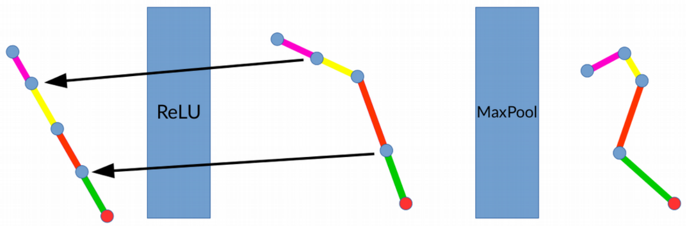


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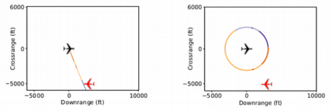

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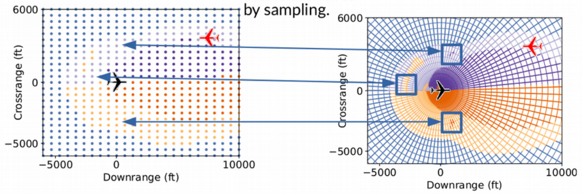


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Legend: Clear-of-Conflict, Weak Right, Strong Right, Strong Left, Weak Left

## Integrated Gradients

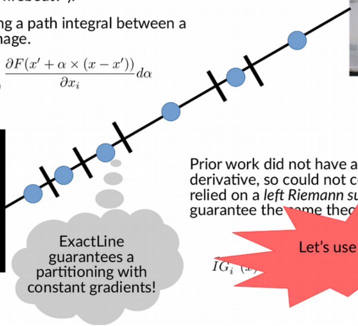
Popular DNN attribution method ("why did the network call this a fireboat?").

Relies on computing a path integral between a *baseline* and the image.

$$IG_i(x) = (x_i - x'_i) \times \int_{\alpha=0}^{1} \frac{\partial F(x' + \alpha \times (x - x'))}{\partial x_i} d\alpha$$

"Fireboat"

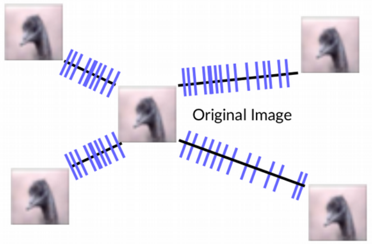
Black Baseline




Prior work did not have an analytic form for the partial derivative, so could not compute this integral. Instead, relied on a *left Riemann sum approximation* that does not guarantee the same theoretical properties.

## Investigating the Linearity Hypothesis

- Q1: Is the area around input points linear?
- A1: No!




We will draw blue lines to delineates different linear partitions (i.e. show where non-linearities are introduced).

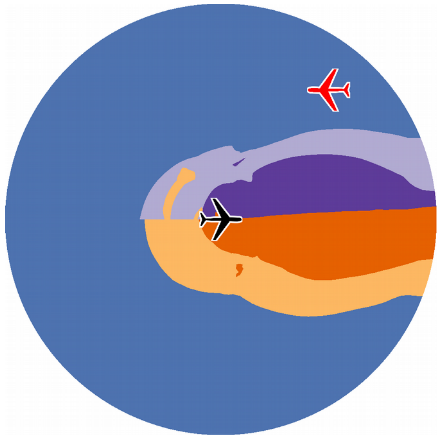
**Prediction:** network is "mostly-linear," so should have few linear partitions.

**Reality:** network is extremely non-linear, with often thousands of different partitions.

## Questions?

# ExactLine Generalization

- In a preprint, we extend ExactLine to 2-dimensional regions.
- Can understand entire decision boundary.
- Can do bounded model checking.
- Can *patch* neural networks.



# Comparison to Other 'White Box' Techniques

**Slow (NP-Hard), But Precise**

- ReluPlex
  - Decision procedure (Y/N)
  - "Is there anyone for whom the model recommends 'no approval?'"
- Linear partitioners
  - "What are *all* the people for whom the model recommends 'no approval?'"

**Fast, But Imprecise**

- ERAN
  - Decision procedure (Y/N)
  - Over-approximation (some Y are N!)
  - "Is it *possible* that there is someone for whom the model recommends 'no approval?'"
- Sampling
  - "What does the model recommend for these N people?"

# Exploring a New Dimension of Analysis

**Prior work:** Speed versus Precision

**ExactLine:** Dimensionality versus (Speed & Precise)