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Intrinsic geometry and constructivity methods for Hilbert's 6th problem

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Abstract The main mathematical work of this paper is to establish a theoretical framework based on a unique basic principle or axiom, so that the major components of theoretical physics can be constructed, and finally the redundant principles and postulates of traditional fundamental physics, as well as artificially introduced equations, can all be turned into theorems which hold automatically in the theory of this paper.

The key ideas are as following. (1)Improve the expression form of Erlangen program, and then generalize Riemannian manifold to geometric manifold. On geometric manifold, bring Riemannian geometry into the geometric framework of improved Erlangen program. (2)Strictly define the general concept of reference-system and generalize the concept of intrinsic geometry, so that the traditional intrinsic geometry based on the first fundamental form becomes a subgeometry of the intrinsic geometry of this paper. (3)Define the concept of simple connection and use it to describe those bending properties that cannot be described by Levi-Civita connection.

Other important ideas are as following. (1)Time metric is defined as the total metric of space. (2)Actual evolution direction is defined as the gradient direction of geometric quantity. (3)Gauge potential is defined as simple connection. (4)Gauge transformation is defined as the transformation of general reference-systems. (5)Energy-momentum of general charge is defined as the absolute derivative of charge tensor, and canonical energy-momentum is defined as the normal derivative. (6)Feynmann propagator and wave function are expressed as the distribution density of actual evolution direction field, which are defined as functions related to measure and become probability after normalization.

The idea of symmetry emphasized in traditional theoretical physics is more convenient to be expressed in the viewpoint of geometry. Concretely, (1)the traditional theory starts from a very large symmetry group, and reduces symmetries in the way of some kind of breaking to approach the target geometry; (2)the theory of this paper starts from the smallest symmetry group $\{e\}$, and adds symmetries in the way of some kind of symmetry conditions to approach the target geometry. These two ways must lead to the same destination. They both go towards the same specific geometry. The way of this paper has more advantages.

Based on these ideas, the concepts of charged lepton, neutrino, down-type color charge, up-type color charge and various gauge potentials are all distinguished by constructive definitions, so that the asymmetric characteristic of chirality of weak interaction, the MNS mixing of leptons and the CKM mixing of color charges hold automatically. There is no need to artificially set up these postulates like the standard model.

Keywords Erlangen program · geometric manifold · Riemannian geometry · intrinsic geometry · reference-system · simple connection · time metric · actual evolution

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0 Introduction

The purpose of Hilbert's 6th problem is to axiomatize the physics. Theoretical physics at the most basic level is an important aspect about it. The unity of the physical world has always been a belief held by many people. The history of theoretical physics is a process that the unity expands step by step.

In the 17th century, the establishment of Newtonian mechanics completed the unified description of the motion laws of macro-low-speed mass point system for the first time, which was marked by the publication of *Mathematical Principles of Natural Philosophy* by Isaac Newton in 1687.

In the 18th century, Lagrangian mechanics, which describes the evolution of mass point system in configuration space, was discovered on the basis of Newtonian mechanics. This was marked by Joseph-Louis Lagrange's publication of *Analytical Mechanics* in 1788. In 1834, William Rowan Hamilton transformed the Euler-Lagrange equation into canonical form, thus establishing Hamiltonian mechanics describing the evolution of mass point system in phase space. Later, Lagrangian mechanics and Hamiltonian mechanics evolved into two equivalent abstract theoretical frameworks for the evolution of material-motion, which can be transformed into each other by Legendre transformation.

In the 19th century, the establishment of Maxwell's electromagnetics completed the unified description of classical electromagnetic laws. It summarizes more order of the material world presented by electromagnetic phenomena, which is marked by James Clerk Maxwell's publication of *A Treatise on Electricity and Magnetism* in 1873. But what is the essential unity between mechanics and electromagnetics? In electrodynamics, the relationship between mechanics and electromagnetics can only be established by Lorentz force formula $\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$ which is obtained from experiments. As for the fundamental origin of Lorentz force, it was not clear at that time and could not be explained by electrodynamics. Lorentz force formula is regarded as a principle.

At the beginning of the 20th century, the establishment of special relativity completed the unified description of the motion laws of macro-low-speed mass point system and macro-high-speed mass point system in inertial system, and perfectly consistent with electrodynamics. It presents the order of material world in a more general form, which is marked by Albert Einstein's *On the Electrodynamics of Moving Bodies* [15] published in 1905. One obvious problem with this theory is that Newton's law of gravitation is incompatible with the mechanics of special relativity. Einstein's general relativity [16, 17], published around 1916, solved this problem. Based on the equivalence principle, a new equation of gravitational field is introduced by using Riemannian geometry as a mathematical tool, and a unified description of the mechanical laws of macroscopic mass point system and macroscopic electromagnetic field in inertial and non-inertial systems is developed. Nevertheless, the energy-momentum tensor of electromagnetic field is still based on Lorentz force formula. Although the energy-momentum tensors of the electromagnetic field and the particle are unified in form, it does not mean that the particle system and the electromagnetic field have reached the essential unity. General relativity does not answer the question of the intrinsic relationship between electromagnetic field and particle, but avoid it. The problem is still concealed in the energy-momentum tensor that unified in form.

Since the 1920s, the establishment of quantum mechanics [4–6, 10, 11, 55–60] has correctly described the motion laws of micro-low-speed mass point system in inertial system. Subsequently, the establishment of relativistic quantum mechanics meant that the motion laws of micro-high-speed mass point system in inertial system was also taken into account. Prior to this, physical theory was based on mass point as a model of physical reality. Beginning with relativistic quantum mechanics, there are more and more signs that something called a "field", which reflects the spatial distribution of physical properties, has a greater advantage as a model of physical reality. For example, the original quantum mechanics cannot explain the physical process of particle number change, light quantum, and negative energy state. By treating relativistic wave equation as field equation and wave function as linear operator, these problems can be solved successfully by acting on more abstract wave function. Combined with renormalization, quantum electrodynamics and quantum field theory were developed finally. In 1948, Richard Feynman proposed the path integral theory [20], which revealed the essence of quantum mechanics from a deeper perspective, and finally improved the quantum field theory. The combination of quantum field theory and Yang-Mills theory [70] proposed by Chen Ning Yang and Robert Mills in 1954 eventually led to Glashow-Weinberg-Salam's unified theory of weak electricity [18, 27, 36, 39–41, 43, 54, 63], quantum chromodynamics [3, 7, 19, 22, 23, 25, 29–31, 53, 62] and various great unified theories [12, 21, 24, 47–49].

Although quantum field theory is the most successful theory to describe the motion laws of microscopic material systems in inertial system up to now, it fails to incorporate the motion laws in noninertial system in a coordinated manner.

In recent years, based on the framework of quantum field theory, Yue-Liang Wu [65–69] described gravitational field by the expression of *locally flat noncoordinate gravifield spacetime* on *globally flat Minkowski spacetime of coordinates*, according to the local equivalence between noninertial system and gravitational field. And then the gravitational field is regarded as a quantum field in the *globally flat Minkowski spacetime of coordinates*. Thus, the unified description of the motion laws of gravitational field and other quantum fields in inertial system is developed, and the motion laws in noninertial system are reflected and explained equivalently, which promotes the development of quantum field theory. What is more noteworthy is that it already contains some more abundant geometric contents than traditional quantum field theory, and has the idea of using these geometric contents to achieve a unified field. Although the meaning of geometry in these literatures is not clear enough, it contains very positive things, and the gravitational field is quantized, which is an important breakthrough in quantum field theory.

Nevertheless, quantum field theory still fails to fully describe the unified structure of matter. Specifically, quantum field theory and the standard model of particle physics can only recognize that particle systems and interaction fields are so-called "fields", and distinguish them into "particle fields" and "gauge fields", but fail to go deep into the root of the unity between them. And it has not been fully explained what kind of inherent relations exist among various particle fields.

Early Kaluza-Klein theory [42, 45, 46] and later string theory as well as superstring theory attempted to provide a unified explanation of this problem in high dimensional space. However, Kaluza-Klein theory transplants the gauge potential forcibly into the metric. It does seems that the gravitational field equation and the gauge field equation can be obtained in a unified way, but the gauge potential and the metric are not the same thing after all. It must be inconsistent to force them together. String theory and superstring theory are also nice attempts, but their representative views [2, 13, 26, 28, 32–35, 37, 38, 52, 61, 64] still cannot be seen as a success.

The details of the above theories will not be discussed here. What should be emphasized is that high dimensional space will be considered in a different way from Kaluza-Klein theory, string theory and superstring theory. A new approach will be used to describe the essence of the unity of various kinds of matter-motion and to explain the inherent relations among various particle fields.

In order to achieve the unity of physical theories at the most basic level, this paper will start from a unique basic principle, and strictly deduce the framework of theoretical physics with constructive mathematical theory, and at the most basic level turn the redundant principles and postulates, as well as artificially introduced equations, into theorems.

The main difficulties of researching Hilbert's 6th problem of theoretical physics at the most basic level have the following aspects.

1. Ontological reality and epistemological concept are not explicitly divided in traditional theories of physics.

They usually do not specify clearly whether a terminology refers to an ontological reality or an epistemological concept, but offten mix ontological reality and epistemological concept together into the meaning of a terminology. Such a practice does not make convenience to axiomatization.

In the theory of this paper, ontological reality and epistemological concept will always be distinguished explicitly, the way of which is very simple, that is, the vast majority of discussions in this paper will just focus on strictly defined epistemological concepts and carry out strict mathematical deduction. Ontological realities will just be mentioned in the discussions concerning physical laws and in the intuitive descriptions connecting with traditional theory of physics.

2. The framework of evolution dynamics in traditional theory of physics is abstract and lacks concrete constructivity.

There are two approaches to develop mathematical theory, one is the approach of concrete constructivity based on set theory, the other is the approach of abstract structure based on category theory. Although the effectiveness of these two research approaches is the same, without either of them, the cognition to this mathematical intuition is not complete.

For example, consider the concept of real number. From the approach of abstract structure, some conventions as the connotation of abstract structure are combined to form axiomatized definitions of real number field, i.e. complete archimedean total ordered field. From the approach of concrete constructivity, natural numbers are constructed from empty set, then integers and rational numbers are constructed, and then irrational numbers are constructed from Dedekind cut to form the real number set. The two concepts defined in two approaches of abstract structure and concrete constructivity respectively reflect the same mathematical intuition. Such two theories of real number provide a complete cognition for the concept of real number.

The framework of evolution dynamics in traditional physics has two equivalent forms, which are Hamilton form and Lagrange form. Based on the above viewpoint, it can be noticed that both Hamiltonian function and Lagrangian function are abstract objects, and there is no any specific connotation given in the sense of concrete constructivity in traditional theory. Even if the certain expression of Lagrangian function is written, the various field functions composing this expression are still abstract. On one hand, the traditional theory describes gauge field with abstract concept of connection on a fibre bundle, without defining the concrete content of the connection. On the other hand, the spinor field, which is composed of several complex-valued functions, is sometimes used to refer to a charged lepton field, and sometimes a neutrino field. It is not clear in traditional theory that how to distinguish field functions by concrete constructivity between charged lepton field and neutrino field, both of which equally satisfy Dirac equation. Therefore, the concepts of such field functions are indeed abstract. So the evolution dynamics of traditional theory is not complete in theory and lacks content of concrete constructivity.

This paper will give a way of concrete constructivity to distinguish concepts such as charged lepton and neutrino, so as to supplement the achievements of traditional theory.

3. The understanding and application of the concept of geometry in traditional theory of physics are not enough.

(1) In 1872, Felix Klein proposed the famous Erlangen program. Based on the idea of Erlangen program, starting from the second half of the 20th century, theoretical physics began to emphasize the notion of *symmetry* and research it with the concept of group extensively. It is right, but easy to cause a kind of misunderstanding, that is, symmetry and group are regarded as equivalent things.

In fact, the essential idea of symmetry is the invariance under transformations, rather than the relationship between transformations. The former is a geometric property, and the latter is an algebraic property. The properties of group just exactly belong to the latter. Geometric property and algebraic property are two opposite and unified aspects about the concept of transformation. Therefore, symmetry and group should not be confused.

The essential connotation of symmetry is geometric property, however traditional theory does not clarify it completely and clearly. In order to solve this problem, the expression form of Erlangen program will be improved in this paper.

(2) In 19th century, Friedrich Gauss and Bernhard Riemann developed intrinsic geometry, which is a great achievement . This theory of intrinsic geometry researches the geometric properties determined by the first fundamental form on manifold. However, in the development process of general relativity, there are indications that such a theory of intrinsic geometry is still not complete.

The traditional intrinsic geometry will be generalized in this paper, based on the improved Erlangen program. This is an indispensable step for researching Hilbert's 6th problem of theoretical physics at the most basic level.

In order to understand the concept to be defined in this paper more conveniently, the intuition of intrinsic geometry must be concisely explained here in another way.

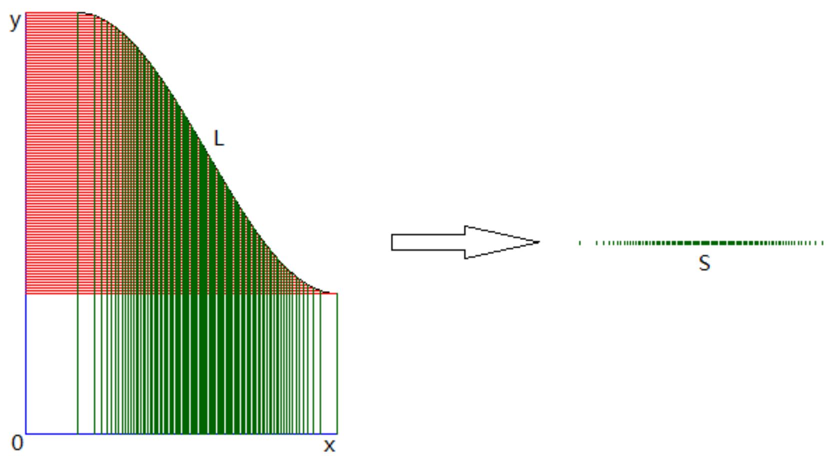

Fig. 1 The intuition of intrinsic geometry of curve

First, consider the case of one-dimension, that is, the intuition of intrinsic geometry of curve. As shown in Figure 1. Select a curve L in the plane rectangular coordinate system. Project the coordinates of the y axis onto the curve L continuously and uniformly, and then onto the x axis.

In this way, the original continuous and uniform coordinate distribution becomes a continuous and ununiform distribution through the medium of curve L, forming an interval S with some continuous and ununiform distribution shown in the right figure of Figure 1. This is actually the intuition of intrinsic geometry of curve L. It can be said that curve S is curve L in intrinsic geometry.

This intuition of intrinsic geometry can be described strictly in the following way.

Let S be a one-dimensional manifold, which is homeomorphic to an Euclidean straight line. Take two coordinate cards (S, x) and (S, y) on S to satisfy the coordinate relation $y = y(x)$.

As shown in the figure above, near every point on S, it shows a kind of intuition reflecting the degree of slackness and tightness of coordinate distribution of y axis in x axis. This degree of slackness and tightness can be strictly described by $\frac{dy}{dx}$. Then the one-dimensional manifold S given the degree of slackness and tightness $\frac{dy}{dx}$ is the curve L defined in the way of intrinsic geometry.

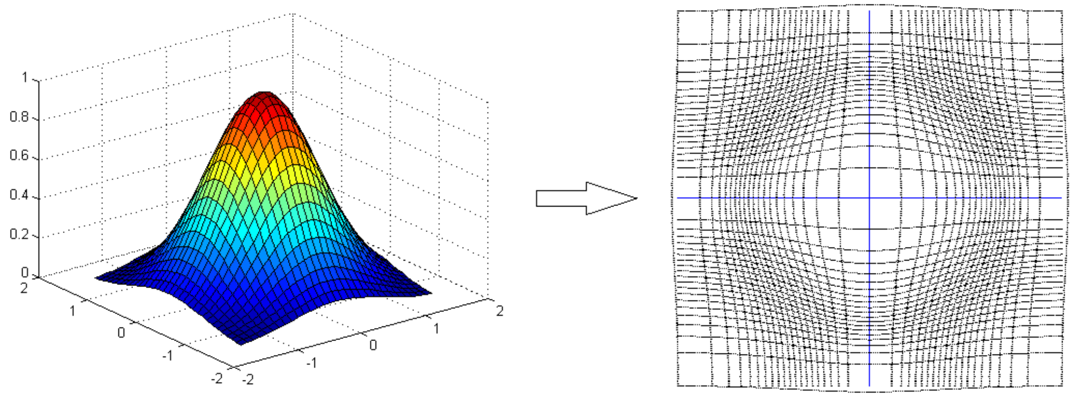

Fig. 2 The intuition of intrinsic geometry of surface

The case of two-dimensional surface is similar. The intuition of intrinsic geometry of the surface in the left figure of Figure 2 can be shown by the degree of slackness and tightness of coordinate network $\left(u^1, u^2\right)$ in coordinate system $\left(x^1, x^2\right)$ at each point in the right figure of Figure 2. This degree of slackness and tightness can be strictly described by $\frac{\partial u^k}{\partial x^i}$ $(i, k = 1, 2)$. It can be said that the degree of slackness and tightness $\frac{\partial u^k}{\partial x^i}$ of the right figure defines the surface of the left figure in the way of intrinsic geometry.

This is an intuitive description of the two simple cases about one and two dimensions, emphasizing the central role of the degree of slackness and tightness $\frac{\partial u^k}{\partial x^i}$ determined by two coordinate systems in reflecting the intuition of intrinsic geometry. In this way, the general concept of intrinsic geometry will be defined strictly in this paper.

The above discussions summarize some important problems existing in traditional theories, and at the same time introduce some viewpoints, thoughts and basic principles of constructing theories to be adopted by the theory described in this paper.

In the following sections, the improved expression of Erlangen program, the mathematical foundation of theoretical physics and the various forms of matter-motion will be discussed strictly. They will form a layer-by-layer dependent and inseparable whole.

1 Improved expression of Erlangen program

The purpose of this paper is to establish the mathematical foundation of theoretical physics at the most basic level by constructivity method. For this purpose, the expression form of Erlangen program [44] must be improved firstly.

The original idea of this improvement has been referred to in literatures [14, 50], but they have not expressed this idea as a strict definition of the general concept of geometry in an explicit form. Such a definition is given below.

Definition 1.1. Let C be a set and $\sim$ a relation of equivalence. The classification $C/\sim$ is called a **geometry** about $\sim$ on C. If any subset of the relation of equivalence $\sim$ constitutes a transformation, it is called an **equivalent transformation**. The whole of all equivalent transformations about $\sim$ on C is called an **equivalent transformation set** about $\sim$ on C.

The equivalent transformation set T and the relation of equivalence $\sim$ are mutually determined. In fact for any relation of equivalence $\sim$, $T \triangleq \{f|(f \subseteq\sim) \wedge (f \text{ is a transformation})\}$, whereas $\forall T \subseteq \{f|(f : C \to C) \wedge (f$ is a transformation$)\}$ the relation of equivalence can be defined as $\sim$: $\forall f \in T,\ (a, b) \in f \Leftrightarrow a \sim b$. So the geometry about $\sim$ on C can be called the geometry about T on C, and $C/\sim$ can be denoted by C/T.

Each equivalence class $[c]$ in geometry C/T is called a **geometric object** about the equivalent transformation set T. Each element in equivalence class $[c]$ is called a **geometric instance** of the geometric object $[c]$. Denote $S \triangleq \bigcup\limits_{c\in C} c$, each element in S is called a **point**, each subset of S is called a **geometric figure**, and (S, T) is called a kind of **geometric theory**.

Let H be a set. If the mapping $h : C \to H$ satisfies $\forall c_1, c_2 \in C,\ c_1 \sim c_2 \Leftrightarrow h(c_1) = h(c_2)$, h is called a **geometric property** of the geometric instances on C. The mapping $\tilde{h} : C/\sim\to H,\ [c] \mapsto h(c)$ induced by h about $\sim$ is called a **geometric property** of the geometric object on C. Each of h and $\tilde{h}$ is called a **geometric property** on C. The image of h and $\tilde{h}$ in H is called the **value** of geometric property.

Suppose there are two relations of equivalence $\sim_a$ and $\sim_b$ on C. If $\sim_a\subset\sim_b$, the relation of equivalence $\sim_a$ is called **stronger** than $\sim_b$, and $\sim_b$ **weaker** than $\sim_a$. In this case it must be true that $\forall[c] \in C/\sim_a,\ \exists[d] \in C/\sim_b$, such that $[c] \subset [d]$. Thus, the classification $C/\sim_a$ is called **more exquisite** than $C/\sim_b$, and the classification $C/\sim_b$ **more rough** than $C/\sim_a$. The geometry $C/\sim_a$ is called **larger** than $C/\sim_b$, and the geometry $C/\sim_b$ **smaller** than $C/\sim_a$. More conveniently $C/\sim_b$ is called a **subgeometry** of $C/\sim_a$.

Remark 1.1.

(1) The above definition is equivalent to the traditional expression of Erlangen program. Fundamentally, the significance of geometry is that it can characterize the specific essence at a specific level. The geometric property is the property reflecting the fundamental difference between one class and another by different values.

(2) The above definition does not follow the traditional form of Erlangen program. Concretely, it adopts relation of equivalence, rather than group, to characterize geometry.

Why a new definition should be adopted? It is because that in the case where some group is difficult to expressed in an explicit form due to its complicated or uncertain structure, it is very inconvenient to describe geometry in the traditional form of Erlangen program. But the above definition in this paper is more convenient for the later application in such cases.

In the past, Erlangen program was used to deal with groups with simple structure. The corresponding geometry was confined to either local of the manifold or homogeneous manifold such as constant curvature manifold. The Riemannian geometry was not brought into the framework of Erlangen program in traditional way. However, based on the expression form in this paper, the definition of geometry can bring Riemannian geometry into the framework of improved Erlangen program completely, and is more convenient for building the foundation of theoretical physics later. It is further discussed in section 2.2.1.3 .

To say the least, if the group structure has to be emphasized, the following additional definition is needed.

The elements in equivalent transformation set T about $\sim$ on C naturally imply a group structure about composite operation of mappings. The group T is called the **tranformation group of geometry** C/T, and C/T is called the **geometry of group** T. Therefore, the group structure exists on the equivalent transformation set naturally, and it is not necessary to make explicit requirements in the definition of geometry as the traditional form of Erlangen program.

Suppose transformation group T_1 acting on S_1 and transformation group T_2 acting on S_2 are isomorphic. (S_1, T_1) and (S_2, T_2) are called **the same kind of geometric theory**. If T_1 is a proper subgroup of T_2, T_1 is called **smaller** than T_2, and T_2 **larger** than T_1. Obviously, the smaller the group, the larger the geometry; conversely, the larger the group, the smaller the geometry.

Now, define another useful concept.

Definition 1.2. On any set C, there must be a special geometry, which has only one equivalence class, that is C itself. This geometry is called a **universal geometry**. The set C is the only geometric object in universal geometry, and it is called a **universal geometric object**. Each geometric property in universal geometry is called a **universal geometric property**, and also called a **geometric invariance** on C. Each universal geometric property with its unique value is called a **geometric identity** on C.

2 Construction of mathematical foundation of theoretical physics

2.1 Axiom for Hilbert's 6th problem

In this paper, physical contents at the most basic level are attributed to a unique physical principle. Except this, the vast majority of the following discussions are mathematical deductions.

The basic principle of theoretical physics: physical reality in ontology is cognized by using the concept of reference-system in epistemology.

This can also be regarded as the unique axiom for Hilbert's 6th problem about theoretical physics at the most basic level.The following section will start discussions from the strict definition of the concept of reference-system.

2.2 Reference-system and geometric manifold

2.2.1 Definition of reference-system

Definition 2.2.1.1. Let M be a $\mathfrak{D}$-dimensional connected orientable smooth real manifold. $\forall p \in M$, let U be a neighborhood of p. For any two C^∞-compatible coordinate charts (V, φ_V) and (W, φ_W) containing U, each of the homeomorphic mappings $\varphi_V : V \to \mathbb{R}^{\mathfrak{D}}$ and $\varphi_W : W \to \mathbb{R}^{\mathfrak{D}}$ is called a **coordinate mapping** on the coordinate chart. If $\varphi_V|_U = \varphi_W|_U$, then by this condition, a relation of equivalence $\varphi_V \sim \varphi_W$ between coordinate mappings can be defined. The equivalence class φ_U determined by this relation of equivalence is called a **coordinate frame** on neighborhood U of point p.

For convenience, $\forall q \in U$, denote $\varphi_U(q) \triangleq \varphi_V|_U(q)$, $\varphi_U^{-1}(x) \triangleq \varphi_V|_U^{-1}(x)$.

For any two coordinate frames φ_U and ψ_U on neighborhood U of point p, if $f_p \triangleq \varphi_U^{-1} \circ \psi_U$ is a smooth homeomorphism between nonvoid open sets $\varphi_U(U)$ and $\psi_U(U)$ in $\mathbb{R}^{\mathfrak{D}}$, f_p is called a **(local) reference-system** on neighborhood U of point p, where ψ_U is the **basis coordinate frame** of the reference-system f_p, and φ_U is called the **performance coordinate frame** of the reference-system f_p.

For any reference-systems f_p and g_p at point p, if the coordinate frames of f_p and the coordinate frames of g_p are C^∞-compatible, the reference-systems f_p and g_p are called **C^∞-compatible**. The whole of the reference-systems that are C^∞-compatible on neighborhood U of point p is called a **reference-system space** on neighborhood U of point p, and denoted by $REF_p(U)$ or REF_p. The whole of all the reference-systems with ψ_U as the basis coordinate frame is denoted by $REF_p(U, \psi_U)$.

Denote $REF \triangleq \bigcup\limits_{p \in M} REF_p$, where $\forall p, q \in M$ the elements in REF_p and REF_q are C^∞-compatible.

If the mapping $f : M \to REF,\ p \mapsto f(p) \in REF_p$ satisfies that the slack-tights B_M^A and C_A^M in definition 2.2.8.1 are all smooth real functions on manifold M, the mapping f is called a **reference-system** on manifold M. The whole of all reference-systems on manifold M is denoted by REF_M.

Definition 2.2.1.2. Let there be two reference-systems f and g on manifold M, if $\forall p \in M$, the reference-systems $f(p) \triangleq \varphi_U^{-1} \circ \psi_U$ and $g(p) \triangleq \varphi_U^{-1} \circ \rho_U$ on neighborhood U of point p have the same performance coordinate frame φ_U, namely it can be intuitively expressed as chart $\psi_U(U) \xleftarrow{f(p)} \varphi_U(U) \xrightarrow{g(p)} \rho_U(U)$, we say reference-systems f and g on manifold M **motion relatively and interact mutually**.

Remark 2.2.1.1. According to the definition above, it is obvious that f and g are motioning relatively and interacting mutually if and only if g and f are motioning relatively and interacting mutually. In the reference-system of classical spacetime defined in section 5.2.1 , according to section 1.2 , it can naturally induce a generalized Newton's third law.

Definition 2.2.1.3. $\forall p \in M$, on neighborhood U of point p, any $\psi_U^{-1} \circ \rho_U \in REF_p(U)$ can induce a transformation $F_{\psi_U^{-1} \circ \rho_U} : REF_p(U, \psi_U) \to REF_p(U, \rho_U),\ \varphi_U^{-1} \circ \psi_U \mapsto (\varphi_U^{-1} \circ \psi_U) \circ (\psi_U^{-1} \circ \rho_U) = \varphi_U^{-1} \circ \rho_U$. $F_{\psi_U^{-1} \circ \rho_U}$ is called the **reference-system transformation** from $\varphi_U^{-1} \circ \psi_U$ to $\varphi_U^{-1} \circ \rho_U$ induced by reference-system $\psi_U^{-1} \circ \rho_U$ on neighborhood U of point p.

$\forall f \in REF_M$, $\forall p \in M$, let $F_{f(p)}$ be a reference-system transformation induced by reference-system $f(p)$ on neighborhood U of point p. The mapping $F_f : p \mapsto F_f(p) \triangleq F_{f(p)}$ is called a **reference-system transformation** on manifold M.

Remark 2.2.1.2. Suppose there is a reference-system f on manifold M. Construct reference-system e in the following way: $\forall p \in M$, on neighborhood U of point p, take the basis coordinate frame of f as the basis coordinate frame of $e(p)$, and take the same basis coordinate frame of f as the performance coordinate frame of $e(p)$. Reference-system e is defined as completely stationary reference-system in section 2.5.2.8 . Thus, reference-system transformation F_f transforms e to f just right. For convenience, it always means reference-system transformation F_f when saying reference-system transformation f.

Definition 2.2.1.4. A differential manifold M with a reference-system f is called a **geometric manifold** given shape by f, and denoted by (M, f).

Remark 2.2.1.3. The geometry determined by general concept of differential homeomorphism is somehow rough on the intuition. There is only intuition of differential topological structure on general differential manifold, and is no any intuition of concrete shape. When giving a reference-system, the differential manifold would obtain a kind of concrete shape. The curve in Figure 1 and the surface in Figure 2 of introduction section are two simple and visualizable examples of geometric manifold. The shape of geometric manifold is completely determined by reference-system. The following will define the geometry concerning concrete shape.

2.2.2 Intrinsic geometry

Definition 2.2.2.1. Inherent geometry of reference-system.

(1) Inherent geometry of local reference-system.

$\forall f_p, g_p \in REF_p(U)$, let slack-tights (see Definition 2.2.8.1) of f_p and g_p be $(b_f)^A_M$ and $(b_g)^A_M$ respectively.

Define a relation of equivalence $\cong$ of reference-systems on $REF_p(U)$, such that $f_p \cong g_p$ if and only if $(b_f)^A_M = (b_g)^A_M$ are all true at each point of the neighborhood U of point p. Thus, the geometry $REF_p(U)/\cong$ is called the **inherent geometry** on the reference-system space $REF_p(U)$. The geometric object $[f_p]$ in the inherent geometry $REF_p(U)/\cong$ is called the **inherence** of reference-system f_p.

(2) Inherent geometry of reference-system on manifold.

$\forall f, g \in REF_M$, let slack-tights (see Definition 2.2.8.1) of f and g be $(B_f)^A_M$ and $(B_g)^A_M$ respectively.

Define a relation of equivalence $\equiv$ on REF_M, such that $f \equiv g$ if and only if $f(q) \cong g(q)$ is true on the neighborhood of each point q on manifold M. Thus, the geometry $REF_M/\equiv$ is called the **strict inherent geometry** on reference-system space REF_M.

Define a relation of equivalence $\cong$ on REF_M, such that $f \cong g$ if and only if $(B_f)^A_M = (B_g)^A_M$ are all true at each point of manifold M. Thus, the geometry $REF_M/\cong$ is called the **inherent geometry** on reference-system space REF_M. The geometric object $[f]$ in the inherent geometry $REF_M/\cong$ is called the **inherence** of reference-system f. Because of the one to one correspondence between mappings $[f] \triangleq [p \mapsto f(p)]$ and $p \mapsto [f(p)]$, the inherence of f can also be expressed as $[f] : p \mapsto [f(p)]$.

(3) Each geometric property on inherent geometry is called an **inherent geometric property**. Slack-tight is the most basic inherent geometric property on the reference-system space.

Definition 2.2.2.2. Intrinsic geometry on geometric manifold.

The whole of all the geometric manifolds on differential manifold M is denoted by $\mathcal{M}(M)$. The relation of equivalence $\cong$ of reference-systems induces a relation of equivalence of geometric manifolds, such that $(M, f) \cong (M, g)$ if and only if $f \cong g$. The classification $\mathcal{C}(M) \triangleq \mathcal{M}(M)/\cong$ determined by this relation of equivalence on $\mathcal{M}(M)$ is called the **intrinsic geometry** on geometric manifolds. Each equivalence class $(M, [f])$ is called an **intrinsic geometric manifold** given shape by $[f]$.

Each geometric property on intrinsic geometry is called an **intrinsic geometric property** on geometric manifolds. According to Definition .1, the value of each intrinsic geometric property completely depends on the inherence of reference-system, and thereby depends on the slack-tight B^A_M or C^M_A of reference-system.

Discussion 2.2.2.1. Geometric manifold is a more basic expression than Riemannian manifold.

According to the viewpoint of Riemannian geometry, the ultimate origin of its geometric property is metric. According to the viewpoint of geometric manifold, the geometric property has more basic origin, which ultimately boils down to reference-system and its slack-tight B^A_M or C^M_A.

(1) In history, the slack-tight is called a semimetric in traditional theory of Riemannian geometry. Physicists noticed long ago that when researching interactions between gravitational field and elementary particles, especially problems about spinor field, it can only be described by adopting semimetric representation, and it does not work by using metric representation. However, they did not realize that it means the connotation of traditional intrinsic geometry needs to be generalized.

(2) On connotation. On one hand, it can be seen from Definition 2.2.8.4 that the slack-tight on geometric manifold determines the metric on Riemannian manifold. On the other hand, even if the coefficients of metric tensors of two geometric manifolds are completely the same, their slack-tights are not necessary to be the same. These two aspects indicate that the theory of intrinsic geometry on geometric manifold has richer geometric properties than the traditional theory of intrinsic geometry on Riemannian manifold.

(3) In addition, the expression form of geometric manifold is more convenient for deducing the fundermental framework of theoretical physics with a uniform foundation. For example, the concept in Definition 2.2.1.2 and various concepts of typical gauge fields defined later can be elegantly expressed with the form of geometric manifold theory, but it is difficult to do so with the form of Riemannian manifold theory. In Discussion 2.2.2.3 , the two expressions of intrinsic geometry are compared further more.

In view of the above reasons, in the following text, the essence of Riemannian geometry will always be expressed based on the viewpoint of geometric manifold.

Definition 2.2.2.3. Inherent transformation of reference-systems.

(1) Inherent transformation of local reference-systems. Let f_p be a reference-system at point p on manifold M. The inherence $[f_p]$ of f_p induces an equivalence class $[F_{f_p}]$ of reference-system transformation F_{f_p}. $[F_{f_p}]$ is called an **inherent transformation** of reference-systems at point p.

(2) Inherent transformation of reference-systems on manifold. Let $f : p \mapsto f(p)$ and $h : p \mapsto h(p)$ be two reference-systems on manifold M. The inherence $[f]$ of f induces an inherent transformation of reference-systems $[F_{f_p}] : h(p) \mapsto h \circ f_h(p)$ in each local of manifold M, where $f_h \in [f]$ and f_h has the same basis coordinate frame with h at any point p. Denote $h \circ f_h$ with $h \circ [f]$, there exists a transformation $[F_{f(p)}] : h(p) \mapsto h \circ [f](p)$.

Thus on the manifold there exists a transformation $[F_f] : h \mapsto h \circ [f]$. Further more, there exists a transformation $F_{[f]} : [h] \mapsto [h] \circ [f] \triangleq [h \circ [f]]$.

$F_{[f]}$ is called an **inherent transformation** of reference-systems on M, or an **intrinsic transformation** of geometric manifolds, or a **transformation of general gauge fields**. Correspondingly, any inherence $[h]$ can also be called a **general gauge field**.

Discussion 2.2.2.2. Transformation group of reference-systems.

(1) Locally, on the neighborhood of any point p on manifold M, the slack-tights B^A_M or C^M_A of a reference-system constitute a $\mathfrak{D}$-order invertible square matrix. The inherent geometry $REF_p(U)/\cong$ is isomorphic to the general linear group $GL(\mathfrak{D}, \mathbb{R})$.

(2) On manifold M, an inherent transformation tansforms an intrinsic geometric manifold to another intrinsic geometric manifold. The group structure of the inherent geometry $REF_M/\cong$ is $GL(M) \triangleq \bigotimes_{p\in M} GL(\mathfrak{D}, \mathbb{R})_p$.

Suppose S is a subgroup of $GL(M)$. The group structure of S is generally complicated and its description must be cumbersome, it is because transformation groups at various points are generally different from each other. This is an important reason why at the beginning Riemannian geometry was not brought into the framework of Erlangen program. According to the original form of Erlangen program, geometry depends on group, that is to say, if group structure does not described clearly, geometry could not be established.

However, that is easy to be described if it is based on the concept of geometry of this paper. Just like Definition 2.2.2.1 and Definition 2.2.2.2 , in order to research a geometry on manifold, it just needs to construct a kind of relation of equivalence about reference-systems, or take some constraints for slack-tights, and it is not necessary to specify detail informations of transformation group. This will bring convenience for the research of the following sections. There is a further discussion in Remark 7.2.2 .

Specially, for the case that the transformation groups at different points of manifold are isomorphic to each other, the following definition is needed.

Definition 2.2.2.4. The general linear group $GL(\mathfrak{D}, \mathbb{R})$ is also called the **intrinsic transformation group**, or **transformation group of general gauge fields**.

Let f be a reference-system on manifold M, and let S be a subgroup of $GL(\mathfrak{D}, \mathbb{R})$. In the sense of isomorphism of groups, for any reference-system f, if the following two conditions are satisfied: (1) $\forall p \in$

M, $[f(p)] \in S$; (2) for any subgroup T of S, $\exists q \in M$, $[f(q)] \notin T$; then objects determined by f, such as f, $(M, [f])$, $F_{[f]}$, etc., are all called **generated by group** S.

As the equivalent transformation set, the whole of all intrinsic geometric transformations generated by group S is used to define the relation of equivalence $\sim_S$, so as to define the geometry $\mathcal{M}(M)/\sim_S$, which is called the **geometry generated by group** S, also denoted by $\mathcal{M}(M)/S$.

Discussion 2.2.2.3. Some important viewpoints must be emphasized:

(1) The comparison of two expressions of intrinsic geometry.

Consider Definition 2.2.8.4 , let G_{AB} be the metric tensor coefficients about basis coordinate frames on manifold M. It is because of the fixed conditions $G_{AB} = \Delta_{AB}$ about basis coordinate frames, that the metric tensor coefficients $G_{MN} = G_{AB}B_M^A B_N^B$ about performance coordinate frames can describe intrinsic geometric properties.

This is like fixing the coordinate of the first endpoint of a segment to zero on the real axis, and the coordinate of the other endpoint can describe the length of the segment. The coordinate of the fixed endpoint has no decisive influence on the length of the segment. Even if the coordinate origin of real axis is moved away from the first endpoint, the length of the segment remains unchanged, except that the coordinate expression of the length changes from the coordinate value of the other endpoint to the difference between the coordinate values of the two endpoints.

In the same way, when the slack-tights B_M^A of reference-system remain unchanged, if the base metric G_{AB} does not be selected as the fixing Δ_{AB}, the expressions of metrics G_{AB} and G_{MN} may have changes, but there is no influence on intrinsic geometry at all.

In the case of fixing the base endpoint, it is certainly correct and feasible to define the length of the segment by the ONE coordinate value of the other endpoint, but it fits in more precisely with the essence of geometric property to define the length of the segment by the difference between the coordinate values of the TWO endpoints.

In this sense, it is certainly correct and feasible to define the intrinsic geometry by the first fundamental form about ONE coordinate frame on Riemannian manifold, but it fits in more precisely with the essence of geometric property to define the intrinsic geometry by reference-system reflecting the relative relationship of TWO coordinate frames on geometric manifold, and the connotation is more comprehensive.

(2) The selection of torsion connection and torsion-free connection.

When describing intrinsic geometry, what is really significant is the relative relationship between the basis coordinate frame and the performance coordinate frame which are determined by reference-system on manifold, and has no essential relationship with the absolute values of affine connection coefficients on some one coordinate frame. So there is completely no difference in sense of intrinsic geometry between selecting torsion connection and selecting torsion-free connection. In other words, the torsion is not important at all for intrinsic geometry.

In Riemannian geometry usually only considering the torsion-free connection when researching intrinsic geometric properties, the rationality and effectiveness of which are guaranteed by the reason above.

(3) Re-examine the necessity of complex-valued expression.

In early years, H.Weyl re-examined the fundamentality of metric, established the concept of affine connection independent with metric, and proposed the research idea of gauge transformation in field of real numbers. However, due to some problems related to quantum measurement, physicists brought the research to complex field. Thus, to a certain extent, the fact is concealed that invariances under gauge transformation reflect intrinsic geometric properties. In fact, complex-valued expressions are necessary only for convenience of discussing problems associated with quantum measurements. The research for gauge transformation does not depend on complex field, but on intrinsic geometry, because all geometries that gauge transformation theory concerned can be regarded as subgeometries of intrinsic geometry.

Therefore, in order to highlight the essence of each concept more precisely, in section 4.3.1 and so on, complex-valued expressions will be deliberately avoided in the further discussion of gauge transformation. Complex-valued expressions will only be used for convenience when discussing problems associated with quantum measurements.

(4) The intrinsic geometric essence and expression forms of gauge field theory.

The intrinsic geometry on manifold is completely determined by reference-system. It has nothing to do with the selection of affine connections on tangent bundle, and also with the selection of arbitrary connections on arbitrary vector bundle. The common transformation groups in traditional gauge field theory are usually compact topological groups such as unitary group $U(n)$, special unitary group $SU(n)$ and special orthogonal group $SO(n)$, which are all subgroups of general linear group. The concept of slack-tight of intrinsic geometry can be used to generally deal with an arbitrary symmetry of general linear group, so it surely can be used to deal with the symmetries of such typical subgroups.

Therefore, when the transformation group is a subgroup of the general linear group, there are only differences in expression form between the tansformation of abstract connection on abstract fibre bundle and the tansformation of affine connection on tangent bundle. The essences they reflect are both intrinsic geometry's subgeometry associated with the transformation group, and have nothing to do with what kind of connection is adopted. In other words, the research content of traditional gauge field theory is nothing more than intrinsic geometric property, which is the fundamental reason of the rationality and effectiveness of the fact that the abstract connection on abstract fibre bundle can be used to research the matter-motion. In order to unify all kinds of theories, this viewpoint is indispensable.

In a word, the intrinsic geometry problems which can be researched in traditional gauge field theory by using abstract connection can also be researched by using affine connection. Moreover, in content, affine connection can be more concrete, and in form, it has a natural unity with the gravitational theory expressed in spacetime coordinates. Therefore, in the following text, affine connection will be adopted to re-express the traditional gauge field theory, so as to achieve the unification of various theoretical forms.

2.2.3 Kernal geometry

Definition 2.2.3.1. Let k be a reference-system on manifold M. Its slack-tights B^A_M (see Definition 2.2.8.1) are constants independent of position on manifold. The inherent transformation $F_{[k]}$ induced by k is called a **flat transformation** of reference-systems. If $\det[B^A_M] = 1$ is satisfied as well, $F_{[k]}$ is called a **unimodular flat transformation** of reference-systems, or a **global gauge transformation**.

Definition 2.2.3.2. Let there be intrinsic geometric manifolds $(M, [f])$ and $(M, [g])$. Define relation of equivalence $\simeq$, such that $[f] \simeq [g]$ and $(M, [f]) \simeq (M, [g])$ if and only if there exists a flat transformation $F_{[k]}$ such that $F_{[k]}([f]) = [g]$. The equivalence classes are denoted by $|f|$ and $(M, |f|)$ respectively, and $|f|$ is called the **kernal** of reference-system f. The geometry $\mathcal{C}(M)/\simeq$ about relation of equivalence $\simeq$ on intrinsic geometry $\mathcal{C}(M)$ is called the **kernal geometry** on geometric manifolds. The element $(M, |f|)$ in kernal geometry $\mathcal{C}(M)/\simeq$ is called a **kernal geometric manifold**.

Specially, if $F_{[k]}$ is a unimodular flat transformation of reference-system, the equivalence classes are denoted by $\|f\|$ and $(M, \|f\|)$ respectively, and $\|f\|$ is called the **regular kernal** of f. In this case, the geometry $\mathcal{C}(M)/\simeq$ is called the **regular kernal geometry** on geometric manifolds, or **regular geometry** for short. The element $(M, \|f\|)$ in regular geometry $\mathcal{C}(M)/\simeq$ is called a **regular geometric manifold**.

Remark 2.2.3.1. It can be understood intuitively as following. Consider Figure1 in the introduction section. Fix axis and scale, and rotate the whole curve L by an angle. The intrinsic geometric curve S' now is different from the intrinsic geometric curve S before. However, the major bending characteristics remain unchanged after the rotation. At this time, what being used to describe these invariant characteristics is the equivalence class $[S]$ determined by the unimodular flat transformation of reference-systems. $[S]$ can be called regular kernal geometric curve. Regular kernal geometry is the geometry which determines these major bending characteristics in intrinsic geometry strictly. Various bending characteristics are described by various regular kernal geometric properties of $[S]$.

2.2.4 Riemannian geometry

Definition 2.2.4.1. Let k be a reference-system on manifold M. Its slack-tights B^A_M (see Definition 2.2.8.1) satisfy $\varDelta_{AB} B^A_M B^B_N = \mathrm{E}_{MN}$. The inherent transformation $F_{[k]}$ induced by k is called an **orthogonal transformation** of reference-system.

Definition 2.2.4.2. Let there be intrinsic geometric manifolds $(M, [f])$ and $(M, [g])$, and their slack-tights be $(B_f)^A_M$ and $(B_g)^A_M$.

Define relation of equivalence $\simeq_O$, such that $[f] \simeq_O [g]$ and $(M, [f]) \simeq_O (M, [g])$ if and only if there exists an orthogonal transformation $F_{[k]}$ such that $F_{[k]}([f]) = [g]$. The equivalence classes are denoted by $[f]_O$ and $(M, [f]_O)$ respectively, and $[f]_O$ is called the **Riemannian core** of f. The geometry $\mathcal{C}(M)/\simeq_O$ about relation of equivalence $\simeq_O$ on intrinsic geometry $\mathcal{C}(M)$ is called the **Riemannian geometry** on geometric manifolds. The element $(M, [f]_O)$ in Riemannian geometry $\mathcal{C}(M)/\simeq_O$ is called a **Riemannian manifold**.

Remark 2.2.4.1. Noticed it is an obvious fact that $[f] \simeq_O [g]$ if and only if $\Delta_{AB}(B_f)^A_M(B_f)^B_N = \Delta_{AB}(B_g)^A_M(B_g)^B_N$, so this definition is consistent with the traditional definition of Riemannian manifold.

Definition 2.2.4.3. Let k be a reference-system on manifold M. If the inherent transformation $F_{[k]}$ induced by k is both a flat transformation and an orthogonal transformation of reference-system, $F_{[k]}$ is called a **general inertial transformation**.

Remark 2.2.4.2. The general inertial transformation will behave as the Lorentz transformation in Mincovski form of reference-system of classical spacetime. The details will be discussed in section 6.3.8.2 .

2.2.5 Universal geometry

Discussion 2.2.5.1. Let there be geometric manifolds (M, f) and (M, g). Define relation of equivalence $\sim$, satisfies that $(M, f) \sim (M, g)$ if and only if there exists an inherent transformation $F_{[k]}$ such that $F_{[k]}([f]) = [g]$. In fact this transformation always exists, which is $F_{[f^{-1}\circ g]}$. Therefore, $\mathcal{M}(M)$ becomes the only equivalence class in the geometry $\tilde{\mathcal{M}}(M) \triangleq \mathcal{M}(M)/\sim$ determined by relation of equivalence $\sim$. It makes $\tilde{\mathcal{M}}(M)$ the universal geometry on geometric manifolds. Because each universal geometry on geometric manifold is independent of the selection of reference-system, it can also be called a **universal geometry of reference-system**.

Discussion 2.2.5.2. Each differential topological property defined on differential manifold is as the same for all geometric manifolds in universal geometry $\tilde{\mathcal{M}}(M)$. In other words, each differential topological property is independent of the selection of reference-system, so as to be a universal geometric property on geometric manifold.

2.2.6 Several corollaries

Discussion 2.2.6.1. Make a summary for geometries of reference-system.

(1) An intrinsic geometric property on geometric manifold is an invariant property under identical inherent transformation of reference-systems. A kernal geometric property on geometric manifold is an invariant property under flat transformation of reference-systems. A Riemannian geometric property on geometric manifold is an invariant property under orthogonal transformation of reference-systems. A universal geometric property on geometric manifold is an invariant property under arbitrary transformation of reference-systems.

These geometries are all subgeometries of intrinsic geometry. For any subgeometry of intrinsic geometry, we say its geometric property is **a class of intrinsic geometric property**.

(2) Consider in the sense of Definition 2.2.2.4 . Let e be the unity element of $GL(\mathfrak{D}, \mathbb{R})$. According to Remark .1, $\{e\}$ as the transformation group of intrinsic geometry is the smallest transformation group on geometric manifold, and $GL(\mathfrak{D}, \mathbb{R})$ as the transformation group of universal geometry is the largest

transformation group on geometric manifold. In other words, intrinsic geometry is the largest geometry on geometric manifold, and universal geometry is the smallest geometry on geometric manifold.

Discussion 2.2.6.2. Now that we have those concepts of geometries of the previous sections, then the basic principle of theoretical physics in section 1.2 naturally has several obvious corollaries as following.

Corollary 1. A specific physical property in ontology is cognized by using a specific intrinsic geometric property on geometric manifold in epistemology.

Corollary 2. Any kind of physical property in ontology is cognized by using a class of intrinsic geometric property on geometric manifold in epistemology.

Corollary 3. A universal physical property in ontology is cognized by using a universal geometric property on geometric manifold in epistemology.

For convenience, Corollary 3 is called the **principle of universal relativity**.

Discussion 2.2.6.3. According to the above discussions, now it can be commented that:

(1) Universal geometry has the universal applicability for cognizing universal property of matter-motion;

(2) Intrinsic geometry has the universal applicability for cognizing specific property of matter-motion;

(3) Other geometries between universal geometry and intrinsic geometry have specific applicability for cognizing some other properties of matter-motion.

2.2.7 Coordinate representation of reference-system

Definition 2.2.7.1. $\forall p \in M$, let U be a neighborhood of p. $\forall q \in U$, denote $\xi \triangleq \psi_U(q) \in \mathbb{R}^{\mathfrak{D}}$, $x \triangleq \varphi_U(q) \in \mathbb{R}^{\mathfrak{D}}$. Now coordinate frames ψ_U and φ_U can be denoted by (U, ξ) and (U, x), or as component forms $\{\xi^A\}$ and $\{x^M\}$.

If no confusion, a reference-system can also be denoted by $\xi \triangleq \varphi_U^{-1} \circ \psi_U \in REF_p(U)$, $x \triangleq \psi_U^{-1} \circ \varphi_U \in REF_p(U)$.

Based on these two kinds of notations, the **coordinate representation** of referece-systems $\varphi_U^{-1} \circ \psi_U$ and $\psi_U^{-1} \circ \varphi_U$ can be written as

$$\xi = \xi(x),\ x = x(\xi)$$

or as component form

$$\xi^A = \xi^A(x^M),\ x^M = x^M(\xi^A).$$

Definition 2.2.7.2. For convenience, some index symbols have to be specified. In the absence of a special declaration, the indices used below are valued in the following range:

(1)for coordinate frame (U, ξ), indices $A, B, C, D, E = 1, 2, \cdots, \mathfrak{D}$, such as ξ^A;

(2)for coordinate frame (U, x), indices $M, N, P, Q, R = 1, 2, \cdots, \mathfrak{D}$, such as x^M.

2.2.8 Basis vectors and metrics of reference-system

Definition 2.2.8.1. Let (M, f) be a geometric manifold. $\forall p \in M$, on a neighborhood U of point p, let the coordinate representation of local reference-system $f(p)$ be $\xi^A = \xi^A(x^M)$, $x^M = x^M(\xi^A)$. Their derived functions

$$b_M^A \triangleq \frac{\partial \xi^A}{\partial x^M}, \quad c_A^M \triangleq \frac{\partial x^M}{\partial \xi^A}$$

on U are called the **slackness and tightness** of the local reference-system $f(p)$ on neighborhood U of point p, or called the **slack-tights** for short.

If need to emphasize the local reference-system $f(p)$ explicitly, b_M^A and c_A^M can be denoted by $(b_{f(p)})_M^A$ and $(c_{f(p)})_A^M$.

Corresponding to the two coordinate frames of local reference-system $f(p)$, the tangent space T_p at point p has two sets of natural base $\frac{\partial}{\partial \xi^A}, \frac{\partial}{\partial x^M} \in T_p$, and the cotangent space T_p^* also has two sets of natural base $d\xi^A, dx^M \in T_p^*$. If need to emphasize the point p explicitly, the tangent vector base $\frac{\partial}{\partial \xi^A}$ and $\frac{\partial}{\partial x^M}$ are denoted by $\left.\frac{\partial}{\partial \xi^A}\right|_p$ and $\left.\frac{\partial}{\partial x^M}\right|_p$ respectively, and the cotangent vector base $d\xi^A$ and dx^M are denoted by $\left.d\xi^A\right|_p$ and $\left.dx^M\right|_p$ respectively.

Define two sets of smooth real functions on manifold M:

$$\begin{cases} B_M^A : M \to \mathbb{R}, \quad p \mapsto B_M^A(p) \triangleq \left\langle \left.\frac{\partial}{\partial x^M}\right|_p, \left.d\xi^A\right|_p \right\rangle = (b_{f(p)})_M^A(p) \\ C_A^M : M \to \mathbb{R}, \quad p \mapsto C_A^M(p) \triangleq \left\langle \left.\frac{\partial}{\partial \xi^A}\right|_p, \left.dx^M\right|_p \right\rangle = (c_{f(p)})_A^M(p) \end{cases},$$

then call B_M^A and C_A^M the **slack-tights** of reference-system f on manifold M.

Definition 2.2.8.2. Let T_s^r be a (r,s)-type tensor bundle on manifold M. The mapping $h : M \to T_s^r, \quad p \mapsto h(p) \in T_s^r(p)$ is called a **section** of tensor bundle T_s^r, or called a **tensor field** on M. Moreover, if h is a smooth mapping, h is called a **smooth section** or a **smooth tensor field**. Specially, a 1-order tensor field is called a **vector field**.

Suppose on neighborhood U of point p there are natural basis vector fields $\left.d\xi^A\right|_U$, $\left.dx^M\right|_U$, $\left.\frac{\partial}{\partial \xi^A}\right|_U$, $\left.\frac{\partial}{\partial x^M}\right|_U$ determined by coordinate frames (U, ξ^A) and (U, x^M) of local reference-system $f(p)$, satisfying $\left.d\xi^A\right|_U(p) = \left.d\xi^A\right|_p$, $\left.dx^M\right|_U(p) = \left.dx^M\right|_p$, $\left.\frac{\partial}{\partial \xi^A}\right|_U(p) = \left.\frac{\partial}{\partial \xi^A}\right|_p$, $\left.\frac{\partial}{\partial x^M}\right|_U(p) = \left.\frac{\partial}{\partial x^M}\right|_p$ at this specific point p. The slack-tight of $f(p)$ can be expressed with these natural basis vector fields, as

$$\begin{cases} b_M^A = \left\langle \left.\frac{\partial}{\partial x^M}\right|_U, \left.d\xi^A\right|_U \right\rangle \\ c_A^M = \left\langle \left.\frac{\partial}{\partial \xi^A}\right|_U, \left.dx^M\right|_U \right\rangle \end{cases}.$$

Suppose on manifold M there are $\mathfrak{D}$ cotangent vector fields λ^A and $\mathfrak{D}$ cotangent vector fields ω^M, satisfying $\lambda^A(p) = \left.d\xi^A\right|_p$ and $\omega^M(p) = \left.dx^M\right|_p$ at any point p on M. If no confusion, the vector fields λ^A and ω^M can be denoted by $d\xi^A$ and dx^M, which are called the **coordinate forms** determined by reference-system f on M. In the same way there are tangent vector fields $\frac{\partial}{\partial \xi^A}$ and $\frac{\partial}{\partial x^M}$, satisfying $\frac{\partial}{\partial \xi^A}(p) = \left.\frac{\partial}{\partial \xi^A}\right|_p$ and $\frac{\partial}{\partial x^M}(p) = \left.\frac{\partial}{\partial x^M}\right|_p$ at any point p on M. Now the slack-tight of f can be expressed with these vector fields, as

$$\begin{cases} B_M^A = \left\langle \frac{\partial}{\partial x^M}, d\xi^A \right\rangle \\ C_A^M = \left\langle \frac{\partial}{\partial \xi^A}, dx^M \right\rangle \end{cases}.$$

Definition 2.2.8.3. Denote

$$\varepsilon_{MN} = \varepsilon^{MN} = \varepsilon_N^M \triangleq \begin{cases} 1, & M = N \\ 0, & M \neq N \end{cases}, \quad \delta_{AB} = \delta^{AB} = \delta_B^A \triangleq \begin{cases} 1, & A = B \\ 0, & A \neq B \end{cases}.$$

The reason of distinguishing the notation ε and δ is to avoid confusion when expressing concrete indices in the following sections.

Definition 2.2.8.4. Let (M, f) be a geometric manifold. $\forall p \in M$, let U be a neighborhood of point p.

(1) The coordinate frames (U, ξ^A) and (U, x^M) of local reference-system $f(p)$ respectively inherit metric tensor fields

$$\begin{cases} \mathbf{g} \triangleq \delta_{AB} d\xi^A \otimes d\xi^B = g_{MN} dx^M \otimes dx^N \\ \mathbf{h} \triangleq \varepsilon_{MN} dx^M \otimes dx^N = h_{AB} d\xi^A \otimes d\xi^B \end{cases}$$

from $\mathbb{R}^{\mathfrak{D}}$, where $d\xi^A$ and dx^M are natural basis vector fields determined by the two coordinate frames (U, ξ^A) and (U, x^M) of $f(p)$. Obviously,

$$\begin{cases} g_{MN} = \delta_{AB} b_M^A b_N^B \\ h_{AB} = \varepsilon_{MN} c_A^M c_B^N \end{cases},$$

which are determined by local reference-system $f(p)$ completely.

If need to emphasize the local reference-system $f(p)$ explicitly, g and h can be expressed as $g_{f(p)}$ and $h_{f(p)}$, then g_{MN} and h_{AB} and be expressed as $(g_{f(p)})_{MN}$ and $(h_{f(p)})_{AB}$.

(2) On manifold M, define two sets of $\mathfrak{D} \times \mathfrak{D}$ smooth real functions

$$\begin{cases} G_{MN} : M \to \mathbb{R}, \quad p \mapsto G_{MN}(p) \triangleq (g_{f(p)})_{MN}(p) \\ H_{AB} : M \to \mathbb{R}, \quad p \mapsto H_{AB}(p) \triangleq (h_{f(p)})_{AB}(p) \end{cases},$$

and define

$$\begin{cases} \varDelta_{AB} : M \to \mathbb{R}, \quad p \mapsto \varDelta_{AB}(p) \triangleq \delta_{AB} \\ \mathrm{E}_{MN} : M \to \mathbb{R}, \quad p \mapsto \mathrm{E}_{MN}(p) \triangleq \varepsilon_{MN} \end{cases}.$$

Thus on the entire manifold M, two metric tensor fields are constructed:

$$\begin{cases} \mathbf{G} = \varDelta_{AB} d\xi^A \otimes d\xi^B = G_{MN} dx^M \otimes dx^N \\ \mathbf{H} = \mathrm{E}_{MN} dx^M \otimes dx^N = H_{AB} d\xi^A \otimes d\xi^B \end{cases},$$

where $d\xi^A$ and dx^M are the coordinate forms determined by f on M. Obviously,

$$\begin{cases} G_{MN} = \varDelta_{AB} B_M^A B_N^B \\ H_{AB} = \mathrm{E}_{MN} C_A^M C_B^N \end{cases},$$

which are determined by reference-system f completely.

(3) The Riemannian manifolds $(M, \mathbf{G})$ and $(M, \mathbf{H})$ determined by f are called dual mutually.

Remark 2.2.8.1.

(1) The local coordinate transformation of metric tensor on manifold: $\forall p \in M$, on the neighbornood U of point p, let the slack-tights of $f(p)$ are b_M^A and c_A^M. $f(p)$ induces a local coordinate transformation $F_{f(p)}$ on U. By $F_{f(p)}$, the restriction of G_{MN} on U is to be transformed to

$$G_{MN} \mapsto G'_{AB} = G_{MN} c_A^M c_B^N = \varDelta_{CD} B_M^C B_N^D c_A^M c_B^N.$$

Notice that for any point q on U, $B^A_M(q) \neq b^A_M(q)$ generally. It is only at point p that $B^A_M(p) = b^A_M(p)$ is true.

So it can be seen that the curved shape determined by slack-tight B^A_M or metric $G_{MN} = \Delta_{CD} B^C_M B^D_N$ will not vanish under the influence of local transformation c^M_A.

(2) For the sake of simplicity, if no confusion, δ_{AB} as a general notation and Δ_{AB} as coefficients of metric tensor are not to be distinguished, and so are ε_{MN} and E_{MN}. They are mainly expressed as the notations δ_{AB} and ε_{MN} uniformly.

Discussion 2.2.8.1. Suppose in the two coordinate frames of local reference-system,

$$\begin{cases} h \triangleq \varepsilon_{MN} dx^M \otimes dx^N = h_{AB} d\xi^A \otimes d\xi^B \\ g \triangleq \delta_{AB} d\xi^A \otimes d\xi^B = g_{MN} dx^M \otimes dx^N \end{cases},$$

where

$$\begin{cases} h_{AB} = \varepsilon_{MN} c^M_A c^N_B \\ g_{MN} = \delta_{AB} b^A_M b^B_N \end{cases}.$$

Denote

$$\begin{cases} d\xi_A \triangleq h_{AB} d\xi^B \\ dx_M \triangleq g_{MN} dx^N \end{cases},$$

two new coordinate frames (U, ξ_A) and (U, x_M) can be determined in the degree of only an integration constant difference. Notice that $d\xi_A$ and dx_M can also become natural basis vectors induced by new coordinate frames on cotangent space. Correspondingly, let the natural vectors induced by new coordinate frames on tangent space be $\frac{\partial}{\partial \xi_A}$ and $\frac{\partial}{\partial x_M}$. These basis vectors are all independent of integration constant. According to Proposition 2.2.8.1, they necessarily satisfy $\left\langle \frac{\partial}{\partial \xi_B}, d\xi_A \right\rangle = \delta^B_A$ and $\left\langle \frac{\partial}{\partial x_N}, dx_M \right\rangle = \varepsilon^N_M$.

Definition 2.2.8.5. Similar to Definition 2.2.8.1 , define

$$\begin{cases} c^{MA} \triangleq \dfrac{\partial x^M}{\partial \xi_A} \\ b^{AM} \triangleq \dfrac{\partial \xi^A}{\partial x_M} \end{cases}, \quad \begin{cases} c_{AM} \triangleq \dfrac{\partial x_M}{\partial \xi^A} \\ b_{MA} \triangleq \dfrac{\partial \xi_A}{\partial x^M} \end{cases}, \quad \begin{cases} \bar{b}^M_A \triangleq \dfrac{\partial \xi_A}{\partial x_M} \\ \bar{c}^A_M \triangleq \dfrac{\partial x_M}{\partial \xi_A} \end{cases}.$$

It should be noted that for the convenience of distinguishing between $\frac{\partial \xi^\tau}{\partial x^\tau}$ and $\frac{\partial \xi_\tau}{\partial x_\tau}$, and between $\frac{\partial x^\tau}{\partial \xi^\tau}$ and $\frac{\partial x_\tau}{\partial \xi_\tau}$, b^M_A and c^A_M are specially denoted by $\bar{b}^M_A$ and $\bar{c}^A_M$.

The transformation relations about $\{b^A_M, c^M_A\}$, $\{b^{AM}, c_{AM}\}$, $\{b_{MA}, c^{MA}\}$, $\{\bar{b}^M_A, \bar{c}^A_M\}$:

$$\begin{cases} dx^M = c^M_A d\xi^A = c^{MA} d\xi_A \\ d\xi^A = b^A_M dx^M = b^{AM} dx_M \end{cases}, \quad \begin{cases} dx_M = \bar{c}^A_M d\xi_A = c_{AM} d\xi^A \\ d\xi_A = \bar{b}^M_A dx_M = b_{MA} dx^M \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial x^M} = b^A_M \dfrac{\partial}{\partial \xi^A} = b_{MA} \dfrac{\partial}{\partial \xi_A} \\ \dfrac{\partial}{\partial \xi^A} = c^M_A \dfrac{\partial}{\partial x^M} = c_{AM} \dfrac{\partial}{\partial x_M} \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial x_M} = \bar{b}^M_A \dfrac{\partial}{\partial \xi_A} = b^{AM} \dfrac{\partial}{\partial \xi^A} \\ \dfrac{\partial}{\partial \xi_A} = \bar{c}^A_M \dfrac{\partial}{\partial x_M} = c^{MA} \dfrac{\partial}{\partial x^M} \end{cases}$$

are called the **basis vector representation** of local reference-systems $\xi = \xi(x)$ and $x = x(\xi)$.

These local relations are also true on the entire manifold. More concretely, similar to the slack-tights $\{B_M^A, C_A^M\}$ of Definition 2.2.8.1 , $\{B^{AM}, C_{AM}\}$ $\{B_{MA}, C^{MA}\}$ $\{\bar{B}_A^M, \bar{C}_M^A\}$ can also be defined.

Take $\{B_M^A, C_A^M\}$ for example, the **basis vector representation** of reference-system on manifold about cotangent vector fields dx^M, $d\xi^A$ and tangent vector fields $\frac{\partial}{\partial x^M}$, $\frac{\partial}{\partial \xi^A}$ can be written as

$$\begin{cases} dx^M = C_A^M d\xi^A \\ d\xi^A = B_M^A dx^M \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial x^M} = B_M^A \dfrac{\partial}{\partial \xi^A} \\ \dfrac{\partial}{\partial \xi^A} = C_A^M \dfrac{\partial}{\partial x^M} \end{cases}. \tag{1}$$

Definition 2.2.8.6. Based on these coefficients of base transformations, some metrics of local reference-system can be defined. The two of them have been defined in Discussion 2.2.8.1 . They are g_{MN} and h_{AB}. Now all four **basic metric tensors** concerning the local reference-system are defined collectively as following:

$$\begin{cases} \mathbf{g} \triangleq \delta_{AB} d\xi^A \otimes d\xi^B = g_{MN} dx^M \otimes dx^N = g^{MN} dx_M \otimes dx_N \\ \mathbf{h} \triangleq \varepsilon_{MN} dx^M \otimes dx^N = h_{AB} d\xi^A \otimes d\xi^B = h^{AB} d\xi_A \otimes d\xi_B \\ \mathbf{k} \triangleq \delta^{AB} d\xi_A \otimes d\xi_B = k^{MN} dx_M \otimes dx_N = k_{MN} dx^M \otimes dx^N \\ \mathbf{l} \triangleq \varepsilon^{MN} dx_M \otimes dx_N = l^{AB} d\xi_A \otimes d\xi_B = l_{AB} d\xi^A \otimes d\xi^B \end{cases}.$$

Correspondingly, there are another four tensors:

$$\begin{cases} \mathbf{x} \triangleq \delta^{AB} \dfrac{\partial}{\partial \xi^A} \otimes \dfrac{\partial}{\partial \xi^B} = x^{MN} \dfrac{\partial}{\partial x^M} \otimes \dfrac{\partial}{\partial x^N} = x_{MN} \dfrac{\partial}{\partial x_M} \otimes \dfrac{\partial}{\partial x_N} \\ \mathbf{y} \triangleq \varepsilon^{MN} \dfrac{\partial}{\partial x^M} \otimes \dfrac{\partial}{\partial x^N} = y^{AB} \dfrac{\partial}{\partial \xi^A} \otimes \dfrac{\partial}{\partial \xi^B} = y_{AB} \dfrac{\partial}{\partial \xi_A} \otimes \dfrac{\partial}{\partial \xi_B} \\ \mathbf{v} \triangleq \delta_{AB} \dfrac{\partial}{\partial \xi_A} \otimes \dfrac{\partial}{\partial \xi_B} = v_{MN} \dfrac{\partial}{\partial x_M} \otimes \dfrac{\partial}{\partial x_N} = v^{MN} \dfrac{\partial}{\partial x^M} \otimes \dfrac{\partial}{\partial x^N} \\ \mathbf{w} \triangleq \varepsilon_{MN} \dfrac{\partial}{\partial x_M} \otimes \dfrac{\partial}{\partial x_N} = w_{AB} \dfrac{\partial}{\partial \xi_A} \otimes \dfrac{\partial}{\partial \xi_B} = w^{AB} \dfrac{\partial}{\partial \xi^A} \otimes \dfrac{\partial}{\partial \xi^B} \end{cases}.$$

The coefficients

$$\begin{cases} g_{MN} = \delta_{AB} b_M^A b_N^B \\ g^{MN} = \delta_{AB} b^{AM} b^{BN} \end{cases}, \quad \begin{cases} h_{AB} = \varepsilon_{MN} c_A^M c_B^N \\ h^{AB} = \varepsilon_{MN} c^{MA} c^{NB} \end{cases}, \quad \begin{cases} k^{MN} = \delta^{AB} \bar{b}_A^M \bar{b}_B^N \\ k_{MN} = \delta^{AB} b_{MA} b_{NB} \end{cases}, \quad \begin{cases} l^{AB} = \varepsilon^{MN} \bar{c}_M^A \bar{c}_N^B \\ l_{AB} = \varepsilon^{MN} c_{AM} c_{BN} \end{cases},$$

$$\begin{cases} x_{MN} = \delta^{AB} c_{AM} c_{BN} \\ x^{MN} = \delta^{AB} c_A^M c_B^N \end{cases}, \quad \begin{cases} y_{AB} = \varepsilon^{MN} b_{MA} b_{NB} \\ y^{AB} = \varepsilon^{MN} b_M^A b_N^B \end{cases}, \quad \begin{cases} v^{MN} = \delta_{AB} c^{MA} c^{NB} \\ v_{MN} = \delta_{AB} \bar{c}_M^A \bar{c}_N^B \end{cases}, \quad \begin{cases} w^{AB} = \varepsilon_{MN} b^{AM} b^{BN} \\ w_{AB} = \varepsilon_{MN} \bar{b}_A^M \bar{b}_B^N \end{cases}$$

are called the **metric representation** of local reference-systems $\xi = \xi(x)$ and $x = x(\xi)$.

Remark 2.2.8.2. The metric representation of reference-system on the entire manifold can also be defined. It only needs to replace the metric tensors of local reference-system to the one on manifold. Their expression forms are the same, so they will not be described repeatly.

Not all of these tensors will be used in this paper. First, according to Discussion 2.2.8.4 , **g**, **h**, **x**, **y** have better properties than **k**, **l**, **v**, **w**. Second, according to Proposition 2.2.8.5 , the coefficients of **g** and **x** are equal completely, and the coefficients of **h** and **y** are equal completely. Third, because of the properties of evolution in Remark 2.4.4.1 . Therefore, only the tensors **g** and **h** are usually needed to concern. The rest contents of this section will discuss this concretely.

Proposition 2.2.8.1. On the neighborhood U of point p on manifold M, the following equations hold:

$$\begin{cases} h_{CA}h^{AB} = \delta_C^B \\ g_{PM}g^{MN} = \varepsilon_P^N \end{cases}, \quad \begin{cases} d\xi^A = h^{AB}d\xi_B \\ dx^M = g^{MN}dx_N \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial \xi_A} = h^{AB}\dfrac{\partial}{\partial \xi^B} \\ \dfrac{\partial}{\partial x_M} = g^{MN}\dfrac{\partial}{\partial x^N} \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial \xi^A} = h_{AB}\dfrac{\partial}{\partial \xi_B} \\ \dfrac{\partial}{\partial x^M} = g_{MN}\dfrac{\partial}{\partial x_N} \end{cases}.$$

Proof.

The first, $c^{MA}d\xi_A = dx^M \Leftrightarrow c^{MA}h_{AB}d\xi^B = dx^M$. In consideration of $c_B^M d\xi^B = dx^M$, then $c^{MA}h_{AB} = c_B^M$. Then, $h^{AB} = \varepsilon_{MN}c^{MA}c^{NB} \Rightarrow h_{AC}h^{AB}h_{BD} = \varepsilon_{MN}(c^{MA}h_{AC})(c^{NB}h_{BD}) = \varepsilon_{MN}c_C^M c_D^N = h_{CD}$, which is a product of invertible square matrices $h_{CA}h^{AB}h_{BD} = h_{CD}$. Therefore $h_{CA}h^{AB} = \delta_C^B$, and it means they are inverse matrices of each other.

The second, $h_{AB}d\xi^B = d\xi_A \Rightarrow h^{AC}h_{AB}d\xi^B = h^{AC}d\xi_A \Leftrightarrow \delta_B^C d\xi^B = h^{AC}d\xi_A \Leftrightarrow d\xi^C = h^{AC}d\xi_A$.

The third and fourth, $c^{MA}h_{AB} = c_B^M \Leftrightarrow \left\langle \frac{\partial}{\partial \xi_A}, dx^M \right\rangle h_{AB} = \left\langle \frac{\partial}{\partial \xi^B}, dx^M \right\rangle \Leftrightarrow \left\langle h_{AB}\frac{\partial}{\partial \xi_A}, dx^M \right\rangle = \left\langle \frac{\partial}{\partial \xi^B}, dx^M \right\rangle \Leftrightarrow h_{AB}\frac{\partial}{\partial \xi_A} = \frac{\partial}{\partial \xi^B} \Rightarrow h^{BC}h_{AB}\frac{\partial}{\partial \xi_A} = h^{BC}\frac{\partial}{\partial \xi^B} \Leftrightarrow \delta_A^C\frac{\partial}{\partial \xi_A} = h^{BC}\frac{\partial}{\partial \xi^B} \Leftrightarrow \frac{\partial}{\partial \xi_C} = h^{BC}\frac{\partial}{\partial \xi^B}$.

The above is a proof about **h**. For **g** it is the same. $g_{MP}g^{MN} = \varepsilon_P^N$, $dx^P = g^{MP}dx_M$, $\frac{\partial}{\partial x^N} = g_{MN}\frac{\partial}{\partial x_M}$, $\frac{\partial}{\partial x_P} = g^{NP}\frac{\partial}{\partial x^N}$ also hold. □

Proposition 2.2.8.2. On the neighborhood U of point p on manifold M, the following equations hold:

$$\begin{cases} k^{MP}v_{PN} = \varepsilon_N^M \\ v^{MP}k_{PN} = \varepsilon_N^M \end{cases}, \quad \begin{cases} l^{AC}w_{CB} = \delta_B^A \\ w^{AC}l_{CB} = \delta_B^A \end{cases}.$$

Proof.

$k^{MP}v_{PN} = \delta^{AB}\bar{b}_A^M\bar{b}_B^P\delta_{CD}\bar{c}_P^C\bar{c}_N^D = \delta^{AB}\bar{b}_A^M\delta_B^C\delta_{CD}\bar{c}_N^D = \delta^{AC}\delta_{CD}\bar{b}_A^M\bar{c}_N^D = \delta_D^A\bar{b}_A^M\bar{c}_N^D = \bar{b}_D^M\bar{c}_N^D = \varepsilon_N^M$.

$v^{MP}k_{PN} = \delta_{AB}c^{MA}c^{PB}\delta^{CD}b_{PC}b_{ND} = \delta_{AB}c^{MA}\delta_C^B\delta^{CD}b_{ND} = \delta_{AC}\delta^{CD}c^{MA}b_{ND} = \delta_A^D c^{MA}b_{ND} = \varepsilon_N^M$.

$w^{AC}l_{CB} = \varepsilon_{MN}b^{AM}b^{CN}\varepsilon^{PQ}c_{CP}c_{BQ} = \varepsilon_{MN}b^{AM}\delta_P^N\varepsilon^{PQ}c_{BQ} = \varepsilon_{MP}\varepsilon^{PQ}b^{AM}c_{BQ} = \varepsilon_M^Q b^{AM}c_{BQ} = \delta_B^A$.

$l^{AC}w_{CB} = \varepsilon^{MN}\bar{c}_M^A\bar{c}_N^C\varepsilon_{PQ}\bar{b}_C^P\bar{b}_B^Q = \varepsilon^{MN}\bar{c}_M^A\varepsilon_N^P\varepsilon_{PQ}\bar{b}_B^Q = \varepsilon^{MP}\varepsilon_{PQ}\bar{c}_M^A\bar{b}_B^Q = \varepsilon_Q^M\bar{c}_M^A\bar{b}_B^Q = \bar{c}_Q^A\bar{b}_B^Q = \delta_B^A$. □

Remark 2.2.8.3. Generally, $k^{MP}k_{PN} = \varepsilon_N^M$ does not hold. A sufficient condition is that if $\delta^{AB}h_{AE}h_{BF} = \delta_{EF}$, then $k^{MP}k_{PN} = \varepsilon_N^M$. There are similar conclusions for $l^{AC}l_{CB}$, $v^{MP}v_{PN}$, $w^{AC}w_{CB}$.

Proposition 2.2.8.3. On the neighborhood U of point p on manifold M, the following equations hold:

$$\begin{cases} c^{MA}h_{AB} = c_B^M \\ c^{MA}g_{MN} = \bar{c}_N^A \end{cases}, \quad \begin{cases} c_{AM}h^{AB} = \bar{c}_M^B \\ c_{AM}g^{MN} = c_A^N \end{cases}, \quad \begin{cases} b^{AM}g_{MN} = b_N^A \\ b^{AM}h_{AB} = \bar{b}_B^M \end{cases}, \quad \begin{cases} b_{MA}g^{MN} = \bar{b}_A^N \\ b_{MA}h^{AB} = b_M^B \end{cases},$$

$$\begin{cases} c^{MA} = c_B^M h^{AB} \\ c^{MA} = \bar{c}_N^A g^{MN} \end{cases}, \quad \begin{cases} c_{AM} = \bar{c}_M^B h_{AB} \\ c_{AM} = c_A^N g_{MN} \end{cases}, \quad \begin{cases} b^{AM} = b_N^A g^{MN} \\ b^{AM} = \bar{b}_B^M h^{AB} \end{cases}, \quad \begin{cases} b_{MA} = \bar{b}_A^N g_{MN} \\ b_{MA} = b_M^B h_{AB} \end{cases}.$$

Proof. Start from Definition 2.2.8.5 , and according to Proposition 2.2.8.1, we have:

$c^{MA}d\xi_A = dx^M \Leftrightarrow c^{MA}h_{AB}d\xi^B = dx^M$, due to $c_B^M d\xi^B = dx^M$, so $c^{MA}h_{AB} = c_B^M$, $c^{MA} = c_B^M h^{AB}$.

$c_{AM}d\xi^A = dx_M \Leftrightarrow c_{AM}h^{AB}d\xi_B = dx_M$, due to $\bar{c}_M^B d\xi_B = dx_M$, so $c_{AM}h^{AB} = \bar{c}_M^B$, $c_{AM} = \bar{c}_M^B h_{AB}$.

$b^{AM}dx_M = d\xi^A \Leftrightarrow b^{AM}g_{MN}dx^N = d\xi^A$, due to $b_N^A dx^N = d\xi^A$, so $b^{AM}g_{MN} = b_N^A$, $b^{AM} = b_N^A g^{MN}$.

$b_{MA}dx^M = d\xi_A \Leftrightarrow b_{MA}g^{MN}dx_N = d\xi_A$, due to $\bar{b}_A^N dx_N = d\xi_A$, so $b_{MA}g^{MN} = \bar{b}_A^N$, $b_{MA} = \bar{b}_A^N g_{MN}$.
$c^{MA}\frac{\partial}{\partial x^M} = \frac{\partial}{\partial \xi_A} \Leftrightarrow c^{MA}g_{MN}\frac{\partial}{\partial x_N} = \frac{\partial}{\partial \xi_A}$, due to $\bar{c}_N^A \frac{\partial}{\partial x_N} = \frac{\partial}{\partial \xi_A}$, so $c^{MA}g_{MN} = \bar{c}_N^A$, $c^{MA} = \bar{c}_N^A g^{MN}$.
$c_{AM}\frac{\partial}{\partial x_M} = \frac{\partial}{\partial \xi^A} \Leftrightarrow c_{AM}g^{MN}\frac{\partial}{\partial x^N} = \frac{\partial}{\partial \xi^A}$, due to $c_A^N \frac{\partial}{\partial x^N} = \frac{\partial}{\partial \xi^A}$, so $c_{AM}g^{MN} = c_A^N$, $c_{AM} = c_A^N g_{MN}$.
$b^{AM}\frac{\partial}{\partial \xi^A} = \frac{\partial}{\partial x_M} \Leftrightarrow b^{AM}h_{AB}\frac{\partial}{\partial \xi_B} = \frac{\partial}{\partial x_M}$, due to $\bar{b}_B^M \frac{\partial}{\partial \xi_B} = \frac{\partial}{\partial x_M}$, so $b^{AM}h_{AB} = \bar{b}_B^M$, $b^{AM} = \bar{b}_B^M h^{AB}$.
$b_{MA}\frac{\partial}{\partial \xi_A} = \frac{\partial}{\partial x^M} \Leftrightarrow b_{MA}h^{AB}\frac{\partial}{\partial \xi^B} = \frac{\partial}{\partial x^M}$, due to $b_M^B \frac{\partial}{\partial \xi^B} = \frac{\partial}{\partial x^M}$, so $b_{MA}h^{AB} = b_M^B$, $b_{MA} = b_M^B h_{AB}$. □

Proposition 2.2.8.4. On the neighborhood U of point p on manifold M, the following equations hold:

$$\begin{cases} g_{MN}c_C^M c_D^N = \delta_{CD} \\ g^{MN}b_M^C b_N^D = \delta^{CD} \end{cases}, \quad \begin{cases} h_{AB}b_P^A b_Q^B = \varepsilon_{PQ} \\ h^{AB}c_A^P c_B^Q = \varepsilon^{PQ} \end{cases}, \quad \begin{cases} v_{MN}\bar{b}_C^M \bar{b}_D^N = \delta_{CD} \\ k^{MN}\bar{c}_M^C \bar{c}_N^D = \delta^{CD} \end{cases}, \quad \begin{cases} w_{AB}\bar{c}_P^A \bar{c}_Q^B = \varepsilon_{PQ} \\ l^{AB}\bar{b}_A^P \bar{b}_B^Q = \varepsilon^{PQ} \end{cases}.$$

Proof.

$g_{MN}c_C^M c_D^N = \delta_{AB}b_M^A b_N^B c_C^M c_D^N = \delta_{AB}(b_M^A c_C^M)(b_N^B c_D^N) = \delta_{AB}\delta_C^A\delta_D^B = \delta_{CD}$.
$\delta^{CD} = \delta_A^C\delta^{AD} = b_N^C c_A^N \delta^{AD} = \varepsilon_N^M b_M^C c_A^N \delta^{AD} = g^{MP}g_{PN}b_M^C c_A^N \delta^{AD} = g^{MP}\delta_{BE}b_P^B b_N^E b_M^C c_A^N \delta^{AD} = g^{MP}b_M^C b_P^D$.
$h_{AB}b_P^A b_Q^B = \varepsilon_{MN}c_A^M c_B^N b_P^A b_Q^B = \varepsilon_{MN}(c_A^M b_P^A)(c_B^N b_Q^B) = \varepsilon_{MN}\varepsilon_P^M\varepsilon_Q^N = \varepsilon_{PQ}$.
$\varepsilon^{PQ} = \varepsilon_M^P\varepsilon^{MQ} = c_B^P b_M^B \varepsilon^{MQ} = \delta_B^A c_A^P b_M^B \varepsilon^{MQ} = h^{AC}h_{CB}c_A^P b_M^B \varepsilon^{MQ} = h^{AC}\varepsilon_{NR}c_C^N c_B^R c_A^P b_M^B \varepsilon^{MQ} = h^{AC}c_A^P c_C^Q$.
$k^{MN}\bar{c}_M^C \bar{c}_N^D = \delta^{AB}\bar{b}_A^M \bar{b}_B^N \bar{c}_M^C \bar{c}_N^D = \delta^{AB}(\bar{b}_A^M \bar{c}_M^C)(\bar{b}_B^N \bar{c}_N^D) = \delta^{AB}\delta_A^C\delta_B^D = \delta^{CD}$.
$l^{AB}\bar{b}_A^P \bar{b}_B^Q = \varepsilon^{MN}\bar{c}_M^A \bar{c}_N^B \bar{b}_A^P \bar{b}_B^Q = \varepsilon^{MN}(\bar{c}_M^A \bar{b}_A^P)(\bar{c}_N^B \bar{b}_B^Q) = \varepsilon^{MN}\varepsilon_M^P\varepsilon_N^Q = \varepsilon^{PQ}$.
$v_{MN}\bar{b}_C^M \bar{b}_D^N = \delta_{AB}\bar{c}_M^A \bar{c}_N^B \bar{b}_C^M \bar{b}_D^N = \delta_{AB}(\bar{c}_M^A \bar{b}_C^M)(\bar{c}_N^B \bar{b}_D^N) = \delta_{AB}\delta_C^A\delta_D^B = \delta_{CD}$.
$w_{AB}\bar{c}_P^A \bar{c}_Q^B = \varepsilon_{MN}\bar{b}_A^M \bar{b}_B^N \bar{c}_P^A \bar{c}_Q^B = \varepsilon_{MN}(\bar{b}_A^M \bar{c}_P^A)(\bar{b}_B^N \bar{c}_Q^B) = \varepsilon_{MN}\varepsilon_P^M\varepsilon_Q^N = \varepsilon_{PQ}$. □

Proposition 2.2.8.5. On the neighborhood U of point p on manifold M, the following equations hold:

$$\begin{cases} g_{MN} \triangleq \delta_{AB}b_M^A b_N^B = \delta^{AB}c_{AM}c_{BN} \triangleq x_{MN} \\ g^{MN} \triangleq \delta_{AB}b^{AM}b^{BN} = \delta^{AB}c_A^M c_B^N \triangleq x^{MN} \end{cases}, \quad \begin{cases} h_{AB} \triangleq \varepsilon_{MN}c_A^M c_B^N = \varepsilon^{MN}b_{MA}b_{NB} \triangleq y_{AB} \\ h^{AB} \triangleq \varepsilon_{MN}c^{MA}c^{NB} = \varepsilon^{MN}b_M^A b_N^B \triangleq y^{AB} \end{cases}.$$

Proof.

According to Proposition 2.2.8.4 , $g^{MP}b_M^C b_P^D = \delta^{CD} \Rightarrow g^{MN}b_M^C b_N^D c_C^P c_D^Q = \delta^{CD}c_C^P c_D^Q \Rightarrow g^{PQ} = x^{PQ}$.

$x_{MN} \triangleq \delta^{AB}c_{AM}c_{BN} \Rightarrow g^{PM}x_{MN}g^{NQ} = g^{PM}\delta^{AB}c_{AM}c_{BN}g^{NQ} = \delta^{AB}(c_{AM}g^{PM})(c_{BN}g^{NQ}) = \delta^{AB}c_A^P c_B^Q = g^{PQ} \Rightarrow g^{PM}x_{MN} = \varepsilon_N^P \Rightarrow x_{MN} = g_{MN}$.

According to Proposition 2.2.8.4 , $h^{AB}c_A^P c_B^Q = \varepsilon^{PQ} \Rightarrow h^{AB}c_A^P c_B^Q b_P^C b_Q^D = \varepsilon^{PQ}b_P^C b_Q^D \Rightarrow h^{CD} = y^{CD}$.

$y_{AB} \triangleq \varepsilon^{MN}b_{MA}b_{NB} \Rightarrow h^{CA}y_{AB}h^{BD} = h^{CA}\varepsilon^{MN}b_{MA}b_{NB}h^{BD} = \varepsilon^{MN}(b_{MA}h^{CA})(b_{NB}h^{BD}) = \varepsilon^{MN}b_M^C b_N^D = h^{CD} \Rightarrow h^{CA}y_{AB} = \delta_B^C \Rightarrow y_{AB} = h_{AB}$. □

Discussion 2.2.8.2.

1. By transformation of basis vector, tensors **g**, **h**, **x**, **y** not only induce automorphisms of tangent space or cotangent space, but also induce isomorphisms between tangent space and cotangent space. For example:

(1)By the coordinate transformation $x^M \mapsto x_N \triangleq x_N(x^M)$, tensor **g** induces an automorphism of tangent space:

$$id_g : T \to T, \quad T^M\frac{\partial}{\partial x^M} \mapsto id_g(T^M\frac{\partial}{\partial x^M}) \triangleq T^M g_{MN}\frac{\partial}{\partial x_N} \triangleq T_N\frac{\partial}{\partial x_N},$$

and induces an isomorphism between cotangent space and tangent space:

$$G : T^* \to T, \quad T_M dx^M \mapsto G(T_M dx^M) \triangleq T_M\frac{\partial}{\partial x_M} = T_M g^{MN}\frac{\partial}{\partial x^N} \triangleq T^N\frac{\partial}{\partial x^N}.$$

(2)Similarly, by the coordinate transformation $x^M \mapsto x_N \triangleq x_N(x^M)$, tensor x induces an automorphism of cotangent space:

$$id_x^* : T^* \to T^*, \quad T_M dx^M \mapsto id_x^*(T_M dx^M) \triangleq (x^{MN} T_M) dx_N \triangleq T^N dx_N,$$

and induces an isomorphism between tangent space and cotangent space:

$$X : T \to T^*, \quad T^M \frac{\partial}{\partial x^M} \mapsto X(T^M \frac{\partial}{\partial x^M}) \triangleq T^M dx_M = T^M x_{MN} dx^N \triangleq T_N dx^N.$$

2. Moreover, the isomorphism between tangent space and cotangent space induces an isomorphism between 2-order covariant tensor space $T \triangleq \{T_{MN} dx^M \otimes dx^N | T_{MN} \in \mathbb{R}\}$ and 2-order contravariant tensor space $I \triangleq \{I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} | I^{MN} \in \mathbb{R}\}$:

$$G : I \to T, \quad I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} \mapsto G(I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N}) \triangleq g_{MP} g_{NQ} I^{MN} dx^P \otimes dx^Q \triangleq I_{PQ} dx^P \otimes dx^Q,$$

$$G^{-1} : T \to I, \quad T_{MN} dx^M \otimes dx^N \mapsto G^{-1}(T_{MN} dx^M \otimes dx^N) \triangleq g^{MP} g^{NQ} T_{MN} \frac{\partial}{\partial x^P} \otimes \frac{\partial}{\partial x^Q} \triangleq T^{PQ} \frac{\partial}{\partial x^P} \otimes \frac{\partial}{\partial x^Q}.$$

3. Taken together, the above discussion can be expressed as a chart:

$$\begin{array}{ccc} I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} & \overset{G}{\underset{G^{-1}}{\longleftrightarrow}} & I_{MN} dx^M \otimes dx^N \\ \downarrow id_g & & \downarrow id_x^* \\ I_{MN} \frac{\partial}{\partial x_M} \otimes \frac{\partial}{\partial x_N} & \overset{G}{\underset{G^{-1}}{\longleftrightarrow}} & I^{MN} dx_M \otimes dx_N \end{array}.$$

Discussion 2.2.8.3.

1. Tensors **k**, **l**, **v**, **w** also induce an automorphism of tangent space or cotangent space, but the way has some different from tensors **g**, **h**, **x**, **y**. It is because there are some differences between the reversible relation of tensor coefficient square matrix in Proposition 2.2.8.1 and the one in Proposition 2.2.8.2 . For example:

(1) Let coordinate k_M be defined by $dk_M \triangleq k_{MN} dx^N$. Due to $\left\langle \frac{\partial}{\partial k_M}, dk_N \right\rangle = \varepsilon_N^M$, then $\frac{\partial}{\partial k_M} = v^{MN} \frac{\partial}{\partial x^N}$. Further, $v^{MN} dk_M = dx^N$ and $k_{MN} \frac{\partial}{\partial k_M} = \frac{\partial}{\partial x^N}$ are obtained.

By coordinate transformation $x^M \mapsto k_N \triangleq k_N(x^M)$, tensor **v** induces an automorphism of cotangent space:

$$id_v^* : T^* \to T^*, \quad T_M dx^M \mapsto id_v^*(T_M dx^M) \triangleq T_M v^{MN} dk_N,$$

and tensor k induces an automorphism of tangent space:

$$id_k : T \to T, \quad T^M \frac{\partial}{\partial x^M} \mapsto id_k(T^M \frac{\partial}{\partial x^M}) \triangleq T^M k_{MN} \frac{\partial}{\partial k_N}.$$

(2) Let coordinate v_M be defined by $dv_M \triangleq v_{MN} dx^N$. Due to $\left\langle \frac{\partial}{\partial v_M}, dv_N \right\rangle = \varepsilon_N^M$, then $\frac{\partial}{\partial v_M} \triangleq k^{MN} \frac{\partial}{\partial x^N}$. Further, $k^{MN} dv_M = dx^N$ and $v_{MN} \frac{\partial}{\partial v_M} = \frac{\partial}{\partial x^N}$ are obtained.

By coordinate transformation $x^M \mapsto v_N \triangleq v_N(x^M)$, tensor **k** induces an automorphism of cotangent space:

$$id_k^* : T^* \to T^*, \quad T_M dx^M \mapsto id_k^*(T_M dx^M) \triangleq T_M k^{MN} dv_N,$$

and tensor v induces an automorphism of tangent space:

$$id_v : T \to T, \quad T^M \frac{\partial}{\partial x^M} \mapsto id_v(T^M \frac{\partial}{\partial x^M}) \triangleq T^M v_{MN} \frac{\partial}{\partial v_N}.$$

2. Tensors **k**, **l**, **v**, **w** induce an isomorphism between tangent space and cotangent space. For example, tensor k induces an isomorphism $K : T^* \to T,\ \ T_M dx^M \mapsto K(T_M dx^M) \triangleq T_M \frac{\partial}{\partial v_M} = T_M k^{MN} \frac{\partial}{\partial x^N}$, etc. Thus the isomorphisms between 2-order covariant tensor space T and 2-order contravariant tensor space I can be induced:

$$K : I \to T, \quad I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} \mapsto K(I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N}) \triangleq I^{MN} dk_M \otimes dk_N \triangleq k_{MP} k_{NQ} I^{MN} dx^P \otimes dx^Q,$$

$$K^{-1} : T \to I, \quad T_{MN} dx^M \otimes dx^N \mapsto K^{-1}(T_{MN} dx^M \otimes dx^N) \triangleq T_{MN} \frac{\partial}{\partial k_M} \otimes \frac{\partial}{\partial k_N} \triangleq v^{MP} v^{NQ} T_{MN} \frac{\partial}{\partial x^P} \otimes \frac{\partial}{\partial x^Q},$$

$$V : I \to T, \quad I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} \mapsto V(I^{MN} \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N}) \triangleq I^{MN} dv_M \otimes dv_N \triangleq v_{MP} v_{NQ} I^{MN} dx^P \otimes dx^Q,$$

$$V^{-1} : T \to I, \quad T_{MN} dx^M \otimes dx^N \mapsto V^{-1}(T_{MN} dx^M \otimes dx^N) \triangleq T_{MN} \frac{\partial}{\partial v_M} \otimes \frac{\partial}{\partial v_N} \triangleq k^{MP} k^{NQ} T_{MN} \frac{\partial}{\partial x^P} \otimes \frac{\partial}{\partial x^Q}.$$

3. Tensors **g** and **x** satisfy $g_{MN} = x_{MN}$ and $g^{MN} = x^{MN}$, so the coordinate g_M defined by $dg_M \triangleq g_{MN} dx^N$ and the coordinate x_M defined by $dx_M \triangleq x_{MN} dx^N$ are the same (only have a difference of an irrelevant integration constant). dg_M and dx_M can be denoted by dx_M uniformly. And also because of this, the four isomorphisms X, X^{-1}, G, G^{-1} between covariant tensor space and contravariant tensor space in Discussion 2.2.8.2 simply merged into two, namely G and G^{-1}.

However, tensors **k** and **v** do not have that relation as **g** and **x**. So k_M and v_M are essentially different coordinates. The four isomorphisms K^{-1}, K and V^{-1}, V are unable to simply merged into two. According to Proposition 2.2.8.2 , tensors **l** and **w** have the similar case with tensors **k** and **v**, so they will not be described repeatly.

Discussion 2.2.8.4. In a word, tensors **g**, **h**, **x**, **y** have better properties than tensors **k**, **l**, **v**, **w**. Therefore, the following sections will only adopt coordinates x^M, ξ^A, x_M, ξ_A to research the properties of reference-system, rather than v^A, v_A, k^A, k_A, l^M, l_M, w^M, w_M, etc.

2.2.9 Connections of reference-system

The essence of connection is to establish differentiation of vector on manifold. In this paper the definition of the well-known concept of affine connection is expressed as following form.

Definition 2.2.9.1. A connection D on tangent bundle or cotangent bundle is called an **affine connection**. Let $\Gamma^M_{NP} \triangleq \Gamma^M{}_{NP}$ are smooth real functions on manifold M. $\forall p \in M$, dx^M and $\frac{\partial}{\partial x^M}$ are natual basis vector fields in coordinate frame (U, x^M) of local reference-system $f(p)$. Consider the restriction of smooth real functions Γ^M_{NP} on U, affine connection can be expressed as:

$$\begin{cases} D\dfrac{\partial}{\partial x^N} \triangleq \Gamma^M_{NP} dx^P \otimes \dfrac{\partial}{\partial x^M} \\ Ddx^N \triangleq -\Gamma^N_{MP} dx^P \otimes dx^M \end{cases}, \tag{2}$$

where Γ^M_{NP} are called **affine connection coefficients** of D about coordinate form dx^M.

Discussion 2.2.9.1. In order to enable affine connection to describe intrinsic geometry, Γ^M_{NP} need to be defined as the one depending on the slack-tight B^A_M or C^M_A of reference-system on manifold, such as Levi-Civita connection

$$\Gamma^M_{NP} \triangleq \frac{1}{2}G^{MQ}\left(\frac{\partial G_{NQ}}{\partial x^P} + \frac{\partial G_{PQ}}{\partial x^N} - \frac{\partial G_{NP}}{\partial x^Q}\right),$$

or other forms.

According to Discussion 2.2.2.3 , when describing intrinsic geometry, any connection form selected is effective, as long as it depends on the slack-tight of the same reference-system on manifold. Considered that in a large quantity of experiment data, the simpler the characteristics, the easier they are to detect. So the simpler the connection form, the easier the theoretical form fits in with those characteristics and laws observed and induced from a large quantity of experiment data.

Levi-Civita connection is the unique torsion-free and metric-compatible connection. However it is regretful that it does not fit in with describing gauge potential in the way of section 6.4.3 and section 7.3.1 , on one hand it is because the description of intrinsic geometry by metric is not comprehensive enough, on the other hand Levi-Civita connection is not simple enough. Fortunately, the significant simplest affine connection that fits in with describing gauge potential does exist.

For Levi-Civita connection, the torsion-free condition is very helpful to simplify the theoretical form, but the metric-compatible condition restricts the further simplification of connection form. Considered that the metric-compatible condition $D\mathbf{G} = 0$ was introduced to establish the intuition of Levi-Civita parallel displacement, but it is not the condition that more general concept of parallel displacement must rely on. Therefore, in order to simplify connection further, it can be imagined that the torsion-free condition remains and the metric-compatible condition is given up. A nice choice is to adopt the following definition.

Definition 2.2.9.2. Let there be an affine connection D, which is expressed as equation (2) on performance coordinate frame (U, x^M). If the connection coefficients are defined as

$$\Gamma^M_{NP} \triangleq \frac{1}{2}C^M_A\left(\frac{\partial B^A_N}{\partial x^P} + \frac{\partial B^A_P}{\partial x^N}\right), \tag{3}$$

D is called a **simple connection**.

Discussion 2.2.9.2. Now it is needed to prove the simple connection is really a connection.

Let there be a reference-system f on manifold M. And let there be a local reference-system t_p, which induces a coordinate transformation $y^R = y^R(x^M)$ and a reference-system transformation F_{t_p}, and F_{t_p} transforms reference-system $f(p)$ whose coordinate representation is $x^M = x^M(\xi^A)$ to reference-system $h(p) \triangleq f(p) \circ t_p$ whose coordinate representation is $y^R = y^R(\xi^A)$. (U, ξ^A) is the common basis coordinate frame of $f(p)$ and $h(p)$. They can be expressed as a chart:

$$\begin{array}{ccc} (U,\xi^A) & \xrightarrow{h(p)} & (U,y^R) \\ {\scriptstyle f(p)}\downarrow & \nearrow t_p & \\ (U,x^M) & & \end{array}$$

Let the slack-tights of t_p be

$$\lambda_M^R \triangleq \frac{\partial y^R}{\partial x^M}, \quad \theta_R^M \triangleq \frac{\partial x^M}{\partial y^R}.$$

Let the slack-tights of f be B_M^A and C_A^M. For the restriction of them on U, the slack-tights after doing the transformation are

$$B'^A_R = \theta_R^M B_M^A, \quad C'^R_A = \lambda_M^R C_A^M.$$

According to Definition 2.2.9.2 , the simple connections before and after doing the transformation are respectively

$$\varGamma_{NP}^M \triangleq \frac{1}{2} C_A^M (\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N}), \quad \varGamma'^R_{ST} \triangleq \frac{1}{2} C'^R_A (\frac{\partial B'^A_S}{\partial y^T} + \frac{\partial B'^A_T}{\partial y^S}).$$

Calculate the local transformation relation of the simple connection:

$$\begin{aligned}
&\varGamma'^R_{ST} \triangleq \frac{1}{2} C'^R_A \left(\frac{\partial B'^A_S}{\partial y^T} + \frac{\partial B'^A_T}{\partial y^S} \right) = \frac{1}{2} \lambda_M^R C_A^M \left(\frac{\partial \left(\theta_S^N B_N^A \right)}{\partial y^T} + \frac{\partial \left(\theta_T^P B_P^A \right)}{\partial y^S} \right) \\
&= \frac{1}{2} \lambda_M^R C_A^M \left(\frac{\partial \theta_S^N}{\partial y^T} B_N^A + \theta_S^N \frac{\partial B_N^A}{\partial y^T} + \frac{\partial \theta_T^P}{\partial y^S} B_P^A + \theta_T^P \frac{\partial B_P^A}{\partial y^S} \right) \\
&= \frac{1}{2} \lambda_M^R C_A^M \left(\theta_S^N \frac{\partial B_N^A}{\partial y^T} + \theta_T^P \frac{\partial B_P^A}{\partial y^S} \right) + \frac{1}{2} \lambda_M^R C_A^M \left(\frac{\partial \theta_S^N}{\partial y^T} B_N^A + \frac{\partial \theta_T^P}{\partial y^S} B_P^A \right) \\
&= \frac{1}{2} \lambda_M^R C_A^M \left(\theta_S^N \theta_T^P \frac{\partial B_N^A}{\partial x^P} + \theta_T^P \theta_S^N \frac{\partial B_P^A}{\partial x^N} \right) + \frac{1}{2} \lambda_M^R C_A^M \left(\theta_T^P \frac{\partial \theta_S^N}{\partial x^P} B_N^A + \theta_S^N \frac{\partial \theta_T^P}{\partial x^N} B_P^A \right) \\
&= \frac{1}{2} \lambda_M^R C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right) \theta_S^N \theta_T^P + \frac{1}{2} \left(\lambda_M^R \frac{\partial \theta_S^M}{\partial x^P} \theta_T^P + \lambda_M^R \frac{\partial \theta_T^M}{\partial x^N} \theta_S^N \right) \\
&= \lambda_M^R \varGamma_{NP}^M \theta_S^N \theta_T^P + \lambda_M^R \frac{\partial \theta_S^M}{\partial y^T}.
\end{aligned}$$

This is completely consistent with the general tansformation relation of affine connection. So it has been proved that the simple connection $\varGamma_{NP}^M \triangleq \frac{1}{2} C_A^M \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right)$ is really a connection, and obviously it is torsion-free.

Remark 2.2.9.1. There are two obvious properties about simple connection.

(1) If defining $\varGamma_{MNP} \triangleq G_{MM'} \varGamma_{NP}^{M'}$, then

$$\varGamma_{MNP} = \frac{1}{2} \delta_{AB} B_M^B \left(\frac{\partial B_N^A}{\partial x^P} + \frac{\partial B_P^A}{\partial x^N} \right).$$

Now it is easy to validate that

$$\varGamma_{MNP} + \varGamma_{NPM} + \varGamma_{PMN} = \frac{1}{2} \left(\frac{\partial G_{MN}}{\partial x^P} + \frac{\partial G_{NP}}{\partial x^M} + \frac{\partial G_{PM}}{\partial x^N} \right).$$

(2) It is obvious that when G_{MN} are all constants, Levi-Civita connection must be zero, and the corresponding Riemannian curvature tensor also must be zero. However, in this case, simple connection is not necessary to be zero, and the corresponding Riemannian curvature tensor is also not necessary to be zero. This indicates that simple connection reflects much more bending properties of manifold than Levi-Civita connection.

2.3 Time metric and space metric

2.3.1 Definition of time metric

Definition 2.3.1.1. On a neighborhood U of point p on geometric manifold (M, f), as Definition 2.2.8.4 says, the two coordinate frames (U, ξ^A) and (U, x^M) of reference-system $f(p)$ inherit Euclidean metric tensors $\mathbf{g} \triangleq \delta_{AB} d\xi^A \otimes d\xi^B$ and $\mathbf{h} \triangleq \varepsilon_{MN} dx^M \otimes dx^N$ respectively. Then there are two metrics defined as

$$\begin{cases} (d\xi^0)^2 \triangleq \delta_{AB} d\xi^A d\xi^B = g_{MN} dx^M dx^N \\ (dx^0)^2 \triangleq \varepsilon_{MN} dx^M dx^N = h_{AB} d\xi^A d\xi^B \end{cases} \tag{4}$$

on U. The $d\xi^0$ and dx^0 are respectively called the **total space metrics** of coordinate frames (U, ξ^A) and (U, x^M), or called the **time metrics** on local coordinate frame.

On geometric manifold (M, f) there are metric tensors constructed in Definition 2.2.8.4 as

$$\begin{cases} \mathbf{G} \triangleq \varDelta_{AB} d\xi^A \otimes d\xi^B = G_{MN} dx^M \otimes dx^N \\ \mathbf{H} \triangleq \mathrm{E}_{MN} dx^M \otimes dx^N = H_{AB} d\xi^A \otimes d\xi^B \end{cases}, \quad \begin{cases} G_{MN} = \varDelta_{AB} B^A_M B^B_N \\ H_{AB} = \mathrm{E}_{MN} C^M_A C^N_B \end{cases}.$$

The $d\xi^0$ and dx^0 defined according to differential forms

$$\begin{cases} (d\xi^0)^2 \triangleq \varDelta_{AB} d\xi^A d\xi^B = G_{MN} dx^M dx^N \\ (dx^0)^2 \triangleq \mathrm{E}_{MN} dx^M dx^N = H_{AB} d\xi^A d\xi^B \end{cases} \tag{5}$$

are respectively called the **total space metrics** about coordinate forms $d\xi^A$ and dx^M on M, or called the **time metrics** on M.

2.3.2 Definition of space metric

Definition 2.3.2.1. Let $1 \leqslant q \leqslant \mathfrak{D}$. On $\mathfrak{D}$-dimensional geometric manifold (M, f), each q-dimensional Euclidean subspace metric that inherits from coordinate frames (U, ξ^A) and (U, x^M) is called a **space metric** of coordinate frames (U, ξ^A) and (U, x^M).

Remark 2.3.2.1. The various space metrics on coordinate frames (U, ξ^A) and (U, x^M) can be uniformly expressed as

$$\begin{cases} (d\xi^{(A_1 A_2 \cdots A_q)})^2 \triangleq \sum\limits_{a=A_1, A_2, \cdots, A_q} (d\xi^a)^2 \\ (dx^{(M_1 M_2 \cdots M_q)})^2 \triangleq \sum\limits_{m=M_1, M_2, \cdots, M_q} (dx^m)^2 \end{cases}$$

or as

$$\begin{cases} d\xi^{(A_1 A_2 \cdots A_q)} \triangleq \pm \sqrt{\sum\limits_{a=A_1, A_2, \cdots, A_q} (d\xi^a)^2} \\ dx^{(M_1 M_2 \cdots M_q)} \triangleq \pm \sqrt{\sum\limits_{m=M_1, M_2, \cdots, M_q} (dx^m)^2} \end{cases},$$

where $1 \leqslant q \leqslant \mathfrak{D}$. Specially, the metrics on 1-dimensional manifold are $d\xi^{(A)} \triangleq \pm d\xi^{A}$ and $dx^{(M)} \triangleq \pm dx^{M}$. It can be seen in section 2.3.2.2 that the metric form is closely related to the evolution of reference-system. Usually the definition of metric adopts only positive sign, but here both positive and negative signs still remain, because the signs actually mark the two opposite directions of evolution, and they will bring convenience for expression.

Definition 2.3.2.2. Let P and N be closed submanifolds of manifold $M = P \times N$. Denote $r \triangleq \dim P$. Let $s, i = 1, \cdots, r$ and $a, m = r+1, \cdots, \mathfrak{D}$. Select some proper coordinate frames $\{\xi^A\}$ and $\{x^M\}$ such that on P there are coordinate frames $\{\xi^s\}$ and $\{x^i\}$ inherited from M, and on N there are coordinate frames $\{\xi^a\}$ and $\{x^m\}$ inherited from M. Correspondingly, two subspace metrics can be defined on the coordinate neighborhoods on P and N respectively:

$$\begin{cases} (d\xi^{(P)})^2 \triangleq \sum\limits_{s=1}^{r} (d\xi^s)^2 = \delta_{st} d\xi^s d\xi^t \\ (dx^{(P)})^2 \triangleq \sum\limits_{i=1}^{r} (dx^i)^2 = \varepsilon_{ij} dx^i dx^j \end{cases}, \qquad \begin{cases} (d\xi^{(N)})^2 \triangleq \sum\limits_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2 = \delta_{ab} d\xi^a d\xi^b \\ (dx^{(N)})^2 \triangleq \sum\limits_{m=r+1}^{\mathfrak{D}} (dx^m)^2 = \varepsilon_{mn} dx^m dx^n \end{cases}.$$

$d\xi^{(N)}$ and $dx^{(N)}$ are called the **propertime metrics** about coordinate frames $\{\xi^a\}$ and $\{x^m\}$ on N. For convenience, N is called an **internal space submanifold** and P is called an **external space submanifold**.

Remark 2.3.2.2. As differential forms, time metric and space metric are universal geometric properties on geometric manifold. According to the principle of universal relativity in section 2.2.5.2 , they necessarily can be used to cognize some universal physical properties of physical reality, which can be understood as the ontological time interval and space interval.

According to the viewpoint of section 5.2.1 , the evolution of light in vacuum can be understood as an evolution totally pointing to external space of geometric manifold. The value of the internal space metric is identically zero on this direction of evolution, and the external space metric is identically equal to the time metric. Thus, Einstein's principle of constancy of light velocity is implied in it automatically. In this section, time metric and space metric have strict definitions, based on which, total space metric is actually time metric, so it can be considered that the origins and essences of time and space are the same. The light velocity c in vacuum is only a superficial constant, which becomes explicit just when time metric and space metric are endowed with different dimensions such as second and meter. The selection of the dimensions cannot divide the connotation of concept.

2.4 Evolution direction and actual evolution

2.4.1 Definition of evolution

Definition 2.4.1.1. Let there be two reference-systems f and g on manifold M. If f and g motioning relatively and interacting mutually, namely $\forall p \in M$ such that

$$\psi_U(U) \xleftarrow{f(p)} \varphi_U(U) \xrightarrow{g(p)} \rho_U(U),$$

it is called that there is an **evolution** of reference-system f in referece-system g, or there is an **evolution** of reference-system f on geometric manifold (M, g), or f **evolves** on (M, g), or f **evolves** in g, for short. Meanwhile, it also can be say that g evolves in f, or g evolves on (M, f).

Definition 2.4.1.2. On manifold M, each smooth tangent vector field X determines a one-parameter group of diffeomorphisms $\varphi_X : M \times \mathbb{R} \to M$. φ_X is called **a set of evolution paths** on M, and X is called an **evolution direction field** on M.

Let $T \subseteq \mathbb{R}$ be an interval. If a smooth mapping $L_p : T \to M$ constitutes a regular submanifold of M, there exists a smooth tangent vector field X such that L_p is on the orbit $\varphi_{X,p}(t)$ of φ_X through point p. Now the mapping L_p is called an **evolution path** through p on M. The image set $L_p \triangleq L_p(T)$ is called a **world line** through p. The tangent vector $\frac{d}{dt} \triangleq [L_p]$ is called an **evolution direction** at p, or **direction** for short.

If it does not need to emphasize the point p, L_p can be denoted by L simply.

2.4.2 Coordinate form of evolution

Discussion 2.4.2.1. Denote $x \triangleq \varphi_U(p)$, $\xi \triangleq \psi_U(p)$, $\zeta \triangleq \rho_U(p)$. Let the time metrics on coordinate frames (U, φ_U), (U, ψ_U), (U, ρ_U) be respectively dx^0, $d\xi^0$, $d\zeta^0$.

The coordinate representations of reference-system $f(p) \triangleq \varphi_U^{-1} \circ \psi_U$ and $f^{-1}(p) \triangleq \psi_U^{-1} \circ \varphi_U$ are

$$\xi^A = \xi^A(x^M), \quad x^M = x_f^M(\xi^A),$$

and the coordinate representations of reference-system $g(p) \triangleq \varphi_U^{-1} \circ \rho_U$ and $g^{-1}(p) \triangleq \rho_U^{-1} \circ \varphi_U$ are

$$\zeta^A = \zeta^A(x^M), \quad x^M = x_g^M(\zeta^A).$$

If no confusion, x_f^M and x_g^M are uniformly denoted by x^M. Thus the coordinate representation of reference-systems $f(p)$ and $f^{-1}(p)$ are

$$\xi^A = \xi^A(x^M), \quad x^M = x^M(\xi^A),$$

and the coordinate representation of reference-systems $g(p)$ and $g^{-1}(p)$ are

$$\zeta^A = \zeta^A(x^M), \quad x^M = x^M(\zeta^A).$$

Definition 2.4.2.1. Consider reference-system f. Each world line L_p is a 1-dimensional regular submanifold of M, therefore on the open set $U_L \triangleq U \cap L_p$ of p there exist coordinate frames (U_L, φ_{UL}) and (U_L, ψ_{UL}) such that the regular embedding

$$\pi : L_p \to M, q \mapsto q \tag{6}$$

induce **coordinate mappings**

$$\begin{cases} \psi_{UL}^{-1} \circ \pi \circ \psi_U : \mathbb{R} \to \mathbb{R}^{\mathfrak{D}}, (\xi^0) \mapsto (\xi^A), \xi^A = \xi^A(\xi^0) \\ \varphi_{UL}^{-1} \circ \pi \circ \varphi_U : \mathbb{R} \to \mathbb{R}^{\mathfrak{D}}, (x^0) \mapsto (x^M), x^M = x^M(x^0) \end{cases},$$

which satisfy

$$\begin{cases} \sum_{A=1}^{\mathfrak{D}} \left(\frac{d\xi^A}{d\xi^0} \right)^2 = 1 \\ \sum_{M=1}^{\mathfrak{D}} \left(\frac{dx^M}{dx^0} \right)^2 = 1 \end{cases},$$

then the coordinates (ξ^A) and (ξ^0) are called **equivalent** on U_L. The equations

$$\begin{cases} \xi^A = \xi^A(\xi^0) \\ x^M = x^M(x^0) \end{cases}$$

describing the above coordinate mappings are called the **original parameter equation** of world line L_p.

Definition 2.4.2.2. By the action of the regular embedding π, the reference-systems on the neighborhood U of manifold M

$$\xi \triangleq f(p) = \varphi_U^{-1} \circ \psi_U,\ x \triangleq f^{-1}(p) = \psi_U^{-1} \circ \varphi_U \in REF_p(U)$$

induce **reference-systems on the neighborhood** U_L **of 1-dimensional regular submanifold** L_p

$$\xi^0 \triangleq f_L(p) \triangleq \varphi_{UL}^{-1} \circ \psi_{UL},\ x^0 \triangleq f_L^{-1}(p) \triangleq \psi_{UL}^{-1} \circ \varphi_{UL} \in REF_p(U_L).$$

$\forall q \in U_L$, if no confusion, denote $\xi^0 \triangleq \psi_{UL}(q)$, $x^0 \triangleq \varphi_{UL}(q)$, then the coordinate representations of reference-systems on U_L are

$$\begin{cases} \xi^0 = \xi^0(x^0) \\ x^0 = x^0(\xi^0) \end{cases}.$$

Using this representation, the original parameter equations become

$$\begin{cases} \xi^A = \xi^A(\xi^0) = \xi^A(\xi^0(x^0)) \triangleq \xi_L^A(x^0) \\ x^M = x^M(x^0) = x^M(x^0(\xi^0)) \triangleq x_L^M(\xi^0) \end{cases}.$$

The equations

$$\begin{cases} \xi^A = \xi_L^A(x^0) \\ x^M = x_L^M(\xi^0) \end{cases} \tag{7}$$

are called the **standard parameter equation** of world line L_p.

Definition 2.4.2.3. Put the above reference-systems and parameter equations together, it will be

$$\begin{cases} \xi^A = \xi^A(x^M) = \xi_L^A(x^0) \\ \xi^0 = \xi^0(x^0) \end{cases}, \quad \begin{cases} x^M = x^M(\xi^A) = x_L^M(\xi^0) \\ x^0 = x^0(\xi^0) \end{cases}, \tag{8}$$

and is called the **coordinate form of evolution** of f and f^{-1} in **arbitrary direction** $\frac{d}{dt}$, or called **coordinate evolution equation**. If no confusion, ξ^A and ξ_L^A as well as x^M and x_L^M are not to be distinguished, and are directly denoted by

$$\begin{cases} \xi^A = \xi^A(x^M) = \xi^A(x^0) \\ \xi^0 = \xi^0(x^0) \end{cases}, \quad \begin{cases} x^M = x^M(\xi^A) = x^M(\xi^0) \\ x^0 = x^0(\xi^0) \end{cases}. \tag{9}$$

Remark 2.4.2.1. Considered generally, for any evolution path $L_p \in \frac{d}{dt}$ defined on M and through p, if regular embedding π induces a mapping $\tilde{\pi} : h \to h_L$ or $\tilde{\pi} : h_L \to h$ between universal geometric properties on M and on its regular submanifold L_p, denoted by $h \simeq h_L$, then $h \simeq h_L$ is called the h **form of evolution** in arbitrary direction $\frac{d}{dt}$ at p, or called the h **evolution equation** in arbitrary direction $\frac{d}{dt}$.

2.4.3 Basis vector form of evolution

Discussion 2.4.3.1. Let L be an evolution path on manifold M, $\forall p \in L$. Suppose $T_p(M)$ and $T_p(L)$ are the tangent spaces at p respectively on M and L, and $T_p^*(M)$ and $T_p^*(L)$ are the cotangent spaces.

$\forall p \in L$, the regular embedding $\pi : L \to M,\ q \mapsto q$ induces tangent mapping

$$\pi_*|_p : T_p(L) \to T_p(M), \quad [\gamma_L] \mapsto [\gamma_L \circ \pi] \tag{10}$$

and cotangent mapping

$$\pi^*|_p : T_p^*(M) \to T_p^*(L), \quad df \mapsto d(\pi \circ f). \tag{11}$$

Restricting on evolution path L, the tangent mapping is an injection, and the cotangent mapping is a surjection.

(1) For tangent mapping, the reference-system is essentially a nonsingular coordinate transformation, the Jacobian determinant of which is non-zero, so the tangent mapping $\pi_*|_p$ is an injection [9].

(2) For cotangent mapping, $\forall w_0 dx^0 \in T_p^*(L),\ \exists w_M dx^M \in T_p^*(M), \quad \pi^*\big|_p(w_M dx^M) = w_0 dx^0$. In fact, when $w_M = w_0 \frac{\partial \xi^A}{\partial x^M} \frac{dx^0}{d\xi^A}\Big|_L$, we have

$$\pi^*|_p\,(w_M dx^M) = w_0 \frac{\partial \xi^A}{\partial x^M}\frac{dx^0}{d\xi^A}\bigg|_L \pi^*|_p\,(dx^M) = w_0 \frac{\partial \xi^A}{\partial x^M}\frac{dx^0}{d\xi^A}\bigg|_L \frac{dx^M}{dx^0}\bigg|_L dx^0 = w_0 dx^0.$$

So $\pi^*|_p$ is a surjection.

In addition, generally, the tangent mapping is not a surjection, and the cotangent mapping is not an injection. The former is true obviously. For the latter, let $w_M dx^M, v_M dx^M \in T_p^*(M)$ satisfy $\pi^*|_p\,(w_M dx^M) = \pi^*|_p\,(v_M dx^M)$ on L, namely $w_M\ \frac{dx^M}{dx^0}\Big|_L dx^0 = v_M\ \frac{dx^M}{dx^0}\Big|_L dx^0$. The dx^0 is the basic vector of contangent space $T_p^*(L)$, so the coefficients satisfy $w_M\ \frac{dx^M}{dx^0}\Big|_L = v_M\ \frac{dx^M}{dx^0}\Big|_L$. This holds only on the specific evolution path L, however it does not hold generally on any other evolution paths. So $w_M - v_M = 0$ does not hold generally, and then $\pi^*|_p$ is not an injection.

Definition 2.4.3.1. $\forall \frac{d}{dt_L} \in T_p(L),\ \forall df \in T_p^*(M)$, denote

$$\frac{d}{dt} \triangleq \pi_*|_p \left(\frac{d}{dt_L}\right) \in T_p(M), \quad df_L \triangleq \pi^*|_p\,(df) \in T_p^*(L). \tag{12}$$

$\frac{d}{dt}$ and $\frac{d}{dt_L}$ are called being **equivalent**. df and df_L are called being **homomorphic**. They are denoted by

$$\frac{d}{dt} \cong \frac{d}{dt_L}, \quad df \simeq df_L \tag{13}$$

called an **equivalence** and a **homomorphism** respectively.

The above locally defined concepts can also be applied to the entire manifold.

Suppose $\varGamma(T(M))$ and $\varGamma(T(L))$ are sets of all sections of tangent bundle respectively on M and on L. $\varGamma(T^*(M))$ and $\varGamma(T^*(L))$ are sets of all sections of cotangent bundle respectively on M and on L. Let there be vector fields $\frac{d}{dt_L} \in \varGamma(T(L)),\ df \in \varGamma(T^*(M)),\ \frac{d}{dt} \in \varGamma(T(M)),\ df_L \in \varGamma(T^*(L))$.

If $\forall p \in L,\ \frac{d}{dt}\big|_p \cong \frac{d}{dt_L}\Big|_p$, say $\frac{d}{dt}$ and $\frac{d}{dt_L}$ are **equivalent** on L, which is denoted by

$$\frac{d}{dt} \cong \frac{d}{dt_L}, \tag{14}$$

called an **equivalence**. Thus, the tangent mapping $\pi_*|_p$ at p induces a tangent mapping

$$\pi_* : \varGamma(T(L)) \to \varGamma(T(M)),\ \frac{d}{dt_L} \mapsto \frac{d}{dt} \tag{15}$$

on the entire path L.

If $\forall p \in L$, $df|_p \simeq df_L|_p$, say df and df_L are **homomorphic** on L, which is denoted by

$$df \simeq df_L, \tag{16}$$

called a **homomorphism**. Thus, the cotangent mapping $\pi^*|_p$ at p induces a cotangent mapping

$$\pi^* : \varGamma(T^*(M)) \to \varGamma(T^*(L)),\ df \mapsto df_L \tag{17}$$

on the entire path L.

Remark 2.4.3.1. If $\frac{d}{dt} \cong \frac{d}{dt_L}$ and $df \simeq df_L$, it is easy to know that $\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle$, denoted by $\frac{df}{dt} = \frac{df_L}{dt_L}$.

In fact, at any point, the tangent vectors $\frac{d}{dt}$ and $\frac{d}{dt_L}$ are respectively defined as equivalence classes $[\gamma]$ and $[\gamma_L]$ of parameter curves, which satisfy

$$\gamma = \gamma_L \circ \pi.$$

The cotengent vectors df and df_L are respectively defined as equivalence classes $[f]$ and $[f_L]$ of smooth functions, which satisfy

$$f_L = \pi \circ f.$$

Thus,

$$\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle \Leftrightarrow \langle [\gamma], [f] \rangle = \langle [\gamma_L], [f_L] \rangle \Leftrightarrow \frac{d(\gamma \circ f)}{dt} = \frac{d(\gamma_L \circ f_L)}{dt},$$

where

$$\gamma \circ f = (\gamma_L \circ \pi) \circ f, \quad \gamma_L \circ f_L = \gamma_L \circ (\pi \circ f).$$

Obviously, $\gamma \circ f = \gamma_L \circ f_L$, which guarantees $\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle$ is true.

Definition 2.4.3.2. The tangent mapping on tangent space induces a tangent mapping on 2-order tensor product space

$$\pi_* : \varGamma(T_p^*(M) \otimes T_p(L)) \to \varGamma(T_p^*(M) \otimes T_p(M)), \quad df \otimes \frac{d}{dt_L} \mapsto df \otimes \frac{d}{dt}.$$

tensor products $df \otimes \frac{d}{dt}$ and $df \otimes \frac{d}{dt_L}$ are called being **equivalent**, denoted by $df \otimes \frac{d}{dt} \cong df \otimes \frac{d}{dt_L}$ called an **equivalence**.

The cotangent mapping on cotangent space also induces a cotangent mapping on 2-order tensor product space

$$\pi^* : \Gamma(T_p^*(M) \otimes T_p(M)) \to \Gamma(T_p^*(L) \otimes T_p(M)), \quad df \otimes \frac{d}{dt} \mapsto df_L \otimes \frac{d}{dt},$$

tensor products $df \otimes \frac{d}{dt}$ and $df_L \otimes \frac{d}{dt}$ are called being **homomorphic**, denoted by $df \otimes \frac{d}{dt} \simeq df_L \otimes \frac{d}{dt}$ called a **homomorphism**.

Remark 2.4.3.2. This kind of concepts of being **equivalent** or **homomorphic** can be transplanted without hindrance to **any-order tensor product space** generated by tangent bundle and cotangent bundle. Therefore, in the following sections, definitions of similar concepts about equivalent or homomorphic tensor products will not be given one by one any more.

Definition 2.4.3.3. Suppose the coordinate forms of evolution of local reference-systems $f(p)$ and $f^{-1}(p)$ in direction $\frac{d}{dt}$ on evolution path L are respectively

$$\begin{cases} \xi^A = \xi^A(x^M) = \xi^A(x^0) \\ \xi^0 = \xi^0(x^0) \end{cases}, \quad \begin{cases} x^M = x^M(\xi^A) = x^M(\xi^0) \\ x^0 = x^0(\xi^0) \end{cases}.$$

$\forall p \in L$, on coordinate neighborhood U_L of point p, define

$$b_0^A \triangleq \frac{d\xi^A}{dx^0}, \quad b_0^0 \triangleq \frac{d\xi^0}{dx^0}, \quad c_0^M \triangleq \frac{dx^M}{d\xi^0}, \quad c_0^0 \triangleq \frac{dx^0}{d\xi^0},$$

$$\varepsilon_0^M \triangleq \frac{dx^M}{dx^0} = b_0^0 c_0^M = b_0^A c_A^M, \quad \delta_0^A \triangleq \frac{d\xi^A}{d\xi^0} = c_0^0 b_0^A = c_0^M b_M^A.$$

They determine smooth functions on the entire evolution path L:

$$\begin{cases} B_0^A : L \to \mathbb{R}, \quad p \mapsto B_0^A(p) \triangleq (b_{f(p)})_0^A(p) \\ C_0^M : L \to \mathbb{R}, \quad p \mapsto C_0^M(p) \triangleq (c_{f(p)})_0^M(p) \end{cases}, \quad \begin{cases} B_0^0 : L \to \mathbb{R}, \quad p \mapsto B_0^0(p) \triangleq (b_{f(p)})_0^0(p) \\ C_0^0 : L \to \mathbb{R}, \quad p \mapsto C_0^0(p) \triangleq (c_{f(p)})_0^0(p) \end{cases}.$$

For convenience, if no confusion, still using notations ε and δ, there are smooth functions:

$$\varepsilon_0^M \triangleq B_0^0 C_0^M = B_0^A C_A^M, \quad \delta_0^A \triangleq C_0^0 B_0^A = C_0^M B_M^A.$$

Define

$$d\xi_0 \triangleq \frac{dx^0}{d\xi^0} dx^0, \quad dx_0 \triangleq \frac{d\xi^0}{dx^0} d\xi^0.$$

$d\xi_0$ and dx_0 determine two new coordinate frames (U_L, ξ_0) and (U_L, x_0) in the degree of only an intergration constant difference. $d\xi_0$ and dx_0 become new natual basis vectors induced on cotangent space by new coordinate frames. Let the natual basis vectors induced on tangent space by new coordinate frames be $\frac{d}{d\xi_0}$ and $\frac{d}{dx_0}$, satisfying $\left\langle \frac{d}{d\xi_0}, d\xi_0 \right\rangle = \delta_0^0 = 1$ and $\left\langle \frac{d}{dx_0}, dx_0 \right\rangle = \varepsilon_0^0 = 1$. These basis vectors are all independent of integration constant.

On U_L, define

$$\bar{b}_A^0 \triangleq \frac{d\xi_A}{dx_0}, \quad \bar{b}_0^0 \triangleq \frac{d\xi_0}{dx_0}, \quad \bar{c}_M^0 \triangleq \frac{dx_M}{d\xi_0}, \quad \bar{c}_0^0 \triangleq \frac{dx_0}{d\xi_0},$$

$$\bar{\varepsilon}_M^0 \triangleq \frac{dx_M}{dx_0} = \bar{b}_0^0 \bar{c}_M^0 = \bar{b}_A^0 \bar{c}_M^A, \quad \bar{\delta}_A^0 \triangleq \frac{d\bar{\xi}_A}{d\bar{\xi}_0} = \bar{c}_0^0 \bar{b}_A^0 = \bar{c}_M^0 \bar{b}_A^M.$$

They determine smooth functions on the entire evolution path L:

$$\begin{cases}\bar{B}_A^0 : L \to \mathbb{R}, \quad p \mapsto \bar{B}_A^0(p) \triangleq (\bar{b}_{f(p)})_A^0(p) \\ \bar{C}_M^0 : L \to \mathbb{R}, \quad p \mapsto \bar{C}_M^0(p) \triangleq (\bar{c}_{f(p)})_M^0(p)\end{cases}, \quad \begin{cases}\bar{B}_0^0 : L \to \mathbb{R}, \quad p \mapsto \bar{B}_0^0(p) \triangleq (\bar{b}_{f(p)})_0^0(p) \\ \bar{C}_0^0 : L \to \mathbb{R}, \quad p \mapsto \bar{C}_0^0(p) \triangleq (\bar{c}_{f(p)})_0^0(p)\end{cases}.$$

For convenience, if no confusion, still using notations $\bar{\varepsilon}$ and $\bar{\delta}$, there are smooth functions:

$$\bar{\varepsilon}_M^0 \triangleq \bar{B}_0^0 \bar{C}_M^0 = \bar{B}_A^0 \bar{C}_M^A, \quad \bar{\delta}_A^0 \triangleq \bar{C}_0^0 \bar{B}_A^0 = \bar{C}_M^0 \bar{B}_A^M.$$

Remark 2.4.3.3. On U_L, the following equations hold:

$$\bar{\varepsilon}_M^0 \varepsilon_0^M = \frac{dx_M}{dx_0}\frac{dx^M}{dx^0} = \frac{d\xi^0 d\xi^0}{d\xi^0 d\xi^0} = 1, \quad \bar{\delta}_A^0 \delta_0^A = \frac{d\xi_A}{d\xi_0}\frac{d\xi^A}{d\xi^0} = \frac{dx^0 dx^0}{dx^0 dx^0} = 1.$$

Proposition 2.4.3.1. (Evolution lemma). Let there be an evolution path L on manifold M. $\forall w^M \frac{\partial}{\partial x^M},\ \bar{w}_M \frac{\partial}{\partial x_M} \in \mathit{\Gamma}(T(M)),\ \ \forall w^0 \frac{d}{dx^0},\ \bar{w}_0 \frac{d}{dx_0} \in \mathit{\Gamma}(T(L)),\ \ \forall w_M dx^M,\ \bar{w}^M dx_M \in \mathit{\Gamma}(T^*(M))$, and $\forall w_0 dx^0,\ \bar{w}^0 dx_0 \in \mathit{\Gamma}(T^*(L))$, the following conclusions hold:

$$\begin{cases} w^M \dfrac{\partial}{\partial x^M} \cong w^0 \dfrac{d}{dx^0} \Leftrightarrow w^M = w^0 \varepsilon_0^M \\ w_M dx^M \simeq w_0 dx^0 \Leftrightarrow \varepsilon_0^M w_M = w_0 \end{cases}, \quad \begin{cases} \bar{w}_M \dfrac{\partial}{\partial x_M} \cong \bar{w}_0 \dfrac{d}{dx_0} \Leftrightarrow \bar{w}_M = \bar{w}_0 \bar{\varepsilon}_M^0 \\ \bar{w}^M dx_M \simeq \bar{w}^0 dx_0 \Leftrightarrow \bar{\varepsilon}_M^0 \bar{w}^M = \bar{w}^0 \end{cases}.$$

Proof. The following locally discussion can also be applied on the entire manifold.

1. Consider the case that basis vectors are dx^M and $\frac{\partial}{\partial x^M}$.

For tangent vector,

$$\pi_*\left(\frac{d}{dx^0}\right) = \frac{dx^M}{dx^0}\frac{\partial}{\partial x^M} \Leftrightarrow \frac{dx^M}{dx^0}\frac{\partial}{\partial x^M} \cong \frac{d}{dx^0} \Leftrightarrow \varepsilon_0^M \frac{\partial}{\partial x^M} \cong \frac{d}{dx^0} \Leftrightarrow w^0 \varepsilon_0^M \frac{\partial}{\partial x^M} \cong w^0 \frac{d}{dx^0}.$$

Because the tangent mapping is an injection, then

$$w^M \frac{\partial}{\partial x^M} \cong w^0 \frac{d}{dx^0} \Leftrightarrow w^M = w^0 \varepsilon_0^M.$$

For cotangent vector, $dx^M \simeq \varepsilon_0^M dx^0 \Rightarrow w_M dx^M \simeq \varepsilon_0^M w_M dx^0$, then $w_M dx^M \simeq w_0 dx^0 \Leftrightarrow \varepsilon_0^M w_M = w_0$.

2. Consider the case that basis vectors are dx_M and $\frac{\partial}{\partial x_M}$.

For tangent vector,

$$\pi_*\left(\frac{d}{dx_0}\right) = \frac{dx_M}{dx_0}\frac{\partial}{\partial x_M} \Leftrightarrow \frac{dx_M}{dx_0}\frac{\partial}{\partial x_M} \cong \frac{d}{dx_0} \Leftrightarrow \bar{\varepsilon}_M^0 \frac{\partial}{\partial x_M} \cong \frac{d}{dx_0} \Leftrightarrow \bar{w}_0 \bar{\varepsilon}_M^0 \frac{\partial}{\partial x_M} \cong \bar{w}_0 \frac{d}{dx_0}.$$

Because the tangent mapping is an injection, then

$$\bar{w}_M \frac{\partial}{\partial x_M} \cong \bar{w}_0 \frac{d}{dx_0} \Leftrightarrow \bar{w}_M = \bar{w}_0 \bar{\varepsilon}_M^0.$$

For cotangent vector, $dx_M \simeq \bar{\varepsilon}_M^0 dx_0 \Rightarrow \bar{w}^M dx_M \simeq \bar{\varepsilon}_M^0 \bar{w}^M dx_0$, then $\bar{w}^M dx_M \simeq \bar{w}^0 dx_0 \Leftrightarrow \bar{\varepsilon}_M^0 \bar{w}^M = \bar{w}^0$.

□

Proposition 2.4.3.2. On the evolution path L, the following equations hold:

$$C_0^M = \delta_0^A C_A^M = \varepsilon_0^M C_0^0, \quad B_0^A = \varepsilon_0^M B_M^A = \delta_0^A B_0^0, \quad \bar{C}_M^0 = \bar{\delta}_A^0 \bar{C}_M^A = \bar{\varepsilon}_M^0 \bar{C}_0^0, \quad \bar{B}_A^0 = \bar{\delta}_A^0 \bar{B}_0^0 = \bar{\varepsilon}_M^0 \bar{B}_A^M.$$

Proof. The differential rule of real function of several variables guarantees these equations are obviously true on any neighborhood. This local conclusion can be applied on the entire evolution path L. □

Definition 2.4.3.4. The conclusions of Proposition 2.4.3.2 are called **basis vector evolution equations** of reference-system f on arbitrary evolution path of geometric manifold (M, f). Due to the expression form of slack-tight, they are also called **slack-tight evolution equations** of f.

Discussion 2.4.3.2. According to the evolution lemma of Proposition 2.4.3.1 and the slack-tight evolution equations of Proposition 2.4.3.2 , on arbitrary evolution path of geometric manifold (M, f), the following conclusions hold:

$$\begin{cases} C_0^M \dfrac{\partial}{\partial x^M} \cong C_0^0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \Leftrightarrow C_0^M = C_0^0 \varepsilon_0^M \\ d\xi^A = B_M^A dx^M \simeq B_0^A dx^0 \Leftrightarrow \varepsilon_0^M B_M^A = B_0^A \end{cases}, \quad \begin{cases} \bar{C}_M^0 \dfrac{\partial}{\partial x_M} \cong \bar{C}_0^0 \dfrac{d}{dx_0} = \dfrac{d}{d\xi_0} \Leftrightarrow \bar{C}_M^0 = \bar{C}_0^0 \bar{\varepsilon}_M^0 \\ d\xi_A = \bar{B}_A^M dx_M \simeq \bar{B}_A^0 dx_0 \Leftrightarrow \bar{\varepsilon}_M^0 \bar{B}_A^M = \bar{B}_A^0 \end{cases},$$

$$\begin{cases} B_0^A \dfrac{\partial}{\partial \xi^A} \cong B_0^0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0} \Leftrightarrow B_0^A = B_0^0 \delta_0^A \\ dx^M = C_A^M d\xi^A \simeq C_0^M d\xi^0 \Leftrightarrow \delta_0^A C_A^M = C_0^M \end{cases}, \quad \begin{cases} \bar{B}_A^0 \dfrac{\partial}{\partial \xi_A} \cong \bar{B}_0^0 \dfrac{d}{d\xi_0} = \dfrac{d}{dx_0} \Leftrightarrow \bar{B}_A^0 = \bar{B}_0^0 \bar{\delta}_A^0 \\ dx_M = \bar{C}_M^A d\xi_A \simeq \bar{C}_M^0 d\xi_0 \Leftrightarrow \bar{\delta}_A^0 \bar{C}_M^A = \bar{C}_M^0 \end{cases}.$$

So there is a definition as following.

Definition 2.4.3.5. The conclusions

$$\begin{cases} d\xi^A = B_M^A dx^M \simeq B_0^A dx^0 \\ C_0^M \dfrac{\partial}{\partial x^M} \cong C_0^0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \end{cases}, \quad \begin{cases} dx^M = C_A^M d\xi^A \simeq C_0^M d\xi^0 \\ B_0^A \dfrac{\partial}{\partial \xi^A} \cong B_0^0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0} \end{cases}$$

are called the **basis vector form of evolution** of reference-system f on arbitrary evolution path of geometric manifold (M, f). They can also be equivalently expressed as

$$\begin{cases} d\xi_A = \bar{B}_A^M dx_M \simeq \bar{B}_A^0 dx_0 \\ \bar{C}_M^0 \dfrac{\partial}{\partial x_M} \cong \bar{C}_0^0 \dfrac{d}{dx_0} = \dfrac{d}{d\xi_0} \end{cases}, \quad \begin{cases} dx_M = \bar{C}_M^A d\xi_A \simeq \bar{C}_M^0 d\xi_0 \\ \bar{B}_A^0 \dfrac{\partial}{\partial \xi_A} \cong \bar{B}_0^0 \dfrac{d}{d\xi_0} = \dfrac{d}{dx_0} \end{cases}.$$

2.4.4 Metric form of evolution

Definition 2.4.4.1. On neighborhood U_L, define

$$g_{00} \triangleq \frac{dx_0}{dx^0} = b_0^0 b_0^0, \quad g^{00} \triangleq \frac{dx^0}{dx_0} = c_0^0 c_0^0, \quad h_{00} \triangleq \frac{d\xi_0}{d\xi^0} = c_0^0 c_0^0, \quad h^{00} \triangleq \frac{d\xi^0}{d\xi_0} = b_0^0 b_0^0.$$

They determine the following smooth functions on evolution path L in the way of Definition 2.2.8.4 :

$$G_{00} \triangleq B_0^0 B_0^0, \quad G^{00} \triangleq C_0^0 C_0^0, \quad H_{00} \triangleq C_0^0 C_0^0, \quad H^{00} \triangleq B_0^0 B_0^0.$$

Proposition 2.4.4.1. On evolution path L, the following equations hold:

$$\frac{d}{dx_0} = G^{00} \frac{d}{dx^0}, \quad \frac{d}{d\xi_0} = H^{00} \frac{d}{d\xi^0}.$$

Proof. At any point, tangent vector $\frac{d}{dx_0}$ is expanded as $\frac{d}{dx_0} = X \frac{d}{dx^0}$ about basis $\frac{d}{dx^0}$, and tangent vector $\frac{d}{d\xi_0}$ is expanded as $\frac{d}{d\xi_0} = Y \frac{d}{d\xi^0}$ about basis $\frac{d}{d\xi^0}$.

$$\left\langle \frac{d}{dx_0}, dx_0 \right\rangle = 1 \Leftrightarrow \left\langle X \frac{d}{dx^0}, g_{00} dx^0 \right\rangle = 1 \Leftrightarrow X g_{00} = 1 \Leftrightarrow X = \frac{1}{g_{00}} = g^{00} \Rightarrow \frac{d}{dx_0} = g^{00} \frac{d}{dx^0},$$

$$\left\langle \frac{d}{d\xi_0}, d\xi_0 \right\rangle = 1 \Leftrightarrow \left\langle Y\frac{d}{d\xi^0}, h_{00}d\xi^0 \right\rangle = 1 \Leftrightarrow Yh_{00} = 1 \Leftrightarrow Y = \frac{1}{h_{00}} = h^{00} \Rightarrow \frac{d}{d\xi_0} = h^{00}\frac{d}{d\xi^0}.$$

This local conclusion can be applied on the entire evolution path, so $\frac{d}{dx_0} = G^{00}\frac{d}{dx^0}$ and $\frac{d}{d\xi_0} = H^{00}\frac{d}{d\xi^0}$ hold on L. □

Proposition 2.4.4.2. On evolution path L, the following equations hold:

$$H_{00} = H_{AB}\delta_0^A\delta_0^B, \quad G_{00} = G_{MN}\varepsilon_0^M\varepsilon_0^N.$$

Proof. On a coordinate neighborhood U_L of any point on L,

$$\begin{cases} h_{AB}\delta_0^A\delta_0^B = \varepsilon_{MN}c_A^M c_B^N\delta_0^A\delta_0^B = \varepsilon_{MN}\dfrac{dx^M}{d\xi^0}\dfrac{dx^N}{d\xi^0} = \dfrac{dx^0}{d\xi^0}\dfrac{dx^0}{d\xi^0} = h_{00} \\ g_{MN}\varepsilon_0^M\varepsilon_0^N = \delta_{AB}b_M^A b_N^B\varepsilon_0^M\varepsilon_0^N = \delta_{AB}\dfrac{d\xi^A}{dx^0}\dfrac{d\xi^B}{dx^0} = \dfrac{d\xi^0}{dx^0}\dfrac{d\xi^0}{dx^0} = g_{00} \end{cases}.$$

So $H_{00} = H_{AB}\delta_0^A\delta_0^B$ and $G_{00} = G_{MN}\varepsilon_0^M\varepsilon_0^N$ hold on the entire evolution path L. □

Discussion 2.4.4.1. Due to Proposition 2.4.4.2 , $G_{MN}\varepsilon_0^M\varepsilon_0^N = G_{00}$. And denote $G_{0N} \triangleq G_{MN}\varepsilon_0^M$, $G_{M0} \triangleq G_{MN}\varepsilon_0^N$. According to Definition 2.4.3.2 , on evolution path L, tensor $G = G_{MN}dx^M \otimes dx^N$ has the following homomorphisms of tensor products:

$$G_{MN}dx^M \otimes dx^N \simeq G_{0N}dx^0 \otimes dx^N \simeq G_{M0}dx^M \otimes dx^0 \simeq G_{00}dx^0 \otimes dx^0.$$

Similarly, on L there exist the following homomorphisms of tensor products:

$$G^{MN}dx_M \otimes dx_N \simeq G^{0N}dx_0 \otimes dx_N \simeq G^{M0}dx_M \otimes dx_0 \simeq G^{00}dx_0 \otimes dx_0,$$

$$H_{AB}d\xi^A \otimes d\xi^B \simeq H_{0B}d\xi^0 \otimes d\xi^B \simeq H_{A0}d\xi^A \otimes d\xi^0 \simeq H_{00}d\xi^0 \otimes d\xi^0,$$

$$H^{AB}d\xi_A \otimes d\xi_B \simeq H^{0B}d\xi_0 \otimes d\xi_B \simeq H^{A0}d\xi_A \otimes d\xi_0 \simeq H^{00}d\xi_0 \otimes d\xi_0.$$

Discussion 2.4.4.2. In a word, the homomorphisms of covariant metric tensors alway exist. However, generally, the below equivalences of contravariant tensors do not exist:

$$X^{MN}\frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N} \cong X^{M0}\frac{\partial}{\partial x^M} \otimes \frac{d}{dx^0} \cong X^{0N}\frac{d}{dx^0} \otimes \frac{\partial}{\partial x^N} \cong X^{00}\frac{d}{dx^0} \otimes \frac{d}{dx^0}.$$

It is because on a neighborhood of any point on L, when $\mathfrak{D} > 1$, x^{MN} cannot be expressed as the form like $y\varepsilon_0^M\varepsilon_0^N$. Otherwise, let $x^{MN} = y\varepsilon_0^M\varepsilon_0^N$, then:

$$\begin{cases} dx_M dx^M = x^{MN}dx_M dx_N = (y\varepsilon_0^M\varepsilon_0^N)dx_M dx_N = y\dfrac{dx^M dx_M dx^N dx_N}{dx^0 dx^0} = yg_{00}dx_M dx^M \Rightarrow y = \dfrac{1}{g_{00}} = g^{00}, \\ \mathfrak{D} = x^{MN}x_{MN} = (y\varepsilon_0^M\varepsilon_0^N)g_{MN} = y(g_{MN}\varepsilon_0^M\varepsilon_0^N) = yg_{00} \Rightarrow y = \dfrac{\mathfrak{D}}{g_{00}} = \mathfrak{D}g^{00}, \end{cases}$$

which contradict with each other. In the same way, the following equivalences of tensors do not exist:

$$X_{MN}\frac{\partial}{\partial x_M} \otimes \frac{\partial}{\partial x_N} \cong X_{0N}\frac{d}{dx_0} \otimes \frac{\partial}{\partial x_N} \cong X_{M0}\frac{\partial}{\partial x_M} \otimes \frac{d}{dx_0} \cong X_{00}\frac{d}{dx_0} \otimes \frac{d}{dx_0},$$

$$Y^{AB}\frac{\partial}{\partial \xi^A} \otimes \frac{\partial}{\partial \xi^B} \cong Y^{A0}\frac{\partial}{\partial \xi^A} \otimes \frac{d}{d\xi^0} \cong Y^{0B}\frac{d}{d\xi^0} \otimes \frac{\partial}{\partial \xi^B} \cong Y^{00}\frac{d}{d\xi^0} \otimes \frac{d}{d\xi^0},$$

$$Y_{AB}\frac{\partial}{\partial \xi_A}\otimes\frac{\partial}{\partial \xi_B}\cong Y_{0B}\frac{d}{d\xi_0}\otimes\frac{\partial}{\partial \xi_B}\cong Y_{A0}\frac{\partial}{\partial \xi_A}\otimes\frac{d}{d\xi_0}\cong Y_{00}\frac{d}{d\xi_0}\otimes\frac{d}{d\xi_0}.$$

Remark 2.4.4.1. It has been seen that although metric tensors G and X have relations $G_{MN}=X_{MN}$ and $G^{MN}=X^{MN}$ and metric tensors H and Y have relations $H_{AB}=Y_{AB}$ and $H^{AB}=Y^{AB}$, if considering the evolution induced by the regular embedding of evolution path, tensors G and H have better properties than tensors X and Y. Therefore, the evolutions of only G and H, instead of X and Y, will be used in the following sections.

Definition 2.4.4.2. The homomorphisms

$$G_{MN}dx^M\otimes dx^N\simeq G_{0N}dx^0\otimes dx^N\simeq G_{M0}dx^M\otimes dx^0\simeq G_{00}dx^0\otimes dx^0$$

$$H_{AB}d\xi^A\otimes d\xi^B\simeq H_{0B}d\xi^0\otimes d\xi^B\simeq H_{A0}d\xi^A\otimes d\xi^0\simeq H_{00}d\xi^0\otimes d\xi^0$$

are called the **metric form of evolution** of reference-system f on arbitrary evolution path of geometric manifold (M,f).

2.4.5 Definition of actual evolution

Definition 2.4.5.1. Let $\mathbb{V}^n$ be the set of all sections of n-order tensor bundle generated by tangent bundle $T(M)$ and cotangent bundle $T^*(M)$. $\forall \mathbf{T}\triangleq t^{\bullet}_{\bullet}\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}\in\mathbb{V}^n$, the absolute differential of $\mathbf{T}$ is

$$D\mathbf{T}\triangleq Dt^{\bullet}_{\bullet}\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}\triangleq t^{\bullet}_{\bullet;Q}dx^Q\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}.$$

On evolution path L, $t_L{}^{\bullet}_{\bullet}\triangleq\pi\circ t^{\bullet}_{\bullet}$ is a smooth real function induced by regular embedding π. Define

$$\begin{aligned}
&\mathbf{T}_L\triangleq t_L{}^{\bullet}_{\bullet}\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\},\\
&t_L{}^{\bullet}_{\bullet;0}\triangleq t^{\bullet}_{\bullet;P}\frac{dx^P}{dx^0},\\
&D_L\mathbf{T}_L\triangleq D_L t_L{}^{\bullet}_{\bullet}\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}\triangleq t_L{}^{\bullet}_{\bullet;0}dx^0\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}.
\end{aligned}$$

Define operators

$$\begin{aligned}
&\nabla:\mathbb{V}^n\to\mathbb{V}^{n+1},\quad \mathbf{T}\mapsto\nabla\mathbf{T}\triangleq\left\langle\frac{\partial}{\partial x^P}\frac{\partial}{\partial x_P},D\mathbf{T}\right\rangle\triangleq t^{\bullet}_{\bullet;P}\frac{\partial}{\partial x_P}\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\},\\
&\nabla_L:\mathbb{V}^n\to\mathit{\Gamma}(T(L))\otimes\mathbb{V}^n,\quad \mathbf{T}_L\mapsto\nabla_L\mathbf{T}_L\triangleq\left\langle\frac{d}{dx^0}\frac{d}{dx_0},D_L\mathbf{T}_L\right\rangle\triangleq t_L{}^{\bullet}_{\bullet;0}\frac{d}{dx_0}\otimes\{\frac{\partial}{\partial x^{\bullet}}\otimes dx^{\bullet}\}.
\end{aligned}\tag{18}$$

They are uniformly called the **(absolute) gradient operators** about connection D on manifold M. $\nabla\mathbf{T}$ and $\nabla_L\mathbf{T}_L$ are uniformly called the **(absolute) gradient** of tensor T, where

$$\nabla t^{\bullet}_{\bullet}\triangleq t^{\bullet}_{\bullet;P}\frac{\partial}{\partial x_P},\quad \nabla_L t_L{}^{\bullet}_{\bullet}\triangleq t_L{}^{\bullet}_{\bullet;0}\frac{d}{dx_0}$$

are uniformly called the **(absolute) gradient direction** of components of tensor $\mathbf{T}$.

Specially, for a smooth function as a zero-order tensor on M, the definition of gradient operator degenerates into

$$\nabla : C^{\infty}(M) \to \Gamma(T(M)), \quad f \mapsto \nabla f \triangleq \left\langle \frac{\partial}{\partial x^P}\frac{\partial}{\partial x_P}, df \right\rangle \triangleq \frac{\partial f}{\partial x_P}\frac{\partial}{\partial x^P} = \frac{\partial f}{\partial x^P}\frac{\partial}{\partial x_P},$$
$$\nabla_L : C^{\infty}(L) \to \Gamma(T(L)), \quad f_L \mapsto \nabla_L f_L \triangleq \left\langle \frac{d}{dx^0}\frac{d}{dx_0}, df_L \right\rangle \triangleq \frac{df_L}{dx_0}\frac{d}{dx^0} = \frac{df_L}{dx^0}\frac{d}{dx_0},$$

where ∇f and $\nabla_L f_L$ are uniformly called the **gradient direction** of smooth function f.

Remark 2.4.5.1. The gradient operator is a universal geometric property on geometric manifold.

Remark 2.4.5.2. According to evolution lemma, the homomorphism of cotangent vector field

$$Dt^{\bullet}_{\bullet} \triangleq t^{\bullet}_{\bullet;P}dx^P \simeq t_L{}^{\bullet}_{\bullet;0}dx^0 \triangleq D_L t_L{}^{\bullet}_{\bullet}$$

holds. Further more, the homomorphism of tensor product

$$D\mathbf{T} \triangleq Dt^{\bullet}_{\bullet} \otimes \{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \simeq D_L t_L{}^{\bullet}_{\bullet} \otimes \{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \triangleq D_L \mathbf{T}_L$$

holds. Specially, for the smooth function as a zero-order tensor, the homomorphism $df \simeq df_L$ holds.

It is especially significant that the following propositions hold.

Proposition 2.4.5.1. $\forall \mathbf{T} \in \mathbb{V}^n$, $\mathbf{T} \triangleq t^{\bullet}_{\bullet}\{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\}$. Let L be an evolution path on any orbit of the one-parameter group of diffeomorphisms determined by smooth gradient field $t^{\bullet}_{\bullet;M}\frac{\partial}{\partial x_M}$ on manifold. The following equivalence of tensor products holds on L necessarily:

$$\nabla\mathbf{T} \triangleq t^{\bullet}_{\bullet;M}\frac{\partial}{\partial x_M} \otimes \{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \cong t_L{}^{\bullet}_{\bullet;0}\frac{d}{dx_0} \otimes \{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \triangleq \nabla_L \mathbf{T}_L,$$

denoted by $\nabla \cong \nabla_L$, where $t_L{}^{\bullet}_{\bullet} \triangleq \pi \circ t^{\bullet}_{\bullet}$.

Proof. Because the tangent mapping is an injection, tangent vector field $t^{\bullet}_{\bullet;M}\frac{\partial}{\partial x_M} \in \Gamma(T(M))$ uniquely corresponds to a tangent vector field $X\frac{d}{dx_0} \in \Gamma(T(L))$ such that

$$t^{\bullet}_{\bullet;M}\frac{\partial}{\partial x_M} \cong X\frac{d}{dx_0}.$$

According to the evolution lemma,

$$t^{\bullet}_{\bullet;M} = X\left.\frac{dx_M}{dx_0}\right|_L, \quad dx^M \simeq \left.\frac{dx^M}{dx^0}\right|_L dx^0.$$

So there is a homomorphism

$$t^{\bullet}_{\bullet;M}dx^M \simeq X\left.\frac{dx_M}{dx_0}\right|_L \left.\frac{dx^M}{dx^0}\right|_L dx^0.$$

According to Definition 2.4.2.1 , the coordinate mapping induced by the regular embedding satisfies $(d\xi^0)^2 = \sum\limits_{A=1}^{\mathfrak{D}}(d\xi^A)^2$, further more, which is $dx_0dx^0 = dx_Mdx^M$ on evolution path L. Substitute it into the above homomorphism, then $t^{\bullet}_{\bullet;M}dx^M \simeq Xdx^0$. Due to the evolution lemma, $X = t^{\bullet}_{\bullet;M}\frac{dx^M}{dx^0} = t_L{}^{\bullet}_{\bullet;0}$. □

Proposition 2.4.5.2. For any smooth real function f on manifold M, let L be an evolution path on any orbit of the one-parameter group of diffeomorphisms determined by smooth gradient field $\frac{\partial f}{\partial x^M}\frac{\partial}{\partial x_M}$ on manifold. The following equivalence of tensor products holds on L necessarily:

$$\nabla f \triangleq \frac{\partial f}{\partial x^M}\frac{\partial}{\partial x_M} \cong \frac{df_L}{dx^0}\frac{d}{dx_0} \triangleq \nabla_L f_L,$$

where $f_L \triangleq \pi \circ f$.

Proof. This is a special case of Proposition 2.4.5.1 , so it holds obviously. □

Definition 2.4.5.2. (Actual evolution). The gradient operator is called the **actual evolution** on manifold. A gradient direction is called an **actual evolution direction**. A gradient direction field is called an **actual evolution direction field**. An evolution path on gradient line is called an **actual evolution path**.

Proposition 2.4.5.3. (Actual evolution theorem).

(1) $\forall \mathbf{T} \triangleq t^{\bullet}_{\bullet}\{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \in \mathbb{V}^n$, equations $t^{\bullet}_{\bullet;M} = t_L{}^{\bullet}_{\bullet;0}\bar{\varepsilon}^0_M$ and $t^{\bullet;M}_{\bullet} = t_L{}^{\bullet;0}_{\bullet}\varepsilon^M_0$ hold on manifold M if and only if their evolution direction field is the actual evolution direction field of $t^{\bullet}_{\bullet}$.

(2) For any smooth real function f, equations $\frac{\partial f}{\partial x^M} = \frac{df}{dx^0}\frac{dx_M}{dx_0}$ and $\frac{\partial f}{\partial x_M} = \frac{df}{dx_0}\frac{dx^M}{dx^0}$ hold on manifold M if and only if their evolution direction field is the actual evolution direction field of f.

Proof. (1) is a direct corollary of the evolution lemma and Proposition 2.4.5.1 . (2) is a special case of (1), and also is a direct corollary of the evolution lemma and Proposition 2.4.5.2 . □

Definition 2.4.5.3. (Actual evolution equation). Equation

$$t^{\bullet}_{\bullet;M} = t_L{}^{\bullet}_{\bullet;0}\bar{\varepsilon}^0_M \quad \text{or} \quad t^{\bullet;M}_{\bullet} = t_L{}^{\bullet;0}_{\bullet}\varepsilon^M_0$$

is called the **actual evolution equation** of $t^{\bullet}_{\bullet}$. Equation

$$\frac{\partial f}{\partial x^M} = \frac{df}{dx^0}\frac{dx_M}{dx_0} \quad \text{or} \quad \frac{\partial f}{\partial x_M} = \frac{df}{dx_0}\frac{dx^M}{dx^0}$$

is called the **actual evolution equation** of f.

Abstractly, in Proposition 2.4.5.1 , the equivalence of gradient operator $\nabla \cong \nabla_L$ induced by the regular embedding is called the **most general actual evolution equation**.

Remark 2.4.5.3. The actual evolution direction field always satisfies

$$\begin{cases} D\mathbf{T} \simeq D_L\mathbf{T}_L \\ \nabla\mathbf{T} \cong \nabla_L\mathbf{T}_L \end{cases}, \tag{19}$$

or written as

$$\begin{cases} t^{\bullet}_{\bullet;Q}dx^Q \simeq t^{\bullet}_{\bullet;0}dx^0 \\ t^{\bullet}_{\bullet;Q}\dfrac{\partial}{\partial x_Q} \cong t^{\bullet}_{\bullet;0}\dfrac{d}{dx_0} \end{cases}. \tag{20}$$

Noticed that for any smooth tensor product $U \triangleq u^{\bullet}_{\bullet Q}\{\frac{\partial}{\partial x^{\bullet}} \otimes dx^{\bullet}\} \otimes dx^Q$, system of 1-order nonhomogeneous linear equations $t^{\bullet}_{\bullet;Q} = u^{\bullet}_{\bullet Q}$ about $t^{\bullet}_{\bullet}$ always has a solution, thus U can necessarily determine an actual evolution direction field $\nabla t^{\bullet}_{\bullet}$ satisfying

$$\begin{cases} u^{\bullet}_{\bullet Q}dx^Q \simeq u^{\bullet}_{\bullet 0}dx^0 \\ u^{\bullet}_{\bullet Q}\dfrac{\partial}{\partial x_Q} \cong u^{\bullet}_{\bullet 0}\dfrac{d}{dx_0} \end{cases}. \tag{21}$$

Say that the actual evolution direction field $u^{\bullet}_{\bullet Q}\frac{\partial}{\partial x_Q}$ or $u^{\bullet}_{\bullet 0}\frac{d}{dx_0}$ is **determined by evolution form $u^{\bullet}_{\bullet Q}dx^Q$ of tensor product U**.

Remark 2.4.5.4. Now that the concepts about actual evolution have strict definitions, next it may be discussed that what more we can say about actual evolution. In fact, for any universal geometric property defined in form of tensor product on geometric manifold, including smooth function as zero-order tensor, its actual evolution may anyway be discussed.

In order to better connect with the traditional theory, only two important cases will be discussed about their actual evolutions in this paper. The one is the actual evolution of potential field of reference-system itself. The other is the case that the general charge of one reference-system evolves in another reference-system.

2.4.6 Actual evolution of potential field of reference-system

Definition 2.4.6.1. Let there be an evolution of reference-system f in reference-system g, namely $(U,\xi^A) \xleftarrow{f(p)} (U,x^M) \xrightarrow{g(p)} (U,\zeta^A)$. In the same coordinate frame (U,x^M), if not specified in the following sections, the notations here will always be adopted.

(1) Colon ":" is used to express the absolute derivative about connection Λ^M_{NP} of f on geometric manifold (M,f), and semicolon ";" is used to express the absolute derivative about connection Γ^M_{NP} of g on geometric manifold (M,g), such as

$$u^Q{}_{:P} = \frac{\partial u^Q}{\partial x^P} + u^H \Lambda^Q_{HP},$$

and

$$u^Q{}_{;P} = \frac{\partial u^Q}{\partial x^P} + u^H \Gamma^Q_{HP}.$$

A connection is also called a **potential field**, or **potential** for short.

(2) The notation

$$K^M_{NPQ} \triangleq \frac{\partial \Lambda^M_{NQ}}{\partial x^P} - \frac{\partial \Lambda^M_{NP}}{\partial x^Q} + \Lambda^H_{NQ}\Lambda^M_{HP} - \Lambda^H_{NP}\Lambda^M_{HQ}$$

is used to express the coefficients of Riemannian curvature of reference-system f, and the notation

$$R^M_{NPQ} \triangleq \frac{\partial \Gamma^M_{NQ}}{\partial x^P} - \frac{\partial \Gamma^M_{NP}}{\partial x^Q} + \Gamma^H_{NQ}\Gamma^M_{HP} - \Gamma^H_{NP}\Gamma^M_{HQ}$$

is used to express the coefficients of Riemannian curvature of reference-system g.

(3) The values of indices of internal space and external space in this section are according to Definition 6.1.1.1 .

Discussion 2.4.6.1. In order to describe the intrinsic geometry of gauge field, Λ^M_{NP} can be either the Levi-Civita connection or the simple connection in Definition 2.2.9.2 . For describing intrinsic geometry, the effectivenesses of them are the same and the simple connection just reflects some more bending properties.

In order to connect the form of traditional gauge theory, here let Λ^M_{NP} and Γ^M_{NP} be simple connections.

Moreover, for reference-system f, the Riemannian curvature tensor satisfies

$$K^M_{NPQ} = \Lambda^M_{NQ:P} - \Lambda^M_{NP:Q},$$

and its absolute divergence about index P is

$$K^M_{NPQ}{}^{:P} = \Lambda^M_{NQ:P}{}^{:P} - \Lambda^M_{NP:Q}{}^{:P} - K^M_{HPQ}\Lambda^{HP}_N.$$

According to Remark 2.4.5.3 , suppose $\frac{d}{dx_0}$ is along the actual evolution direction determined by curvature divergence form $K^M_{NPQ}{}^{:P}dx^Q$, that means

$$\begin{cases} K^M_{NPQ}{}^{:P}dx^Q \simeq \rho^M_{N0}dx^0 \\ K^M_{NPQ}{}^{:P}\dfrac{\partial}{\partial x_Q} \cong \rho^M_{N0}\dfrac{d}{dx_0} \end{cases} \tag{22}$$

holds, where

$$\rho^M_{N0} \triangleq K^M_{NPQ}{}^{:P}\varepsilon^Q_0. \tag{23}$$

Now according to the evolution lemma in Proposition 2.4.3.1 , we get the actual evolution equation

$$K^M_{NPQ}{}^{:P} = \rho^M_{N0}\bar{\varepsilon}^0_Q.$$

Denote

$$j^M_{NQ} \triangleq \rho^M_{N0}\bar{\varepsilon}^0_Q, \tag{24}$$

the actual evolution equation becomes

$$K^M_{NPQ}{}^{:P} = j^M_{NQ}. \tag{25}$$

Due to the above discussion or according to the actual evolution theorem in Proposition 2.4.5.3 , the following proposition is directly deduced.

Proposition 2.4.6.1. (General gauge field evolution theorem). The evolution equation $K^M_{NPQ}{}^{:P} = j^M_{NQ}$ holds on M if and only if its evolution direction field is the actual evolution direction field determined by form $K^M_{NPQ}{}^{:P}dx^Q$.

Definition 2.4.6.2. $K^M_{NPQ}{}^{:P} = j^M_{NQ}$ is called the **general Yang-Mills field equation** of reference-system f.

Definition 2.4.6.3. Except ρ^M_{N0}, there are $\rho^{M0}_N \triangleq G^{00}\rho^M_{N0}$, $\rho_{MN0} \triangleq G_{MM'}\rho^{M'}_{N0}$ and $\rho_{MN}{}^0 \triangleq G^{00}\rho_{MN0}$. Each of them is called a **general charge (density field)**, or **charge** for short. If no confusion, they are denoted by ρ^M_N or ρ_{MN}, or simply denoted by ρ for convenience.

2.4.7 Actual evolution of general charge of reference-system

Discussion 2.4.7.1. In order to better connect with the traditional theory, when discussing the acutal evolution of general charge of f evolving on geometric manifold (M, g), without loss of generality, only the case of charge tensor $\mathbf{F}^0 \triangleq \rho_{MN}{}^0 dx^M \otimes dx^N$ will be considered, and $\rho_{MN}{}^0$ is denoted by ρ_{MN} simply.

Definition 2.4.7.1. On manifold M, suppose reference-system f evolves in reference-system g, namely $(U, \xi^A) \xleftarrow{f(p)} (U, x^M) \xrightarrow{g(p)} (U, \zeta^A)$.

On geometric manifold (M, g), the absolute differential of tensor $\mathbf{F}^0$ is defined as $D\mathbf{F}^0 \triangleq \rho_{MN;R}dx^R \otimes dx^M \otimes dx^N$, where D is the simple connection of g. Denote $D\rho_{MN} \triangleq \rho_{MN;R}dx^R$ and call it the **charge differential form** of f evolving on (M, g).

Discussion 2.4.7.2. On (M, g), the absolute gradient of tensor $\mathbf{F}^0$ is $\nabla\mathbf{F}^0 \triangleq \rho_{MN;R}\frac{\partial}{\partial x_R} \otimes dx^M \otimes dx^N$. Then the absolute gradient direction $\nabla\rho_{MN} \triangleq \rho_{MN;R}\frac{\partial}{\partial x_R}$ is the actual evolution direction of ρ_{MN} of f evolving in g.

According to Definition 2.4.3.1 , on the actual evolution path L through any point p on (M,g), the charge ρ_{MN} has the following homomorphism and equivalence induced by regular embedding $\pi : L \to M$,

$$\begin{cases} D\rho_{MN} \triangleq \rho_{MN;R}dx^R \simeq \rho_{MN;0}dx^0 \triangleq D_L\rho_{MN}, \\ \nabla\rho_{MN} \triangleq \rho_{MN;R}\dfrac{\partial}{\partial x_R} \cong \rho_{MN;0}\dfrac{d}{dx_0} \triangleq \nabla_L\rho_{MN}, \end{cases}$$

due to which, we get Proposition 2.4.7.1 .

Definition 2.4.7.2. Formulas

$$\begin{cases} \rho_{MN;R}dx^R \simeq \rho_{MN;0}dx^0 \\ \rho_{MN;R}\dfrac{\partial}{\partial x_R} \cong \rho_{MN;0}\dfrac{d}{dx_0} \end{cases} \tag{26}$$

or

$$\begin{cases} \rho_{MN}{}^{;R}dx_R \simeq \rho_{MN}{}^{;0}dx_0 \\ \rho_{MN}{}^{;R}\dfrac{\partial}{\partial x^R} \cong \rho_{MN}{}^{;0}\dfrac{d}{dx^0} \end{cases} \tag{27}$$

are called the **actual evolution equation** of ρ_{MN} evolving in g, or called the **charge evolution equation**.

Proposition 2.4.7.1. (General charge evolution theorem). The charge evolution equation holds if and only if its evolution direction field is the actual evolution direction field of the charge.

Remark 2.4.7.1. According to Definition 2.4.5.2 , the actual evolution is a universal geometric property on geometric manifold. Due to the principle of universal relativity, the concept of actual evolution can be used to cognized an ontological universal physical property, which can be understood as the objective universal evolution of physical reality.

Like the viewpoint of section 2.2.9.1 that the origins and essences of time and space are the same, in the next section it will be seen that for the energy, mass, momentum, kinetic energy and potential energy, their origins and essences can also be regarded as the same. For example, the classical rest-mass is actually the total energy-momentum in direction of internal space. The strict connotations of these concepts will be described by the definitions of the next section.

2.4.8 Energy-momentum equation

Definition 2.4.8.1. For the evolution of reference-system f in reference-system g, the concepts about energy and momentum of general charge can be defined. For convenience, omitting some index notations, ρ_{MN} is denoted by ρ.

(1) $E^0 \triangleq \rho^{;0} \triangleq \rho^{;R}\bar{\varepsilon}^0_R$ and $E_0 \triangleq \rho_{;0} \triangleq \rho_{;R}\varepsilon^R_0$ are called the **energy (density)** of charge ρ evolving in g, or called **total energy**, **total momentum**, **total mass**, **total kinetic energy** or **total energy-momentum**. These terminologies are used to refer to the same concept. They are no difference essentially.

(2) $p^R \triangleq \rho^{;R}$ and $p_R \triangleq \rho_{;R}$ are called the **momentum (density)** of charge ρ evolving in g, or called **kinetic energy**, **energy** or **energy-momentum**.

(3) $H^0 \triangleq \frac{d\rho}{dx_0}$ and $H_0 \triangleq \frac{d\rho}{dx^0}$ are called the **canonical energy (density)** of charge ρ evolving in g, or called **canonical total momentum**, **canonical total kinetic energy**, etc.

(4) $P^R \triangleq \frac{\partial\rho}{\partial x_R}$ and $P_R \triangleq \frac{\partial\rho}{\partial x^R}$ are called the **canonical momentum (density)** of charge ρ evolving in g, or called **canonical kinetic energy**, etc.

(5) $V^0 \triangleq E^0 - H^0$ and $V_0 \triangleq E_0 - H_0$ are called the **scalar potential energy (density)** of interaction, or called **field action kinetic energy**.

(6) $V^R \triangleq p^R - P^R$ and $V_R \triangleq p_R - P_R$ are called the **vector potential energy (density)** of interaction, or called **field action momentum**.

Proposition 2.4.8.1. If and only if the evolution direction of ρ evolving in g is the actural evolution direction, equation

$$E_0 E^0 = p_R p^R$$

holds.

Proof. With the concepts of energy and momentum, the actual evolution equation of charge ρ can be expressed as

$$\begin{cases} E_0 dx^0 \simeq p_R dx^R \\ E_0 \dfrac{d}{dx_0} \cong p_R \dfrac{\partial}{\partial x_R} \end{cases}, \quad \begin{cases} E^0 dx_0 \simeq p^R dx_R \\ E^0 \dfrac{d}{dx^0} \cong p^R \dfrac{\partial}{\partial x^R} \end{cases}. \tag{28}$$

The conjugation between the actual evolution direction and the charge differential form is the directional derivative of ρ in the actual evolution direction, i.e.:

$$\frac{D_L\rho}{dt_{L\rho}} \triangleq \left\langle \frac{d}{dt_{L\rho}}, D_L\rho \right\rangle = \left\langle \frac{d}{dt_\rho}, D\rho \right\rangle \triangleq \frac{D\rho}{dt_\rho},$$

more explicitly,

$$\left\langle E_0 \frac{d}{dx_0}, E_0 dx^0 \right\rangle = \left\langle p_R \frac{\partial}{\partial x_R}, p_M dx^M \right\rangle,$$

which is $G^{00}E_0E_0 = G^{RM}p_R p_M$, i.e. $E_0 E^0 = p_R p^R$. □

Definition 2.4.8.2. Equation

$$E_0 E^0 = p_R p^R$$

is called the **general energy-momentum equation** of charge ρ of f evolving in g.

Remark 2.4.8.1. Specially, if g is a complete inertial reference-system defined later, the energy-momentum equation becomes

$$E_0^2 = \sum_{R=1}^{\mathfrak{D}} p_R^2$$

or

$$(E^0)^2 = \sum_{R=1}^{\mathfrak{D}} (p^R)^2.$$

Further more, the total energy-momentum in partial direction can be defined similarly, such as

$$E_{part} dx^{(N)} \triangleq \sum_{m=i_1,\cdots i_k;1\leqslant k\leqslant\mathfrak{D}} p_m dx^m, \quad E_{part}^2 \triangleq \sum_{m=i_1,\cdots i_k;1\leqslant k\leqslant\mathfrak{D}} p_m^2.$$

Proposition 2.4.8.2. The relations about energy-momentum of ρ

$$p^R = E^0 \frac{dx^R}{dx^0}, \quad p_R = E_0 \frac{dx_R}{dx_0} \tag{29}$$

hold if and only if the evolution direction of ρ is its actual evolution direction.

Proof. Starting from the equivalences $p^R\frac{\partial}{\partial x^R}\cong E^0\frac{d}{dx^0}$ and $p_R\frac{\partial}{\partial x_R}\cong E_0\frac{d}{dx_0}$ determined by the actual evolution, according to the evolution lemma, $p^R=E^0\frac{dx^R}{dx^0}$ and $p_R=E_0\frac{dx_R}{dx_0}$ are deduced immediately. □

Remark 2.4.8.2. This proposition can also be regarded as a corollary of the actual evolution theorem in Proposition 2.4.5.3 . In the actual evolution direction, the conclusion above is completely consistent with the classical definition

$$p=mv$$

of momentum in traditional theory.

2.4.9 Conservation of energy-momentum of charge

This section will deduce the conservation of energy-momentum of charge by calculating step by step.

Definition 2.4.9.1. Denote

$$\begin{cases}[\rho\Gamma_G]\triangleq\dfrac{\partial\rho}{\partial x^G}-\rho_{;G}\triangleq\dfrac{\partial\rho_{MN}}{\partial x^G}-\rho_{MN;G}=\rho_{MH}\Gamma^H_{NG}+\rho_{HN}\Gamma^H_{MG},\\[\rho\Gamma_0]\triangleq\dfrac{d\rho}{dx^0}-\rho_{;0}\triangleq\dfrac{d\rho_{MN}}{dx^0}-\rho_{MN;0}=\rho_{MH}\Gamma^H_{N0}+\rho_{HN}\Gamma^H_{M0},\end{cases}\qquad\begin{cases}[\rho\Gamma^Q]\triangleq G^{GQ}[\rho\Gamma_G],\\[\rho\Gamma^0]\triangleq G^{00}[\rho\Gamma_0],\end{cases}$$

therefore

$$\begin{cases}[\rho\Gamma_G]=P_G-p_G,\\[\rho\Gamma_0]=H_0-E_0\ ,\end{cases}\qquad\begin{cases}[\rho\Gamma^G]=P^G-p^G,\\[\rho\Gamma^0]=H^0-E^0\ .\end{cases}$$

And denote

$$\begin{cases}[\rho B_{PQ}]\triangleq\rho_{MH}\left(\dfrac{\partial\Gamma^H_{NQ}}{\partial x^P}-\dfrac{\partial\Gamma^H_{NP}}{\partial x^Q}\right)+\rho_{HN}\left(\dfrac{\partial\Gamma^H_{MQ}}{\partial x^P}-\dfrac{\partial\Gamma^H_{MP}}{\partial x^Q}\right),\\[\rho R_{PQ}]\triangleq\rho_{MH}R^H_{NPQ}+\rho_{HN}R^H_{MPQ},\end{cases}$$

then denote

$$\begin{cases}[\rho F_{PQ}]\triangleq\dfrac{\partial[\rho\Gamma_Q]}{\partial x^P}-\dfrac{\partial[\rho\Gamma_P]}{\partial x^Q},\\[\rho E_{PQ}]\triangleq[\rho\Gamma_Q]_{;P}-[\rho\Gamma_P]_{;Q}.\end{cases}$$

Proposition 2.4.9.1. The following two equations hold:

$$(1)[\rho F_{PQ}]=[\rho E_{PQ}];$$

$$(2)[\rho F_{PQ}]-[\rho B_{PQ}]=\left(\rho_{MH,P}\Gamma^H_{NQ}-\rho_{MH,Q}\Gamma^H_{NP}\right)+\left(\rho_{HN,P}\Gamma^H_{MQ}-\rho_{HN,Q}\Gamma^H_{MP}\right).$$

Proof.

$$\begin{aligned}&[\rho E_{PQ}]=[\rho\Gamma_Q]_{;P}-[\rho\Gamma_P]_{;Q}\\=&\left(\rho_{MH}\Gamma^H_{NQ}+\rho_{HN}\Gamma^H_{MQ}\right)_{;P}-\left(\rho_{MH}\Gamma^H_{NP}+\rho_{HN}\Gamma^H_{MP}\right)_{;Q}\\=&\left(\rho_{MH}\Gamma^H_{NQ}\right)_{;P}+\left(\rho_{HN}\Gamma^H_{MQ}\right)_{;P}-\left(\rho_{MH}\Gamma^H_{NP}\right)_{;Q}-\left(\rho_{HN}\Gamma^H_{MP}\right)_{;Q}\\=&\rho_{MH;P}\Gamma^H_{NQ}+\rho_{MH}\Gamma^H_{NQ;P}+\rho_{HN;P}\Gamma^H_{MQ}+\rho_{HN}\Gamma^H_{MQ;P}-\rho_{MH;Q}\Gamma^H_{NP}-\rho_{MH}\Gamma^H_{NP;Q}-\rho_{HN;Q}\Gamma^H_{MP}-\rho_{HN}\Gamma^H_{MP;Q}\end{aligned}$$

$$
\begin{aligned}
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH;P}\Gamma^H_{NQ} - \rho_{MH;Q}\Gamma^H_{NP}\right) + \left(\rho_{HN;P}\Gamma^H_{MQ} - \rho_{HN;Q}\Gamma^H_{MP}\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) \\
&\quad + \left(\left(\rho_{MH,P} - \rho_{MG}\Gamma^G_{HP} - \rho_{GH}\Gamma^G_{MP}\right)\Gamma^H_{NQ} - \left(\rho_{MH,Q} - \rho_{MG}\Gamma^G_{HQ} - \rho_{GH}\Gamma^G_{MQ}\right)\Gamma^H_{NP}\right) \\
&\quad + \left(\left(\rho_{HN,P} - \rho_{HG}\Gamma^G_{NP} - \rho_{GN}\Gamma^G_{HP}\right)\Gamma^H_{MQ} - \left(\rho_{HN,Q} - \rho_{HG}\Gamma^G_{NQ} - \rho_{GN}\Gamma^G_{HQ}\right)\Gamma^H_{MP}\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad + \left(\left(-\rho_{MG}\Gamma^G_{HP} - \rho_{GH}\Gamma^G_{MP}\right)\Gamma^H_{NQ} - \left(-\rho_{MG}\Gamma^G_{HQ} - \rho_{GH}\Gamma^G_{MQ}\right)\Gamma^H_{NP}\right) \\
&\quad + \left(\left(-\rho_{HG}\Gamma^G_{NP} - \rho_{GN}\Gamma^G_{HP}\right)\Gamma^H_{MQ} - \left(-\rho_{HG}\Gamma^G_{NQ} - \rho_{GN}\Gamma^G_{HQ}\right)\Gamma^H_{MP}\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad + \left(\left(-\rho_{MG}\Gamma^G_{HP}\Gamma^H_{NQ} - \rho_{GH}\Gamma^G_{MP}\Gamma^H_{NQ}\right) - \left(-\rho_{MG}\Gamma^G_{HQ}\Gamma^H_{NP} - \rho_{GH}\Gamma^G_{MQ}\Gamma^H_{NP}\right)\right) \\
&\quad + \left(\left(-\rho_{HG}\Gamma^G_{NP}\Gamma^H_{MQ} - \rho_{GN}\Gamma^G_{HP}\Gamma^H_{MQ}\right) - \left(-\rho_{HG}\Gamma^G_{NQ}\Gamma^H_{MP} - \rho_{GN}\Gamma^G_{HQ}\Gamma^H_{MP}\right)\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad - \left(\rho_{MG}\Gamma^G_{HP}\Gamma^H_{NQ} - \rho_{MG}\Gamma^G_{HQ}\Gamma^H_{NP}\right) - \left(\rho_{GH}\Gamma^G_{MP}\Gamma^H_{NQ} - \rho_{GH}\Gamma^G_{MQ}\Gamma^H_{NP}\right) \\
&\quad - \left(\rho_{HG}\Gamma^G_{NP}\Gamma^H_{MQ} - \rho_{HG}\Gamma^G_{NQ}\Gamma^H_{MP}\right) - \left(\rho_{GN}\Gamma^G_{HP}\Gamma^H_{MQ} - \rho_{GN}\Gamma^G_{HQ}\Gamma^H_{MP}\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad - \left(\rho_{MG}\Gamma^G_{HP}\Gamma^H_{NQ} - \rho_{MG}\Gamma^G_{HQ}\Gamma^H_{NP}\right) - \left(\rho_{GN}\Gamma^G_{HP}\Gamma^H_{MQ} - \rho_{GN}\Gamma^G_{HQ}\Gamma^H_{MP}\right) \\
&= \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q}\right) + \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad - \left(\rho_{MH}\Gamma^H_{GP}\Gamma^G_{NQ} - \rho_{MH}\Gamma^H_{GQ}\Gamma^G_{NP}\right) - \left(\rho_{HN}\Gamma^H_{GP}\Gamma^G_{MQ} - \rho_{HN}\Gamma^H_{GQ}\Gamma^G_{MP}\right) \\
&= \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) \\
&\quad + \rho_{MH}\left(\Gamma^H_{NQ;P} - \Gamma^H_{NP;Q} + \Gamma^H_{GQ}\Gamma^G_{NP} - \Gamma^H_{GP}\Gamma^G_{NQ}\right) + \rho_{HN}\left(\Gamma^H_{MQ;P} - \Gamma^H_{MP;Q} + \Gamma^H_{GQ}\Gamma^G_{MP} - \Gamma^H_{GP}\Gamma^G_{MQ}\right) \\
&= \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{HN,P}\Gamma^H_{MQ}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{HN,Q}\Gamma^H_{MP}\right) + \rho_{MH}\left(\frac{\partial\Gamma^H_{NQ}}{\partial x^P} - \frac{\partial\Gamma^H_{NP}}{\partial x^Q}\right) + \rho_{HN}\left(\frac{\partial\Gamma^H_{MQ}}{\partial x^P} - \frac{\partial\Gamma^H_{MP}}{\partial x^Q}\right) \\
&= \left(\rho_{MH,P}\Gamma^H_{NQ} - \rho_{MH,Q}\Gamma^H_{NP}\right) + \left(\rho_{HN,P}\Gamma^H_{MQ} - \rho_{HN,Q}\Gamma^H_{MP}\right) + \rho_{MH}\left(\frac{\partial\Gamma^H_{NQ}}{\partial x^P} - \frac{\partial\Gamma^H_{NP}}{\partial x^Q}\right) + \rho_{HN}\left(\frac{\partial\Gamma^H_{MQ}}{\partial x^P} - \frac{\partial\Gamma^H_{MP}}{\partial x^Q}\right) \\
&= \left(\rho_{MH,P}\Gamma^H_{NQ} - \rho_{MH,Q}\Gamma^H_{NP}\right) + \left(\rho_{HN,P}\Gamma^H_{MQ} - \rho_{HN,Q}\Gamma^H_{MP}\right) + [\rho B_{PQ}] \\
&= \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{MH}\frac{\partial\Gamma^H_{NQ}}{\partial x^P}\right) + \left(\rho_{HN,P}\Gamma^H_{MQ} + \rho_{HN}\frac{\partial\Gamma^H_{MQ}}{\partial x^P}\right) \\
&\quad - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{MH}\frac{\partial\Gamma^H_{NP}}{\partial x^Q}\right) - \left(\rho_{HN,Q}\Gamma^H_{MP} + \rho_{HN}\frac{\partial\Gamma^H_{MP}}{\partial x^Q}\right) \\
&= \frac{\partial\left(\rho_{MH}\Gamma^H_{NQ}\right)}{\partial x^P} + \frac{\partial\left(\rho_{HN}\Gamma^H_{MQ}\right)}{\partial x^P} - \frac{\partial\left(\rho_{MH}\Gamma^H_{NP}\right)}{\partial x^Q} - \frac{\partial\left(\rho_{HN}\Gamma^H_{MP}\right)}{\partial x^Q} \\
&= \frac{\partial}{\partial x^P}\left(\rho_{MH}\Gamma^H_{NQ} + \rho_{HN}\Gamma^H_{MQ}\right) - \frac{\partial}{\partial x^Q}\left(\rho_{MH}\Gamma^H_{NP} + \rho_{HN}\Gamma^H_{MP}\right) \\
&= \frac{\partial[\rho\Gamma_Q]}{\partial x^P} - \frac{\partial[\rho\Gamma_P]}{\partial x^Q} = [\rho F_{PQ}]. \quad \square
\end{aligned}
$$

Proposition 2.4.9.2. The following two equations hold:

$$(1)\frac{\partial p_P}{\partial x^Q}-\frac{\partial p_Q}{\partial x^P}-[\rho F_{PQ}]=0;$$
$$(2)\frac{dp_P}{dx^0}-\frac{\partial E_0}{\partial x^P}+p_Q\frac{\partial \varepsilon_0^Q}{\partial x^P}-[\rho F_{PQ}]\varepsilon_0^Q=0.$$

Proof. Accoring to Definition 2.4.8.1 ,

$$\frac{\partial P_P}{\partial x^Q}-\frac{\partial P_Q}{\partial x^P}=0\Leftrightarrow\frac{\partial p_P}{\partial x^Q}-\frac{\partial p_Q}{\partial x^P}+\frac{\partial[\rho\Gamma_P]}{\partial x^Q}-\frac{\partial[\rho\Gamma_Q]}{\partial x^P}=0\Leftrightarrow\frac{\partial p_P}{\partial x^Q}-\frac{\partial p_Q}{\partial x^P}-[\rho F_{PQ}]=0.$$

The cotangent mapping π^* induced by the regular embedding of evolution path maps

$$\frac{\partial p_P}{\partial x^Q}dx^Q-\frac{\partial p_Q}{\partial x^P}dx^Q-[\rho F_{PQ}]dx^Q$$

to the evolution path:

$$\pi^*:\frac{\partial p_P}{\partial x^Q}dx^Q\mapsto\frac{\partial p_P}{\partial x^Q}\frac{dx^Q}{dx^0}dx^0=\frac{dp_P}{dx^0}dx^0,$$
$$\pi^*:\frac{\partial p_Q}{\partial x^P}dx^Q\mapsto\frac{\partial p_Q}{\partial x^P}\frac{dx^Q}{dx^0}dx^0=\frac{\partial\left(p_Q\frac{dx^Q}{dx^0}\right)}{\partial x^P}dx^0-p_Q\frac{\partial}{\partial x^P}\left(\frac{dx^Q}{dx^0}\right)dx^0=\frac{\partial E_0}{\partial x^P}dx^0-p_Q\frac{\partial\varepsilon_0^Q}{\partial x^P}dx^0,$$
$$\pi^*:[\rho F_{PQ}]dx^Q\mapsto[\rho F_{PQ}]\frac{dx^Q}{dx^0}dx^0=[\rho F_{PQ}]\varepsilon_0^Q dx^0.$$

Then

$$\frac{dp_P}{dx^0}dx^0-\frac{\partial E_0}{\partial x^P}dx^0+p_Q\frac{\partial\varepsilon_0^Q}{\partial x^P}dx^0-[\rho F_{PQ}]\varepsilon_0^Q dx^0=0,$$

finally

$$\frac{dp_P}{dx^0}-\frac{\partial E_0}{\partial x^P}+p_Q\frac{\partial\varepsilon_0^Q}{\partial x^P}-[\rho F_{PQ}]\varepsilon_0^Q=0.\ \square$$

Proposition 2.4.9.3. With torsion-free connection, the following two equations hold:

$$(1)p_{P;Q}-p_{Q;P}-[\rho B_{PQ}]=0;$$
$$(2)p_{P;0}-E_{0;P}+p_Q\varepsilon_{0;P}^Q-[\rho B_{PQ}]\varepsilon_0^Q=0.$$

Proof. According to equation (2) of Proposition 2.4.9.2 , $\frac{\partial p_P}{\partial x^Q}-\frac{\partial p_Q}{\partial x^P}-[\rho F_{PQ}]=0$. Substitute equation (2) of Proposition 2.4.9.1 into this equation, then we get

$$\frac{\partial p_P}{\partial x^Q}-\frac{\partial p_Q}{\partial x^P}-\left(\rho_{MH,P}\Gamma_{NQ}^H-\rho_{MH,Q}\Gamma_{NP}^H\right)-\left(\rho_{HN,P}\Gamma_{MQ}^H-\rho_{HN,Q}\Gamma_{MP}^H\right)=[\rho B_{PQ}]$$
$$\Leftrightarrow\frac{\partial\rho_{MN;P}}{\partial x^Q}-\frac{\partial\rho_{MN;Q}}{\partial x^P}-\left(\rho_{MH,P}\Gamma_{NQ}^H-\rho_{MH,Q}\Gamma_{NP}^H\right)-\left(\rho_{HN,P}\Gamma_{MQ}^H-\rho_{HN,Q}\Gamma_{MP}^H\right)=[\rho B_{PQ}]$$
$$\Leftrightarrow\left(\frac{\partial\rho_{MN;P}}{\partial x^Q}-\rho_{MH,P}\Gamma_{NQ}^H-\rho_{HN,P}\Gamma_{MQ}^H\right)-\left(\frac{\partial\rho_{MN;Q}}{\partial x^P}-\rho_{MH,Q}\Gamma_{NP}^H-\rho_{HN,Q}\Gamma_{MP}^H\right)=[\rho B_{PQ}]$$
$$\Leftrightarrow\left(\frac{\partial\rho_{MN;P}}{\partial x^Q}-\rho_{MH,P}\Gamma_{NQ}^H-\rho_{HN,P}\Gamma_{MQ}^H-\rho_{MN;H}\Gamma_{PQ}^H\right)$$
$$-\left(\frac{\partial\rho_{MN;Q}}{\partial x^P}-\rho_{MH,Q}\Gamma_{NP}^H-\rho_{HN,Q}\Gamma_{MP}^H-\rho_{MN;H}\Gamma_{QP}^H\right)+\rho_{MN;H}\left(\Gamma_{PQ}^H-\Gamma_{QP}^H\right)=[\rho B_{PQ}]$$
$$\Leftrightarrow\rho_{MN;P;Q}-\rho_{MN;Q;P}=[\rho B_{PQ}]$$
$$\Leftrightarrow p_{P;Q}-p_{Q;P}-[\rho B_{PQ}]=0.$$

The cotangent mapping π^* induced by the regular embedding of evolution path maps

$$p_{P;Q}dx^Q - p_{Q;P}dx^Q - [\rho B_{PQ}]dx^Q$$

to the evolution path:

$$\pi^* : p_{P;Q}dx^Q \mapsto p_{P;Q}\frac{dx^Q}{dx^0}dx^0 = p_{P;0}dx^0,$$
$$\pi^* : p_{Q;P}dx^Q \mapsto p_{Q;P}\frac{dx^Q}{dx^0}dx^0 = \left(\left(p_Q\frac{dx^Q}{dx^0}\right)_{;P} - p_Q\left(\frac{dx^Q}{dx^0}\right)_{;P}\right)dx^0 = E_{0;P}dx^0 - p_Q\varepsilon^Q_{0;P}dx^0,$$
$$\pi^* : [\rho B_{PQ}]dx^Q \mapsto [\rho B_{PQ}]\frac{dx^Q}{dx^0}dx^0 = [\rho B_{PQ}]\varepsilon^Q_0 dx^0.$$

Then

$$p_{P;0}dx^0 - E_{0;P}dx^0 + p_Q\varepsilon^Q_{0;P}dx^0 - [\rho B_{PQ}]\varepsilon^Q_0 dx^0 = 0,$$

finally

$$p_{P;0} - E_{0;P} + p_Q\varepsilon^Q_{0;P} - [\rho B_{PQ}]\varepsilon^Q_0 = 0. \quad \square$$

Proposition 2.4.9.4. With torsion-free connection, the following three equations hold:

$$(1) p_{P;Q} - p_{Q;P} - [\rho R_{PQ}] = 0;$$
$$(2) p_{P;0} - E_{0;P} + p_Q\varepsilon^Q_{0;P} - [\rho R_{PQ}]\varepsilon^Q_0 = 0;.$$
$$(3) [\rho B_{PQ}] = [\rho R_{PQ}].$$

Proof. The covariant derivative of $p_P \triangleq \rho_{;P} \triangleq \rho_{MN;P} = \rho_{MN,P} - \rho_{MH}\Gamma^H_{NP} - \rho_{HN}\Gamma^H_{MP}$ is:

$$p_{P;Q} = \rho_{MN;P;Q} = \rho_{MN;P,Q} - \rho_{MH;P}\Gamma^H_{NQ} - \rho_{HN;P}\Gamma^H_{MQ} - \rho_{MN;H}\Gamma^H_{PQ},$$
$$p_{Q;P} = \rho_{MN;Q;P} = \rho_{MN;Q,P} - \rho_{MH;Q}\Gamma^H_{NP} - \rho_{HN;Q}\Gamma^H_{MP} - \rho_{MN;H}\Gamma^H_{QP}.$$

Substract them:

$$\begin{aligned}
& p_{P;Q} - p_{Q;P} \\
&= \left(\rho_{MN;P,Q} - \rho_{MH;P}\Gamma^H_{NQ} - \rho_{HN;P}\Gamma^H_{MQ} - \rho_{MN;H}\Gamma^H_{PQ}\right) - \left(\rho_{MN;Q,P} - \rho_{MH;Q}\Gamma^H_{NP} - \rho_{HN;Q}\Gamma^H_{MP} - \rho_{MN;H}\Gamma^H_{QP}\right) \\
&= \left(\rho_{MN;P,Q} - \rho_{MN;Q,P}\right) + \left(\rho_{MH;Q}\Gamma^H_{NP} - \rho_{MH;P}\Gamma^H_{NQ}\right) + \left(\rho_{HN;Q}\Gamma^H_{MP} - \rho_{HN;P}\Gamma^H_{MQ}\right) \\
&+ \left(\rho_{MN;H}\Gamma^H_{QP} - \rho_{MN;H}\Gamma^H_{PQ}\right) \\
&= \left(\rho_{MN;P,Q} - \rho_{MN;Q,P}\right) + \left(\rho_{MH;Q}\Gamma^H_{NP} - \rho_{MH;P}\Gamma^H_{NQ}\right) + \left(\rho_{HN;Q}\Gamma^H_{MP} - \rho_{HN;P}\Gamma^H_{MQ}\right) \\
&= \left(\rho_{MN,P} - \rho_{MH}\Gamma^H_{NP} - \rho_{HN}\Gamma^H_{MP}\right)_{,Q} - \left(\rho_{MN,Q} - \rho_{MH}\Gamma^H_{NQ} - \rho_{HN}\Gamma^H_{MQ}\right)_{,P} \\
&\quad + \left(\rho_{MH,Q} - \rho_{MG}\Gamma^G_{HQ} - \rho_{GH}\Gamma^G_{MQ}\right)\Gamma^H_{NP} - \left(\rho_{MH,P} - \rho_{MG}\Gamma^G_{HP} - \rho_{GH}\Gamma^G_{MP}\right)\Gamma^H_{NQ} \\
&\quad + \left(\rho_{HN,Q} - \rho_{HG}\Gamma^G_{NQ} - \rho_{GN}\Gamma^G_{HQ}\right)\Gamma^H_{MP} - \left(\rho_{HN,P} - \rho_{HG}\Gamma^G_{NP} - \rho_{GN}\Gamma^G_{HP}\right)\Gamma^H_{MQ} \\
&= \left(\left(\rho_{MH}\Gamma^H_{NQ}\right)_{,P} + \left(\rho_{HN}\Gamma^H_{MQ}\right)_{,P}\right) - \left(\left(\rho_{MH}\Gamma^H_{NP}\right)_{,Q} + \left(\rho_{HN}\Gamma^H_{MP}\right)_{,Q}\right) \\
&\quad + \left(\rho_{MH,Q} - \rho_{MG}\Gamma^G_{HQ} - \rho_{GH}\Gamma^G_{MQ}\right)\Gamma^H_{NP} - \left(\rho_{MH,P} - \rho_{MG}\Gamma^G_{HP} - \rho_{GH}\Gamma^G_{MP}\right)\Gamma^H_{NQ} \\
&\quad + \left(\rho_{HN,Q} - \rho_{HG}\Gamma^G_{NQ} - \rho_{GN}\Gamma^G_{HQ}\right)\Gamma^H_{MP} - \left(\rho_{HN,P} - \rho_{HG}\Gamma^G_{NP} - \rho_{GN}\Gamma^G_{HP}\right)\Gamma^H_{MQ}
\end{aligned}$$

$$= \left(\rho_{MH,P}\Gamma^H_{NQ} + \rho_{MH}\Gamma^H_{NQ,P} + \rho_{HN,P}\Gamma^H_{MQ} + \rho_{HN}\Gamma^H_{MQ,P}\right) - \left(\rho_{MH,Q}\Gamma^H_{NP} + \rho_{MH}\Gamma^H_{NP,Q} + \rho_{HN,Q}\Gamma^H_{MP} + \rho_{HN}\Gamma^H_{MP,Q}\right)$$
$$+ \left(\rho_{MH,Q}\Gamma^H_{NP} - \rho_{MG}\Gamma^G_{HQ}\Gamma^H_{NP} - \rho_{GH}\Gamma^G_{MQ}\Gamma^H_{NP}\right) - \left(\rho_{MH,P}\Gamma^H_{NQ} - \rho_{MG}\Gamma^G_{HP}\Gamma^H_{NQ} - \rho_{GH}\Gamma^G_{MP}\Gamma^H_{NQ}\right)$$
$$+ \left(\rho_{HN,Q}\Gamma^H_{MP} - \rho_{HG}\Gamma^G_{NQ}\Gamma^H_{MP} - \rho_{GN}\Gamma^G_{HQ}\Gamma^H_{MP}\right) - \left(\rho_{HN,P}\Gamma^H_{MQ} - \rho_{HG}\Gamma^G_{NP}\Gamma^H_{MQ} - \rho_{GN}\Gamma^G_{HP}\Gamma^H_{MQ}\right)$$
$$= \left(\rho_{MH}\Gamma^H_{NQ,P} + \rho_{HN}\Gamma^H_{MQ,P}\right) - \left(\rho_{MH}\Gamma^H_{NP,Q} + \rho_{HN}\Gamma^H_{MP,Q}\right)$$
$$+ \left(-\rho_{MG}\Gamma^G_{HQ}\Gamma^H_{NP} - \rho_{GH}\Gamma^G_{MQ}\Gamma^H_{NP}\right) - \left(-\rho_{MG}\Gamma^G_{HP}\Gamma^H_{NQ} - \rho_{GH}\Gamma^G_{MP}\Gamma^H_{NQ}\right)$$
$$+ \left(-\rho_{HG}\Gamma^G_{NQ}\Gamma^H_{MP} - \rho_{GN}\Gamma^G_{HQ}\Gamma^H_{MP}\right) - \left(-\rho_{HG}\Gamma^G_{NP}\Gamma^H_{MQ} - \rho_{GN}\Gamma^G_{HP}\Gamma^H_{MQ}\right)$$
$$= \left(\rho_{MH}\Gamma^H_{NQ,P} + \rho_{HN}\Gamma^H_{MQ,P}\right) - \left(\rho_{MH}\Gamma^H_{NP,Q} + \rho_{HN}\Gamma^H_{MP,Q}\right)$$
$$+ \left(-\rho_{MH}\Gamma^H_{GQ}\Gamma^G_{NP} - \rho_{GH}\Gamma^G_{MQ}\Gamma^H_{NP}\right) - \left(-\rho_{MH}\Gamma^H_{GP}\Gamma^G_{NQ} - \rho_{GH}\Gamma^G_{MP}\Gamma^H_{NQ}\right)$$
$$+ \left(-\rho_{GH}\Gamma^H_{NQ}\Gamma^G_{MP} - \rho_{HN}\Gamma^H_{GQ}\Gamma^G_{MP}\right) - \left(-\rho_{GH}\Gamma^H_{NP}\Gamma^G_{MQ} - \rho_{HN}\Gamma^H_{GP}\Gamma^G_{MQ}\right)$$
$$= \rho_{MH}(\Gamma^H_{NQ,P} - \Gamma^H_{NP,Q} + \Gamma^H_{GP}\Gamma^G_{NQ} - \Gamma^H_{GQ}\Gamma^G_{NP}) + \rho_{HN}(\Gamma^H_{MQ,P} - \Gamma^H_{MP,Q} + \Gamma^H_{GP}\Gamma^G_{MQ} - \Gamma^H_{GQ}\Gamma^G_{MP})$$
$$= \rho_{MH}R^H_{NPQ} + \rho_{HN}R^H_{MPQ} = [\rho R_{PQ}].$$

That is $p_{P;Q} - p_{Q;P} - [\rho R_{PQ}] = 0$. And compare it with equation (1) of Proposition 2.4.9.3 , then $[\rho B_{PQ}] = [\rho R_{PQ}]$ is obtained. Finally, due to equation (2) of Proposition 2.4.9.3 , $p_{P;0} - E_{0;P} + p_Q\varepsilon^Q_{0;P} - [\rho R_{PQ}]\varepsilon^Q_0 = 0$ holds. □

Definition 2.4.9.2. Equations

$$\frac{dp_P}{dx^0} - \frac{\partial E_0}{\partial x^P} + p_Q\frac{\partial \varepsilon^Q_0}{\partial x^P} - [\rho F_{PQ}]\varepsilon^Q_0 = 0 \tag{30}$$

and

$$p_{P;0} - E_{0;P} + p_Q\varepsilon^Q_{0;P} - [\rho R_{PQ}]\varepsilon^Q_0 = 0 \tag{31}$$

are called the **conservation of energy-momentum** of charge ρ of f evolving in g.

Remark 2.4.9.1. For the way of consideration of conserved quantity in this paper, see Remark 2.4.11.1

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Definition 2.4.9.3. Formula

$$F_P \triangleq \frac{dp_P}{dx^0} = \frac{\partial E_0}{\partial x^P} - p_Q\frac{\partial \varepsilon^Q_0}{\partial x^P} + [\rho F_{PQ}]\varepsilon^Q_0 \tag{32}$$

is called the **interaction force (density)** on charge ρ. Formula

$$f_P \triangleq p_{P;0} = E_{0;P} - p_Q\varepsilon^Q_{0;P} + [\rho R_{PQ}]\varepsilon^Q_0 \tag{33}$$

is called the **absolute interaction force (density)** on charge ρ. These two formulas are uniformly called the **general Lorentz force equations**.

Remark 2.4.9.2. Lorentz force equation has a status as principle in traditional theory, but there is no need to have such a principle in the theory of this paper, because it automatically holds due to the definition of energy-momentum. Further more, it will transition to the traditional form on the following conditions.

Definition 2.4.9.4. The following two conditions are uniformly called the **traditional standard conditions**.

(1) $dE_0 = 0$ is called the **constant mass condition**.

(2) $\mathit{\Gamma}^M_{N0} \triangleq \mathit{\Gamma}^M_{NP}\varepsilon^P_0 = 0$ is called the **canonical mass condition**.

Remark 2.4.9.3. A few explanations for traditional standard conditions are given below.

1. Constant mass condition. In the Minkowski coordinate frame defined in section 6.1.3.1 , the evolution parameter x^0 becomes $\tilde{x}^\tau$, and the constant mass condition is correspondingly re-defined to $d\tilde{m}_\tau = 0$, where $\tilde{m}_\tau$ is the rest-mass. This fits in with the physical intuition of mass point model of traditional theory.

Now return to the general coordinate frame, according to Proposition 2.4.13.1 , on the constant mass condition $dE_0 = 0$, the actual evolution path of ρ is a geodesic line on (M, g), so the geodesic equations

$$\left(\frac{dx^M}{dx^0}\right)_{;N} = 0, \qquad \left(\frac{dx_M}{dx_0}\right)^{;N} = 0$$

hold. That is to say that on the constant mass condition, the motion of ρ in potential field $\mathit{\Gamma}^M_{NP}$ is equivalent to a free motion of ρ on geometric manifold (M, g).

2. Canonical mass condition. In the Minkowski coordinate frame, the canonical mass condition will be re-defined to $\tilde{\mathit{\Gamma}}^\mu_{\nu\tau} \triangleq \tilde{\mathit{\Gamma}}^\mu_{\nu\rho}\tilde{\varepsilon}^\rho_\tau = 0$.

(1) Take the electrodynamics for example. With natural units, the canonical energy-momentums of electric charged particle are

$$H = E + q\varphi, \qquad \boldsymbol{P} = \boldsymbol{p} + q\boldsymbol{A}.$$

Noticed that there is no concept of canonical mass $\tilde{M}_\tau$ in traditional theory. If defining

$$\tilde{A}_\tau \triangleq \varphi\gamma + \boldsymbol{A}\cdot\boldsymbol{u}, \qquad \tilde{M}_\tau \triangleq \tilde{m}_\tau + q\tilde{A}_\tau,$$

the canonical mass condition is actually

$$\varphi\gamma + \boldsymbol{A}\cdot\boldsymbol{u} = 0, \qquad \tilde{M}_\tau = \tilde{m}_\tau,$$

which can be understood as that when a charge ρ evolves in an electromagnetic potential field $(\varphi, \boldsymbol{A})$, the energy-momentum flow of $(\varphi, \boldsymbol{A})$ contributes just to the energy and momentum of ρ, but nothing to the rest-mass of ρ. This does fit in with the traditional physical intuition.

Based on the above reason and the following two reasons, at least it can be considered that the traditional theory regards $\tilde{M}_\tau$ and $\tilde{m}_\tau$ as the same by default.

(2) In the Minkowski coordinate frame, it will be seen later that the canonical mass condition which makes $\tilde{M}_\tau = \tilde{m}_\tau$ hold is the premise of the Legendre transformation and Euler-Lagrange equation remaining their traditional forms.

(3) In the general coordinate frame, on the constant mass condition and the canonical mass condition, the following conclusions hold.

$$\frac{\partial}{\partial x^N}\left(\frac{dx^M}{dx^0}\right) = 0, \quad \frac{\partial}{\partial x_N}\left(\frac{dx_M}{dx_0}\right) = 0, \qquad DE_0 = 0,$$

which make the general Lorentz force equation simplified to

$$\begin{cases} F_P \triangleq \dfrac{dp_P}{dx^0} = [\rho F_{PQ}]\varepsilon_0^Q, \\ f_P \triangleq p_{P;0} = [\rho R_{PQ}]\varepsilon_0^Q. \end{cases}.$$

In the Minkowski coordinate form defined later, it will be re-expressed as the traditional forms of Lorentz force equation:

$$\begin{cases} \tilde{F}_\rho \triangleq \dfrac{d\tilde{p}_\rho}{d\tilde{x}^\tau} = [\tilde{\rho}\tilde{F}_{\rho\sigma}]\tilde{\varepsilon}_\tau^\sigma, \\ \tilde{f}_\rho \triangleq \tilde{p}_{\rho;\tau} = [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}_\tau^\sigma. \end{cases}.$$

In a word, the two conditions in Definition 2.4.9.4 is necessary for the general theoretical form of this paper to transition to the traditional theoretical form.

2.4.10 Conservation of energy-momentum flow of potential field

Discussion 2.4.10.1. If there exists a symmetric tensor Y_{MP} the divergence of which satisfies

$$Y_{MP}{}^{;M} = -G^{00}(E_{0;P} - p_Q\varepsilon_{0;P}^Q + [\rho R_{PQ}]\varepsilon_0^Q),$$

then

$$p_P{}^{;0} = -Y_{MP}{}^{;M}.$$

Definition 2.4.10.1. Y_{MP} is called the **energy-momentum flow of potential field** of reference-system g. $p_P{}^{;0} = -Y_{MP}{}^{;M}$ is called the **conservation of energy-momentum flow of potential field** of g.

2.4.11 Conservation of total energy-momentum flow

Definition 2.4.11.1. $W_{MN} \triangleq E_0\frac{dx_M}{dx_0}\frac{dx_N}{dx_0}$ is called the **energy-momentum flow** of ρ, or called the **energy-momentum tensor** of ρ.

Discussion 2.4.11.1. In the actual evolution direction of ρ, consider

$$W_{MN} = E_0\frac{dx_M}{dx_0}\frac{dx_N}{dx_0} = \frac{dx_M}{dx_0}p_N,$$

then

$$W_{MN}{}^{;M} = \left(\frac{dx_M}{dx_0}p_N\right)^{;M} = p_N{}^{;M}\frac{dx_M}{dx_0} = p_N{}^{;0} = -Y_{MN}{}^{;M},$$

thus

$$(W_{MN} + Y_{MN})^{;M} = 0.$$

Denote $T_{MN} \triangleq W_{MN} + Y_{MN}$, then

$$T_{MN}{}^{;M} = 0.$$

Definition 2.4.11.2. T_{MN} is called the **total energy-momentum flow** of the actual evolution of ρ of f evolving in g, and also called the **total energy-momentum tensor**. ${T_{MN}}^{;M} = 0$ is called the **conservation of total energy-momentum flow**.

Remark 2.4.11.1. A conserved quantity is a geometric property. The most general abstract theory about conserved quantity is the Neother theorem, which relies on a concept of action defined by an abstract way. When the connotations of the functions in the abstract expression of action have no concrete construction, the Neother theorem would never tell us the concrete connotations of those conserved quantities determined by the action. One of the important purposes of this paper is exactly to solve the problem of the absence of concrete connotation, by the way of constructive definition.

From this perspective, the Neother theorem does not meet the needs of this paper. In other words, it is not enough to just content with the abstract way like the traditional theory to research conserved quantities.

Therefore, as a supplement to traditional theory, in this paper, Definition 2.4.8.1 and Definition 2.4.9.2 and so on do not consider conserved quantities in the abstract way, but based on the concept of reference-system, directly construct concrete connotations for the specific conserved quantities, then prove these concrete connotations make the conservations hold automatically.

The significance of the concrete connotations of these concepts is no less than the significance of the abstract summary of Neother theorem. They are two sides of the same thing.

2.4.12 General gravitational field equation

Discussion 2.4.12.1. Consider the actual evolution of reference-system f in reference-system g.

(1) Let $C_{(x)MN}$ be a 0-order or 2-order symmetric tensor satisfying ${C_{(x)MN}}^{;M} = 0$ only depending on g. Different tensors are distinguished by index (x). For any (x), let $c^{(x)} \in \mathbb{R}$ is a constant, then

$$\left(\sum_x c^{(x)} C_{(x)MN}\right)^{;M} = 0,$$

where the summation traverses all the 0-order or 2-order symmetric tensors with zero divergence depending only on g.

(2) Let ${T_{(\rho)MN}}^{;M} = 0$ be the conservation of total energy-momentum flow of charge ρ of f. Different charges are distinguished by index (ρ). For any (ρ), let $c^{(\rho)} \in \mathbb{R}$ is a constant, then

$$\left(\sum_\rho c^{(\rho)} T_{(\rho)MN}\right)^{;M} = 0,$$

where the summation traverses all the total energy-momentum tensors determined by various indices of charge $\rho \triangleq \rho_{PQ}$.

Therefore, $\forall c^{(x)}, c^{(\rho)} \in \mathbb{R}$,

$$\left(\sum_x c^{(x)} C_{(x)MN} + \sum_\rho c^{(\rho)} T_{(\rho)MN}\right)^{;M} = 0.$$

If the ergodic ranges of the summations are sufficiently large, it is deduced directly that

$$\sum_x c^{(x)} C_{(x)MN} + \sum_\rho c^{(\rho)} T_{(\rho)MN} = 0.$$

Definition 2.4.12.1. Equation $\sum\limits_x c^{(x)} C_{(x)MN} + \sum\limits_\rho c^{(\rho)} T_{(\rho)MN} = 0$ is called the **general gravitational field equation** of the actual evolution of reference-system f in reference-sytem g, where the dimensions among various terms are harmonized by constants $c^{(x)}, c^{(\rho)}$.

2.4.13 Evolution quantity

Suppose charge ρ of f evolves in g. On manifold M take a as the start point and b as the end point of an evolution path. Let $\mathbb{L}$ be the set of all evolution paths from a to b.

Definition 2.4.13.1. Let dx^0 be the time metric and satisfies $t_a \triangleq x^0(a) < x^0(b) \triangleq t_b$. $\forall L_\rho \in \mathbb{L}$, denote

$$s_{\rho W}\left(L_\rho\right) \triangleq \int_{L_\rho} D\rho = \int_{t_a}^{t_b} E_0 dx^0 = \int_{t_a}^{t_b} p_R dx^R = \int_{t_a}^{t_b} W_{MN} \frac{dx^M}{dx^0} \frac{dx^N}{dx^0} dx^0,$$

The functional $s_{\rho W}\left(L_\rho\right)$ about path L_ρ is called the **general evolution quantity (density functional)** of ρ evolving along path L_ρ.

Remark 2.4.13.1. In the Minkowski coordinate frame defined later, the evolution quantity will be expressed in form of well-known action, for example, evolution quantity $\int_{t_a}^{t_b} E_0 dx^0$ will be re-defined to $\int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tau$ in the Minkowski coordinate frame. In the actual evolution direction, the integrand of evolution quantity is a directional derivative in gradient direction, so the actual evolution path as the integral curve of gradient directions should satisfy the following proposition. In addition, it will be seen later that for $\int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tau$ there is also a concept of gradient direction in the Minkowski coordinate frame. Thus, the least action principle, which has a status as principle in traditional theory, becomes a theorem in this paper. First, a proposition in general coordinate form is given as below.

Proposition 2.4.13.1. (General evolution quantity extreme value theorem). For the charge ρ of f evolving in g, an evolution path L_ρ is exactly the actual evolution path if and only if $\delta s_{\rho W} = 0$.

Proof. Let the parameter equation of evolution path L_ρ be

$$x^R = x^R(x^0), \quad t_a \leqslant x^0 \leqslant t_b,$$

and let the parameter equation of evolution path $L_\rho + \delta L_\rho$ be

$$x^R = x^R(x^0) + \delta x^R(x^0), \quad t_a \leqslant x^0 \leqslant t_b, \quad \delta x^R(t_a) = \delta x^R(t_b) = 0.$$

Let the unit tangent vector of path L_ρ at any x^0 be

$$X \triangleq \pi_* \left(\frac{d}{dx^0}\right) \triangleq \left.\frac{dx^R}{dx^0}\right|_{x^0} \frac{\partial}{\partial x^R} = \varepsilon_0^R\left(x^0\right) \frac{\partial}{\partial x^R},$$

and let the unit tangent vector of path $L_\rho + \delta L_\rho$ be

$$X + \delta X \triangleq \left.\frac{d\left(x^R + \delta x^R\right)}{dx^0}\right|_{x^0} \frac{\partial}{\partial x^R} = \left.\left(\frac{dx^R}{dx^0} + \delta \frac{dx^R}{dx^0}\right)\right|_{x^0} \frac{\partial}{\partial x^R} = \left(\varepsilon_0^R\left(x^0\right) + \delta\varepsilon_0^R\left(x^0\right)\right) \frac{\partial}{\partial x^R}.$$

Then consider the variation of $s_{\rho W}(L_\rho)=\int_{L_\rho}E_0dx^0=\int_{L_\rho}p_R\varepsilon_0^Rdx^0$.

$$\varDelta s_{\rho W}(L_\rho)=\varDelta\int_{L_\rho}p_R\varepsilon_0^Rdx^0=\int_{L_\rho+\delta L_\rho}p_R\varepsilon_0^Rdx^0-\int_{L_\rho}p_R\varepsilon_0^Rdx^0=\int_{L_\rho+\delta L_\rho}\rho_{;R}\varepsilon_0^Rdx^0-\int_{L_\rho}\rho_{;R}\varepsilon_0^Rdx^0$$
$$=\int_{L_\rho+\delta L_\rho}\langle X,D\rho\rangle\,dx^0-\int_{L_\rho}\langle X,D\rho\rangle\,dx^0$$
$$=\int_{t_a}^{t_b}\left\langle X+\delta X,D\rho\left(x^R+\delta x^R\right)\right\rangle dx^0-\int_{t_a}^{t_b}\left\langle X,D\rho\left(x^R\right)\right\rangle dx^0$$
$$=\int_{t_a}^{t_b}\left\langle X+\delta X,D\rho(x^R)+\frac{\partial D\rho(x^R)}{\partial x^M}\delta x^M+o(\delta x)\right\rangle dx^0-\int_{t_a}^{t_b}\left\langle X,D\rho(x^R)\right\rangle dx^0$$
$$=\int_{t_a}^{t_b}\left(\langle X+\delta X,D\rho\rangle+\left\langle X+\delta X,\frac{\partial D\rho}{\partial x^M}\delta x^M\right\rangle\right)dx^0-\int_{t_a}^{t_b}\langle X,D\rho\rangle\,dx^0+o\left(\delta x\right)$$
$$=\int_{t_a}^{t_b}\left(\langle\delta X,D\rho\rangle+\left\langle X,\frac{\partial D\rho}{\partial x^M}\delta x^M\right\rangle\right)dx^0+o\left(\delta x\right)$$
$$=\int_{t_a}^{t_b}\left(\langle\delta X,D\rho\rangle+\langle X,\delta D\rho\rangle\right)dx^0+o\left(\delta x\right)$$
$$=\int_{t_a}^{t_b}\langle\delta X,D\rho\rangle\,dx^0+\int_{t_a}^{t_b}\delta D\rho+o\left(\delta x\right)$$
$$=\int_{t_a}^{t_b}\langle\delta X,D\rho\rangle\,dx^0+o\left(\delta x\right).$$

Thus we get

$$\delta s_{\rho W}=\int_{t_a}^{t_b}\langle\delta X,D\rho\rangle\,dx^0.$$

When point $b\to a$, $\delta ds_{\rho W}=\langle\delta X,D\rho\rangle\,dx^0$. The directional derivative $\langle X,D\rho\rangle=\rho_{;0}\cos\theta$, where θ is the included angle between the evolution direction X and the gradient direction. Take the directional variation,

$$\langle\delta X,D\rho\rangle=\rho_{;0}\delta\cos\theta=-\rho_{;0}\sin\theta\delta\theta.$$

Thus, the evolution quantity variation of ρ is

$$\delta ds_{\rho W}=-\rho_{;0}\sin\theta\delta\theta dx^0.$$

For general ρ, $\delta ds_{\rho W}=0$ if and only if $\sin\theta=0$, namely the evolution direction at this point is exactly the actual evolution direction (take the positive direction without loss of generality).

Take integration from a to b, then $\delta\int_{t_a}^{t_b}ds_{\rho W}=0$ if and only if the evolution direction at each point of integral curve L_ρ is the actual evolution direction of ρ. In other words, $\delta s_{\rho W}=0$ if and only if L_ρ is the actual evolution path of ρ. □

Remark 2.4.13.2. Compare the actual evolution equations of charge and potential field

$$\begin{cases}p_Rdx^R\simeq E_0dx^0,\\ p_R\dfrac{\partial}{\partial x_R}\cong E_0\dfrac{d}{dx_0},\end{cases}\qquad\begin{cases}{K_{NPQ}^M}^{;P}dx^Q\simeq\rho_{N0}^Mdx^0,\\ {K_{NPQ}^M}^{;P}\dfrac{\partial}{\partial x_Q}\cong\rho_{N0}^M\dfrac{d}{dx_0},\end{cases}$$

due to the expression form of the evolution quantity

$$s_{\rho W}=\int E_0dx^0=\int p_R\varepsilon_0^Rdx^0,$$

one may naturally associate the definition of evolution quantity of the potential field with

$$s_{\rho Y}=\int \rho^M_{N0}dx^0=\int K^{M}_{NPQ}{}^{:P}\varepsilon^Q_0 dx^0.$$

It is effective, because in the gradient direction $\frac{d}{dt}$ determined by the above evolution equation according to Remark 2.4.5.3 , $\delta s_{\rho Y}=0$ holds necessarily, which describes the actual evolution of potential field of f (in form of curvature divergence). Meanwhile it is remarkable that in the same direction $\frac{d}{dt}$, the form of $s_{\rho Y}$ satisfying $\delta s_{\rho Y}=0$ is not unique.

Traditional theory has already told us that the forms of action (evolution quantity) can be diverse, and different evolution quantities can be used to describe the same actual evolution. According to a concrete Yang-Mills field equation determined by a concrete reference-system f, correspondingly, kinds of evolution quantities $s_{\rho Y}$ about potential fields Λ^M_{NP} and Γ^M_{NP} can anyway be constructed such that the Yang-Mills field equation holds if and only if $\delta s_{\rho Y}=0$ holds.

The feasibility of these constructions makes it sure that when $s_\rho \triangleq s_{\rho W}+s_{\rho Y}$ is defined, no matter which effective form of $s_{\rho Y}$ is adopted, $\delta s_\rho=0$ can always be used to uniformly express the actual evolutions of both the charge and the potential field of f.

In addition, it has to be noticed that the effectiveness of the traditional action in the form that is similar to

$$\sum_{m,n=r+1,\cdots,\mathfrak{D}}\frac{1}{4}K^m_{nPQ}K^{mPQ}_n$$

is due to some occasionality. It is because it actually should be strictly written as

$$\sum_{m,n=r+1,\cdots,\mathfrak{D}}\frac{1}{4}K^m_{nPQ}R^{mPQ}_n,$$

which is however not appropriate to be used to deduce the Yang-Mills field equation. This is a complicated problem, which should be discussed in detail in further articles rather than here.

2.4.14 Evolution equations of quantum mechanics

Discussion 2.4.14.1. For any two smooth tangent vector fields X and Y on manifold M, let L_Y be the Lie derivative operator induced by the one-parameter group of diffeomorphisms φ_Y determined by Y. According to a well-known theorem [9], Lie derivative equation $[X,Y]=L_YX$ holds.

On one hand, suppose H is the unit tangent vector field along the actual evolution directions of ρ, and φ_H is the one-parameter group of diffeomorphisms determined by H, and the the parameter of φ_H is x^0. The Lie derivative equation induced by φ_H is $[X,H]=L_HX$. Lie derivative operator L_H and tangent vector field $\frac{d}{dx^0}$ are both uniquely determined by H, so it can be denoted that $\frac{d}{dx^0}X\triangleq L_HX$. Thus, the Lie derivative equation becomes $[X,H]=\frac{d}{dx^0}X$.

On the other hand, according to Remark 2.4.3.1 and the equivalence $H\cong H_L$ induced by the regular embedding of evolution path, for any smooth function f, equation $\langle H,df\rangle=\langle H_L,df_L\rangle$ holds. Notice that H_L and $\frac{d}{dx^0}$ are the same, so $Hf=\frac{d}{dx^0}f_L$ holds.

In a word, L_H and H_L are both uniquely determined by the actual evolution direction unit field H. Due to the above discussion, the following proposition holds immediately.

Proposition 2.4.14.1. Let H be the actual evolution direction unit field, for any X and any f, equations

$$[X,H]=\frac{d}{dx^0}X,\qquad Hf=\frac{d}{dx^0}f_L \tag{34}$$

hold if and only if $\frac{d}{dx^0}$ is the actual evolution direction unit field.

Definition 2.4.14.1. Equation $[X,H]=\frac{d}{dx^0}X$ is called the **general Heisenberg equation**. Equation $Hf=\frac{d}{dx^0}f_L$ is called the **general Schrödinger equation**

Discussion 2.4.14.2. Both the two equations describe the actual evolution. H can be defined as the actual evolution direction determined not only by charge differential form $D\rho$ like Discussion 2.4.14.1 , but also by curvature divergence form $R^{M}_{NPQ}{}^{;P}dx^Q$, or even by any other differential form.

The actual evolution is a universal geometric property on geometric manifold, so these two equations are applicable to arbitrary reference-system.

The essences of the quantum mechanics and the theory of this paper are the same. Both of them describe the actual evolution, just expression forms have differences.

(1) Heisenberg's matrix mechanics and Schrödinger's wave mechanics are two analysis theories about two mutual dual linear spaces. They describe the same actual evolution. However, the selection of mutual dual spaces is not unique. In quantum mechanics they are abstract operator space and state space, and in this paper they are concrete tangent bundle and cotangent bundle. The mutual transformation between Lie derivative operator L_H and tangent vector field H_L represents the mutual transformation between two pictures of mechanics. What remain unchanged during the transformation are their geometric properties. As such a geometric property, the actual evolution direction field is the common meaning of different pictures of mechanics.

(2) Notice that Definition 2.4.14.1 is not expressed in form of complex value. It is not important, because what equations in form of complex value describe is none other than the actual evolution. So equations in form of complex value necessarily can be deduced from a form of real value in a certain way, no matter for wave function or for field function. Such as the concrete deductive process of complex-valued Schrödinger equation of charge field function, see section 6.3.7.1 . It can be said that the value of a specific actual evolution direction is determined by intrinsic geometry, and has nothing to do with the form of either real value or complex value, the effects of which for describing intrinsic geometry are the same. During the transformation between the two theoretical forms, what remain unchanged are their geometric properties. As such a geometric property, the actual evolution direction field is the common meaning of different theoretical forms of quantum mechanics.

In a word, there is no need to be constrained on theoretical forms, and the actual evolution is the very essence should be grasped. Heisenberg equation and Schrödinger equation do not rely on complex form essentially, and they hold not only in the quantum mechanics.

The only necessity of using complex form is that it is most convenient for describing the coherent superposition of propagator. However, it is a different problem with that of this section and it will be specifically discussed in the next section. In order to achieve the purpose of clarifying concepts, it is beneficial to separate the two equations here from the coherent superposition of propagators of the next section.

2.5 Measurement and evolution distribution

2.5.1 Definition of measurement

Definition 2.5.1.1. If reference-system f evolves in reference-system g, we say that g makes a **measurement** of f.

Remark 2.5.1.1. For ontological measurement, there are usually two aspects to consider. One is to measure the objective position, the other is to measure the objective evolution.

(1) The basic principle of theoretical physics tells us a physical reality is cognized by using a reference-system, rather than using a point. That means it is hard to cognize the full picture of the objective positions of physical reality by using the coordinate of a single point.

(2) Generally, an ontological measurement is always accomplished by the interaction of a physical reality A on another physical reality B. Physical reality A is specific after all, the interaction of A on B is necessarily inclined to a certain orientation, rather than omni-directional. That means it is hard to get a glimpse of the full picture of the objective evolution of reality B by a single measurement.

In a word, the cognition will be more comprehensive to research the distribution of positions or distribution of evolution directions.

Remark 2.5.1.2. As Remark 2.4.5.4 said, for any universal geometric property ρ defined in form of tensor on geometric manifold, its actual evolution can anyway be discussed. Similarly, for any universal geometric property ρ, the distribution of its evolution can anyway be discussed.

(1) Intuitively, if g is a completely stationary reference-system defined later, the actual evolution direction field of ρ of f distributes uniformly on the completely flat geometric manifold (M, g).

(2) If g is non-trivial, the potential field of g would effect the distribution of the actual evolution directions of ρ of f, in other words, the shape of geometric manifold (M, g) would effect the distribution of the actual evolution directions of ρ of f.

In order to describe the effects, it has significance to research this distribution, which will be discussed in detail in the next section.

2.5.2 Constructions of propagator and wave function

Discussion 2.5.2.1. Abstractly, propagator is defined as the Green function of evolution equation. Concretely, propagator still needs a constructive definition.

One method is to construct with Feynman path integral $\int_{x_a}^{x_b} e^{iS}\mathfrak{D}x(t)$, which is expressed in form of functional integral. However, until now the functional integral has strict definition only in some special cases, and the strict definition in general case is still an unsolved problem.

This paper adopts another method to strictly construct propagator.

Definition 2.5.2.1. Let $\mathfrak{T}$ be the set of all flat transformations of reference-system defined in section 2.2.2.3 . Any geometric property ρ determined by reference-system f is a universal geometric property on geometric manifold (M, g). Let H be an actual evolution direction field of ρ on (M, g).

Let an element $T \in \mathfrak{T}$ act on reference-system f, then it induces $T_*\rho$ of Tf and the actual evolution direction field T_*H of $T_*\rho$ on (M, g). The set

$$|\rho| \triangleq \{\rho_T \triangleq T_*\rho | T \in \mathfrak{T}\},$$

which is determined by the kernal $|f|$ of f, is called the kernal of ρ. The set

$$|H| \triangleq \{H_T \triangleq T_*H | T \in \mathfrak{T}\}$$

is called the actual evolution direction field of $|\rho|$ on (M, g).

Let φ_H be the one-parameter group of diffeomorphisms induced by H as a smooth tangent vector field on M. $\forall a \in M$, the actual evolution path determined by H and starting from point a is denoted by $\varphi_{H,a}$. Suppose $a = \varphi_{H,a}(0)$, the set

$$\varphi_{|H|,a} \triangleq \{\varphi_{X,a} | X \in |H|\}$$

is called the actual evolution path of $|\rho|$ starting from a. $\forall t \in \mathbb{R}^+$, the set

$$\varphi_{|H|,a}(t) \triangleq \{\varphi_{X,a}(t) | X \in |H|\}$$

is called the **evolution image** of point a at time t.

$\forall \Omega \subseteq \mathfrak{T}$, the set

$$|H_\Omega| \triangleq \{T_*H | T \in \Omega\}$$

is a subset of $|H|$, and the set

$$\varphi_{|H_\Omega|,a} \triangleq \{\varphi_{X,a} | X \in |H_\Omega|\}$$

is a subset of $\varphi_{|H|,a}$. Correspondingly, $\forall t \in \mathbb{R}^+$, the evolution image

$$\varphi_{|H_\Omega|,a}(t) \triangleq \{\varphi_{H,a}(t) | X \in |X_\Omega|\}$$

of a at t is a subset of $\varphi_{|H|,a}(t)$.

$\forall a \in M$, the restrictions of $|H|$ and $|H_\Omega|$ at point a are respectively denoted by

$$|H(a)| \triangleq \{T_*H(a) | T \in \mathfrak{T}\}, \quad |H_\Omega(a)| \triangleq \{T_*H(a) | T \in \Omega\}.$$

Remark 2.5.2.1. When $t = 0$, intuitively, the actual evolution directions $|H(a)|$ of $|\rho|$ start from a and point to all directions around a uniformly. Affected by the potential field of reference-system g, when they evolve to a certain $t > 0$, the distribution of the actual evolution directions on $\varphi_{|H|,a}(t)$ are no longer as uniform as around a.

To exactly characterize this kind of uniformity provides a way of describing the effects of potential field. The following definition is needed.

Definition 2.5.2.2. (Evolution distribution). Take the inverse transformation $F_{g^{-1}}$ of g, we get a trivial reference-system $e \triangleq F_{g^{-1}}(g)$. Now (M, g) is transformed to a completely flat geometric manifold (M, e). The actual evolution direction field $|H|$ of $|\rho|$ on (M, g) is transformed to an actual evolution direction field $|O|$ on (M, e). Correspondingly, $\varphi_{|H|,a}(t)$ is transformed to $\varphi_{|O|,a}(t)$. In a word, $F_{g^{-1}}$ induces the following two mappings:

$$g_*^{-1} : |H| \mapsto |O|, \qquad g_{**}^{-1} : \varphi_{|H|,a} \mapsto \varphi_{|O|,a}.$$

$\forall |H_\Omega| \subseteq |H|$, denote

$$|O_\Omega| \triangleq g_*^{-1}\left(|H_\Omega|\right) \subseteq |O|, \qquad \varphi_{|O_\Omega|,a} \triangleq g_{**}^{-1}\left(\varphi_{|H_\Omega|,a}\right).$$

Further, $\forall t \in \mathbb{R}^+$, the measure $P\left(\varphi_{|O_\Omega|,a}(t)\right) = P\left(g_{**}^{-1}\left(\varphi_{|H_\Omega|,a}(t)\right)\right)$ of $\varphi_{|O_\Omega|,a}(t)$ is called the **actual evolution distribution** of $\varphi_{|H_\Omega|,a}(t)$, or called the **actual evolution distribution** of $|\rho|$ starting from a along $|H_\Omega|$ at time t, or called **evolution distribution** for short.

Due to $\mathfrak{T} \cong SL(\mathfrak{D}, \mathbb{R})$, for convenience, take Ω as a neighborhood of any element $T \in SL(\mathfrak{D}, \mathbb{R})$. Now at the start point a, $|H_\Omega(a)|$ is called an **evolution neighborhood** of $H_T(a)$, and

$$|O_\Omega(a)| \triangleq g_*^{-1}(|H_\Omega(a)|)$$

is called an **evolution neighborhood** of

$$O_T(a) \triangleq g_*^{-1}(H_T(a)).$$

When the neighborhood Ω is sufficiently small, the evolution neighborhood $|H_\Omega(a)|$ and $|O_\Omega(a)|$ are both sufficiently small, and $\forall t \in \mathbb{R}^+$ the sets $\varphi_{|H_\Omega|,a}(t)$ and $\varphi_{|O_\Omega|,a}(t)$ are also sufficiently small.

Concretely, when the neighborhood Ω approach to T, $|H_\Omega|$ will approach to $H_T = \lim\limits_{\Omega \to T} |H_\Omega|$. Therefore, the evolution neighborhood $|H_\Omega(a)|$ at start point a will approach to the evolution direction $H_T(a) = \lim\limits_{\Omega \to T} |H_\Omega(a)|$, and the set of evolution images $\varphi_{|H_\Omega|,a}(t)$ of a at time t will approach to a point

$$b_T \triangleq \varphi_{H_T,a}(t) = \lim_{|H_\Omega(a)| \to H_T(a)} \varphi_{|H_\Omega|,a}(t)$$

on manifold M.

The limit

$$w_a(b_T) \triangleq \frac{dV_{O_T}}{dV_{H_T}} \triangleq \lim_{\Omega \to T} \frac{P\left(\varphi_{|O_\Omega|,a}(t)\right)}{P\left(\varphi_{|H_\Omega|,a}(t)\right)} = \lim_{\Omega \to T} \frac{P\left(g_{**}^{-1}\left(\varphi_{|H_\Omega|,a}(t)\right)\right)}{P\left(\varphi_{|H_\Omega|,a}(t)\right)} \tag{35}$$

is called the **actual evolution distribution density** of $|\rho|$ at point b_T about the start point a, or called the **evolution distribution density** for short.

Remark 2.5.2.2. Radon-Nikodym theorem [51] guarantees the existence of such a limit.

Remark 2.5.2.3. For any two points a and b on manifold M, we can anyway talk about the actual evolution path of ρ from a to b. It is because even if the actual evolution path of ρ starting from a does not go through b, it only needs to properly adjust the initial momentum of ρ so that the path exactly goes through b. This is the same way of consideration as traditional theory. Generally, it only needs to take a flat transformation of reference-system defined in section 2.2.2.3 for f so that the purpose of adjusting initial momentum can be achieved. During the transformation of initial momentum of ρ, the value of kernal geometric property of f remains unchanged, so the geometric essences about the curved shape reflected by ρ before and after the transformation are the same.

More strictly, according to Definition 2.5.2.1 , let $|\rho|$ be the kernal of ρ. For any points a and b selected, it is always meaningful to discuss the actual evolution path of $|\rho|$ from a to b, because there certainly exists an element $\rho' \in |\rho|$ such that a and b are exactly both on the actual evolution path $L(b,a)$ of ρ'. Therefore, it can be said broadly that $L(b,a)$ is an actual evolution path of $|\rho|$.

If necessary, the connotation of ρ may be re-defined as ρ', we can now talk about the actual evolution path of ρ from a to b. However, usually it is not necessary to do so, because it is very convenient to discuss by using the kernal $|\rho|$.

To say it informally, $|\rho|$ enbodies the common essence of ρ and ρ' in different motion directions. $|\rho|$ can be regarded as the particle ITSELF, and a particle $|\rho|$ is respectively denoted by ρ and ρ' in two different directions. No matter how the motion direction changes, the particle as a motion subject is unique.

That is what geometry does. It can characterize the specific essence at a specific level. The kernal geometry describes a particle at the level that is independent of overall directions.

Definition 2.5.2.3. (Evolutor). $\forall a, b \in M$, let $L(b,a)$ be an actual evolution path of ρ from a to b on (M,g), and $w_a(b)$ is the actual evolution distribution density at b about a on this path.

$$r_L(b,a) \triangleq \sqrt{w_a(b)}$$

is called the **real-valued evolutor** of ρ about $L(b,a)$ on (M,g). Let $s_L(b,a)$ be the evolution quantity of $L(b,a)$.

$$R_L(b,a) \triangleq r_L(b,a)e^{is_L(b,a)}$$

is called the **complex-valued evolutor** about $L(b,a)$, or called **evolutor** for short.

Remark 2.5.2.4. The definition here put the evolution quantity density $s_L(b,a)$ in the exponent. Traditional theory customarily put the volume integral $S_L(b,a) \triangleq \int s_L(b,a)dV$ of evolution quantity density in the exponent. It will be seen from section 6.3.7.1 that the ways of using $S_L(b,a)$ and using $s_L(b,a)$ have no essential difference for describing the evolution. Therefore, in order to be consistent with traditional theory, in the discussions of some following sections, $R_L(b,a) \triangleq r_L(b,a)e^{iS_L(b,a)}$ will be used as evolutor indiscriminately. But in this section, in order to clarify the essential form of the theory, the evolutor is expressed by using $s_L(b,a)$, rather that $S_L(b,a)$.

Remark 2.5.2.5. Feynman path integral takes the summation of all paths from a to b. It is difficult to get a general and strict definition, and it is not necessary. Now all we have to do is to reduce the scope of summation to the set of all actual evolution paths from a to b. In some special case, the actual evolution path of $|\rho|$ from a to b is unique, such as the case of free particle, but in general case they are not unique.

Definition 2.5.2.4. (Propagator). Let $\mathfrak{L}(b,a)$ be the set of all actual evolution paths of $|\rho|$ from a to b. $\forall L(b,a) \in \mathfrak{L}(b,a)$, let $R_L(b,a)$ be the evolutor about $L(b,a)$. Then

$$K(b,a) \triangleq \sum_{L\in\mathfrak{L}(b,a)} R_L(b,a) \tag{36}$$

is called the **propagator** of $|\rho|$ from a to b.

Remark 2.5.2.6. As the simplest example, consider the propagator of free particle.

On the completely flat geometric manifold (M, g), in the sense of Remark 2.5.2.1 , intuitively, the actual evolution directions of $|\rho|$ starting from a spread uniformly in all directions around a. No matter where b is, the actual evolution distribution density $w_a(b)$ at b about a is identically equal to 1.

Then for a fixed b, take the actual evolution path $L(b,a)$, the corresponding evolutor of ρ is $R_L(b,a) = r_L(b,a)e^{is_L(b,a)} = \sqrt{w_a(b)}e^{is_L(b,a)} = e^{is_L(b,a)}$. Because $|\rho|$ is a free particle, there is only one element in $\mathfrak{L}(b,a)$, which is $L(b,a)$. So the propagator is $K(b,a) = R_L(b,a) = e^{is_L(b,a)}$.

Of course, it has not been normalized on the wavefront, otherwise there would be a coefficient of normalization.

Remark 2.5.2.7. For the propagator of non-free particle, there may be multiple elements in $\mathfrak{L}(b,a)$. Now that the superposition of the evolutors about these different actual evolution paths has been defined, why could it be in form of complex number?

That is because ρ is determined by the potential field of reference-system f. The wave of the potential field of f determines the wave of ρ, and the coherent superposition of the potential field of f determines the coherent superposition of ρ. We know any coherent superposition can be described in form of complex number.

Concretely, the evolution distribution is determined by two aspects. (i) The shape of geometric manifold (M, g) makes the evolution distribution of ρ deformed. (ii) The coherent superposition of the potential field of f makes ρ itself changed, and finally effects the evolution distribution.

In order to understand conveniently, review the electromagnetic wave in Maxwell theory. Let an electromagnetic wave f propagates on geometric manifold (M, g). The propagation direction $\boldsymbol{c}$ of f and the potential vector direction $\boldsymbol{A}$ of f are completely not the same thing. The wave direction is $\boldsymbol{c}$, rather than $\boldsymbol{A}$. On (M, g), $\boldsymbol{c}$ determines an evolution path L_c. There is a phase difference of $\boldsymbol{A}$ between any two different positions on L_c. This kind of phase difference can make $\boldsymbol{A}$ be coherently superposed and cause the coherent superposition of the distribution of $\boldsymbol{c}$. This is what the interference of light is.

From this analogy we may think that on (M, g), the actual evolution direction $\boldsymbol{v}$ of ρ and the potential field direction $\boldsymbol{\Lambda}$ of f are also not the same thing. On (M, g), $\boldsymbol{v}$ determines an evolution path L_v. If there is a phase difference of the slack-tight of f between any two different positions on L_v, there is also a phase difference of potential field $\boldsymbol{\Lambda}$. This kind of phase difference can make $\boldsymbol{\Lambda}$ be coherently superposed and cause the coherent superposition of the distribution of $\boldsymbol{v}$. This is what the essence of quantum interference of ρ is.

Traditional theory cannot clarify the origin of coherent superposition of particle. It is only considered as a probability wave, obviously which is not enough. However, the above discussion clearly illustrates this origin. The probability wave is not the most fundamental understanding about particle, and is not the only way of description, because there are more fundamental $|\rho|$ and the most fundamental f.

According to the general form of Fourier series, although the coherent superposition can be completely described in form of real number, the expression form of complex number is also feasible and convenient. This is the answer of the above question.

Definition 2.5.2.5. (Wave function). Let a be a point on geometric manifold (M, g). $\forall a_0 \in M$, $d(a, a_0)$ is the geodesic distance between a_0 and a. Denote $\Sigma_a(a_0) \triangleq \{q \in M | d(a, q) = d(a, a_0)\}$. Let $(b_0)_M^A$ be the slack-tight of g at a_0 and satisfy $\lim\limits_{d(a,a_0)\to\infty} (b_0)_M^A = \delta_M^A$. If complex-valued function $\psi : M \to \mathbb{C}$ satisfies both the following two conditions, then ψ is called a **wave function** of actual evolution of ρ on (M, g).

(1) $\exists r > 0$ such that $\lim\limits_{d(a,a_0)\to r} \psi(a_0) = c$ or $\lim\limits_{d(a,a_0)\to\infty} \psi(a_0) = c$ holds, where c is a constant.

(2) $\forall a_0, a \in M$,

$$\psi(a) = \int_{\Sigma_a(a_0)} K(a, q)\psi(q) d\sigma_q.$$

Remark 2.5.2.8. Propagator and wave function describes the distribution of the same actual evolution directions in different ways, so their effectivenesses are the same.

Based on the above basic concepts and discussions related to propagator and wave function, it must be able to expand the whole quantum mechanics and quantum field theory in the way of constructivity. Although it may be formally a little different from traditional theory expressed in the way of abstraction, the essences of them are the same. This paper only focuses on the theoretical foundation at the most basic level, and the geometric viewpoints for further development have already been established in section 2.4.13.2 and section 2.4.14.2 , so the construction in this paper about quantum theory stops here. Further development and formal comparison need to be researched in other articles.

2.6 Summary of this section

This section mainly discusses the following contents.

1. An axiom is established for Hilbert's 6th problem of theoretical physics.

2. Based on the concept of reference-system, Riemannian manifold is generalized to geometric manifold.

3. Based on the concept of reference-ssytem, the concept of intrinsic geometry is generalized.

4. The concept of simple connection, which can be used to describe some more bending properties of manifold than Levi-Civita connectdion, is defined.

5. Those having a status as principle in traditional theory, such as Yang-Mills field equation, Lorentz force equation, conservation law of energy-momentum, gravitational field equation, least action principle, Schrödinger equation, Heisenberg equation, Dirac equation(see section 6.3.7.1), etc., become theorems that automatically hold in the theory of this paper. The purpose of removing redundant principles and postulates has been achieved.

6. This section adopts the most general coordinate form, whose evolution parameter is time. In section 6.1.3.1 , the Minkowski coordinate will be constructed, whose evolution parameter is proper-time. It has to be emphasized that no matter what coordinate forms are adopted, their geometric essences are the same, and the physical properties described by them are also the same.

Based on the theoretical foundation established in this section, various kinds of concrete reference-systems can be discussed in the following sections, which are used to cognize various matter-motions.

3 Trivial interaction and relative motion

Definition 3.1. On manifold M, if the slack-tights B^A_M and C^M_A of reference-system f are all constants independent of positions, f is called a **completely flat reference-system** on M. Specially, if the metric tensor satisfies orthogonal condition $G_{MN} = \Delta_{AB} B^A_M B^B_N = \mathrm{E}_{MN}$, f is called a **completely inertial reference-system**. More specially, if $B^A_M = \delta^A_M$ and $C^M_A = \delta^M_A$, f is called a **completely static reference-system**.

Discussion 3.1. Let f be a completely flat reference-system on M, g be an arbitrary reference-system on M, and f evolves in g, i.e. $\forall p \in M$ $\psi_U(U) \xleftarrow{f(p)} \varphi_U(U) \xrightarrow{g(p)} \rho_U(U)$. And let the coordinate forms determined by coordinate mappings ψ, φ, ρ are respectively $d\xi^A$, dx^M, $d\zeta^A$.

On a neighborhood U of point p on M, let the coordinate representation of local reference-system $f(p)$ is $x^M = c^M_A \xi^A + a^M$, where c^M_A and a^M are all constant functions on U. And let the basis vector representation of f is $d\xi^A = B^A_M dx^M$ and $dx^M = C^M_A d\xi^A$, where B^A_M and C^M_A are all constant functions on M.

Thus, the simple connection of f is $\Lambda^M_{NP} = 0$, and the curvature tensor is $K^M_{NPQ} = 0$. The charge of f evolving in g is $\rho_{MN} = 0$.

Discussion 3.2. Conversely, let f is an arbitrary reference-system, and g is a completely flat reference-system. Similarly, consider the evolution of f in g. Then the simple connection of g is $\Gamma^M_{NP} = 0$. For any charge ρ of f, the actual evolution direction of ρ on (M, g) is

$$\frac{d}{dt_\rho} = \rho_{;R} \frac{d}{dx_R} = \left(\frac{d\rho}{dx^R} + [\rho \Gamma_R] \right) \frac{d}{dx_R} = \frac{d\rho}{dx^R} \frac{d}{dx_R}.$$

That is to say, ρ moves freely in g, and there is no interaction.

Remark 3.1. According to the basic principle of theoretical physics, the completely flat reference-system points to a physical reality. In fact, it is the physical reality in the ideal case of trivial relative motion and no interaction. In ontology, trivial relative motion is the same thing as no interaction, they are both finally cognized by using the uniform concept of completely flat reference-system in epistemology.

4 Inversion interaction and relative motion

4.1 Coordinate inversion transformation

Remark 4.1.1. In this section, the index values of internal space and external space are taken according to Definition 6.1.1.1 .

Definition 4.1.1. Suppose the coordinate representation of each local reference-system on M is $\xi^A = \xi^A(x^M)$ such that

(1) On the internal space N, the coordinate frame inheriting from M satisfies $\xi^a = \delta^a_m x^m$;

(2) On the external space P, the coordinate frame inheriting from M satisfies $\xi^s = -\delta^s_i x^i$.

The transformation of reference-system induced by such a reference-system is called the **external space coordinate inversion transformation** on M, or the **parity transformation**, denoted by P.

Definition 4.1.2. Suppose the coordinate representation of each local reference-system on M is $\xi^A = \xi^A(x^M)$ such that

(1) On the internal space N, the coordinate frame inheriting from M satisfies $\xi^a = -\delta^a_m x^m$;

(2) On the external space P, the coordinate frame inheriting from M satisfies $\xi^s = \delta^s_i x^i$.

The transformation of reference-system induced by such a reference-system is called the **internal space coordinate inversion transformation** on M, or the **charge conjugate transformation**, denoted by C.

Definition 4.1.3. Suppose the coordinate representation of each local reference-system on M is $\xi^A = -\delta^A_M x^M$, and the slack-tights satisfy $B^A_M = -\delta^A_M$. Then the transformation of reference-system induced by such a reference-system is called the **total space coordinate inversion transformation** on M, denoted by PC or CP.

4.2 Metric inversion transformation

According to the definition in section 2.2.9.1 , the positive or negative sign of metric is independent of the sign of cooridnate. The time metric and the space metric may be either positive or negative. They reflect two opposite directions of evolution.

Definition 4.2.1. Let N be a closed submanifold of M, and let its metric about submanifold be $dx^{(N)}$. The transformation $dx^{(N)} \to -dx^{(N)}$ is called the **space metric single inversion transformation** on N, denoted by $T_0^{(N)}$.

Specially, when $N = M$, the transformation $T_0^{(M)} : dx^0 \to -dx^0$ is called the **total space metric single inversion transformation**, or **time metric single inversion transformation**, denoted by T_0 for short.

Definition 4.2.2. Let N is a closed submanifold of M. The set of all closed submanifolds of N is denoted by $\mathfrak{B}(N)$. For every $B \in \mathfrak{B}(N)$, take a space metric single inversion transformation $T_0^{(B)} : dx^{(B)} \to -dx^{(B)}$ on B, then the transformation

$$T^{(N)} \triangleq \prod_{B \in \mathfrak{B}(N)} T_0^{(B)}$$

is called the **space metric complete inversion transformation** on N.

Specially, when $N = M$, $T^{(M)}$ is called the **total space metric complete inversion transformation**, or **time metric complete inversion transformation**, or **time inversion transforamtion** for short, denoted by T.

Remark 4.2.1. Consider in the premise that the positive and negative signs of all coordinates remain unchanged.

(1) According to the definition, no matter metrics inverse or not, the signs of E_0 and p_R remain unchanged.

(2) A space metric single inversion transformation may change the sign of the specific term of evolution quantity $ds = E_0 dx^0 = p_R dx^R$ about a certain value of index R.

(3) The time metric complete inversion transformation T changes the signs of all the terms of evolution quantity, that is

$$E_0 dx^0 \mapsto -E_0 dx^0, \quad p_R dx^R \mapsto -p_R dx^R.$$

So it is not difficult to understand that here is exactly the origin of the complex conjugate in the time inversion transformation T of traditional theory.

4.3 Space-time inversion transformation

Definition 4.3.1. The joint transformation of total space coordinate inversion transformation CP and total space metric complete inversion transformation T is called the **space-time inversion transformation**, denoted by CPT.

Remark 4.3.1. In the case where coordinates and metrics are all inversed, it is not difficult to understand the intuitive meanings of well-known conclusions such as physical laws remain unchanged during CPT, there is a difference of CPT between a particle field and its antiparticle field, etc.

5 Typical gauge interaction and relative motion

Section 2.2.1.3 has defined the concepts of general gauge field and tansformation of general gauge field. Now some concepts related to typical gauge field have to be defined.

5.1 Typical gauge reference-system

Definition 5.1.1. Suppose each local coordinate representation of reference-system f on M is $\xi^A = \xi^A(x^M)$ such that

(1) the internal coordinates ξ^a satisfy $\xi^a = \xi^a(x^m)$;

(2) the external coordinates ξ^s satisfy $\xi^s = \delta^s_i x^i$.

Such a reference-system f is called an **internal gauge reference-system** on M. The transformation F_f induced by f is called an **internal gauge transformation**, or a **gauge transformation of traditional gauge field**.

Definition 5.1.2. Suppose each local coordinate representation of reference-system f on M is $\xi^A = \xi^A(x^M)$ such that

(1) the internal coordinates ξ^a satisfy $\xi^a = \delta^a_m x^m$;

(2) the external coordinates ξ^s satisfy $\xi^s = \xi^s(x^i)$.

Such a reference-system f is called an **external gauge reference-system** on M. The transformation F_f induced by f is called an **external gauge transformation**.

Definition 5.1.3. Suppose each local coordinate representation of reference-system f on M is $\xi^A = \xi^A(x^M)$ such that

(1) the internal coordinates ξ^a satisfy $\xi^a = \xi^a(x^m)$;

(2) the external coordinates ξ^s satisfy $\xi^s = \xi^s(x^i)$.

Such a reference-system f is called a **typical gauge reference-system** on M. The transformation F_f induced by f is called a **typical gauge transformation**.

5.2 Typical gauge field reference-system

Definition 5.2.1. Suppose each local coordinate representation of reference-system f on M is $\xi^A = \xi^A(x^M)$ such that

(1) the internal coordinates ξ^a satisfy $\xi^a = \xi^a(x^M)$;

(2) the external coordinates ξ^s satisfy $\xi^s = \xi^s(x^i)$.

Such a reference-system f is called a **typical gauge field reference-system** on M, or called a **typical gauge field** for short. The transformation F_f induced by f is called a **typical gauge field transformation**, or **gauge transformation of traditional gravitational field**.

Definition 5.2.2. A reference-system which satisfies the **externally flat conditions** $C^M_s = \delta^M_s$ and $B^A_i = \delta^A_i$ is called an **externally flat reference-system**. A refernce-system which satisfies the **internally standard conditions** $G_{mn} = const$, and $G_{mn} = 0$ when $m \neq n$, is called an **internally standard reference-system**.

Remark 5.2.1. In ontology, the gauge fields and particles which are observed so far can be cognized by using the typical gauge field reference-system such that: external space indices satisfy $i, s = 1, 2, \cdots, r$ and internal space indices satisfy $m, a = \mathfrak{D} - 4, \mathfrak{D} - 3, \mathfrak{D} - 2, \mathfrak{D} - 1, \mathfrak{D}$, where $r = 3$ and $\mathfrak{D} = 8$.

In order to connect the standard model of traditional theory more clearly, when discussing concrete typical gauge field reference-systems in section 6.4.3 and section 7.3.1 , the electromagnetic-weak interaction and relative motion as well as the strong interaction and relative motion will be discussed respectively first, and then the reference-system of electromagnetic-weak-strong unified field will be discussed, and the externally flat conditions and internally standard conditions will always be adopted.

Before that, we should strictly discuss the relation between general coordinate and classical spacetime coordinate, as well as the classical expression forms of some important contents of section 1.2 .

6 Classical spacetime interaction and relative motion

6.1 Regular form of evolution of classical spacetime reference-system

6.1.1 Regular coordinate form

Definition 6.1.1.1. Consider the external space submanifold P and internal space submanifold N defined in Definition 2.3.2.2 , and their coordinate frames $\{\xi^s\}\{x^i\}$ and $\{\xi^a\}\{x^m\}$ inheriting from M.

For convenience, if there is no special declaration, the values of the internal space indices and the external space indices are as following.

(1) The external space indices in coordinate frame (U,ξ) are $s,t,u,v=1,2,\cdots,r$, and the external space indices in coordinate frame (U,x) are $i,j,k,l=1,2,\cdots,r$.

(2) The internal space indices in coordinate frame (U,ξ) are $a,b,c,d=r+1,r+2,\cdots,\mathfrak{D}$, and the internal space indices in coordinate frame (U,x) are $m,n,p,q=r+1,r+2,\cdots,\mathfrak{D}$.

(3) The regular simplified indices in coordinate frame (U,ξ) are $S,T,U,V=1,2,\cdots,r,\tau$, and the regular simplified indices in coordinate frame (U,x) are $I,J,K,L=1,2,\cdots,r,\tau$.

(4) For the definition of Minkowski indices, see Remark 6.2.1.1 .

Definition 6.1.1.2. On (M,f), if tangent vector

$$\frac{d}{dt}\triangleq\alpha^A\frac{\partial}{\partial\xi^A}=\beta^M\frac{\partial}{\partial x^M}$$

satisfies that components α^a and β^m about internal space indices are not all zero, such a $\frac{d}{dt}$ is called an **internal-directed evolution direction**. Let there be a smooth tangent vector field X on M. If $\forall p\in M$, $X(p)$ is an internal-directed evolution direction, then X is called an **internal-directed evolution direction field**.

According to Definition 2.3.2.2 , let $M=P\times N$ and P be the external space submanifold of M. $\forall q\in M$, parameter equations $x^m=x^m_q(m=r+1,\cdots,\mathfrak{D})$ can define a closed submanifold $P\times\{q\}$ through q on M. $\forall p\in P\times\{q\}$, the closed submanifold $P\times\{q\}$ can also be denoted by $P\times\{p\}$.

Let $\varphi_X:M\times\mathbb{R}\to M$ be the one-parameter group of diffeomorphisms determined by an internal-directed evolution direction field X on M. The restriction of φ_X on $P\times\{p\}$ is

$$\varphi_X|_{P\times\{p\}}:P\times\{p\}\times\{t\}\mapsto P'\times\{p'\},$$

where points p and p' are on the same orbit $L_p\triangleq\varphi_{X,p}$, and $P\times\{p\}$ and $P\times\{p'\}$ are both homeomorphic to P. If not to distinguish P and P', we have

$$\varphi_X|_{P\times\{p\}}:P\times\{p\}\times\mathbb{R}\to P\times L_p.$$

Then considering all of such $\{p\}$ on the entire orbit L_p, we obtain a mapping

$$\varphi_X|_{P\times L_p}: P\times L_p\times\mathbb{R}\to P\times L_p.$$

$\forall p\in P$, L_p is homeomorphic to $\mathbb{R}$, this is to say, each orbit L_p can be expressed as a parameter equation. All of them can take the metric parameter x^0 on M as their common parameter. Thus, for every orbits of φ_X we obtain the following mapping:

$$\varphi_X|_{P\times\mathbb{R}}: P\times\mathbb{R}\times\mathbb{R}\to P\times\mathbb{R}.$$

Denote $\tilde{M}\triangleq P\times\mathbb{R}$. $\tilde{M}$ is called the **classical spacetime submanifold** of M along evolution direction field X. Then

$$\varphi_{\tilde{X}}\triangleq\varphi_X|_{\tilde{M}}:\tilde{M}\times\mathbb{R}\to\tilde{M}$$

constitutes a one-parameter group of diffeomorphisms on $\tilde{M}$. Natually, $\varphi_{\tilde{X}}$ determines a smooth tangent vector field $\tilde{X}$ on $\tilde{M}$.

Accoring to the above construction, the tangent mapping

$$\gamma_*: T(\tilde{M})\to T(M),\quad \tilde{X}\mapsto X$$

induced by regular embedding mapping $\gamma:\tilde{M}\to M$ is an injection and generally not a surjection, so $\gamma_*\tilde{X}$ is a subset of X, and vectors in $\tilde{X}$ correspond one-to-one to vectors in the restriction of X on $\tilde{M}$. For convenience, on the classical spacetime submanifold $\tilde{M}$, X and $\tilde{X}$ are usually not distinguished, which are uniformly denoted by X.

Each evolution path $L:T\to M,\quad t\mapsto p$ determined by X on M induces an evolution path

$$\tilde{L}=L\circ\gamma^{-1}:T\to\tilde{M},t\mapsto p$$

determined by $\tilde{X}$ on $\tilde{M}$. Obviously, the image sets of L and $\tilde{L}$ are the same, i.e. $L(T)=\tilde{L}(T)$. For convenience, on the classical spacetime submanifold $\tilde{M}$, usually L and $\tilde{L}$ are uniformly denoted by L.

It is seen that the classical spacetime submanifold $\tilde{M}$ is not independent of M, and determined by the evolution direction field X on M. So a part of the properties on M can be expressed as properties on $\tilde{M}$, anyway which is for only some properties on M, not all.

Now these properties should be researched on the classical spacetime submanifold $\tilde{M}$. First, consider the relationship between the parameter equations of L and $\tilde{L}$, and establish the coordinate representation of classial spacetime reference-system.

Definition 6.1.1.3. Let $\frac{d}{dt}\triangleq X(p)$, and let $L_p\in\frac{d}{dt}$ be an evolution path on orbit $\varphi_{X,p}$ through p on M. And the evolution direction at each point on L_p is an internal-directed evolution direction. Thus, about the metric parameters ξ^τ and x^τ on the internal space submanifold N, there is a kind of parameter equation of $\tilde{L}_p$:

$$\begin{cases}\xi^A=\xi^A_\tau(\xi^\tau)\\ x^M=x^M_\tau(x^\tau)\end{cases}.$$

Substitute this parameter equation into the coordinate form of evolution in Definition 2.4.2.3 :

$$\begin{cases}\xi^A=\xi^A(x^M)=\xi^A_L(x^0)\\ \xi^0=\xi^0(x^0)\end{cases},\quad \begin{cases}x^M=x^M(\xi^A)=x^M_L(\xi^0)\\ x^0=x^0(\xi^0)\end{cases},$$

then it is obtained that

$$\begin{cases}\xi^A=\xi^A\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^A_L\left(x^0\right)\\ \xi^0=\xi^0\left(x^0\right)\end{cases}\quad \begin{cases}x^M=x^M\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^M_L\left(\xi^0\right)\\ x^0=x^0\left(\xi^0\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u=\xi^u\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^u_L\left(x^0\right)\\ \xi^a=\xi^a\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^a_L\left(x^0\right)\\ \xi^0=\xi^0\left(x^0\right)\end{cases}\quad \begin{cases}x^k=x^k\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^k_L\left(\xi^0\right)\\ x^m=x^m\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^m_L\left(\xi^0\right)\\ x^0=x^0\left(\xi^0\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u=\xi^u\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^u_L\left(x^0\right)\\ \xi^a_\tau\left(\xi^\tau\right)=\xi^a\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^a_L\left(x^0\right)\\ \xi^0=\xi^0\left(x^0\right)\end{cases}\quad \begin{cases}x^k=x^k\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^k_L\left(\xi^0\right)\\ x^m_\tau\left(x^\tau\right)=x^m\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^m_L\left(\xi^0\right)\\ x^0=x^0\left(\xi^0\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u=\xi^u\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^u_L\left(x^0\right)\\ \xi^\tau=\xi^a\circ\left(\xi^a_\tau\right)^{-1}\left(x^k,x^m_\tau\left(x^\tau\right)\right)=\xi^a_L\circ\left(\xi^a_\tau\right)^{-1}\left(x^0\right)\\ \xi^0=\xi^0\left(x^0\right)\end{cases}\quad \begin{cases}x^k=x^k\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^k_L\left(\xi^0\right)\\ x^\tau=x^m\circ\left(x^m_\tau\right)^{-1}\left(\xi^u,\xi^a_\tau\left(\xi^\tau\right)\right)=x^m_L\circ\left(x^m_\tau\right)^{-1}\left(\xi^0\right)\\ x^0=x^0\left(\xi^0\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u=\xi^u_\tau\left(x^k,x^\tau\right)=\xi^u_L\left(x^0\right)\\ \xi^\tau=\xi^\tau\left(x^k,x^\tau\right)=\xi^\tau_L\left(x^0\right)\\ \xi^0=\xi^0\left(x^0\right)\end{cases}\quad \begin{cases}x^k=x^k_\tau\left(\xi^u,\xi^\tau\right)=x^k_L\left(\xi^0\right)\\ x^\tau=x^\tau\left(\xi^u,\xi^\tau\right)=x^\tau_L\left(\xi^0\right)\\ x^0=x^0\left(\xi^0\right)\end{cases}.$$

Now these two systems of equations are called the **regular simplified coordinate form of reference-system and its evolution about proper-time parameter in direction** $\frac{d}{dt}$, or **regular simplified coordinate form of evolution of reference-system**. For convenience, they can also be called the **regular coordinate form** for short. If no confusion, they are also be denoted by

$$\begin{cases}\xi^u=\xi^u(x^k,x^\tau)=\xi^u(x^0)\\ \xi^\tau=\xi^\tau(x^k,x^\tau)=\xi^\tau(x^0)\\ \xi^0=\xi^0(x^0)\end{cases},\quad \begin{cases}x^k=x^k(\xi^u,\xi^\tau)=x^k(\xi^0)\\ x^\tau=x^\tau(\xi^u,\xi^\tau)=x^\tau(\xi^0)\\ x^0=x^0(\xi^0)\end{cases},\tag{37}$$

more concisely, denoted by

$$\begin{cases}\xi^U=\xi^U(x^K)=\xi^U(x^0)\\ \xi^0=\xi^0(x^0)\end{cases},\quad \begin{cases}\xi^U=\xi^U(x^K)=\xi^U(x^0)\\ \xi^0=\xi^0(x^0)\end{cases},\tag{38}$$

where $\xi^U=\xi^U(x^K)$ and $x^K=x^K(\xi^U)$ are called the **(local) classical spacetime reference-systems** at p, denoted by $f(p)$ and $f^{-1}(p)$. The coordinate frames $(\tilde{U},\xi^U)$ and $(\tilde{U},x^K)$ on a neighborhood $\tilde{U}$ of point p on $\tilde{M}$ are called two **classical spacetime regular coordinate frames**.

According to Definition 2.2.1.1 , these local classical space time reference-systems $f(p)$ constitute a **classical spacetime reference-system** f on $\tilde{M}$, and f can also be called a **classical gravitational field reference-system**, or **gravitational field** for short.

Remark 6.1.1.1. Generally, the simplified coordinate representation of reference-system is not equivalent to the original reference-system. It packs the properties of internal space of the original reference-system, so it cannot reflect all the geometric details of internal space of the original reference-system.

6.1.2 Regular basis vector form

Definition 6.1.2.1. For a classical spacetime reference-system, the complete coordinate representation

$$\begin{cases} \xi^A = \xi^A(x^M) \\ x^M = x^M(\xi^A) \end{cases}$$

transitions to simplified coordinate representation

$$\begin{cases} \xi^S = \xi^S(x^I) \\ x^I = x^I(\xi^S) \end{cases}.$$

Correspondingly, the basis vector representation

$$\begin{cases} d\xi^A = b^A_M dx^M \\ dx^M = c^M_A d\xi^A \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial \xi^A} = c^M_A \dfrac{\partial}{\partial x^M} \\ \dfrac{\partial}{\partial x^M} = b^A_M \dfrac{\partial}{\partial \xi^A} \end{cases}$$

also transitions to

$$\begin{cases} d\xi^S = b^S_I dx^I \\ dx^I = c^I_S d\xi^S \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial \xi^S} = c^I_S \dfrac{\partial}{\partial x^I} \\ \dfrac{\partial}{\partial x^I} = b^S_I \dfrac{\partial}{\partial \xi^S} \end{cases},$$

where the internal space basis vectors are packed in the following way:

$$\begin{cases} \dfrac{\partial \xi^S}{\partial x^\tau} dx^\tau \simeq \dfrac{\partial \xi^S}{\partial x^m} dx^m \\ \dfrac{\partial x^I}{\partial \xi^\tau} d\xi^\tau \simeq \dfrac{\partial x^I}{\partial \xi^a} d\xi^a \end{cases}, \quad \begin{cases} \dfrac{\partial \xi_S}{\partial x_\tau} \dfrac{\partial}{\partial x^\tau} \cong \dfrac{\partial x^m}{\partial \xi^S} \dfrac{\partial}{\partial x^m} \\ \dfrac{\partial x_I}{\partial \xi_\tau} \dfrac{\partial}{\partial \xi^\tau} \cong \dfrac{\partial \xi^a}{\partial x^I} \dfrac{\partial}{\partial \xi^a} \end{cases},$$

$\frac{\partial \xi^S}{\partial x^\tau}$, $\frac{\partial x^I}{\partial \xi^\tau}$, $\frac{\partial \xi_S}{\partial x_\tau}$, $\frac{\partial x_I}{\partial \xi_\tau}$ on $\tilde{M}$ and $\frac{d\xi^S}{dx^\tau}$, $\frac{dx^I}{d\xi^\tau}$, $\frac{d\xi_S}{dx_\tau}$, $\frac{dx_I}{d\xi_\tau}$ on the evolution path of internal space submanifold N of M are equal respectively.

Similar to Definition 2.2.8.1 , there is a basis vector representation on $\tilde{M}$:

$$\begin{cases} d\xi^S = B^S_I dx^I \\ dx^I = C^I_S d\xi^S \end{cases}, \quad \begin{cases} \dfrac{\partial}{\partial \xi^S} = C^I_S \dfrac{\partial}{\partial x^I} \\ \dfrac{\partial}{\partial x^I} = B^S_I \dfrac{\partial}{\partial \xi^S} \end{cases}. \tag{39}$$

By transplanting Definition 2.4.3.3 and Definition 2.4.3.5 , the **regular basis vector form** of evolution of classical spacetime reference-system on $\tilde{M}$ can be defined as

$$\begin{cases} d\xi^S = B^S_I dx^I \simeq B^S_0 dx^0 \\ C^I_0 \dfrac{\partial}{\partial x^I} \cong C^0_0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \end{cases}, \quad \begin{cases} dx^I = C^I_S d\xi^S \simeq C^I_0 d\xi^0 \\ B^S_0 \dfrac{\partial}{\partial \xi^S} \cong B^0_0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0} \end{cases}, \tag{40}$$

or as

$$\begin{cases} d\xi_S = \bar{B}_S^I dx_I \simeq \bar{B}_S^0 dx_0 \\ \bar{C}_I^0 \frac{\partial}{\partial x_I} \cong \bar{C}_0^0 \frac{d}{dx_0} = \frac{d}{d\xi_0} \end{cases}, \quad \begin{cases} dx_I = \bar{C}_I^S d\xi_S \simeq \bar{C}_I^0 d\xi_0 \\ \bar{B}_S^0 \frac{\partial}{\partial \xi_S} \cong \bar{B}_0^0 \frac{d}{d\xi_0} = \frac{d}{dx_0} \end{cases}. \tag{41}$$

6.1.3 Regular metric form

The regular metric form can be directly transitioned from section 2.2.7.2 , which will not be discussed repeatly. The concept of time metric in Definition 2.3.1.1 is expressed as the following form on classical spacetime submanifold.

Definition 6.1.3.1. On a neighborhood $\tilde{U}$ of p on geometric manifold $(\tilde{M}, f)$, similar to Definition 2.2.8.4 , the two coordinate frames $(\tilde{U}, \xi^S)$ and $(\tilde{U}, x^I)$ of $f(p)$ respectively interit Euclidian metric tensors $\mathbf{g} \triangleq \delta_{ST} d\xi^S \otimes d\xi^T$ and $\mathbf{h} \triangleq \varepsilon_{IJ} dx^I \otimes dx^J$ from $\mathbb{R}^{r+1}$. On $\tilde{U}$, two kinds of metrics are defined according to

$$\begin{cases} (d\xi^0)^2 \triangleq \sum_{s=1}^{r} (d\xi^s)^2 + (d\xi^\tau)^2 = \delta_{ST} d\xi^S d\xi^T = g_{IJ} dx^I dx^J \\ (dx^0)^2 \triangleq \sum_{i=1}^{r} (dx^i)^2 + (dx^\tau)^2 = \varepsilon_{IJ} dx^I dx^J = h_{ST} d\xi^S d\xi^T \end{cases}.$$

Obviously, such $d\xi^0$ and dx^0 are consistent with the $d\xi^0$ and dx^0 in Definition 2.3.1.1 . It is because the internal space metrics satisfy

$$(d\xi^\tau)^2 \triangleq \sum_{a=r+1}^{\mathfrak{D}} (d\xi^a)^2, \quad (dx^\tau)^2 \triangleq \sum_{m=r+1}^{\mathfrak{D}} (dx^m)^2.$$

The above $d\xi^0$ and dx^0 are called the **total space metrics** of classical spacetime coordinate frames $(\tilde{U}, \xi^S)$ and $(\tilde{U}, x^I)$, or called the **time metrics**. $d\xi^\tau$ and dx^τ are called the **proper-time metrics** of coordinate frames $(\tilde{U}, \xi^S)$ and $(\tilde{U}, x^I)$.

Similar to Definition 2.2.8.4 , there are metric tensors $\mathbf{G} \triangleq G_{IJ} dx^I \otimes dx^J$ and $\mathbf{H} \triangleq H_{ST} d\xi^S \otimes d\xi^T$ on $\tilde{M}$. The differential forms $d\xi^0$ and dx^0 determined by

$$\begin{cases} (d\xi^0)^2 \triangleq G_{IJ} dx^I dx^J \\ (dx^0)^2 \triangleq H_{ST} d\xi^S d\xi^T \end{cases}$$

are called the **total space metrics** of $\tilde{M}$ about coordinate form dx^I and $d\xi^S$, or called the **time metrics**.

6.2 Minkowski form of evolution of classical spacetime reference-system

6.2.1 Minkowski coordinate form

Remark 6.2.1.1. Due to historical reasons, some concepts in traditional physics such as coordinate, vector, connection, curvature, etc. are not based on the regular coordinate (coordinate indices take values in $1, 2, \cdots, r, \tau$ and the value of evolution parameter index is 0) defined in Definition 6.1.1.1 , but based on the Minkowski coordinate (coordinate indices take values in $0, 1, 2, \cdots, r$ and the value of evolution parameter index is τ). If the evolution direction is internal-directed, such a Minkowski coordinate always exists, which actually can be constructed based on the regular coordinate in the following way.

For convenience, if not specified in the following sections, the Minkowski indices take values in the following range.

(1) $\alpha,\beta,\gamma,\delta = 0,1,2,\cdots,r$ in coordinate frame $(\tilde{U},\tilde{\xi})$.

(2) $\mu,\nu,\rho,\sigma = 0,1,2,\cdots,r.$ in coordinate frame $(\tilde{U},\tilde{x})$.

Definition 6.2.1.1. The Minkowski coordinate representation should be constructed from the regular coordinate representation in Definition 6.1.1.3 .

$$(1)\begin{cases}\xi^u = \xi^u(x^k,x^\tau) = \xi^u_L(x^0)\\ \xi^\tau = \xi^\tau(x^k,x^\tau) = \xi^\tau_L(x^0)\\ \xi^0 = \xi^0(x^0)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u = \xi^u\left(x^k, x^\tau_L\left(\xi^0\right)\right) = \xi^u_L\left(x^0\left(\xi^0\right)\right)\\ \xi^\tau_L\left(x^0\right) = \xi^\tau\left(x^k, x^\tau_L\left(\xi^0\right)\right) = \xi^\tau_L\left(x^0\left(\xi^0\right)\right)\\ \xi^0\left(x^0\right) = \xi^0\end{cases}\Leftrightarrow\begin{cases}\xi^u = \xi^u\left(x^k, x^\tau_L\left(\xi^0\left(x^0\right)\right)\right) = \xi^u_L\left(x^0\left({x^\tau_L}^{-1}\left(x^\tau\right)\right)\right)\\ \xi^\tau_L\left(x^0\left(\xi^0\right)\right) = \xi^\tau\left(x^k, x^\tau_L\left(\xi^0\left(x^0\right)\right)\right) = \xi^\tau_L\left(x^0\left({x^\tau_L}^{-1}\left(x^\tau\right)\right)\right)\\ \xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\right)\right) = {x^\tau_L}^{-1}\left(x^\tau\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}\xi^u = \xi^u\left(x^k, x^\tau_L\left(\xi^0\left(x^0\right)\right)\right) = \xi^u_L\left(x^0\left({x^\tau_L}^{-1}\left(x^\tau\right)\right)\right)\\ \xi^0 = \xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\left(x^k, x^\tau_L\left(\xi^0\left(x^0\right)\right)\right)\right)\right) = {x^\tau_L}^{-1}\left(x^\tau\right)\\ \xi^\tau = \xi^\tau_L\left(x^0\left({x^\tau_L}^{-1}\left(x^\tau\right)\right)\right)\end{cases}, \text{ abbreviated as }\begin{cases}\xi^u = \tilde{\xi}^u(x^k,x^0) = \tilde{\xi}^u_L(x^\tau)\\ \xi^0 = \tilde{\xi}^0(x^k,x^0) = \tilde{\xi}^0_L(x^\tau)\\ \xi^\tau = \tilde{\xi}^\tau(x^\tau)\end{cases}.$$

$$(2)\begin{cases}x^k = x^k(\xi^u,\xi^\tau) = x^k_L(\xi^0)\\ x^\tau = x^\tau(\xi^u,\xi^\tau) = x^\tau_L(\xi^0)\\ x^0 = x^0(\xi^0)\end{cases}$$

$$\Leftrightarrow\begin{cases}x^k = x^k\left(\xi^u, \xi^\tau_L\left(x^0\right)\right) = x^k_L\left(\xi^0\left(x^0\right)\right)\\ x^\tau_L\left(\xi^0\right) = x^\tau\left(\xi^u, \xi^\tau_L\left(x^0\right)\right) = x^\tau_L\left(\xi^0\left(x^0\right)\right)\\ x^0\left(\xi^0\right) = x^0\end{cases}\Leftrightarrow\begin{cases}x^k = x^k\left(\xi^u, \xi^\tau_L\left(x^0\left(\xi^0\right)\right)\right) = x^k_L\left(\xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\right)\right)\right)\\ x^\tau_L\left(\xi^0\left(x^0\right)\right) = x^\tau\left(\xi^u, \xi^\tau_L\left(x^0\left(\xi^0\right)\right)\right) = x^\tau_L\left(\xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\right)\right)\right)\\ x^0\left({x^\tau_L}^{-1}\left(x^\tau\right)\right) = {\xi^\tau_L}^{-1}\left(\xi^\tau\right)\end{cases}$$

$$\Leftrightarrow\begin{cases}x^k = x^k\left(\xi^u, \xi^\tau_L\left(x^0\left(\xi^0\right)\right)\right) = x^k_L\left(\xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\right)\right)\right)\\ x^0 = x^0\left({x^\tau_L}^{-1}\left(x^\tau\left(\xi^u, \xi^\tau_L\left(x^0\left(\xi^0\right)\right)\right)\right)\right) = {\xi^\tau_L}^{-1}\left(\xi^\tau\right)\\ x^\tau = x^\tau_L\left(\xi^0\left({\xi^\tau_L}^{-1}\left(\xi^\tau\right)\right)\right)\end{cases}, \text{ abbreviated as }\begin{cases}x^k = \tilde{x}^k\left(\xi^u,\xi^0\right) = \tilde{x}^k_L\left(\xi^\tau\right)\\ x^0 = \tilde{x}^0\left(\xi^u,\xi^0\right) = \tilde{x}^0_L\left(\xi^\tau\right)\\ x^\tau = \tilde{x}^\tau\left(\xi^\tau\right)\end{cases}.$$

Define Minkowski coordinate

$$\begin{cases}\tilde{\xi}^s \triangleq \xi^s\\ \tilde{\xi}^\tau \triangleq \xi^\tau\\ \tilde{\xi}^0 \triangleq \xi^0\end{cases},\quad\begin{cases}\tilde{x}^i \triangleq x^i\\ \tilde{x}^\tau \triangleq x^\tau\\ \tilde{x}^0 \triangleq x^0\end{cases},$$

therefore

$$\begin{cases}\xi^u = \tilde{\xi}^u\left(x^k,x^0\right) = \tilde{\xi}^u_L\left(x^\tau\right)\\ \xi^0 = \tilde{\xi}^0\left(x^k,x^0\right) = \tilde{\xi}^0_L\left(x^\tau\right)\\ \xi^\tau = \tilde{\xi}^\tau\left(x^\tau\right)\end{cases}\Leftrightarrow\begin{cases}\tilde{\xi}^u = \tilde{\xi}^u\left(\tilde{x}^k,\tilde{x}^0\right) = \tilde{\xi}^u_L\left(\tilde{x}^\tau\right)\\ \tilde{\xi}^0 = \tilde{\xi}^0\left(\tilde{x}^k,\tilde{x}^0\right) = \tilde{\xi}^0_L\left(\tilde{x}^\tau\right)\\ \tilde{\xi}^\tau = \tilde{\xi}^\tau\left(\tilde{x}^\tau\right)\end{cases}, \text{abbreviated as}\begin{cases}\tilde{\xi}^u = \tilde{\xi}^u\left(\tilde{x}^k,\tilde{x}^0\right) = \tilde{\xi}^u\left(\tilde{x}^\tau\right)\\ \tilde{\xi}^0 = \tilde{\xi}^0\left(\tilde{x}^k,\tilde{x}^0\right) = \tilde{\xi}^0\left(\tilde{x}^\tau\right)\\ \tilde{\xi}^\tau = \tilde{\xi}^\tau\left(\tilde{x}^\tau\right)\end{cases}.$$

$$\begin{cases} x^k = \tilde{x}^k\left(\xi^u, \xi^0\right) = \tilde{x}^k_L\left(\xi^\tau\right) \\ x^0 = \tilde{x}^0\left(\xi^u, \xi^0\right) = \tilde{x}^0_L\left(\xi^\tau\right) \\ x^\tau = \tilde{x}^\tau\left(\xi^\tau\right) \end{cases} \Leftrightarrow \begin{cases} \tilde{x}^k = \tilde{x}^k\left(\tilde{\xi}^u, \tilde{\xi}^0\right) = \tilde{x}^k_L\left(\xi^\tau\right) \\ \tilde{x}^0 = \tilde{x}^0\left(\tilde{\xi}^u, \tilde{\xi}^0\right) = \tilde{x}^0_L\left(\xi^\tau\right) \\ \tilde{x}^\tau = \tilde{x}^\tau\left(\tilde{\xi}^\tau\right) \end{cases}, \text{abbreviated as} \begin{cases} \tilde{x}^k = \tilde{x}^k\left(\tilde{\xi}^u, \tilde{\xi}^0\right) = \tilde{x}^k\left(\xi^\tau\right) \\ \tilde{x}^0 = \tilde{x}^0\left(\tilde{\xi}^u, \tilde{\xi}^0\right) = \tilde{x}^0\left(\xi^\tau\right) \\ \tilde{x}^\tau = \tilde{x}^\tau\left(\tilde{\xi}^\tau\right) \end{cases}.$$

Using Minkowski indices, they can be concisely denoted by

$$\begin{cases} \tilde{\xi}^\alpha = \tilde{\xi}^\alpha\left(\tilde{x}^\mu\right) = \tilde{\xi}^\alpha\left(\tilde{x}^\tau\right) \\ \tilde{\xi}^\tau = \tilde{\xi}^\tau\left(\tilde{x}^\tau\right) \end{cases}, \quad \begin{cases} \tilde{x}^\mu = \tilde{x}^\mu\left(\tilde{\xi}^\alpha\right) = \tilde{x}^\mu\left(\tilde{\xi}^\tau\right) \\ \tilde{x}^\tau = \tilde{x}^\tau\left(\tilde{\xi}^\tau\right) \end{cases}, \tag{42}$$

which are called the **Minkowski coordinate form** of evolution of classical spacetime reference-system, determined by the regular coordinate form

$$\begin{cases} \xi^S = \xi^S\left(x^I\right) = \xi^S\left(x^0\right) \\ \xi^0 = \xi^0\left(x^0\right) \end{cases}, \quad \begin{cases} x^I = x^I\left(\xi^S\right) = x^I\left(\xi^0\right) \\ x^0 = x^0\left(\xi^0\right) \end{cases}.$$

On the classical spacetime submanifold $\tilde{M}$, $\forall p \in \tilde{M}$, the **Minkowski coordinate frames** on a neighborhood $\tilde{U}$ of p are $(\tilde{U}, \tilde{\xi}^\alpha)$ and $(\tilde{U}, \tilde{x}^\mu)$.

Remark 6.2.1.2. It is seen that when describing internal-directed evolution, the Minkowski coordinate frame is not independent of the regular coordinate frame, but is uniquely determined by the regular coordinate frame. The effectiveness of discussions of geometric property in Minkowski coordinate frame is as same as that in regular coordinate frame.

It must be noted that the Minkowski coordinate frame is not suitable to express the parameter equation of the evolution path not internal-directed, such as the parameter equation of the actual evolution path of light in vacuum. By contrast, the regular coordinate frame is suitable for this.

6.2.2 Minkowski vector form

This section at first defines Minkowski tangent vector and Minkowski cotangent vector which are equivalent to the regular tangent vector and regular cotangent vector, then constructs the Minkowski vector form of evolution, and at last obtains the evolution lemma in Minkowski form.

Definition 6.2.2.1. $\forall \left.\frac{d}{dt_L}\right|_p \in T_p(L)$, $\forall \left.\frac{d}{ds}\right|_p \in T_p(\tilde{M})$, each element of direct sum space $T_p(L) \oplus T_p(\tilde{M})$ is denoted by $\left.\frac{d}{dt_L}\right|_p + \left.\frac{d}{ds}\right|_p$. Let $V_p(\tilde{M})$ be a 1-dimensional linear subspace of tangent space $T_p(\tilde{M})$. And each $\pi_*\left(\left.\frac{d}{dt_L}\right|_p\right)$ has a unique vector projection $P\left(\left.\frac{d}{dt_L}\right|_p\right) \in V_p\left(\tilde{M}\right)$. Then, there exists a unique $\left.\frac{d}{ds}\right|_p \triangleq \pi_*\left(\left.\frac{d}{dt_L}\right|_p\right) - P\left(\left.\frac{d}{dt_L}\right|_p\right) \in T_p(\tilde{M})$, and moreover there exists a unique $\left.\frac{d}{dt_L}\right|_p - \left.\frac{d}{ds}\right|_p \in T_p\left(L\right) \oplus T_p(\tilde{M})$. Thus, there exists an injection

$$i : T_p\left(L\right) \to T_p\left(L\right) \oplus T_p\left(\tilde{M}\right), \quad \left.\frac{d}{dt_L}\right|_p \mapsto i\left(\left.\frac{d}{dt_L}\right|_p\right) \triangleq \left.\frac{d}{dt_L}\right|_p - \left(\pi_*\left(\left.\frac{d}{dt_L}\right|_p\right) - P\left(\left.\frac{d}{dt_L}\right|_p\right)\right).$$

Specially, if $V_p(\tilde{M})$ is spanned by the basis vector $\{\frac{\partial}{\partial x^\tau}\}$, then $i\left(\frac{d}{dt_L}\Big|_p\right)$ and $P\left(\frac{d}{dt_L}\Big|_p\right)$ are uniformly called the **Minkowski tangent vector** of $\frac{d}{dt_L}\Big|_p$, respectively denoted by $\frac{d}{d\tilde{t}}\big|_p$ and $\frac{d}{d\tilde{t}_L}\Big|_p$. The set of all such Minkowski vectors $\frac{d}{d\tilde{t}}\big|_p$ and $\frac{d}{d\tilde{t}_L}\Big|_p$ at p is respectively denoted by $\tilde{W}_p(\tilde{M})$ and $\tilde{W}_p(L)$.

The injection

$$\tilde{\pi}_*|_p : \tilde{W}_p\left(L\right) \to \tilde{W}_p\left(\tilde{M}\right), \quad P\left(\frac{d}{dt_L}\Big|_p\right) \mapsto i\left(\frac{d}{dt_L}\Big|_p\right)$$

induced by $\frac{d}{dt_L}\Big|_p$ is called the **Minkowski tangent mapping** at p, and say the Minkowski tangent vectors $i\left(\frac{d}{dt_L}\Big|_p\right)$ and $P\left(\frac{d}{dt_L}\Big|_p\right)$ are **equivalent** at p, which is denoted by

$$i\left(\frac{d}{dt_L}\Big|_p\right) \cong P\left(\frac{d}{dt_L}\Big|_p\right) \text{ or } \frac{d}{d\tilde{t}}\Big|_p \cong \frac{d}{d\tilde{t}_L}\Big|_p.$$

The above is a local definition. These local concepts exist on the entire evolution path L. If a tangent vector field $\frac{d}{dt_L}$ on 1-dimensional manifold L satisfies $i\left(\frac{d}{dt_L}\Big|_q\right) \cong P\left(\frac{d}{dt_L}\Big|_q\right)$ at each point q on L, then say the **Minkowski tangent vector fields** $i\left(\frac{d}{dt_L}\right)$ and $P\left(\frac{d}{dt_L}\right)$ are **equivalent** on L, which is denoted by

$$i\left(\frac{d}{dt_L}\right) \cong P\left(\frac{d}{dt_L}\right) \text{ or } \frac{d}{d\tilde{t}} \cong \frac{d}{d\tilde{t}_L}.$$

The corresponding **Minkowski tangent mapping** on L is denoted by $\tilde{\pi}_* : P\left(\frac{d}{dt_L}\right) \mapsto i\left(\frac{d}{dt_L}\right)$ or $\frac{d}{d\tilde{t}_L} \mapsto \frac{d}{d\tilde{t}}$.

More concretely, let

$$\frac{d}{dt_L} \triangleq A^0 \frac{d}{dx^0}, \quad \frac{d}{dt} \triangleq \pi_*(\frac{d}{dt_L}) \triangleq A^i \frac{\partial}{\partial x^i} + A^\tau \frac{\partial}{\partial x^\tau},$$

then

$$P\left(\frac{d}{dt_L}\right) = A^\tau \frac{\partial}{\partial x^\tau}, \quad i\left(\frac{d}{dt_L}\right) = A^0 \frac{d}{dx^0} - A^i \frac{\partial}{\partial x^i},$$

now we have the equivalence between the Minkowski tangent vectors:

$$A^0 \frac{d}{dx^0} - A^i \frac{\partial}{\partial x^i} \cong A^\tau \frac{\partial}{\partial x^\tau}.$$

Define notations

$$\frac{\partial}{\partial \tilde{x}^i} \triangleq -\frac{\partial}{\partial x^i}, \quad \frac{d}{d\tilde{x}^\tau} \triangleq \frac{\partial}{\partial x^\tau}, \quad \frac{\partial}{\partial \tilde{x}^0} \triangleq \frac{d}{dx^0},$$

thus

$$A^0 \frac{\partial}{\partial \tilde{x}^0} + A^i \frac{\partial}{\partial \tilde{x}^i} \cong A^\tau \frac{d}{d\tilde{x}^\tau}.$$

And denote

$$\tilde{A}^0 \triangleq A^0, \quad \tilde{A}^i \triangleq A^i, \quad \tilde{A}^\tau \triangleq A^\tau,$$

thus

$$\tilde{A}^0 \frac{\partial}{\partial \tilde{x}^0} + \tilde{A}^i \frac{\partial}{\partial \tilde{x}^i} \cong \tilde{A}^\tau \frac{d}{d\tilde{x}^\tau}.$$

Using Minkowski indices, this equivalence can be simply denoted by

$$\tilde{A}^\mu \frac{\partial}{\partial \tilde{x}^\mu} \cong \tilde{A}^\tau \frac{d}{d\tilde{x}^\tau}.$$

Remark 6.2.2.1. Applying the metric tensor $\tilde{G}_{\mu\nu}$ defined later, define $\frac{\partial}{\partial \tilde{x}_\nu} \triangleq \tilde{G}^{\mu\nu} \frac{\partial}{\partial \tilde{x}^\mu}$, $\frac{d}{d\tilde{x}_\tau} \triangleq \tilde{G}^{\tau\tau} \frac{d}{d\tilde{x}^\tau}$, $\tilde{A}_\nu \triangleq \tilde{G}_{\mu\nu} \tilde{A}^\mu$, $\tilde{A}_\tau \triangleq \tilde{G}_{\tau\tau} \tilde{A}^\tau$. Then we have

$$\tilde{A}_\mu \frac{\partial}{\partial \tilde{x}_\mu} \cong \tilde{A}_\tau \frac{d}{d\tilde{x}_\tau}.$$

Remark 6.2.2.2. Similar to the above definitions about Minkowski tangent vector, there are concepts about Minkowski cotangent vector, which are defined as following.

Definition 6.2.2.2. $\forall\ df|_p\,, ds|_p \in T_p^*(\tilde{M})$, each element of direct sum space $T_p^*(L) \oplus T_p^*(\tilde{M})$ is denoted by $\pi^* \left(df|_p \right) + ds|_p$. Let $V_p^*(\tilde{M})$ be a 1-dimensional linear subspace of cotangent space $T_p^*(\tilde{M})$. And each $df|_p$ has a unique vector projection $P\left(df|_p \right) \in V_p^* \left(\tilde{M} \right)$. Then, there exists a unique $ds|_p \triangleq df|_p - P\left(df|_p \right) \in T_p^* \left(\tilde{M} \right)$, moreover there exists a unique $\pi^* \left(df|_p \right) - ds|_p \in T_p^*\left(L \right) \oplus T_p^* \left(\tilde{M} \right)$. Thus, there exists an injection

$$i : T_p^* \left(\tilde{M} \right) \to T_p^* \left(L \right) \oplus T_p^* \left(\tilde{M} \right), \quad df|_p \mapsto i \left(df|_p \right) \triangleq \pi^* \left(.df|_p \right) - \left(df|_p - P\left(df|_p \right) \right).$$

Specially, if $V_p^*(\tilde{M})$ is spanned by the basis vector $\{dx^\tau\}$, then $i\left(df|_p \right)$ and $P\left(df|_p \right)$ are uniformly called the **Minkowski cotangent vector** of $df|_p$, respectively denoted by $d\tilde{f}\Big|_p$ and $d\tilde{f}_L\Big|_p$. The set of all such Minkowski vectors $d\tilde{f}\Big|_p$ and $d\tilde{f}_L\Big|_p$ at p is respectively denoted by $\tilde{W}_p^*(\tilde{M})$ and $\tilde{W}_p^*(L)$.

The surjection

$$\tilde{\pi}^*|_p : \tilde{W}_p^*(\tilde{M}) \to \tilde{W}_p^*(L), \quad i\left(df|_p \right) \mapsto P\left(df|_p \right)$$

induced by $df|_p$ is called the **Minkowski cotangent mapping** at p, and say the Minkowski cotangent vectors $i\left(df|_p \right)$ and $P\left(df|_p \right)$ are homomorphic at p, which is denoted by

$$i\left(df|_p \right) \simeq P\left(df|_p \right) \text{ or } d\tilde{f}\Big|_p \simeq d\tilde{f}_L\Big|_p.$$

The above is a local definition. These local concepts exist on the entire evolution path L. If a cotangent vector field df on 1-dimensional manifold L satisfies $i\left(df|_q \right) \cong P\left(df|_q \right)$ at each point q on L, then say the **Minkowski cotangent vector fields** $i(df)$ and $P(df)$ are homomorphic on L, which is denoted by

$$i(df) \simeq P(df) \text{ or } d\tilde{f} \simeq d\tilde{f}_L.$$

The corresponding **Minkowski cotangent mapping** on L is denoted by $\tilde{\pi}^* : i(df) \mapsto P(df)$.

More concretely, let $df \triangleq B_i dx^i + B_\tau dx^\tau, \quad df_L \triangleq \pi^*(df) \triangleq B_0 dx^0$ then

$$P(df) = B_\tau dx^\tau, \quad i(df) = B_0 dx^0 - B_i dx^i,$$

now we have the homomorphism between the Minkowski cotangent vectors:

$$B_0 dx^0 - B_i dx^i \simeq B_\tau dx^\tau.$$

Define notations

$$d\tilde{x}^i \triangleq dx^i, \quad d\tilde{x}^\tau \triangleq dx^\tau, \quad d\tilde{x}^0 \triangleq dx^0,$$

thus

$$B_0 d\tilde{x}^0 - B_i d\tilde{x}^i \simeq B_\tau d\tilde{x}^\tau.$$

And denote

$$\tilde{B}_0 \triangleq B_0, \quad \tilde{B}_i \triangleq -B_i, \quad \tilde{B}_\tau \triangleq B_\tau,$$

thus

$$\tilde{B}_0 d\tilde{x}^0 + \tilde{B}_i d\tilde{x}^i \simeq \tilde{B}_\tau d\tilde{x}^\tau.$$

Using Minkowski indices, this homomorphism can be simply denoted by

$$\tilde{B}_\mu d\tilde{x}^\mu \simeq \tilde{B}_\tau d\tilde{x}^\tau.$$

Remark 6.2.2.3. Applying the metric tensor $\tilde{G}_{\mu\nu}$ defined later, define $d\tilde{x}_\nu \triangleq \tilde{G}_{\mu\nu} d\tilde{x}^\mu$, $d\tilde{x}_\tau \triangleq \tilde{G}_{\tau\tau} d\tilde{x}^\tau$, $\tilde{B}^\nu \triangleq \tilde{G}^{\mu\nu}\tilde{B}_\mu$, $\tilde{B}^\tau \triangleq \tilde{G}^{\tau\tau}\tilde{B}_\tau$. Then we have

$$\tilde{B}^\mu d\tilde{x}_\mu \simeq \tilde{B}^\tau d\tilde{x}_\tau.$$

Definition 6.2.2.3. $\forall \frac{d}{d\tilde{t}} \triangleq \tilde{A}^\mu \frac{\partial}{\partial \tilde{x}^\mu}$, $d\tilde{f} \triangleq \tilde{B}_\mu d\tilde{x}^\mu$, $\frac{d}{d\tilde{t}_L} \triangleq \tilde{A}^\tau \frac{d}{d\tilde{x}^\tau}$, $d\tilde{f}_L \triangleq \tilde{B}_\tau d\tilde{x}^\tau$, define the conjugation of Minkowski tangent vector and Minkowski cotangent vector:

$$\left\langle \frac{d}{d\tilde{t}}, d\tilde{f} \right\rangle = \left\langle \tilde{A}^\mu \frac{\partial}{\partial \tilde{x}^\mu}, \tilde{B}_\mu d\tilde{x}^\mu \right\rangle \triangleq \tilde{A}^\mu \tilde{B}_\mu, \quad \left\langle \frac{d}{d\tilde{t}_L}, d\tilde{f}_L \right\rangle = \left\langle \tilde{A}^\tau \frac{d}{d\tilde{x}^\tau}, \tilde{B}_\tau d\tilde{x}^\tau \right\rangle \triangleq \tilde{A}^\tau \tilde{B}_\tau.$$

Proposition 6.2.2.1. If

$$\frac{d}{d\tilde{t}} \cong \frac{d}{d\tilde{t}_L} \quad d\tilde{f} \simeq d\tilde{f}_L,$$

then

$$\left\langle \frac{d}{d\tilde{t}}, d\tilde{f} \right\rangle = \left\langle \frac{d}{d\tilde{t}_L}, d\tilde{f}_L \right\rangle.$$

Proof. Let

$$\frac{d}{dt_L} \triangleq A^0 \frac{d}{dx^0}, \qquad \frac{d}{dt} = \pi_* \left(\frac{d}{dt_L} \right) \triangleq A^I \frac{\partial}{\partial x^I},$$

$$df \triangleq B_I dx^I, \qquad df_L = \pi^*(df) \triangleq B_0 dx^0.$$

And let

$$\frac{d}{d\tilde{t}} \triangleq \tilde{A}^\mu \frac{\partial}{\partial \tilde{x}^\mu}, \qquad \frac{d}{d\tilde{t}_L} \triangleq \tilde{A}^\tau \frac{d}{d\tilde{x}^\tau}$$

are the Minkowski tangent vectors of $\frac{d}{dt_L}$, and

$$d\tilde{f} \triangleq \tilde{B}_\mu d\tilde{x}^\mu, \qquad d\tilde{f}_L \triangleq \tilde{B}_\tau d\tilde{x}^\tau$$

are the Minkowski cotangent vectors of df.

According to Definition 6.2.2.1 and Definition 6.2.2.2 we obtain

$$\frac{d}{d\tilde{t}} \cong \frac{d}{d\tilde{t}_L} \Leftrightarrow \frac{d}{dt} \cong \frac{d}{dt_L}, \qquad d\tilde{f} \simeq d\tilde{f}_L \Leftrightarrow df \simeq df_L.$$

Now according to the equation $\left\langle \frac{d}{dt}, df \right\rangle = \left\langle \frac{d}{dt_L}, df_L \right\rangle$ of Remark 2.4.3.1 ,

$$\left\langle A^I \frac{\partial}{\partial x^I}, B_I dx^I \right\rangle = \left\langle A^0 \frac{d}{dx^0}, B_0 dx^0 \right\rangle,$$

that is $A^I B_I = A^0 B_0$, or can be expressed as $A^i B_i + A^\tau B_\tau = A^0 B_0$, then $A^0 B_0 - A^i B_i = A^\tau B_\tau$, further more, $\tilde{A}^0 \tilde{B}_0 + \tilde{A}^i \tilde{B}_i = \tilde{A}^\tau \tilde{B}_\tau$, i.e. $\tilde{A}^\mu \tilde{B}_\mu = \tilde{A}^\tau \tilde{B}_\tau$, that is to say

$$\left\langle \frac{d}{d\tilde{t}}, d\tilde{f} \right\rangle = \left\langle \frac{d}{d\tilde{t}_L}, d\tilde{f}_L \right\rangle . \square$$

Remark 6.2.2.4. With the above concepts about Minkowski tangent vector and Minkowski cotangent vector, the Minkowski form of Definition 6.2.2.5 can finally be constructed. Now do it step by step.

Definition 6.2.2.4. According to the Minkowski coordinate form of evolution of classical spacetime reference-system on a neighborhood of point p,

$$\begin{cases} \tilde{\xi}^\alpha = \tilde{\xi}^\alpha(\tilde{x}^\mu) = \tilde{\xi}^\alpha(\tilde{x}^\tau) \\ \tilde{\xi}^\tau = \tilde{\xi}^\tau(\tilde{x}^\tau) \end{cases}, \quad \begin{cases} \tilde{x}^\mu = \tilde{x}^\mu(\tilde{\xi}^\alpha) = \tilde{x}^\mu(\tilde{\xi}^\tau) \\ \tilde{x}^\tau = \tilde{x}^\tau(\tilde{\xi}^\tau) \end{cases}.$$

The partial derivatives

$$\tilde{b}^\alpha_\mu \triangleq \frac{\partial \tilde{\xi}^\alpha}{\partial \tilde{x}^\mu}, \quad \tilde{c}^\mu_\alpha \triangleq \frac{\partial \tilde{x}^\mu}{\partial \tilde{\xi}^\alpha}$$

are called the **Minkowski slack-tights** on the neighborhood of p. In addition it can also be defined that

$$\tilde{b}^\alpha_\tau \triangleq \frac{d\tilde{\xi}^\alpha}{d\tilde{x}^\tau}, \quad \tilde{b}^\tau_\tau \triangleq \frac{d\tilde{\xi}^\tau}{d\tilde{x}^\tau},$$

$$\tilde{c}^\mu_\tau \triangleq \frac{d\tilde{x}^\mu}{d\tilde{\xi}^\tau}, \quad \tilde{c}^\tau_\tau \triangleq \frac{d\tilde{x}^\tau}{d\tilde{\xi}^\tau}.$$

Similar to Definition 2.4.3.3 , they determine smooth functions $\tilde{B}^\alpha_\mu$, $\tilde{C}^\mu_\alpha$, $\tilde{B}^\alpha_\tau$, $\tilde{B}^\tau_\tau$, $\tilde{C}^\mu_\tau$, $\tilde{C}^\tau_\tau$ on $\tilde{M}$. Then denote

$$\tilde{\varepsilon}^\mu_\nu \triangleq \tilde{C}^\mu_\alpha \tilde{B}^\alpha_\nu, \quad \tilde{\delta}^\alpha_\beta = \tilde{B}^\alpha_\mu \tilde{C}^\mu_\beta,$$

$$\tilde{\varepsilon}^\mu_\tau \triangleq \tilde{B}^\tau_\tau \tilde{C}^\mu_\tau = \tilde{B}^\alpha_\tau \tilde{C}^\mu_\alpha, \quad \tilde{\delta}^\alpha_\tau \triangleq \tilde{C}^\tau_\tau \tilde{B}^\alpha_\tau = \tilde{C}^\mu_\tau \tilde{B}^\alpha_\mu.$$

Proposition 6.2.2.2. There are the following relationships between Minkowski slack-tight and regular slack-tight.

$$\begin{cases} \tilde{B}^s_i = -B^s_i \\ \tilde{B}^0_i = -\dfrac{B^\tau_i}{\delta^\tau_0} \\ \tilde{C}^i_\tau = \dfrac{C^i_0}{\delta^\tau_0} \end{cases}, \quad \begin{cases} \tilde{B}^s_\tau = B^s_\tau \\ \tilde{B}^0_\tau = \dfrac{B^\tau_\tau}{\delta^\tau_0} \\ \tilde{C}^\tau_\tau = \dfrac{C^\tau_0}{\delta^\tau_0} \end{cases}, \quad \begin{cases} \tilde{B}^s_0 = B^s_0 \\ \tilde{B}^0_0 = \dfrac{B^\tau_0}{\delta^\tau_0} \\ \tilde{C}^0_\tau = \dfrac{C^0_0}{\delta^\tau_0} \end{cases}; \quad \begin{cases} \tilde{C}^i_s = -C^i_s \\ \tilde{C}^0_s = -\dfrac{C^\tau_s}{\varepsilon^\tau_0} \\ \tilde{B}^s_\tau = \dfrac{B^s_0}{\varepsilon^\tau_0} \end{cases}, \quad \begin{cases} \tilde{C}^i_\tau = C^i_\tau \\ \tilde{C}^0_\tau = \dfrac{C^\tau_\tau}{\varepsilon^\tau_0} \\ \tilde{B}^\tau_\tau = \dfrac{B^\tau_0}{\varepsilon^\tau_0} \end{cases}, \quad \begin{cases} \tilde{C}^i_0 = C^i_0 \\ \tilde{C}^0_0 = \dfrac{C^\tau_0}{\varepsilon^\tau_0} \\ \tilde{B}^0_\tau = \dfrac{B^0_0}{\varepsilon^\tau_0} \end{cases}.$$

Proof. It can be directly calculated from Definition 6.2.1.1 according to the derivative rule of multivariate compound function. On any local region of manifold, the complete expression of the Minkowski coordinate form

$$\begin{cases} \tilde{\xi}^s = \tilde{\xi}^s\left(\tilde{x}^i, \tilde{x}^0\right) = \tilde{\xi}^s_L\left(\tilde{x}^\tau\right) \\ \tilde{\xi}^0 = \tilde{\xi}^0\left(\tilde{x}^i, \tilde{x}^0\right) = \tilde{\xi}^0_L\left(\tilde{x}^\tau\right) \\ \tilde{\xi}^\tau = \tilde{\xi}^\tau\left(\tilde{x}^\tau\right) \end{cases}, \quad \begin{cases} \tilde{x}^i = \tilde{x}^i\left(\tilde{\xi}^s, \tilde{\xi}^0\right) = \tilde{x}^i_L\left(\tilde{\xi}^\tau\right) \\ \tilde{x}^0 = \tilde{x}^0\left(\tilde{\xi}^s, \tilde{\xi}^0\right) = \tilde{x}^0_L\left(\tilde{\xi}^\tau\right) \\ \tilde{x}^\tau = \tilde{x}^\tau\left(\tilde{\xi}^\tau\right) \end{cases}$$

is

$$\begin{cases} \xi^s = \xi^s\left(x^i, x_L^\tau\left(\xi^0\left(x^0\right)\right)\right) = \xi_L^s\left(x^0\left({x_L^\tau}^{-1}\left(x^\tau\right)\right)\right) \\ \xi^0 = \xi^0\left({\xi_L^\tau}^{-1}\left(\xi^\tau\left(x^i, x_L^\tau\left(\xi^0\left(x^0\right)\right)\right)\right)\right) = {x_L^\tau}^{-1}\left(x^\tau\right) \\ \xi^\tau = \xi_L^\tau\left(x^0\left({x_L^\tau}^{-1}\left(x^\tau\right)\right)\right) \end{cases}.$$

$$\begin{cases} x^i = x^i\left(\xi^s, \xi_L^\tau\left(x^0\left(\xi^0\right)\right)\right) = x_L^i\left(\xi^0\left({\xi_L^\tau}^{-1}\left(\xi^\tau\right)\right)\right) \\ x^0 = x^0\left({x_L^\tau}^{-1}\left(x^\tau\left(\xi^s, \xi_L^\tau\left(x^0\left(\xi^0\right)\right)\right)\right)\right) = {\xi_L^\tau}^{-1}\left(\xi^\tau\right) \\ x^\tau = x_L^\tau\left(\xi^0\left({\xi_L^\tau}^{-1}\left(\xi^\tau\right)\right)\right) \end{cases}.$$

Then

$$\begin{cases} \tilde{b}_i^s \triangleq \dfrac{\partial\tilde{\xi}^s}{\partial\tilde{x}^i} = -\dfrac{\partial\tilde{\xi}^s}{\partial x^i} = -\dfrac{\partial\xi^s}{\partial x^i} = -b_i^s \\ \tilde{b}_i^0 \triangleq \dfrac{\partial\tilde{\xi}^0}{\partial\tilde{x}^i} = -\dfrac{\partial\tilde{\xi}^0}{\partial x^i} = -\dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau}\dfrac{\partial\xi^\tau}{\partial x^i} = -\dfrac{d\xi^0}{d\xi^\tau}\dfrac{\partial\xi^\tau}{\partial x^i} = -\dfrac{b_i^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^i \triangleq \dfrac{d\tilde{x}_L^i}{d\tilde{\xi}^\tau} = \dfrac{d\tilde{x}_L^i}{d\xi^\tau} = \dfrac{dx_L^i}{d\xi^0}\dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx_L^i}{d\xi^\tau} = \dfrac{d\xi^0}{d\xi^\tau}\dfrac{dx_L^i}{d\xi^0} = \dfrac{c_0^i}{\delta_0^\tau} \end{cases},$$

$$\begin{cases} \tilde{b}_\tau^s \triangleq \dfrac{d\tilde{\xi}_L^i}{d\tilde{x}^\tau} = \dfrac{d\tilde{\xi}_L^i}{dx^\tau} = \dfrac{d\xi_L^i}{dx^0}\dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi_L^i}{dx^\tau} = b_\tau^s \\ \tilde{b}_\tau^0 \triangleq \dfrac{d\tilde{\xi}_L^0}{d\tilde{x}^\tau} = \dfrac{d\tilde{\xi}_L^0}{dx^\tau} = \dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi^0}{d\xi_L^\tau}\dfrac{d\xi_L^\tau}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi^0}{d\xi_L^\tau}\dfrac{d\xi_L^\tau}{dx^\tau} = \dfrac{b_\tau^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^\tau \triangleq \dfrac{d\tilde{x}^\tau}{d\tilde{\xi}^\tau} = \dfrac{d\tilde{x}^\tau}{d\xi^\tau} = \dfrac{dx_L^\tau}{d\xi^0}\dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx_L^\tau}{d\xi^\tau} = \dfrac{d\xi^0}{d\xi^\tau}\dfrac{dx_L^\tau}{d\xi^0} = \dfrac{c_0^\tau}{\delta_0^\tau} \end{cases},$$

$$\begin{cases} \tilde{b}_0^s \triangleq \dfrac{\partial\tilde{\xi}^s}{\partial\tilde{x}^0} = \dfrac{\partial\tilde{\xi}^s}{\partial x^0} = \dfrac{\partial\xi^s}{\partial x^i}\dfrac{dx_L^i}{dx^0} + \dfrac{\partial\xi^s}{\partial x^\tau}\dfrac{dx_L^\tau}{d\xi^0}\dfrac{d\xi^0}{dx^0} = \dfrac{\partial\xi^s}{\partial x^i}\dfrac{dx_L^i}{dx^0} + \dfrac{\partial\xi^s}{\partial x^\tau}\dfrac{dx_L^\tau}{dx^0} = \dfrac{d\xi^s}{dx^0} = b_0^s \\ \tilde{b}_0^0 \triangleq \dfrac{\partial\tilde{\xi}^0}{\partial\tilde{x}^0} = \dfrac{\partial\tilde{\xi}^0}{\partial x^0} = \dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau}\left(\dfrac{\partial\xi^\tau}{\partial x^i}\dfrac{dx_L^i}{dx^0} + \dfrac{\partial\xi^\tau}{\partial x^\tau}\dfrac{dx_L^\tau}{d\xi^0}\dfrac{d\xi^0}{dx^0}\right) = \dfrac{d\xi^0}{d\xi^\tau}\dfrac{d\xi^\tau}{dx^0} = \dfrac{b_0^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^0 \triangleq \dfrac{d\tilde{x}_L^0}{d\tilde{\xi}^\tau} = \dfrac{d\tilde{x}_L^0}{d\xi^\tau} = \dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx^0}{d\xi^0}\dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx^0}{d\xi^0}\dfrac{d\xi^0}{d\xi^\tau} = \dfrac{c_0^0}{\delta_0^\tau} \end{cases};$$

$$\begin{cases} \tilde{c}_s^i \triangleq \dfrac{\partial\tilde{x}^i}{\partial\tilde{\xi}^s} = -\dfrac{\partial\tilde{x}^i}{\partial\xi^s} = -\dfrac{\partial x^i}{\partial\xi^s} = -c_s^i \\ \tilde{c}_s^0 \triangleq \dfrac{\partial\tilde{x}^0}{\partial\tilde{\xi}^s} = -\dfrac{\partial\tilde{x}^0}{\partial\xi^s} = -\dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau}\dfrac{\partial x^\tau}{\partial\xi^s} = -\dfrac{dx^0}{dx^\tau}\dfrac{\partial x^\tau}{\partial\xi^s} = -\dfrac{c_s^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^s \triangleq \dfrac{d\tilde{\xi}_L^s}{d\tilde{x}^\tau} = \dfrac{d\tilde{\xi}_L^s}{dx^\tau} = \dfrac{d\xi_L^s}{dx^0}\dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi_L^s}{dx^\tau} = \dfrac{dx^0}{dx^\tau}\dfrac{d\xi_L^s}{dx^0} = \dfrac{b_0^s}{\varepsilon_0^\tau} \end{cases},$$

$$\begin{cases} \tilde{c}_\tau^i \triangleq \dfrac{d\tilde{x}_L^i}{d\tilde{\xi}^\tau} = \dfrac{d\tilde{x}_L^i}{d\xi^\tau} = \dfrac{dx_L^i}{d\xi^0}\dfrac{d\xi^0}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx_L^i}{d\xi^\tau} = c_\tau^i \\ \tilde{c}_\tau^0 \triangleq \dfrac{d\tilde{x}_L^0}{d\tilde{\xi}^\tau} = \dfrac{d\tilde{x}_L^0}{d\xi^\tau} = \dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx^0}{dx_L^\tau}\dfrac{dx_L^\tau}{dx^0}\dfrac{d\left(\xi_L^\tau\right)^{-1}}{d\xi^\tau} = \dfrac{dx^0}{dx_L^\tau}\dfrac{dx_L^\tau}{d\xi^\tau} = \dfrac{c_\tau^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^\tau \triangleq \dfrac{d\tilde{\xi}^\tau}{d\tilde{x}^\tau} = \dfrac{d\tilde{\xi}^\tau}{dx^\tau} = \dfrac{d\xi^\tau}{dx^0}\dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi^\tau}{dx^\tau} = \dfrac{dx^0}{dx^\tau}\dfrac{d\xi^\tau}{dx^0} = \dfrac{b_0^\tau}{\varepsilon_0^\tau} \end{cases}$$

$$\begin{cases}\tilde{c}_0^i \triangleq \dfrac{\partial \tilde{x}^i}{\partial \tilde{\xi}^0} = \dfrac{\partial \tilde{x}^i}{\partial \xi^0} = \dfrac{\partial x^i}{\partial \xi^s}\dfrac{d\xi_L^s}{\partial \xi^0} + \dfrac{\partial x^i}{\partial \xi^\tau}\dfrac{d\xi_L^\tau}{dx^0}\dfrac{dx^0}{\partial \xi^0} = \dfrac{\partial x^i}{\partial \xi^s}\dfrac{d\xi_L^s}{\partial \xi^0} + \dfrac{\partial x^i}{\partial \xi^\tau}\dfrac{d\xi_L^\tau}{\partial \xi^0} = \dfrac{dx^i}{d\xi^0} = c_0^i \\ \tilde{c}_0^0 \triangleq \dfrac{\partial \tilde{x}^0}{\partial \tilde{\xi}^0} = \dfrac{\partial \tilde{x}^0}{\partial \xi^0} = \dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau}\left(\dfrac{\partial x^\tau}{\partial \xi^s}\dfrac{d\xi_L^s}{d\xi^0} + \dfrac{\partial x^\tau}{\partial \xi^\tau}\dfrac{d\xi_L^\tau}{dx^0}\dfrac{dx^0}{d\xi^0}\right) = \dfrac{dx^0}{dx^\tau}\dfrac{dx^\tau}{d\xi^0} = \dfrac{c_0^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^0 \triangleq \dfrac{d\tilde{\xi}^0}{d\tilde{x}^\tau} = \dfrac{d\tilde{\xi}^0}{dx^\tau} = \dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi^0}{dx^0}\dfrac{dx^0}{d\xi^0}\dfrac{d\left(x_L^\tau\right)^{-1}}{dx^\tau} = \dfrac{d\xi^0}{dx^0}\dfrac{dx^0}{dx^\tau} = \dfrac{b_0^0}{\varepsilon_0^\tau}\end{cases}.$$

Thus, the following conclusions are proved on the neighborhood of any point on the manifold.

$$\begin{cases}\tilde{b}_i^s = -b_i^s \\ \tilde{b}_i^0 = -\dfrac{b_i^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^i = \dfrac{c_0^i}{\delta_0^\tau}\end{cases},\quad \begin{cases}\tilde{b}_\tau^s = b_\tau^s \\ \tilde{b}_\tau^0 = \dfrac{b_\tau^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^\tau = \dfrac{c_0^\tau}{\delta_0^\tau}\end{cases},\quad \begin{cases}\tilde{b}_0^s = b_0^s \\ \tilde{b}_0^0 = \dfrac{b_0^\tau}{\delta_0^\tau} \\ \tilde{c}_\tau^0 = \dfrac{c_0^0}{\delta_0^\tau}\end{cases};\quad \begin{cases}\tilde{c}_s^i = -c_s^i \\ \tilde{c}_s^0 = -\dfrac{c_s^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^s = \dfrac{b_0^s}{\varepsilon_0^\tau}\end{cases},\quad \begin{cases}\tilde{c}_\tau^i = c_\tau^i \\ \tilde{c}_\tau^0 = \dfrac{c_\tau^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^\tau = \dfrac{b_0^\tau}{\varepsilon_0^\tau}\end{cases},\quad \begin{cases}\tilde{c}_0^i = c_0^i \\ \tilde{c}_0^0 = \dfrac{c_0^\tau}{\varepsilon_0^\tau} \\ \tilde{b}_\tau^0 = \dfrac{b_0^0}{\varepsilon_0^\tau}\end{cases}.$$

Therefore, there are the following conclusions on the entire manifold.

$$\begin{cases}\tilde{B}_i^s = -B_i^s \\ \tilde{B}_i^0 = -\dfrac{B_i^\tau}{\delta_0^\tau} \\ \tilde{C}_\tau^i = \dfrac{C_0^i}{\delta_0^\tau}\end{cases},\quad \begin{cases}\tilde{B}_\tau^s = B_\tau^s \\ \tilde{B}_\tau^0 = \dfrac{B_\tau^\tau}{\delta_0^\tau} \\ \tilde{C}_\tau^\tau = \dfrac{C_0^\tau}{\delta_0^\tau}\end{cases},\quad \begin{cases}\tilde{B}_0^s = B_0^s \\ \tilde{B}_0^0 = \dfrac{B_0^\tau}{\delta_0^\tau} \\ \tilde{C}_\tau^0 = \dfrac{C_0^0}{\delta_0^\tau}\end{cases};\quad \begin{cases}\tilde{C}_s^i = -C_s^i \\ \tilde{C}_s^0 = -\dfrac{C_s^\tau}{\varepsilon_0^\tau} \\ \tilde{B}_\tau^s = \dfrac{B_0^s}{\varepsilon_0^\tau}\end{cases},\quad \begin{cases}\tilde{C}_\tau^i = C_\tau^i \\ \tilde{C}_\tau^0 = \dfrac{C_\tau^\tau}{\varepsilon_0^\tau} \\ \tilde{B}_\tau^\tau = \dfrac{B_0^\tau}{\varepsilon_0^\tau}\end{cases},\quad \begin{cases}\tilde{C}_0^i = C_0^i \\ \tilde{C}_0^0 = \dfrac{C_0^\tau}{\varepsilon_0^\tau} \\ \tilde{B}_\tau^0 = \dfrac{B_0^0}{\varepsilon_0^\tau}\end{cases}.$$

Discussion 6.2.2.1. Now starting from the regular basis vector form

$$\begin{cases}d\xi^S = b_I^S dx^I \simeq b_0^S dx^0 \\ c_0^I \dfrac{\partial}{\partial x^I} \cong c_0^0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0}\end{cases},\quad \begin{cases}dx^I = c_S^I d\xi^S \simeq c_0^I d\xi^0 \\ b_0^S \dfrac{\partial}{\partial \xi^S} \cong b_0^0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0}\end{cases},$$

the corresponding Minkowski basis vector form can be constructed. The regular basis vector form can be expanded as

$$\begin{cases}d\xi^s = b_i^s dx^i + b_\tau^s dx^\tau \simeq b_0^s dx^0 \\ d\xi^\tau = b_i^\tau dx^i + b_\tau^\tau dx^\tau \simeq b_0^\tau dx^0 \\ c_0^i \dfrac{\partial}{\partial x^i} + c_0^\tau \dfrac{\partial}{\partial x^\tau} \cong c_0^0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0}\end{cases} \Rightarrow \begin{cases}d\xi^s = b_i^s dx^i + b_\tau^s dx^\tau \simeq b_0^s dx^0 \\ \dfrac{b_i^\tau}{\delta_0^\tau} dx^i + \dfrac{b_\tau^\tau}{\delta_0^\tau} dx^\tau \simeq \dfrac{b_0^\tau}{\delta_0^\tau} dx^0 = d\xi^0 \\ \dfrac{c_0^i}{\delta_0^\tau}\dfrac{\partial}{\partial x^i} + \dfrac{c_0^\tau}{\delta_0^\tau}\dfrac{\partial}{\partial x^\tau} \cong \dfrac{c_0^0}{\delta_0^\tau}\dfrac{d}{dx^0} = \dfrac{1}{\delta_0^\tau}\dfrac{d}{d\xi^0}\end{cases} \Rightarrow \begin{cases}b_0^s dx^0 - b_i^s dx^i \simeq b_\tau^s dx^\tau \\ \dfrac{b_0^\tau}{\delta_0^\tau} dx^0 - \dfrac{b_i^\tau}{\delta_0^\tau} dx^i \simeq \dfrac{b_\tau^\tau}{\delta_0^\tau} dx^\tau \\ \dfrac{c_0^0}{\delta_0^\tau}\dfrac{d}{dx^0} - \dfrac{c_0^i}{\delta_0^\tau}\dfrac{\partial}{\partial x^i} \cong \dfrac{c_0^\tau}{\delta_0^\tau}\dfrac{\partial}{\partial x^\tau}\end{cases}$$

$$\Rightarrow \begin{cases}\tilde{b}_0^s dx^0 + \tilde{b}_i^s dx^i \simeq \tilde{b}_\tau^s dx^\tau \\ \tilde{b}_0^0 dx^0 + \tilde{b}_i^0 dx^i \simeq \tilde{b}_\tau^0 dx^\tau \\ \tilde{c}_\tau^0 \dfrac{d}{dx^0} - \tilde{c}_\tau^i \dfrac{\partial}{\partial x^i} \cong \tilde{c}_\tau^\tau \dfrac{\partial}{\partial x^\tau}\end{cases} \Rightarrow \begin{cases}\tilde{b}_0^s d\tilde{x}^0 + \tilde{b}_i^s d\tilde{x}^i \simeq \tilde{b}_\tau^s d\tilde{x}^\tau \\ \tilde{b}_0^0 d\tilde{x}^0 + \tilde{b}_i^0 d\tilde{x}^i \simeq \tilde{b}_\tau^0 d\tilde{x}^\tau \\ \tilde{c}_\tau^0 \dfrac{\partial}{\partial \tilde{x}^0} + \tilde{c}_\tau^i \dfrac{\partial}{\partial \tilde{x}^i} \cong \tilde{c}_\tau^\tau \dfrac{d}{d\tilde{x}^\tau}\end{cases} \Leftrightarrow \begin{cases}\tilde{b}_\mu^\alpha d\tilde{x}^\mu \simeq \tilde{b}_\tau^\alpha d\tilde{x}^\tau \\ \tilde{c}_\tau^\mu \dfrac{\partial}{\partial \tilde{x}^\mu} \cong \tilde{c}_\tau^\tau \dfrac{d}{d\tilde{x}^\tau}\end{cases}, \qquad (*)$$

$$\begin{cases}dx^i = c_s^i d\xi^s + c_\tau^i d\xi^\tau \simeq c_0^i d\xi^0 \\ dx^\tau = c_s^\tau d\xi^s + c_\tau^\tau d\xi^\tau \simeq c_0^\tau d\xi^0 \\ b_0^s \dfrac{\partial}{\partial \xi^s} + b_0^\tau \dfrac{\partial}{\partial \xi^\tau} \cong b_0^0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0}\end{cases} \Rightarrow \begin{cases}dx^i = c_s^i d\xi^s + c_\tau^i d\xi^\tau \simeq c_0^i d\xi^0 \\ \dfrac{c_s^\tau}{\varepsilon_0^\tau} d\xi^s + \dfrac{c_\tau^\tau}{\varepsilon_0^\tau} d\xi^\tau \simeq \dfrac{c_0^\tau}{\varepsilon_0^\tau} d\xi^0 = dx^0 \\ \dfrac{b_0^s}{\varepsilon_0^\tau}\dfrac{\partial}{\partial \xi^s} + \dfrac{b_0^\tau}{\varepsilon_0^\tau}\dfrac{\partial}{\partial \xi^\tau} \cong \dfrac{b_0^0}{\varepsilon_0^\tau}\dfrac{d}{d\xi^0} = \dfrac{1}{\varepsilon_0^\tau}\dfrac{d}{dx^0}\end{cases} \Rightarrow \begin{cases}c_0^i d\xi^0 - c_s^i d\xi^s \simeq c_\tau^i d\xi^\tau \\ \dfrac{c_0^\tau}{\varepsilon_0^\tau} d\xi^0 - \dfrac{c_s^\tau}{\varepsilon_0^\tau} d\xi^s \simeq \dfrac{c_\tau^\tau}{\varepsilon_0^\tau} d\xi^\tau \\ \dfrac{b_0^0}{\varepsilon_0^\tau}\dfrac{d}{d\xi^0} - \dfrac{b_0^s}{\varepsilon_0^\tau}\dfrac{\partial}{\partial \xi^s} \cong \dfrac{b_0^\tau}{\varepsilon_0^\tau}\dfrac{\partial}{\partial \xi^\tau}\end{cases}$$

$$\Rightarrow \begin{cases}\tilde{c}_0^i d\xi^0 + \tilde{c}_s^i d\xi^s \simeq \tilde{c}_\tau^i d\xi^\tau \\ \tilde{c}_0^0 d\xi^0 + \tilde{c}_s^0 d\xi^s \simeq \tilde{c}_\tau^0 d\xi^\tau \\ \tilde{b}_\tau^0 \dfrac{d}{d\xi^0} - \tilde{b}_\tau^s \dfrac{\partial}{\partial \xi^s} \cong \tilde{b}_\tau^\tau \dfrac{\partial}{\partial \xi^\tau}\end{cases} \Rightarrow \begin{cases}\tilde{c}_0^i d\tilde{\xi}^0 + \tilde{c}_s^i d\tilde{\xi}^s \simeq \tilde{c}_\tau^i d\tilde{\xi}^\tau \\ \tilde{c}_0^0 d\tilde{\xi}^0 + \tilde{c}_s^0 d\tilde{\xi}^s \simeq \tilde{c}_\tau^0 d\tilde{\xi}^\tau \\ \tilde{b}_\tau^0 \dfrac{\partial}{\partial \tilde{\xi}^0} + \tilde{b}_\tau^s \dfrac{\partial}{\partial \tilde{\xi}^s} \cong \tilde{b}_\tau^\tau \dfrac{d}{d\tilde{\xi}^\tau}\end{cases} \Leftrightarrow \begin{cases}\tilde{c}_\alpha^\mu d\tilde{\xi}^\alpha \simeq \tilde{c}_\tau^\mu d\tilde{\xi}^\tau \\ \tilde{b}_\tau^\alpha \dfrac{\partial}{\partial \tilde{\xi}^\alpha} \cong \tilde{b}_\tau^\tau \dfrac{d}{d\tilde{\xi}^\tau}\end{cases}. \qquad (**)$$

(1) On one hand, $\tilde{b}^\alpha_\mu d\tilde{x}^\mu,\ \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \in \tilde{W}^*_p(\tilde{M})$ and $\tilde{c}^\mu_\tau \frac{\partial}{\partial\tilde{x}^\mu},\ \tilde{b}^\alpha_\tau \frac{\partial}{\partial\tilde{\xi}^\alpha} \in \tilde{W}_p(\tilde{M})$ in (*) and (**) are Minkowski cotangent vectors and Minkowski tangent vectors, respectively.

(2) On the other hand, in cotangent space $\tilde{T}^*_p(\tilde{M})$ and tangent space $\tilde{T}_p(\tilde{M})$ determined by the Minkowski coordinate frames $\{\tilde{\xi}^\alpha\}$ and $\{\tilde{x}^\mu\}$, there are cotangent vectors $d\tilde{\xi}^\alpha = \tilde{b}^\alpha_\mu d\tilde{x}^\mu,\ d\tilde{x}^\mu = \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \in \tilde{T}^*_p(\tilde{M})$ and tangent vectors $\tilde{c}^\mu_\tau \frac{\partial}{\partial\tilde{x}^\mu},\ \tilde{b}^\alpha_\tau \frac{\partial}{\partial\tilde{\xi}^\alpha} \in \tilde{T}_p(\tilde{M})$.

The above (1) and (2) determine the below (3) and (4).

(3) The Minkowski cotangent vectors $\tilde{b}^\alpha_\mu d\tilde{x}^\mu,\ \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \in \tilde{W}^*_p(\tilde{M})$ uniquely correspond to the cotangent vectors $\tilde{b}^\alpha_\mu d\tilde{x}^\mu,\ \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \in \tilde{T}^*_p(\tilde{M})$. Such an injection defines a relation of equivalence $\equiv$ such that

$$\tilde{b}^\alpha_\mu d\tilde{x}^\mu \equiv \tilde{b}^\alpha_\mu d\tilde{x}^\mu, \quad \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \equiv \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha.$$

The Minkowski tangent vectors in $\tilde{W}_p(\tilde{M})$ and the tangent vectors in $\tilde{T}_p(\tilde{M})$ have a similar relation to the above, which is denoted by

$$\tilde{c}^\mu_\tau \frac{\partial}{\partial\tilde{x}^\mu} \equiv \tilde{c}^\mu_\tau \frac{\partial}{\partial\tilde{x}^\mu}, \quad \tilde{b}^\alpha_\tau \frac{\partial}{\partial\tilde{\xi}^\alpha} \equiv \tilde{b}^\alpha_\tau \frac{\partial}{\partial\tilde{\xi}^\alpha}.$$

(4) The Minkowski cotangent vectors $\tilde{b}^\alpha_\tau d\tilde{x}^\tau,\ \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \in \tilde{W}^*_p(L)$ uniquely correspond to the cotangent vectors $\tilde{b}^\alpha_\tau d\tilde{x}^\tau,\ \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \in \tilde{T}^*_p(L)$, which is denoted by

$$\tilde{b}^\alpha_\tau d\tilde{x}^\tau \equiv \tilde{b}^\alpha_\tau d\tilde{x}^\tau, \quad \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \equiv \tilde{c}^\mu_\tau d\tilde{\xi}^\tau.$$

The Minkowski tangent vectors in $\tilde{W}_p(L)$ and the tangent vectors in $\tilde{T}_p(L)$ also have a similar relation to the above, which is denoted by

$$\tilde{c}^\tau_\tau \frac{d}{d\tilde{x}^\tau} \equiv \tilde{c}^\tau_\tau \frac{d}{d\tilde{x}^\tau}, \quad \tilde{b}^\tau_\tau \frac{d}{d\tilde{\xi}^\tau} \equiv \tilde{b}^\tau_\tau \frac{d}{d\tilde{\xi}^\tau}.$$

It has to be noticed that the mappings of equivalence from $\tilde{W}_p(\tilde{M})$ to $\tilde{T}_p(\tilde{M})$ in (3) and (4) are injections, and not surjections. $\tilde{W}_p(\tilde{M})$ is not a linear space spanned by $\{d\tilde{x}^\mu\}$ as a basis. When $\tilde{A}_0 = 0$ and $\tilde{A}_i \neq 0$, although $\tilde{A}_0 d\tilde{x}^0 + \tilde{A}_i d\tilde{x}^i$ is a cotangent vector in $\tilde{T}_p(\tilde{M})$, it is not a Minkowski cotangent vector in $\tilde{W}_p(\tilde{M})$. For example, consider the energy-momentum cotangent vector $\tilde{E}_0 d\tilde{x}^0 - \tilde{p}_i d\tilde{x}^i \in \tilde{T}_p(\tilde{M})$. If $\tilde{E}_0 = 0$ and $\tilde{p}_i \neq 0$, $\tilde{E}_0 d\tilde{x}^0 - \tilde{p}_i d\tilde{x}^i$ is not a Minkowski cotangent vector in $\tilde{W}_p(\tilde{M})$, and as an exception case it just exactly has no corresponding physical intuition. Obviously, any formula including $\equiv$ constructed in the way of (3) and (4) is absolutely not such an exception case.

All of the above equations together can be written as formula

$$\begin{cases} d\tilde{\xi}^\alpha = \tilde{b}^\alpha_\mu d\tilde{x}^\mu \equiv \tilde{b}^\alpha_\mu d\tilde{x}^\mu \simeq \tilde{b}^\alpha_\tau d\tilde{x}^\tau \equiv \tilde{b}^\alpha_\tau d\tilde{x}^\tau \\ \tilde{c}^\mu_\tau \dfrac{\partial}{\partial\tilde{x}^\mu} \equiv \tilde{c}^\mu_\tau \dfrac{\partial}{\partial\tilde{x}^\mu} \cong \tilde{c}^\tau_\tau \dfrac{d}{d\tilde{x}^\tau} \equiv \tilde{c}^\tau_\tau \dfrac{d}{d\tilde{x}^\tau} = \dfrac{d}{d\tilde{\xi}^\tau} \end{cases}, \quad \begin{cases} d\tilde{x}^\mu = \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \equiv \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \simeq \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \equiv \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \\ \tilde{b}^\alpha_\tau \dfrac{\partial}{\partial\tilde{\xi}^\alpha} \equiv \tilde{b}^\alpha_\tau \dfrac{\partial}{\partial\tilde{\xi}^\alpha} \cong \tilde{b}^\tau_\tau \dfrac{d}{d\tilde{\xi}^\tau} \equiv \tilde{b}^\tau_\tau \dfrac{d}{d\tilde{\xi}^\tau} = \dfrac{d}{d\tilde{x}^\tau} \end{cases}.$$

In consideration of the fact that no matter for the Minkowski tangent vector and Minkowski cotangent vector in $\tilde{W}_p(\tilde{M})$ and $\tilde{W}^*_p(\tilde{M})$ or for the tangent vector and cotangent vector in $\tilde{T}_p(\tilde{M})$ and $\tilde{T}^*_p(\tilde{M})$, the basic conclusion of Proposition 6.2.2.1 holds for the same, so these two expression ways are consistent with each other. In this sense, the above formula can alway be concisely denoted by

$$\begin{cases} d\tilde{\xi}^\alpha = \tilde{b}^\alpha_\mu d\tilde{x}^\mu \simeq \tilde{b}^\alpha_\tau d\tilde{x}^\tau \\ \tilde{c}^\mu_\tau \dfrac{\partial}{\partial\tilde{x}^\mu} \cong \tilde{c}^\tau_\tau \dfrac{d}{d\tilde{x}^\tau} = \dfrac{d}{d\tilde{\xi}^\tau} \end{cases}, \quad \begin{cases} d\tilde{x}^\mu = \tilde{c}^\mu_\alpha d\tilde{\xi}^\alpha \simeq \tilde{c}^\mu_\tau d\tilde{\xi}^\tau \\ \tilde{b}^\alpha_\tau \dfrac{\partial}{\partial\tilde{\xi}^\alpha} \cong \tilde{b}^\tau_\tau \dfrac{d}{d\tilde{\xi}^\tau} = \dfrac{d}{d\tilde{x}^\tau} \end{cases}.$$

The above discussions can be applied to the entire manifold, therefore on $\tilde{M}$ it is obtained that

$$\begin{cases} d\tilde{\xi}^\alpha = \tilde{B}^\alpha_\mu d\tilde{x}^\mu \simeq \tilde{B}^\alpha_\tau d\tilde{x}^\tau \\ \tilde{C}^\mu_\tau \dfrac{\partial}{\partial \tilde{x}^\mu} \cong \tilde{C}^\tau_\tau \dfrac{d}{d\tilde{x}^\tau} = \dfrac{d}{d\tilde{\xi}^\tau} \end{cases}, \quad \begin{cases} d\tilde{x}^\mu = \tilde{C}^\mu_\alpha d\tilde{\xi}^\alpha \simeq \tilde{C}^\mu_\tau d\tilde{\xi}^\tau \\ \tilde{B}^\alpha_\tau \dfrac{\partial}{\partial \tilde{\xi}^\alpha} \cong \tilde{B}^\tau_\tau \dfrac{d}{d\tilde{\xi}^\tau} = \dfrac{d}{d\tilde{x}^\tau} \end{cases}. \tag{43}$$

Definition 6.2.2.5. The fomula (43) is called the **Minkowski basis vector form** determined by the regular basis vector form

$$\begin{cases} d\xi^S = B^S_I dx^I \simeq B^S_0 dx^0 \\ C^I_0 \dfrac{\partial}{\partial x^I} \cong C^0_0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \end{cases}, \quad \begin{cases} dx^I = C^I_S d\xi^S \simeq C^I_0 d\xi^0 \\ B^S_0 \dfrac{\partial}{\partial \xi^S} \cong B^0_0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0} \end{cases}$$

of evolution of classical spacetime reference-system on $\tilde{M}$.

Similar to Definition 2.4.3.3 , for the Minkowski form, define

$$d\tilde{\xi}_\tau \triangleq \frac{d\tilde{x}^\tau}{d\tilde{\xi}^\tau} d\tilde{x}^\tau, \qquad d\tilde{x}_\tau \triangleq \frac{d\tilde{\xi}^\tau}{d\tilde{x}^\tau} d\tilde{\xi}^\tau,$$

and define $\frac{d}{d\tilde{\xi}_\tau}$ and $\frac{d}{d\tilde{x}_\tau}$ such that

$$\left\langle \frac{d}{d\tilde{\xi}_\tau}, d\tilde{\xi}_\tau \right\rangle = 1, \qquad \left\langle \frac{d}{d\tilde{x}_\tau}, d\tilde{x}_\tau \right\rangle = 1.$$

Denote

$$\tilde{\tilde{b}}^\tau_\alpha \triangleq \frac{d\tilde{\xi}_\alpha}{d\tilde{x}_\tau}, \quad \tilde{\tilde{b}}^\tau_\tau \triangleq \frac{d\tilde{\xi}_\tau}{d\tilde{x}_\tau}, \quad \tilde{\tilde{c}}^\tau_\mu \triangleq \frac{d\tilde{x}_\mu}{d\tilde{\xi}_\tau}, \quad \tilde{\tilde{c}}^\tau_\tau \triangleq \frac{d\tilde{x}_\tau}{d\tilde{\xi}_\tau},$$

which determine smooth functions $\tilde{\tilde{B}}^\tau_\alpha$, $\tilde{\tilde{B}}^\tau_\tau$, $\tilde{\tilde{C}}^\tau_\mu$, $\tilde{\tilde{C}}^\tau_\tau$ on $\tilde{M}$. Moreover, denote

$$\tilde{\tilde{\varepsilon}}^\mu_\nu \triangleq \tilde{\tilde{B}}^\mu_\alpha \tilde{\tilde{C}}^\alpha_\nu = \varepsilon^\mu_\nu, \quad \tilde{\tilde{\delta}}^\alpha_\beta \triangleq \tilde{\tilde{C}}^\alpha_\mu \tilde{\tilde{B}}^\mu_\beta = \delta^\alpha_\beta, \quad \tilde{\tilde{\varepsilon}}^\tau_\mu \triangleq \tilde{\tilde{B}}^\tau_\tau \tilde{\tilde{C}}^\tau_\mu = \tilde{\tilde{B}}^\tau_\alpha \tilde{\tilde{C}}^\alpha_\mu, \quad \tilde{\tilde{\delta}}^\tau_\alpha \triangleq \tilde{\tilde{C}}^\tau_\tau \tilde{\tilde{B}}^\tau_\alpha = \tilde{\tilde{C}}^\tau_\mu \tilde{\tilde{B}}^\mu_\alpha.$$

Thus, the Minkowski basis vector form can also be expressed as

$$\begin{cases} d\tilde{\xi}_\alpha = \tilde{\tilde{B}}^\mu_\alpha d\tilde{x}_\mu \simeq \tilde{\tilde{B}}^\tau_\alpha d\tilde{x}_\tau \\ \tilde{\tilde{C}}^0_\mu \dfrac{\partial}{\partial \tilde{x}_\mu} \cong \tilde{\tilde{C}}^\tau_\tau \dfrac{d}{d\tilde{x}_\tau} = \dfrac{d}{d\tilde{\xi}_\tau} \end{cases}, \quad \begin{cases} d\tilde{x}_\mu = \tilde{\tilde{C}}^\alpha_\mu d\tilde{\xi}_\alpha \simeq \tilde{\tilde{C}}^\tau_\mu d\tilde{\xi}_\tau \\ \tilde{\tilde{B}}^\tau_\alpha \dfrac{\partial}{\partial \tilde{\xi}_\alpha} \cong \tilde{\tilde{B}}^\tau_\tau \dfrac{d}{d\tilde{\xi}_\tau} = \dfrac{d}{d\tilde{x}_\tau} \end{cases}.$$

Proposition 6.2.2.3. The inverse transformations of Proposition 6.2.2.2 are

$$\begin{cases} B^s_0 = \tilde{B}^s_0 \\ B^\tau_0 = \dfrac{\tilde{B}^0_0}{\tilde{\delta}^0_\tau} \\ C^0_0 = \dfrac{\tilde{C}^0_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} B^s_i = -\tilde{B}^s_i \\ B^\tau_i = -\dfrac{\tilde{B}^0_i}{\tilde{\delta}^0_\tau} \\ C^i_0 = -\dfrac{\tilde{C}^i_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} B^s_\tau = \tilde{B}^s_\tau \\ B^\tau_\tau = \dfrac{\tilde{B}^0_\tau}{\tilde{\delta}^0_\tau} \\ C^\tau_0 = \dfrac{\tilde{C}^\tau_\tau}{\tilde{\delta}^0_\tau} \end{cases}; \quad \begin{cases} C^i_0 = \tilde{C}^i_0 \\ C^\tau_0 = \dfrac{\tilde{C}^0_0}{\tilde{\varepsilon}^0_\tau} \\ B^0_0 = \dfrac{\tilde{B}^0_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} C^i_s = -\tilde{C}^i_s \\ C^\tau_s = -\dfrac{\tilde{C}^0_s}{\tilde{\varepsilon}^0_\tau} \\ B^s_0 = \dfrac{\tilde{B}^s_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} C^i_\tau = \tilde{C}^i_\tau \\ C^\tau_\tau = \dfrac{\tilde{C}^0_\tau}{\tilde{\varepsilon}^0_\tau} \\ B^\tau_0 = \dfrac{\tilde{B}^\tau_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}.$$

Proof. First, consider the local case. On a neighborhood of a point on $\tilde{M}$:

$$\begin{cases} d\xi^s = b^s_i dx^i + b^s_\tau dx^\tau \simeq b^s_0 dx^0 \\ d\xi^\tau = b^\tau_i dx^i + b^\tau_\tau dx^\tau \simeq b^\tau_0 dx^0 \\ c^i_0 \dfrac{\partial}{\partial x^i} + c^\tau_0 \dfrac{\partial}{\partial x^\tau} \cong c^0_0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \end{cases} \Rightarrow \begin{cases} b^s_0 dx^0 - b^s_i dx^i \simeq b^s_\tau dx^\tau \\ b^\tau_0 dx^0 - b^\tau_i dx^i \simeq b^\tau_\tau dx^\tau \\ c^0_0 \dfrac{d}{dx^0} - c^i_0 \dfrac{\partial}{\partial x^i} \cong c^\tau_0 \dfrac{\partial}{\partial x^\tau} \end{cases}, \tag{44}$$

$$\begin{cases} dx^i = c^i_s d\xi^s + c^i_\tau d\xi^\tau \simeq c^i_0 d\xi^0 \\ dx^\tau = c^\tau_s d\xi^s + c^\tau_\tau d\xi^\tau \simeq c^\tau_0 d\xi^0 \\ b^s_0 \frac{\partial}{\partial \xi^s} + b^\tau_0 \frac{\partial}{\partial \xi^\tau} \cong b^0_0 \frac{d}{d\xi^0} = \frac{d}{dx^0} \end{cases} \Rightarrow \begin{cases} c^i_0 d\xi^0 - c^i_s d\xi^s \simeq c^i_\tau d\xi^\tau \\ c^\tau_0 d\xi^0 - c^\tau_s d\xi^s \simeq c^\tau_\tau d\xi^\tau \\ b^0_0 \frac{d}{d\xi^0} - b^s_0 \frac{\partial}{\partial \xi^s} \cong b^\tau_0 \frac{\partial}{\partial \xi^\tau} \end{cases}, \tag{45}$$

$$\begin{cases} d\tilde{\xi}^s = \tilde{b}^s_0 d\tilde{x}^0 + \tilde{b}^s_i d\tilde{x}^i \simeq \tilde{b}^s_\tau d\tilde{x}^\tau \\ d\tilde{\xi}^0 = \tilde{b}^0_0 d\tilde{x}^0 + \tilde{b}^0_i d\tilde{x}^i \simeq \tilde{b}^0_\tau d\tilde{x}^\tau \\ \tilde{c}^0_\tau \frac{\partial}{\partial \tilde{x}^0} + \tilde{c}^i_\tau \frac{\partial}{\partial \tilde{x}^i} \cong \tilde{c}^\tau_\tau \frac{d}{d\tilde{x}^\tau} = \frac{d}{d\tilde{\xi}^\tau} \end{cases} \Rightarrow \begin{cases} d\tilde{\xi}^s = \tilde{b}^s_0 d\tilde{x}^0 + \tilde{b}^s_i d\tilde{x}^i \simeq \tilde{b}^s_\tau d\tilde{x}^\tau \\ \frac{1}{\tilde{\delta}^0_\tau} d\tilde{\xi}^0 = \frac{\tilde{b}^0_0}{\tilde{\delta}^0_\tau} d\tilde{x}^0 + \frac{\tilde{b}^0_i}{\tilde{\delta}^0_\tau} d\tilde{x}^i \simeq \frac{\tilde{b}^0_\tau}{\tilde{\delta}^0_\tau} d\tilde{x}^\tau \\ \frac{\tilde{c}^0_\tau}{\tilde{\delta}^0_\tau} \frac{\partial}{\partial \tilde{x}^0} + \frac{\tilde{c}^i_\tau}{\tilde{\delta}^0_\tau} \frac{\partial}{\partial \tilde{x}^i} \cong \frac{\tilde{c}^\tau_\tau}{\tilde{\delta}^0_\tau} \frac{d}{d\tilde{x}^\tau} = \frac{1}{\tilde{\delta}^0_\tau} \frac{d}{d\tilde{\xi}^\tau} \end{cases}, \tag{46}$$

$$\begin{cases} d\tilde{x}^i = \tilde{c}^i_0 d\tilde{\xi}^0 + \tilde{c}^i_s d\tilde{\xi}^s \simeq \tilde{c}^i_\tau d\tilde{\xi}^\tau \\ d\tilde{x}^0 = \tilde{c}^0_0 d\tilde{\xi}^0 + \tilde{c}^0_s d\tilde{\xi}^s \simeq \tilde{c}^0_\tau d\tilde{\xi}^\tau \\ \tilde{b}^0_\tau \frac{\partial}{\partial \tilde{\xi}^0} + \tilde{b}^s_\tau \frac{\partial}{\partial \tilde{\xi}^s} \cong \tilde{b}^\tau_\tau \frac{d}{d\tilde{\xi}^\tau} = \frac{d}{d\tilde{x}^\tau} \end{cases} \Rightarrow \begin{cases} d\tilde{x}^i = \tilde{c}^i_0 d\tilde{\xi}^0 + \tilde{c}^i_s d\tilde{\xi}^s \simeq \tilde{c}^i_\tau d\tilde{\xi}^\tau \\ \frac{1}{\tilde{\varepsilon}^0_\tau} d\tilde{x}^0 = \frac{\tilde{c}^0_0}{\tilde{\varepsilon}^0_\tau} d\tilde{\xi}^0 + \frac{\tilde{c}^0_s}{\tilde{\varepsilon}^0_\tau} d\tilde{\xi}^s \simeq \frac{\tilde{c}^0_\tau}{\tilde{\varepsilon}^0_\tau} d\tilde{\xi}^\tau \\ \frac{\tilde{b}^0_\tau}{\tilde{\varepsilon}^0_\tau} \frac{\partial}{\partial \tilde{\xi}^0} + \frac{\tilde{b}^s_\tau}{\tilde{\varepsilon}^0_\tau} \frac{\partial}{\partial \tilde{\xi}^s} \cong \frac{\tilde{b}^\tau_\tau}{\tilde{\varepsilon}^0_\tau} \frac{d}{d\tilde{\xi}^\tau} = \frac{1}{\tilde{\varepsilon}^0_\tau} \frac{d}{d\tilde{x}^\tau} \end{cases}. \tag{47}$$

Compare (44) and (46), it is obtained that

$$\begin{cases} b^s_0 = \tilde{b}^s_0 \\ b^\tau_0 = \frac{\tilde{b}^0_0}{\tilde{\delta}^0_\tau} \\ c^0_0 = \frac{\tilde{c}^0_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} b^s_i = -\tilde{b}^s_i \\ b^\tau_i = -\frac{\tilde{b}^0_i}{\tilde{\delta}^0_\tau} \\ c^i_0 = -\frac{\tilde{c}^i_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} b^s_\tau = \tilde{b}^s_\tau \\ b^\tau_\tau = \frac{\tilde{b}^0_\tau}{\tilde{\delta}^0_\tau} \\ c^\tau_0 = \frac{\tilde{c}^\tau_\tau}{\tilde{\delta}^0_\tau} \end{cases}.$$

Compare (45) and (47), it is obtained that

$$\begin{cases} c^i_0 = \tilde{c}^i_0 \\ c^\tau_0 = \frac{\tilde{c}^0_0}{\tilde{\varepsilon}^0_\tau} \\ b^0_0 = \frac{\tilde{b}^0_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} c^i_s = -\tilde{c}^i_s \\ c^\tau_s = -\frac{\tilde{c}^0_s}{\tilde{\varepsilon}^0_\tau} \\ b^s_0 = \frac{\tilde{b}^s_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} c^i_\tau = \tilde{c}^i_\tau \\ c^\tau_\tau = \frac{\tilde{c}^0_\tau}{\tilde{\varepsilon}^0_\tau} \\ b^\tau_0 = \frac{\tilde{b}^\tau_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}.$$

The above conclusions are proved on a neighborhood of an arbitrary point on $\tilde{M}$, so there are the following conclusions on the entire manifold.

$$\begin{cases} B^s_0 = \tilde{B}^s_0 \\ B^\tau_0 = \frac{\tilde{B}^0_0}{\tilde{\delta}^0_\tau} \\ C^0_0 = \frac{\tilde{C}^0_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} B^s_i = -\tilde{B}^s_i \\ B^\tau_i = -\frac{\tilde{B}^0_i}{\tilde{\delta}^0_\tau} \\ C^i_0 = -\frac{\tilde{C}^i_\tau}{\tilde{\delta}^0_\tau} \end{cases}, \quad \begin{cases} B^s_\tau = \tilde{B}^s_\tau \\ B^\tau_\tau = \frac{\tilde{B}^0_\tau}{\tilde{\delta}^0_\tau} \\ C^\tau_0 = \frac{\tilde{C}^\tau_\tau}{\tilde{\delta}^0_\tau} \end{cases}; \quad \begin{cases} C^i_0 = \tilde{C}^i_0 \\ C^\tau_0 = \frac{\tilde{C}^0_0}{\tilde{\varepsilon}^0_\tau} \\ B^0_0 = \frac{\tilde{B}^0_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} C^i_s = -\tilde{C}^i_s \\ C^\tau_s = -\frac{\tilde{C}^0_s}{\tilde{\varepsilon}^0_\tau} \\ B^s_0 = \frac{\tilde{B}^s_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}, \quad \begin{cases} C^i_\tau = \tilde{C}^i_\tau \\ C^\tau_\tau = \frac{\tilde{C}^0_\tau}{\tilde{\varepsilon}^0_\tau} \\ B^\tau_0 = \frac{\tilde{B}^\tau_\tau}{\tilde{\varepsilon}^0_\tau} \end{cases}.$$

Proposition 6.2.2.4. (The Minkowski form of the evolution lemma). For any Minkowski tangent vector field $w^\mu \frac{\partial}{\partial \tilde{x}^\mu}$, $\bar{w}_\mu \frac{\partial}{\partial \tilde{x}_\mu}$ and any Minkowski cotangent vector field $w^\tau \frac{d}{d\tilde{x}^\tau}$, $\bar{w}_\tau \frac{d}{d\tilde{x}_\tau}$, the following conclusions hold.

$$\begin{cases} w^\mu \frac{\partial}{\partial \tilde{x}^\mu} \cong w^\tau \frac{d}{d\tilde{x}^\tau} \Leftrightarrow w^\mu = w^\tau \tilde{\varepsilon}^\mu_\tau \\ w_\mu d\tilde{x}^\mu \simeq w_\tau d\tilde{x}^\tau \Leftrightarrow \tilde{\varepsilon}^\mu_\tau w_\mu = w_\tau \end{cases}, \quad \begin{cases} \bar{w}_\mu \frac{\partial}{\partial \tilde{x}_\mu} \cong \bar{w}_\tau \frac{d}{d\tilde{x}_\tau} \Leftrightarrow \bar{w}_\mu = \bar{w}_\tau \tilde{\bar{\varepsilon}}^\tau_\mu \\ \bar{w}^\mu d\tilde{x}_\mu \simeq \bar{w}^\tau d\tilde{x}_\tau \Leftrightarrow \tilde{\bar{\varepsilon}}^\tau_\mu \bar{w}^\mu = \bar{w}^\tau \end{cases}.$$

Proof. Similar to the proof of Proposition 2.4.3.1 , the following local discussions can be applied on the entire manifold.

1. Consider the case that basis vectors are $d\tilde{x}^\mu$ and $\frac{\partial}{\partial\tilde{x}^\mu}$.

For Minkowski tangent vector,

$$\tilde{\pi}_*\left(\frac{d}{d\tilde{x}^\tau}\right)=\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{\partial}{\partial\tilde{x}^\mu}\Leftrightarrow\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{\partial}{\partial\tilde{x}^\mu}\cong\frac{d}{d\tilde{x}^\tau}\Leftrightarrow\tilde{\varepsilon}^\mu_\tau\frac{\partial}{\partial\tilde{x}^\mu}\cong\frac{d}{d\tilde{x}^\tau}\Leftrightarrow w^\tau\tilde{\varepsilon}^\mu_\tau\frac{\partial}{\partial\tilde{x}^\mu}\cong w^\tau\frac{d}{d\tilde{x}^\tau}.$$

Because the tangent mapping is an injection, then

$$w^\mu\frac{\partial}{\partial\tilde{x}^\mu}\cong w^\tau\frac{d}{d\tilde{x}^\tau}\Leftrightarrow w^\mu=w^\tau\tilde{\varepsilon}^\mu_\tau.$$

For Minkowski cotangent vector, $d\tilde{x}^\mu\simeq\tilde{\varepsilon}^\mu_\tau d\tilde{x}^\tau\Rightarrow w_\mu d\tilde{x}^\mu\simeq\tilde{\varepsilon}^\mu_\tau w_\mu d\tilde{x}^\tau$, then $w_\mu d\tilde{x}^\mu\simeq w_\tau d\tilde{x}^\tau\Leftrightarrow\tilde{\varepsilon}^\mu_\tau w_\mu=w_\tau$.

2. Consider the case that basis vectors are $d\tilde{x}_\mu$ and $\frac{\partial}{\partial\tilde{x}_\mu}$.

For Minkowski tangent vector,

$$\tilde{\pi}_*\left(\frac{d}{d\tilde{x}_\tau}\right)=\frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}\frac{\partial}{\partial\tilde{x}_\mu}\Leftrightarrow\frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}\frac{\partial}{\partial\tilde{x}_\mu}\cong\frac{d}{d\tilde{x}_\tau}\Leftrightarrow\bar{\tilde{\varepsilon}}^\tau_\mu\frac{\partial}{\partial\tilde{x}_\mu}\cong\frac{d}{d\tilde{x}_\tau}\Leftrightarrow\bar{w}_\tau\bar{\tilde{\varepsilon}}^\tau_\mu\frac{\partial}{\partial\tilde{x}_\mu}\cong\bar{w}_\tau\frac{d}{d\tilde{x}_\tau}.$$

Because the tangent mapping is an injection, then

$$\bar{w}_\mu\frac{\partial}{\partial\tilde{x}_\mu}\cong\bar{w}_\tau\frac{d}{d\tilde{x}_\tau}\Leftrightarrow\bar{w}_\mu=\bar{w}_\tau\bar{\tilde{\varepsilon}}^\tau_\mu.$$

For Minkowski cotangent vector, $d\tilde{x}_\mu\simeq\bar{\tilde{\varepsilon}}^\tau_\mu d\tilde{x}_\tau\Rightarrow\bar{w}^\mu d\tilde{x}_\mu\simeq\bar{\tilde{\varepsilon}}^\tau_\mu\bar{w}^\mu d\tilde{x}_\tau$, then $\bar{w}^\mu d\tilde{x}_\mu\simeq\bar{w}^\tau d\tilde{x}_\tau\Leftrightarrow\bar{\tilde{\varepsilon}}^\tau_\mu\bar{w}^\mu=\bar{w}^\tau$. □

6.2.3 Other Minkowski forms

Some concepts related to Minkowski metric representation of classical spacetime reference-system can be tranplanted from the general concepts in section 2.2.7.2 and section 2.4.3.5 .

Definition 6.2.3.1. Denote

$$\tilde{\varepsilon}_{\mu\nu}=\tilde{\varepsilon}^{\mu\nu}\triangleq\begin{cases}1\quad\mu=\nu=0\\-1,\mu=\nu\neq0\\0,\quad\mu\neq\nu\end{cases},\quad\tilde{\delta}_{\alpha\beta}=\tilde{\delta}^{\alpha\beta}\triangleq\begin{cases}1\quad\alpha=\beta=0\\-1,\alpha=\beta\neq0\\0,\quad\alpha\neq\beta\end{cases}.$$

Discussion 6.2.3.1. According to Definition 6.1.3.1 , on a neighborhood of any point p on $\tilde{M}$,

$$\begin{cases}(d\xi^0)^2=\sum_{s=1}^r(d\xi^s)^2+(d\xi^\tau)^2\\(dx^0)^2=\sum_{i=1}^r(dx^i)^2+(dx^\tau)^2\end{cases}\Rightarrow\begin{cases}(d\xi^\tau)^2=(d\xi^0)^2-\sum_{s=1}^r(d\xi^s)^2\\(dx^\tau)^2=(dx^0)^2-\sum_{i=1}^r(dx^i)^2\end{cases}\Rightarrow\begin{cases}(d\tilde{\xi}^\tau)^2=(d\tilde{\xi}^0)^2-\sum_{s=1}^r(d\tilde{\xi}^s)^2\\(d\tilde{x}^\tau)^2=(d\tilde{x}^0)^2-\sum_{i=1}^r(d\tilde{x}^i)^2\end{cases}$$

$$\Rightarrow\begin{cases}(d\tilde{\xi}^\tau)^2=\tilde{\delta}_{\alpha\beta}d\tilde{\xi}^\alpha d\tilde{\xi}^\beta\\(d\tilde{x}^\tau)^2=\tilde{\varepsilon}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu\end{cases}.$$

Substitute the local Minkowski basis vector form of Discussion 6.2.2.1 to the above equations,

$$\begin{cases}(d\tilde{\xi}^{\tau})^2=\tilde{\delta}_{\alpha\beta}d\tilde{\xi}^{\alpha}d\tilde{\xi}^{\beta}=\tilde{\delta}_{\alpha\beta}\tilde{b}^{\alpha}_{\mu}\tilde{b}^{\beta}_{\nu}d\tilde{x}^{\mu}d\tilde{x}^{\nu}\\(d\tilde{x}^{\tau})^2=\tilde{\varepsilon}_{\mu\nu}d\tilde{x}^{\mu}d\tilde{x}^{\nu}=\tilde{\varepsilon}_{\mu\nu}\tilde{c}^{\mu}_{\alpha}\tilde{c}^{\nu}_{\beta}d\tilde{\xi}^{\alpha}d\tilde{\xi}^{\beta}\end{cases}.$$

Denote

$$\begin{cases}\tilde{g}_{\mu\nu}\triangleq\tilde{\delta}_{\alpha\beta}\tilde{b}^{\alpha}_{\mu}\tilde{b}^{\beta}_{\nu}\\\tilde{h}_{\alpha\beta}\triangleq\tilde{\varepsilon}_{\mu\nu}\tilde{c}^{\mu}_{\alpha}\tilde{c}^{\nu}_{\beta}\end{cases},$$

then

$$\begin{cases}(d\tilde{\xi}^{\tau})^2=\tilde{\delta}_{\alpha\beta}d\tilde{\xi}^{\alpha}d\tilde{\xi}^{\beta}=\tilde{g}_{\mu\nu}d\tilde{x}^{\mu}d\tilde{x}^{\nu}\\(d\tilde{x}^{\tau})^2=\tilde{\varepsilon}_{\mu\nu}d\tilde{x}^{\mu}d\tilde{x}^{\nu}=\tilde{h}_{\alpha\beta}d\tilde{\xi}^{\alpha}d\tilde{\xi}^{\beta}\end{cases}.$$

For the same reasons as Discussion 2.2.8.4 , here we do not define other metric tensors like Definition 2.2.8.6 . It is enough to just consider metric tensors $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$.

Definition 6.2.3.2. Define the Minkowski metric tensors

$$\begin{cases}\tilde{\mathbf{g}}\triangleq\tilde{\delta}_{\alpha\beta}d\tilde{\xi}^{\alpha}\otimes d\tilde{\xi}^{\beta}=\tilde{g}_{\mu\nu}d\tilde{x}^{\mu}\otimes d\tilde{x}^{\nu}=\tilde{g}^{\mu\nu}d\tilde{x}_{\mu}\otimes d\tilde{x}_{\nu}\\\tilde{\mathbf{h}}\triangleq\tilde{\varepsilon}_{\mu\nu}d\tilde{x}^{\mu}\otimes d\tilde{x}^{\nu}=\tilde{h}_{\alpha\beta}d\tilde{\xi}^{\alpha}\otimes d\tilde{\xi}^{\beta}=\tilde{h}^{\alpha\beta}d\tilde{\xi}_{\alpha}\otimes d\tilde{\xi}_{\beta}\end{cases}$$

of classical spacetime reference-system on a neighborhood of p, where

$$\begin{cases}\tilde{g}_{\mu\nu}=\tilde{\delta}_{\alpha\beta}\tilde{b}^{\alpha}_{\mu}\tilde{b}^{\beta}_{\nu}=\tilde{\delta}^{\alpha\beta}\tilde{c}_{\alpha\mu}\tilde{c}_{\beta\nu}\\\tilde{g}^{\mu\nu}=\tilde{\delta}_{\alpha\beta}\tilde{b}^{\alpha\mu}\tilde{b}^{\beta\nu}=\tilde{\delta}^{\alpha\beta}\tilde{c}^{\mu}_{\alpha}\tilde{c}^{\nu}_{\beta}\end{cases},\quad\begin{cases}\tilde{h}_{\alpha\beta}=\tilde{\varepsilon}_{\mu\nu}\tilde{c}^{\mu}_{\alpha}\tilde{c}^{\nu}_{\beta}=\tilde{\varepsilon}^{\mu\nu}\tilde{b}_{\mu\alpha}\tilde{b}_{\nu\beta}\\\tilde{h}^{\alpha\beta}=\tilde{\varepsilon}_{\mu\nu}\tilde{c}^{\mu\alpha}\tilde{c}^{\nu\beta}=\tilde{\varepsilon}^{\mu\nu}\tilde{b}^{\alpha}_{\mu}\tilde{b}^{\beta}_{\nu}\end{cases}.$$

Similar to Definition 2.2.8.4 , define the Minkowski metric tensors

$$\begin{cases}\tilde{\mathbf{G}}\triangleq\tilde{\Delta}_{\alpha\beta}d\tilde{\xi}^{\alpha}\otimes d\tilde{\xi}^{\beta}=\tilde{G}_{\mu\nu}d\tilde{x}^{\mu}\otimes d\tilde{x}^{\nu}\\\tilde{\mathbf{H}}\triangleq\tilde{\mathrm{E}}_{\mu\nu}d\tilde{x}^{\mu}\otimes d\tilde{x}^{\nu}=\tilde{H}_{\alpha\beta}d\tilde{\xi}^{\alpha}\otimes d\tilde{\xi}^{\beta}\end{cases}$$

on $\tilde{M}$, where

$$\begin{cases}\tilde{G}_{\mu\nu}=\tilde{\Delta}_{\alpha\beta}\tilde{B}^{\alpha}_{\mu}\tilde{B}^{\beta}_{\nu}=\tilde{\Delta}^{\alpha\beta}\tilde{C}_{\alpha\mu}\tilde{C}_{\beta\nu}\\\tilde{G}^{\mu\nu}=\tilde{\Delta}_{\alpha\beta}\tilde{B}^{\alpha\mu}\tilde{B}^{\beta\nu}=\tilde{\Delta}^{\alpha\beta}\tilde{C}^{\mu}_{\alpha}\tilde{C}^{\nu}_{\beta}\end{cases},\quad\begin{cases}\tilde{H}_{\alpha\beta}=\tilde{\mathrm{E}}_{\mu\nu}\tilde{C}^{\mu}_{\alpha}\tilde{C}^{\nu}_{\beta}=\tilde{\mathrm{E}}^{\mu\nu}\tilde{B}_{\mu\alpha}\tilde{B}_{\nu\beta}\\\tilde{H}^{\alpha\beta}=\tilde{\mathrm{E}}_{\mu\nu}\tilde{C}^{\mu\alpha}\tilde{C}^{\nu\beta}=\tilde{\mathrm{E}}^{\mu\nu}\tilde{B}^{\alpha}_{\mu}\tilde{B}^{\beta}_{\nu}\end{cases}.$$

Definition 6.2.3.3. Denote $\tilde{G}_{\tau\tau}\triangleq\tilde{B}^{\tau}_{\tau}\tilde{B}^{\tau}_{\tau}$, $\tilde{G}^{\tau\tau}\triangleq\tilde{C}^{\tau}_{\tau}\tilde{C}^{\tau}_{\tau}$, $\tilde{H}_{\tau\tau}\triangleq\tilde{C}^{\tau}_{\tau}\tilde{C}^{\tau}_{\tau}$, $\tilde{H}^{\tau\tau}\triangleq\tilde{B}^{\tau}_{\tau}\tilde{B}^{\tau}_{\tau}$.

Proposition 6.2.3.1. On the evolution path L,

$$\frac{d}{d\tilde{x}_{\tau}}=\tilde{G}^{\tau\tau}\frac{d}{d\tilde{x}^{\tau}},\quad\frac{d}{d\tilde{\xi}_{\tau}}=\tilde{H}^{\tau\tau}\frac{d}{d\tilde{\xi}^{\tau}}.$$

Proof. It is similar to Proposition 2.4.4.1 .

Expand tangent vector $\frac{d}{d\tilde{x}_{\tau}}$ as $\frac{d}{d\tilde{x}_{\tau}}=X\frac{d}{d\tilde{x}^{\tau}}$ about basis $\frac{d}{d\tilde{x}^{\tau}}$, and expand tangent vector $\frac{d}{d\tilde{\xi}_{\tau}}$ as $\frac{d}{d\tilde{\xi}_{\tau}}=Y\frac{d}{d\tilde{\xi}^{\tau}}$ about basis $\frac{d}{d\tilde{\xi}_{\tau}}$.

$$\left\langle\frac{d}{d\tilde{x}_{\tau}},d\tilde{x}_{\tau}\right\rangle=1\Leftrightarrow\left\langle X\frac{d}{d\tilde{x}^{\tau}},\tilde{G}_{\tau\tau}d\tilde{x}^{\tau}\right\rangle=1\Leftrightarrow X\tilde{G}_{\tau\tau}=1\Leftrightarrow X=\frac{1}{\tilde{G}_{\tau\tau}}=\tilde{G}^{\tau\tau}\Rightarrow\frac{d}{d\tilde{x}_{\tau}}=\tilde{G}^{\tau\tau}\frac{d}{d\tilde{x}^{\tau}}.$$
$$\left\langle\frac{d}{d\tilde{\xi}_{\tau}},d\tilde{\xi}_{\tau}\right\rangle=1\Leftrightarrow\left\langle Y\frac{d}{d\tilde{\xi}^{\tau}},\tilde{H}_{\tau\tau}d\tilde{\xi}^{\tau}\right\rangle=1\Leftrightarrow Y\tilde{H}_{\tau\tau}=1\Leftrightarrow Y=\frac{1}{\tilde{H}_{\tau\tau}}=\tilde{H}^{\tau\tau}\Rightarrow\frac{d}{d\tilde{\xi}_{\tau}}=\tilde{H}^{\tau\tau}\frac{d}{d\tilde{\xi}^{\tau}}.\quad\square$$

Proposition 6.2.3.2. The following conclusions hold.

$$\tilde{G}_{\tau\tau}=\frac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}G_{00},\quad \tilde{H}_{\tau\tau}=\frac{\varepsilon_0^\tau\varepsilon_0^\tau}{\delta_0^\tau\delta_0^\tau}H_{00};\quad G_{00}=\frac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\tilde{G}_{\tau\tau},\quad H_{00}=\frac{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}\tilde{H}_{\tau\tau}.$$

Proof. The following local discussions can be applied to the entire evolution path.

$$\tilde{g}_{\tau\tau}=\frac{d\tilde{\xi}^\tau d\tilde{\xi}^\tau}{d\tilde{x}^\tau d\tilde{x}^\tau}=\frac{d\xi^\tau d\xi^\tau}{dx^\tau dx^\tau}=\frac{\delta_0^\tau\delta_0^\tau d\xi^0 d\xi^0}{\varepsilon_0^\tau\varepsilon_0^\tau dx^0 dx^0}=\frac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}g_{00}\Rightarrow g_{00}=\frac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\tilde{g}_{\tau\tau},$$
$$\tilde{h}_{\tau\tau}=\frac{d\tilde{x}^\tau d\tilde{x}^\tau}{d\tilde{\xi}^\tau d\tilde{\xi}^\tau}=\frac{dx^\tau dx^\tau}{d\xi^\tau d\xi^\tau}=\frac{\varepsilon_0^\tau\varepsilon_0^\tau dx^0 dx^0}{\delta_0^\tau\delta_0^\tau d\xi^0 d\xi^0}=\frac{\varepsilon_0^\tau\varepsilon_0^\tau}{\delta_0^\tau\delta_0^\tau}h_{00}\Rightarrow h_{00}=\frac{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}\tilde{h}_{\tau\tau}.\quad\square$$

Proposition 6.2.3.3. The transformation relationships between regular metric G_{IJ} and Minkowski metric $\tilde{G}_{\mu\nu}$ are

$$\begin{cases}G_{\tau\tau}=\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}G_{00}=\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{\tau\tau}\\ G_{i\tau}=-\dfrac{\tilde{G}_{i0}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{i0}\\ G_{\tau j}=-\dfrac{\tilde{G}_{0j}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{0j}\\ G_{ij}=-\dfrac{\tilde{G}_{ij}}{\tilde{G}_{00}}G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{ij}\end{cases},\quad \begin{cases}\tilde{G}_{00}=\dfrac{G_{00}}{G_{\tau\tau}}\tilde{G}_{\tau\tau}=\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{00}\\ \tilde{G}_{i0}=-\dfrac{G_{i\tau}}{G_{\tau\tau}}\varepsilon_0^\tau\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{i\tau}\\ \tilde{G}_{0j}=-\dfrac{G_{\tau j}}{G_{\tau\tau}}\varepsilon_0^\tau\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{\tau j}\\ \tilde{G}_{ij}=-\dfrac{G_{ij}}{G_{\tau\tau}}\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{ij}\end{cases}.$$

Proof. The regular metric and the Minkowski metric are

$$\begin{aligned}G_{00}(dx^0)^2=(d\xi^0)^2\triangleq G_{IJ}dx^Idx^J&=G_{\tau\tau}(dx^\tau)^2+G_{i\tau}dx^idx^\tau+G_{\tau j}dx^\tau dx^j+G_{ij}dx^idx^j\\&=G_{\tau\tau}(dx^\tau)^2+G_{i\tau}\varepsilon_0^\tau dx^idx^0+G_{\tau j}\varepsilon_0^\tau dx^0dx^j+G_{ij}dx^idx^j,\\ \tilde{G}_{\tau\tau}(d\tilde{x}^\tau)^2=(d\tilde{\xi}^\tau)^2\triangleq\tilde{G}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu&=\tilde{G}_{00}(d\tilde{x}^0)^2+\tilde{G}_{i0}d\tilde{x}^id\tilde{x}^0+\tilde{G}_{0j}d\tilde{x}^0d\tilde{x}^j+\tilde{G}_{ij}d\tilde{x}^id\tilde{x}^j\\&=\tilde{G}_{00}(d\tilde{x}^0)^2+\tilde{G}_{i0}\tilde{\varepsilon}_\tau^0d\tilde{x}^id\tilde{x}^\tau+\tilde{G}_{0j}\tilde{\varepsilon}_\tau^0d\tilde{x}^\tau d\tilde{x}^j+\tilde{G}_{ij}d\tilde{x}^id\tilde{x}^j.\end{aligned}$$

The transformation relationship between G_{IJ} and $\tilde{G}_{\mu\nu}$ can be obtained in the following way.

$$\begin{cases}(dx^0)^2=\dfrac{G_{\tau\tau}}{G_{00}}(dx^\tau)^2+\dfrac{G_{i\tau}}{G_{00}}dx^idx^\tau+\dfrac{G_{\tau j}}{G_{00}}dx^\tau dx^j+\dfrac{G_{ij}}{G_{00}}dx^idx^j\\ (d\tilde{x}^0)^2=\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}(d\tilde{x}^\tau)^2-\dfrac{\tilde{G}_{i0}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0d\tilde{x}^id\tilde{x}^\tau-\dfrac{\tilde{G}_{0j}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0d\tilde{x}^\tau d\tilde{x}^j-\dfrac{\tilde{G}_{ij}}{\tilde{G}_{00}}d\tilde{x}^id\tilde{x}^j\end{cases}\Rightarrow\begin{cases}G_{\tau\tau}=\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}G_{00}=\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{\tau\tau}\\ G_{i\tau}=-\dfrac{\tilde{G}_{i0}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{i0}\\ G_{\tau j}=-\dfrac{\tilde{G}_{0j}}{\tilde{G}_{00}}\tilde{\varepsilon}_\tau^0G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{0j}\\ G_{ij}=-\dfrac{\tilde{G}_{ij}}{\tilde{G}_{00}}G_{00}=-\dfrac{\tilde{\delta}_\tau^0\tilde{\delta}_\tau^0}{\tilde{\varepsilon}_\tau^0\tilde{\varepsilon}_\tau^0}\dfrac{\tilde{G}_{\tau\tau}}{\tilde{G}_{00}}\tilde{G}_{ij}\end{cases},$$

$$\begin{cases}(dx^\tau)^2=\dfrac{G_{00}}{G_{\tau\tau}}(dx^0)^2-\dfrac{G_{i\tau}}{G_{\tau\tau}}\varepsilon_0^\tau dx^idx^0-\dfrac{G_{\tau j}}{G_{\tau\tau}}\varepsilon_0^\tau dx^0dx^j-\dfrac{G_{ij}}{G_{\tau\tau}}dx^idx^j\\ (d\tilde{x}^\tau)^2=\dfrac{\tilde{G}_{00}}{g_{\tau\tau}}(d\tilde{x}^0)^2+\dfrac{\tilde{G}_{i0}}{\tilde{G}_{\tau\tau}}d\tilde{x}^id\tilde{x}^0+\dfrac{\tilde{G}_{0j}}{\tilde{G}_{\tau\tau}}d\tilde{x}^0d\tilde{x}^j+\dfrac{\tilde{G}_{ij}}{\tilde{G}_{\tau\tau}}d\tilde{x}^id\tilde{x}^j\end{cases}\Rightarrow\begin{cases}\tilde{G}_{00}=\dfrac{G_{00}}{G_{\tau\tau}}\tilde{G}_{\tau\tau}=\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{00}\\ \tilde{G}_{i0}=-\dfrac{G_{i\tau}}{G_{\tau\tau}}\varepsilon_0^\tau\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{i\tau}\\ \tilde{G}_{0j}=-\dfrac{G_{\tau j}}{G_{\tau\tau}}\varepsilon_0^\tau\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{\tau j}\\ \tilde{G}_{ij}=-\dfrac{G_{ij}}{G_{\tau\tau}}\tilde{G}_{\tau\tau}=-\dfrac{\delta_0^\tau\delta_0^\tau}{\varepsilon_0^\tau\varepsilon_0^\tau}\dfrac{G_{00}}{G_{\tau\tau}}G_{ij}\end{cases}.$$

□

Definition 6.2.3.4. Similar to Definition 2.2.9.1 , we can define an affine connection $\tilde{D}$ on $\tilde{M}$, called a **Minkowski affine connection**. Take smooth real functions $\tilde{\Gamma}^{\mu}_{\nu\rho}$ on $\tilde{M}$. Using the restriction of them on a neighborhood $\tilde{U}$ of p, the Minkowski affine connection can be expressed as

$$\begin{cases} \tilde{D}\dfrac{\partial}{\partial\tilde{x}^{\nu}} \triangleq \tilde{\Gamma}^{\mu}_{\nu\rho} d\tilde{x}^{\rho} \otimes \dfrac{\partial}{\partial\tilde{x}^{\mu}} \\ \tilde{D}d\tilde{x}^{\nu} \triangleq -\tilde{\Gamma}^{\nu}_{\mu\rho} d\tilde{x}^{\rho} \otimes d\tilde{x}^{\mu} \end{cases}.$$

The **Minkowski-Riemannian curvature tensor** is defined as

$$\tilde{\mathbf{\Omega}} \triangleq \tilde{R}^{\mu}_{\nu\rho\sigma} \frac{\partial}{\partial\tilde{x}^{\mu}} \otimes d\tilde{x}^{\nu} \otimes d\tilde{x}^{\rho} \otimes d\tilde{x}^{\sigma}, \quad \tilde{R}^{\mu}_{\nu\rho\sigma} \triangleq \frac{\partial\tilde{\Gamma}^{\mu}_{\nu\sigma}}{\partial\tilde{x}^{\rho}} - \frac{\partial\tilde{\Gamma}^{\mu}_{\nu\rho}}{\partial\tilde{x}^{\sigma}} + \tilde{\Gamma}^{\lambda}_{\nu\sigma}\tilde{\Gamma}^{\mu}_{\lambda\rho} - \tilde{\Gamma}^{\lambda}_{\nu\rho}\tilde{\Gamma}^{\mu}_{\lambda\sigma}.$$

6.3 Actual evolution of classical spacetime reference-system

6.3.1 Actual evolution direction

The general theory about the actual evolution of general reference-system in section 2.4.4.2 can be transplanted to the regular form and Minkowski form of classical spacetime reference-system. The forms of regular concepts such as regular gradient, regular actual evolution equation, etc. are as same as the general forms. This section mainly discuss their Minkowski forms.

Definition 6.3.1.1. Let $\tilde{\mathbb{V}}^n$ be the set of all sections of n-order tensor bundle generated by tangent bundle $\tilde{T}(\tilde{M})$ and cotangent bundle $\tilde{T}^*(\tilde{M})$. $\forall \tilde{\mathbf{T}} \triangleq \tilde{t}^{\bullet}_{\bullet}\{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\} \in \tilde{\mathbb{V}}^n$, the absolute differential of $\tilde{\mathbf{T}}$ is

$$\tilde{D}\tilde{\mathbf{T}} \triangleq \tilde{t}^{\bullet}_{\bullet;\sigma} d\tilde{x}^{\sigma} \otimes \{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\}.$$

On evolution path L, $\tilde{t}^{\bullet}_{L\bullet} \triangleq \pi \circ \tilde{t}^{\bullet}_{\bullet}$ is a smooth real function induced by regular embedding π. Define

$$\tilde{\mathbf{T}}_L \triangleq \tilde{t}^{\bullet}_{L\bullet}\{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\},$$

$$\tilde{t}^{\bullet}_{L\bullet;\tau} \triangleq \tilde{t}^{\bullet}_{\bullet;\sigma} \frac{d\tilde{x}^{\sigma}}{d\tilde{x}^{\tau}},$$

$$\tilde{D}_L\tilde{\mathbf{T}}_L \triangleq \tilde{t}^{\bullet}_{L\bullet;\tau} d\tilde{x}^{\tau} \otimes \{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\}.$$

Define operators

$$\tilde{\nabla} : \tilde{\mathbb{V}}^n \to \tilde{\mathbb{V}}^{n+1}, \tilde{\mathbf{T}} \mapsto \tilde{\nabla}\tilde{\mathbf{T}} \triangleq \tilde{t}^{\bullet}_{\bullet;\sigma} \frac{\partial}{\partial\tilde{x}_{\sigma}} \otimes \{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\},$$
$$\tilde{\nabla}_L : \tilde{\mathbb{V}}^n \to \Gamma(\tilde{T}(L)) \otimes \tilde{\mathbb{V}}^n, \tilde{\mathbf{T}}_L \mapsto \tilde{\nabla}_L\tilde{\mathbf{T}}_L \triangleq \tilde{t}^{\bullet}_{L\bullet;\tau} \frac{d}{d\tilde{x}_{\tau}} \otimes \{\frac{\partial}{\partial\tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\}.$$

They are uniformly called the **Minkowski (absolute) gradient operators** about connection$\tilde{D}$ on manifold $\tilde{M}$. $\tilde{\nabla}\tilde{T}$ and $\tilde{\nabla}_L\tilde{T}_L$ are uniformly called the **Minkowski (absolute) gradient** of tensor $\tilde{T}$, where

$$\tilde{\nabla}\tilde{t}^{\bullet}_{\bullet} \triangleq \tilde{t}^{\bullet}_{\bullet;\mu} \frac{\partial}{\partial\tilde{x}_{\mu}}, \quad \tilde{\nabla}_L\tilde{t}^{\bullet}_{L\bullet} \triangleq \tilde{t}^{\bullet}_{L\bullet;\tau} \frac{d}{d\tilde{x}_{\tau}}$$

are uniformly called the **Minkowski (absolute) gradient direction** of components of tensor $\tilde{T}$.

Proposition 6.3.1.1. $\forall \tilde{\mathbf{T}} \in \tilde{\mathbb{V}}^n$, $\tilde{T} \triangleq \tilde{t}^{\bullet}_{\bullet}\{\frac{\partial}{\partial \tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\}$. Let L be an evolution path on any orbit of the one-parameter group of diffeomorphisms determined by smooth gradient field $\tilde{t}^{\bullet}_{\bullet;\mu}\frac{\partial}{\partial \tilde{x}_\mu}$ on manifold $\tilde{M}$. The following equivalence of tensor products holds on L necessarily:

$$\tilde{\nabla}\tilde{\mathbf{T}} \triangleq \tilde{t}^{\bullet}_{\bullet;\mu}\frac{\partial}{\partial \tilde{x}_\mu} \otimes \{\frac{\partial}{\partial \tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\} \cong \tilde{t}^{\bullet}_{L\bullet;\tau}\frac{d}{d\tilde{x}_\tau} \otimes \{\frac{\partial}{\partial \tilde{x}^{\bullet}} \otimes d\tilde{x}^{\bullet}\} \triangleq \tilde{\nabla}_L \tilde{\mathbf{T}}_L,$$

denoted by $\tilde{\nabla} \cong \tilde{\nabla}_L$, where $\tilde{t}^{\bullet}_{L\bullet} \triangleq \pi \circ \tilde{t}^{\bullet}_{\bullet}$.

Proof. It is similar to Proposition 2.4.5.1 .

Because the tangent mapping is an injection, tangent vector field $\tilde{t}^{\bullet}_{\bullet;\mu}\frac{\partial}{\partial \tilde{x}_\mu} \in \Gamma\left(\tilde{T}\left(\tilde{M}\right)\right)$ along L uniquely correspond to a tangent vector field $X\frac{d}{d\tilde{x}_\tau} \in \Gamma\left(\tilde{T}\left(L\right)\right)$ such that

$$\tilde{t}^{\bullet}_{\bullet;\mu}\frac{\partial}{\partial \tilde{x}_\mu} \cong X\frac{d}{d\tilde{x}_\tau}$$

holds in sense of the equivalence of Minkowski tangent vectors. According to the evolution lemma,

$$\tilde{t}^{\bullet}_{\bullet;\mu} = X\left.\frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}\right|_L, \quad d\tilde{x}^\mu \simeq \left.\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\right|_L d\tilde{x}^\tau.$$

So there is a homomorphism

$$\tilde{t}^{\bullet}_{\bullet;\mu}d\tilde{x}^\mu \simeq X\left.\frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}\right|_L \left.\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\right|_L d\tilde{x}^\tau.$$

According to Definition 2.4.2.1 , the coordinate mapping induced by the regular embedding satisfies $\left(d\xi^0\right)^2 = \sum_{s=1}^{r}\left(d\xi^s\right)^2 + \left(d\xi^\tau\right)^2$, i.e. $(d\tilde{\xi}^\tau)^2 = (d\tilde{\xi}^0)^2 - \sum_{s=1}^{r}(d\tilde{\xi}^s)^2$, further more, which is $d\tilde{x}_\tau d\tilde{x}^\tau = d\tilde{x}_\mu d\tilde{x}^\mu$ on evolution path L. Substitute it into the above homomorphism, then $\tilde{t}^{\bullet}_{\bullet;\mu}d\tilde{x}^\mu \simeq X d\tilde{x}^\tau$. Due to the evolution lemma, $X = \tilde{t}^{\bullet}_{\bullet;\mu}\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau} = \tilde{t}^{\bullet}_{L\bullet;\tau}$. $\square$

Definition 6.3.1.2. The Minkowski gradient operator is called the **(Minkowski) classical spacetime actual evolution**. A Minkowski gradient direction is called a **(Minkowski) classical spacetime actual evolution direction**. An evolution path on Minkowki gradient line is called a **(Minkowski) classical spacetime actual evolution path**. Equation

$$\tilde{t}^{\bullet}_{\bullet;\mu} = \tilde{t}^{\bullet}_{L\bullet;\tau}\tilde{\bar{\varepsilon}}^{\tau}_{\mu}, \quad \tilde{t}^{\bullet;\mu}_{\bullet} = \tilde{t}^{\bullet;\tau}_{L\bullet}\tilde{\varepsilon}^{\mu}_{\tau}$$

is called the **(Minkowski) classical spacetime actual evolution equation** of $\tilde{t}^{\bullet}_{\bullet}$.

6.3.2 Actual evolutions of potential field and charge

Discussion 6.3.2.1. Similar to Discussion 2.4.6.1 , we have **Minkowski Yang-Mills field equation**

$$\tilde{K}^{\mu}_{\nu\rho\sigma}{}^{:\rho} = \tilde{j}^{\mu}_{\nu\sigma},$$

where the Minkowski charge and flow are defined as

$$\tilde{\rho}^{\mu}_{\nu\tau} \triangleq \tilde{K}^{\mu}_{\nu\rho\sigma}{}^{:\rho}\tilde{\varepsilon}^{\sigma}_{\tau}, \qquad \tilde{j}^{\mu}_{\nu\sigma} \triangleq \tilde{\rho}^{\mu}_{\nu\tau}\tilde{\bar{\varepsilon}}^{\tau}_{\sigma}.$$

Consider the externally flat case, only the internal charge $\tilde{\rho}^0_{0\tau}$ does not vanish, and the Minkowski Yang-Mills field equation becomes

$$\tilde{K}^{0}_{0\rho\sigma}{}^{;\rho} = \tilde{j}^0_{0\sigma},$$

where

$$\tilde{\rho}^0_{0\tau} \triangleq \tilde{K}^{0}_{0\rho\sigma}{}^{;\rho}\,\tilde{\varepsilon}^{\sigma}_{\tau}, \qquad \tilde{j}^0_{0\sigma} \triangleq \tilde{\rho}^0_{0\tau}\tilde{\bar{\varepsilon}}^{\tau}_{\sigma}.$$

Similar to the discussions of section 2.4.6.3 , there is the actual evolution equation of Minkowski charge $\tilde{\rho}$ of classical spacetime reference-system f evolving in classical spacetime reference-system g, as following:

$$\begin{cases} \tilde{\rho}_{;\theta} d\tilde{x}^{\theta} \simeq \tilde{\rho}_{;\tau} d\tilde{x}^{\tau} \\ \tilde{\rho}_{;\theta} \dfrac{\partial}{\partial \tilde{x}_{\theta}} \cong \tilde{\rho}_{;\tau} \dfrac{d}{d\tilde{x}_{\tau}} \end{cases}.$$

Now there is a problem. After the encapsulation of classical spacetime, several internal space dimensions in general reference-system become only one. Several internal charges ρ^m_{n0} describing by the general Yang-Mills field equation also become only one, that is $\rho^{\tau}_{\tau 0}$ in regular form, and is $\tilde{\rho}^0_{0\tau}$ in Minkowski form. Thus, the original geometric properties of internal space cannot be completely described.

On the premise of not abandoning the four-dimensional spacetime, if we want to describe gauge fields, the only way is to put those degrees of freedom of internal space to the phase of complex-valued field function. This way is effective, but not natural at all.

The logically more natural way is to abandon the framework of four-dimensional spacetime. We should put internal space and external space together to describe their unified intrinsic geometry, rather than based on the rigid intuition of four-dimensional spacetime, artificially setting up several abstract degrees of freedom which are irrelevant to the concept of time and space to describe the so called gauge fields.

No matter for gauge fields or for gravitational fields, their concepts of time and space should be unified. The gravitational fields are described by the intrinsic geometry of external space, and the gauge fields are described by the intrinsic geometry of internal space. They are unified in intrinsic geometry.

Now we have to know that the complex-valued expression form of traditional gauge field theory is a historical necessity, but not a logical necessity. It can be seen later that as long as expanding those encapsulated dimensions, only real-valued expression form will be needed to clarify the goemetric properties of internal space. Especially, if understanding in the way of intrinsic geometry, some man-made postulates of standard model of particle physics will be unnecessary, because they will hold automatically.

The details will be discussed in section 6.4.3 and section 7.3.1 based on section 5.1.3 .

6.3.3 Energy-momentum equation

Discussion 6.3.3.1. (Regular form). Let f and g be classical spacetime reference-systems, and f evolves in g. As same as section 2.4.7.1 , here simply denote regular charge ρ_{IJ} by ρ.

The energy (density) of regular charge ρ evolving in g is $E^0 \triangleq \rho^{;0} \triangleq \rho^{;K}\bar{\varepsilon}^0_K$ or $E_0 \triangleq \rho_{;0} \triangleq \rho_{;K}\varepsilon^K_0$. The momentum (density) is $p^K \triangleq \rho^{;K}$ or $p_K \triangleq \rho_{;K}$. The canonical energy (density) is $H^0 \triangleq \frac{d\rho}{dx_0}$ or $H_0 \triangleq \frac{d\rho}{dx^0}$.

The canonical momentum (density) is $P^K \triangleq \frac{\partial\rho}{\partial x_K}$ or $P_K \triangleq \frac{\partial\rho}{\partial x^K}$. Usually denote

$$m^\tau \triangleq p^\tau, \quad m_\tau \triangleq p_\tau, \quad M^\tau \triangleq -P^\tau, \quad M_\tau \triangleq -P_\tau.$$

Similar to Proposition 2.4.8.1 and Proposition 2.4.8.2 , in the actual evolution direction of ρ, there is an energy-momentum equation $E_0 E^0 = p_K p^K$, i.e.

$$E_0 E^0 = p_k p^k + m_\tau m^\tau.$$

And in the actual evolution direction, the energy and momentum satisfy relations $p^K = E^0 \varepsilon_0^K$ and $p_K = E_0 \bar{\varepsilon}_K^0$.

Definition 6.3.3.1. (Minkowski form). Let f and g be classical spacetime reference-systems, and f evolves in g. Simply denote Minkowski charge $\tilde{\rho}_{\mu\nu}$ by $\tilde{\rho}$.

(1) $\tilde{m}^\tau \triangleq \tilde{\rho}^{;\tau}$ and $\tilde{m}_\tau \triangleq \tilde{\rho}_{;\tau}$ are called the **rest-mass (density)** of $\tilde{\rho}$ evolving in g.

(2) $\tilde{p}^\mu \triangleq \tilde{\rho}^{;\mu}$ and $\tilde{p}_\mu \triangleq \tilde{\rho}_{;\mu}$ are called the **energy-momentum (density)** of $\tilde{\rho}$ evolving in g.

(3) $\tilde{M}^\tau \triangleq \frac{d\tilde{\rho}}{d\tilde{x}_\tau}$ and $\tilde{M}_\tau \triangleq \frac{d\tilde{\rho}}{d\tilde{x}^\tau}$ are called the **canonical rest-mass (density)** of $\tilde{\rho}$ evolving in g.

(4) $\tilde{P}^\mu \triangleq \frac{\partial\tilde{\rho}}{\partial\tilde{x}_\mu}$ and $\tilde{P}_\mu \triangleq \frac{\partial\tilde{\rho}}{\partial\tilde{x}^\mu}$ are called the **canonical energy-momentum (density)** of $\tilde{\rho}$ evolving in g.

Usually denote

$$\tilde{E}^0 \triangleq \tilde{p}^0, \quad \tilde{E}_0 \triangleq \tilde{p}_0, \quad \tilde{H}^0 \triangleq -\tilde{P}^0, \quad \tilde{H}_0 \triangleq -\tilde{P}_0.$$

Proposition 6.3.3.1. If and only if the evolution direction of $\tilde{\rho}$ evolving in g is the actual evolution direction, equation

$$\tilde{m}_\tau \tilde{m}^\tau = \tilde{p}_\mu \tilde{p}^\mu$$

holds.

Proof. Similar to section 2.4.7.1 , with the concepts of energy and momentum, the actual evolution equation of charge $\tilde{\rho}$ can be expressed as

$$\begin{cases} \tilde{m}^\tau d\tilde{x}_\tau \simeq \tilde{p}^\mu d\tilde{x}_\mu \\ \tilde{m}^\tau \dfrac{d}{d\tilde{x}^\tau} \cong \tilde{p}^\mu \dfrac{\partial}{\partial\tilde{x}^\mu} \end{cases} \text{or} \quad \begin{cases} \tilde{m}_\tau d\tilde{x}^\tau \simeq \tilde{p}_\mu d\tilde{x}^\mu \\ \tilde{m}_\tau \dfrac{d}{d\tilde{x}_\tau} \cong \tilde{p}_\mu \dfrac{\partial}{\partial\tilde{x}_\mu} \end{cases}.$$

The conjugation between the actual evolution direction and the charge differential form is the directional derivative of $\tilde{\rho}$ in the acutal evolution direction, i.e.:

$$\frac{\tilde{D}_L\tilde{\rho}}{d\tilde{t}_{L\tilde{\rho}}} \triangleq \left\langle \frac{d}{d\tilde{t}_{L\tilde{\rho}}}, \tilde{D}_L\tilde{\rho} \right\rangle = \left\langle \frac{d}{d\tilde{t}_{\tilde{\rho}}}, \tilde{D}\tilde{\rho} \right\rangle \triangleq \frac{\tilde{D}\tilde{\rho}}{d\tilde{t}_{\tilde{\rho}}},$$

more explicitly,

$$\left\langle \tilde{m}_\tau \frac{d}{d\tilde{x}_\tau}, \tilde{m}_\tau d\tilde{x}^\tau \right\rangle = \left\langle \tilde{p}_\mu \frac{\partial}{\partial\tilde{x}_\mu}, \tilde{p}_\mu d\tilde{x}^\mu \right\rangle,$$

which is $\tilde{G}^{\tau\tau}\tilde{m}_\tau\tilde{m}_\tau = \tilde{G}^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu$, i.e. $\tilde{m}_\tau\tilde{m}^\tau = \tilde{p}_\mu\tilde{p}^\mu$ □

Definition 6.3.3.2. Equation

$$\tilde{m}_\tau \tilde{m}^\tau = \tilde{p}_\mu \tilde{p}^\mu$$

is called the **energy-momentum equation** of Minkowski charge $\tilde{\rho}$ evolving in g.

Remark 6.3.3.1. The energy-momentum equation can also be written as $\tilde{m}_\tau \tilde{m}^\tau = \tilde{p}_0 \tilde{p}^0 + \tilde{p}_k \tilde{p}^k$ or denoted by

$$\tilde{m}_\tau \tilde{m}^\tau = \tilde{E}_0 \tilde{E}^0 + \tilde{p}_k \tilde{p}^k.$$

Specially, if g is an inertial reference-system defined later, the energy-momentum equation of $\tilde{\rho}$ becomes

$$\tilde{m}_\tau^2 = \tilde{E}_0^2 - \tilde{p}_k^2.$$

Proposition 6.3.3.2. The relations about energy-momentum of $\tilde{\rho}$

$$\tilde{p}^\mu = \tilde{m}^\tau \frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}, \qquad \tilde{p}_\mu = \tilde{m}_\tau \frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}$$

hold if and only if the evolution direction of $\tilde{\rho}$ is its actual evolution direction.

Proof. Starting from the equivalences $\tilde{p}^\mu \frac{\partial}{\partial \tilde{x}^\mu} \cong \tilde{m}^\tau \frac{d}{d\tilde{x}^\tau}$ and $\tilde{p}_\mu \frac{\partial}{\partial \tilde{x}_\mu} \cong \tilde{m}_\tau \frac{d}{d\tilde{x}_\tau}$ determined by the actual evolution, according to the evolution lemma, $\tilde{p}^\mu = \tilde{m}^\tau \frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}$ and $\tilde{p}_\mu = \tilde{m}_\tau \frac{d\tilde{x}_\mu}{d\tilde{x}_\tau}$ are deduced immediately. □

Remark 6.3.3.2. Similar to Remark 2.4.8.2 , in the actual evolution direction, the conclusion of this proposition is consistent with the classical definition of momentum in traditional theory.

6.3.4 Conservation of Energy-momentum

Similar to section 2.4.8.2 , we can define the conservation of energy-momentum in classical spacetime reference-system. Now talk about it briefly.

First of all, the traditional standard conditions of Definition 2.4.9.4 will be expressed as following in classical spacetime reference-system.

Definition 6.3.4.1. (Regular form) The following two conditions are called the **(regular) traditional standard conditions**.

(1) $dE_0 = 0$ is called the **(regular) constant mass condition**.

(2) $\varGamma^M_{N0} \triangleq \varGamma^M_{NP} \varepsilon^P_0 = 0$ is called the **(regular) canonical mass condition**.

Definition 6.3.4.2. (Minkowski form) The following two conditions are called the **(Minkowski) traditional standard conditions**.

(1) $d\tilde{m}_\tau = 0$ is called the **(Minkowski) constant mass condition**.

(2) $\tilde{\varGamma}^\mu_{\nu\tau} \triangleq \tilde{\varGamma}^\mu_{\nu\rho} \tilde{\varepsilon}^\rho_\tau = 0$ is called the **(Minkowski) canonical mass condition**.

Discussion 6.3.4.1. (Regular form) It is similar to section 2.4.8.2 .

The conservations of energy-momentum of the actual evolution of regular charge ρ of classical spacetime reference-system f are $\frac{dp_K}{dx^0} - \frac{\partial E_0}{\partial x^K} + p_L \frac{\partial \varepsilon^L_0}{\partial x^K} - [\rho F_{KL}] \varepsilon^L_0 = 0$ and $p_{K;0} - E_{0;K} + p_L \varepsilon^L_{0\ ;K} - [\rho R_{KL}] \varepsilon^L_0 = 0$. On the traditional standard conditions, $\frac{dp_K}{dx^0} - [\rho F_{KL}] \varepsilon^L_0 = 0$, $p_{K;0} - [\rho R_{KL}] \varepsilon^L_0 = 0$.

The regular Lorentz force equations are $F_K \triangleq \frac{dp_K}{dx^0} = \frac{\partial E_0}{\partial x^K} - p_L \frac{\partial \varepsilon^L_0}{\partial x^K} + [\rho F_{KL}] \varepsilon^L_0$ and $f_K \triangleq p_{K;0} = E_{0;K} - p_L \varepsilon^L_{0;K} + [\rho R_{KL}] \varepsilon^L_0$. On the traditional approxitmate conditions, $F_K = [\rho F_{KL}] \varepsilon^L_0$, $f_K = [\rho R_{KL}] \varepsilon^L_0$.

Similar to discussions in section 2.4.10.1 , there is a conservation of total energy-momentum flow $T_{IJ}{}^{;I} = 0$.

Discussion 6.3.4.2. (Minkowski form) The strict deduction of the conservation of energy-momentum in Minkowski form is similar to the discussions in section 2.4.8.2 . Here only list the results. Denote $\tilde{\rho} \triangleq \tilde{\rho}_{\mu\nu}$ and

$$\begin{cases} [\tilde{\rho}\tilde{\Gamma}_\omega] \triangleq \dfrac{\partial \tilde{\rho}}{\partial \tilde{x}^\omega} - \tilde{\rho}_{;\omega} \triangleq \dfrac{\partial \tilde{\rho}_{\mu\nu}}{\partial \tilde{x}^\omega} - \tilde{\rho}_{\mu\nu;\omega} = \tilde{\rho}_{\mu\chi}\tilde{\Gamma}^\chi_{\nu\omega} + \tilde{\rho}_{\chi\nu}\tilde{\Gamma}^\chi_{\mu\omega} \\ [\tilde{\rho}\tilde{\Gamma}_\tau] \triangleq \dfrac{d\tilde{\rho}}{d\tilde{x}^\tau} - \tilde{\rho}_{;\tau} \triangleq \dfrac{d\tilde{\rho}_{\mu\nu}}{d\tilde{x}^\tau} - \tilde{\rho}_{\mu\nu;\tau} = \tilde{\rho}_{\mu\chi}\tilde{\Gamma}^\chi_{\nu\tau} + \tilde{\rho}_{\chi\nu}\tilde{\Gamma}^\chi_{\mu\tau} \end{cases},$$

and denote

$$\begin{cases} [\tilde{\rho}\tilde{\Gamma}^\omega] \triangleq g^{\chi\omega}[\tilde{\rho}\tilde{\Gamma}_\chi] \\ [\tilde{\rho}\tilde{\Gamma}^\tau] \triangleq g^{\tau\tau}[\tilde{\rho}\tilde{\Gamma}_\tau] \end{cases},$$

therefore

$$\begin{cases} [\tilde{\rho}\tilde{\Gamma}_\omega] = \tilde{P}_\omega - \tilde{p}_\omega \\ [\tilde{\rho}\tilde{\Gamma}_\tau] = \tilde{M}_\tau - \tilde{m}_\tau \end{cases}, \quad \begin{cases} [\tilde{\rho}\tilde{\Gamma}^\omega] = \tilde{P}^\omega - \tilde{p}^\omega \\ [\tilde{\rho}\tilde{\Gamma}^\tau] = \tilde{M}^\tau - \tilde{m}^\tau \end{cases}.$$

Then denote

$$[\tilde{\rho}\tilde{B}_{\rho\sigma}] \triangleq \tilde{\rho}_{\mu\chi}\left(\frac{\partial \tilde{\Gamma}^\chi_{\nu\sigma}}{\partial \tilde{x}^\rho} - \frac{\partial \tilde{\Gamma}^\chi_{\nu\rho}}{\partial \tilde{x}^\sigma}\right) + \tilde{\rho}_{\chi\nu}\left(\frac{\partial \tilde{\Gamma}^\chi_{\mu\sigma}}{\partial \tilde{x}^\rho} - \frac{\partial \tilde{\Gamma}^\chi_{\mu\rho}}{\partial \tilde{x}^\sigma}\right), \quad [\tilde{\rho}\tilde{R}_{\rho\sigma}] \triangleq \tilde{\rho}_{\mu\chi}\tilde{R}^\chi_{\nu\rho\sigma} + \tilde{\rho}_{\chi\nu}\tilde{R}^\chi_{\mu\rho\sigma},$$

$$[\tilde{\rho}\tilde{F}_{\rho\sigma}] \triangleq \frac{\partial[\tilde{\rho}\tilde{\Gamma}_\sigma]}{\partial \tilde{x}^\rho} - \frac{\partial[\tilde{\rho}\tilde{\Gamma}_\rho]}{\partial \tilde{x}^\sigma}, \quad [\tilde{\rho}\tilde{E}_{\rho\sigma}] \triangleq [\tilde{\rho}\tilde{\Gamma}_\sigma]_{;\rho} - [\tilde{\rho}\tilde{\Gamma}_\rho]_{;\sigma}.$$

The conservation of energy-momentum of $\tilde{\rho}$ is

$$\begin{cases} \dfrac{d\tilde{p}_\rho}{d\tilde{x}^\tau} - \dfrac{\partial \tilde{m}_\tau}{\partial \tilde{x}^\rho} + \tilde{p}_\sigma \dfrac{\partial \tilde{\varepsilon}^\sigma_\tau}{\partial \tilde{x}^\rho} - [\tilde{\rho}\tilde{F}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau = 0 \\ \tilde{p}_{\rho;\tau} - \tilde{m}_{\tau;\rho} + \tilde{p}_\sigma \tilde{\varepsilon}^\sigma_{\tau;\rho} - [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau = 0 \end{cases},$$

on the traditional standard conditions,

$$\begin{cases} \dfrac{d\tilde{p}_\rho}{d\tilde{x}^\tau} - [\tilde{\rho}\tilde{F}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau = 0 \\ \tilde{p}_{\rho;\tau} - [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau = 0 \end{cases}.$$

The Minkowski Lorentz force is

$$\begin{cases} \tilde{F}_\rho \triangleq \dfrac{d\tilde{p}_\rho}{d\tilde{x}^\tau} = \dfrac{\partial \tilde{m}_\tau}{\partial \tilde{x}^\rho} - \tilde{p}_\sigma \dfrac{\partial \tilde{\varepsilon}^\sigma_\tau}{\partial \tilde{x}^\rho} + [\tilde{\rho}\tilde{F}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau \\ \tilde{f}_\rho \triangleq \tilde{p}_{\rho;\tau} = \tilde{m}_{\tau;\rho} - \tilde{p}_\sigma \tilde{\varepsilon}^\sigma_{\tau;\rho} + [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau \end{cases},$$

on the traditional standard conditions,

$$\begin{cases} \tilde{F}_\rho = [\tilde{\rho}\tilde{F}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau \\ \tilde{f}_\rho = [\tilde{\rho}\tilde{R}_{\rho\sigma}]\tilde{\varepsilon}^\sigma_\tau \end{cases}.$$

Similar to section 2.4.9.3 , there is a conservation of total energy-momentum flow $\tilde{T}_{\mu\nu}{}^{;\mu} = 0$.

6.3.5 Gravitational field equation

Discussion 6.3.5.1. Similar to section 2.4.11.1 , for the evolution of classical spacetime reference-system f in classical spacetime reference-system g, the following gravitational field equations are obtained.

(1) Regular gravitational field equation is $\sum\limits_{x} c^{(x)} C_{(x)IJ} + \sum\limits_{\rho} c^{(\rho)} T_{(\rho)IJ} = 0$, where the dimensions among various terms are harmomized by constants $c^{(x)}, c^{(\rho)}$.

(2) Minkowski gravitational field equation is $\sum\limits_{\tilde{x}} c^{(\tilde{x})} \tilde{C}_{(\tilde{x})\mu\nu} + \sum\limits_{\tilde{\rho}} c^{(\tilde{\rho})} \tilde{T}_{(\tilde{\rho})\mu\nu} = 0$, where the dimensions among various terms are harmomized by constants $c^{(\tilde{x})}, c^{(\tilde{\rho})}$.

6.3.6 Evolution quantity

The general discussion about evolution quantity in section 2.4.12.1 can be transplanted to classical spacetime reference-system. Now talk about it briefly.

Discussion 6.3.6.1. (Regular form) The regular form is similar to the general form.

Let $L_\rho \in \mathbb{L}$ be an evolution path of regular charge ρ of f evolving in g, and dx^0 be the time metric and satisfy $t_a \triangleq x^0(a) < x^0(b) \triangleq t_b$.

The regular evolution quantity (density functional) of ρ from a to b along path L_ρ is

$$s_{\rho W}\left(L_\rho\right) \triangleq \int_{L_\rho} D\rho = \int_{t_a}^{t_b} E_0 dx^0 = \int_{t_a}^{t_b} p_R dx^R = \int_{t_a}^{t_b} \frac{dx^0}{dx^\tau}\left(H_0 - [\rho\mathit{\Gamma}_R]\varepsilon_0^R\right) dx^\tau.$$

Similar to Proposition 2.4.13.1 , because in the actual evolution direction, the integrand of evolution quantity is a directional derivative in gradient direction, so the actual evolution path as the integral curve of gradient directions should satisfy the following proposition. The proof is as same as Proposition 2.4.13.1 , so it does not be discussed repeatly.

Proposition 6.3.6.1. (Regular evolution quantity extreme value theorem). Suppose charge ρ of f evolves in g. For the charge ρ, an evolution path L_ρ is exactly the actual evolution path if and only if $\delta s_{\rho W} = 0$.

Definition 6.3.6.1. (Minkowski form) Let $d\tilde{x}^\tau$ be the proper-time metric and satisfy $\tau_a \triangleq \tilde{x}^\tau(a) < \tilde{x}^\tau(b) \triangleq \tau_b$. $\forall L_{\tilde{\rho}} \in \tilde{\mathbb{L}}$, denote

$$\tilde{s}_{\tilde{\rho}\tilde{W}}(L_{\tilde{\rho}}) \triangleq \int_{L_{\tilde{\rho}}} D\tilde{\rho} = \int_{\tau_a}^{\tau_b} \tilde{m}_\tau d\tilde{x}^\tau = \int_{\tau_a}^{\tau_b} \tilde{p}_\mu d\tilde{x}^\mu = \int_{\tau_a}^{\tau_b} \frac{d\tilde{x}^\tau}{d\tilde{x}^0}\left(\tilde{M}_\tau - [\tilde{\rho}\tilde{\mathit{\Gamma}}_\sigma]\tilde{\varepsilon}_\tau^\sigma\right) d\tilde{x}^0.$$

The functional $\tilde{s}_{\tilde{\rho}\tilde{W}}(L_{\tilde{\rho}})$ about path $\forall L_{\tilde{\rho}}$ is called the **Minkowski evolution quantity (density functional)** of $\tilde{\rho}$ evolving along path $L_{\tilde{\rho}}$. Usually it is also called the **action** of $\tilde{\rho}$ evolving in potential fields.

Proposition 6.3.6.2. (Minkowski evolution quantity extreme value theorem). For the Minkowski charge $\tilde{\rho}$ of classical spacetime reference-system f evolving in classical spacetime reference-system g, an evolution path $L_{\tilde{\rho}}$ is exactly the actual evolution path if and only if $\delta\tilde{s}_{\tilde{\rho}\tilde{W}} = 0$.

Proof. It is similar to Proposition 2.4.13.1 . The only difference is that the gradient about general coordinate is replaced by the gradient about Minkowski coordinate.

Let the parameter equation of evolution path $L_{\tilde{\rho}}$ be

$$\tilde{x}^\sigma = \tilde{x}^\sigma(\tilde{x}^\tau), \quad \tau_a \leqslant \tilde{x}^\tau \leqslant \tau_b,$$

and let the parameter equation of evolution path $L_{\tilde{\rho}} + \delta L_{\tilde{\rho}}$ be

$$\tilde{x}^{\sigma} = \tilde{x}^{\sigma}(\tilde{x}^{\tau}) + \delta\tilde{x}^{\sigma}(\tilde{x}^{\tau}), \quad \tau_a \leqslant \tilde{x}^{\tau} \leqslant \tau_b, \quad \delta\tilde{x}^{\sigma}(\tau_a) = \delta\tilde{x}^{\sigma}(\tau_b) = 0.$$

Let the unit tangent vector of path $L_{\tilde{\rho}}$ at any $\tilde{x}^{\tau}$ be

$$\tilde{X} \triangleq \tilde{\pi}_* \left(\frac{d}{d\tilde{x}^{\tau}}\right) \triangleq \left.\frac{d\tilde{x}^{\sigma}}{d\tilde{x}^{\tau}}\right|_{\tilde{x}^{\tau}} \frac{\partial}{\partial\tilde{x}^{\sigma}} \triangleq \tilde{\varepsilon}^{\sigma}_{\tau}\left(\tilde{x}^{\tau}\right) \frac{\partial}{\partial\tilde{x}^{\sigma}},$$

and let the unit tangent vector of $L_{\tilde{\rho}} + \delta L_{\tilde{\rho}}$ be

$$\tilde{X} + \delta\tilde{X} \triangleq \left.\frac{d\left(\tilde{x}^{\sigma} + \delta\tilde{x}^{\sigma}\right)}{d\tilde{x}^{\tau}}\right|_{\tilde{x}^{\tau}} \frac{\partial}{\partial\tilde{x}^{\sigma}} = \left.\left(\frac{d\tilde{x}^{\sigma}}{d\tilde{x}^{\tau}} + \delta\frac{d\tilde{x}^{\sigma}}{d\tilde{x}^{\tau}}\right)\right|_{\tilde{x}^{\tau}} \frac{\partial}{\partial\tilde{x}^{\sigma}} = \left(\tilde{\varepsilon}^{\sigma}_{\tau}\left(\tilde{x}^{\tau}\right) + \delta\tilde{\varepsilon}^{\sigma}_{\tau}\left(\tilde{x}^{\tau}\right)\right) \frac{\partial}{\partial\tilde{x}^{\sigma}}.$$

Then consider the variation of $\tilde{s}_{\tilde{\rho}\tilde{W}}(L_{\tilde{\rho}}) = \int_{L_{\tilde{\rho}}} \tilde{m}_{\tau} d\tilde{x}^{\tau} = \int_{L_{\tilde{\rho}}} \tilde{p}_{\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau}$.

$$\begin{aligned}
\varDelta\tilde{s}_{\tilde{\rho}\tilde{W}}(L_{\tilde{\rho}}) &= \varDelta\int_{L_{\tilde{\rho}}} \tilde{p}_{\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau} = \int_{L_{\tilde{\rho}}+\delta L_{\tilde{\rho}}} \tilde{p}_{\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau} - \int_{L_{\tilde{\rho}}} \tilde{p}_{\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau} = \int_{L_{\tilde{\rho}}+\delta L_{\tilde{\rho}}} \tilde{\rho}_{;\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau} - \int_{L_{\tilde{\rho}}} \tilde{\rho}_{;\sigma}\tilde{\varepsilon}^{\sigma}_{\tau} d\tilde{x}^{\tau} \\
&= \int_{L_{\tilde{\rho}}+\delta L_{\tilde{\rho}}} \left\langle \tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau} - \int_{L_{\tilde{\rho}}} \left\langle \tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau} \\
&= \int_{\tau_a}^{\tau_b} \left\langle \tilde{X} + \delta\tilde{X}, \tilde{D}\tilde{\rho}\left(\tilde{x}^{\sigma} + \delta\tilde{x}^{\sigma}\right)\right\rangle d\tilde{x}^{\tau} - \int_{\tau_a}^{\tau_b} \left\langle \tilde{X}, \tilde{D}\tilde{\rho}\left(\tilde{x}^{\sigma}\right)\right\rangle d\tilde{x}^{\tau} \\
&= \int_{\tau_a}^{\tau_b} \left\langle \tilde{X} + \delta\tilde{X}, \tilde{D}\tilde{\rho}(\tilde{x}^{\sigma}) + \frac{\partial\tilde{D}\tilde{\rho}(\tilde{x}^{\sigma})}{\partial\tilde{x}^{\mu}}\delta\tilde{x}^{\mu} + o(\delta\tilde{x})d\tilde{x}^{\sigma}\right\rangle d\tilde{x}^{\tau} - \int_{\tau_a}^{\tau_b} \left\langle \tilde{X}, \tilde{D}\tilde{\rho}(\tilde{x}^{\sigma})\right\rangle d\tilde{x}^{\tau} \\
&= \int_{\tau_a}^{\tau_b} \left(\left\langle \tilde{X} + \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle + \left\langle \tilde{X} + \delta\tilde{X}, \frac{\partial\tilde{D}\tilde{\rho}}{\partial\tilde{x}^{\mu}}\delta\tilde{x}^{\mu}\right\rangle\right) d\tilde{x}^{\tau} - \int_{\tau_a}^{\tau_b} \left\langle \tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau} + o\left(\delta\tilde{x}\right) \\
&= \int_{\tau_a}^{\tau_b} \left(\left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle + \left\langle \tilde{X}, \frac{\partial\tilde{D}\tilde{\rho}}{\partial\tilde{x}^{\mu}}\delta\tilde{x}^{\mu}\right\rangle\right) d\tilde{x}^{\tau} + o\left(\delta\tilde{x}\right) = \int_{\tau_a}^{\tau_b} \left(\left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle + \left\langle \tilde{X}, \delta\tilde{D}\tilde{\rho}\right\rangle\right) d\tilde{x}^{\tau} + o\left(\delta\tilde{x}\right) \\
&= \int_{\tau_a}^{\tau_b} \left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau} + \int_{\tau_a}^{\tau_b} \delta\tilde{D}\tilde{\rho} + o\left(\delta\tilde{x}\right) \\
&= \int_{\tau_a}^{\tau_b} \left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau} + o(\delta\tilde{x}),
\end{aligned}$$

Thus we get

$$\delta\tilde{s}_{\tilde{\rho}\tilde{W}} = \int_{\tau_a}^{\tau_b} \left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau}.$$

When point $b \to a$, $\delta d\tilde{s}_{\tilde{\rho}\tilde{W}} = \left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle d\tilde{x}^{\tau}$. The directional derivative $\left\langle \tilde{X}, \tilde{D}\tilde{\rho}\right\rangle = \tilde{\rho}_{;\tau}\cos\theta$, where θ is the included angle between the evolution direction $\tilde{X}$ and the gradient direction. Take the directional variation,

$$\left\langle \delta\tilde{X}, \tilde{D}\tilde{\rho}\right\rangle = \tilde{\rho}_{;\tau}\delta\cos\theta = -\tilde{\rho}_{;\tau}\sin\theta\delta\theta,$$

Thus, the evolution quantity variation of $\tilde{\rho}$ is

$$\delta d\tilde{s}_{\tilde{\rho}\tilde{W}} = -\tilde{\rho}_{;\tau}\sin\theta\delta\theta dx^0.$$

For arbitrary $\tilde{\rho}$, $\delta d\tilde{s}_{\tilde{\rho}\tilde{W}} = 0$ if and only if $\sin\theta = 0$, namely the evolution direction at this point is exactly the actual evolution direction (take the positive direction without loss of generality).

Take integration from a to b, then $\delta\int_{\tau_a}^{\tau_b} d\tilde{s}_{\tilde{\rho}\tilde{W}}=0$ if and only if the evolution direction at each point of integral curve $L_{\tilde{\rho}}$ is the actual evolution direction of $\tilde{\rho}$. In other words, $\delta\tilde{s}_{\tilde{\rho}\tilde{W}}=0$ if and only if $L_{\tilde{\rho}}$ is the actual evolution path of $\tilde{\rho}$. □

6.3.7 Legendre transformation and equation of motion

This section does not discuss the general abstract theory of Legendre transformation, but discusses the relationship between energy-momentum equation and the concrete construction of Legendre transformation.

Definition 6.3.7.1. (Regular form). Denote

$$L_\tau \triangleq E_0\frac{dx^0}{dx^\tau}=\frac{dx^0}{dx^\tau}\left(H_0-[\rho\mathit{\Gamma}_K]\varepsilon_0^K\right),$$

$$\mathcal{L}_\tau \triangleq H_0\frac{dx^0}{dx^\tau}=\frac{dx^0}{dx^\tau}\left(E_0+[\rho\mathit{\Gamma}_K]\varepsilon_0^K\right).$$

Obviously on the canonical mass condition,

$$\mathcal{L}_\tau=L_\tau.$$

On the canonical mass condition, $\mathcal{L}_\tau$ is called the **regular Lagrangian densigy** of the regular charge ρ of f evolving in g.

According to Discussion 6.3.3.1 , the canonical energy-momentums satisfy $H_0dx^0=P_kdx^k-M_\tau dx^\tau$. As a result,

$$M_\tau=P_k\frac{dx^k}{dx^\tau}-H_0\frac{dx^0}{dx^\tau}=P_k\frac{dx^k}{dx^\tau}-\mathcal{L}_\tau.$$

M_τ is called the **regular Hamiltonian density** of regular charge ρ of f evolving in g. Such a transformation between M_τ and $\mathcal{L}_\tau$ is called the **regular Legendre transformation**.

Definition 6.3.7.2. (Minkowski form). Denote

$$\tilde{L}_0 \triangleq \tilde{m}_\tau\frac{d\tilde{x}^\tau}{d\tilde{x}^0}=\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\left(\tilde{M}_\tau-[\tilde{\rho}\tilde{\mathit{\Gamma}}_\sigma]\tilde{\varepsilon}_\tau^\sigma\right),$$

$$\tilde{\mathcal{L}}_0 \triangleq \tilde{M}_\tau\frac{d\tilde{x}^\tau}{d\tilde{x}^0}=\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\left(\tilde{m}_\tau+[\tilde{\rho}\tilde{\mathit{\Gamma}}_\sigma]\tilde{\varepsilon}_\tau^\sigma\right).$$

Obviously on the canonical mass condition,

$$\tilde{\mathcal{L}}_0=\tilde{L}_0.$$

On the canonical mass condition, $\tilde{\mathcal{L}}_0$ is called the **(Minkowski) Lagrangian density** of Minkowski charge $\tilde{\rho}$ of f evolving in g.

According to Definition 6.3.3.1 , the canonical energy-momentums satisfy $\tilde{M}_\tau d\tilde{x}^\tau=\tilde{P}_kd\tilde{x}^k-\tilde{H}_0d\tilde{x}^0$. As a result,

$$\tilde{H}_0=\tilde{P}_k\frac{d\tilde{x}^k}{d\tilde{x}^0}-\tilde{M}_\tau\frac{d\tilde{x}^\tau}{d\tilde{x}^0}=\tilde{P}_k\frac{d\tilde{x}^k}{d\tilde{x}^0}-\tilde{\mathcal{L}}_0.$$

$\tilde{H}_0$ is called the **(Minkowski) Hamiltonian density** of Minkowski charge $\tilde{\rho}$ of f evolving in g. Such a transformation between $\tilde{H}_0$ and $\tilde{\mathcal{L}}_0$ is called the **(Minkowski) Legendre transformation**.

Discussion 6.3.7.1. (Euler-Lagrange equation) On the traditional standard conditions, Euler-Lagrange equation can be obtained directly from the definition of Lagrangian density. Concretely,

$$\mathcal{L}_\tau \triangleq H_0 \frac{dx^0}{dx^\tau} = \frac{d\rho}{dx^0}\frac{dx^0}{dx^\tau} = \frac{\partial\rho}{\partial x^M}\frac{dx^M}{dx^0}\frac{dx^0}{dx^\tau},$$

so

$$\frac{\partial\mathcal{L}_\tau}{\partial x^K} = \frac{\partial}{\partial x^K}\left(\frac{\partial\rho}{\partial x^M}\frac{dx^M}{dx^0}\frac{dx^0}{dx^\tau}\right).$$

According to Remark 2.4.9.3 , in the actual evolution direction,

$$\frac{\partial\mathcal{L}_\tau}{\partial x^K} = \frac{dx^M}{dx^0}\frac{dx^0}{dx^\tau}\frac{\partial}{\partial x^K}\left(\frac{\partial\rho}{\partial x^M}\right) = \frac{dx^M}{dx^0}\frac{dx^0}{dx^\tau}\frac{\partial}{\partial x^M}\left(\frac{\partial\rho}{\partial x^K}\right) = \frac{dx^M}{dx^0}\frac{dx^0}{dx^\tau}\frac{\partial P_K}{\partial x^M} = \frac{dP_K}{dx^\tau},$$

Thus we get

$$\frac{dP_k}{dx^\tau} - \frac{\partial\mathcal{L}_\tau}{\partial x^k} = 0,$$

which is the regular Euler-lagrange equation.

There is a similar deduction in the Minkowski coordinate form. On the traditioinal standard conditions,

$$\tilde{\mathcal{L}}_0 \triangleq \tilde{M}_\tau \frac{d\tilde{x}^\tau}{d\tilde{x}^0} = \frac{d\tilde{\rho}}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0} = \frac{\partial\tilde{\rho}}{\partial\tilde{x}^\mu}\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0},$$

so

$$\frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{x}^\sigma} = \frac{\partial}{\partial\tilde{x}^\sigma}\left(\frac{\partial\tilde{\rho}}{\partial\tilde{x}^\mu}\frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\right).$$

Similar to Remark 2.4.9.3 , in the actual evolution direction,

$$\frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{x}^\sigma} = \frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\frac{\partial}{\partial\tilde{x}^\sigma}\left(\frac{\partial\tilde{\rho}}{\partial\tilde{x}^\mu}\right) = \frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\frac{\partial}{\partial\tilde{x}^\mu}\left(\frac{\partial\tilde{\rho}}{\partial\tilde{x}^\sigma}\right) = \frac{d\tilde{x}^\mu}{d\tilde{x}^\tau}\frac{d\tilde{x}^\tau}{d\tilde{x}^0}\frac{\partial\tilde{P}_\sigma}{\partial\tilde{x}^\mu} = \frac{d\tilde{P}_\sigma}{d\tilde{x}^0},$$

Thus we get

$$\frac{d\tilde{P}_k}{d\tilde{x}^0} - \frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{x}^k} = 0,$$

which is the Minkowski Euler-lagrange equation.

Denote $\tilde{v}^k \triangleq \frac{d\tilde{x}^k}{d\tilde{x}^0}$. In traditional theory $\tilde{\mathcal{L}}_0 = \tilde{P}_k\tilde{v}^k - \tilde{H}_0$ is usually regarded as a function $\tilde{\mathcal{L}}_0(\tilde{x}^k, \tilde{v}^k)$, then $\tilde{P}_k = \frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{v}^k}$. Consequently the Minkowski Euler-Lagrange equation $\frac{d\tilde{P}_k}{d\tilde{x}^0} - \frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{x}^k} = 0$ can be written as

$$\frac{d}{d\tilde{x}^0}\left(\frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{v}^k}\right) - \frac{\partial\tilde{\mathcal{L}}_0}{\partial\tilde{x}^k} = 0,$$

which is as same as the traditional Euler-Lagrange equation.

6.3.8 Evolution equation in complex-valued form

This section concisely discusses the construction of evolution equation in complex-valued form. In order to connect traditional theory, we consider the Minkowski form directly.

Discussion 6.3.8.1. Take a very simple field function $\psi(\tilde{x}^\mu) = fe^{i\tilde{S}}$ for example. Let real function $f(\tilde{x}^\mu)$ satisfy

$$\int f^2 dV = 1, \qquad \frac{df}{d\tilde{x}^\tau} \triangleq \tilde{\varepsilon}^\mu_\tau \frac{\partial f}{\partial\tilde{x}^\mu} = 0.$$

And let

$$\tilde{S} \triangleq \int \tilde{s} dV, \qquad \tilde{s} = \int \tilde{\mathcal{L}}_0 d\tilde{x}^0 = f^2 \tilde{S}.$$

Then let

$$\tilde{\mathbb{M}}_\tau \triangleq \int \tilde{m}_\tau dV, \qquad \tilde{m}_\tau = f^2 \tilde{\mathbb{M}}_\tau.$$

Further more let

$$[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] \triangleq \int [\tilde{\rho}\tilde{\Gamma}_\mu] dV, \qquad [\tilde{\rho}\tilde{\Gamma}_\mu] = f^2 [\tilde{\mathrm{P}}\tilde{\Gamma}_\mu].$$

According to Discussion 6.3.2.1 , in the actual evolution direction of $\tilde{\rho}$,

$$\begin{cases} \tilde{\rho}_{;\mu} d\tilde{x}^\mu \simeq \tilde{\rho}_{;\tau} d\tilde{x}^\tau \\ \tilde{\rho}_{;\mu} \dfrac{\partial}{\partial \tilde{x}_\mu} \cong \tilde{\rho}_{;\tau} \dfrac{d}{d\tilde{x}_\tau} \end{cases},$$

then we obtain energy-momentum equation $\tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau}$, so the above actual evolution equation can be equivalently witten as

$$\begin{cases} \tilde{\rho}_{;\mu} \tilde{\varepsilon}^\mu_\tau = \tilde{\rho}_{;\tau} \\ \tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau} \end{cases}.$$

Without loss of generality, it can be denoted that either $\tilde{m}_\tau \triangleq \tilde{\rho}_{;\tau}$ or $\tilde{m}_\tau \triangleq -\tilde{\rho}_{;\tau}$. For convenience, adopt the latter. On the canonical mass condition, $\tilde{\mathcal{L}}_0 = \tilde{L}_0$. It is obtained that

$$\begin{cases} \tilde{\varepsilon}^\mu_\tau \dfrac{\partial \tilde{s}}{\partial x^\mu} = -\tilde{m}_\tau \\ \tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau} \end{cases} \Rightarrow \begin{cases} \tilde{\varepsilon}^\mu_\tau \dfrac{\partial \tilde{S}}{\partial x^\mu} = -\tilde{\mathbb{M}}_\tau \\ \tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau} \end{cases}.$$

On the canonical mass condition, $[\tilde{\rho}\tilde{\Gamma}_\mu]\tilde{\varepsilon}^\mu_\tau = 0$, and the first equation becomes

$$\begin{aligned} &\tilde{\varepsilon}^\mu_\tau \left(\frac{\partial \tilde{S}}{\partial x^\mu} - [\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] \right) = -\tilde{\mathbb{M}}_\tau \\ \Leftrightarrow\ & i\tilde{\varepsilon}^\mu_\tau \left(i\frac{\partial \tilde{S}}{\partial x^\mu} f e^{i\tilde{S}} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] f e^{i\tilde{S}} \right) = \tilde{\mathbb{M}}_\tau f e^{i\tilde{S}} \Leftrightarrow i\tilde{\varepsilon}^\mu_\tau \left(\frac{\partial f}{\partial \tilde{x}^\mu} e^{i\tilde{S}} + i\frac{\partial \tilde{S}}{\partial x^\mu} f e^{i\tilde{S}} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] f e^{i\tilde{S}} \right) = \tilde{\mathbb{M}}_\tau f e^{i\tilde{S}} \\ \Leftrightarrow\ & i\tilde{\varepsilon}^\mu_\tau \left(\frac{\partial \left(f e^{i\tilde{S}} \right)}{\partial \tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] f e^{i\tilde{S}} \right) = \tilde{\mathbb{M}}_\tau f e^{i\tilde{S}} \Leftrightarrow i\tilde{\varepsilon}^\mu_\tau \left(\frac{\partial \psi}{\partial \tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu]\psi \right) = \tilde{\mathbb{M}}_\tau \psi \\ \Leftrightarrow\ & i\tilde{\varepsilon}^\mu_\tau \left(\frac{\partial}{\partial \tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu] \right) \psi = \tilde{\mathbb{M}}_\tau \psi, \end{aligned}$$

which is concisely denoted by $i\tilde{\varepsilon}^\mu_\tau \tilde{D}_\mu \psi = \tilde{\mathbb{M}}_\tau \psi$, where $\tilde{D}_\mu \triangleq \frac{\partial}{\partial \tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\Gamma}_\mu]$. Thus, the evolution equation becomes

$$\begin{cases} i\tilde{\varepsilon}^\mu_\tau \tilde{D}_\mu \psi = \tilde{\mathbb{M}}_\tau \psi \\ \tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau} \end{cases},$$

which is the actual evolution equation in complex-valued form.

Discussion 6.3.8.2. The above equation is similar to Dirac equation, but not the same. $\tilde{\rho}_{;\mu}\tilde{\rho}^{;\mu} = \tilde{\rho}_{;\tau}\tilde{\rho}^{;\tau}$ can also be written as $\tilde{G}^{\mu\nu}\tilde{\rho}_{;\mu}\tilde{\rho}_{;\nu} = \tilde{m}^2_\tau$, where $\tilde{G}^{\mu\nu}$ is the metric tensor of g. Then define the well-known **general Dirac algebra** such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\tilde{G}^{\mu\nu}.$$

No matter what kind of matrix representation of γ^μ is, directly substitutes it into $\tilde{G}^{\mu\nu}\tilde{\rho}_{;\mu}\tilde{\rho}_{;\nu} = \tilde{m}_\tau^2$, then

$$(\gamma^\mu\tilde{\rho}_{;\mu})(\gamma^\nu\tilde{\rho}_{;\nu}) + (\gamma^\nu\tilde{\rho}_{;\nu})(\gamma^\mu\tilde{\rho}_{;\mu}) = 2\tilde{m}_\tau^2.$$

If g satisfies that $\mu \neq \nu \Rightarrow \tilde{G}^{\mu\nu} = 0$, then $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ holds for different μ and ν. In this case the above equation can be writter as $(\gamma^\mu\tilde{\rho}_{;\mu})(\gamma^\nu\tilde{\rho}_{;\nu}) = \tilde{m}_\tau^2$, moreover we obtain $\gamma^\mu\tilde{\rho}_{;\mu} = -\tilde{m}_\tau$ (be consistent with Discussion 6.3.8.1).

Now according to the same thoughts in Discussion 6.3.8.1 , add an extra condition of flow conservation $\gamma^\mu\frac{\partial f}{\partial\tilde{x}^\mu} = 0$ and a similar canonical mass condition $\gamma^\mu[\tilde{\rho}\tilde{\varGamma}_\mu] = 0$. Then

$$\begin{aligned}
&\gamma^\mu\tilde{\rho}_{;\mu} = -\tilde{m}_\tau \Leftrightarrow \gamma^\mu\frac{\partial\tilde{s}}{\partial x^\mu} = -\tilde{m}_\tau \Leftrightarrow \gamma^\mu\frac{\partial\tilde{S}}{\partial x^\mu} = -\tilde{\mathbb{M}}_\tau \Leftrightarrow \gamma^\mu\left(\frac{\partial\tilde{S}}{\partial x^\mu} - [\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]\right) = -\tilde{\mathbb{M}}_\tau \\
&\Leftrightarrow i\gamma^\mu\left(i\frac{\partial\tilde{S}}{\partial x^\mu}fe^{i\tilde{S}} - i[\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]fe^{i\tilde{S}}\right) = \tilde{\mathbb{M}}_\tau fe^{i\tilde{S}} \Leftrightarrow i\gamma^\mu\left(\frac{\partial f}{\partial\tilde{x}^\mu}e^{i\tilde{S}} + i\frac{\partial\tilde{S}}{\partial x^\mu}fe^{i\tilde{S}} - i[\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]fe^{i\tilde{S}}\right) = \tilde{\mathbb{M}}_\tau fe^{i\tilde{S}} \\
&\Leftrightarrow i\gamma^\mu\left(\frac{\partial\left(fe^{i\tilde{S}}\right)}{\partial\tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]fe^{i\tilde{S}}\right) = \tilde{\mathbb{M}}_\tau fe^{i\tilde{S}} \Leftrightarrow i\gamma^\mu\left(\frac{\partial\psi}{\partial\tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]\psi\right) = \tilde{\mathbb{M}}_\tau\psi \\
&\Leftrightarrow i\gamma^\mu\left(\frac{\partial}{\partial\tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\varGamma}_\mu]\right)\psi = \tilde{\mathbb{M}}_\tau\psi,
\end{aligned}$$

which can be denoted by

$$i\gamma^\mu\tilde{D}_\mu\psi = \tilde{\mathbb{M}}_\tau\psi,$$

where the definition of $\tilde{D}_\mu$ is as same as that in Discussion 6.3.8.1 . Thus the **Dirac equation** has been constructed. If adopting the matrix representation of γ^μ, it just needs to use a four-component field consisting of such four ψ.

However, that is not all. Noticed that the above Dirac equation requires that the metric tensor of g is a diagonal matrix. For a general g, the metric tensor is not necessarily a diagonal matrix. At this time we should take another way based on the above result to obtain a general Dirac equation.

Concretely, consider the evolution of a charge $\tilde{\rho}$ of an arbitrary classical spacetime reference-system f in another arbitrary classical spacetime reference-system g. Review Remark 2.2.1.2 , and define a completely static reference-system e such that $F_g(e) = g$. There is a traditional Dirac algebra $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\tilde{\delta}^{\alpha\beta}$ strictly defined on e. Then using the above result, it is obtained that the Dirac equation of $\tilde{\rho}$ evolving in e is $i\gamma^\alpha\tilde{D}_\alpha\psi_e = \tilde{\mathbb{M}}_\tau\psi_e$, where $\tilde{D}_\alpha \triangleq \frac{\partial}{\partial\tilde{\xi}^\alpha} - i[\tilde{\mathrm{P}}\tilde{\varDelta}_\alpha]$

When the reference-system transformation F_g transforms e to g, the transformation of slack-tight is

$$\eta_\beta^\alpha \mapsto \tilde{B}_\mu^\alpha,$$

and the transformation of basis vector $\frac{\partial}{\partial\tilde{\xi}^\alpha}$ is

$$\frac{\partial}{\partial\tilde{\xi}^\beta} = \eta_\beta^\alpha\frac{\partial}{\partial\tilde{\xi}^\alpha} \mapsto \frac{\partial}{\partial\tilde{x}^\mu} = \tilde{B}_\mu^\alpha\frac{\partial}{\partial\tilde{\xi}^\alpha},$$

and correspondingly let the transformation of γ^α be

$$\gamma^\beta = \eta_\alpha^\beta\gamma^\alpha \mapsto \gamma^\mu \triangleq \tilde{C}_\alpha^\mu\gamma^\alpha.$$

According to the definition of simple connection, the transformation of $[\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha]$ can be expressed as

$$[\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha] \mapsto [\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu] = [\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha]\tilde{B}^\alpha_\mu + r_\mu,$$

and correspondingly the transformation of $\tilde{D}_\alpha$ is

$$\tilde{D}_\alpha \triangleq \frac{\partial}{\partial\tilde{\xi}^\alpha} - i[\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha] \mapsto \tilde{D}_\mu \triangleq \frac{\partial}{\partial\tilde{x}^\mu} - i[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu] = \tilde{B}^\alpha_\mu \frac{\partial}{\partial\tilde{\xi}^\alpha} - i[\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha]\tilde{B}^\alpha_\mu - ir_\mu = \tilde{B}^\alpha_\mu \tilde{D}_\alpha - ir_\mu.$$

Thus the transformation of ψ_e is

$$\psi_e = fe^{i\int\left(\tilde{\mathrm{P}}_{;\alpha} + [\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha]\right)d\tilde{\xi}^\alpha} \mapsto \psi = fe^{i\int\left(\tilde{\mathrm{P}}_{;\mu} + [\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\right)d\tilde{x}^\mu} = fe^{i\int\left(\tilde{\mathrm{P}}_{;\mu} + [\tilde{\mathrm{P}}\tilde{\mathit{\Delta}}_\alpha]\tilde{B}^\alpha_\mu + r_\mu\right)d\tilde{x}^\mu},$$

i.e.

$$\psi_e \mapsto \psi = \psi_e e^{i\int r_\mu d\tilde{x}^\mu}.$$

Denote

$$\theta \triangleq \int r_\mu d\tilde{x}^\mu, \qquad r_\mu = \partial_\mu\theta,$$

then

$$\psi_e \mapsto \psi = \psi_e e^{i\theta}.$$

The Dirac equation $i\gamma^\alpha\tilde{D}_\alpha\psi_e = \tilde{\mathbb{M}}_\tau\psi_e$ correspondingly becomes

$$i\left(\tilde{B}^\alpha_\mu\gamma^\mu\right)\left(\tilde{C}^\nu_\alpha(\tilde{D}_\nu + i\partial_\nu\theta)\right)\left(\psi e^{-i\theta}\right) = \tilde{\mathbb{M}}_\tau\left(\psi e^{-i\theta}\right),$$

$$i\gamma^\mu(\tilde{D}_\mu + i\partial_\mu\theta)\left(\psi e^{-i\theta}\right) = \tilde{\mathbb{M}}_\tau\psi e^{-i\theta},$$

$$i\gamma^\mu\left(\partial_\mu - i[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu] + i\partial_\mu\theta\right)\left(\psi e^{-i\theta}\right) = \tilde{\mathbb{M}}_\tau\psi e^{-i\theta},$$

$$i\gamma^\mu\partial_\mu\left(\psi e^{-i\theta}\right) + \gamma^\mu[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\psi e^{-i\theta} - \gamma^\mu\psi e^{-i\theta}\partial_\mu\theta = \tilde{\mathbb{M}}_\tau\psi e^{-i\theta},$$

$$i\gamma^\mu\partial_\mu\psi e^{-i\theta} + i\gamma^\mu\psi\partial_\mu e^{-i\theta} + \gamma^\mu[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\psi e^{-i\theta} - \gamma^\mu\psi e^{-i\theta}\partial_\mu\theta = \tilde{\mathbb{M}}_\tau\psi e^{-i\theta},$$

$$i\gamma^\mu\partial_\mu\psi e^{-i\theta} + \gamma^\mu\psi\partial_\mu\theta e^{-i\theta} + \gamma^\mu[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\psi e^{-i\theta} - \gamma^\mu\psi e^{-i\theta}\partial_\mu\theta = \tilde{\mathbb{M}}_\tau\psi e^{-i\theta},$$

$$i\gamma^\mu\partial_\mu\psi + \gamma^\mu\psi\partial_\mu\theta + \gamma^\mu[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\psi - \gamma^\mu\psi\partial_\mu\theta = \tilde{\mathbb{M}}_\tau\psi,$$

$$i\gamma^\mu\partial_\mu\psi + \gamma^\mu[\tilde{\mathrm{P}}\tilde{\mathit{\Gamma}}_\mu]\psi = \tilde{\mathbb{M}}_\tau\psi,$$

finally that is

$$i\gamma^\mu\tilde{D}_\mu\psi = \tilde{\mathbb{M}}_\tau\psi.$$

It is shown that a reference-system transformation F_g brings a transformation of slack-tight, further more it brings a kind of very general gauge transformation. In this sense, it is not difficult to understand the terminologies in Definition 2.2.2.3 . The essence of the so-called gauge transformation is actually a transformation of reference-system.

Thus, the Dirac equation $i\gamma^\mu\tilde{D}_\mu\psi = \tilde{\mathbb{M}}_\tau\psi$ about arbitrary classical spacetime reference-system has been constructed.

Now the Dirac equation becomes a theorem, and is not a principle anymore. In consideration of that the Dirac equation is constructed from the actual evolution equation, so it also reflects the concept of gradient direction, and it thereby describes the effects of intrinsic geometry of manifold on the gradient direction. In this sense, the Dirac equation in epistemology can be used to cognize the actual evolution of physical reality in ontology. This is the origin of the effectiveness of Dirac equation in traditional physics.

6.4 Classical spacetime inertial reference-system

In order to connect traditional theory, directly consider the Minkowski form.

Definition 6.4.1. Let g be a classical spacetime reference-system on manifold $\tilde{M}$.

(1) If g satisfies $d\tilde{\zeta}^\tau = d\tilde{x}^\tau$ on $\tilde{M}$, g is called a **classical spacetime orthogonal reference-system**, and geometric manifold $(\tilde{M}, g)$ is called an **isotropic spacetime**. For convenience, $d\tilde{\zeta}^\tau$ and $d\tilde{x}^\tau$ are uniformly denoted by $d\tau$. According to Definition 6.2.3.2 , the metric tensors $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{H}}$ of g satisfy $\tilde{G}_{\mu\nu} = \tilde{\varDelta}_{\alpha\beta}\tilde{B}^\alpha_\mu \tilde{B}^\beta_\nu = \tilde{\mathrm{E}}_{\mu\nu}$ and $\tilde{H}_{\alpha\beta} = \tilde{\mathrm{E}}_{\mu\nu}\tilde{C}^\mu_\alpha \tilde{C}^\nu_\beta = \tilde{\varDelta}_{\alpha\beta}$.

(2) If the slack-tights $\tilde{B}^\alpha_\mu$ and $\tilde{C}^\mu_\alpha$ of g are both constants independent of positions on manifold $\tilde{M}$, g is called a **classical spacetime flat reference-system**, and $(\tilde{M}, g)$ is called a **flat spacetime**.

(3) If g is not only a classical spacetime orthogonal reference-system but also a classical spacetime flat reference-system, g is called a **classical spacetime inertial reference-system**, or an **inertial-system** for short, and $(\tilde{M}, g)$ is called an **isotropic and flat spacetime**.

Definition 6.4.2. Let F_g is a reference-system transformation induced by g.

(1) If g is a classical spacetime orthogonal reference-system, F_g is called a **classical spacetime orthogonal transformation**.

(2) If g is a classical spacetime flat reference-system, F_g is called a **classical spacetime flat transformation**.

(3) If g is a classical inertial reference-system, F_g is called a **classical spacetime inertial transformation**, or an **inertial transformation** for short, or called a **Lorentz transformation**.

Remark 6.4.1. Reviewing section 2.2.2.3 and section 2.2.3.1 , a property which remains unchanged during a Lorentz transformation is not only a classical spacetime Riemannian geometric property but also a kernal geometric property. It can be called a **Minkowski geometric property**.

Discussion 6.4.1. The cotangent vector field form of the evolution of an inertial-system g is

$$\begin{cases} d\tilde{\zeta}^s = \tilde{B}^s_0 d\tilde{x}^0 + \tilde{B}^s_i d\tilde{x}^i \simeq \tilde{B}^s_\tau d\tau \\ d\tilde{\zeta}^0 = \tilde{B}^0_0 d\tilde{x}^0 + \tilde{B}^0_i d\tilde{x}^i \simeq \tilde{B}^0_\tau d\tau \end{cases}, \quad \begin{cases} d\tilde{x}^i = \tilde{C}^i_0 d\tilde{\zeta}^0 + \tilde{C}^i_s d\tilde{\zeta}^s \simeq \tilde{C}^i_\tau d\tau \\ d\tilde{x}^0 = \tilde{C}^0_0 d\tilde{\zeta}^0 + \tilde{C}^0_s d\tilde{\zeta}^s \simeq \tilde{C}^0_\tau d\tau \end{cases}.$$

Denote

$$\tilde{v}^\alpha \triangleq \tilde{\delta}^\alpha_\tau = \frac{d\tilde{\zeta}^\alpha}{d\tau}, \quad \tilde{u}^\mu \triangleq \tilde{\varepsilon}^\mu_\tau = \frac{d\tilde{x}^\mu}{d\tau}.$$

They are called **proper-time velocities**, and obviously satisfy

$$\begin{cases} \tilde{v}^s = \tilde{B}^s_0 \tilde{u}^0 + \tilde{B}^s_i \tilde{u}^i \\ \tilde{v}^0 = \tilde{B}^0_0 \tilde{u}^0 + \tilde{B}^0_i \tilde{u}^i \end{cases}, \quad \begin{cases} \tilde{u}^i = \tilde{C}^i_0 \tilde{v}^0 + \tilde{C}^i_s \tilde{v}^s \\ \tilde{u}^0 = \tilde{C}^0_0 \tilde{v}^0 + \tilde{C}^0_s \tilde{v}^s \end{cases}.$$

Discussion 6.4.2. Consider the evolution of a charge $\tilde{\rho}$ of an arbitrary classical spacetime reference-system f in an inertial-system g.

(1) In the basis coordinate frame $\{\tilde{\zeta}^\alpha\}$ of g, let the actual evolution direction of $\tilde{\rho}$ completely point to the internal space, that is to say, on the evolution path the external metrics satisfy $d\tilde{\zeta}^s = 0$. So $d\tilde{\zeta}^0 = d\tau$, the time $\tilde{\zeta}^0$ is exactly the **proper-time**. Therefore, the proper-time velocities of $\tilde{\rho}$ are $\tilde{v}^s = \frac{d\tilde{\zeta}^s}{d\tau} = 0$ and $\tilde{v}^0 = \frac{d\tilde{\zeta}^0}{d\tau} = 1$.

(2) In the performance coordinate frame $\{\tilde{x}^\mu\}$ of g, on the evolution path the time metric of $\tilde{\rho}$ satisfies $d\tilde{x}^0 = \tilde{C}^0_0 d\tilde{\zeta}^0 = \tilde{C}^0_\tau d\tau$, and $\tilde{x}^0$ is called the **coordinate time** of $\tilde{\rho}$. The proper-time velocities of $\tilde{\rho}$ are $\tilde{u}^s = \tilde{C}^i_0$ and $\tilde{u}^0 = \tilde{C}^0_0 = \tilde{C}^0_\tau$, which are all constants.

That is to say those two coordinate frames have a constant relative velocity with each other, which is exactly the typical intuition of traditional inertial-system.

Thus, the evolution of $\tilde{\rho}$ performs as **relative rest** in the basis coordinate frame $\{\tilde{\zeta}^\alpha\}$ and performs as isotropic uniform linear **relative motion** in the performance coordinate frame $\{\tilde{x}^\mu\}$. The relative rest and the isotropic uniform linear relative motion are uniformly called the **isotropic and flat relative motion**, or the **inertial relative motion**.

Remark 6.4.2. The relative rest is not a stopped evolution, but an evolution in a specific direction.

Remark 6.4.3. On one hand, a non-trivial interaction should be described with non-trivial slack-tight. On the other hand, if the slack-tights are not constants or does not satisfy the orthogonal condition, the evolution will not perform as an isotropic and flat relative motion, at this time it can also be said that some classical spacetime interaction happens.

In a word, the relative motion and the interaction are just two different statements for the same thing. Specially, the trivial relative motion and no interaction are just two different statements for the same thing.

7 Weak and electromagnetic interaction and relative motion

7.1 Weak-electromagnetic reference-system and its general evolution form

Definition 7.1.1. Let $\mathfrak{D} = r + 2$, and on a $\mathfrak{D}$-dimensional smooth manifold M there be a typical gauge field reference-system f defined in Definition 5.2.1 , such that on a neighborhood U of each point p, the coordinate representation of $f(p)$ is

$$\begin{cases} \xi^s = \xi^s(x^i) \\ \xi^a = \xi^a(x^M) \end{cases}, \quad \begin{cases} x^i = x^i(\xi^s) \\ x^m = x^m(\xi^A) \end{cases}; \quad 1 \leqslant s, i \leqslant r \ ; \quad a, m = \mathfrak{D} - 1, \mathfrak{D}.$$

Then f is called a **typical weak-electromagnetic interaction reference-system**, or a **weak-electromagnetic reference-system** for short.

Discussion 7.1.1. The coordinate form of the evolution of weak-electromagnetic reference-system is

$$\begin{cases} \xi^s = \xi^s(x^i) = \xi^s(x^0) \\ \xi^a = \xi^a(x^M) = \xi^a(x^0) \ , \\ \xi^0 = \xi^0(x^0) \end{cases} \quad \begin{cases} x^i = x^i(\xi^s) = x^i(\xi^0) \\ x^m = x^m(\xi^A) = x^m(\xi^0) \ , \\ x^0 = x^0(\xi^0) \end{cases}$$

and the basis vector form is

$$\begin{cases} d\xi^s = B^s_i dx^i \simeq B^s_0 dx^0 \\ d\xi^a = B^a_M dx^M \simeq B^a_0 dx^0 \\ C^M_0 \dfrac{\partial}{\partial x^M} \cong C^0_0 \dfrac{d}{dx^0} = \dfrac{d}{d\xi^0} \end{cases}, \quad \begin{cases} dx^i = C^i_s d\xi^s \simeq C^i_0 d\xi^0 \\ dx^m = C^m_A d\xi^A \simeq C^m_0 d\xi^0 \\ B^A_0 \dfrac{\partial}{\partial \xi^A} \cong B^0_0 \dfrac{d}{d\xi^0} = \dfrac{d}{dx^0} \end{cases},$$

where

$$B^s_m = 0, \quad C^i_a = 0.$$

For the sake of simplicity, the effects of gravitational field should be excluded. So let f be externally flat and internally standard. According to Definition 5.2.2 , it is required that

$$B^s_i = \delta^s_i, \quad B^a_i = 0, \quad C^i_s = \delta^i_s, \quad C^m_s = 0.$$

$$G_{mn} = 0 (m \neq n), \quad G_{mn} = const.$$

Definition 7.1.2. For convenience, indices should be specified first of all. Based on Definition 6.1.1.1 , if not specified in other sections, the values of internal indices are as following.

The internal indices are $a, b, c, d, e = \mathfrak{D} - 1, \mathfrak{D}$ in coordinate frame ξ and $m, n, p, q, r = \mathfrak{D} - 1, \mathfrak{D}$ in coordinate frame x.

Discussion 7.1.2. Calculate the metric tensor of weak-electromagnetic reference-system f.

$$G_{MN} \triangleq \delta_{AB} B^A_M B^B_N = \delta_{st} B^s_M B^t_N + \delta_{ab} B^a_M B^b_N \Rightarrow \begin{cases} G_{ij} = \delta_{st} B^s_i B^t_j + \delta_{ab} B^a_i B^b_j = \delta_{st} \delta^s_i \delta^t_j = \delta_{ij} \\ G_{in} = \delta_{st} B^s_i B^t_n + \delta_{ab} B^a_i B^b_n = 0 \\ G_{mj} = \delta_{st} B^s_m B^t_j + \delta_{ab} B^a_m B^b_j = 0 \\ G_{mn} = \delta_{st} B^s_m B^t_n + \delta_{ab} B^a_m B^b_n = B^{\mathfrak{D}-1}_m B^{\mathfrak{D}-1}_n + B^{\mathfrak{D}}_m B^{\mathfrak{D}}_n \end{cases},$$

$$G^{MN} = \delta^{AB} C^M_A C^N_B = \delta^{st} C^M_s C^N_t + \delta^{ab} C^M_a C^N_b \Rightarrow \begin{cases} G^{ij} = \delta^{st} C^i_s C^j_t = \delta^{st} \delta^i_s \delta^j_t = \delta^{ij} \\ G^{in} = \delta^{st} C^i_s C^n_t = 0 \\ G^{mj} = \delta^{st} C^m_s C^j_t = 0 \\ G^{mn} = \delta^{st} C^m_s C^n_t + \delta^{ab} C^m_a C^n_b = C^m_{\mathfrak{D}-1} C^n_{\mathfrak{D}-1} + C^m_{\mathfrak{D}} C^n_{\mathfrak{D}} \end{cases}.$$

Discussion 7.1.3. Calculate the connection of weak-electromagnetic reference-system f.

For the simple connection defined in Definition 2.2.9.2 , it is obtained that

$$\varLambda^M_{NP} \triangleq \frac{1}{2} C^M_A \left(\frac{\partial B^A_N}{\partial x^P} + \frac{\partial B^A_P}{\partial x^N}\right) = 0 + \frac{1}{2} C^M_a \left(\frac{\partial B^a_N}{\partial x^P} + \frac{\partial B^a_P}{\partial x^N}\right) \Rightarrow \begin{cases} \varLambda^i_{NP} = 0 \\ \varLambda^m_{jk} = 0 \\ \varLambda^m_{nP} = \dfrac{1}{2} C^m_a \left(\dfrac{\partial B^a_n}{\partial x^P} + \dfrac{\partial B^a_P}{\partial x^n}\right) \\ \varLambda^m_{Np} = \dfrac{1}{2} C^m_a \left(\dfrac{\partial B^a_N}{\partial x^p} + \dfrac{\partial B^a_p}{\partial x^N}\right) \end{cases}.$$

$$\Lambda_{MNP}\triangleq G_{MM'}\Lambda^{M'}_{NP}\Rightarrow\begin{cases}\Lambda_{iNP}=G_{iM'}\Lambda^{M'}_{NP}=G_{ii'}\Lambda^{i'}_{NP}=0\\ \Lambda_{mjk}=G_{mM'}\Lambda^{M'}_{jk}=G_{mm'}\Lambda^{m'}_{jk}=0\\ \Lambda_{mnP}=\frac{1}{2}\delta_{AB}B^B_m\left(\frac{\partial B^A_n}{\partial x^P}+\frac{\partial B^A_P}{\partial x^n}\right)=\frac{1}{2}\delta_{ab}B^b_m\left(\frac{\partial B^a_n}{\partial x^P}+\frac{\partial B^a_P}{\partial x^n}\right)\\ \Lambda_{mNp}=\frac{1}{2}\delta_{AB}B^B_m\left(\frac{\partial B^A_N}{\partial x^p}+\frac{\partial B^A_p}{\partial x^N}\right)=\frac{1}{2}\delta_{ab}B^b_m\left(\frac{\partial B^a_N}{\partial x^p}+\frac{\partial B^a_p}{\partial x^N}\right)\end{cases}.$$

Definition 7.1.3. The affine connection components Λ_{mnP}, i.e. $\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}$, $\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}$, $\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}$, $\Lambda_{\mathfrak{D}\mathfrak{D}P}$, are called the **general original gauge potential** of the weak-electromagnetic reference-system f.

Discussion 7.1.4. Calculate the curvature coefficients of f.

$$K^m_{nPQ}\triangleq\frac{\partial\Lambda^m_{nQ}}{\partial x^P}-\frac{\partial\Lambda^m_{nP}}{\partial x^Q}+\Lambda^m_{HP}\Lambda^H_{nQ}-\Lambda^H_{nP}\Lambda^m_{HQ}$$

$$\Rightarrow\begin{cases}K^{\mathfrak{D}-1}_{(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}}{\partial x^Q}+\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)Q}-\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}Q}\\ K^{\mathfrak{D}-1}_{\mathfrak{D}PQ}=\frac{\partial\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P}}{\partial x^Q}+\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}Q}+\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}}_{\mathfrak{D}Q}-\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)Q}-\Lambda^{\mathfrak{D}}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}Q}\\ K^{\mathfrak{D}}_{(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}}{\partial x^Q}+\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)Q}+\Lambda^{\mathfrak{D}}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)Q}-\Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)Q}-\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}}_{\mathfrak{D}Q}\\ K^{\mathfrak{D}}_{\mathfrak{D}PQ}=\frac{\partial\Lambda^{\mathfrak{D}}_{\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda^{\mathfrak{D}}_{\mathfrak{D}P}}{\partial x^Q}+\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}Q}-\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P}\Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)Q}\end{cases}.$$

$$K_{mnPQ}\triangleq G_{mM'}K^{M'}_{nPQ}=G_{mm'}K^{m'}_{nPQ}=\frac{\partial\Lambda_{mnQ}}{\partial x^P}-\frac{\partial\Lambda_{mnP}}{\partial x^Q}+\Lambda_{mHP}\Lambda^H_{nQ}-\Lambda^H_{nP}\Lambda_{mHQ}$$
$$=\frac{\partial\Lambda_{mnQ}}{\partial x^P}-\frac{\partial\Lambda_{mnP}}{\partial x^Q}+G^{HG}\Lambda_{mHP}\Lambda_{GnQ}-G^{HG}\Lambda_{GnP}\Lambda_{mHQ}$$
$$=\frac{\partial\Lambda_{mnQ}}{\partial x^P}-\frac{\partial\Lambda_{mnP}}{\partial x^Q}+G^{hg}\left(\Lambda_{mhP}\Lambda_{gnQ}-\Lambda_{gnP}\Lambda_{mhQ}\right)$$

$$\Rightarrow\begin{cases}K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q}+G^{hg}\left(\Lambda_{(\mathfrak{D}-1)hP}\Lambda_{g(\mathfrak{D}-1)Q}-\Lambda_{g(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)hQ}\right)\\ K_{\mathfrak{D}(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}}{\partial x^Q}+G^{hg}\left(\Lambda_{\mathfrak{D}hP}\Lambda_{g(\mathfrak{D}-1)Q}-\Lambda_{g(\mathfrak{D}-1)P}\Lambda_{\mathfrak{D}hQ}\right)\\ K_{(\mathfrak{D}-1)\mathfrak{D}PQ}=\frac{\partial\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}}{\partial x^Q}+G^{hg}\left(\Lambda_{(\mathfrak{D}-1)hP}\Lambda_{g\mathfrak{D}Q}-\Lambda_{g\mathfrak{D}P}\Lambda_{(\mathfrak{D}-1)hQ}\right)\\ K_{\mathfrak{D}\mathfrak{D}PQ}=\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q}+G^{hg}\left(\Lambda_{\mathfrak{D}hP}\Lambda_{g\mathfrak{D}Q}-\Lambda_{g\mathfrak{D}P}\Lambda_{\mathfrak{D}hQ}\right)\end{cases}$$

$$\Rightarrow\begin{cases}K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q}+G^{\mathfrak{D}\mathfrak{D}}\left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}-\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}\right)\\ K_{\mathfrak{D}(\mathfrak{D}-1)PQ}=\frac{\partial\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}}{\partial x^Q}+G^{\mathfrak{D}\mathfrak{D}}\left(\Lambda_{\mathfrak{D}\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}-\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{\mathfrak{D}\mathfrak{D}Q}\right)\\ \qquad+G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}-\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}\right)\\ K_{(\mathfrak{D}-1)\mathfrak{D}PQ}=\frac{\partial\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}}{\partial x^Q}+G^{\mathfrak{D}\mathfrak{D}}\left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}\mathfrak{D}Q}-\Lambda_{\mathfrak{D}\mathfrak{D}P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}\right)\\ \qquad+G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}-\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}\right)\\ K_{\mathfrak{D}\mathfrak{D}PQ}=\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q}+G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}-\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}\right)\end{cases}.$$

Definition 7.1.4. In order to conveniently compare with the traditional Glashow-Weinberg-Salam theory, define

$$\begin{cases} B_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda^{\mathfrak{D}}_{\mathfrak{D}P} + \Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\right) \\ A^3_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda^{\mathfrak{D}}_{\mathfrak{D}P} - \Lambda^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\right) \end{cases}, \quad \begin{cases} A^1_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P} + \Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) \\ A^2_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda^{\mathfrak{D}-1}_{\mathfrak{D}P} - \Lambda^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) \end{cases}.$$

And define

$$\begin{cases} B_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}}_{\mathfrak{D}PQ} + K^{\mathfrak{D}-1}_{(\mathfrak{D}-1)PQ}\right) \\ F^3_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}}_{\mathfrak{D}PQ} - K^{\mathfrak{D}-1}_{(\mathfrak{D}-1)PQ}\right) \end{cases}, \quad \begin{cases} F^1_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}-1}_{\mathfrak{D}PQ} + K^{\mathfrak{D}}_{(\mathfrak{D}-1)PQ}\right) \\ F^2_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}-1}_{\mathfrak{D}PQ} - K^{\mathfrak{D}}_{(\mathfrak{D}-1)PQ}\right) \end{cases}.$$

Discussion 7.1.5. By direct calculation we obtain

$$\begin{cases} B_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}}_{\mathfrak{D}PQ} + K^{\mathfrak{D}-1}_{(\mathfrak{D}-1)PQ}\right) = \dfrac{\partial B_Q}{\partial x^P} - \dfrac{\partial B_P}{\partial x^Q} \\ F^3_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}}_{\mathfrak{D}PQ} - K^{\mathfrak{D}-1}_{(\mathfrak{D}-1)PQ}\right) = \dfrac{\partial A^3_Q}{\partial x^P} - \dfrac{\partial A^3_P}{\partial x^Q} + \sqrt{2}\left(A^1_P A^2_Q - A^2_P A^1_Q\right) \\ F^1_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}-1}_{\mathfrak{D}PQ} + K^{\mathfrak{D}}_{(\mathfrak{D}-1)PQ}\right) = \dfrac{\partial A^1_Q}{\partial x^P} - \dfrac{\partial A^1_P}{\partial x^Q} + \sqrt{2}\left(A^2_P A^3_Q - A^3_P A^2_Q\right) \\ F^2_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K^{\mathfrak{D}-1}_{\mathfrak{D}PQ} - K^{\mathfrak{D}}_{(\mathfrak{D}-1)PQ}\right) = \dfrac{\partial A^2_Q}{\partial x^P} - \dfrac{\partial A^2_P}{\partial x^Q} - \sqrt{2}\left(A^3_P A^1_Q - A^1_P A^3_Q\right) \end{cases}.$$

Noticed that there is a difference of positive and negative sign between F^2_{PQ} here and that in Glashow-Weinberg-Salam theory. Fortunately this difference is not very important, which does not hinder further discussions.

And noticed that there is a difference of coupling constant between the $F^1_{PQ}, F^2_{PQ}, F^3_{PQ}$ above and those in Glashow-Weinberg-Salam theory. The results of Discussion 7.1.4 indicate that the coupling constant is determined by G^{hg}. Concretely, consider a condition $G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}$, denote $g \triangleq \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2}$, and replace Definition 7.1.4 with

$$\begin{cases} B_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda_{\mathfrak{D}\mathfrak{D}P} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \\ A^3_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda_{\mathfrak{D}\mathfrak{D}P} - \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \end{cases}, \quad \begin{cases} A^1_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} + \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \\ A^2_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P} - \Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \end{cases},$$

$$\begin{cases} B_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right) \\ F^3_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} - K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right) \end{cases}, \quad \begin{cases} F^1_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K_{(\mathfrak{D}-1)\mathfrak{D}PQ} + K_{\mathfrak{D}(\mathfrak{D}-1)PQ}\right) \\ F^2_{PQ} \triangleq \dfrac{1}{\sqrt{2}}\left(K_{(\mathfrak{D}-1)\mathfrak{D}PQ} - K_{\mathfrak{D}(\mathfrak{D}-1)PQ}\right) \end{cases}.$$

Thus we obtain

$$\begin{aligned} B_{PQ} &\triangleq \frac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ} + K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right) = \frac{1}{\sqrt{2}}\frac{\partial\left(\Lambda_{\mathfrak{D}\mathfrak{D}Q} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}\right)}{\partial x^P} - \frac{1}{\sqrt{2}}\frac{\partial\left(\Lambda_{\mathfrak{D}\mathfrak{D}P} + \Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right)}{\partial x^Q} \\ &= \frac{\partial B_Q}{\partial x^P} - \frac{\partial B_P}{\partial x^Q}. \end{aligned}$$

$$
\begin{aligned}
F_{PQ}^3 &\triangleq \frac{1}{\sqrt{2}}\left(K_{\mathfrak{D}\mathfrak{D}PQ}-K_{(\mathfrak{D}-1)(\mathfrak{D}-1)PQ}\right)\\
&=\frac{1}{\sqrt{2}}\left(\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}Q}}{\partial x^P}-\frac{\partial\Lambda_{\mathfrak{D}\mathfrak{D}P}}{\partial x^Q}+G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\left(\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}-\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}\right)\right)\\
&-\frac{1}{\sqrt{2}}\left(\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)Q}}{\partial x^P}-\frac{\partial\Lambda_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}}{\partial x^Q}+G^{\mathfrak{D}\mathfrak{D}}\left(\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}-\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}\right)\right)\\
&=\frac{\partial A_Q^3}{\partial x^P}-\frac{\partial A_P^3}{\partial x^Q}+g\left(\Lambda_{\mathfrak{D}(\mathfrak{D}-1)P}\Lambda_{(\mathfrak{D}-1)\mathfrak{D}Q}-\Lambda_{(\mathfrak{D}-1)\mathfrak{D}P}\Lambda_{\mathfrak{D}(\mathfrak{D}-1)Q}\right)\\
&=\frac{\partial A_Q^3}{\partial x^P}-\frac{\partial A_P^3}{\partial x^Q}+g\left(A_P^1A_Q^2-A_P^2A_Q^1\right).
\end{aligned}
$$

Similarly we also obtain

$$
\begin{aligned}
F_{PQ}^1&=\frac{\partial A_Q^1}{\partial x^P}-\frac{\partial A_P^1}{\partial x^Q}+g\left(A_P^2A_Q^3-A_P^3A_Q^2\right),\\
F_{PQ}^2&=\frac{\partial A_Q^2}{\partial x^P}-\frac{\partial A_P^2}{\partial x^Q}-g\left(A_P^3A_Q^1-A_P^1A_Q^3\right).
\end{aligned}
$$

If considering in the way of traditional theory, substitute them into $-\frac{1}{4}B_{PQ}B^{PQ}-\frac{1}{4}\sum_{k=1}^{3}F_{PQ}^kF^{kPQ}$. By mixing them with Weinberg angle θ_W, then define the mixture of gauge potentials

$$
\begin{cases}
Z_P\triangleq -B_P\sin\theta_W+A_P^3\cos\theta_W\\
A_P\triangleq B_P\cos\theta_W+A_P^3\sin\theta_W
\end{cases},\quad
\begin{cases}
W_P^+\triangleq \frac{1}{\sqrt{2}}\left(A_P^1-iA_P^2\right)\\
W_P^-\triangleq \frac{1}{\sqrt{2}}\left(A_P^1+iA_P^2\right)
\end{cases}.
$$

Thus, we can obtain the Lagrangian of Glashow-Weinberg-Salam theory in the degree of just some differences of positive and negative signs. The mass term will be explained later.

In this sense, beyond doubt it is feasible and effective for the weak-electromagnetic reference-system f to describe the weak and electromagnetic interactions. More significantly, the gauge potentials are no longer connections abstractly defined in traditional theory, but affine connections concretely constructed by the slack-tight on geometric manifold, therefore they are intrinsic geometric properties now. The coupling constant is also no longer introduced artificially like that in traditional theory, but a natural reflection of intrinsic geometric property.

7.2 Actual evolution of weak-electromagnetic charges

Discussion 7.2.1. Now calculate the evolution forms of charges $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}\mathfrak{D}}$, $\rho_{\mathfrak{D}(\mathfrak{D}-1)}$, $\rho_{(\mathfrak{D}-1)\mathfrak{D}}$ of f evolving in g, where f and g are two weak-electromagnetic reference-systems.

First, generally, $\rho_{mn;P}=\partial_P\rho_{mn}-\rho_{Hn}\Gamma_{mP}^H-\rho_{mH}\Gamma_{nP}^H=\partial_P\rho_{mn}-\rho_{hn}\Gamma_{mP}^h-\rho_{mh}\Gamma_{nP}^h$. Concretely, by calculation we obtain

$$
\begin{cases}
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P}=\partial_P\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}-2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1}-\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)\mathfrak{D}}\right)\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}\\
\rho_{\mathfrak{D}\mathfrak{D};P}=\partial_P\rho_{\mathfrak{D}\mathfrak{D}}-2\rho_{\mathfrak{D}\mathfrak{D}}\Gamma_{\mathfrak{D}P}^{\mathfrak{D}}-\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)}+\rho_{(\mathfrak{D}-1)\mathfrak{D}}\right)\Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1}\\
\rho_{\mathfrak{D}(\mathfrak{D}-1);P}=\partial_P\rho_{\mathfrak{D}(\mathfrak{D}-1)}-\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1}+\rho_{\mathfrak{D}\mathfrak{D}}\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}\right)-\rho_{\mathfrak{D}(\mathfrak{D}-1)}\left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1}+\Gamma_{\mathfrak{D}P}^{\mathfrak{D}}\right)\\
\rho_{(\mathfrak{D}-1)\mathfrak{D};P}=\partial_P\rho_{(\mathfrak{D}-1)\mathfrak{D}}-\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma_{\mathfrak{D}P}^{\mathfrak{D}-1}+\rho_{\mathfrak{D}\mathfrak{D}}\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}}\right)-\rho_{(\mathfrak{D}-1)\mathfrak{D}}\left(\Gamma_{(\mathfrak{D}-1)P}^{\mathfrak{D}-1}+\Gamma_{\mathfrak{D}P}^{\mathfrak{D}}\right)
\end{cases}.
$$

Definition 7.2.1. Denote $l \triangleq (\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)},\ \rho_{\mathfrak{D}\mathfrak{D}})$, which is called a **charged lepton**. Denote $\nu \triangleq (\rho_{\mathfrak{D}(\mathfrak{D}-1)},\ \rho_{(\mathfrak{D}-1)\mathfrak{D}})$, which is called a **neutrino**. l and ν are uniformly called **leptons**, denoted by L.

$L\frac{1}{\sqrt{2}}\binom{1}{1}$ is called a **left-handed lepton**, and $L\frac{1}{\sqrt{2}}\binom{1}{-1}$ is called a **right-handed lepton**

$L\frac{1}{\sqrt{2}}\binom{-1}{-1}$ is called a **right-handed anti-lepton**, and $L\frac{1}{\sqrt{2}}\binom{-1}{1}$ is called a **left-handed anti-lepton**.

Thus, left-handed and right-handed charged leptons are

$$l_L \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right), \quad l_R \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right).$$

Left-handed and right-handed neutrinos are

$$\nu_L \triangleq \frac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right), \quad \nu_R \triangleq \frac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right).$$

l_L and l_R are uniformly called the **implicit polarity representation of charged leptons**.

Remark 7.2.1. The reason for defining chirality in this way is that in traditional theory the essence of helicity is a concept reflecting the relative relation of phases of components. It is true for both the polarization of electromagnetic wave and the spin polarization of lepton. Different relative relation modes of phases of components represent different motion states. The existence of different components is the premise of the existence of such a degree of freedom of motion.

Take the charged lepton $l \triangleq \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)},\ \rho_{\mathfrak{D}\mathfrak{D}}\right)$ for example. Denote $\angle\theta \triangleq (\cos\theta,\ \sin\theta)$. With two independent components, l can make various orthogonal decompositions.

(1) If the orthogonal basis is chosen to be $\angle\frac{\pi}{2}$ and $\angle 0$, then

$$l = \rho_{\mathfrak{D}\mathfrak{D}}\angle\frac{\pi}{2} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\angle 0.$$

(2) If the orthogonal basis is chosen to be $\angle\left(+\frac{\pi}{4}\right)$ and $\angle\left(-\frac{\pi}{4}\right)$, then it is regarded as the orthogonal decomposition reflecting two opposite helicity directions that

$$l = l_L\angle\left(+\frac{\pi}{4}\right) + l_R\angle\left(-\frac{\pi}{4}\right),$$

where

$$l_L \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right), \quad l_R \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right).$$

The concept of chirality in Definition 7.2.1 exactly comes from such a consideration.

Although here is no complex-valued operator analysis like the traditional quantum mechanics, actually a similar effect appears. The above orthogonal basis $\angle\left(+\frac{\pi}{4}\right)$ and $\angle\left(-\frac{\pi}{4}\right)$ correspond to the eigenvectors of helicity operator of traditional theory, and the above parameters $+\frac{\pi}{4}$ and $-\frac{\pi}{4}$ of orthogonal basis correspond to the eigenvalues of helicity operator. Anyway there must be a way to describe the degree of freedom of helicity for each theoretical form.

Except this, it also can be seen from the evolution forms of leptons deduced later that such a definition of chirality is reasonable.

Definition 7.2.2. Denote

$$\begin{cases} W_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} + \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \\ W_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} - \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \end{cases}, \quad \begin{cases} W_P^+ \triangleq \frac{1}{\sqrt{2}}\left(W_P^1 - iW_P^2\right) \\ W_P^- \triangleq \frac{1}{\sqrt{2}}\left(W_P^1 + iW_P^2\right) \end{cases},$$

which are uniformly called the **W potential of weak interaction**, and the former is called the **implicit polarity representation of W potential**, and the latter is called the **explicit polarity representation of W potential**.

Discussion 7.2.2. According to Definition 7.2.2 , there are

$$\begin{cases} W_P^1 = \frac{1}{\sqrt{2}}\left(W_P^+ + W_P^-\right) \\ W_P^2 = \frac{i}{\sqrt{2}}\left(W_P^+ - W_P^-\right) \end{cases}, \quad \begin{cases} \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \frac{1}{\sqrt{2}}\left(W_P^1 + W_P^2\right) \\ \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} = \frac{1}{\sqrt{2}}\left(W_P^1 - W_P^2\right) \end{cases}.$$

Moreover,

$$\begin{cases} \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(W_P^+ + W_P^-\right) + \frac{i}{\sqrt{2}}\left(W_P^+ - W_P^-\right)\right) \\ \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(W_P^+ + W_P^-\right) - \frac{i}{\sqrt{2}}\left(W_P^+ - W_P^-\right)\right) \end{cases}$$
$$\Rightarrow \begin{cases} \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \frac{1+i}{2}W_P^+ + \frac{1-i}{2}W_P^- \\ \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} = \frac{1-i}{2}W_P^+ + \frac{1+i}{2}W_P^- \end{cases},$$

$$\begin{cases} W_P^+ = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) - i\frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right)\right) \\ W_P^- = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) + i\frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right)\right) \end{cases}$$

$$\Rightarrow \begin{cases} W_P^+ = \frac{1-i}{2}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \frac{1+i}{2}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} \\ W_P^- = \frac{1+i}{2}\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \frac{1-i}{2}\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} \end{cases}.$$

Substitute them into the arranged result of Discussion 7.2.1 , it is obtained that

$$\begin{cases} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P} = \partial_P\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - 2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right)\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \\ \rho_{\mathfrak{D}\mathfrak{D};P} = \partial_P\rho_{\mathfrak{D}\mathfrak{D}} - 2\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right)\Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} \\ \rho_{\mathfrak{D}(\mathfrak{D}-1);P} = \partial_P\rho_{\mathfrak{D}(\mathfrak{D}-1)} - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P} = \partial_P\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) - \rho_{(\mathfrak{D}-1)\mathfrak{D}}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P} = \partial_P\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - 2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} - \frac{g}{\sqrt{2}}\nu_L\left(W_P^1 - W_P^2\right) \\ \rho_{\mathfrak{D}\mathfrak{D};P} = \partial_P\rho_{\mathfrak{D}\mathfrak{D}} - 2\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} - \frac{g}{\sqrt{2}}\nu_L\left(W_P^1 + W_P^2\right) \\ \rho_{\mathfrak{D}(\mathfrak{D}-1);P} = \partial_P\rho_{\mathfrak{D}(\mathfrak{D}-1)} - \frac{1}{\sqrt{2}}\left(\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right)\frac{g}{\sqrt{2}}W_P^1 + \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right)\frac{g}{\sqrt{2}}W_P^2\right) \\ \qquad - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P} = \partial_P\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \frac{1}{\sqrt{2}}\left(\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right)\frac{g}{\sqrt{2}}W_P^1 + \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right)\frac{g}{\sqrt{2}}W_P^2\right) \\ \qquad - \rho_{(\mathfrak{D}-1)\mathfrak{D}}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right) \end{cases}$$

$$\Rightarrow\begin{cases}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P}=\partial_P\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}-2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}-\dfrac{g}{\sqrt{2}}\nu_L\left(W^1_P-W^2_P\right)\\ \rho_{\mathfrak{D}\mathfrak{D};P}=\partial_P\rho_{\mathfrak{D}\mathfrak{D}}-2\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}-\dfrac{g}{\sqrt{2}}\nu_L\left(W^1_P+W^2_P\right)\\ \rho_{\mathfrak{D}(\mathfrak{D}-1);P}=\partial_P\rho_{\mathfrak{D}(\mathfrak{D}-1)}-\dfrac{g}{2}\left(\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}+\rho_{\mathfrak{D}\mathfrak{D}}\right)W^1_P+\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}-\rho_{\mathfrak{D}\mathfrak{D}}\right)W^2_P\right)\\ \qquad\qquad\quad-\rho_{\mathfrak{D}(\mathfrak{D}-1)}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P}=\partial_P\rho_{(\mathfrak{D}-1)\mathfrak{D}}-\dfrac{g}{2}\left(\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}+\rho_{\mathfrak{D}\mathfrak{D}}\right)W^1_P+\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}-\rho_{\mathfrak{D}\mathfrak{D}}\right)W^2_P\right)\\ \qquad\qquad\quad-\rho_{(\mathfrak{D}-1)\mathfrak{D}}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\end{cases}$$

$$\Rightarrow\begin{cases}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P}=\partial_P\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}-2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}-\dfrac{g}{\sqrt{2}}\nu_L\left(W^1_P-W^2_P\right)\\ \rho_{\mathfrak{D}\mathfrak{D};P}=\partial_P\rho_{\mathfrak{D}\mathfrak{D}}-2\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}-\dfrac{g}{\sqrt{2}}\nu_L\left(W^1_P+W^2_P\right)\\ \rho_{\mathfrak{D}(\mathfrak{D}-1);P}=\partial_P\rho_{\mathfrak{D}(\mathfrak{D}-1)}-\dfrac{g}{\sqrt{2}}l_LW^1_P-\dfrac{g}{\sqrt{2}}l_RW^2_P-\rho_{\mathfrak{D}(\mathfrak{D}-1)}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P}=\partial_P\rho_{(\mathfrak{D}-1)\mathfrak{D}}-\dfrac{g}{\sqrt{2}}l_LW^1_P-\dfrac{g}{\sqrt{2}}l_RW^2_P-\rho_{(\mathfrak{D}-1)\mathfrak{D}}\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\end{cases}$$

$$\Rightarrow\begin{cases}\begin{cases}l_{L;P}=\partial_Pl_L-\sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}-\sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}-g\nu_LW^1_P\\ l_{R;P}=\partial_Pl_R-\sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}+g\nu_LW^2_P\end{cases}\\ \begin{cases}\nu_{L;P}=\partial_P\nu_L-gl_LW^1_P-gl_RW^2_P-\nu_L\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\\ \nu_{R;P}=\partial_P\nu_R-\nu_R\left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}+\Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right)\end{cases}\end{cases}.$$

The above result has already shown an obvious evidence of asymmetric chirality appearing automatically. But that is not all. In order to obtain further results, the geometric essence of symmetry breaking has to be clarified first of all.

Remark 7.2.2. Generally, the essence of symmetry breaking is that the geometry is too small and its ability of describing shapes is not strong enough, so the geometry needs to be made larger to enhance the alility of describing shapes. Concretely it can be illustrated in the following way.

Let R and S be two subgroups of the general linear group. According to Definition 2.2.2.4 , let f on M be a reference-system generated by R.

(1) If S is a subgroup of R, according to Remark .1, geometry $\mathcal{M}/S$ is larger than geometry $\mathcal{M}/R$. $\mathcal{M}/S$ has a richer description of geometric shapes than $\mathcal{M}/R$. The geometric shapes of f generated by R of course can also be described by $\mathcal{M}/S$. In other words, all of those properties of f remaining unchanged during transformations of R will surely also remain unchanged during transformations of S. At this time we usually say f has a symmetry of S group, or say $\mathcal{M}/S$ has the ability of describing all the details of the shape of f.

(2) If S is not a subgroup of R, S may not have a richer description of geometric shapes than $\mathcal{M}/R$. So we cannot promise all of those properties of f remaining unchanged during transformations of R still remain unchanged during transformations of S. There must exist some geometric shapes of f, we cannot find any geometric property of $\mathcal{M}/S$ to be able to describe them. From the point of view of S, they are so irregular that any reference-system transformation generated by S cannot eliminate them anyway. At this time we

usually say the symmetry of group S is broken, actually it is because S is not small enough or say $\mathcal{M}/S$ is not larger enough and $\mathcal{M}/S$ has no enough ability to describe those shapes that seem irregular to $\mathcal{M}/S$.

In a word, it is more intuitive to say the geometric shape cannot be completely described by $\mathcal{M}/S$, or $\mathcal{M}/S$ has no enough alility to clarify all the details of the shape of f, than to say f has a breaking symmetry of S.

Then, which kind of geometry on geometric manifold has the strongest ability of describing shapes? The answer is instrinsic geometry. According to section 2.2.5.2 , the intrinsic geometry is the largest geometry on geometric manifold. It just requires the smallest symmetry. Its symmetry group is the subgroup $\{e\}$ consisting of just one element which is the unit element e of the general linear group $GL(\mathfrak{D}, \mathbb{R})$. It represents the most trivial symmetry. In other words, the symmetry of intrinsic geometry will never be broken, because it is too small to continue to break. Therefore, each geometry on geometric manifold is a subgeometry of the intrinsic geometry. And the intrinsic geometry is defined by the slack-tight of reference-system, so no matter how irregular the shape is, it can be exactly described by the slack-tight. Other kinds of geometries can be regarded as those obtained from the restrictions of the slack-tight by some symmetry conditions, just like what section 2.2.2.3 and section 2.2.3.1 do.

Now we have to know:

(1) The traditional theory starts from a very large symmetry group, and reduces symmetries in the way of some kind of breaking to approach the target geometry.

(2) The theory of this paper starts from the smallest symmetry group $\{e\}$, and adds symmetries in the way of some kind of symmetry conditions to approach the target geometry.

These two ways must lead to the same destination. They both go towards the same specific geometry.

It is remarkable that not all the transformation groups of the geometries determined by symmetry conditions used to restrict slack-tight are simple and easy to describe. It is nice that if the transformation group is too complicated, we can choose not to describe its structure but directly describe the geometry in the way of adding symmetry conditions restricting the slack-tight. When discussing in this way, it does not matter what the group looks like.

Thus it can be seen that for the general matter-motion, it is better to focus on geometry than to focus on symmetry group. And it is more convenient to study how to add symmetry conditions to intrinsic geometry than to study how to break a symmetry group.

Discussion 7.2.3. Return to the evolution of charges. The evolution form

$$\begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} - \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P} - g\nu_L W^1_P \\ l_{R;P} = \partial_P l_R - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P} + g\nu_L W^2_P \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g l_L W^1_P - g l_R W^2_P - \nu_L \left(\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ \nu_{R;P} = \partial_P \nu_R - \nu_R \left(\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \end{cases} \end{cases}$$

is deduced in the sense of the most general intrinsic geometry, and it is determined by the slack-tight of reference-system. During the process of deduction, except externally flat condition, internally standard condition and $G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}$, the slack-tights have not been restricted by any other conditions.

Now we have to introduce a more symmetry to the shape of geometric manifold (M, g). Concretely, if adding a symmetry condition $\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} = \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}$ to reference-system g, then $W_P^2 = 0$, and the above result becomes

$$\begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} - \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P} - g\nu_L W_P^1 \\ l_{R;P} = \partial_P l_R - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P} \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - \nu_L\left(\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) - g l_L W_P^1 \\ \nu_{R;P} = \partial_P \nu_R - \nu_R\left(\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \end{cases} \end{cases}.$$

Definition 7.2.3. An externally flat and internally standard weak-electromagnetic reference-system which satisfies $\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} = \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}$ and $G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}$ is called a **typical weak-electromagnetic gauge field**, or **weak-electromagnetic field** for short.

Remark 7.2.3. Noticed that on the symmetry condition $\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} = \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}$, $W_P^2 = 0$. The W potential defined in Definition 7.2.2 satisfies $W_P^+ = W_P^-$. Superficially, it seems that W potential loses a degree of freedom. However, this lost degree of freedom is actually insignificant.

(1) In traditional theory, the W potential can also be written as $W_P^{\pm} = W_P e^{\mp i\frac{\pi}{4}}$. It is one real function W_P rather than two real functions W_P^1 and W_P^2 that effects the interaction. The complex exponent just has an effect on passively marking the polarity, and the dynamic effect of polarity is completely determined by the charge, rather than by the complex exponent $e^{\mp i\frac{\pi}{4}}$. So the positive and negative signs of the complex exponent indeed just has an effect on passively marking polarity, and this mark may or may not be needed.

(2) In traditional theory, the propagators of W_P^+ and W_P^- are completely the same, which are independent of $\tan^{-1}\frac{W_P^2}{W_P^1}$. The relative relation between W_P^1 and W_P^2 cannot bring any degree of distinguishment between the propagators of W_P^+ and W_P^-.

In summary, even in traditional theory, it is enough for W_P^+ and W_P^- to be described by just one degree of freedom, and the other degree of freedom is redundant essentially. Therefore, on the symmetry condition $\mathit{\Gamma}_{(\mathfrak{D}-1)\mathfrak{D}P} = \mathit{\Gamma}_{\mathfrak{D}(\mathfrak{D}-1)P}$, it is reasonable to cognize the properties of W field with just one real function W_P^1.

Discussion 7.2.4. Reviewing Remark 7.1.5 and Definition 7.1.4 , there the electromagnetic potential and Z potential are defined in the way of Glashow-Weinberg-Salam theory, that is

$$\begin{cases} Z_P \triangleq -B_P \sin\theta_W + A_P^3 \cos\theta_W \\ A_P \triangleq B_P \cos\theta_W + A_P^3 \sin\theta_W \end{cases},$$

and for reference-system g,

$$\begin{cases} B_P \triangleq \dfrac{1}{\sqrt{2}}\left(\mathit{\Gamma}_{\mathfrak{D}\mathfrak{D}P} + \mathit{\Gamma}_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \\ A_P^3 \triangleq \dfrac{1}{\sqrt{2}}\left(\mathit{\Gamma}_{\mathfrak{D}\mathfrak{D}P} - \mathit{\Gamma}_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \end{cases}.$$

This definition is very suitable for the traditional theory with Higgs mechanism. However, according to section 7.2.4 , Higgs mechanism is more of a phenomenological theory. In this paper, the way of attributing the rest-mass to matter-motion in internal space is more natural than Higgs mechanism. Higgs field is probably not fundamental and the Higgs mechanism is probably an equivalent theory brought by the group behavior of zero-spin neutrino pairs at a higher level of theory, which is the reason why the electromagnetic coupling constant and weak coupling constant become different. In a word, Higgs field does not necessarily have enough importance at the most basic level.

Thus, at the most basic level, there is no sufficient reason to stick to the expression form of traditional theory with Higgs mechanism, so it is not necessary to stick to the traditional definition of the electromagnetic potential and Z potential. The result of Discussion 7.2.3 implies that the following definition might be reasonable in theory of this paper.

Definition 7.2.4. Define the **Z potential of weak interaction** and the **electomagnetic potential**

$$\begin{cases} Z_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \\ A_P \triangleq \dfrac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \end{cases},$$

correspondingly,

$$\begin{cases} \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} = \dfrac{1}{\sqrt{2}}\left(Z_P + A_P\right) \\ \Gamma_{\mathfrak{D}\mathfrak{D}P} = \dfrac{1}{\sqrt{2}}\left(Z_P - A_P\right) \end{cases}.$$

Discussion 7.2.5. Substitute the above definition into the result of Discussion 7.2.3 and obtain that

$$\begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\dfrac{g}{\sqrt{2}}\dfrac{1}{\sqrt{2}}\left(Z_P + A_P\right) - \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\dfrac{g}{\sqrt{2}}\dfrac{1}{\sqrt{2}}\left(Z_P - A_P\right) - g\nu_L W_P^1 \\ l_{R;P} = \partial_P l_R - \sqrt{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\dfrac{g}{\sqrt{2}}\dfrac{1}{\sqrt{2}}\left(Z_P + A_P\right) + \sqrt{2}\rho_{\mathfrak{D}\mathfrak{D}}\dfrac{g}{\sqrt{2}}\dfrac{1}{\sqrt{2}}\left(Z_P - A_P\right) \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - g l_L W_P^1 \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P \end{cases} \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right)\dfrac{g}{\sqrt{2}}Z_P - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right)\dfrac{g}{\sqrt{2}}A_P - g\nu_L W_P^1 \\ l_{R;P} = \partial_P l_R - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right)\dfrac{g}{\sqrt{2}}Z_P - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right)\dfrac{g}{\sqrt{2}}A_P \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - g l_L W_P^1 \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P \end{cases} \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g\nu_L W_P^1 \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - g l_L W_P^1 \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P \end{cases} \end{cases}.$$

Definition 7.2.5. The above equation is called the **evolution form of weak-electromagnetic interaction of leptons**.

Discussion 7.2.6. The following actual evolution equations of charges completely describe the dynamics of weak-electromagnetic interaction of leptons.

$$\begin{cases} l_{L;P}dx^P \simeq l_{L;0}dx^0 \\ l_{L;P}\dfrac{\partial}{\partial x_P} \cong l_{L;0}\dfrac{d}{dx_0} \end{cases}, \quad \begin{cases} l_{R;P}dx^P \simeq l_{R;0}dx^0 \\ l_{R;P}\dfrac{\partial}{\partial x_P} \cong l_{R;0}\dfrac{d}{dx_0} \end{cases}, \quad \begin{cases} \nu_{L;P}dx^P \simeq \nu_{L;0}dx^0 \\ \nu_{L;P}\dfrac{\partial}{\partial x_P} \cong \nu_{L;0}\dfrac{d}{dx_0} \end{cases}, \quad \begin{cases} \nu_{R;P}dx^P \simeq \nu_{R;0}dx^0 \\ \nu_{R;P}\dfrac{\partial}{\partial x_P} \cong \nu_{R;0}\dfrac{d}{dx_0} \end{cases}.$$

Moreover, they can also deduce the forms similar to Dirac equation in the way of section 6.3.7.1 ,and then imagined that complex-valued Lagrangian and action can be constructed, and further more the theory similar to QFT can be developed in the sense of section 2.4.14.2 , etc. However, these are beyond the subject of this paper and they will not be discussed in detail.

Discussion 7.2.7. The above evolution form is expressed as the implicit polarity form, now its explicit polarity form is discussed as following.

Definition 7.2.6. Define

$$l^- \triangleq \left(\frac{1+i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \quad \frac{1-i}{2}\rho_{\mathfrak{D}\mathfrak{D}}\right), \qquad l^+ \triangleq \left(\frac{1-i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \quad \frac{1+i}{2}\rho_{\mathfrak{D}\mathfrak{D}}\right)$$

and its left-handed representation and right-handed representation

$$\begin{cases} l_L^- \triangleq \dfrac{1+i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \dfrac{1-i}{2}\rho_{\mathfrak{D}\mathfrak{D}} \\ l_L^+ \triangleq \dfrac{1-i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \dfrac{1+i}{2}\rho_{\mathfrak{D}\mathfrak{D}} \end{cases}, \quad \begin{cases} l_R^- \triangleq \dfrac{1+i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \dfrac{1-i}{2}\rho_{\mathfrak{D}\mathfrak{D}} \\ l_R^+ \triangleq \dfrac{1-i}{2}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \dfrac{1+i}{2}\rho_{\mathfrak{D}\mathfrak{D}} \end{cases}.$$

l_L^-, l_L^+, l_R^-, l_R^+ are uniformly called the **explicit polarity representation of charged leptons**.

Discussion 7.2.8. Due to the definition, it is obtained that

$$\begin{cases} l_L = \dfrac{1}{\sqrt{2}}\left(l_L^+ + l_L^-\right) = \dfrac{i}{\sqrt{2}}\left(l_R^- - l_R^+\right) \\ l_R = \dfrac{i}{\sqrt{2}}\left(l_L^+ - l_L^-\right) = \dfrac{1}{\sqrt{2}}\left(l_R^- + l_R^+\right) \end{cases}, \quad \begin{cases} l_L^- = \dfrac{1}{\sqrt{2}}\left(l_L + il_R\right) \\ l_L^+ = \dfrac{1}{\sqrt{2}}\left(l_L - il_R\right) \end{cases}, \quad \begin{cases} l_R^- \triangleq \dfrac{1}{\sqrt{2}}\left(l_R + il_L\right) \\ l_R^+ \triangleq \dfrac{1}{\sqrt{2}}\left(l_R - il_L\right) \end{cases}.$$

Thus, starting from the implicit polarity evolution form of weak-electromagnetic interaction, we can deduce its explicit polarity form. Concretely,

$$\begin{cases} \begin{cases} l_{L;P} = \partial_P l_L - gl_L Z_P - gl_R A_P - g\nu_L W_P^1 \\ l_{R;P} = \partial_P l_R - gl_R Z_P - gl_L A_P \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - gl_L W_P^1 \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P \end{cases} \end{cases} \Leftrightarrow \begin{cases} \begin{cases} l_{L;P}^- = \partial_P l_L^- - gl_L^- Z_P - gl_R^- A_P - g\nu_L W_P^- \\ l_{L;P}^+ = \partial_P l_L^+ - gl_L^+ Z_P - gl_R^+ A_P - g\nu_L W_P^+ \end{cases} \\ \begin{cases} l_{R;P}^- = \partial_P l_R^- - gl_R^- Z_P - gl_L^- A_P - ig\nu_L W_P^- \\ l_{R;P}^+ = \partial_P l_R^+ - gl_R^+ Z_P - gl_L^+ A_P + ig\nu_L W_P^+ \end{cases} \\ \begin{cases} \nu_{L;P} = \partial_P \nu_L - g\nu_L Z_P - gl_L^+ W_P^- - gl_L^- W_P^+ \\ \nu_{R;P} = \partial_P \nu_R - g\nu_R Z_P \end{cases} \end{cases}.$$

Thus the evolution forms of leptons have been expressed in the two ways of implicit polarity and explicit polarity. Because these two ways are equivalent, from the former we know the terms $-ig\nu_L W_P^-$ and $+ig\nu_L W_P^+$ in the latter with imaginary unit do not reflect any ontological observable physical effect.

Remark 7.2.4. Why do leptons have three generations? The typical weak-electromagnetic reference-system in Definition 7.1.1 has no enough ability to describe the differences of three generations of leptons. It requires a more complete reference-system in Definition 9.1 .

7.3 Rest-mass problem of fermion and boson

Discussion 7.3.1. It is well-known that traditional theory has solved the rest-mass problem in the way of Higgs mechanism. The physical reality cognized by the concept of Higgs field of traditional theory can be detected in the experiment of LHC [1,8], but it does not mean the concept of Higgs field of traditioal theory is necessarily the most appropriate concept to cognize such a physical reality. Higgs mechanism may be just a phenomenological equivalent theory.

According to the viewpoint of this paper, the physical reality is finally matter-motion. Any viewpoint that cannot be attributed to matter-motion is finally a matter of expediency. So an abstract concept of Higgs field is unsatisfactory.

1. The rest-mass of fermion.

In the traditional non-Abelian gauge field theory, the appearance of mass term of fermion field breaks the gauge invariance. By Yukawa coupling and Higgs mechanism, Glashow-Weinberg-Salam theory tells us that the broken is just superficial, and is caused by the non-commutative theoretical form, and essentially the gauge invariance is not broken. This explanation is successful.

However, it has to be emphasized that a theory fundamental enough should never need extra explanation to maintain its rationality. That is to say, the theoretical form of traditional gauge field is not fundamental. In the case of without adding Higgs mechanism, it has no ability to fit in with the geometric essence precisely. Even if adding Higgs mechanism such that it fits in, the theoretical form would be distorted. Moreover the Higgs mechanism is not based on the viewpoint of motion, so it is not natural enough.

From the viewpoint of motion, let f evolve in g. According to Discussion 6.3.2.1 , the evolution equation of charge ρ of f is

$$\begin{cases} \tilde{p}_\mu d\tilde{x}^\mu \simeq \tilde{m}_\tau d\tilde{x}^\tau \\ \tilde{p}_\mu \dfrac{\partial}{\partial \tilde{x}_\mu} \cong \tilde{m}_\tau \dfrac{d}{d\tilde{x}_\tau} \end{cases},$$

where $\tilde{x}^\tau$ is the evolution parameter, and $\frac{d}{d\tilde{x}_\tau}$ is completely internal-directed. The rest-mass $\tilde{m}_\tau \triangleq \tilde{\rho}_{;\tau}$ is the total energy-momentum in internal space direction of the evolution of ρ. This is the explanation to the essence of rest-mass of fermion from the viewpoint of matter-motion.

In this viewpoint:

(1) The mass term $\tilde{\rho}_{;\tau} d\tilde{x}^\tau$ is an expression form of geometric property $\tilde{\rho}_{;\mu} d\tilde{x}^\mu$, caused due to the regular embedding of evolution path.

(2) Now that the universal geometric property $D\tilde{\rho}$ is independent of the selection of reference-system, then it is of course invariant when reference-system transforms. According to Discussion 6.3.8.2 , the essence of gauge transformation is exactly a reference-system transformation, then $\tilde{\rho}_{;\mu}d\tilde{x}^{\mu}$ is gauge-invariant on manifold $\tilde{M}$ and $\tilde{\rho}_{;\tau}d\tilde{x}^{\tau}$ is also gauge-invariant on evolution path L.

It is seen that now the theoretical form fits in precisely with the geometric essence, without adding an extra Higgs mechanism.

2. The rest-mass of boson.

Noticed that the ${K^{m}_{nPQ}}^{:P}$ in general Yang-Mills field equation does not contain any mass term. Then how the rest-mass of the gauge boson of weak interaction appears?

In traditional theory, the rest-mass of gauge field is brought by Higgs field. In the Lagrangian

$$\mathcal{L}_H \triangleq (D^{\mu}\phi)^{+}D_{\mu}\phi - V(\phi)$$

of Higgs field of Glashow-Weinberg-Salam theory,

$$V(\phi) \triangleq -\mu^2\phi^{+}\phi + \lambda(\phi^{+}\phi)^2,$$

$$D_{\mu} \triangleq \partial_{\mu} - igT_iA^i_{\mu} - i\frac{g'}{2}YB_{\mu}.$$

The rest-mass of boson is contained in the ground state of ϕ.

Higgs field is a complex-valued doublets $\begin{pmatrix}\phi_+\\ \phi_0\end{pmatrix}$. $V(\phi)$ determines that the amplitude of ground state with spontaneous symmetry breaking is $\binom{u}{v}$, such that $\sqrt{u^2+v^2} = \sqrt{\frac{\mu^2}{2\lambda}}$. At this time, it needs to be artificially specified that $u = 0$ and $v \neq 0$. Concretely, the purpose of the specified $u = 0$ is to make A_{μ} never couple with Higgs field, and the purpose of the specified $v \neq 0$ is to make Z_{μ} and $W^{\pm}_{\mu}$ necessarily couple with Higgs field.

(1) Such designations are artifical and not natural. Why an abstractly defined Higgs field distributing all over the space does not couple with the electromagnetic potential A_{μ} and not couple with the gluon potential G_{μ} but just only couple with the potentials Z_{μ} and $W^{\pm}_{\mu}$ of weak interaction? We cannot just specify them but not ask why.

(2) $V(\phi) \triangleq -\mu^2\phi^{+}\phi + \lambda(\phi^{+}\phi)^2$ is a potential energy term. Generally, there is a concept of potential energy just only for interactions. The fact that there is a potential energy term in Lagrangian of Higgs field indicates that the Higgs field consists of several particles more fundamental.

(3) According to Remark 7.2.2 , the symmetry breaking indicates that the geometry is not large enough. Noticed that there is no mass term in ${K^{m}_{nPQ}}^{:P}$ described by intrinsic geometry, and the symmetry breaking can lead to the mass term. This fact tells us that the intrinsic geometry of one reference-system is not large enough for describing such a mass term. If making the geometry larger, the only way is to use the intrinsic geometry of more reference-systems.

In a word, although the rest-mass is made out by the Higgs mechanism, the concrete form of internal motion does not explained clearly by Higgs mechanism. In the viewpoint of matter-motion, the only probable rational explanation is that the Higgs boson consists of at least two particles more fundamental, which only participate in weak interaction and their total spin is zero.

A bold but reasonable idea is that these particles more fundamental is exactly neutrinos. The zero-spin neutrino pair is exactly the matter-motion form having such characteristics.

It can be imagined that Z field and W field propagate in the medium consisting of large quantities of neutrino pairs. Z field and W field interact with neutrino pairs and they have interaction potential energy, which makes Z field and W field act like fields with rest-mass.

Simply make an analogy. If an electromagnetic wave propagates in a dielectric medium and its group velocity v is less than the light velocity c in vacuum, the proper-time metric of evolution is

$$d\tau = \sqrt{(dx^0)^2 - \sum_{k=1}^{r}(dx_k)^2} = \sqrt{c^2dt^2 - \sum_{k=1}^{r}(dx_k)^2} = \sqrt{c^2 - v^2}dt \neq 0.$$

According to Definition 6.1.1.2 , the evolution direction of the electromagnetic wave is internal-directed. This can be explained as that the interaction potential energy between the electromagnetic wave and the medium make the electromagnetic wave act like a field with equivalent rest-mass.

Such pictures at least provides a more natural intuition of the rest-mass of Z field and W field.

The group behavior of particles does not belong to the research scope at the most basic level, and it should be described by a theory at a higher level, so in this paper it is judged that Higgs field is a concept at a higher level, like the Cooper pair about superconductor. Higgs field is not fundamental, so there is no need to take the equivalent rest-mass caused by the reason at such a high level into the general Yang-Mills field equation at the most basic level.

To say the least, it does not matter at all even if the rest-mass is contained, because the equation would still be strictly gauge-invariant at the most basic level.

Make a summary concisely.

(1) The rest-mass of fermion is the total kinetic energy in the internal evolution direction of charge.

(2) The rest-mass of boson is the total potential energy of the interaction of gauge field in medium.

(3) Higgs boson is not fundamental, which is probably a zero-spin neutrino pair.

8 Strong interaction and relative motion

8.1 Strong reference-system

Definition 8.1.1. Let $\mathfrak{D} = r + 3$, and on a $\mathfrak{D}$-dimensional smooth manifold M there be a typical gauge field reference-system f defined in Definition 5.2.1 , such that on a neighborhood U of each point p, the coordinate representation of $f(p)$ is

$$\begin{cases} \xi^s = \xi^s(x^i) \\ \xi^a = \xi^a(x^M) \end{cases}, \quad \begin{cases} x^i = x^i(\xi^s) \\ x^m = x^m(\xi^A) \end{cases}; \quad 1 \leqslant s, i \leqslant r \ ; \quad a, m = \mathfrak{D} - 2, \mathfrak{D} - 1, \mathfrak{D}.$$

Then f is called a **typical strong interaction reference-system**, or a **strong reference-system**.

According to the definition,

$$B_m^s = 0, \quad C_a^i = 0.$$

For the sake of simplicity, the effects of gravitational field should be excluded. So let f be externally flat and internally standard. According to Definition 5.2.2 , it is required that

$$B_i^s = \delta_i^s, \quad B_i^a = 0, \quad C_s^i = \delta_s^i, \quad C_s^m = 0.$$

$$G_{mn} = 0 (m \neq n), \quad G_{mn} = const.$$

In addition, a symmetry condition $G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}$ is required, and denote $g_s \triangleq \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2}$.

For convenience, indices should be specified first of all. Based on Definition 6.1.1.1 , if not specified in other sections, the values of internal indices are as following.

The internal indices are $a, b, c, d, e = \mathfrak{D}-2, \mathfrak{D}-1, \mathfrak{D}$ in coordinate frame ξ and $m, n, p, q, r = \mathfrak{D}-2, \mathfrak{D}-1, \mathfrak{D}$ in coordinate frame x.

8.2 Actual evolution of strong charges

Discussion 8.2.1. Let f and g be two strong reference-systems. Now focus on calculating the evolution forms of charges $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}\mathfrak{D}}$, $\rho_{\mathfrak{D}(\mathfrak{D}-1)}$, $\rho_{(\mathfrak{D}-1)\mathfrak{D}}$, $\rho_{\mathfrak{D}(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-2)\mathfrak{D}}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}$ of f in g.

$$\begin{aligned}
&\rho_{mn;P} = \partial_P \rho_{mn} - \rho_{Hn} \Gamma^H_{mP} - \rho_{mH} \Gamma^H_{nP} \\
&= \partial_P \rho_{mn} - \rho_{(\mathfrak{D}-2)n} \Gamma^{\mathfrak{D}-2}_{mP} - \rho_{(\mathfrak{D}-1)n} \Gamma^{\mathfrak{D}-1}_{mP} - \rho_{\mathfrak{D}n} \Gamma^{\mathfrak{D}}_{mP} - \rho_{m(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{nP} - \rho_{m(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{nP} - \rho_{m\mathfrak{D}} \Gamma^{\mathfrak{D}}_{nP}.
\end{aligned}$$

Concretely,

$$\begin{aligned}
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - 2\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}\right) \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} \\
&\quad - \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P}, \\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - 2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} \\
&\quad - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\right) \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P}, \\
\rho_{\mathfrak{D}\mathfrak{D};P} &= \partial_P \rho_{\mathfrak{D}\mathfrak{D}} - 2\rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{\mathfrak{D}P} - \left(\rho_{(\mathfrak{D}-2)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-2)}\right) \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} - \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P}, \\
\rho_{\mathfrak{D}(\mathfrak{D}-1);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-1)} - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) - \rho_{\mathfrak{D}(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\right) \\
&\quad - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} - \rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P}, \\
\rho_{(\mathfrak{D}-1)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-1)\mathfrak{D}} - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) - \rho_{(\mathfrak{D}-1)\mathfrak{D}} \left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\
&\quad - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} - \rho_{(\mathfrak{D}-2)\mathfrak{D}} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P}, \\
\rho_{\mathfrak{D}(\mathfrak{D}-2);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-2)} - \left(\rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) - \rho_{\mathfrak{D}(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P}\right) \\
&\quad - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} - \rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P}, \\
\rho_{(\mathfrak{D}-2)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-2)\mathfrak{D}} - \left(\rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) - \rho_{(\mathfrak{D}-2)\mathfrak{D}} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P}\right) \\
&\quad - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} - \rho_{(\mathfrak{D}-1)\mathfrak{D}} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P},
\end{aligned}$$

$$\begin{aligned}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P}\right) \\ &\quad - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\right) - \rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} - \rho_{(\mathfrak{D}-1)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P}, \\ \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P}\right) \\ &\quad - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P}\right) - \rho_{(\mathfrak{D}-2)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} - \rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P}.\end{aligned}$$

Definition 8.2.1. Denote

$$\begin{cases} d_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_2 \triangleq \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \quad \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_3 \triangleq \left(\rho_{\mathfrak{D}\mathfrak{D}}, \quad \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases}, \quad \begin{cases} u_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_2 \triangleq \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \quad \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_3 \triangleq \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases}.$$

d_1 and u_1 are called **red charges**, d_2 and u_2 are called **blue charges**, and d_3 and u_3 are called **green charges**.

$-d_1$ and $-u_1$ are called **anti-red charges**, $-d_2$ and $-u_2$ are called **anti-blue charges**, and $-d_3$ and $-u_3$ are called **anti-green charges**.

d_1, d_2, d_3 are called **down-type color charges**, uniformly denoted by d.

u_1, u_2, u_3 are called **up-type color charges**, uniformly denoted by u.

d and u are uniformly called **color charges**, denoted by q.

$q\frac{1}{\sqrt{2}}\binom{1}{1}$ is called the **left-handed color charge**, and $q\frac{1}{\sqrt{2}}\binom{1}{-1}$ is called the **right-handed color charge**.

$q\frac{1}{\sqrt{2}}\binom{-1}{-1}$ is called the **right-handed anti-color charge**, and $q\frac{1}{\sqrt{2}}\binom{-1}{1}$ is called the **left-handed anti-color charge**.

Concretely, left-handed and right-handed down-type color charges are

$$\begin{cases} d_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases}, \quad \begin{cases} d_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases}.$$

Left-handed and right-handed up-type color charges are

$$\begin{cases} u_{1L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_{2L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_{3L} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases}, \quad \begin{cases} u_{1R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_{2R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_{3R} \triangleq \dfrac{1}{\sqrt{2}}\left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases}.$$

Remark 8.2.1. The terminology of quark is deliberately avoided here, and the terminology of color charge is adopted instead. It is because the connotation referred to by the terminology of quark in traditional theory contains not only property of color charge but also properties of electric charge and weak charge, and this connotation is different from the concept in Definition 8.2.1 . In order to avoid confusion, the terminology of color charge is adopted to clearly refer to the connotation of the above concepts.

Definition 8.2.2. Denote

$$\begin{cases} U_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \\ V_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \end{cases}, \quad \begin{cases} X_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right) \\ Y_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right) \end{cases},$$

$$\begin{cases} U_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \\ V_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \end{cases}, \quad \begin{cases} X_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \\ Y_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \end{cases},$$

$$\begin{cases} U_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right) \\ V_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right) \end{cases}, \quad \begin{cases} X_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right) \\ Y_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right) \end{cases}.$$

They are called the **strong interaction potentials**.

Discussion 8.2.2. From Definition 8.2.1 , it is obtained that

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \frac{1}{\sqrt{2}}\left(d_{1L} - d_{2L} + d_{3L}\right) = \frac{1}{\sqrt{2}}\left(d_{1L} + d_{1R}\right) = \frac{1}{\sqrt{2}}\left(d_{3L} - d_{3R}\right) \\ \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \frac{1}{\sqrt{2}}\left(d_{1L} + d_{2L} - d_{3L}\right) = \frac{1}{\sqrt{2}}\left(d_{2L} + d_{2R}\right) = \frac{1}{\sqrt{2}}\left(d_{1L} - d_{1R}\right) \\ \rho_{\mathfrak{D}\mathfrak{D}} = \frac{1}{\sqrt{2}}\left(-d_{1L} + d_{2L} + d_{3L}\right) = \frac{1}{\sqrt{2}}\left(d_{3L} + d_{3R}\right) = \frac{1}{\sqrt{2}}\left(d_{2L} - d_{2R}\right) \end{cases}.$$

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} = \frac{1}{\sqrt{2}}(u_{1L} + u_{1R}) \\ \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} = \frac{1}{\sqrt{2}}(u_{1L} - u_{1R}) \end{cases}, \quad \begin{cases} \rho_{(\mathfrak{D}-1)\mathfrak{D}} = \frac{1}{\sqrt{2}}(u_{2L} + u_{2R}) \\ \rho_{\mathfrak{D}(\mathfrak{D}-1)} = \frac{1}{\sqrt{2}}(u_{2L} - u_{2R}) \end{cases}, \quad \begin{cases} \rho_{\mathfrak{D}(\mathfrak{D}-2)} = \frac{1}{\sqrt{2}}(u_{3L} + u_{3R}) \\ \rho_{(\mathfrak{D}-2)\mathfrak{D}} = \frac{1}{\sqrt{2}}(u_{3L} - u_{3R}) \end{cases}.$$

From Definition 8.2.2 , it is obtained that

$$\begin{cases} \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} = \frac{1}{\sqrt{2}}(U_P^1 + V_P^1) = \frac{1}{\sqrt{2}}(U_P^3 - V_P^3) \\ \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} = \frac{1}{\sqrt{2}}(U_P^2 + V_P^2) = \frac{1}{\sqrt{2}}(U_P^1 - V_P^1) \\ \Gamma_{\mathfrak{D}\mathfrak{D}P} = \frac{1}{\sqrt{2}}(U_P^3 + V_P^3) = \frac{1}{\sqrt{2}}(U_P^2 - V_P^2) \end{cases}.$$

$$\begin{cases} \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} = \frac{1}{\sqrt{2}}(X_P^{23} + Y_P^{23}) \\ \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P} = \frac{1}{\sqrt{2}}(X_P^{23} - Y_P^{23}) \end{cases}, \quad \begin{cases} \Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} = \frac{1}{\sqrt{2}}(X_P^{12} + Y_P^{12}) \\ \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P} = \frac{1}{\sqrt{2}}(X_P^{12} - Y_P^{12}) \end{cases}, \quad \begin{cases} \Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} = \frac{1}{\sqrt{2}}(X_P^{31} + Y_P^{31}) \\ \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P} = \frac{1}{\sqrt{2}}(X_P^{31} - Y_P^{31}) \end{cases}.$$

Discussion 8.2.3. According to the result of Discussion 8.2.1 , by calculation we obtain

$$\begin{cases} d_{1L;P} = \partial_P d_{1L} - g_s d_{1L} U_P^1 - g_s d_{1R} V_P^1 - g_s u_{1L} X_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} + \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} \\ d_{2L;P} = \partial_P d_{2L} - g_s d_{2L} U_P^2 - g_s d_{2R} V_P^2 - g_s u_{2L} X_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} + \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} \\ d_{3L;P} = \partial_P d_{3L} - g_s d_{3L} U_P^3 - g_s d_{3R} V_P^3 - g_s u_{3L} X_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} + \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} \end{cases}.$$

$$\begin{cases} d_{1R;P} = \partial_P d_{1R} - g_s d_{1R} U_P^1 - g_s d_{1L} V_P^1 + g_s u_{1L} Y_P^{23} + \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} \\ d_{2R;P} = \partial_P d_{2R} - g_s d_{2R} U_P^2 - g_s d_{2L} V_P^2 + g_s u_{2L} Y_P^{31} + \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} \\ d_{3R;P} = \partial_P d_{3R} - g_s d_{3R} U_P^3 - g_s d_{3L} V_P^3 + g_s u_{3L} Y_P^{12} + \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} \end{cases} .$$

$$\begin{cases} u_{1L;P} = \partial_P u_{1L} - g_s u_{1L} U_P^1 - \frac{g_s}{2} u_{2L} X_P^{12} - \frac{g_s}{2} u_{2L} Y_P^{12} - \frac{g_s}{2} u_{3L} X_P^{31} + \frac{g_s}{2} u_{3L} Y_P^{31} - g_s d_{1L} X_P^{23} - g_s d_{1R} Y_P^{23} \\ u_{2L;P} = \partial_P u_{2L} - g_s u_{2L} U_P^2 - \frac{g_s}{2} u_{3L} X_P^{23} - \frac{g_s}{2} u_{3L} Y_P^{23} - \frac{g_s}{2} u_{1L} X_P^{12} + \frac{g_s}{2} u_{1L} Y_P^{12} - g_s d_{2L} X_P^{31} - g_s d_{2R} Y_P^{31} \\ u_{3L;P} = \partial_P u_{3L} - g_s u_{3L} U_P^3 - \frac{g_s}{2} u_{1L} X_P^{31} - \frac{g_s}{2} u_{1L} Y_P^{31} - \frac{g_s}{2} u_{2L} X_P^{23} + \frac{g_s}{2} u_{2L} Y_P^{23} - g_s d_{3L} X_P^{12} - g_s d_{3R} Y_P^{12} \end{cases} .$$

$$\begin{cases} u_{1R;P} = \partial_P u_{1R} - g_s u_{1R} U_P^1 + \frac{g_s}{2} u_{2R} X_P^{12} + \frac{g_s}{2} u_{2R} Y_P^{12} + \frac{g_s}{2} u_{3R} X_P^{31} - \frac{g_s}{2} u_{3R} Y_P^{31} \\ u_{2R;P} = \partial_P u_{2R} - g_s u_{2R} U_P^2 + \frac{g_s}{2} u_{3R} X_P^{23} + \frac{g_s}{2} u_{3R} Y_P^{23} + \frac{g_s}{2} u_{1R} X_P^{12} - \frac{g_s}{2} u_{1R} Y_P^{12} \\ u_{3R;P} = \partial_P u_{3R} - g_s u_{3R} U_P^3 + \frac{g_s}{2} u_{1R} X_P^{31} + \frac{g_s}{2} u_{1R} Y_P^{31} + \frac{g_s}{2} u_{2R} X_P^{23} - \frac{g_s}{2} u_{2R} Y_P^{23} \end{cases} .$$

Discussion 8.2.4. According Definition 8.2.2 , U, V, X, Y fields are generated by nine internal connections $\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}$, $\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}$, $\Gamma_{\mathfrak{D}\mathfrak{D}P}$, $\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P}$, $\Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}$, $\Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}$, $\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P}$, $\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P}$, $\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}$, so there are nine independent strong interaction potential fields. This actually corresponds to the transformation group $GL(3,\mathbb{R})$ of slack-tight of internal space. $GL(3,\mathbb{R})$ has nine generators. On a proper condition, it is completely feasible to use $GL(3,\mathbb{R})$ to describe the algebraic properties of $SU(3)$. Now consider the algebraic corresponding relationship between the above connections and the gluon potentials of QCD.

Noticed that

$$\begin{cases} U_P^2 - U_P^3 + V_P^1 = 0 \\ U_P^3 - U_P^1 + V_P^2 = 0 \\ U_P^1 - U_P^2 + V_P^3 = 0 \end{cases} ,$$

so there are only three independent potentials in U_P^1, U_P^2, U_P^3, V_P^1, V_P^2, V_P^3. Without loss of generality, let

$$\begin{cases} R_P \triangleq a_R U_P^1 + b_R U_P^2 + c_R U_P^3 \\ S_P \triangleq a_S U_P^1 + b_S U_P^2 + c_S U_P^3 \\ T_P \triangleq a_T U_P^1 + b_T U_P^2 + c_T U_P^3 \end{cases}$$

and its inverse transformation

$$\begin{cases} U_P^1 \triangleq \alpha_R R_P + \alpha_S S_P + \alpha_T T_P \\ U_P^2 \triangleq \beta_R R_P + \beta_S S_P + \beta_T T_P \\ U_P^3 \triangleq \gamma_R R_P + \gamma_S S_P + \gamma_T T_P \end{cases} ,$$

where the coefficient matrix is nonsingular.

If taking $a_R = b_R = c_R \triangleq \frac{1}{3\gamma}$ and $\alpha_R = \beta_R = \gamma_R$, where γ is a constant then according to the viewpoint of QCD, $R_P = \frac{1}{3\gamma}\left(U_P^1 + U_P^2 + U_P^3\right) = \frac{2}{3\gamma}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right)$ can be regarded as a color singlet. Now as long as adding a symmetry condition $\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0$, there is no need to put R_P into $SU(3)$ theory of QCD. Thus, the rest eight gauge potentials S_P, T_P, X_P^{12}, X_P^{23}, X_P^{31}, Y_P^{12}, Y_P^{23}, Y_P^{31} exactly fit in with the generators of $SU(3)$.

Define

$$\begin{cases} A_P^1 \triangleq X_P^{12} \\ A_P^2 \triangleq Y_P^{12} \end{cases}, \quad \begin{cases} A_P^4 \triangleq X_P^{31} \\ A_P^5 \triangleq Y_P^{31} \end{cases}, \quad \begin{cases} A_P^6 \triangleq X_P^{23} \\ A_P^7 \triangleq Y_P^{23} \end{cases}, \quad \begin{cases} A_P^3 \triangleq S_P \\ A_P^8 \triangleq T_P \end{cases}.$$

And define

$$\begin{cases} A_P^{21} \triangleq X_P^{12} + iY_P^{12} \\ A_P^{12} \triangleq X_P^{12} - iY_P^{12} \end{cases}, \quad \begin{cases} A_P^{31} \triangleq X_P^{31} + iY_P^{31} \\ A_P^{13} \triangleq X_P^{31} - iY_P^{31} \end{cases}, \quad \begin{cases} A_P^{32} \triangleq X_P^{23} + iY_P^{23} \\ A_P^{23} \triangleq X_P^{23} - iY_P^{23} \end{cases},$$

$$A_P^{11} \triangleq S_P + \frac{1}{\sqrt{6}} T_P, \quad A_P^{22} \triangleq -S_P + \frac{1}{\sqrt{6}} T_P, \quad A_P^{33} \triangleq -\frac{2}{\sqrt{6}} T_P.$$

Then define the following matrix consisted of the above potentials.

$$\begin{aligned} A_P &\triangleq \frac{1}{2} \begin{pmatrix} A_P^{11} & A_P^{12} & A_P^{13} \\ A_P^{21} & A_P^{22} & A_P^{23} \\ A_P^{31} & A_P^{32} & A_P^{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} S_P + \frac{1}{\sqrt{6}} T_P & X_P^{12} - iY_P^{12} & X_P^{31} - iY_P^{31} \\ X_P^{12} + iY_P^{12} & -S_P + \frac{1}{\sqrt{6}} T_P & X_P^{23} - iY_P^{23} \\ X_P^{31} + iY_P^{31} & X_P^{23} + iY_P^{23} & -\frac{2}{\sqrt{6}} T_P \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} A_P^3 + \frac{1}{\sqrt{6}} A_P^8 & A_P^1 - iA_P^2 & A_P^4 - iA_P^5 \\ A_P^1 + iA_P^2 & -A_P^3 + \frac{1}{\sqrt{6}} A_P^8 & A_P^6 - iA_P^7 \\ A_P^4 + iA_P^5 & A_P^6 + iA_P^7 & -\frac{2}{\sqrt{6}} A_P^8 \end{pmatrix} \\ &= \frac{1}{2} \left(\lambda_1 A_P^1 + \lambda_2 A_P^2 + \lambda_3 A_P^3 + \lambda_4 A_P^4 + \lambda_5 A_P^5 + \lambda_6 A_P^6 + \lambda_7 A_P^7 + \lambda_8 A_P^8 \right) \\ &= T_a A_P^a, \end{aligned}$$

where $T_a \triangleq \frac{1}{2}\lambda_a$ are the generators of $SU(3)$, and λ_a are the well-known Gell-Mann matrices defined as

$$\lambda_1 \triangleq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 \triangleq \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 \triangleq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 \triangleq \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 \triangleq \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Now if defining the notations of color states

$$\left\{ \begin{matrix} / & \bar{r}b \triangleq \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2) & \bar{r}g \triangleq \frac{1}{\sqrt{2}}(\lambda_4 + i\lambda_5) \\ \bar{b}r \triangleq \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2) & / & \bar{b}g \triangleq \frac{1}{\sqrt{2}}(\lambda_6 + i\lambda_7) \\ \bar{g}r \triangleq \frac{1}{\sqrt{2}}(\lambda_4 - i\lambda_5) & \bar{g}b \triangleq \frac{1}{\sqrt{2}}(\lambda_6 - i\lambda_7) & / \end{matrix} \right\}$$

and that on diagonal

$$\begin{cases} \frac{1}{\sqrt{2}}(\bar{r}r - \bar{b}b) \triangleq \lambda_3 \\ \frac{1}{\sqrt{6}}(\bar{r}r + \bar{b}b - 2\bar{g}g) \triangleq \lambda_8 \end{cases},$$

it is obtained that

$$\left\{ \begin{matrix} \lambda_1 = \frac{1}{\sqrt{2}}(\bar{r}b + \bar{b}r) & \lambda_4 = \frac{1}{\sqrt{2}}(\bar{r}g + \bar{g}r) & \lambda_6 = \frac{1}{\sqrt{2}}(\bar{b}g + \bar{g}b) & \lambda_3 = \frac{1}{\sqrt{2}}(\bar{r}r - \bar{b}b) \\ \lambda_2 = -\frac{i}{\sqrt{2}}(\bar{r}b - \bar{b}r) & \lambda_5 = -\frac{i}{\sqrt{2}}(\bar{r}g - \bar{g}r) & \lambda_7 = -\frac{i}{\sqrt{2}}(\bar{b}g - \bar{g}b) & \lambda_8 = \frac{1}{\sqrt{6}}(\bar{r}r + \bar{b}b - 2\bar{g}g) \end{matrix} \right\}.$$

Substitute them back into matrix A_P, then

$$\begin{aligned} 2A_P &= \lambda_1 A_P^1 + \lambda_2 A_P^2 + \lambda_3 A_P^3 + \lambda_4 A_P^4 + \lambda_5 A_P^5 + \lambda_6 A_P^6 + \lambda_7 A_P^7 + \lambda_8 A_P^8 \\ &= \frac{1}{\sqrt{2}}(\bar{r}b + \bar{b}r) X_P^{12} - \frac{i}{\sqrt{2}}(\bar{r}b - \bar{b}r) Y_P^{12} + \frac{1}{\sqrt{2}}(\bar{r}r - \bar{b}b) S_P + \frac{1}{\sqrt{2}}(\bar{r}g + \bar{g}r) X_P^{31} \\ &\quad - \frac{i}{\sqrt{2}}(\bar{r}g - \bar{g}r) Y_P^{31} + \frac{1}{\sqrt{2}}(\bar{b}g + \bar{g}b) X_P^{23} - \frac{i}{\sqrt{2}}(\bar{b}g - \bar{g}b) Y_P^{23} + \frac{1}{\sqrt{6}}(\bar{r}r + \bar{b}b - 2\bar{g}g) T_P. \end{aligned}$$

It is seen from the results of Discussion 8.2.3 that the superscripts of potentials X and Y reflect the color types of interaction, and there are corresponding relations $1 \leftrightarrow red$, $2 \leftrightarrow blue$, $3 \leftrightarrow green$. The results are completely consistent with the traditional theory of QCD.

The above discussions clarifies that the concept of strong interaction reference-system in Definition 8.1.1 contains the key part of QCD, so it is reasonable to use such a concept of strong interaction reference-system to describe the ontological strong interaction.

Discussion 8.2.5. The electromagnetic potential, Z potential and W potential cannot be described in the strong interaction reference-system defined in section 7.3.1 , because they are seperate from the weak-electromagnetic reference-system defined in section 6.4.3 .

In order to completely include the weak-electromagnetic interaction of hadron, the two dimensions of the internal space of weak-electromagnetic reference-system of Definition 7.1.1 has to be put together with the three dimensions of the internal space of strong reference-system of Definition 8.1.1 , and we should consider the case of five dimensions, which will be described strictly in the next section.

Review the GUT based on $SU(5)$, which unifies the electromagnetic, weak and strong interactions together. Of course it benifits from the five dimentions of internal space. Just as section 7.2.4 , the theoretical forms of such GUTs are not fundamental enough, so they are not easy to fit in with the geometric essence precisely, therefore they may cause a prediction of proton decay which may be inconsistent with experiments. Such a prediction will not be caused by the theory of this paper.

This paper expresses in the way of intrinsic geometry, then appropriately makes the geometry smaller by adding symmetry conditions, finally it must be easier than traditional theory to achieve the purpose of approaching the target geometry.

9 Weak-electromagnetic-strong unified reference-system

Definition 9.1. Similar to Definition 7.1.1 and Definition 8.1.1 , let $\mathfrak{D} = r + 5$ and on a $\mathfrak{D}$-dimensional smooth manifold M there be a typical gauge field reference-system f such that on a neighborhood U of each point p, the coordinate representation of $f(p)$ is

$$\begin{cases} \xi^s = \xi^s(x^i) \\ \xi^a = \xi^a(x^M) \end{cases}, \quad \begin{cases} x^i = x^i(\xi^s) \\ x^m = x^m(\xi^A) \end{cases}; \quad 1 \leqslant s, i \leqslant r \ ; \quad a, m = \mathfrak{D}-4, \mathfrak{D}-3, \mathfrak{D}-2, \mathfrak{D}-1, \mathfrak{D}.$$

Then f is called a **Weak-electromagnetic-strong unified reference-system**.

Accoring to the definition,

$$B^s_m = 0, \quad C^i_a = 0.$$

For the sake of simplicity, the effects of gravitational field should be excluded. So let f be externally flat and internally standard. According to Definition 5.2.2 , it is required that

$$B^s_i = \delta^s_i, \quad B^a_i = 0, \quad C^i_s = \delta^i_s, \quad C^m_s = 0.$$

$$G_{mn} = 0 (m \neq n), \quad G_{mn} = const.$$

In addition, symmetry conditions

$$\begin{cases} G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}, \\ G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}}, \end{cases}$$

are required, and denote

$$\begin{cases} g \triangleq \sqrt{\left(G^{(\mathfrak{D}-4)(\mathfrak{D}-4)}\right)^2 + \left(G^{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right)^2}, \\ g_s \triangleq \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right)^2 + \left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2} = \sqrt{\left(G^{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right)^2 + \left(G^{\mathfrak{D}\mathfrak{D}}\right)^2}. \end{cases}$$

Specify the internal indices, which are $a, b, c, d, e = \mathfrak{D}-4, \mathfrak{D}-3, \mathfrak{D}-2, \mathfrak{D}-1, \mathfrak{D}$ in coordinate frame ξ and $m, n, p, q, r = \mathfrak{D}-4, \mathfrak{D}-3, \mathfrak{D}-2, \mathfrak{D}-1, \mathfrak{D}$ in coordinate frame x.

Discussion 9.1. Let f and g be two weak-electromagnetic-strong reference-systems. Now focus on calculating the evolution forms of charges ρ_{mn} of f in g.

$$\begin{aligned} \rho_{mn;P} &= \partial_P \rho_{mn} - \rho_{Hn} \mathit{\Gamma}^H_{mP} - \rho_{mH} \mathit{\Gamma}^H_{nP} \\ &= \partial_P \rho_{mn} - \rho_{(\mathfrak{D}-4)n} \mathit{\Gamma}^{\mathfrak{D}-4}_{mP} - \rho_{(\mathfrak{D}-3)n} \mathit{\Gamma}^{\mathfrak{D}-3}_{mP} - \rho_{(\mathfrak{D}-2)n} \mathit{\Gamma}^{\mathfrak{D}-2}_{mP} - \rho_{(\mathfrak{D}-1)n} \mathit{\Gamma}^{\mathfrak{D}-1}_{mP} - \rho_{\mathfrak{D}n} \mathit{\Gamma}^{\mathfrak{D}}_{mP} \\ &- \rho_{m(\mathfrak{D}-4)} \mathit{\Gamma}^{\mathfrak{D}-4}_{nP} - \rho_{m(\mathfrak{D}-3)} \mathit{\Gamma}^{\mathfrak{D}-3}_{nP} - \rho_{m(\mathfrak{D}-2)} \mathit{\Gamma}^{\mathfrak{D}-2}_{nP} - \rho_{m(\mathfrak{D}-1)} \mathit{\Gamma}^{\mathfrak{D}-1}_{nP} - \rho_{m\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{nP}. \end{aligned}$$

Concretely,

$$\begin{aligned} \rho_{\mathfrak{D}\mathfrak{D};P} &= \partial_P \rho_{\mathfrak{D}\mathfrak{D}} - 2\rho_{\mathfrak{D}\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P} - \left(\rho_{(\mathfrak{D}-2)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-2)}\right) \mathit{\Gamma}^{\mathfrak{D}-2}_{\mathfrak{D}P} - \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \mathit{\Gamma}^{\mathfrak{D}-1}_{\mathfrak{D}P} \\ &- \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}\right) \mathit{\Gamma}^{\mathfrak{D}-4}_{\mathfrak{D}P} - \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}\right) \mathit{\Gamma}^{\mathfrak{D}-3}_{\mathfrak{D}P}, \\ \rho_{(\mathfrak{D}-1)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{(\mathfrak{D}-1)\mathfrak{D}} \left(\mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \mathit{\Gamma}^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{(\mathfrak{D}-1)P}\right) - \left(\rho_{(\mathfrak{D}-2)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \mathit{\Gamma}^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} \mathit{\Gamma}^{\mathfrak{D}-4}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} \mathit{\Gamma}^{\mathfrak{D}-3}_{\mathfrak{D}P}\right), \\ \rho_{(\mathfrak{D}-2)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-2)\mathfrak{D}} - \rho_{(\mathfrak{D}-2)\mathfrak{D}} \left(\mathit{\Gamma}^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{\mathfrak{D}\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \mathit{\Gamma}^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \mathit{\Gamma}^{\mathfrak{D}-1}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \mathit{\Gamma}^{\mathfrak{D}-4}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \mathit{\Gamma}^{\mathfrak{D}-3}_{\mathfrak{D}P}\right), \\ \rho_{(\mathfrak{D}-3)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-3)\mathfrak{D}} - \rho_{(\mathfrak{D}-3)\mathfrak{D}} \left(\mathit{\Gamma}^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{\mathfrak{D}\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} \mathit{\Gamma}^{\mathfrak{D}-3}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-2)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \mathit{\Gamma}^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \mathit{\Gamma}^{\mathfrak{D}-1}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} \mathit{\Gamma}^{\mathfrak{D}-4}_{\mathfrak{D}P} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}\right), \\ \rho_{(\mathfrak{D}-4)\mathfrak{D};P} &= \partial_P \rho_{(\mathfrak{D}-4)\mathfrak{D}} - \rho_{(\mathfrak{D}-4)\mathfrak{D}} \left(\mathit{\Gamma}^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P} + \mathit{\Gamma}^{\mathfrak{D}}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{\mathfrak{D}\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} \mathit{\Gamma}^{\mathfrak{D}-4}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-2)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \mathit{\Gamma}^{\mathfrak{D}-2}_{\mathfrak{D}P}\right) \\ &- \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \mathit{\Gamma}^{\mathfrak{D}-1}_{\mathfrak{D}P}\right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} \mathit{\Gamma}^{\mathfrak{D}-3}_{\mathfrak{D}P} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \mathit{\Gamma}^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}\right) \end{aligned}$$

$$\begin{aligned}
\rho_{\mathfrak{D}(\mathfrak{D}-1);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-1)} - \rho_{\mathfrak{D}(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} \right), \\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - 2\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \right) \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)\mathfrak{D}} \right) \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} \right) \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} \right) \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P}, \\
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} \right), \\
\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \right), \\
\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1);P} &= \partial_P \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \left(\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P} + \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} \right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} \right), \\
\rho_{\mathfrak{D}(\mathfrak{D}-2);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{\mathfrak{D}(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} \right), \\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} \right), \\
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - 2\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} \right) \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}} \right) \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} \\
&- \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \right) \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \right) \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P}, \\
\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} \right), \\
\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2);P} &= \partial_P \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \left(\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P} + \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} \right) \\
&- \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} \right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} \right),
\end{aligned}$$

$$
\begin{aligned}
\rho_{\mathfrak{D}(\mathfrak{D}-3);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-3)} - \rho_{\mathfrak{D}(\mathfrak{D}-3)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P}\right), \\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} \left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P}\right), \\
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \left(\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P}\right), \\
\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3);P} &= \partial_P \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} - 2\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P} \\
&\quad - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}\right)\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\right)\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\right)\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}\right)\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P}, \\
\rho_{(\mathfrak{D}-4)(\mathfrak{D}-3);P} &= \partial_P \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} \left(\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P} + \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P}\right), \\
\rho_{\mathfrak{D}(\mathfrak{D}-4);P} &= \partial_P \rho_{\mathfrak{D}(\mathfrak{D}-4)} - \rho_{\mathfrak{D}(\mathfrak{D}-4)} \left(\Gamma^{\mathfrak{D}}_{\mathfrak{D}P} + \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} + \rho_{\mathfrak{D}\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-2}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-1}_{\mathfrak{D}P} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P}\right), \\
\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4);P} &= \partial_P \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} \left(\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-1)P} + \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-1)P} + \rho_{(\mathfrak{D}-1)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P}\right), \\
\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4);P} &= \partial_P \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \left(\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-2)P} + \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-2)P} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P}\right), \\
\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4);P} &= \partial_P \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} \left(\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-3)P} + \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P}\right) \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P}\right) - \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P} + \rho_{(\mathfrak{D}-3)\mathfrak{D}}\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P}\right), \\
\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4);P} &= \partial_P \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} - 2\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-4)P} \\
&\quad - \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right)\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} - \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}\right)\Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} \\
&\quad - \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}\right)\Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} - \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}}\right)\Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P}.
\end{aligned}
$$

Definition 9.2. Interaction potentials.

Define the **weak-electromagnetic interaction potentials**

$$\begin{cases} Z_P \triangleq \frac{1}{\sqrt{2}}(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}) \\ A_P \triangleq \frac{1}{\sqrt{2}}(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-4)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-3)P}) \end{cases}, \quad \begin{cases} W_P^1 \triangleq \frac{1}{\sqrt{2}}(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} + \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}) \\ W_P^2 \triangleq \frac{1}{\sqrt{2}}(\Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} - \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P}) \end{cases}$$

and the **strong interaction potentials**

$$\begin{cases} U_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \\ V_P^1 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P}\right) \end{cases}, \quad \begin{cases} X_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right) \\ Y_P^{23} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-1)P} - \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-2)P}\right) \end{cases},$$

$$\begin{cases} U_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \\ V_P^2 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} - \Gamma_{\mathfrak{D}\mathfrak{D}P}\right) \end{cases}, \quad \begin{cases} X_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} + \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \\ Y_P^{31} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{(\mathfrak{D}-1)\mathfrak{D}P} - \Gamma_{\mathfrak{D}(\mathfrak{D}-1)P}\right) \end{cases},$$

$$\begin{cases} U_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} + \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right) \\ V_P^3 \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}\mathfrak{D}P} - \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P}\right) \end{cases}, \quad \begin{cases} X_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right) \\ Y_P^{12} \triangleq \frac{1}{\sqrt{2}}\left(\Gamma_{\mathfrak{D}(\mathfrak{D}-2)P} - \Gamma_{(\mathfrak{D}-2)\mathfrak{D}P}\right) \end{cases}.$$

Definition 9.3. Weak-electromagnetic charges and color charges.

Define the **electric weak charge**

$$l \triangleq \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)},\ \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right)$$

and the **neutral weak charge**

$$\nu \triangleq \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)},\ \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right),$$

which are uniformly called the **weak-electromagnetic charges**. Define **down-type color charges**

$$\begin{cases} d_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_2 \triangleq \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}, \quad \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_3 \triangleq \left(\rho_{\mathfrak{D}\mathfrak{D}}, \quad \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases},$$

and **up-type color charges**

$$\begin{cases} u_1 \triangleq \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)}, \quad \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_2 \triangleq \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}}, \quad \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_3 \triangleq \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)}, \quad \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases},$$

which are uniformly called the **color charges**. d_1 and u_1 are called **red charges**, d_2 and u_2 are called **blue charges**, and d_3 and u_3 are called **green charges**. Similar to Definition 7.2.1 and Definition 8.2.1 , define left-handed and right-handed weak-electromagnetic charges

$$\begin{cases} l_L \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right) \\ l_R \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right) \end{cases}, \quad \begin{cases} \nu_L \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right) \\ \nu_R \triangleq \frac{1}{\sqrt{2}}\left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} - \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right) \end{cases},$$

and left-handed and right-handed down-type color charges

$$\begin{cases} d_{1L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_{2L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} + \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_{3L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{\mathfrak{D}\mathfrak{D}} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases}, \quad \begin{cases} d_{1R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)}\right) \\ d_{2R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} - \rho_{\mathfrak{D}\mathfrak{D}}\right) \\ d_{3R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{\mathfrak{D}\mathfrak{D}} - \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)}\right) \end{cases},$$

and left-handed and right-handed up-type color charges

$$\begin{cases} u_{1L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_{2L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_{3L} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases}, \quad \begin{cases} u_{1R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} - \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)}\right) \\ u_{2R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)\mathfrak{D}} - \rho_{\mathfrak{D}(\mathfrak{D}-1)}\right) \\ u_{3R} \triangleq \dfrac{1}{\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-2)} - \rho_{(\mathfrak{D}-2)\mathfrak{D}}\right) \end{cases}.$$

Definition 9.4. Define the symmetry conditions of weak-electromagnetic-strong unified rerference-system.

(1) Basic conditions, No.1:

$$\begin{cases} G^{(\mathfrak{D}-4)(\mathfrak{D}-4)} = G^{(\mathfrak{D}-3)(\mathfrak{D}-3)} \\ G^{(\mathfrak{D}-2)(\mathfrak{D}-2)} = G^{(\mathfrak{D}-1)(\mathfrak{D}-1)} = G^{\mathfrak{D}\mathfrak{D}} \end{cases};$$

(2) Basic conditions, No.2:

$$\begin{cases} \Gamma_{(\mathfrak{D}-3)(\mathfrak{D}-4)P} = \Gamma_{(\mathfrak{D}-4)(\mathfrak{D}-3)P} \\ \Gamma_{(\mathfrak{D}-2)(\mathfrak{D}-2)P} + \Gamma_{(\mathfrak{D}-1)(\mathfrak{D}-1)P} + \Gamma_{\mathfrak{D}\mathfrak{D}P} = 0 \end{cases};$$

(3) MNS mixing conditions of leptons, No.1:

$$\begin{cases} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}-2}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \\ \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}-1}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \\ \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-4)P} = c^{\mathfrak{D}}_{\mathfrak{D}-3}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \end{cases}, \quad \begin{cases} \Gamma^{\mathfrak{D}-2}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}-2}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \\ \Gamma^{\mathfrak{D}-1}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}-1}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \\ \Gamma^{\mathfrak{D}}_{(\mathfrak{D}-3)P} = c^{\mathfrak{D}}_{\mathfrak{D}-4}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \end{cases}, \quad \begin{cases} c^{\mathfrak{D}-2}_{\mathfrak{D}-3} = c^{\mathfrak{D}-2}_{\mathfrak{D}-4} \\ c^{\mathfrak{D}-1}_{\mathfrak{D}-3} = c^{\mathfrak{D}-1}_{\mathfrak{D}-4} \\ c^{\mathfrak{D}}_{\mathfrak{D}-3} = c^{\mathfrak{D}}_{\mathfrak{D}-4} \end{cases};$$

(4) MNS mixing conditions of leptons, No.2:

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \\ \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} \\ \rho_{\mathfrak{D}(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)} \end{cases}, \quad \begin{cases} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \\ \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \\ \rho_{(\mathfrak{D}-3)\mathfrak{D}} = \rho_{(\mathfrak{D}-4)\mathfrak{D}} \end{cases};$$

(5) CKM mixing conditions of color charges, No.1:

$$\begin{cases} \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-2)P} = c^{\mathfrak{D}-4}_{\mathfrak{D}-2}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \\ \Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-1)P} = c^{\mathfrak{D}-4}_{\mathfrak{D}-1}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \\ \Gamma^{\mathfrak{D}-3}_{\mathfrak{D}P} = c^{\mathfrak{D}-4}_{\mathfrak{D}}\Gamma^{\mathfrak{D}-3}_{(\mathfrak{D}-4)P} \end{cases}, \quad \begin{cases} \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-2)P} = c^{\mathfrak{D}-3}_{\mathfrak{D}-2}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \\ \Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-1)P} = c^{\mathfrak{D}-3}_{\mathfrak{D}-1}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \\ \Gamma^{\mathfrak{D}-4}_{\mathfrak{D}P} = c^{\mathfrak{D}-3}_{\mathfrak{D}}\Gamma^{\mathfrak{D}-4}_{(\mathfrak{D}-3)P} \end{cases}, \quad \begin{cases} c^{\mathfrak{D}-4}_{\mathfrak{D}-2} = c^{\mathfrak{D}-4}_{\mathfrak{D}-1} = c^{\mathfrak{D}-4}_{\mathfrak{D}} \\ c^{\mathfrak{D}-3}_{\mathfrak{D}-2} = c^{\mathfrak{D}-3}_{\mathfrak{D}-1} = c^{\mathfrak{D}-3}_{\mathfrak{D}} \end{cases};$$

(6) CKM mixing conditions of color charges, No.2:

$$\begin{cases} \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} = \rho_{\mathfrak{D}(\mathfrak{D}-3)} \\ \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} = \rho_{\mathfrak{D}(\mathfrak{D}-4)} \end{cases}, \quad \begin{cases} \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-3)\mathfrak{D}} \\ \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-4)\mathfrak{D}} \end{cases}.$$

The c_n^m are all real constants. An externally flat and internally standard weak-electromagnetic-strong unified reference-system satisfying all the above conditions is called a **typical weak-electromagnetic-strong unified field**, or an **elementary particle field**.

Definition 9.5. Lepton and hadron.

If there is a typical weak-electrolmagnetic-strong unified field such that $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)\mathfrak{D}} = \rho_{\mathfrak{D}(\mathfrak{D}-1)} = \rho_{\mathfrak{D}(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-2)\mathfrak{D}} = 0$, it is called a **lepton reference-system**, or a **lepton field**. Otherwise, it is called a **hadron reference-system**, or a **hadron field**.

The electric weak charge of lepton reference-system is called an **electric charged lepton**, and the neutral weak charge of lepton reference-system is called a **neutrino**. They are uniformly called **lepton charges**, or **leptons** for short.

If a hadron reference-system satisfies that its three up-type color charges are constantly zero, and two of the three down-type color charges are also constantly zero but the other one is not, such a reference-system is called a **single down-type quark**.

If a hadron reference-system satisfies that its three down-type color charges are constantly zero, and two of the three up-type color charges are also constantly zero but the other one is not, such a reference-system is called a **single up-type quark**.

The single down-type quark and the single up-type quark are uniformly called the **single quarks**.

Proposition 9.1. (Color confinement) Single quarks do not exist.

For the single down-type quark, this proposition is obviously true. Without loss of generality, suppose $u_1 = u_2 = u_3 = 0$ and $d_1 = d_2 = 0$, then $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = 0$, so $d_3 = 0$ holds surely.

For the single up-type quark, without loss of generality, consider the case that $d_1 = d_2 = d_3 = 0$ and $u_1 = u_2 = 0$. In this case, $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)\mathfrak{D}} = \rho_{\mathfrak{D}(\mathfrak{D}-1)} = \rho_{(\mathfrak{D}-2)(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-1)(\mathfrak{D}-1)} = \rho_{\mathfrak{D}\mathfrak{D}} = 0$, and we need to prove $u_3 = 0$, that is $\rho_{\mathfrak{D}(\mathfrak{D}-2)} = \rho_{(\mathfrak{D}-2)\mathfrak{D}} = 0$, where

$$
\begin{aligned}
\rho_{MN} = \rho^0_{MN} = G_{MM'}\rho_N^{M'0} = G_{MM'}G^{00}\rho^{M'}_{N0} = G_{MM'}G^{00}K^{M'}_{NPQ}{}^{:P}\varepsilon^Q_0 \\
= G_{MM'}G^{00}\left(\Lambda^{M'}_{NQ:P}{}^{:P} - \Lambda^{M'}_{NP:Q}{}^{:P} - K^{M'}_{HPQ}\Lambda^{HP}_{N}\right)\varepsilon^Q_0
\end{aligned}
$$

and

$$
\Lambda^M_{NP} \triangleq \frac{1}{2}C^M_A\left(\frac{\partial B^A_N}{\partial x^P} + \frac{\partial B^A_P}{\partial x^N}\right).
$$

It seems probably true, but this paper has not made progress in its proof.

Discussion 9.2. According to the results of calculation of Discussion 9.1 , as well as Definition 9.2 , Definition 9.3 and Definition 9.4 , the evolution forms of weak-electromagnetic charges can be obtained as below.

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu_L W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}-2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) \right] \frac{g}{\sqrt{2}} W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}-1} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) \right] \frac{g}{\sqrt{2}} W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) \right] \frac{g}{\sqrt{2}} W_P^1, \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P, \\ \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l_L W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}-2} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \right) \right] \frac{g}{\sqrt{2}} W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}-1} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}-1} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} \right) \right] \frac{g}{\sqrt{2}} W_P^1 \\ \qquad - \frac{1}{2} \left[c_{\mathfrak{D}-4}^{\mathfrak{D}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) + c_{\mathfrak{D}-3}^{\mathfrak{D}} \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)} \right) \right] \frac{g}{\sqrt{2}} W_P^1, \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P. \end{cases}$$

Denote

$$\begin{cases} l_1' \triangleq \rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)} \\ \quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) \\ l_2' \triangleq \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)} \\ \quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2} \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right) \\ \nu_1' \triangleq \rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)} \\ \quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) \\ \nu_2' \triangleq \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)} \\ \quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2} \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right) \end{cases}.$$

Further more it is obtained that

$$\begin{cases} l_L' = l_L \\ \quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) \\ \quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right) \\ \nu_L' = \nu_L \\ \quad + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-2}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}-1}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-3}^{\mathfrak{D}}}{2\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-4)} + \rho_{(\mathfrak{D}-4)\mathfrak{D}} \right) \\ \quad + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-2}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}-1}}{2\sqrt{2}} \left(\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} \right) + \frac{c_{\mathfrak{D}-4}^{\mathfrak{D}}}{2\sqrt{2}} \left(\rho_{\mathfrak{D}(\mathfrak{D}-3)} + \rho_{(\mathfrak{D}-3)\mathfrak{D}} \right) \end{cases}.$$

Substitute them into the above results of calculation, then it is obtained that

$$\begin{cases} l_{L;P} = \partial_P l_L - g l_L Z_P - g l_R A_P - g \nu'_L W^1_P \\ l_{R;P} = \partial_P l_R - g l_R Z_P - g l_L A_P \\ \nu_{L;P} = \partial_P \nu_L - g \nu_L Z_P - g l'_L W^1_P \\ \nu_{R;P} = \partial_P \nu_R - g \nu_R Z_P \end{cases} . \tag{48}$$

Definition 9.6. The above equations are called the **evolution forms of weak-electromagnetic charges** of weak-electromagnetic-strong reference-system. Specially, for a lepton reference-system, they are called the **evolution forms of leptons**. l' and ν' are called the **eigen charges about weak interaction**, correspondingly l and ν are called the **eigen charges about mass**.

Remark 9.1. For a lepton reference-system, it is seen from the expressions of l' and ν' that the three generations of leptons may be distinguished by charges $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}(\mathfrak{D}-3)}$, $\rho_{(\mathfrak{D}-3)\mathfrak{D}}$, $\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}$, $\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)}$, $\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}$, $\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)}$, $\rho_{\mathfrak{D}(\mathfrak{D}-4)}$, $\rho_{(\mathfrak{D}-4)\mathfrak{D}}$ in l' and ν'.

For example, we can imagine that the definitions of electron and electron-neutrino might be

$$\begin{cases} e \triangleq l = \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-4)}, \quad \rho_{(\mathfrak{D}-3)(\mathfrak{D}-3)}\right) \\ \nu_e \triangleq \nu = \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-4)}, \quad \rho_{(\mathfrak{D}-4)(\mathfrak{D}-3)}\right) \end{cases},$$

and the definitions of muon and muon-neutrino might be

$$\begin{cases} \mu \triangleq a_\mu e + \frac{1}{2}\Big(a_{\mu}{}^{\mathfrak{D}-2}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + a_{\mu}{}^{\mathfrak{D}-1}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + a_{\mu}{}^{\mathfrak{D}}_{\mathfrak{D}-4}\rho_{\mathfrak{D}(\mathfrak{D}-4)}, \\ \qquad a_{\mu}{}^{\mathfrak{D}-2}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + a_{\mu}{}^{\mathfrak{D}-1}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + a_{\mu}{}^{\mathfrak{D}}_{\mathfrak{D}-3}\rho_{\mathfrak{D}(\mathfrak{D}-3)}\Big) \\ \nu_\mu \triangleq b_\mu \nu_e + \frac{1}{2}\Big(b_{\mu}{}^{\mathfrak{D}-2}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} + b_{\mu}{}^{\mathfrak{D}-1}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)} + b_{\mu}{}^{\mathfrak{D}}_{\mathfrak{D}-3}\rho_{\mathfrak{D}(\mathfrak{D}-4)}, \\ \qquad b_{\mu}{}^{\mathfrak{D}-2}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} + b_{\mu}{}^{\mathfrak{D}-1}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)} + b_{\mu}{}^{\mathfrak{D}}_{\mathfrak{D}-4}\rho_{\mathfrak{D}(\mathfrak{D}-3)}\Big) \end{cases},$$

and the definitions of tauon and tauon-neutrino might be

$$\begin{cases} \tau \triangleq a_\tau \mu + \frac{1}{2}\big(a_{\tau}{}^{\mathfrak{D}-2}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + a_{\tau}{}^{\mathfrak{D}-1}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + a_{\tau}{}^{\mathfrak{D}}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-4)\mathfrak{D}}, \\ \qquad a_{\tau}{}^{\mathfrak{D}-2}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + a_{\tau}{}^{\mathfrak{D}-1}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + a_{\tau}{}^{\mathfrak{D}}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-3)\mathfrak{D}}\big) \\ \nu_\tau \triangleq b_\tau \nu_\mu + \frac{1}{2}\Big(b_{\tau}{}^{\mathfrak{D}-2}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + b_{\tau}{}^{\mathfrak{D}-1}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + b_{\tau}{}^{\mathfrak{D}}_{\mathfrak{D}-3}\rho_{(\mathfrak{D}-4)\mathfrak{D}}, \\ \qquad b_{\tau}{}^{\mathfrak{D}-2}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + b_{\tau}{}^{\mathfrak{D}-1}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + b_{\tau}{}^{\mathfrak{D}}_{\mathfrak{D}-4}\rho_{(\mathfrak{D}-3)\mathfrak{D}}\Big) \end{cases},$$

where a_τ, b_τ, $a_{\tau}{}^{m}_{n}$, $b_{\tau}{}^{m}_{n}$ are all constants.

The above discussion indicates that the MNS mixing of leptons is determined by geometric properties of reference-system. It is an issue worthy of further study that how to use this new approach to explain experiment data.

Although the above definitions of electron, muon and tauon are just preliminary ideas, it is sure that when the three generations of leptons and its mixing are constructed in the above way, they will no longer be artificial postulates like those in traditional theory. Such constructive definitions must become our goal.

Discussion 9.3. According to the results of Discussion 9.1 , as well as Definition 9.2 , Definition 9.3 and Definition 9.4 , calculate the evolution forms of color charges and it is obtained that

$$
\begin{aligned}
d_{1L;P} &= \partial_P d_{1L} - g_s d_{1L} U_P^1 + g_s d_{2L} V_P^1 - g_s d_{3L} V_P^1 \\
&- g_s u_{1L} X_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} + \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} \\
&- \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
&- \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
d_{2L;P} &= \partial_P d_{2L} - g_s d_{2L} U_P^2 + g_s d_{3L} V_P^2 - g_s d_{1L} V_P^2 \\
&- g_s u_{2L} X_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} + \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} \\
&- \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
&- \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
d_{3L;P} &= \partial_P d_{3L} - g_s d_{3L} U_P^3 + g_s d_{1L} V_P^3 - g_s d_{2L} V_P^3 \\
&- g_s u_{3L} X_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} + \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} \\
&- \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \qquad , \\
&- \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
d_{1R;P} &= \partial_P d_{1R} - g_s d_{1L} V_P^1 + g_s d_{2L} U_P^1 - g_s d_{3L} U_P^1 \\
&+ g_s u_{1L} Y_P^{23} + \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12}, \\
d_{2R;P} &= \partial_P d_{2R} - g_s d_{2L} V_P^2 + g_s d_{3L} U_P^2 - g_s d_{1L} U_P^2 \\
&+ g_s u_{2L} Y_P^{31} + \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23}, \\
d_{3R;P} &= \partial_P d_{3R} - g_s d_{3L} V_P^3 + g_s d_{1L} U_P^3 - g_s d_{2L} U_P^3 \\
&+ g_s u_{3L} Y_P^{12} + \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31}, \\
u_{1L;P} &= \partial_P u_{1L} - g_s u_{1L} U_P^1 - \frac{g_s}{2} u_{2L} X_P^{12} - \frac{g_s}{2} u_{2L} Y_P^{12} - \frac{g_s}{2} u_{3L} X_P^{31} + \frac{g_s}{2} u_{3L} Y_P^{31} \\
&- g_s d_{1L} X_P^{23} + g_s d_{2L} Y_P^{23} - g_s d_{3L} Y_P^{23} \\
&- \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1, \\
&- \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 \\
u_{2L;P} &= \partial_P u_{2L} - g_s u_{2L} U_P^2 - \frac{g_s}{2} u_{3L} X_P^{23} - \frac{g_s}{2} u_{3L} Y_P^{23} - \frac{g_s}{2} u_{1L} X_P^{12} + \frac{g_s}{2} u_{1L} Y_P^{12} \\
&- g_s d_{2L} X_P^{31} + g_s d_{3L} Y_P^{31} - g_s d_{1L} Y_P^{31} \\
&- \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}\right) \frac{g}{\sqrt{2}} W_P^1, \\
&- \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}\right) \frac{g}{\sqrt{2}} W_P^1
\end{aligned}
$$

$$
\begin{aligned}
u_{3L;P} &= \partial_P u_{3L} - g_s u_{3L} U_P^3 - \frac{g_s}{2} u_{1L} X_P^{31} - \frac{g_s}{2} u_{1L} Y_P^{31} - \frac{g_s}{2} u_{2L} X_P^{23} + \frac{g_s}{2} u_{2L} Y_P^{23} \\
&\quad - g_s d_{3L} X_P^{12} + g_s d_{1L} Y_P^{12} - g_s d_{2L} Y_P^{12} \\
&\quad - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)} \right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-3} \left(\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)} \right) \frac{g}{\sqrt{2}} W_P^1, \\
&\quad - \frac{1}{2} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)} \right) \frac{g}{\sqrt{2}} W_P^1 - \frac{1}{2} c_{\mathfrak{D}}^{\mathfrak{D}-4} \left(\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)} \right) \frac{g}{\sqrt{2}} W_P^1 \\
u_{1R;P} &= \partial_P u_{1R} - g_s u_{1R} U_P^1 + \frac{g_s}{2} u_{2R} X_P^{12} + \frac{g_s}{2} u_{2R} Y_P^{12} + \frac{g_s}{2} u_{3R} X_P^{31} - \frac{g_s}{2} u_{3R} Y_P^{31}, \\
u_{2R;P} &= \partial_P u_{2R} - g_s u_{2R} U_P^2 + \frac{g_s}{2} u_{3R} X_P^{23} + \frac{g_s}{2} u_{3R} Y_P^{23} + \frac{g_s}{2} u_{1R} X_P^{12} - \frac{g_s}{2} u_{1R} Y_P^{12}, \\
u_{3R;P} &= \partial_P u_{3R} - g_s u_{3R} U_P^3 + \frac{g_s}{2} u_{1R} X_P^{31} + \frac{g_s}{2} u_{1R} Y_P^{31} + \frac{g_s}{2} u_{2R} X_P^{23} - \frac{g_s}{2} u_{2R} Y_P^{23}.
\end{aligned}
$$

Discussion 9.4. Similar to Discussion 9.2 , denote

$$
\begin{cases}
d'_{1L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) \\
d'_{2L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) \\
d'_{3L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) \\
u'_{1L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) \\
u'_{2L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-1}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-1)} + \rho_{(\mathfrak{D}-1)(\mathfrak{D}-3)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) \\
u'_{3L} \triangleq \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)\mathfrak{D}} + \rho_{\mathfrak{D}(\mathfrak{D}-3)}) \\
\qquad + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-3} (\rho_{(\mathfrak{D}-4)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-4)}) + \frac{1}{2\sqrt{2}} c_{\mathfrak{D}-2}^{\mathfrak{D}-4} (\rho_{(\mathfrak{D}-3)(\mathfrak{D}-2)} + \rho_{(\mathfrak{D}-2)(\mathfrak{D}-3)})
\end{cases}
.
$$

Substitute them into the results of Discusstion 9.3 , and it is obtained that

$$
\begin{cases}
d_{1L;P} = \partial_P d_{1L} - g_s d_{1L} U_P^1 + g_s d_{2L} V_P^1 - g_s d_{3L} V_P^1 \\
\qquad - g_s u_{1L} X_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} + \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - g u'_{1L} W_P^1 \\
d_{2L;P} = \partial_P d_{2L} - g_s d_{2L} U_P^2 + g_s d_{3L} V_P^2 - g_s d_{1L} V_P^2 \\
\qquad - g_s u_{2L} X_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} + \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - g u'_{2L} W_P^1 \\
d_{3L;P} = \partial_P d_{3L} - g_s d_{3L} U_P^3 + g_s d_{1L} V_P^3 - g_s d_{2L} V_P^3 \\
\qquad - g_s u_{3L} X_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} + \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - g u'_{3L} W_P^1
\end{cases}
,
$$

$$
\begin{cases}
d_{1R;P} = \partial_P d_{1R} - g_s d_{1L} V_P^1 + g_s d_{2L} U_P^1 - g_s d_{3L} U_P^1 \\
\qquad + g_s u_{1L} Y_P^{23} + \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31} - \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} \\
d_{2R;P} = \partial_P d_{2R} - g_s d_{2L} V_P^2 + g_s d_{3L} U_P^2 - g_s d_{1L} U_P^2 \\
\qquad + g_s u_{2L} Y_P^{31} + \frac{g_s}{2} u_{3L} X_P^{12} - \frac{g_s}{2} u_{3L} Y_P^{12} - \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} \\
d_{3R;P} = \partial_P d_{3R} - g_s d_{3L} V_P^3 + g_s d_{1L} U_P^3 - g_s d_{2L} U_P^3 \\
\qquad + g_s u_{3L} Y_P^{12} + \frac{g_s}{2} u_{1L} X_P^{23} - \frac{g_s}{2} u_{1L} Y_P^{23} - \frac{g_s}{2} u_{2L} X_P^{31} - \frac{g_s}{2} u_{2L} Y_P^{31}
\end{cases},
$$

$$
\begin{cases}
u_{1L;P} = \partial_P u_{1L} - g_s u_{1L} U_P^1 - \frac{g_s}{2} u_{2L} X_P^{12} - \frac{g_s}{2} u_{2L} Y_P^{12} - \frac{g_s}{2} u_{3L} X_P^{31} + \frac{g_s}{2} u_{3L} Y_P^{31} \\
\qquad - g_s d_{1L} X_P^{23} + g_s d_{2L} Y_P^{23} - g_s d_{3L} Y_P^{23} - g d'_{1L} W_P^1 \\
u_{2L;P} = \partial_P u_{2L} - g_s u_{2L} U_P^2 - \frac{g_s}{2} u_{3L} X_P^{23} - \frac{g_s}{2} u_{3L} Y_P^{23} - \frac{g_s}{2} u_{1L} X_P^{12} + \frac{g_s}{2} u_{1L} Y_P^{12} \\
\qquad - g_s d_{2L} X_P^{31} + g_s d_{3L} Y_P^{31} - g_s d_{1L} Y_P^{31} - g d'_{2L} W_P^1 \\
u_{3L;P} = \partial_P u_{3L} - g_s u_{3L} U_P^3 - \frac{g_s}{2} u_{1L} X_P^{31} - \frac{g_s}{2} u_{1L} Y_P^{31} - \frac{g_s}{2} u_{2L} X_P^{23} + \frac{g_s}{2} u_{2L} Y_P^{23} \\
\qquad - g_s d_{3L} X_P^{12} + g_s d_{1L} Y_P^{12} - g_s d_{2L} Y_P^{12} - g d'_{3L} W_P^1
\end{cases},
$$

$$
\begin{cases}
u_{1R;P} = \partial_P u_{1R} - g_s u_{1R} U_P^1 + \frac{g_s}{2} u_{2R} X_P^{12} + \frac{g_s}{2} u_{2R} Y_P^{12} + \frac{g_s}{2} u_{3R} X_P^{31} - \frac{g_s}{2} u_{3R} Y_P^{31} \\
u_{2R;P} = \partial_P u_{2R} - g_s u_{2R} U_P^2 + \frac{g_s}{2} u_{3R} X_P^{23} + \frac{g_s}{2} u_{3R} Y_P^{23} + \frac{g_s}{2} u_{1R} X_P^{12} - \frac{g_s}{2} u_{1R} Y_P^{12} \\
u_{3R;P} = \partial_P u_{3R} - g_s u_{3R} U_P^3 + \frac{g_s}{2} u_{1R} X_P^{31} + \frac{g_s}{2} u_{1R} Y_P^{31} + \frac{g_s}{2} u_{2R} X_P^{23} - \frac{g_s}{2} u_{2R} Y_P^{23}
\end{cases}.
$$

Definition 9.7. The above results are called the **evolution forms of color charges**.

Remark 9.2. The charges

$$
d'_{1L}, \quad d'_{2L}, \quad d'_{3L}, \quad u'_{1L}, \quad u'_{2L}, \quad u'_{3L}
$$

in the above discussion indicate that the CKM mixing of color charges are also determined by geometric properties of reference-system. It is also an issue worthy of further study that how to use this new approach to explain experiment data.

Discussion 9.5. An ontological hadron may participate not only weak-electromagnetic interaction but also strong interaction, so in epistemology the evolution form of a hadron charge may generally be expressed as

$$
q_{;P} \triangleq (Ad_{1S} + Bd_{2S} + Cd_{3S} + Du_{1S} + Eu_{2S} + Fu_{3S} + Gl_S + H\nu_S)_{;P} ,
$$

where q represents a hadron charge and S represents left-spin L or right-spin R.

10 Summary and supplement

10.1 A supplement to the logical structure of theoretical physics

This paper argues that the research objects of physics have only two kinds, one is the physical realities that can be actually detected in ontology, the other is the mathematical concepts that can be strictly defined in epistemology, nothing else.

1. In order to clarify the logical structure of theoretical physics, the ontology and epistemology of theoretical physics have to be clarified first. A convenient way is to construct some philosophical starting points as following.

(1) Fundamental philosophical starting point.

(a) Ontological fundamental category: **existence**.

(b) Ontological fundamental principle: the world is existent.

(c) Epistemological fundamental category: **cognition**.

(d) Epistemological fundamental principle: existence is cognizable.

(2) Natural philosophical starting point.

(a) Ontological fundamental existence: **matter in motion and motion of matter**, abbreviated as **matter-motion**, or **matter** for short, or **reality**.

(b) Ontological fundamental principle: the world is existent as matter-motion.

(c) Epistemological fundamental cognition: **theory in practice and practice of theory**, abbreviated as **theory-practice**, or **theory** for short

(d) Epistemological fundamental principle: matter-motion is cognized by using theory-practice.

(3) Starting point of physics.

(a) Ontological fundamental object: **physical reality**.

(b) Ontological fundamental principle: matter-motion is existent as a physical reality.

(c) Epistemological fundamental object: **mathematical concept**.

(d) Epistemological fundamental principle: a physical reality is cognized by using a mathematical concept. All the physical assertions of this paper are expressed according to this principle.

(4) Starting point of theoretical physics.

(a) Ontological fundamental physical reality: **interaction field and relative motion of field**. As matter of motion, the physical reality manifests itself as an interaction field; as motion of matter, the physical reality manifests itself as a relative motion of field. They are unified in physical reality.

(b) Ontological fundamental principle: a physical reality is existent as an interaction field and relative motion of field.

(c) Epistemological fundamental concept: **reference-system**.

(d) Epistemological fundamental principle: The basic principle of theoretical physics.

In a word, physical significances of an axiomatic mathematical theory are obtained just from the above starting points.

2. In order to clarify the logical structure of theoretical physics, the logical structure of mathematics also has to be clarified first. Based on the above starting points, the following viewpoints are reasonable.

(1) **Mathematics** is the theory-practice about concept. Concept is the fundamental philosophical category of mathematics.

(2) **Logics** is the theory-practice about spekulation. Spekulation is the fundamental philosophical category of logics.

(3) Mathematics is not similar to other disciplines to research which specified concept is used to cognize which specified reality, but to research both the concept and reality uniformly in the name of concept. Those above epistemological fundamental principles constitute the ultimate origin of the effectiveness of applications of mathematics to natural sciences.

(4) **Basic principle of mathematics: a concept is established by spekulation, but spekulation do not alway establish a concept.**

(5) A **proposition** is spekulation about concept. A concept in a proposition is called a **class**. A class and a proposition about this class are put together to be called an **identification**. Concretely, if class x satisfies proposition $P(x)$, then the identification is denoted by $x|P(x)$, and say the class x meets the identification $x|P(x)$.

(6) According to (5), further more, a concept can be established by a proposition. Concretely, a concept X can be expressed as the whole of all the classes which meet identification $x|P(x)$. The identification $x|P(x)$ is called the **intension (connotation)** of X. Class X is called the **extension (denotation)** of $x|P(x)$, denoted by $\{x|P(x)\}$.

If class a meets identification $x|P(x)$, say a belongs to X, and say a is an element of X, denoted by $a \in X$. Otherwise, say a does not belong to X, denoted by $a \notin X$.

(7) Now that intension and extension have been defined, the basic principle of mathematics can also be expressed as a conclusion that **a concept always has an intension, but an identification does not always has an extension.** Thus, **it is conditional for an identification to have an extension.** So in epistemology, we have to **appoint which identifications have extensions**, and call it a convention.

(8) Different ways of such conventions reflect the same essence of the world from different perspectives. It makes the conventions obtain sufficient legitimacy.

Concretely, in order to establish the foundation of mathematics, no matter the set theory or the category theory being adopted, we can always establish the following five **basic conventions** first.

(I) Axiom of extensionality. There exists an identification that has extension, and equivalent identifications have the same extension, which is expressed by the notation "=".

(II) Axiom of single element. If identification $x|P(x)$ has extension, denoted by a, then identification $x|x = a$ also has extension, denoted by $\{a\}$.

(III) Axiom of union-class. If both of identifications $x|P_1(x)$ and $x|P_2(x)$ have extensions, then identification $x|P_1(x) \vee P_2(x)$ also has extension.

(IV) Axiom of sub-class. If any of identifications $x|P_1(x)$ and $x|P_2(x)$ has extension, then identification $x|P_1(x) \wedge P_2(x)$ also has extension.

(V) Axiom of power-class. If identification $a|P(a)$ has extension, denoted by x, then identification "$z|\forall w \in z,\ w \in x$" also has extension.

From the perspective of concrete construction, except the above five basic conventions, a pragmatic set theory still requires other conventions of ZFC axiom system. The reason why ZFC system is a good axiom system is that it has enough many extensions and can deduce enough rich concepts.

From the perspective of abstract structure, except the above five basic conventions, there is **no need to require** other **universal** conventions. It just needs to specifically give conventions to the definition of each specified category.

In general, set theory and category theory respectively appoint extensions of the identifications in the two ways of concrete construction and abstract structure, thereby respectively establish the foundation for mathematics from different perspectives.

Take the concept of real numbers for example. From the perspective of abstract structure, it just needs to put some abstract conventions together so that defines the complete archimedean total ordered field, thereby forms an abstract concept of real numbers. From the perspective of concrete construction, it needs to construct natural numbers from empty set, then construct integers and rationals, finally construct irrational numbers by Dedekind cut, thereby forms a concrete concept of real numbers.

Strictly, they are two concepts defined respectively in two different ways, but they reflect the same mathematical intuition and they complement each other. It is such a situation that gives real numbers a complete description. Similarly, set theory and category theory also complement each other, they together give mathematics a complete description.

So does theoretical physics. In epistemology, we should not just satisfy to use the abstract way like traditional theory to discuss energy-momentum and describe gauge fields. We should be able to concretely construct all the concepts such as energy-momentum and gauge fields. It is precisely because mathematics has the logical structure mentioned above that the feasibility of achieving such a target is guaranteed. The theory of this paper has made an effective attempt for this target.

3. In order to achieve the above purpose, we have to make sure that ontological reality and epistemological concept are not confused.

A concept must have a strict definition, and a reality must can be detected. Difference between them is obvious. In traditional physical theory, we usually see such a practice that an ontological reality is used as the connotation to define an epistemological concept. For example, traditional physical theory ususally supposes that e_L is a free electron field and ν_L is a free neutrino field, which both satisfy the free Dirac equation, so e_L and ν_L cannot be distinguished by mathematical connotation. However the traditional theory looks e_L and ν_L as different concepts. This is exactly a practice to define epistemological concepts with ontological realities as their connotations. Such a practice is harmful, because it is easy to cause confusion of cognition and thereby conceals the true connotations of the mathematical concepts of e_L and ν_L.

In this paper it is suggested that a physical theory with a clear logical structure should strictly distinguish ontological objects and epistemological objects in discussion. It is best to execute deductive logic for epistemological concepts and to execute inductive logic for ontological objects. It should be noticed that the mathematical induction is always expressed in form of deduction, so it should be attributed to deductive logic.

1. Take some examples as following.

(1) The concepts of time metric and space metric in section 2.2.9.1 , and the concept of actual evolution in section 2.4.4.2 are all universal geometric properties on geometric manifold. According to the Corollary 3 in section 2.2.5.2 , they can be used to cognize some kinds of ontological universal physical properties.

(2) The concept of gravitational field in section 5.2.1 and the concept of inertial-system in section 6.3.8.2 are both concepts of reference-system. According to the basic principle of theoretical physics in section 1.2 , they can be used to cognize some kinds of physical reality.

The above are all deductive logical discussions.

2. Then, concretely, what kinds of universal physical properties and what kinds of physical realities after all? It can only be dealt with by inductive logics. By ontologically executing inductive logic for physical realities, the following physical assertions can just be obtained.

(1) Time and space law. The time interval and space interval of physical reality in ontology are cognized by using the concepts of time metric and space metric in epistemology.

(2) Evolution law. The actual evolution of physical reality in ontology is cognized by using the concept of actual evolution in epistemology.

(3) Principle of equivalence. The gravitational field as a physical reality in ontology is cognized by using the concept of gravitational field in epistemology.

(4) Newton's first law. The physical reality of inertial relative motion and no classical spacetime interaction in ontology is cognized by using the concept of inertial-system in epistemology.

3. In the sense of deductive logics, we say the basic principle in section 1.2 is the unique axiom of Hilbert's 6th problem for theoretical physics at the most basic level.

4. In the sense of the above cooperation between mathematical deduction and physical induction, we say the basic principle in section 1.2 is the unique fundamental physical principle for theoretical physics at the most basic level.

10.2 Summary

1. This paper gives an improved expression of Erlangen program. It enhances the flexibility of applications of Erlangen program.

2. This paper generalizes Riemannian manifold to geometryic manifold. On geometric manifold, the Riemannian geometry is completely brought into the geometric framework of the improved Erlangen program.

3. This paper strictly defines the concept of reference-system and generalizes the concept of intrinsic geometric, so that the traditional intrinsic geometry based on the first fundamental form becomes a subgeometry of the intrinsic geometry of this paper.

4. This paper defines the concept of simple connection, which reflects more bending properties of manifold than Levi-Civita connection.

5. In this paper it is suggested that a research for a kind of mathematical intuition is regarded as a complete one only if it is from both the ways of abstract structure and concrete construction. This viewpoint is carried out and practiced in the process of constructing the foundation of theoretical physics in this paper. The basic framework of theoretical physics is strictly developed by constructivity methods of mathematics under a unified view of time and space, and based on a unique fundamental principle. All of those redundant principles, equations, and postulates at the most basic level in traditional theories are turned into theorems which hold automatically in the theory of this paper, so that various relative motions and interactions can be described in a unified form.

6. In this paper it is suggested that the research objects of physics have only two kinds, one is the physical realities that can be actually detected in ontology, the other is the mathematical concepts that can be strictly defined in epistemology, nothing else. It is best to carry out deductive logic for epistemological objects and carry out inductive logic for ontological objects, and strictly distinguish ontology and epistemology in discussion. A physical assertion should be expressed as a normalized language structure like "a physical reality is cognized by using a mathematical concept". The theory of this paper practices the above viewpoints, so the logical structure of theory is more rigor and clearer than traditional theory. This paper gives a feasible solution to the problem that traditional physical theory for a long time confused and mixed ontology and epistemology.

7. Except the concept of reference-system on manifold, this paper presents some ideas such as that time metric is the total space metric, actual evolution direction is the gradient direction, propagator and wave function reflect the distribution density of actual evolution directions, typical gauge potentials are described by simple connection, etc., as well as evolution lemma, and they all play key roles in the construction of theory.

8. Concepts such as various charges, gauge potentials, energy-momentum, etc. are defined by constructivity methords in this paper, so that their connotations become more concrete. These are supplements to traditional physics.

There is an obvious difference between the theory of this paper and the traditional theory.

(1) The traditional theory starts from a very large symmetry group, and reduces symmetries in the way of some kind of breaking to approach the target geometry.

(2) The theory of this paper starts from the smallest symmetry group $\{e\}$, and adds symmetries in the way of some kind of symmetry conditions to approach the target geometry.

These two ways must lead to the same destination. They both go towards the same specific geometry. However the way of this paper has more advantages.

(1) The theory of this paper is based on intrinsic geometry. It has a more strong ability of describing shapes. And the theoretical form is more fundametal, which fits in with the geometric essence precisely.

Compared with the abstract way of dicussing the degeneration of group (i.e. symmetry breaking), the construcive way of adding symmetry conditions to intrinsic geometry in this paper is more suitable for the case of complicated group structure. It is easier for the theory of this paper to clarify the essence of many things. Those characteristics which are artificially postulated by traditional theory can naturally appear in the way of intrinsic geometry, such as the constructions of concepts of electric charged leptons and neutrinos, the constructions of concepts of down-type color charges and up-type color charges, the chirality asymmetry of charges, the MNS mixing of leptons, the CKM mixing of color charges, etc.

(2) In the theory of this paper, typical gauge fields and gravitational fields are unified in the viewpoint of spacetime. They are completely consistent and perfectly coordinated. Typical gauge fields are described by the intrinsic geometry about internal space, gravitational fields are described by the intrinsic geometry about external space, and they are unified in intrinsic geometry.

9. Important issues and directions to be further studied.

(1) The discussions of this paper do not involve angular momentum at all. It needs further study for this aspect.

(2) This paper strictly restricts the scope of discussions to the relative motion and interaction with no more than two reference-systems. It does not involve those theories at a higher level with more than two reference-systems. Therefore, with the current degree of development of this theory, some problems, such as the origin of rest-mass of weak interaction gauge field in section 7.2.4 and the relation between zero-spin neutrio pair and the peak [1,8] observed by the LHC near 125 GeV, can be just given qualitative judgements. It needs further development of theory and exploration of experiment.

(3) Further research of the complex-valued evolution equation in section 6.3.7.1 will surely depend on the degree of development of the theory in section 2.4.14.2 . In order to grasp the main line of geometric thought, section 2.4.14.2 just gives the most basic, the most core and the most representative concepts and framework, without developing further more. These contents still need to be developed deeply, so that achieving the degree of being able to concretely calculate the scattering problem.

(4) Proposition 9.1 has not been fully proved yet and needs further study.

(5) The MNS mixing of three generations of leptons and the CKM mixing of three generations of color charges in Discussion 9.2 and Remark 9.1 have been constructed. It is worthy of further research for physicists that how to explain experiment data in this new way.

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