



An Introduction to Neutrosophic Bipolar Vague Topological Spaces

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Abstract: The main objective of this paper is to make known to a new concept of generalised neutrosophic bipolar vague sets and also defined neutrosophic bipolar vague topology in topological spaces. Also, we introduce generalized neutrosophic bipolar vague closed sets and conferred its properties.

Keywords: Bipolar set, Vague set, Neutrosophic set, Neutrosophic Bipolar Vague set, Neutrosophic Bipolar Vague Topological Spaces.

1. Introduction

Levine [24] studied the Generalized closed sets in general topology. Several investigations were conducted on the generalizations of the notion of the fuzzy set, after the introduction of the concept of fuzzy sets by Zadeh [34]. In the traditional fuzzy sets, the membership degree of component ranges over the interval $[0, 1]$. Few types of fuzzy set extensions in the fuzzy set theory are present, for example, intuitionistic fuzzy sets[12], interval-valued fuzzy sets[32], vague sets[30] etc. As a generalization of Zadeh's fuzzy set, the notion of vague set theory was first introduced by Gau W.L and Buehrer D.J [22]. In 1996, H.Bustince & P.Burillo indicated that vague sets are intuitionistic fuzzy sets [15].

Intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets can handle only unfinished information but not the indeterminate and unreliable information which happens normally in actual circumstances. Hence, the conception of a neutrosophic set is very common, and then it can overcome the aforesaid issues on the intuitionistic fuzzy set and the interval-valued intuitionistic fuzzy set. In 1995, the definition of Smarandache's neutrosophic set, neutrosophic sets and neutrosophic logic have been useful in many real applications to handle improbability. Neutrosophy is a branch of philosophy which studies the source, nature and scope of neutralities, as well as their interactions with different ideational scales [31]. The neutrosophic set uses one single value to indicate the truth-membership grade, indeterminacy-membership degree and falsity membership grade of an element in the universe X . The theory has been brought into extensive application in varieties of field [1-6, 8, 10, 11, 14, 17, 23, 27, 33, 35] for dealing with indeterminate and unreliable information in actual domain. The conception of Neutrosophic Topological space was introduced by A.A.Salama and S.A.Alblowi [29].

Bipolar-valued fuzzy sets, which was introduced by Lee [25, 26] is an extension of fuzzy sets whose membership degree range is extended from the interval $[0, 1]$ to $[-1, 1]$. The membership degrees of the Bipolar valued fuzzy sets signify the degree of satisfaction to the property analogous to a fuzzy set and its counter-property in a bipolar valued fuzzy set, if the membership degree is 0 it means that the elements are unrelated to the corresponding property. Furthermore if the membership degree is on $(0, 1]$ it indicates that the elements somewhat fulfil the property, and if the membership degree is on

[-1,0) it indicates that elements somewhat satisfy the entire counter property. After that, Deli et al. [21] announced the concept of bipolar neutrosophic sets, as an extension lead of neutrosophic sets. In the bipolar neutrosophic sets, the positive membership degree $T^+(x), I^+(x), F^+(x)$ signifies the truth membership, indeterminate membership and false membership of an element $x \in X$ analogous to a bipolar neutrosophic set A and the negative membership degree $T^-(x), I^-(x), F^-(x)$ signifies the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implied counter-property analogous to a bipolar neutrosophic set A. There are quite a few extensions of Neutrosophic Bipolar sets such as Neutrosophic Bipolar Soft sets [7] and Rough Neutrosophic Bipolar sets [28].

Neutrosophic vague set is a combination of neutrosophic set and vague set which was well-defined by Shawkat Alkhazaleh [30]. Neutrosophic vague theory is a useful tool to practise incomplete, indeterminate and inconsistent information. In this paper, we introduced the perception of a neutrosophic bipolar vague set as a combination of neutrosophic set, Bipolar set and vague set and we also define the concept of generalised Neutrosophic Bipolar Vague set.

2. Preliminaries

Definition 2.1[16]: Let X be the universe. Then a bipolar valued fuzzy sets, A on X is defined by positive membership function $\mu_A^+ : X \rightarrow [0,1]$ and a negative membership function $\mu_A^- : X \rightarrow [-1,0]$. For sake of easiness, we shall practice the symbol $A = \{ \langle x, \mu_A^+(x), \mu_A^-(x) \rangle : x \in X \}$.

Definition 2.2[18]: Let A and B be two bipolar valued fuzzy sets then their union, intersection and complement are well-defined as follows:

- (i) $\mu_{A \cup B}^+(x) = \max \{ \mu_A^+(x), \mu_B^+(x) \}$.
- (ii) $\mu_{A \cup B}^-(x) = \min \{ \mu_A^-(x), \mu_B^-(x) \}$.
- (iii) $\mu_{A \cap B}^+(x) = \min \{ \mu_A^+(x), \mu_B^+(x) \}$.
- (iv) $\mu_{A \cap B}^-(x) = \max \{ \mu_A^-(x), \mu_B^-(x) \}$.
- (v) $\mu_A^+(x) = 1 - \mu_A^-(x)$ and $\mu_A^-(x) = -1 - \mu_A^+(x)$ for all $x \in X$.

Definition 2.3[15]: A vague set A in the universe of discourse U is a pair (t_A, f_A) where $t_A : U \rightarrow [0,1]$, $f_A : U \rightarrow [0,1]$ denote the mapping such that $t_A + f_A \leq 1$ for all $u \in U$. The function t_A and f_A are called true membership function and false membership function respectively. The interval $[t_A, 1-f_A]$ is called the vague value of u in A, and denoted by $v_A(u)$, i.e $v_A(u) = [t_A, 1-f_A]$.

Definition 2.4[15]: Let A be a non-empty set and the vague set A and B in the form $A = \{ \langle x, t_A, 1-f_A \rangle : x \in X \}$, $B = \{ \langle x, t_B, 1-f_B \rangle : x \in X \}$.

Then

- (i) $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $1-f_B(x) \leq 1-f_A(x)$.
- (ii) $A \cup B = \{ \langle \max(t_A(x), t_B(x)), \max(1-f_A(x), 1-f_B(x)) \rangle : x \in X \}$.
- (iii) $A \cap B = \{ \langle \min(t_A(x), t_B(x)), \min(1-f_A(x), 1-f_B(x)) \rangle : x \in X \}$.
- (iv) $\bar{A} = \{ \langle x, f_A(x), 1-t_A(x) \rangle : x \in X \}$.

Definition 2.5[14]: Let X be a universe of discourse. Then a neutrosophic set is well-defined as: $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$, which is categorized by a truth-membership function $T_A : X \rightarrow]0-, 1+[$, an indeterminacy membership function $I_A : X \rightarrow]0-, 1+[$ and a falsity-membership function $F_A : X \rightarrow]0-, 1+[$. There is no restriction to the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) \leq \sup I_A(x) \leq \sup F_A(x) \leq 3^+$.

Definition 2.6[30]: A neutrosophic vague set A_{NBV} (NVS in short) on the universe of discourse X written as,

$A_{NBV} = \{ \langle \hat{T}_{NBV}(x), \hat{I}_{NBV}(x), \hat{F}_{NBV}(x) \rangle : x \in X \}$ whose truth-membership, indeterminacy-membership and falsity-membership functions is defined as,
 $\hat{T}_{NBV}(x) = [T^-, T^+]$, $\hat{I}_{NBV}(x) = [I^-, I^+]$, $\hat{F}_{NBV}(x) = [F^-, F^+]$ where
 $T^+ = 1 - F^-$, $F^+ = 1 - I^-$ and $T^-, 0 \leq T^- + I^- + F^- \leq 2^+$.

3. Bipolar Neutrosophic Vague Set:

Under this division, we present and well-defined the notion of neutrosophic bipolar vague set and its operations.

Definition 3.1: If $A = \langle x, [T_A^-, T_A^+]^+, [I_A^-, I_A^+]^+, [F_A^-, F_A^+]^+, [T_A^-, T_A^+]^-, [I_A^-, I_A^+]^-, [F_A^-, F_A^+]^- \rangle$ and $B = \langle x, [T_B^-, T_B^+]^+, [I_B^-, I_B^+]^+, [F_B^-, F_B^+]^+, [T_B^-, T_B^+]^-, [I_B^-, I_B^+]^-, [F_B^-, F_B^+]^- \rangle$ where $T^+ = 1 - F^-$, $F^+ = 1 - T^-$, and $T^+, I^+, F^+ : X \rightarrow [2, 0]$ and $T^-, I^-, F^- : X \rightarrow [-2, 0]$ are two neutrosophic bipolar vague sets then their union, intersection and complement are well-defined as follows:

1. $A \cup B = \{ \max[T_A^-, T_B^-]^+, \max[T_A^+, T_B^+]^+, \min[I_A^-, I_B^-]^+, \min[I_A^+, I_B^+]^+, \min[F_A^-, F_B^-]^+, \min[F_A^+, F_B^+]^+, \min[T_A^-, T_B^-]^-, \min[T_A^+, T_B^+]^-, \max[I_A^-, I_B^-]^-, \max[I_A^+, I_B^+]^-, \max[F_A^-, F_B^-]^-, \max[F_A^+, F_B^+]^- \}$.
2. $A \cap B = \{ \min[T_A^-, T_B^-]^+, \min[T_A^+, T_B^+]^+, \max[I_A^-, I_B^-]^+, \max[I_A^+, I_B^+]^+, \max[F_A^-, F_B^-]^+, \max[F_A^+, F_B^+]^+, \max[T_A^-, T_B^-]^-, \max[T_A^+, T_B^+]^-, \min[I_A^-, I_B^-]^-, \min[I_A^+, I_B^+]^-, \min[F_A^-, F_B^-]^-, \min[F_A^+, F_B^+]^- \}$.
3. $\bar{A} = \langle [F_A^-, F_A^+]^+, [1 - I_A^-, 1 - I_A^+]^+, [T_A^-, T_A^+]^+, [F_A^-, F_A^+]^-, [1 - I_A^-, 1 - I_A^+]^-, [T_A^-, T_A^+]^- \rangle$.

Definition 3.2: Suppose A and B be two neutrosophic bipolar vague sets defined over a universe of disclosure X. We say that $A \subseteq B$ if and only if $[T_A^- \leq T_B^-]^+, [T_A^+ \leq T_B^+]^+, [I_A^- \geq I_B^-]^+, [I_A^+ \geq I_B^+]^+, [F_A^- \geq F_B^-]^+, [F_A^+ \geq F_B^+]^+, [T_A^- \geq T_B^-]^-, [T_A^+ \geq T_B^+]^-, [I_A^- \leq I_B^-]^-, [I_A^+ \leq I_B^+]^-, [F_A^- \leq F_B^-]^-, [F_A^+ \leq F_B^+]^-$.

Definition 3.3: A bipolar vague topology NBVT on a nonempty set X is a family NBV_τ of Neutrosophic bipolar vague set in X sustaining the following axioms:

1. $0, 1 \in NBV_\tau$.
2. $G_1 \cap G_2 \in NBV_\tau$, for any $G_1, G_2 \in NBV_\tau$.
3. $\cup G_i \in NBV_\tau$ for any arbitrary family $\{ G_i : G_i \in NBV_\tau, i \in I \}$.

Under such case the pair (X, NBV_τ) is known as the neutrosophic bipolar vague topological space and any NBVS in NBV_τ is known as bipolar vague open set in X. The complement \bar{A} of a neutrosophic bipolar vague open set (NBVOS) A in a neutrosophic bipolar vague topological space (X, NBV_τ) is referred as a neutrosophic bipolar vague closed (NBVCS) in X.

Example 3.4: Assume $X = \{u, v\}$,

$$A_{NBV} = \left\{ \frac{u}{[0.5, 0.7][0.5, 0.5][0.3, 0.5] [-0.4, -0.1] [-0.5, -0.6] [1.1, 1.4]}, \frac{v}{[0.3, 0.6][0.4, 0.4][0.4, 0.7] [-0.2, -0.2] [-0.6, -0.8] [1.2, 1.2]} \right\},$$

$$B_{NBV} = \left\{ \frac{u}{[0.5, 0.9][0.3, 0.3][0.1, 0.5] [-0.4, -0.3] [-0.4, -0.4] [1.3, 1.4]}, \frac{v}{[0.4, 0.6][0.2, 0.2][0.4, 0.6] [-0.5, -0.3] [-0.5, -0.5] [1.3, 1.5]} \right\}.$$

Then the family $NBV_\tau = \{0, 1, A, B\}$ of neutrosophic bipolar vague sets in X is a NBVT on X.

Definition 3.5: Suppose (X, NBV_τ) is a neutrosophic bipolar vague topological space and

$A = \langle x, [T_A^-, T_A^+]^+, [I_A^-, I_A^+]^+, [F_A^-, F_A^+]^+, [T_A^-, T_A^+]^-, [I_A^-, I_A^+]^-, [F_A^-, F_A^+]^- \rangle$ be a NBVS in X. Then the neutrosophic bipolar vague interior and neutrosophic bipolar vague closure of A are well-defined by,

$$NBVcl(A) = \bigcap \{ K : K \text{ is a NBVCS in } X \text{ and } A \subseteq K \},$$

$$NBVint(A) = \bigcup \{ G : G \text{ is a NBVOS in } X \text{ and } G \subseteq A \}.$$

Note that $NBVcl(A)$ is a NBVCS and $NBVint(A)$ is a NBVOS in X. Further,

1. A is a NBVCS in X iff $NBVcl(A) = A$
2. A is a NBVOS in X iff $NBVint(A) = A$.

Example 3.6: Assume that $X = \{a, b\}$,

$$A = \left\{ x, \frac{a}{[0.5, 0.7][0.5, 0.5][0.3, 0.5] [-0.4, -0.1] [-0.5, -0.6] [1.1, 1.4]}, \frac{b}{[0.3, 0.6][0.4, 0.4][0.4, 0.7] [-0.2, -0.2] [-0.6, -0.8] [1.2, 1.2]} \right\}$$

$$B = \left\{ \frac{a}{[0.5, 0.9][0.3, 0.3][0.1, 0.5] [-0.4, -0.3] [-0.4, -0.4] [1.3, 1.4]}, \frac{b}{[0.4, 0.6][0.2, 0.2][0.4, 0.6] [-0.5, -0.3] [-0.5, -0.5] [1.3, 1.5]} \right\}.$$

Then the family $NBV_\tau = \{0, 1, A, B\}$ of a neutrosophic bipolar vague sets in X is NBVT on X. If,

$$F = \left\{ \langle x, \frac{a}{[0.5, 0.4][0.5, 0.5][0.6, 0.5] [-0.6, -0.4] [-0.3, -0.3] [1.4, 1.6]}, \frac{b}{[0.5, 0.7][0.1, 0.1][0.3, 0.5] [-0.3, -0.4] [-0.2, -0.2] [1.4, 1.3]} \rangle \right\}$$

Then, $NBVint(A)=\cup\{G:G \text{ is a NBVOS in } X \text{ and } G\subseteq F\}=0$ and $NBVcl(A)=\cap\{K:K \text{ is a NBVCS in } X \text{ and } F\subseteq K\}=1$.

Proposition 3.7: For any NBVS A in (X, NBV_t) we have,

1. $NBVcl(\bar{A})=\overline{NBVint(A)}$
2. $NBVint(\bar{A})=\overline{NBVcl(A)}$.

Proof: Let $A=\{< x, [T_A^-, T_A^+]^+, [I_A^-, I_A^+]^+, [F_A^-, F_A^+]^+, [T_A^-, T_A^+]^-, [I_A^-, I_A^+]^-, [F_A^-, F_A^+]^- >\}$ and suppose that NBVOS's contained in A are indexed by the family

$\{< x, [T_{G_i}^-, T_{G_i}^+]^+, [I_{G_i}^-, I_{G_i}^+]^+, [F_{G_i}^-, F_{G_i}^+]^+, [T_{G_i}^-, T_{G_i}^+]^-, [I_{G_i}^-, I_{G_i}^+]^-, [F_{G_i}^-, F_{G_i}^+]^- >: i \in J\}$. Then

$$NBVint(A)=\{< x, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cap[I_{G_i}^-, I_{G_i}^+]^+, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cap[T_{G_i}^-, T_{G_i}^+]^-, \cup[I_{G_i}^-, I_{G_i}^+]^-, \cup[F_{G_i}^-, F_{G_i}^+]^- >\} \text{ and hence}$$

$$\overline{NBVint(A)} = \{< x, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cup[1 - I_{G_i}^-, 1 - I_{G_i}^+]^+, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cup[F_{G_i}^-, F_{G_i}^+]^-, \cap[1 - I_{G_i}^-, 1 - I_{G_i}^+]^-, \cap[T_{G_i}^-, T_{G_i}^+]^- >\} \text{ (1)}$$

Since,

$\bar{A}=\{< [F_A^-, F_A^+]^+, [1 - I_A^-, 1 - I_A^+]^+, [T_A^-, T_A^+]^+, [F_A^-, F_A^+]^-, [1 - I_A^-, 1 - I_A^+]^-, [T_A^-, T_A^+]^- >\}$. Where $[T_{G_i}^-, T_{G_i}^+]^+ \subseteq [T_A^-, T_A^+]^+, [I_{G_i}^-, I_{G_i}^+]^+ \supseteq [I_A^-, I_A^+]^+, [F_{G_i}^-, F_{G_i}^+]^+ \supseteq [F_A^-, F_A^+]^+, [T_{G_i}^-, T_{G_i}^+]^- \supseteq [T_A^-, T_A^+]^-$, $[I_{G_i}^-, I_{G_i}^+]^- \subseteq [I_A^-, I_A^+]^-$, $[F_{G_i}^-, F_{G_i}^+]^- \subseteq [F_A^-, F_A^+]^-$ for every $i \in J$ we obtain that

$\{< x, [F_{G_i}^-, F_{G_i}^+]^+, [1 - I_{G_i}^-, 1 - I_{G_i}^+]^+, [T_{G_i}^-, T_{G_i}^+]^+, [F_{G_i}^-, F_{G_i}^+]^-, [1 - I_{G_i}^-, 1 - I_{G_i}^+]^-, [T_{G_i}^-, T_{G_i}^+]^- >: i \in J\}$

is the family of NBVS's containing \bar{A} , that is,

$$NBVcl(\bar{A}) = \{< x, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cup[1 - I_{G_i}^-, 1 - I_{G_i}^+]^+, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cup[F_{G_i}^-, F_{G_i}^+]^-, \cap[1 - I_{G_i}^-, 1 - I_{G_i}^+]^-, \cap[T_{G_i}^-, T_{G_i}^+]^- >\} \text{ (2)}$$

Hence from (1) and (2) we get $NBVcl(\bar{A})=\overline{NBVint(A)}$

(2) follows from (1).

Proposition 3.8: If (X, NBV_t) is a NBVTS and A, B be are NBVS's in X . Then the following properties hold:

1. $NBVint(A) \subseteq A$
2. $A \subseteq NBVcl(A)$
3. $A \subseteq B \Rightarrow NBVint(A) \subseteq NBVint(B)$
4. $A \subseteq B \Rightarrow NBVcl(A) \subseteq NBVcl(B)$
5. $NBVint(NBVint(A))=NBVint(A)$
6. $NBVcl(NBVcl(A))=NBVcl(A)$
7. $NBVint(A \cap B)=NBVint(A) \cap NBVint(B)$
8. $NBVcl(A \cup B)=NBVcl(A) \cup NBVcl(B)$
9. $NBVint(1)=1$
10. $NBVcl(0)=0$

Definition 3.9: Suppose (X, NBV_t) and (Y, NBV_σ) be two neutrosophic bipolar vague topological spaces and $\psi: X \rightarrow Y$ be a function. Then ψ is referred to be a neutrosophic bipolar vague continuous iff the preimage of each neutrosophic bipolar vague open set in Y is a neutrosophic bipolar vague open set in X .

Proposition 3.10: Suppose $A, \{A_i: i \in J\}$ be a neutrosophic bipolar vague set in X , and $B, \{B_j: j \in K\}$ be a neutrosophic bipolar vague set in Y , and let $\psi: X \rightarrow Y$ be a function. Then,

- (a) $A_1 \subseteq A_2 \Leftrightarrow \psi(A_1) \subseteq \psi(A_2)$
- (b) $B_1 \subseteq B_2 \Leftrightarrow \psi^{-1}(B_1) \subseteq \psi^{-1}(B_2)$
- (c) $\psi^{-1}(\cup B_i) = \cup \psi^{-1}(B_i)$ and $\psi^{-1}(\cap B_i) = \cap \psi^{-1}(B_i)$

Proof: Obvious.

Proposition 3.11: The subsequent are equivalent to each other.

1. $\psi: X \rightarrow Y$ is neutrosophic bipolar vague continuous.
2. $\psi^{-1}(NBVint(B)) \subseteq NBVint(\psi^{-1}(B))$ for each NBVOS B in Y .
3. $NBVcl(\psi^{-1}(B)) \subseteq \psi^{-1}(NBVcl(B))$ for each NBVOS B in Y .

Proof: (1) \Rightarrow (2) Given $\psi: X \rightarrow Y$ is neutrosophic bipolar vague continuous.

Then we have to show that $\psi^{-1}(\text{NBVint}(B)) \subseteq \text{NBVint}(\psi^{-1}(B))$ for each NBVOS B in Y.

Let $B = \langle x, [T_B^-, T_B^+]^+, [I_B^-, I_B^+]^+, [F_B^-, F_B^+]^+, [T_B^-, T_B^+]^-, [I_B^-, I_B^+]^-, [F_B^-, F_B^+]^- \rangle$ be NBVOS in Y.

$\text{NBVint}(B) =$

$$\{ \langle x, \cup [T_{H_i}^-, T_{H_i}^+]^+, \cap [I_{H_i}^-, I_{H_i}^+]^+, \cap [F_{H_i}^-, F_{H_i}^+]^+, \cap [T_{H_i}^-, T_{H_i}^+]^-, \cup [I_{H_i}^-, I_{H_i}^+]^-, \cup [F_{H_i}^-, F_{H_i}^+]^- \rangle : i \in I \}$$

Where,

$$[T_{H_i}^-, T_{H_i}^+]^+ \leq [T_B^-, T_B^+]^+, [I_{H_i}^-, I_{H_i}^+]^+ \geq [I_B^-, I_B^+]^+, [F_{H_i}^-, F_{H_i}^+]^+ \geq [F_B^-, F_B^+]^+, [T_{H_i}^-, T_{H_i}^+]^- \geq [T_B^-, T_B^+]^-, [I_{H_i}^-, I_{H_i}^+]^- \leq [I_B^-, I_B^+]^-, [F_{H_i}^-, F_{H_i}^+]^- \leq [F_B^-, F_B^+]^- \text{ for every } i \in I.$$

By the definition of continuity $\psi^{-1}(\text{NBVint}(B))$ is a neutrosophic bipolar vague open set in NBV_τ . Now,

$$\psi^{-1}(\text{NBVint}(B)) = \{$$

$$\psi^{-1}(\langle x, \cup [T_{H_i}^-, T_{H_i}^+]^+, \cap [I_{H_i}^-, I_{H_i}^+]^+, \cap [F_{H_i}^-, F_{H_i}^+]^+, \cap [T_{H_i}^-, T_{H_i}^+]^-, \cup [I_{H_i}^-, I_{H_i}^+]^-, \cup [F_{H_i}^-, F_{H_i}^+]^- \rangle)\}$$

$$= \{ \langle x, \psi^{-1}(\cup [T_{H_i}^-, T_{H_i}^+]^+), \psi^{-1}(\cap [I_{H_i}^-, I_{H_i}^+]^+), \psi^{-1}(\cap [F_{H_i}^-, F_{H_i}^+]^+), \psi^{-1}(\cap [T_{H_i}^-, T_{H_i}^+]^-), \psi^{-1}(\cup [I_{H_i}^-, I_{H_i}^+]^-), \psi^{-1}(\cup [F_{H_i}^-, F_{H_i}^+]^-) \rangle \}$$

$$= \{ \langle x, \cup [\psi^{-1} [T_{H_i}^-, T_{H_i}^+]^+], \cap [\psi^{-1} [I_{H_i}^-, I_{H_i}^+]^+], \cap [\psi^{-1} [F_{H_i}^-, F_{H_i}^+]^+], \cap [\psi^{-1} [T_{H_i}^-, T_{H_i}^+]^-], \cup [\psi^{-1} [I_{H_i}^-, I_{H_i}^+]^-], \cup [\psi^{-1} [F_{H_i}^-, F_{H_i}^+]^-] \rangle \}$$

$$\subseteq \text{NBVint}(\psi^{-1}(B))$$

(2) \Rightarrow (1). Given $\psi^{-1}(\text{NBVint}(B)) \subseteq \text{NBVint}(\psi^{-1}(B))$ for each NBVOS B in Y. Let

$B = \langle x, [T_B^-, T_B^+]^+, [I_B^-, I_B^+]^+, [F_B^-, F_B^+]^+, [T_B^-, T_B^+]^-, [I_B^-, I_B^+]^-, [F_B^-, F_B^+]^- \rangle$ be NBVOS in Y. We know that B is a neutrosophic bipolar vague open in Y if and only if $\text{NBVint}(B) = B$ and hence $\psi^{-1}(\text{NBVint}(B)) = \psi^{-1}(B)$.

But according to our supposition $\psi^{-1}(\text{NBVint}(B)) \subseteq \text{NBVint}(\psi^{-1}(B))$, therefore we get $\psi^{-1}(B) \subseteq \text{NBVint}(\psi^{-1}(B))$, i.e., $\psi^{-1}(B)$ is a NBVS in X and thus ψ is a neutrosophic bipolar vague continuous.

(1) \Rightarrow (3) Given $\psi: X \rightarrow Y$ is neutrosophic bipolar vague continuous.

Suppose $B = \langle x, [T_B^-, T_B^+]^+, [I_B^-, I_B^+]^+, [F_B^-, F_B^+]^+, [T_B^-, T_B^+]^-, [I_B^-, I_B^+]^-, [F_B^-, F_B^+]^- \rangle$ be NBVOS in Y.

Also suppose $\text{NBVcl}(B) =$

$$\{ \langle x, \cap [T_{K_i}^-, T_{K_i}^+]^+, \cup [I_{K_i}^-, I_{K_i}^+]^+, \cup [F_{K_i}^-, F_{K_i}^+]^+, \cup [T_{K_i}^-, T_{K_i}^+]^-, \cap [I_{K_i}^-, I_{K_i}^+]^-, \cap [F_{K_i}^-, F_{K_i}^+]^- \rangle : i \in I \}, \text{ where}$$

$$[T_{K_i}^-, T_{K_i}^+]^+ \leq [T_B^-, T_B^+]^+, [I_{K_i}^-, I_{K_i}^+]^+ \geq [I_B^-, I_B^+]^+, [F_{K_i}^-, F_{K_i}^+]^+ \geq [F_B^-, F_B^+]^+,$$

$[T_{K_i}^-, T_{K_i}^+]^- \geq [T_B^-, T_B^+]^-$, $[I_{K_i}^-, I_{K_i}^+]^- \leq [I_B^-, I_B^+]^-$, $[F_{K_i}^-, F_{K_i}^+]^- \leq [F_B^-, F_B^+]^-$ for every $i \in I$. Since ψ is a neutrosophic bipolar vague continuous iff the inverse image of each NBVCS in Y is a NBVCS in X, therefore $\psi^{-1}(\text{NBVcl}(B))$ is a NBVCS in X.

$$\text{Now, } \psi^{-1}(\text{NBVcl}(B)) = \{ \psi^{-1}(\langle x, \cap [T_{K_i}^-, T_{K_i}^+]^+, \cup [I_{K_i}^-, I_{K_i}^+]^+, \cup [F_{K_i}^-, F_{K_i}^+]^+, \cup [T_{K_i}^-, T_{K_i}^+]^-, \cap [I_{K_i}^-, I_{K_i}^+]^-, \cap [F_{K_i}^-, F_{K_i}^+]^- \rangle)\}$$

$$= \{ \langle x, \psi^{-1}(\cap [T_{K_i}^-, T_{K_i}^+]^+), \psi^{-1}(\cup [I_{K_i}^-, I_{K_i}^+]^+), \psi^{-1}(\cup [F_{K_i}^-, F_{K_i}^+]^+), \psi^{-1}(\cup [T_{K_i}^-, T_{K_i}^+]^-), \psi^{-1}(\cap [I_{K_i}^-, I_{K_i}^+]^-), \psi^{-1}(\cap [F_{K_i}^-, F_{K_i}^+]^-) \rangle \}$$

$$= \{ \langle x, \cap [\psi^{-1} [T_{K_i}^-, T_{K_i}^+]^+], \cup [\psi^{-1} [I_{K_i}^-, I_{K_i}^+]^+], \cup [\psi^{-1} [F_{K_i}^-, F_{K_i}^+]^+], \cup [\psi^{-1} [T_{K_i}^-, T_{K_i}^+]^-], \cap [\psi^{-1} [I_{K_i}^-, I_{K_i}^+]^-], \cap [\psi^{-1} [F_{K_i}^-, F_{K_i}^+]^-] \rangle \}$$

$$\supseteq \text{NBVcl}(\psi^{-1}(B))$$

(3) \Rightarrow (1)

Given $\text{NBVcl}(\psi^{-1}(B)) \subseteq \psi^{-1}(\text{NBVcl}(B))$, for each NBVOS B in Y. Let

$B = \langle x, [T_B^-, T_B^+]^+, [I_B^-, I_B^+]^+, [F_B^-, F_B^+]^+, [T_B^-, T_B^+]^-, [I_B^-, I_B^+]^-, [F_B^-, F_B^+]^- \rangle$ be NBVCS in Y. Since $\text{NBVcl}(B) = B$.

But it is given that $\text{NBVcl}(\psi^{-1}(B)) \subseteq \psi^{-1}(\text{NBVcl}(B))$, hence $\text{NBVcl}(\psi^{-1}(B)) \subseteq \psi^{-1}(B)$. Hence $\psi^{-1}(B) = \text{NBVcl}(\psi^{-1}(B))$, i.e., $\psi^{-1}(B)$ is a NBVCS in X and this proves that ψ is a neutrosophic bipolar vague continuous.

4. Generalized Neutrosophic Bipolar Vague Closed Sets:

Definition 4.1: Suppose if (X, NBV_τ) be a neutrosophic bipolar vague topological space. A neutrosophic bipolar vague set A in (X, NBV_τ) is referred to be a generalized neutrosophic bipolar vague closed set if $\text{NBVcl}(A) \subseteq G$ whenever $A \subseteq G$ and G is a neutrosophic bipolar vague open. The complement of a generalized neutrosophic bipolar vague closed set is generalized neutrosophic bipolar vague open set.

Definition 4.2: Suppose let (X, NBV_τ) be a neutrosophic bipolar vague topological space and let A be a neutrosophic bipolar vague set in X . The generalized neutrosophic bipolar vague closure (GNBVcl for short) and the generalized neutrosophic bipolar vague interior (GNBVint for short) of A are well-defined by,

- 1) $GNBVcl(A) = \bigcap \{G : G \text{ is a generalized neutrosophic bipolar vague closed sets in } X \text{ and } A \subseteq G\}$,
- 2) $GNBVint(A) = \bigcup \{G : G \text{ is a generalized neutrosophic bipolar vague open sets in } X \text{ and } A \supseteq G\}$.

Remark 4.3: Every NBVCS is generalized neutrosophic bipolar vague closed but not conversely.

Example 4.4: Assume that $X = \{u, v\}$ and $NBV_\tau = \{0, 1, F\}$ is a NBVT on X where,

$$F = \langle x, \overline{\overline{u}}_{[0.5, 0.9][0.3, 0.3][0.1, 0.5] [-0.4, -0.3] [-0.4, -0.4] [1.3, 1.4]}, \overline{\overline{v}}_{[0.4, 0.6][0.2, 0.2][0.4, 0.6] [-0.5, -0.3] [-0.5, -0.5] [1.5, 1.3]} \rangle$$

Then the neutrosophic bipolar vague set,

$$A = \langle x, \overline{\overline{u}}_{[0.5, 0.7][0.5, 0.5][0.3, 0.5] [-0.4, -0.1] [-0.5, -0.6] [1.1, 1.4]}, \overline{\overline{v}}_{[0.3, 0.6][0.4, 0.4][0.4, 0.7] [-0.2, -0.2] [-0.6, -0.8] [1.2, 1.2]} \rangle \text{ is a generalized neutrosophic bipolar vague closed but not NBVC in } X.$$

Proposition 4.5: Suppose that (X, NBV_τ) be a neutrosophic bipolar vague topological space. If A is a generalized neutrosophic bipolar vague closed set and $A \subseteq B \subseteq NBVcl(A)$, then B is a generalized neutrosophic bipolar vague closed set.

Proof: Suppose let G be a neutrosophic bipolar vague open set in (X, NBV_τ) , such that $B \subseteq G$. Since $A \subseteq B$, $A \subseteq G$. Now A is a generalized neutrosophic bipolar vague closed set and $NBVcl(A) \subseteq G$. But $NBVcl(B) \subseteq NBVcl(A)$. Since $NBVcl(B) \subseteq NBVcl(A) \subseteq G$, $NBVcl(B) \subseteq G$. Hence B is a generalized neutrosophic bipolar vague closed set.

Proposition 4.6: Suppose if A is a neutrosophic bipolar vague open set and generalized neutrosophic bipolar vague closed set in (X, NBV_τ) , then A is said to be a neutrosophic bipolar vague closed set in X .

Proof: Assume that A is a neutrosophic bipolar vague open set in X . Since $A \subseteq A$, by hypothesis $NBVcl(A) \subseteq A$. Then from definition $A \subseteq NBVcl(A)$. Therefore $NBVcl(A) = A$. Hence A is neutrosophic bipolar vague closed set in X . **Proposition 4.7:** Suppose that $NBVint(A) \subseteq B \subseteq A$ and assume A is a generalized neutrosophic bipolar vague open set then B is also a generalized neutrosophic bipolar vague open set.

Proof: Now, $\bar{A} \subseteq \bar{B} \subseteq \overline{NBVint(A)} = NBVcl(\bar{A})$. As A is a generalized neutrosophic bipolar vague open, \bar{A} is a generalized neutrosophic bipolar vague closed set. By proposition 4.5, \bar{B} generalized neutrosophic bipolar vague closed set. That is, B is also a generalized neutrosophic bipolar vague open set.

Definition 4.8: Suppose (X, NBV_τ) and (Y, NBV_σ) be any two neutrosophic bipolar vague topological spaces.

1. A map $\psi : (X, NBV_\tau) \rightarrow (Y, NBV_\sigma)$ is referred to be a generalized neutrosophic bipolar vague continuous if the inverse image of every neutrosophic bipolar vague open set in (Y, NBV_σ) is a generalized neutrosophic bipolar vague open set in (X, NBV_τ) .
2. A map $\psi : (X, NBV_\tau) \rightarrow (Y, NBV_\sigma)$ is called as a generalized neutrosophic bipolar vague irresolute if the inverse image of every generalized neutrosophic bipolar vague open set in (Y, NBV_σ) is a generalized neutrosophic bipolar vague open set in (X, NBV_τ) .

Proposition 4.9: Suppose (X, NBV_τ) and (Y, NBV_σ) be any two neutrosophic bipolar vague topological spaces. A mapping $\psi : (X, NBV_\tau) \rightarrow (Y, NBV_\sigma)$ is referred to be generalized neutrosophic bipolar vague continuous function mapping. Then for every neutrosophic bipolar vague set A in X , $\psi(GNBVcl(A)) \subseteq NBVcl(\psi(A))$.

Proof: Assume A to be a neutrosophic bipolar vague set in (X, NBV_τ) . Since $NBVcl(\psi(A))$ is a neutrosophic bipolar vague closed set and since ψ is a generalized neutrosophic bipolar vague continuous mapping, the set $\psi^{-1}(NBVcl(\psi(A)))$ is a generalized neutrosophic bipolar vague closed set and thus $\psi^{-1}(NBVcl(\psi(A))) \supseteq A$.

Now, $GNBVcl(A) \subseteq \psi^{-1}(NBVcl(\psi(A)))$. Therefore $\psi(GNBVcl(A)) \subseteq NBVcl(\psi(A))$.

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Received: June 03, 2019. Accepted: October 17, 2019