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Special Elements In Semigroups

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Abstract

This paper defines the notion of special elements like Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents for semigroups. This is the first time such study has been carried out on semigroups.

Key words: Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents.

1 Introduction

This paper has three sections. Section one is introductory in nature. Section two defines the notion of special elements like Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents for semigroups. The conclusions are given in the final section.

2 Special Elements in Semigroups :

For the first time this paper defines the notion of special elements like Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents for semigroups whenever applicable. These concepts are introduced and studied in case of rings and semirings[3,4]. These concepts are illustrated by examples. Conditions for these elements to exist in a semigroup is determined.

Definition 2.1: Let S be a semigroup with unit and zero divisors. x, y S is said to be a Smarandache zero divisor (S-zero divisor) if $x \cdot y = 0$ and there exists $a, b \in S \setminus \{x, y, 0\}$ with

1) $xa = 0$ or $ax = 0$,
2) $yb = 0$ or $by = 0$ and
3) $ab \neq 0$ or $ba \neq 0$.

Examples of S-zero divisors are given only in case of $S = \{Z_n,\times\}$ for $S(n)$ the symmetric semigroup has no zero divisors[1,2].

Example 2.1: Let $S = \{Z_{20}, \times\}$ be the semigroup.

10, 16 $\in S$ are zero divisors as $10 \times 16 = 0$ (mod 20) and is also a S-zero divisor for 5, 6 $\in Z_{20} \setminus \{0, 10, 16\}$ is such that

$5 \times 16 = 0$ (mod 20), $6 \times 10 = 0$ (mod 20) and $6 \times 5 \neq 0$ (mod 20).

It is important to note all semigroups built using $\{Z_n, \times\}$, n a composite number has zero divisors but it need not in general be S-zero divisors.

Example 2.2: Let $S = Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the semigroup under $\times$. $2, 5 \in Z_{10}$ is such that $2 \times 5 = 0 \pmod{10}$ is a zero divisor and is not a S-zero divisor.

In view of this the following result is true:

Proposition 2.1: Let S be a semigroup. Every S-zero divisor is a zero divisor but a zero divisor in general is not a S-zero divisor.

Proof: One way is evident from the definition of a S-zero divisor. Example 2.2 proves the other part of the result.

Consider *S(n);* this is a semigroup which has no zero divisors; so S-zero divisor has no relevance to this semigroup *S(n)*.

Next the notion of S-units is defined for semigroups.

Definition 2.2: Let S be a semigroup with unit (monoid). $x \in S \setminus \{1\}$ *is defined as the Smarandache unit (S-unit) if there exists* $y \in S$ *with*

1) $xy = 1$ *there exist* $a, b \in S \setminus \{x, y, 1\}$.
2) i) $xa = y$ *or* $ax = y$ *or*
   ii) $yb = x$ *or* $by = x$ *and*
   iii) $ab = 1$.

(2(i) or 2(ii) is satisfied it is enough to make a S-unit).

This is represented by the following examples:

Example 2.3: Let $S = \{Z_{15}, \times\}$ be the semigroup.

Now $2 \in Z_{15}$; $2.8 = 1 \pmod{15}$. Consider $4 \in Z_{15}$; $4^2 \equiv 1$ and $2.4 = 8$. Thus (2, 8) is a S-unit of the semigroup S.

Proposition 2.2: Every S-unit in a semigroup S is a unit. However all units in general are not S-units in S.

Proof: Consider $4 \in Z_{15}$ in the above example 2.3 which is a unit in Z_{15}; but 4 is not a S-unit for in this case $x = y = 4$. $4a \equiv 4$ or $4b \equiv 4$ with $a \cdot b = 1$.

In view of this, as in case of S-units in a ring[3] the following result is proved for semigroups.

Theorem 2.1: Let S be a monoid. If $x \in S \setminus \{1\}$ *is a S-unit;* $xy = 1$ *then* $x \neq y$.

Proof: The proof is similar to rings. Let $x \in S \setminus \{0\}$ be a S-unit, this implies $xy = 1$ with $xa = y$ or $ax = y$ ($by = x$ or $yb = x$) and $ab = 1$ if $x = y$ then $x^2 = 1$; $xa = x$; $x^2 a = x^2$ forcing $a = 1$; as $x^2 = 1$ a contradiction.

Now for the first time the notion of S-idempotents in rings is adopted to semigroups in this paper.

Definition 2.3: Let S be a semigroup. $x \in S \setminus \{0, 1\}$ *is defined as a Smarandache idempotent of S if* $x^2 = x$ *and there exist* $y \in S \setminus \{0, 1, x\}$ *such that* $y^2 = x$ *and* $yx = x$ *or* $xy = y$. *y is defined as the Smarandache coidempotent (S coidempotent) and the pair is denoted by (x, y).*

Example 2.4: Let $S = \{Z_{12}, \times\}$ be the semigroup. $4 \in S$ is such that $4^2 = 4 \pmod 8$. $8^2 = 4$ and $8 \times 4 = 8$ so 4 is a S-idempotent. Clearly if x is an idempotent.

But every idempotent in a semigroup need not be a S-idempotent.

Example 2.5: Let $S(4)$ be the symmetric semigroup. $S(4)$ has no zero divisors but has units and idempotents.

Here the study pertains to finding S-idempotents and S-units if any in $S(4)$.

Take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in S(4);$$

clearly $x^2 = x$; let

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} \in S(4).$$

$$y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x$$

and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = x.$

Thus x is an S-idempotent of $S(4)$. Thus the symmetric semigroup $S(4)$ has S-idempotents.

Take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \in S(4).$$

$$x \times y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1.$$

Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \in S(4);$$

$$x \times a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = y.$$

Now

$$a \cdot a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1.$$

Thus x is a S-unit of $S(4)$. Hence $S(4)$ has both S-idempotents and S-units. However as $S(4)$ has no zero divisors. $S(4)$ cannot have S-zero divisors as every S-zero divisor is a zero divisor.

In view of these the following result is proved.

Theorem 2.2: Let S(n) be the symmetric semigroup of degree n.

i) S(n) has no S-zero divisors,
ii) S(n) has S-units and
iii) S(n) has S-idempotents.

Proof : Since $S(n)$ is the symmetric semigroup of degree n and has no zero divisors. Since every S-zero divisor is a zero divisor hence $S(n)$ cannot have S-zero divisors. $S(n)$ has S-units. For take

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix} \in S(n).$$

$$x \circ y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 2 & 3 & 4 & 5 & \dots & n \end{pmatrix} = 1;$$

the identity element of $S(n)$.

Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix} \in S(n)$$

$$x \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 2 & 3 & 4 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 4 & 1 & 2 & 3 & 5 & \dots & n \end{pmatrix} = y$$

and

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 3 & 4 & 1 & 2 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 2 & 3 & 4 & 5 & \dots & n \end{pmatrix} = 1 \in S(n).$$

Thus x is a S-unit of $S(n)$. Hence (ii) is true.

Now to prove S has S-idempotents.

Let

$$x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n). \; x_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1.$$

Hence x_1- is an idempotent of $S(n)$. Take

$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 3 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n). \; y_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1$$

$$y_1 \cdot x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 3 & 1 & 1 & 5 & \dots & n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1.$$

Thus x_1 is an S-idempotent of $S(n)$ hence (iii) is proved.

The next natural question would be; will the co-idempotents in $S(n)$ be unique. The answer is no.

This is proved by the following result:

Proposition 2.3: Let S(n) be the symmetric semigroup of degree n; the S-coidempotents of an S-idempotent in S(n) in general are not unique.

Proof: The result is proved by a counter example.

$$x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n)$$

is an S-idempotent of $S(n)$. The S-coidempotent of x_1 is

$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 3 & 1 & 1 & 5 & \dots & n \end{pmatrix} \text{ in } S(n).$$

Consider

$$y_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 4 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n). \; y_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} \in S(n)$$

and

$$y_2 \cdot x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 4 & 1 & 1 & 5 & \dots & n \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 1 & 1 & 5 & \dots & n \end{pmatrix} = x_1 \in S(n).$$

Thus y_2 is also a S-coidempotent of x_1 in $S(n)$. The coidempotents in general in $S(n)$ for a given S-idempotent is not unique.

However the notion of semi idempotents and S-semi idempotents in case of rings has no relevance to semigroups of finite order under the product operation.

Next the notion of nilpotent elements and S-nilpotent elements are defined in case of semigroups. At the outset it is clear that only semigroups which has zero divisors can have nilpotent elements. Hence the symmetric semigroup $S(n)$ has no zero divisors so has no nilpotents.

Thus the only class of finite non abstract semigroups which has zero divisors is the class of semigroups $S = \{Z_n, \times\}$; n not a prime number.

Definition 2.4: Let S be a semigroup under product with zero divisors. $x \in S \setminus \{0\}$ *is said to be a Smarandache nilpotent element if* $x^n = 0$ *and there exists a* $y \in S \setminus \{0, x\}$ *such that* $x^r y = 0$ *or* $yx^s = 0$, *r, s,* > 0 *and* $y^m \neq 0$ *for any integer* $m > 1$.

First this situation will be described by some examples.

Example 2.6: Let $S = \{Z_{12}, \times\}$ be the semigroup. Clearly $6^2 = 0 \pmod{12}$; $8 \in S$ is such that $6 \times 8 \equiv 0 \pmod{12}$ but $8^m \neq 0 \pmod{12}$ for $m > 1$ as $8^3 \equiv 8 \pmod{12}$. Thus 6 is a S-nilpotent element of *S*.

Example 2.7: Let $S = \{Z_8, \times\}$ be the semigroup. *S* has nilpotents but none of them are S-nilpotents of *S*.

For $2^3 \equiv 0 \pmod 8$; $4^2 \equiv 0 \pmod 8$. There are no S-nilpotents in *S*.

In view of this one has the following result:

Proposition 2.4: Let $\{S, \times\}$ *be a semigroup with nilpotents.*

i) *Every S-nilpotent element of S is a nilpotent element of S.*
ii) *If x is a nilpotent element of S, x need not in general be S-nilpotent.*

Proof: Proof of (i) follows from the very definition of the S-nilpotent element of *S*. Proof of (ii) follows from the above example 2.7 for $2 \in S = \{Z_8, \times\}$ is a nilpotent element of *S* but 2 is not a S-nilpotent of *S*.

Example 2.8: Let $S = \{Z_{27}, \times\}$ be the semigrouop. 3 is a nilpotent element of *S*. 6 is a nilpotent element of *S*. 12 is a nilpotent element of *S*. But *S* has no S-nilpotent elements.

In view of this the following interesting result is proved:

Theorem 2.3: Let $S = \{Z_{p^n}, \times\}$ *where p is a prime n 2; S has no S-nilpotent elements.*

Proof: $x \in S$ is a nilpotent element if and only if p / x and $x^n = (0)$. Further $x^t y = 0$ if and only if p^{n-t} / y and hence $y^m = 0$ for some *m*. Hence it is not possible to find a *y* such that $y^m \neq 0$ and $x^t y = 0$. Hence the claim.

Corollary 2.1: Let $S = \{Z_{p^n}, \times\}$, *p a prime; be a semigroup. Then the nilpotent elements of S are p, 2p, 3p,…,* $(p^{n-1} - 1)p$). *That is there are* $(p^{n-1} - 1)$ *number of nilpotents.*

Proof: Follows from simple number theoretic argument.

This is illustrated by an example.

Example 2.9: Let $S = \{Z_{3^5} = Z_{243}, \times\}$ be a semigroup. The nilpotent elements of *S* are 3, 6, 9, 12, 15, 18, 21, 24, …, 240 = $(3^4 - 1)3$. Thus there are $3^4 - 1$ number of nilpotents in *S* none of them are S-nilpotents of *S*.

Example 2.10: Let $S = \{Z_{5^{10}}, \times\}$ be the semigroup. *S* has $(5^9 - 1)$ number of nilpotents; none of them are S-nilpotents of *S*.

Thus there exists a class of semigroups

which has only nilpotent elements and none of them are S-nilpotents. In fact this class has infinite number of finite semigroups of the form $S = \{Z_{p^n}, \times\}$ where $2 \leq n < \infty$ and p any prime. So for a fixed prime; one has infinite number of such semigroups. Further for the number of primes is also infinite so this class of semigroups has undoubtedly infinite cardinality.

3. Conclusion

In this paper for the first time special elements like Smarandache zero divisors, Smarandache units, Smarandache idempotents and Smarandache nilpotnents are introduced These special elements help in studying the properties and in characterization of these semigroups.

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